Two-component Bose-Hubbard model in an array of cavity polaritons

Yong-Chang Zhang,1,2 Xiang-Fa Zhou,1,2 Xingxiang Zhou,1,2 Guang-Can Guo,1,2 Han Pu,3,4,* and Zheng-Wei Zhou1,2,1

1Key Laboratory of Quantum Information, University of Science and Technology of China, CAS, Hefei, Anhui 230026, People’s Republic of China
2Synergetic Innovation Center of Quantum Information and Quantum Physics, University of Science and Technology of China, Hefei, Anhui 230026, China
3Department of Physics and Astronomy, Rice University, Houston, Texas 77005, USA
4Center for Cold Atom Physics, Chinese Academy of Sciences, Wuhan 430071, China

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A polariton is a kind of bosonic quasiparticle in the light-matter system, and Bose-Einstein condensation (BEC) of microcavity polaritons has been realized in experiment recently [22,23]. In this model, in addition to the density-density interaction (Kerr nonlinearity), there also exist two types of nonlinear coupling between the polariton components. In the current work, we extend this model by consider a one-dimensional array of such a cavity polariton system, which realizes a two-component Bose-Hubbard model. We will calculate the phase diagram of the system and focus on how the interspecies interactions and the nonlinear coupling terms will affect the transition between the Mott phase and the superfluid phase.

This paper is organized as follows. In Sec. II we present the model Hamiltonian that describes an extended two-component Bose-Hubbard model. In Sec. III, we provide the ground-state phase diagram obtained from the mean-field decoupling approach. To gain further insights and examine the effects of different terms in the Hamiltonian, we present a perturbative analysis in Sec. IV. In Sec. V, we give a beyond-mean-field exact diagonalization calculation. In particular, we calculate the number fluctuation per cavity and establish connections between this result and the mean-field result presented earlier. Finally, a summary is presented in Sec. VI.

II. MODEL

Our underlying system is schematically shown in Fig. 1(a). We consider a one-dimensional array of connected optical cavities, with the connection provided by the photon tunneling between neighboring cavities. Inside each cavity, we have an ensemble of bosonic atoms whose level structure is sketched in Fig. 1(b). We label the atomic hyperfine states as $|i\rangle$ with $i = 1,2,\ldots,7$. These of these states (states $|1\rangle$, $|2\rangle$, and $|3\rangle$ belong to the electronic ground manifold, and the other four (states $|4\rangle$, $|5\rangle$, and $|6\rangle$, respectively, the cavity field, with corresponding coupling strengths $g_{14}$, $g_{25}$, and $g_{36}$. Additionally, states $|2\rangle$ and $|3\rangle$ are coupled to $|4\rangle$ by external laser fields with coupling strengths $\Omega_{24}$ and $\Omega_{34}$,
respectively. For details, please see Ref. [21]. Finally, within the excited manifold, states |5⟩ and |6⟩ are coupled to |7⟩ by microwave fields with corresponding coupling strengths Ω1, and Ω2, e, δ, Δ, Δs, and Δ6 are various detunings between the driving field and the corresponding atomic transitions, as labeled in Fig. 1(b).

In the limit of weak excitation where the atomic population in the excited levels is negligible, we can construct two polariton modes for each cavity. The corresponding annihilation operator for the two polariton modes in the ith cavity is given by

\[ P_{1i} = \frac{1}{2} \left( \frac{g_i}{\omega_i} + 1 \right) S_{2i} + \frac{1}{2} \left( \frac{g_i}{\omega_i} - 1 \right) S_{3i} - \frac{\Omega}{\sqrt{2 \omega_i}} a_i, \]

\[ P_{2i} = \frac{1}{2} \left( \frac{g_i}{\omega_i} - 1 \right) S_{2i} + \frac{1}{2} \left( \frac{g_i}{\omega_i} + 1 \right) S_{3i} - \frac{\Omega}{\sqrt{2 \omega_i}} a_i, \]

where \( a_i \) is the cavity photon annihilation operator for the ith cavity, and

\[ S_{2i} = \frac{1}{\sqrt{N_{ai}}} \sum_{j=1}^{N_{ai}} |1⟩_{jj} |2⟩, \]

