V

LACUNARY SERIES[1]

20. Consider an increasing sequence of positive integers $\{\lambda_n\}$. Denote by $\{\lambda'_n\}$ the increasing sequence consisting of the positive integers not contained in $\{\lambda_n\}$. Both sequences are assumed to be infinite. The zero coefficients in the series

$$\sum a_{\lambda_n} x^{\lambda_n} = \sum c_m x^m, \tag{1}$$

where

$$c_m = \begin{cases} a_{\lambda_n} & \text{when } m = \lambda_n, \\ 0 & \text{" } \quad m = \lambda'_n, \end{cases}$$

are called *lacunae*, and a series of the form (1) is a *lacunary series*. Weierstrass first, and later Fredholm, gave examples of such series, in which the circle of convergence is a cut.

It is natural to inquire whether the presence of an infinity of lacunae, distributed according to a definite law, is a characteristic property which may supply information relative to the singularities of the function represented by the series.

The number and, to some extent, the types of singular points of a function are invariant with respect to differentiation and integration. Any property of the coefficients from which conclusions may be reached relative to singularities should therefore be of an invariant character with respect to these operations.

The simplest of such properties is given by the lacunae and their distribution (to within a translation).

[1] Mandelbrojt: Ann. de l'École Norm. Sup., t. 20 (1923), p. 413.

The following general theorem concerning lacunary series is due to Hadamard.[1]

THEOREM 1: *If, for the series* (1), *we have*

$$\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} > 1 + \delta, \tag{3}$$

where $\delta > 0$ *is arbitrary, the circle of convergence is a cut.*

Fabry has shown that the conclusion holds if (3) is replaced by

$$\lim_{n\to\infty} (\lambda_{n+1} - \lambda_n) = \infty, \tag{4}$$

a condition which is implied by (3). These theorems will be considered in Chapter XII. In this chapter we shall study lacunary series from a different point of view.

21. THEOREM 2: *If*

$$\overline{\lim_{n\to\infty}} (\lambda_{n+1} - \lambda_n) = \infty \tag{5}$$

the series (1) *has on the circle of convergence at least one singularity which is not a pole.*

LEMMA: *If*

$$\overline{\lim_{n\to\infty}} (\lambda_{n+1} - \lambda_n) > K, \tag{6}$$

where K is a positive integer, and if the series (1) *has only poles on the circle of convergence, the number of poles is at least* $K + 1$.

The requirement (6) is equivalent to the statement that we can extract a partial sequence $\{\lambda_{n_i}\}$, $i = 1, 2, \cdots$, such that

$$\lambda_{n_i+1} - \lambda_{n_i} > K, \quad i = 1, 2, \cdots.$$

Suppose the lemma false, *i.e.*, we assume that the series, having only poles on the circle of convergence, has

[1] *loc. cit.*, p. 116.

$K' < K + 1$ poles. Then, from Theorem 1, Chapter IV, we have

$$\overline{\lim_{r\to\infty}}\sqrt[r]{|D_{r,K'}|} < \frac{1}{R^{K'+1}}, \tag{7}$$

where

$$D_{r,K'} = \begin{vmatrix} c_r & c_{r+1} & \cdots & c_{r+K'} \\ \cdots & \cdots & \cdots & \cdots \\ c_{r+K'} & \cdots & \cdots & c_{r+2K'} \end{vmatrix}$$

Hence

$$\overline{\lim_{r\to\infty}}\sqrt[r]{|D_{r,o}|} = \overline{\lim_{r\to\infty}}\sqrt[r]{|c_r|} = \frac{1}{R}. \tag{8}$$

Let K'' denote the largest integer such that, for $p = 0, 1, 2, \cdots, K''$, we have

$$\overline{\lim_{r\to\infty}}\sqrt[r]{|D_{r,p}|} = \frac{1}{R^{p+1}}.$$

The existence of K'' follows from (7) and (8); moreover, $0 \leq K'' < K'$. And since

$$\overline{\lim_{r\to\infty}}\sqrt[r]{|D_{r,K''+1}|} < \frac{1}{R^{K''+2}},$$

we have, by Theorem 2, Chapter IV,

$$\lim_{r\to\infty}\sqrt[r]{|D_{r,K''}|} = \frac{1}{R^{K''+1}}, \quad K'' + 1 \leq K',$$

which is a contradiction. For in the determinant

$$D_{\lambda_{n_i}+1,K''} = \begin{vmatrix} c_{\lambda_{n_i}+1} & \cdots & c_{\lambda_{n_i}+K''+1} \\ \cdots & \cdots & \cdots \\ c_{\lambda_{n_i}+K''+1} & \cdots & c_{\lambda_{n_i}+1+2K''} \end{vmatrix},$$

all the elements of the first row belong to the sequence $\{\lambda'_n\}$, since

$$\lambda_{n_i+1} - \lambda_{n_i} > K > K'' + 1.$$

Hence $D_{\lambda_{n_i}+1,K''} = 0$ for all i.

We may now proceed with the proof of Theorem 2. If a function has only poles as singularities on the circle of convergence, there must be a finite number of them. Since there is a subsequence $\{\lambda_{n_i}\}$ such that

$$\lim_{i\to\infty} (\lambda_{n_i+1} - \lambda_{n_i}) = \infty,$$

it follows that, for i sufficiently large,

$$\lambda_{n_i+1} - \lambda_{n_i} > K, \text{ arbitrary.}$$

As we have just seen, the function must therefore have, for each K, at least $K+1$ poles, and consequently the poles are infinite in number. This being impossible, there is at least one singularity other than a pole on the circle of convergence.

