II

ELEMENTARY THEOREMS ON SINGULARITIES

8. The point of view of Méray-Weierstrass mentioned in the preceding chapter is of the nature of an existence theorem: a unique function is defined by the sequence of coefficients in a Taylor's series about a given point. This says nothing about the actual determination of the function, nor gives any information about its properties. Our purpose is, given the Taylor coefficients, to examine the behavior of the function in the entire plane, particularly with reference to the nature and position of the singularities.

THEOREM 1: *The set of singularities of a uniform analytic function is closed.*

Without loss of generality we may take $f(x)$ as given by $\Sigma a_n x^n$. Let P be a limiting point of the set E of singularities of $f(x)$. Suppose P is not a singular point. By definition, there exists a function $g(x)$ holomorphic in a simply connected region D_1 containing P and the circle of convergence C of $\Sigma a_n x^n$ in its interior, and equal to $f(x)$ in C. Hence we can construct a circle C_1 having its center at P, and lying within D_1, such that $f(x)$ has no singularities within C_1. Consequently P is not a limiting point of E.

For the sense of this theorem it is not necessary to consider the point at infinity as a limiting point of an unbounded set; the theorem remains valid, however, if this interpretation is chosen.

If a series converges for all finite x, the problem of locating the singularities ceases to exist. We shall always assume that the radius of convergence is finite.

THEOREM 2: *There exists at least one singular point on the circle of convergence.*

Suppose every point on C is regular. Then each point on C can be made the center of a circle C_1 in which there exists a function $g(x)$, holomorphic in C_1, and equal to $f(x)$ in the region common to C and C_1. By the Borel-Lebesgue theorem, there is a finite number of such circles, each point on C being interior to at least one of them. Hence we can find an annular region D, containing C in its interior, and in which $f(x)$ is regular. It is then possible to construct a circle C_2, concentric with C and of larger radius, such that the circumference of C_2 lies within D. Then $f(x)$ converges within C_2, which is a contradiction.

Thus the radius r of C is the distance from the origin (or the point a if the function is developed about $x = a$) to at least one singular point. In order to determine r when the sequence of coefficients is given we introduce the concept of the *superior limit*.[1]

9. DEFINITION: Given a bounded sequence of real numbers $\{\alpha_n\}$. We say that A is the *superior limit* of the sequence:

$$A = \overline{\lim_{n\to\infty}}\, \alpha_n$$

if, for an arbitrary $\epsilon > 0$, there exists a number m such that

$$\alpha_n < A + \epsilon, \quad n > m,$$

and if there is an infinity of values of n for which

$$\alpha_n > A - \epsilon.$$

If the sequence is unbounded positively, we define

$$\overline{\lim_{n\to\infty}}\, \alpha_n = +\infty.$$

[1] This notion was introduced by Du Bois Reymond.

THEOREM 3: *The radius of convergence is the value r given by*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|a_n|} = A = \frac{1}{r}.$$

Consider the series $\Sigma a_n x^n$. Assuming that A is finite, not zero, we shall prove that the series converges for

$$|x| < \frac{1}{A(1+\epsilon)}, \tag{1}$$

where $\epsilon > 0$ is arbitrary.

There exists, by definition, a number m such that

$$\sqrt[n]{|a_n|} < A\left(1+\frac{\epsilon}{2}\right), \quad n \geq m.$$

Then, for any x which satisfies (1), we have

$$|a_n x^n| < \left(\frac{1+\frac{\epsilon}{2}}{1+\epsilon}\right)^n = \eta^n, \quad n \geq m.$$

The series

$$\sum_{n=m}^{\infty} |a_n x^n|$$

is therefore dominated by the convergent geometric series

$$\sum_{n=m}^{\infty} \eta^n,$$

and converges.

We show next that the series diverges for

$$|x| > \frac{A}{1-\epsilon}.$$

We have, for infinitely many values of n,

$$\sqrt[n]{|a_n|} > A\left(1-\frac{\epsilon}{2}\right).$$

For these values of n,

$$|a_n x^n| > \left(\frac{1 - \frac{\epsilon}{2}}{1 - \epsilon}\right)^n = \xi^n > 1.$$

Hence the terms of the series do not approach zero, and the series diverges.