\[ S_{3i} = \frac{1}{\sqrt{N_{ai}}} \sum_{j=1}^{N_{ai}} |1⟩_{jj} |3⟩, \]

comprise the collective atomic operator, with \( N_{ai} \) being the total atom number in the ith cavity. For simplicity, we have taken \( \Omega_{2s} = \Omega_{3s} = \sqrt{2} \Omega \), \( g_i = \sqrt{N_{ai}} g_{14} \), and \( \omega_i = \sqrt{g_i^2 + \Omega^2} \). The total Hamiltonian of the system reads

\[ H = \sum_i H_i + \sum_{(i,j)} H'_{(i,j)}, \]

where

\[ H_i = \frac{V_1}{2} \left( P_{1i}^2 + P_{2i}^2 \right) + \frac{V_2}{2} \left( P_{1i}^2 + P_{2i}^2 \right) + U P_{1i}^4 P_{2i}^4 + T^+ \left[ P_{1i}^1 P_{1i}^1 P_{1i}^2 P_{2i} \right] + T^- \left[ P_{1i}^1 P_{1i}^1 P_{2i}^1 P_{2i} \right] + T^+ \left[ P_{1i}^2 P_{1i}^2 P_{2i}^1 P_{1i}^1 \right] + T^- \left[ P_{1i}^2 P_{1i}^1 P_{2i}^1 P_{2i} \right] + T^+ \left[ P_{1i}^2 P_{1i}^1 P_{2i}^2 P_{2i} \right] + T^- \left[ P_{1i}^1 P_{1i}^2 P_{2i}^1 P_{2i} \right] \]

\[ -t \langle P_{1i}^1 P_{2i}^1 + P_{2i}^1 P_{1i}^1 \rangle \]

represents the Hamiltonian of a two-component polariton in the ith cavity, the derivation of which can be found in our previous work [21], and

\[ H'_{(i,j)} = -t \langle P_{1i}^1 P_{1j}^1 + P_{2i}^1 P_{2j}^1 \rangle \]

describes the tunneling of polaritons between adjacent cavities [24]. Here, \( P_{1i}^{(1,2)} \) and \( P_{1i}^{(1,2)} \) obey the bosonic commutation relation: [\( P_{1i}^{(1,2)} \), \( P_{1j}^{(1,2)} \)] = 0, [\( P_{1i}^{(1,2)} \), \( P_{1j}^{(1,2)} \)] = 0, and [\( P_{1i}^{(1,2)} \), \( P_{1j}^{(1,2)} \)] = \( \delta_{\alpha i} \delta_{\beta j} \). As shown in Ref. [21], the key parameters in Eq. (2), \( V_1, V_2, U \), and \( T^\pm \), can be tuned over a large extent by appropriately controlling the laser intensities and frequencies. Here, to avoid instability, we only consider repulsive interactions such that \( V_1, V_2, U \) are all positive. In addition, if we simultaneously change the signs of \( T^\pm \), the physics remains unchanged as that sign change can be absorbed by a redefinition of the polariton modes. Hence we will only consider the case with \( T^+ > 0, T^- < 0 \) for simplicity.

In comparison to the single-component BHM, for which there are plenty of theoretical and experimental investigations, our model contains three new key parameters: the interspecies on-site interaction characterized by the interaction strength \( U \), the two nonlinear coupling terms characterized by the coupling strength \( T^\pm \). Our work will focus on elucidating the effects of these terms.

### III. MEAN-FIELD PHASE DIAGRAM

Based on the mean-field decoupling theory [5], we introduce the superfluid order parameters: \( \phi_{1(2)} = \langle P_{1(2)i} \rangle \). By the approximation

\[ P_{1i}^{(1,2)}, P_{1j}^{(1,2)} = \phi_{1(2)} P_{1i}^{(1,2)} P_{1j}^{(1,2)} + \phi^*_{1(2)} P_{1i}^{(1,2)} P_{1j}^{(1,2)} - \phi_{1(2)} \phi_{1(2)} \]

the total Hamiltonian can be decoupled into the following form:

\[ H_0 = -\mu (P_{1i}^1 P_{1i}^1 + P_{2i}^1 P_{2i}^1) + \frac{V_1}{2} P_{1i}^1 P_{1i}^1 + \frac{V_2}{2} P_{2i}^1 P_{2i}^1 + U P_{1i}^4 P_{2i}^4 + T^+ \left[ P_{1i}^1 P_{1i}^1 P_{1i}^2 P_{2i} \right] + T^- \left[ P_{1i}^1 P_{1i}^1 P_{2i}^1 P_{2i} \right] + T^+ \left[ P_{1i}^2 P_{1i}^2 P_{2i}^1 P_{1i}^1 \right] + T^- \left[ P_{1i}^2 P_{1i}^1 P_{2i}^1 P_{2i} \right] + T^+ \left[ P_{1i}^2 P_{1i}^1 P_{2i}^2 P_{2i} \right] + T^- \left[ P_{1i}^1 P_{1i}^2 P_{2i}^1 P_{2i} \right] + zt\left( \phi^*_{1} \phi_{1} + \phi^*_{2} \phi_{2} \right), \]

\[ H_i = -zt\left( \phi^*_{1} P_{1i} + \phi^*_{2} P_{2i} + H.c. \right), \]

where \( \mu \) is the chemical potential and \( z = 2 \) is the number of nearest neighbors. For simplicity, here, the site index is neglected. Since Hamiltonian \( H_0 \) keeps the local particle number
conserved, in principle, we can numerically diagonalize $\mathcal{H}_0$ in a subspace with fixed total number of particles.

As a reference, let us first set the nonlinear coupling strength $T^\pm = 0$, and the tunneling rate $t = 0$. Under this situation $\mathcal{H}_0$ describes a two-component Bose gas with density-density interactions, and it preserves the number of polaritons in each component. The eigenstates therefore correspond to Fock states $|n_1, n_2\rangle$ with definite integer values of $n_1$ and $n_2$, and with corresponding ground-state energy $E(n_1, n_2)$, where $n_{\alpha} (\alpha = 1, 2)$ represents the number of polaritons in component $\alpha$ in each cavity. Note that due to the conservation of polariton numbers in the individual component we need to introduce two chemical potentials $\mu_1$ and $\mu_2$ for the two polariton modes. The ground state is determined by the relative size of $V_1$, $V_2$, and $U$. In Fig. 2 the ground-state phase diagram is shown for the case with equal intraspecies interaction strength, i.e., $V_1 = V_2 = V$. If we focus on the situation with equal chemical potential ($\mu_1 = \mu_2 = \mu$), it is quite straightforward to show that if the interspecies interaction is smaller than the intraspecies one [i.e., $U < V$, see Fig. 2(a)] then we have a single ground state with $n_1 = n_2$ for even total particle number, and two degenerate ground states with $n_1 = 0$ or $n_2 = 0$, while for $U > V$ [see Fig. 2(b)], whether the particle number is even or odd, we have two degenerate ground states with $n_1 = 0$ or $n_2 = 0$.

The Fock state $|n_1, n_2\rangle$ discussed above corresponds to the Mott regime with $\phi_1 = \phi_2 = 0$. As intercavity tunneling is turned on, $\phi_2$ may take finite values and the system enters the superfluid regime. At zero temperature, the transition from the MI to the SF regime represents a quantum phase transition. Our goal is to map out a phase diagram by calculating the MI to the SF regime represents a quantum phase transition.

As we want to focus on the effects of the terms characterized by $U$ and $T^\pm$, for most of our calculation we will choose $V_1 = V_2 = V$ as the units for energy, and consider the situation with $U < V$ and $U > V$, respectively, while $|T^\pm| \ll V$.

Figure 3 shows six examples of the phase diagram obtained from the mean-field decoupling approach with $U < V$ in Figs. 3(a)–3(c) and $U > V$ in Figs. 3(d)–3(f). We have also checked that essentially the same phase diagram can be obtained using the Gutzwiller method. The details of the phase diagram will be described in the following.

