III

THEOREMS OF HADAMARD AND HURWITZ ON THE COMPOSITION OF SINGULARITIES

12. In order to obtain general theorems relative to the singularities of Taylor's series it is necessary to be able to proceed from theorems in which the hypotheses on the coefficients are comparatively simple (as, for example, in Faber's theorem, Chapter VIII) to more complicated cases by means of "theorems of composition." The problem is this: Given two series whose singularities are known, to determine the singularities of series obtained by combining the two given series in various ways. Two theorems of the kind described are those of Hadamard and Hurwitz.

The theorem of Hadamard [1] is the following:

THEOREM 1: *Given the series*

$$\sum a_n x^n, \quad \overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = \frac{1}{r}, \tag{1}$$

$$\sum b_n x^n, \quad \overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|} = \frac{1}{r_1}, \tag{2}$$

representing the uniform functions $f(x)$, $\phi(x)$, respectively. If γ is a singularity of the function defined by

$$\sum a_n b_n x^n, \tag{3}$$

then there exists a singularity α of (1), *and a singularity β of* (2), *such that $\gamma = \alpha\beta$.*

[1] Acta Math., t. xxii (1898), p. 55.

13. In order to follow easily the method of proof, suppose first that $f(x)$ has only isolated singularities α_i, and that $\phi(x)$ has one and only one singularity β. For convenience, suppose that the α_i are ordered according to increasing absolute values: $|\alpha_1| \leq |\alpha_2| \leq \cdots$

With the origin as center, construct a circle C_R of radius R such that no α_i is on the circumference, R being otherwise arbitrary.

Suppose C_R contains $\alpha_1, \alpha_2, \cdots, \alpha_k$. Make α_i, $i \leq k$, the center of a small circle C_i having no point in common with C_j or C_R. Join the circumference of each C_i with C_R by a (Jordan) curve l_i in such a way that l_i does not pass through the origin, and has no point in common with l_j or C_j. Denote by C the boundary of the simply connected region thus obtained.

Construct a curve C_β by multiplying each x of C by β. Then C_β is similar to C. The circles C_i are carried into the circles $C_{i\beta}$ with centers $\alpha_{i\beta}$, $i = 1, 2, \cdots, k$; the curves l_i are carried into curves $l_{i\beta}$. Denote by $C_{R\beta}$ the circle into which C_R is carried.

The function

$$F(z) = \frac{1}{2\pi i}\int_C f(x)\,\phi\left(\frac{z}{x}\right)\frac{dx}{x}$$

is holomorphic within C_β. For if x is on C, $\frac{z}{x} \neq \beta$. Moreover, $f(x)$ is regular on C. Since $\phi\left(\frac{z}{x}\right)$ is continuous in z and x, where z is within C_β, $F(z)$ is continuous. Finally, $\phi\left(\frac{z}{x}\right)$ has a continuous partial derivative with respect to z. Hence $F'(z)$ exists for z within C_β, and $F(z)$ is accordingly holomorphic.

With center at the origin and radius r' construct a small circle C' lying entirely within the region bounded by C. Take z, fixed, interior to C_β and also interior to the circle C'_β:

$$|z| = r'\,|\beta|.$$

Then the function

$$f(x)\ \phi\left(\frac{z}{x}\right)\frac{1}{x}$$

is holomorphic at all points of C' and all points of the open region bounded by C' and C; in fact for all such points

$$\left|\frac{z}{x}\right| \leq \frac{|z|}{r'} < \beta.$$

Hence

$$\int_C f(x)\,\phi\left(\frac{z}{x}\right)\frac{dx}{x} = \int_{C'} f(x)\,\phi\left(\frac{z}{x}\right)\frac{dx}{x}.$$

Moreover

$$\phi\left(\frac{z}{x}\right) = \sum_{n=0}^{\infty} b_n \frac{z^n}{x^n},$$

where the series is absolutely and uniformly convergent in x for x on C', from the above inequality. Since the same is true of the series for $f(x)$, we can evaluate the right hand member directly:

$$\int_{C'}\left(\sum a_k x^k\right)\left(\sum b_n \frac{z^n}{x^n}\right)\frac{dx}{x} = \int_{C'} \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_k b_n z^n x^{k-n-1}\,dx$$

$$= 2\,\pi i \sum_{n=1}^{\infty} a_n b_n z^n.$$

That is, $F(z)$ is the function defined by (3). But we have seen that $F(z)$ is holomorphic within C_β. Moreover, the circles C_i can be made arbitrarily small, and C_R arbitrarily large. The same is therefore true for $C_{i\beta}$ and $C_{R\beta}$ respectively. Finally, the curves $l_{i\beta}$ are arbitrary, since the

l_i are arbitrary. Consequently $F(z)$ is regular except perhaps at the points $\alpha_i\beta$. The conclusion of the theorem is therefore valid for this case.

