MODERN RESEARCHES ON THE SINGULARITIES OF FUNCTIONS DEFINED BY TAYLOR'S SERIES[1]

I

PRELIMINARY THEOREMS[2]

1. A *region* D, lying in a plane, is a set of points having the property that each of its points is an interior point, that is, each point of D can be made the center of a circle which contains in its interior only points of D.

Let P, P' be arbitrary points contained in a set E. If, corresponding to an arbitrary positive number ϵ, the points P and P' can be joined by a polygonal line consisting of a finite number of segments, all of its vertices belonging to E,

[1] Lectures delivered at the Rice Institute during the academic year 1926–27 by S. Mandelbrojt, Docteur ès Sciences (Paris), Lecturer in Mathematics at the Rice Institute. Edited by E. R. C. Miles, M.A. (Harvard), Instructor in Mathematics at the Rice Institute.

[2] This first chapter is devoted to definitions and theorems which may be classed as elementary, but in subsequent chapters the author gives more modern and therefore more specialized theorems. The present type of research, which began with the famous thesis of Hadamard, has broadened considerably during the past few years. Complete proofs of all the theorems cannot be compressed in the space available in a short treatise, so the author treats only of such theorems as seem to give unity to the theory, and of the latter theorems proofs are given, for the greater part, in detail.

The reader will find a large bibliography, as well as an enumeration of nearly all the results on the singularities of Taylor's series in Hadamard and Mandelbrojt: *La Série de Taylor et son prolongement analytique*, Scientia, No. 41, also in the author's volume of the Memorial des Sciences Mathématiques. The present treatise may be regarded as complementary to, as well as an elaboration of, some parts of the works just mentioned.

such that the length of each segment is less than ϵ, the set E is said to be *chained.*

A *Cantor continuum* is a set which is closed and chained.

A region is *connected* if two arbitrary points of the region can be joined by a Cantor continuum, consisting entirely of points of the region. Unless otherwise stated, the word "region" will denote a connected region.

If a region D lies entirely in the finite domain, then there are points which are limiting points of D without belonging to D. The aggregate of such points is called the *boundary* of the region D. If we add to the set D the set of boundary points, we obtain a closed set, or, as we shall say, a *closed region.*

2. The equations

$$x = f(t), \qquad y = \phi(t)$$

are said to define a *regular arc* if in the interval $t_0 \leq t \leq t_1$, the following conditions are satisfied:

(*a*) the functions $f(t)$, $\phi(t)$, $f'(t)$ and $\phi'(t)$ are continuous;

(*b*) the equations $f(t) = f(t')$, $\phi(t) = \phi(t')$ admit the unique solution $t = t'$;

(*c*) $f'^2(t) + \phi'^2(t) \neq 0$.

If a finite number of regular arcs are joined consecutively, the resulting curve is said to be regular. A regular curve has therefore a continuously turning tangent except perhaps at a finite number of points.

A set of points which can be placed in one-to-one and continuous correspondence with the totality of points t, $t_0 \leq t \leq t_1$, constitutes a *Jordan curve.* Such a curve may be represented by the equations

$$x = f(t), \qquad y = \phi(t),$$

where $f(t)$ and $\phi(t)$ are continuous in the interval $t_0 \leq t \leq t_1$.

If $(\bar{x}, \bar{y})$ is a point of the set, the pair of equations

$$\bar{x} = f(t), \quad \bar{y} = \phi(t), \quad t_0 < t < t_1,$$

must admit a unique solution $t = \bar{t}$, $t_0 \leq \bar{t} \leq t_1$. This last property is usually indicated by saying that the curve is *simple*. If $f(t_0) = f(t_1)$ and $\phi(t_0) = \phi(t_1)$, the curve is said to be *closed*.

3. The function $f(z)$ is *holomorphic* in a connected region D of the plane $z = x + yi$ if, in each point of D, the following conditions are satisfied:

(*a*) $f(z)$ is defined;
(*b*) $f(z)$ is continuous;
(*c*) $f'(z)$ exists.

The existence of a holomorphic function in a non-connected region is conceivable, as, for example, in a region consisting of the interior and the exterior of a closed curve. We shall see, however, that if the region is composed of n connected subregions, there will exist, in general, n different holomorphic functions.