A. Small interspecies interaction ($U < V$)

In Fig. 3(a), the nonlinear coupling term is turned off, i.e., $T^\pm = 0$. We again focus on the situation with equal chemical potential ($\mu_1 = \mu_2 = \mu$). Here, the MI regime exhibits an "even-odd effect"; i.e., the MI region with odd occupation (the number of polaritons per cavity $n_{\text{tot}} = 2n + 1$) is smaller than its nearest neighbors with $n_{\text{tot}} = 2n$ and $2(n + 1)$. A qualitative interpretation of this effect can be provided by studying the excitation gap of a MI [14]. For even occupation MI with $n_{\text{tot}} = 2n$, the excitation gap is given by

$$\Delta_{2n} = E(n, n + 1) + E(n - 1, n) - 2E(n, n),$$

while that for $n_{\text{tot}} = 2n + 1$ is given by

$$\Delta_{2n+1} = E(n, n) + E(n + 1, n + 1) - 2E(n, n + 1) = U.$$

Since $V > U$, the MI regime with even occupation will be more robust and hence exists in a larger parameter space. The excitation gap will compete with the tunneling energy $-t \sum_{\alpha=1,2} \langle P_{a_{\alpha}} P_{a_{\alpha+1}} \rangle$ which can be effectively strengthened if there exists degeneracy. As the filling number increases, the tunneling energy grows and the MI regimes with large occupation number become smaller and smaller. However, when the nonlinear coupling terms are present, i.e., $T^\pm \neq 0$, this even-odd effect is weakened and may disappear completely as shown in Fig. 3(c). An interpretation of the $T^\pm$ effect is provided by perturbative analysis in Sec. IV.
Furthermore, the nonlinear coupling terms could change the degeneracy of the ground state, which will be discussed in detail in the following.

As mentioned previously, if $t$ and $T^\pm$ are all zero, the ground states are doubly degenerate when $n_{\text{tot}} = n_1 + n_2$ is odd. If $t$ is switched on, due to the superexchange interaction induced by the intercavity tunneling, the degeneracy would be lifted up and the ground state becomes the superposition state of $n_1 - n_2 = \pm 1$ with equal weight [9]. This is indicated in Fig. 4, where we have defined the average particle numbers per cavity $n_\alpha = \langle P_\alpha \rangle$ ($\alpha = 1, 2$) and the fluctuations $\Delta n_\alpha = \langle (P_\alpha^\dagger P_\alpha)^2 \rangle - \langle P_\alpha \rangle^2$, $\Delta n_{\text{tot}} = \langle (P_1^\dagger P_1 + P_2^\dagger P_2)^2 \rangle - \langle P_1^\dagger P_1 + P_2^\dagger P_2 \rangle^2$. For Figs. 4(a) and 4(c), we take a relatively small chemical potential with $\mu / V = 0.25$, for which the average particle numbers are $n_1 = n_2 = 0.5$. In the absence of the tunneling, the ground state is doubly degenerate and are represented by the Fock states $[1,0]$ and $[0,1]$. For $0 < t < t_c$, where $t_c$ is the critical tunneling strength beyond which the system changes from MI to SF and is indicated in the figure by the vertical dashed line, the fluctuations of the total number per cavity $\Delta n_{\text{tot}}$ vanish, while $\Delta n_1$ and $\Delta n_2$ are both finite, as shown in Fig. 4(c). Therefore, we can conclude that the ground state is an equal-weight superposition state of $[1,0]$ and $[0,1]$. For the case depicted in Figs. 4(b) and 4(d), we used a larger chemical potential $\mu / V = 1$ and the average particle number is $n_{\text{tot}} = 2$. In this case, the MI state is represented by the Fock state $[1,1]$ with vanishing fluctuations in both $\Delta n_{\text{tot}}$ and $\Delta n_{1,2}$.

If $T^-$ is turned on, the ground state for $t < t_c$ would be a superposition state Fock state $[n_1,n_2]$ with fixed $n_1 + n_2 = n_{\text{tot}}$. This state is nondegenerate if $n_{\text{tot}}$ is even, and can be written as

$$|G\rangle = \sum_{n_1,n_2} f_{n_1,n_2} |n_1,n_2\rangle,$$

with $f_{n_1,n_2} = (-1)^{n_1-n_2}$. An example is represented in Figs. 5(a) and 5(c) with $n_{\text{tot}} = 2$. For $t > t_c$, we note that the particle number difference remains zero and $n_{1,2}$ have finite fluctuations while $\Delta n_{\text{tot}}$ vanishes, so the ground state is a superposition of states $[1,1]$, $[0,2]$, and $[2,0]$. By contrast, when $n_{\text{tot}}$ is odd [see Figs. 5(b) and 5(d)], $T^-$ separates the whole Hilbert space into two degenerate subspaces with $n_1 > n_2$ and $n_1 < n_2$, respectively, and the corresponding ground states have the form