22. THEOREM 3: *It is possible to construct a series* $\Sigma a_{\lambda_n} x^{\lambda_n}$, *for which*

$$\overline{\lim_{n\to\infty}} (\lambda_{n+1} - \lambda_n) = \infty, \tag{5}$$

and which has only one singularity in the entire plane.

In order to prove this, we state without proof two theorems on which the demonstration is based.

THEOREM 4 (Leau): *Given an integral function* $g(z)$ *of order* $\leq \delta < 1$, i.e., $|g(z)| < e^{r^{\delta+\epsilon}}$ *for* $r > r_\epsilon$, $r = |z|$; *then the function*

$$f(x) = \Sigma g(n) x^n$$

has the point 1 *as its only singularity, and is accordingly regular at infinity.*[1]

THEOREM 5: *Given a sequence* $\{\alpha_n\}$ *such that* $\sum \frac{1}{|\alpha_n|^{\delta+\epsilon}}$ *converges. It is possible to construct an integral function of order* $\leq \delta$, *having the* α_n *as zeros, and no others.*[2]

[1] See Chapter VIII. [2] Borel: *Fonctions entières*, 2d ed. (1921), p. 56.

Proof of Theorem 3. Let s be chosen so that $\frac{1}{2} < s < 1$. Consider the sequence of series

$$\sum_{n=0}^{\infty} \frac{1}{(n^2+m)^s}, \quad m = 0, 1, 2, \cdots \tag{9}$$

Each of these series, being dominated by the series $\sum \frac{1}{n^{2s}}$, is convergent.

Let

$$\sum_{n=0}^{\infty} \epsilon_n = I$$

be a convergent series of positive terms. For each m, choose n_m such that

$$\sum_{n=n_m}^{\infty} \frac{1}{(n^2+m)^s} < \epsilon_m.$$

Then

$$\sum_{m=0}^{\infty} \left[\sum_{n=n_m}^{\infty} \frac{1}{(n^2+m)^s} \right] < I.$$

Let $u_{n,m} = n^2 + m$, $n \geq n_m$. The series

$$\sum_{m=0}^{\infty} \sum_{n=n_m}^{\infty} \frac{1}{u_{n,m}^s}$$

is absolutely convergent. Sum this series by taking the terms in the order of decreasing magnitude: $\frac{1}{U_1}, \frac{1}{U_2}, \cdots$ (In case two or more terms are equal, we remove all except one of them.) Then the series

$$\sum_{k=1}^{\infty} \frac{1}{U_k^s}$$

converges. By Theorem 5, we can construct an integral function $g(z)$, of order $\leq s$, having the U_k as zeros, and no others. Hence, by Theorem 4, the series

$$\sum_{n=0}^{\infty} g(n)x^n$$

has the point 1 as its only singularity. We write

$$\sum_{n=0}^{\infty} g(n)x^n = \sum_{n=0}^{\infty} b_{\lambda_n} x^{\lambda_n}, \tag{10}$$

and show that the sequence $\{\lambda_n\}$ has the property (5).

Let i be an arbitrary integer. Corresponding to each i choose an integer p_i such that

$$p_i > n_0, n_1, \cdots, n_i.$$

The zeros U_k, being of the form $n^2 + m$, $n \geq n_m$, are given by

$$p_i^2, p_i^2 + 1, \cdots, p_i^2 + i.$$

For if $m \leq i$, n takes on the values n_m, $n_m + 1, \cdots$, thus including $n = p_i$. Hence, for $i = 0, 1, 2, \cdots$, we have

$$\left.\begin{array}{l} g(p_0^2) = 0 \\ g(p_1^2) = 0, \quad g(p_1^2 + 1) = 0 \\ \quad \cdots\cdot \qquad\qquad \cdots\cdot \\ g(p_i^2) = 0, \quad g(p_i^2 + 1) = 0, \cdots, g(p_i^2 + i) = 0 \end{array}\right\} \tag{11}$$

The integers in (11) are the λ'_n of the series (10). Select a partial sequence $\{\lambda_{n_i}\}$ as follows: Let $\lambda_{n_1} = p_0^2 - 1$, $\lambda_{n_2} = p_1^2 - 1$, $\lambda_{n_2+1} = p_1^2 + 2$, and so on; let λ_{n_i} be the last subscript before p_i^2 which is not a zero of $g(n)$ and let λ_{n_i+1} be the first subscript after $p_i^2 + i$ which is not a zero of $g(n)$.

With the λ_{n_i} so chosen, we have

$$\lim_{i\to\infty} (\lambda_{n_i+1} - \lambda_{n_i}) = \infty .$$

In the case just considered, since the singularity is not a pole, and is unique, it cannot be a branch point, and must be an essential singularity.

Faber[1] has given an example of a series for which

$$\overline{\lim_{n\to\infty}} \frac{\lambda_n}{n} = \infty,$$

and the function defined by the series has only the point 1 on the circle of convergence as a singularity, but has the lemniscate

$$|x(x-1)| = 2$$

as a cut.

The theorem just proved shows that properties (4) and (5) are entirely distinct, since a consequence of (4) is that the circle of convergence is a cut. We also note that the condition (5) gives no information relative either to the number or the position of the singularities of a function. In fact, if k is an arbitrary positive integer, we can form a series $\Sigma a_n x^n$ having exactly k singularities, arbitrarily situated, in the entire plane. For if we let

$$a_n = g(n)\left(\frac{1}{\beta_1^n} + \frac{1}{\beta_2^n} + \cdots + \frac{1}{\beta_k^n}\right),$$

where the $g(n)$ is as before, and the β_i are arbitrary, but distinct, then the function defined by

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} c_{\lambda n} x^{\lambda n}$$

has $\beta_1, \beta_2, \cdots, \beta_k$ as essential singularities. On the other hand, the sequence $\{\lambda_n\}$ satisfies (5), since $\Sigma a_n x^n$ is the sum of k series, all of which have the same lacunae.