4. THEOREM 1 (Cauchy-Goursat): *If $f(z)$ is holomorphic in a region D and continuous on the boundary C, consisting of a finite number of closed simple regular curves, then*

$$\int_c f(z)\, dz = 0,$$

the integral being taken along C in the positive sense.

Cauchy's proof of this theorem required the continuity of $f'(z)$. Goursat has given a proof in which only the existence of $f'(z)$ is assumed, as indicated in the statement of the theorem.

THEOREM 2: *If $f(z)$ is holomorphic in D and continuous on the boundary, then, in each interior point of D,*

$$f(z) = \frac{1}{2\pi i}\int_c \frac{f(t)dt}{t - z}. \tag{1}$$

Moreover, the successive derivatives of $f(z)$ may be obtained by differentiating under the integral sign. In fact,

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t-z)^2},$$

$$f''(z) = \frac{2!}{2\pi i}\int_C \frac{f(t)dt}{(t-z)^3},$$

$$\cdots\cdot$$

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(t)dt}{(t-z)^{n+1}}.$$

Hence *a function holomorphic in a region D has continuous derivatives of all orders in D.*

Consider the series

$$\frac{1}{t-z} = \frac{1}{t-a}\,\frac{1}{1-\dfrac{z-a}{t-a}}$$

$$= \sum_{n=0}^{\infty} \frac{(z-a)^n}{(t-a)^{n+1}}, \tag{2}$$

which converges if

$$\left|\frac{z-a}{t-a}\right| < 1. \tag{3}$$

If the point a is within D, then

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-a} + \frac{z-a}{2\pi i}\int_C \frac{f(t)dt}{(t-a)^2} + \cdots. \tag{4}$$

For the series (2) converges uniformly for t on C, and z in a circle K, center at a, lying within D, and containing only points of D in its interior. We may therefore multiply each of its terms by $f(t)dt$ and integrate around C. The resulting series (4) converges for all z in K. We have therefore the following theorem:

THEOREM 3: *The function $f(z)$ can be developed in a Taylor's series about each interior point a of D:*

$$f(z) = a_0 + a_1(z - a) + a_2(z - a)^2 + \cdots,$$

where

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{(t - a)^{n+1}} = \frac{f^{(n)}(a)}{n!}.$$

In particular, a function holomorphic within a circle can be represented by a Taylor's series which converges uniformly in each closed region lying within the circle. Moreover, a Taylor's series which converges in a circle K, converges uniformly in each closed region interior to K, and represents a function which is holomorphic within K.

5. THEOREM 4: *If the functions $f(z)$, $\phi(z)$ are holomorphic in a region D, and if $f(z) = \phi(z)$ in a subregion D' of D, then $f(z) = \phi(z)$ throughout D.*

The proof may be based on a familiar theorem in the analysis of point sets, namely, the Heine-Borel, or Borel-Lebesgue theorem:

Let E be a closed set lying in the finite domain. If all the points of E are enclosed in a family of open sets O, then the set E is contained in a finite number of the sets O.

Suppose a is a point of D'. Let b be a point of D exterior to D'. Join a and b by a Cantor continuum composed of points of D. Each point of this continuum will be the center of a family of concentric circles. The points of the continuum will be enclosed in a finite number of these circles, all lying within D. The continuum will then lie within the region G included within these circles. Incidentally, this shows that two arbitrary points of a region can be joined by a Jordan curve lying within the region.

Construct within G a region formed by a family of equal circles C_0, $C_1, \cdots C_k$, where C_0 has its center at a, the center of C_{i+1} lies within C_i, and b is within C_k. By hypothesis, $f = \phi$ in C_0. Moreover, $f = \phi$ in C_1. For within a circle K concentric with C_1 and lying within C_0 the functions f and ϕ are equal, and are therefore represented by the same Taylor development about the center of K. But since f and ϕ are holomorphic in C_1, the series converges in C_1. Hence the statement is proved. Proceeding in this way we obtain, finally, $f = \phi$ in C_k, hence in b.