This is known as the Cauchy-Hadamard theorem.

If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = 0$, the series converges for all x. If $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|a_n|} = \infty$, the series converges only for $x = 0$. Except when the contrary is stated, it will be assumed hereafter that $r = 1$. For the transformation $z = rx$ replaces a circle of radius r in the z-plane by a circle of radius 1 in the x-plane.

10. Denote by $\Delta_n a_1$ the nth difference of the sequence $\{a_n\}$, viz.,

$$\Delta_n a_1 = a_n - C_n^1 a_{n-1} + \cdots + (-1)^n a_1,$$

where C_n^k is coefficient of the $(k + 1)$-th term in the expansion of $(a + b)^n$. We may state here a theorem which is merely a particular case of one to be proved later.

THEOREM 4:[1] *Given the series $\Sigma a_n x^n$. Let*

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|\Delta_n a_1|} = K.$$

Then at least one of the two points on the circle of convergence C with the angle $\cos^{-1}\left(1 - \frac{K^2}{2}\right)$ is singular, provided the function defined by the series has all its singularities on C and is accordingly regular at ∞.

[1]Mandelbrojt, *La recherche des points singuliers d'une fonction analytique*, Jour. de Liouville, ser. 9, t. 5 (1926), p. 197.

11. It is in general difficult, if not impossible, to determine directly from the sequence of coefficients the position and character of the singular points. As Hadamard has pointed out, it is necessary to begin by selecting particular cases more or less general.[1] We shall study some cases in which the coefficients are of the types most important with regard to singularities.

THEOREM 5 (Hadamard): *If* $a_n > 0$ *for all* n, *then the point* 1 *is singular.*

Let b be an arbitrary point within C:

$$0 < |b| = \rho < 1,$$

$$b = \rho e^{i\phi}.$$

If the point $e^{i\phi}$ on C is regular, there will exist a circle C_1, center at b, containing $e^{i\phi}$ in its interior, such that the series

$$\sum_{n=0}^{\infty} b_n (x - b)^n$$

defines a function which is equal to

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

within the region common to C and C_1, and which converges in C_1. Hence the radius r_1 of C_1 exceeds $1 - \rho$. But in any case, whether $e^{i\phi}$ is regular or not, we have

$$r_1 \geq 1 - \rho,$$

the lower sign holding if $e^{i\phi}$ is singular. Conversely, if $r_1 = 1 - \rho$, then $e^{i\phi}$ is singular. Thus a necessary and sufficient condition that $e^{i\phi}$ be singular is that

$$r_1 = 1 - \rho.$$

[1] Hadamard-Mandelbrojt, *La série de Taylor*, Scientia, 41, 2d edition, Paris (1926), ch. 2.

Since $e^{i\phi}$ must be singular for at least one value of ϕ, say $\phi = \phi_1$, in the interval $0 \leq \phi < 2\pi$, the above equality is verified for $\phi = \phi_1$. Let b_1 denote the point corresponding to ϕ_1, and let b_n^1 denote the corresponding coefficients. Then

$$\overline{\lim_{n\to\infty}} \sqrt[n]{|b_n^1|} > \overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|} \qquad (1)$$

for every sequence of coefficients b_n corresponding to a value of ϕ for which $e^{i\phi}$ is regular. But

$$b_n = \frac{f^{(n)}(b)}{n!},$$

and

$$f^{(m)}(b) = n!\,a_n + (n+1)!\,a_{n+1}b + \cdots + \frac{(n+m)!}{m!}a_{n+m}b^m + \cdots \qquad (2)$$

Since the a_i are positive, the absolute value of (2) is a maximum if b is real and positive. Hence (1) is not true. This means that $\phi = 0$ is a singular point.

In a later chapter, we shall investigate the nature and position of the singularities, given either the function itself, or certain properties of the sequence of coefficients.