14. For the general case, let C_1, containing but not passing through O, be the boundary of an open simply-connected region D which lies within the region of regularity of $f(x)$. Denote by E the set of singularities of $\phi(x)$. Then E is a closed set. Multiply each point of CD (the set complementary to D) by each point of E. Denote by E the resulting set. Then E is closed. The function

$$F(z) = \int_c f(x)\,\phi\left(\frac{z}{x}\right)\frac{dx}{x}$$

is regular for z in CE.

Remark. If D is chosen so that it contains in its interior a circle with center at the origin and radius $r' < r$, the set CE contains at least the interior points of the circle with center at the origin and radius $r'r_1$. Since CE is not a null set, and is open, it is a region.

Denote by E_1 the set of singularities of $f(x)$; E_1 is closed. Multiply each point of E by each point of E_1, obtaining a set P. Since E_1 is closed, it is clear that if z is in CP we may choose the region D in such a way that z is in CE.

Hence $f(z)$ is regular in CP, that is, the function

$$F(x) = \sum a_n b_n x^n$$

is regular at every point except perhaps at the points γ of the form $\gamma = \alpha\beta$.

The proof just given includes the first case. The truth of the theorem depends on the extended definition of a singular point.[1]

[1] In this connection, see Montel, *Séries de Polynomes*, Paris (1910), p. 33.

15. Theorem 2 (Hurwitz [1]): *Given two uniform functions* $f(x)$, $\phi(x)$, *both regular at infinity*:

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{x^{n+1}},$$

$$\phi(x) = \sum_{n=0}^{\infty} \frac{b_n}{x^{n+1}},$$

with radii of convergence r, r_1, *respectively. If* γ *is a singularity of*

$$F(x) = \sum_{n=0}^{\infty} \frac{c_n}{x^{n+1}},$$

where $c_n = a_0 b_n + C_n^1 a_1 b_{n-1} + \cdots + a_n b_0$, *then there exists a singularity* α *of* $f(x)$, *and a singularity* β *of* $\phi(x)$ *such that* $\gamma = \alpha + \beta$.

The proof is analogous to that of Hadamard's theorem. Corresponding to the set E will be the set E' composed of all possible sums of points of E and points of CD, where D is chosen as before. The function

$$F(z) = \frac{1}{2\pi i} \int_c f(x)\, \phi(z - x) dx$$

is regular in CE', hence in CP, where P is the set consisting of all possible sums of points of E and points of E_1.

By analogy, suppose now that $|z| > r' + r$, where $r' > r$. Then

$$F(z) = \frac{1}{2\pi i} \int_{C_{r'}} f(x)\, \phi(z - x) dx, \tag{4}$$

[1] Comptes Rendus, t. 128 (1899), p. 350.

where $C_{r'}$ has its center at the origin. We have now $|z - x| > r_1$. The integral (4) is equivalent to

$$\frac{1}{2\pi i}\int_{C_{r'}} f(x) \sum_{n=0}^{\infty} \frac{b_n}{(z-x)^{n+1}}\,dx$$

$$= \frac{1}{2\pi i}\sum_{n=1}^{\infty}\int_{C_{r'}} f(x)\frac{b_n}{(z-x)^{n+1}}\,dx$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty} b_n \sum_{m=1}^{\infty} a_m \int_{C_{r'}} \frac{dx}{x^{m+1}(z-x)^{n+1}}.$$

Now

$$\frac{1}{2\pi i}\int_{C_{r'}} \frac{dx}{x^{m+1}(z-x)^{n+1}} = \frac{(-1)^n}{n!}\left[\frac{d^n x^{-(m+1)}}{dx^n}\right]_{x=z}$$

$$= \frac{(n+m)!}{n!\,m!}\,\frac{1}{z^{m+n+1}}.$$

Hence

$$F(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_m \frac{(m+n)!}{m!\,n!}\,\frac{1}{z^{m+n+1}}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} a_n b_n C^n_{m+n} z^{-(m+n+1)} \tag{5}$$

$$= \sum_{n=0}^{\infty} (a_0 b_n + C^1_n a_1 b_{n-1} + \cdots + a_n b_0)\frac{1}{z^{n+1}},$$

since (5) converges absolutely. But the only singularities of $F(z)$ are of the form $\alpha + \beta$, and the theorem is proved.

Theorems complementary to the foregoing will be treated in Chapter X.