As Montel[2] has remarked, the author's theorems show that the "magnitude" of the lacunae give information

[1] Sitzungsberichte de l'Ac. de Bavière, t. 36 (1906).

[2] Comptes Rendus, 27 mai, 1925.

about the nature of the singularities, whereas their arithmetical distribution affects the number of singularities. By the *magnitude* of a lacuna of subscript n_i is meant the integer k such that

$$a_{\lambda_{n_i}+1} = 0,\ a_{\lambda_{n_i}+2} = 0, \cdots,\ a_{\lambda_{n_i}+k} = 0,$$

$$a_{\lambda_{n_i}} \neq 0,\ a_{\lambda_{n_i}+k+1} \neq 0.$$

23. The following theorem is a generalization of one of the author's theorems. It is a theorem of Ostrowski [1]; a part of the proof is due to Tchebotareff.

THEOREM 6: *If the series* (1) *has the property that*

$$\overline{\lim_{n\to\infty}}\,(\lambda_{n+1} - 2^p\lambda_n) = \infty, \tag{12}$$

where p is a positive integer, or zero, then the function defined by the series cannot be represented as

$$\frac{\phi(x)}{[P(x)]^{\frac{1}{p+1}}},$$

where P is a polynomial, and the radius of convergence of the series for $\phi(x)$ exceeds that of (1).

If $p = 0$, the theorem states that

$$f(x) \neq \frac{\phi(x)}{P(x)},$$

which is Theorem 2; the function $f(x)$ cannot have poles exclusively on its circle of convergence.

By hypothesis there exists a partial sequence $\{\lambda_{n_i}\}$ for which

$$\lim_{i\to\infty}\,(\lambda_{n_i+1} - 2^p\lambda_{n_i}) = \infty. \tag{13}$$

[1] Jahresbuch der deutschen Math.-Verein., Bd. 35 (1926), 9–21 Heft, p. 269.

Suppose the theorem false. If $P(x)$ has zeros elsewhere as well as on the circle of convergence of (1) we can write

$$f(x) = \frac{\phi(x)}{[P(x)]^{\frac{1}{p+1}}} = \frac{\phi_1(x)}{[P_1(x)]^{\frac{1}{p+1}}},$$

where the radius of convergence of the series for $\phi_1(x)$ exceeds that of (1), and where the roots of $P_1(x)$ are situated only on the circle of convergence of (1).

Case 1. We assume that there is at least one zero, α, of P for which $\phi_1(\alpha) \neq 0$.

LEMMA: *Given*

$$f_1(x) = \sum_{n=0}^{\infty} a'_{\lambda_n} x^{\lambda_n} = \sum_{n=0}^{\infty} c_n x^n,$$

$$f_2(x) = \sum_{n=0}^{\infty} b_{\lambda_n} x^{\lambda_n} = \sum_{n=0}^{\infty} d_n x^n,$$

both series satisfying the requirement (13). *Then*

$$f_1(x) f_2(x) = \sum_{n=0}^{\infty} l_{\mu_n} x^{\mu_n} = \sum_{n=0}^{\infty} k_n x^n,$$

where

$$\varlimsup_{n\to\infty} (\mu_{n+1} - 2^{p-1}\mu_n) = \infty.$$

As before, let $\{\lambda'_n\}$ be the sequence complementary to $\{\lambda_n\}$. Then $c_{\lambda'_n} = d_{\lambda'_n} = 0$. Let $\{m_n\}$ denote an increasing sequence of positive integers. From (13), we have, for each m_j,

$$\lambda_{n_i+1} > 2^p \lambda_{n_i} + m_j, \quad i > i_j.$$

Then every integer l such that $\lambda_{n_i} < l \leq 2^p \lambda_{n_i} + m_j$ belongs to $\{\lambda'_n\}$. Hence

$$c_{\lambda_{n_i}+1} = 0, c_{\lambda_{n_i}+2} = 0, \cdots, c_{2\lambda_{n_i}} = 0, c_{2\lambda_{n_i}+1} = 0, c_{2^p\lambda_{n_i}+m_j} = 0, \quad (14)$$

with a similar set of equations for the d's.

Now
$$k_n = c_n d_0 + c_{n-1} d_1 + \cdots + c_0 d_n,$$
which may be written
$$k_n = c_n d_0 + c_{n-1} d_1 + \cdots + c_{\frac{n}{2}} d_{\frac{n}{2}} + \cdots + c_0 d_n \qquad (15)$$
when n is even, and
$$k_n = c_n d_0 + c_{n-1} d_1 + \cdots + c_{\left[\frac{n}{2}\right]} d_{\left(\frac{n}{2}\right)} + c_{\left(\frac{n}{2}\right)} d_{\left[\frac{n}{2}\right]} + \cdots + c_0 d_n \qquad (16)$$
when n is odd. The symbol $\left(\frac{n}{2}\right)$ denotes the largest integer less than $\frac{n}{2}$, and $\left[\frac{n}{2}\right]$ the smallest integer greater than $\frac{n}{2}$.