The theorem is equivalent to the statement that if f, ϕ are holomorphic in D_1, D_2, respectively, and if $f = \phi$ in a common subregion, then $f = \phi$ throughout the interior of the region common to D_1 and D_2. We say, then, that in the region $D = D_1 + D_2$ we have a *unique* function; the function f is an analytic continuation in D_1 of the function ϕ defined in D_2, and *vice versa.*

6. THEOREM 5: *If the series*

$$\sum_{n=0}^{\infty} a_n(x - a)^n \tag{5}$$

converges for $x = x_0$, *it converges absolutely for all* x *such that* $|x - a| < |x_0 - a|$, *and converges uniformly in each closed region interior to the circle with center at* a *and radius* $|x_0 - a|$. *If the series diverges for* $x = x_1$, *it diverges for all* x *such that* $|x - a| > |x_1 - a|$.

If, therefore, the series (5) converges for at least one $x \neq a$ and diverges for some x, there exists a positive number r such that (5) converges absolutely for $|x - a| < r$, and uniformly in each closed region interior to the circle with center at a, radius r, and diverges for $|x - a| > r$.

As to what happens when $|x - a| = r$, no general statement can be made. The series may converge for all such values of x, or for none; it may converge for some such values, and diverge for others. The number r is called the *radius of convergence*, and the circle with center at a and radius r is called the *circle of convergence.* If the series converges for every value of x, that is, if $r = \infty$, the series is said to define an *integral* function. We shall assume in what follows that the series converges for at least one value of x different from a, that is, $r \neq 0$.

As a consequence of Theorem 4, we can say that if the functions $f(x)$, $\phi(x)$ are expanded into two Taylor's series about the points a, b, respectively, where the circles of convergence C_1, C_2 overlap, and if $f = \phi$ in the region common to C_1 and C_2, we have a unique holomorphic function in the region included within the circles C_1 and C_2. If we have n Taylor's series with overlapping circles of convergence $C_1, C_2, \cdots, C_n$, with $f_k = f_{k+1}$ in the region common to C_k and C_{k+1}, we have a unique holomorphic function in the region included within the circles $C_1, C_2, \cdots, C_n$.

7. DEFINITION 1: Let r, C denote, respectively, the radius and the circle of convergence of the series (5). Suppose the points x and a can be joined by a Jordan curve L, and a sequence of overlapping circles of convergence $C, C_1, \cdots, C_n$ of the corresponding Taylor's series for the functions $f(x), f_1(x), \cdots, f_n(x)$, the circles having their centers on L, x being within C_n, and $f_{k+1} = f_k$ in the common part of C_k and C_{k+1}. Then the function $f(x)$ defined by the series (5) is continued to the point x along the curve L. (Thus, beginning with (5), we have defined a holomorphic function in the extended region, containing the point x, by means of an analytic continuation with respect to the curve L.)

Then the point x is said to be *regular* for the function $f(x)$ defined by (5), relative to the curve L.

Let L' be a second curve joining x and a. Consider a second sequence of Taylor's series for the functions $f(x)$, $\phi_1(x), \cdots, \phi_m(x)$, these functions and their respective circles of convergence having the properties mentioned in the preceding paragraph. Then, in general, $f_n(x) \neq \phi_m(x)$.

DEFINITION 2: Consider all the curves L joining the points x and a, and having the property that if b is an arbitrary point of L different from x, then b is regular for $f(x)$ relative to that part of L from a to b. If x is regular for $f(x)$ relative to every such curve L, then x is said to be *regular for the function* $f(x)$. If, in addition, we have, for a regular point x, and for every pair of these curves, $f_n(x) = \phi_m(x)$, then $f(x)$ is *uniform in the point* x. If $f_n(x) = \phi_m(x)$ for every regular point x, and for every pair of such curves, the function $f(x)$ is said to be *uniform.*

DEFINITION 3: If there exists a curve L' with the property stated in Definition 2, yet such that there is no sequence of overlapping circles of convergence $C, C_1, \cdots, C_n$, centers on L, with $f_{k+1} = f_k$ in the region common to C_{k+1} and C_k, and x in C_n, then the point x is *singular* for $f(x)$, relative to L'. If this condition holds for every curve of the kind described in Definition 2, then x is a *singular point of the function* $f(x)$.

For the purposes of the subsequent investigations, it is convenient to have a more general definition of a singular point.

DEFINITION 4: A point x is *singular for the function* $f(x)$ defined by the series (5) if there exists no curve from a to x such that there is a finite number of circles of convergence $C, C_1, \cdots, C_n$, with centers on the curve, and with $f_{k+1} = f_k$ in the region common to C_{k+1} and C_k, the point x lying within C_n.

