III

THEOREMS OF HADAMARD AND HURWITZ ON THE COMPOSITION OF SINGULARITIES

12. In order to obtain general theorems relative to the singularities of Taylor's series it is necessary to be able to proceed from theorems in which the hypotheses on the coefficients are comparatively simple (as, for example, in Faber's theorem, Chapter VIII) to more complicated cases by means of "theorems of composition." The problem is this: Given two series whose singularities are known, to determine the singularities of series obtained by combining the two given series in various ways. Two theorems of the kind described are those of Hadamard and Hurwitz.

The theorem of Hadamard is the following:

**Theorem 1:** Given the series

\[ \sum a_n x^n, \quad \lim_{n \to \infty} \sqrt[n]{|a_n|} = \frac{1}{r} \]  

\[ \sum b_n x^n, \quad \lim_{n \to \infty} \sqrt[n]{|b_n|} = \frac{1}{r_1} \]

representing the uniform functions \( f(x) \), \( \phi(x) \), respectively. If \( \gamma \) is a singularity of the function defined by

\[ \sum a_n b_n x^n, \]

then there exists a singularity \( \alpha \) of (1), and a singularity \( \beta \) of (2), such that \( \gamma = \alpha \beta \).

\[ ^1 \text{Acta Math., t. xxii (1898), p. 55.} \]

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13. In order to follow easily the method of proof, suppose first that \( f(x) \) has only isolated singularities \( \alpha_i \), and that \( \phi(x) \) has one and only one singularity \( \beta \). For convenience, suppose that the \( \alpha_i \) are ordered according to increasing absolute values: \( |\alpha_1| \leq |\alpha_2| \leq \cdots \).

With the origin as center, construct a circle \( C_R \) of radius \( R \) such that no \( \alpha_i \) is on the circumference, \( R \) being otherwise arbitrary.

Suppose \( C_R \) contains \( \alpha_1, \alpha_2, \ldots, \alpha_k \). Make \( \alpha_i, i \leq k \), the center of a small circle \( C_i \) having no point in common with \( C_j \) or \( C_R \). Join the circumference of each \( C_i \) with \( C_R \) by a (Jordan) curve \( l_i \) in such a way that \( l_i \) does not pass through the origin, and has no point in common with \( l_i \) or \( C_j \). Denote by \( C \) the boundary of the simply connected region thus obtained.

Construct a curve \( C_\beta \) by multiplying each \( x \) of \( C \) by \( \beta \). Then \( C_\beta \) is similar to \( C \). The circles \( C_i \) are carried into the circles \( C_i \beta \) with centers \( \alpha_i \beta, i = 1, 2, \ldots, k \); the curves \( l_i \) are carried into curves \( l_i \beta \). Denote by \( C_R \beta \) the circle into which \( C_R \) is carried.

The function

\[
F(x) = \frac{1}{2 \pi i} \int_C f(x) \phi \left( \frac{z}{x} \right) \frac{dx}{x}
\]

is holomorphic within \( C_\beta \). For if \( x \) is on \( C \), \( \frac{z}{x} \neq \beta \). Moreover, \( f(x) \) is regular on \( C \). Since \( \phi \left( \frac{z}{x} \right) \) is continuous in \( z \) and \( x \), where \( z \) is within \( C_\beta \), \( F(z) \) is continuous. Finally, \( \phi \left( \frac{z}{x} \right) \) has a continuous partial derivative with respect to \( z \). Hence \( F'(z) \) exists for \( z \) within \( C_\beta \), and \( F(z) \) is accordingly holomorphic.
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With center at the origin and radius \( r' \) construct a small circle \( C' \) lying entirely within the region bounded by \( C \). Take \( z \), fixed, interior to \( C_\beta \) and also interior to the circle \( C'_\beta \):

\[
|z| = r'|\beta|.
\]

Then the function

\[
f(x) \phi \left( \frac{z}{x} \right) \frac{1}{x}
\]

is holomorphic at all points of \( C' \) and all points of the open region bounded by \( C' \) and \( C \); in fact for all such points

\[
\left| \frac{z}{x} \right| \leq \frac{|z|}{r'} < \beta.
\]

Hence

\[
\int_{C} f(x) \phi \left( \frac{z}{x} \right) \frac{dx}{x} = \int_{C'} f(x) \phi \left( \frac{z}{x} \right) \frac{dx}{x}.
\]

Moreover

\[
\phi \left( \frac{z}{x} \right) = \sum_{n=0}^{\infty} b_n \frac{z^n}{x^n},
\]

where the series is absolutely and uniformly convergent in \( x \) for \( x \) on \( C' \), from the above inequality. Since the same is true of the series for \( f(x) \), we can evaluate the right hand member directly:

\[
\int_{C'} \left( \sum_{k} a_k x^k \right) \left( \sum_{n} b_n \frac{z^n}{x^n} \right) \frac{dx}{x} = \int_{C} \sum_{k} \sum_{n} a_k b_n z^n x^{k-n-1} \frac{dx}{x}
\]

\[
= 2 \pi i \sum_{n=1}^{\infty} a_n b_n z^n.
\]

That is, \( F(z) \) is the function defined by (3). But we have seen that \( F(z) \) is holomorphic within \( C_\beta \). Moreover, the circles \( C_i \) can be made arbitrarily small, and \( C_R \) arbitrarily large. The same is therefore true for \( C_{i\beta} \) and \( C_{R\beta} \) respectively. Finally, the curves \( l_{i\beta} \) are arbitrary, since the
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are arbitrary. Consequently $F(z)$ is regular except perhaps at the points $\alpha_i\beta$. The conclusion of the theorem is therefore valid for this case.