From (15) and (16), it follows that if we place, consecutively,
$$\left.\begin{aligned} n &= 2\,\lambda_{n_i} + 1, \\ &\cdots\cdots \\ n &= 2^p \lambda_{n_i} + m_j, \end{aligned}\right\} i > i_j, \qquad (17)$$
all the corresponding k_n vanish. For in (15) all the c_q preceding and including $c_{\frac{n}{2}}$ vanish, by (14), and in the remaining terms the d_r vanish. Similarly for (16). Hence all the integers
$$2\,\lambda_{n_i} + 1,\; 2\,\lambda_{n_i} + 2, \cdots, 2^p \lambda_{n_i} + m_j$$
belong to the sequence $\{\mu'_n\}$ complementary to $\{\mu_n\}$.

If we let
$$2\,\lambda_{n_i} = \mu_{n_i},\; 2^p \lambda_{n_i} + m_j + 1 = \mu_{n_i+1},$$
we have
$$2^p \lambda_{n_i} = 2^{p-1} \mu_{n_i},\; 2^{p-1} \mu_{n_i} + m_j + 1 = \mu_{n_i+1},$$
and consequently
$$\mu_{n_i+1} - 2^{p-1} \mu_{n_i} > m_j.$$

Since this inequality holds for all i, we have the conclusion stated in the lemma.

Returning to the proof of the theorem, we have, from the lemma,

$$[f(x)]^2 = \left[\sum a_{\lambda_n} x^{\lambda_n}\right]\left[\sum a_{\lambda_n} x^{\lambda_n}\right] = \sum a''_{\mu_n} x^{\mu_n},$$

where it is possible to find a partial sequence $\{\mu_{n_i}\}$ such that

$$\lim_{i\to\infty} (\mu_{n_i+1} - 2^{p-1}\mu_{n_i}) = \infty .$$

Denote by v'_n the sequences

$$\left.\begin{array}{c} 2\lambda_{n_i} + 1 \\ \cdots\cdots \\ 2^p\lambda_{n_i} + m_j \end{array}\right\} i > i_j,\ j = 1, 2, \cdots,$$

and by v'_n the complementary sequences. Then, by (14) and (17), we may write

$$f(x) = \sum a'_{v_n} x^{v_n},\ [f(x)]^2 = \sum a''_{v_n} x^{v_n}.$$

Now we can find a partial sequence $\{v_{n_i}\}$ such that

$$\lim_{i\to\infty} (v_{n_i+1} - v_{n_i}) = \infty .$$

If we replace p by $p - 1$ in the lemma, we obtain

$$\begin{aligned}[f(x)]^3 = f(x)[f(x)]^2 &= \left[\sum a'_{v_n} x^{v_n}\right][a''_{v_n} x^{v_n}] \\ &= \sum a'''_{K_n} x^{K_n}\end{aligned}$$

where we can form a partial sequence $\{K_{n_i}\}$ such that

$$\lim_{i\to\infty} (K_{n_i+1} - K_{n_i}) = \infty .$$

Proceeding in this way we obtain, finally,

$$[f(x)]^{p+1} = \sum a^{(p+1)}_{\pi_n} x^{\pi_n},$$

where there exists a partial sequence $\{\pi_{n_i}\}$ for which

$$\lim_{i\to\infty} (\pi_{n_i+1} - \pi_{n_i}) = \infty .$$

We have consequently arrived at a contradiction. For we have assumed that

$$[f(x)]^{p+1} = \frac{[\phi_1(x)]^{p+1}}{P_1(x)} = \Phi(x),$$

and since $\phi_1(\alpha) \neq 0$, the function $\Phi(x)$ has the point α as a pole on the circle of convergence. Moreover, $\Phi(x)$ has only poles as singularities. This, however, we have just seen to be impossible, and the theorem is proved.

Case 2. To show that $f(x)$ cannot be represented in the form

$$\frac{P_1(x)\ \phi_2(x)}{[P_1(x)]^{\frac{1}{p+1}}},$$

where the radius of convergence of the series for $\phi_2(x)$ exceeds that for $f(x)$. We now admit multiple zeros of $P_1(x)$, but require that there exist at least one *simple* zero α of $P_1(x)$ which is not a zero of $\phi_2(x)$.

Unless the theorem is true, we have

$$f(x) = [P_1(x)]^r \phi_2(x), \qquad r = \frac{p}{p+1},$$

$$f'(x) = r\,[P_1(x)^{r-1} P_1'(x)\,\phi_2(x) + [P_1(x)]^r \phi_2'(x)$$

$$= \frac{rP_1'(x)\,\phi_2(x) + P_1(x)\,\phi_2'(x)}{[P_1(x)]^{1-r}}$$

$$= \frac{\phi_3(x)}{[P_1(x)]^{\frac{1}{p+1}}},$$

where $\phi_3(x)$ has the same properties as the function $\phi_1(x)$ in Case 1. For, by hypothesis, α is not a zero of $P_1'(x)\,\phi_2(x)$, hence not of $\phi_3(x)$.

On the other hand, the derivative series has lacunae which are characterized by (12).

Case 3. The function $f(x)$ cannot be written in the form

$$\frac{(x-\alpha_i)^{k'_i}\phi_2(x)}{[P_1(x)]^{\frac{1}{p+1}}},$$

i.e., assuming that $\phi_1(x) = (x-\alpha_i)^{k'_i}\phi_2(x)$, where $\phi_2(\alpha_i) \neq 0$, k'_i is an integer, and α_i is a zero of $P_1(x)$ of order $k_i > k'_i$.