We shall say that (r, θ), where r is infinite, represents a single point for all θ, called the point at infinity. Make the transformation $x' = \frac{1}{x}$; the point $x = \infty$ is regular, or singular according as the point $x' = 0$ is regular or singular, respectively, for the transformed function.

DEFINITION 5: If every point of a region D bounded by a closed curve C is a singular point (in the sense of Definition 4) for a uniform function defined by a Taylor's series, then the curve C is called a *cut* for the function defined by the series.

As a consequence of Definition 4, the region D of Definition 5 is always simply connected. Moreover, if every point of a closed curve containing the point a in its interior is singular, so also is every point exterior to the curve.

DEFINITION 6: Let x be a point on the circle of convergence C of (5). Suppose a function $\phi(x)$ is given by a Taylor development which converges in a circle C_1. If x is within C_1, and if $f(x) = \phi(x)$ in the region common to C and C_1, the point x on C is called *regular for the series* (5). If there exists no such circle C_1, such that $f = \phi$ in the common region, then x is *singular for the series* (5).

This definition is equivalent to the previous definition of a regular [singular] point for the function defined by (5), relative to a curve joining a and x, the curve containing only interior points of C.

If the point x, regular [singular] for the function defined by (5), is or is not on C, we shall call x regular [singular] for the series (5) whenever there is no ambiguity, as, for example, when $f(x)$ is uniform.

If the series (5) is such that for every point x of the plane the function $f(x)$ is either regular and uniform in x, or singular in x, then we say that (5) defines an analytic uniform function in the plane, and the set of regular points is called the *region of regularity*, or the *region of existence* of the function. That this set of points actually constitutes a region will be proved in the next chapter.

At a given point, the function, if not uniform, may be regular for some curves, may have different values for different curves, or may be singular for certain curves. All three cases may occur for a point.

Thus, given a Taylor's series, and the notion of analytic continuation, we can characterize the behavior of the function in every point of the plane, by taking the continuation along every curve from the point a to the point in question. This was the point of view of Méray in France and Weierstrass in Germany. That is, the sequence of coefficients in a Taylor's series about a point a gives the complete determination of a function, regular or singular in each point of the plane, relative to various curves.

For example, a point x_0 on the circumference of C is a *pole of order m* for the series (5) if there exist m numbers $A_1, A_2, \cdots, A_m$ ($A_m \neq 0$) such that on placing

$$\sum_{i=1}^{m} \frac{A_i}{(x - x_0)^i} = \sum_{n=0}^{\infty} b_n(x - a)^n,$$

$$\sum_{n=0}^{\infty} a_n(x - a)^n - \sum_{n=0}^{\infty} b_n(x - a)^n = \sum_{n=0}^{\infty} (a_n - b_n)(x - a)^n, \qquad (6)$$

we have x_0 as a regular point for the series on the right-hand side of (6).

If the point x_0 is not on the circumference of C, and the function defined by the series (5) is uniform in all the plane, then x_0 is a pole of order m of the function defined by the series (5) if x_0 is regular for the function defined by the series (6).

If x_0 is regular with respect to a curve L and the series (6), then x_0 is said to be a pole of order m relative to the curve L.

Similarly a point x_0 on the circumference of C is an *essential singularity* for the series (5) if there exists an infinite sequence of numbers $A_1, A_2, \cdots$ such that the series

$$\sum_{n=1}^{\infty} A_n z^n \tag{7}$$

is convergent in all the plane, and such that x_0 is regular for the series

$$\sum_{n=0}^{\infty} (a_n - b_n)(x - a)^n,$$

where

$$\sum_{n=0}^{\infty} b_n x^n = \sum_{i=1}^{\infty} \frac{A_i}{(x - x_0)^i}, \tag{8}$$

that is, the convergence of the series (7) implies the convergence of the series on the right-hand side of (8) for all $x \neq x_0$.

We can proceed to the general case for essential singularities as above in the case of poles, by replacing

$$\sum_{i=1}^{m} \frac{A_i}{(x - x_0)^i} \text{ by } \sum_{i=1}^{\infty} \frac{A_i}{(x - x_0)^i},$$

where (8) converges for all $x \neq x_0$.