14. For the general case, let $C_1$, containing but not passing through $0$, be the boundary of an open simply-connected region $D$ which lies within the region of regularity of $f(x)$. Denote by $E$ the set of singularities of $\phi(x)$. Then $E$ is a closed set. Multiply each point of $CD$ (the set complementary to $D$) by each point of $E$. Denote by $E$ the resulting set. Then $E$ is closed. The function

$$F(z) = \int_0 f(x) \phi\left(\frac{z}{x}\right) \frac{dx}{x}$$

is regular for $z$ in $CE$.

Remark. If $D$ is chosen so that it contains in its interior a circle with center at the origin and radius $r' < r$, the set $CE$ contains at least the interior points of the circle with center at the origin and radius $r' r_1$. Since $CE$ is not a null set, and is open, it is a region.

Denote by $E_1$ the set of singularities of $f(x)$; $E_1$ is closed. Multiply each point of $E$ by each point of $E_1$, obtaining a set $P$. Since $E_1$ is closed, it is clear that if $z$ is in $CP$ we may choose the region $D$ in such a way that $z$ is in $CE$.

Hence $f(z)$ is regular in $CP$, that is, the function

$$F(x) = \sum a_n b_n x^n$$

is regular at every point except perhaps at the points $\gamma$ of the form $\gamma = \alpha \beta$.

The proof just given includes the first case. The truth of the theorem depends on the extended definition of a singular point.\(^1\)

\(^1\) In this connection, see Montel, *Séries de Polynomes*, Paris (1910), p. 33.
15. **Theorem 2 (Hurwitz)**: Given two uniform functions \( f(x), \phi(x) \), both regular at infinity:

\[
f(x) = \sum_{n=0}^{\infty} \frac{a_n}{x^{n+1}},
\]

\[
\phi(x) = \sum_{n=0}^{\infty} \frac{b_n}{x^{n+1}},
\]

with radii of convergence \( r, r_1 \), respectively. If \( \gamma \) is a singularity of

\[
F(x) = \sum_{n=0}^{\infty} \frac{c_n}{x^{n+1}},
\]

where \( c_n = a_0b_n + C_{n}a_1b_{n-1} + \cdots + a_nb_0 \), then there exists a singularity \( \alpha \) of \( f(x) \), and a singularity \( \beta \) of \( \phi(x) \) such that \( \gamma = \alpha + \beta \).

The proof is analogous to that of Hadamard’s theorem. Corresponding to the set \( E \) will be the set \( E' \) composed of all possible sums of points of \( E \) and points of \( CD \), where \( D \) is chosen as before. The function

\[
F(z) = \frac{1}{2\pi i} \int_{c} f(x) \phi(z-x) dx
\]

is regular in \( CE' \), hence in \( CP \), where \( P \) is the set consisting of all possible sums of points of \( E \) and points of \( E_i \).

By analogy, suppose now that \( |z| > r' + r \), where \( r' > r \). Then

\[
F(z) = \frac{1}{2\pi i} \int_{c_{r'}} f(x) \phi(z-x) dx, \quad (1)
\]

\(^1\) Comptes Rendus, t. 128 (1899), p. 350.
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where $C_r$ has its center at the origin. We have now $|z - x| > r_1$. The integral (4) is equivalent to

$$\frac{1}{2\pi i} \oint_{C_r} f(x) \sum_{n=0}^{\infty} \frac{b_n}{(z - x)^{n+1}} dx$$

$$= \frac{1}{2\pi i} \sum_{n=1}^{\infty} \oint_{C_r} f(x) \frac{b_n}{(z - x)^{n+1}} dx$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} b_n \sum_{m=1}^{\infty} a_m \int_{C_r} \frac{dx}{x^{m+1}(z - x)^{n+1}}.$$ 

Now

$$\frac{1}{2\pi i} \int_{C_r} \frac{dx}{x^{m+1}(z - x)^{n+1}} = \frac{(-1)^n}{n!} \left[ \frac{d^n x^{-(m+1)}}{dx^n} \right]_{x=z}$$

$$= \frac{(n+m)!}{n! m!} \frac{1}{z^{m+n+1}}.$$ 

Hence

$$F(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_m \frac{(m+n)!}{m! n!} \frac{1}{z^{m+n+1}}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_n b_n C_{m+n}^n z^{-(m+n+1)}$$

$$= \sum_{n=0}^{\infty} (a_0 b_n + C_n^1 a_1 b_{n-1} + \cdots + a_n b_0) \frac{1}{z^{n+1}},$$

since (5) converges absolutely. But the only singularities of $F(z)$ are of the form $\alpha + \beta$, and the theorem is proved.

Theorems complementary to the foregoing will be treated in Chapter X.