Let $P_1(x) = A(x-\alpha_1)^{k_1}\cdots(x-\alpha_i)^{k_i}\cdots(x-\alpha_i)^{k_r}$, and suppose

$$f(x) = \frac{(x-\alpha_i)^{k'_i}\phi_2(x)}{(x-\alpha_i)^{\frac{k_i}{p+1}}[P_2(x)]^{\frac{1}{p+1}}}$$

$$= (x-\alpha_i)^r \frac{\phi_2(x)}{[P_2(x)]^{\frac{1}{p+1}}}, \quad r = \frac{k_i'(p+1)-k_i}{p+1}.$$

The number r may be taken as positive; otherwise we have Case 1. Suppose for the present that $r < 1$. Then

$$f'(x) = (x-\alpha_i)^{-(1-r)}\frac{\phi_2(x)}{[P_2(x)]^{\frac{1}{p+1}}} + (x-\alpha_i)^r \frac{d}{dx}\frac{\phi_2(x)}{[P_2(x)]^{\frac{1}{p+1}}}$$

$$= \frac{\phi_4(x)}{(x-\alpha_i)^{1-r}[P_2(x)]^{\frac{1}{p+1}}},$$

which reverts to Case 1 by virtue of the statement at the end of Case 2. If $r > 1$, we need only consider $f^{(n+1)}(x)$ instead of $f'(x)$.

By combining these three cases, we have the general theorem.

Tsuji[1] has generalized the lemma, Theorem 2 and Theorem 6, in that "pole" is replaced by "algebraic singularity."

[1] Japanese Journal of Mathematics, vol. iii, no. 2 (1926), p. 69. Generalizations of Theorems 2 and 6 have also been given by Polya, C. R., t. 184 (1921), p. 502, who proved that a series having the property (5) cannot have singularities of the same kind as those of a linear differential equation of Fuchsian type.

For other proofs and different generalizations of the lemma, Theorem 2 and Theorem 6, see Ostrowski, Jahresbuch der deutschen Math.-Vereinigung, Bd. 35 (1926), 9–12 Heft, p. 269, and Obrechkoff, Comptes Rendus, t. 184 (1927), p. 271.

24. THEOREM 7: *If, for the series* (1),

$$\overline{\lim_{n\to\infty}} \frac{\lambda_{n+1}}{\lambda_n} = \infty, \tag{18}$$

the function defined by the series has as singularities only unbounded continua.

We state, without proof, two theorems on which the proof is based. The first is due to Ostrowski, the second to Weierstrass.

THEOREM 8:[1] *If the series* (1) *has the property* (18), i.e., *if there exists a partial sequence* $\{\lambda_{n_i}\}$ *such that*

$$\lim_{i\to\infty} \frac{\lambda_{n_i+1}}{\lambda_{n_i}} = \infty,$$

and if we let

$$S_i(x) = \sum_{n=1}^{n_i} a_{\lambda_n} x^{\lambda_n},$$

then the sequence $\{S_i(x)\}$ *converges uniformly in each closed region interior to the region of existence of* $f(x)$; *moreover,* $f(x)$ *is uniform.*

THEOREM 9:[2] *If a sequence of functions* $f_n(x)$ *holomorphic in a closed region* D *is uniformly convergent on the boundary, then the sequence converges uniformly in the closed region* D. *The limiting function* $F(x)$ *is holomorphic within* D, *and* $F^{(k)}(x) = \lim_{n\to\infty} f_n^{(k)}(x)$.

LEMMA: *Let* E *be a bounded closed set, not a Cantor continuum. Then there exist two regions without common points, and at a distance not zero from each other, such that within each region is at least one point of* E, *and such that each point of* E *belongs to at least one of the regions.*

Construct about each point of E a circle of radius ϵ.

[1] Abh. aus dem Math. Seminär der Hamburgischen Universität (1922), p. 327.

[2] Montel, *Series des polynomes*, Paris (1910), p. 16.

There is a finite number of these circles, $C_1, C_2, \cdots, C_k$, such that each point of E is interior to at least one of them. Denote by D_ϵ the region composed of the interior points of these circles.

Since E is not a continuum, ϵ may be taken so small that D_ϵ is not connected. Otherwise each pair of points of E can be joined by a polygonal line of a finite number of segments, each of length less than 4ϵ, and such that each vertex is in E. Hence E is a Cantor continuum, contrary to hypothesis. Hence D_ϵ consists of at least two connected regions.

There is at least one of these regions which occludes none of the others from the point at ∞. Let this be the first region. The other regions may be connected by polygonal lines which have no point in common with the first region, and are therefore at a finite distance from it. These can be covered by a finite number of overlapping circles of radius less than $\frac{\delta}{2}$, and with centers on these lines; these circles, together with those not of the first region, constitute the second region.

COROLLARY: *If, for every ϵ, a circle of radius ϵ is described about each point of a closed set E, and if, for every ϵ, the region formed by application of the Borel-Lebesgue theorem is connected, it is a continuum.*

THEOREM 10: *Given a closed curve C. Let $C_1, C_2, \cdots$ be a set of polygonal lines within C, all having a common vertex P. Suppose that each C_i has a vertex P_i such that $\lim_{i\to\infty} \eta_i = 0$, where η_i is the distance from P_i to C. Denote by ϵ_i the length of the longest segment of C_i, and suppose $\lim_{i\to\infty} \epsilon_i = 0$. Let E be the set of vertices of the C_i. Then E' is a continuum which contains P and a point of C.*

We note first that E' is not a null set, and is closed; further that it contains the point P, and some point of C. Describe a circle of radius η about each point of E'. Then there exists a finite number of these circles such that each point of E' is interior to at least one of them. Let $T\eta$ be the region composed of all of these circles. This region must contain all the vertices of all the C_i, $i > i_0$, where i_0 is sufficiently large. In fact, if there were a vertex P_i, outside or on the boundary of $T\eta$, for some i arbitrarily large, the points P_i would have a limiting point P', outside or on the boundary of $T\eta$, which is contrary to hypothesis.

The set E' is chained. For if A and B are two of its points, there will be a C_A of sides $< 2\,\eta$, of which one vertex is distant from A by less than $2\,\eta$, and a C_B of sides $< 2\,\eta$, of which one vertex is distant from B by less than $2\,\eta$; moreover, every vertex of C_A will be distant by less than $2\,\eta$ from some point of E', and similarly for C_B, if these are C_i, $i > i_0$, and C_A and C_B both contain P; this proves the statement.

Since E' is closed and chained, it is a continuum.

We proceed to the proof of Theorem 7. Let E denote the set of singularities of $f(x)$. Since, by Theorem 8, $f(x)$ is uniform, E is closed.

Moreover, E is perfect. For suppose an arbitrary point P_0 of E is not a limiting point. With P_0 as center describe a circle C_0 within which P_0 is the only singularity. Let C_1 be a smaller circle concentric with C_0. Within the ring thus formed, $f(x)$ is regular. By Theorem 8, the sequence

$$S_i(x) = \sum_{n=1}^{n_i} a_{\lambda_n} x^{\lambda_n}$$

converges uniformly in the closed ring, hence, by Theorem 9, within C_1, and consequently at P_0, which contradicts the hypothesis.

With center at an arbitrary point P_0 of E, describe a circle C_R of radius R, arbitrarily large. Denote by $\mathcal{E}$ the closed set of points composed of the circumference of C_R plus those points of E which lie within C_R.

The set $\mathcal{E}$ is a Cantor continuum. Otherwise there will be a closed curve K, lying within C_R, such that there is at least one point of $\mathcal{E}$ exterior to K, at least one point of $\mathcal{E}$ interior to K, and no point of $\mathcal{E}$ on K. Hence we can construct a ring surrounding K, lying within C_R, and having the same property. Then $S_i(x)$ converges uniformly within K, so that the points of $\mathcal{E}$ lying within K are regular.

We wish to prove that any point P of E belongs to a Cantor continuum P_C which is not bounded. Choose R large enough so that P is an interior point of the circle C_R. Let Q be a point on the circumference. Then P and Q are points of $\mathcal{E}$ and can be joined by a polygonal line whose vertices belong to $\mathcal{E}$:

$$P = P_1^i,\ P_2^i,\ \cdots,\ P_n^i,\ \cdots,\ P_k^i = Q,$$

where

$$\overline{P_j^i P_{j+1}^i} < \epsilon_i,\ j = 1, 2, \cdots, k-1,$$

and where $P_1^i, P_2^i, \cdots, P_n^i$ are all within and not on C_R, and P_n^i is the last such vertex. Then a point P of E may be joined to some point P' of E on C_R by a continuum consisting of points of E. The theorem is therefore proved, since P and R are arbitrary.

25. We have seen that a series for which the condition

$$\lim_{i\to\infty} (\lambda_{n_i+1} - \lambda_{n_i}) = \infty$$

is verified, has, on the circle of convergence, at least one singularity which is not a pole. But *a priori* there may also be poles on the circle of convergence. In this connection the following theorem is useful:

THEOREM 11: *If the series* $\Sigma a_n x^{\lambda_n}$ *represents a function having exactly one singularity on the circle of convergence, then the series* $\Sigma b_n x^{\lambda'_n}$, *where the* b_n *are arbitrary, can not have a pole with principal part*

$$\frac{A_p}{(x - x_0)^p}.$$

In particular, there are no simple poles on the circle of convergence.[1]

The proof is based on the following theorem:

THEOREM 12: *If, corresponding to the series*

$$\sum_{n=0}^{\infty} x^{\lambda_n},$$

we have a series

$$\sum_{n=0}^{\infty} c_n x^{\lambda'_n},$$

with unit radius of convergence, having the property that the series

$$\sum_{m=0}^{\infty} d_m x^m = \sum_{n=0}^{\infty} c_n x^{\lambda'_n} + \sum_{n=0}^{\infty} x^{\lambda_n},$$

where $d_{\lambda'_n} = c_n$, $d_{\lambda_n} = 1$, *is regular at the point* 1, *then the function*

$$\psi(x) = \sum_{n=0}^{\infty} a_n x^{\lambda_n},$$

where the a_n *are subject only to the condition that the series shall have unit radius of convergence, has at least two singular points on the circle of convergence.*

[1] Polya has generalized this theorem by showing that on the circle of convergence the function can have neither an algebraico-logarithmic point nor an isolated singular point about which the function remains uniform. Comptes Rendus, t. 184 (1927), p. 504. See also Gergen, American Journal of Mathematics, vol. 49 (1927), p. 407, for a generalization by means of "generalized lacunae," a concept introduced by the author. Bull. de la Soc. Math. de France, t. 53 (1925), p. 235.

This theorem, as we have stated, implies the preceding. Otherwise there will exist a series

$$\sum_{n=0}^{\infty} a_n x^{\lambda_n}$$

having one and only one singularity, and a corresponding series

$$\sum_{n=0}^{\infty} b_n x^{\lambda'_n} \tag{19}$$

having a pole with principal part

$$\frac{A_p}{(x - x_0)^p}.$$

Integrate (19) $p - 1$ times. The resulting series,

$$\sum_{n=0}^{\infty} b'_n x^{\lambda'_n + p - 1},$$

which we assume to have unit radius of convergence, will have a simple pole at a point $x_0 = e^{i\phi}$. The series

$$\frac{1}{x^{p-1}} \sum_{n=0}^{\infty} b'_n e^{ni\phi} x^{\lambda'_n + p - 1} = \sum_{n=0}^{\infty} b''_n x^{\lambda'_n} \tag{20}$$

has a simple pole at $x = 1$, and may therefore be written in the form

$$\frac{A}{1 - x} + \sum_{n=0}^{\infty} k_n x^n, \tag{21}$$

the last series being regular at $x = 1$. On the other hand, its radius of convergence is unity. For the series (20) has, on the unit circle, at least one singularity other than the pole $x = 1$. Otherwise we have an immediate contradiction. For $\overline{\lim}\, (\lambda'_{n+1} - \lambda'_n) > 1$, which means that there must be at least two poles on the circle of convergence. Hence every singularity of (20), other than the pole $x = 1$, is also a singularity of the series in (21), that is, the latter series has a unit radius of convergence. We have the following relations:

$$-\frac{1}{A}\sum_{n=0}^{\infty} b''_n x^{\lambda'_n} = \sum_{n=0}^{\infty} b'''_n x^{\lambda'_n} = \frac{-1}{1-x} - \sum_{n=0}^{\infty} \frac{k_n}{A} x^n,$$

$$\sum_{n=0}^{\infty} b'''_n x^{\lambda'_n} + \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{\lambda_n} + \sum_{n=0}^{\infty} (1 + b'''_n) x^{\lambda'_n}$$

$$= -\frac{1}{A}\sum_{n=0}^{\infty} k_n x^n.$$

The series Σx^{λ_n} accordingly satisfies the hypothesis of Theorem 5, where $c_n = 1 + b'''_n$. Hence the series $\Sigma a_n x^{\lambda_n}$ has, contrary to hypothesis, at least two singularities on the circle of convergence.

We proceed to the proof of Theorem 12. If the theorem is false, there will exist a series $\Sigma a'_{\lambda_n} x^{\lambda_n}$ with unit radius of convergence, having only one singular point, say $e^{i\phi}$, on the unit circle.

Denote by H the operation by which $\Sigma a_n b_n x^n$ is obtained from $\Sigma a_n x^n$ and $\Sigma b_n x^n$:

$$H\left(\sum_{n=0}^{\infty} a_n x^n \sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} a_n b_n x^n.$$

Since, by hypothesis, $\Sigma d_n x^n$ is regular at $x = 1$, we may write

$$H\left(\sum_{n=0}^{\infty} a'_{\lambda_n} x^{\lambda_n} \sum_{n=0}^{\infty} d_n x^n\right) = \sum k_n x^n,$$

a series which, by Hadamard's theorem,[1] has, on the unit circle, singularities only of the form $\gamma = e^{i\phi}e^{i\theta}$, where $e^{i\theta}$

[1] The proof of Hadamard's theorem given in Chapter III assumes that the functions $f(x)$, $\phi(x)$ involved are uniform. But here all three series in question have radius of convergence 1. Hence it may be shown that the curves C, C_β may be drawn so that if x_0 is a regular point on the circumference for $f(x)$, and x_1 for $\phi(x)$, then the point x_0x_1 will be included in the region of regularity for $F(z)$, from which the desired conclusion follows. [EDITOR.]

is a singularity of $\Sigma d_n x^n$. But we have

$$\sum_{n=0}^{\infty} k_n x^n = \sum_{n=0}^{\infty} a'_{\lambda_n} x^{\lambda_n}, \tag{22}$$

from the way in which the d_n are defined. Consequently

$$e^{i\phi} = e^{i\theta} e^{i\phi},\ e^{i\theta} = 1,$$

which contradicts the hypothesis that $x = 1$ is a regular point for $\Sigma d_n x^n$.

From the fact that the series (22) has at least two singularities on the unit circle, it follows that if $e^{i\phi_0}$ is one of them, there will be another, say $e^{i\phi_1}$, such that

$$e^{i\phi_0} = e^{i\theta} e^{i\phi_1}. \tag{23}$$

Thus for example, if the sequence $\{\lambda_n\}$ contains only a finite number of multiples of an integer p, the series $\Sigma a_n x^{\lambda_n}$ has at least two singular points on the circle of convergence.[1] For consider

$$-\frac{p}{1-x^p} = -p\sum_{n=0}^{\infty} x^{np} = -\frac{1}{1-x} + \phi(x),$$

where $\phi(x)$ is regular in the point $x = 1$. We may write

$$\begin{aligned}\sum_{n=0}^{\infty} d_n x^n &= -p\sum_{n=0}^{\infty} x^{np} + \sum_{n=0}^{\infty} x^n \\ &= \sum_{n=0}^{\infty} x^{\lambda_n} + \sum_{n=0}^{\infty} (1-p)x^{\lambda'_n} \\ &= \phi(x),\end{aligned}$$

where np has been replaced by λ'_n. This function has poles at the points $e^{\frac{2\pi qi}{p}}$, $0 < q < p$, that is, the series Σx^{λ_n} satisfies the hypothesis of Theorem 12.

[1] See also Ostrowski: Jahresbuch der deutschen Math.-Verein., Bd. 35 (1926), 9–12 Heft, p. 269, and Jour. London Math. Soc., vol. 1, part 4, Oct., 1926, p. 236.

From the hypothesis concerning the λ_n, the θ of (23) must be of the form $\frac{2\pi qi}{p}$, $0 < q < p$. Hence if $e^{i\phi_0}$ is a singularity of $\Sigma a_{\lambda_n} x^{\lambda_n}$, there will be a singularity $e^{i\phi_1}$, of the same series such that

$$e^{i\phi_0} = e^{\frac{2\pi qi}{p}} e^{i\phi_1} \tag{24}$$

26. THEOREM 13: *Given a sequence* $\{\lambda_n\}$ *containing only a finite number of multiples of each* p_i *of an infinite sequence of prime numbers* $\{p_n\}$. *Then the series* $\Sigma a_{\lambda_n} x^{\lambda_n}$ *has an irreducible set of singularities on the circle of convergence.*

Without loss of generality, we may assume that $x = 1$ is a singular point. From (24) the set E of singularities on the unit circle consists of the points

$$e^{\frac{2\pi q_1 i}{p_1}}, \; e^{\frac{2\pi q_2 i}{p_2}}, \; \cdots, \; 0 < q_j < p_j,$$

and these points are distinct, since the p_j are prime numbers. Consequently E' is not a null set.

We shall prove that if $E^{(n)}$ is not a null set, and if to each $p_0^{(n)}$ of $E^{(n)}$, and for each p_j, there exists a number q_j such that

$$p_0^{(n)} e^{\frac{2\pi q_j^{(n)} i}{p_j}} = p_1^{(n)}, \tag{25}$$

then $E^{(n+1)}$ is not a null set, and for each $p_0^{(n+1)}$ we have

$$p_0^{(n+1)} e^{\frac{2\pi q_j^{(n+1)} i}{p_j}} = p_1^{(n+1)}.$$

The existence of $E^{(n+1)}$ follows at once from (25), since there is an infinity of distinct points $p_1^{(n)}$.

There exists a sequence of points $p_k^{(n)}$, $k = 1, 2, \cdots$ for which $\lim_{k\to\infty} p_k^{(n)} = p_0^{(n+1)}$. For an arbitrary j, the set of points

$$p_k^{(n)} e^{\frac{2\pi q_j^{(n)} i}{p_j}}, \; k = 0, 1, 2, \cdots$$

has at least one limiting point:

$$p_1^{(n+1)} = p_0^{(n+1)} e^{\frac{2\pi q_j^{(n+1)} i}{p_j}}.$$

In fact, since j is fixed, one of the factors $e^{\frac{2\pi q_j^{(n)} i}{p_j}}$ must be repeated an infinite number of times. This $q_j^{(n)}$ may be taken as $q_j^{(n+1)}$; hence the point $p_1^{(n+1)}$ thus given is distinct from $p_0^{(n+1)}$.

Generalizations of theorems similar to the preceding have been given by Ostrowski.[1] One of his theorems is the following:

If $f(x) = \Sigma a_n x^{\lambda_n}$, where the sequence $\{\lambda_n\}$ does not contain any member of the form

$$\left.\begin{array}{ll} l_1 + nq, & l_1 < q, \\ l_2 + nq, & l_2 < q, \\ \cdots\cdots & \cdots\cdots \\ l_k + nq & l_k < q, \end{array}\right\} \quad n = 1, 2, \cdots$$

where q is a prime number, and where the integers l_i are distinct, then $f(x)$ has at least $k + 1$ singular points on its circle of convergence. If $x = \alpha$ is a singular point on the circle of convergence, then $f(x)$ has, on the circle of convergence, at least k other singularities $\alpha_1, \alpha_2, \cdots, \alpha_k$ of the form

$$\alpha_j = \alpha e^{\frac{2\pi r_j i}{q}},$$

where r_j is an integer, $0 < r_j < q$.

27. We have seen the importance of determining whether the expression $\sqrt[m]{|D_{m,p+1}|}$ approaches 1 regularly or not. For a series $\Sigma a_n x^n$ having an infinity of lacunae, this regularity does not hold for $p = 1$, but since there is a partial sequence of coefficients $a_{n_1}, a_{n_2}, \cdots$ such that $\lim_{i\to\infty} \sqrt[n_i]{|a_{n_i}|} = 1$, we say that the sequence $\sqrt[n]{|a_n|}$ approaches 1 irregularly.

[1] *loc. cit.*

It is also natural to say that the series $\Sigma a_n x^n$ has more lacunae than the series $\Sigma b_n x^n$, or that $\sqrt[n]{|b_n|}$ approaches 1 more regularly than $\sqrt[n]{|a_n|}$ if, whatever the sequence of positive integers n_i, such that $\lim_{i\to\infty} \sqrt[n_i]{|a_{n_i}|} = 1$, it is also true that $\lim_{i\to\infty} \sqrt[n_i]{|b_{n_i}|} = 1$, but not *vice versa* (*i.e.*, there exists a sequence $\{n'_i\}$ such that $\lim_{i\to\infty} \sqrt[n'_i]{|b_{n'_i}|} = 1$, and $\lim_{i\to\infty} \sqrt[n'_i]{|a_{n'_i}|} < 1$).

This terminology has the advantage of permitting the formulation in a general manner of the law expressing the influence of the lacunae on the corresponding singularities, their nature and their number.

If the sequence $\sqrt[n]{|b_n|}$ approaches 1 irregularly so that the singularities of $\Sigma b_n x^n$ fail to satisfy an arbitrary property (A), then, from the foregoing theorems, there corresponds to the sequence $\sqrt[n]{|a_n|}$ the series $\Sigma a_n x^n$ whose singularities can not satisfy the property (B) less restrictive than (A), that is, every set of singularities satisfying (A) also satisfies (B), whereas there are sets satisfying (B) which do not satisfy (A). Briefly, the set of singularities satisfying (A) is in general contained in the set satisfying (B).

Conclusions of this kind are, moreover, the only ones that can be given when the hypotheses refer only to the absolute values of the coefficients. In particular, a theorem of Fatou states that it suffices to change the signs of an infinity of coefficients in order that the circle of convergence shall become a cut.