$$|G_+\rangle = \sum_{n_1,n_2} c_{n_1,n_2}^+ |n_1,n_2\rangle,$$

and

$$|G_-\rangle = \sum_{n_1,n_2} c_{n_1,n_2}^- |n_1,n_2\rangle,$$

with $c_{n_1,n_2}^\pm = (-1)^{n_1-n_2} c_{n_1,n_2}$. One can readily verify that the $T^-$ term has a vanishing matrix element between $|G_+\rangle$ and $|G_-\rangle$. Thus the double degeneracy of the ground state is preserved even if $T^-$ is turned on, and the even-odd effect remains, as shown in Fig. 3(b).

However, if we turn on $T^+$, the degeneracy will be lifted, as the $T^+$ term could couple the two degenerate subspaces; i.e., it has nonvanishing matrix elements between $|G_+\rangle$ and $|G_-\rangle$. This explains the weakening of the even-odd effect as shown in Fig. 3(c).

**B. Large interspecies interaction ($U > V$)**

In Fig. 2(b), we have known that the ground states, i.e., $|0,n_{\text{tot}}\rangle$ and $|n_{\text{tot}},0\rangle$, are doubly degenerate for both the even and the odd occupation when $U > V$, $T^\pm = 0$, and $\mu_1 = \mu_2 = \mu$ at $t = 0$. In Fig. 6, we plot the occupation numbers and their fluctuations as functions of tunneling rate $t$. We can see that, in the MI region ($0 < t < t_c$), the population difference is $n_1 - n_2 = \pm n_{\text{tot}}$ while the number fluctuations $\Delta n_{\text{tot}}$ and $\Delta n_{1,2}$ are all zero. It tells us that the ground state is either $|0,n_{\text{tot}}\rangle$ or $|n_{\text{tot}},0\rangle$, regardless of whether $n_{\text{tot}}$ is even or odd. In addition, the excitation gap is independent of $n_{\text{tot}}$, i.e., $\Delta = V$.
FIG. 6. (Color online) The average particle numbers and their fluctuations obtained by the Gutzwiller method. For all the subplots, we have $V_1 = V_2 = V, U = 1.5$, and $T^+ = 0$. The chemical potential is taken as $\mu = 0.5$ for (a) and (c); $\mu = 1.5$ for (b) and (d). The vertical dashed lines indicate the critical tunneling rate $t_c$ beyond which the system changes from MI to SF.

However, when $T^-$ is turned on, the even-odd effect emerges as can be seen in Fig. 3(e). This is because for finite $T^-$ the ground state in the Mott insulator region is nondegenerate when $n_{\text{tot}}$ is even, and takes a similar form as in Eq. (5). An example is shown in Figs. 7(a) and 7(c). By contrast, for odd $n_{\text{tot}}$, $T^-$ can preserve the ground state’s double degeneracy and the two degenerate ground states take similar forms as in Eqs. (6) and (7), but with $c_{-n_1,n_2} = (-1)^n c_{n_1,n_2}$. It is easy to find that the tunneling energy in this nondegenerate state is much lower than that in $|0,n_{\text{tot}}\rangle$ or $|n_{\text{tot}},0\rangle$ while the excitation gap decrease is very small. Thus the MI region with even occupation is enlarged by $T^-$ and the even-odd effect reappears.

Finally, Fig. 3(f) shows that the even-odd effect vanishes again if $T^+$ is also turned on. The explanation for this is quite similar to the case when $U < V$: the $T^+$ term lifts the degeneracy for odd $n_{\text{tot}}$ as its matrix element between $|G_+\rangle$ and $|G_-\rangle$ is nonzero.

We comment in passing that, from Figs. 3(a)–3(f), we may notice that the MI region with $n_{\text{tot}} = 1$ remains unchanged for different values of $T^\pm$. This is because the $T^\pm$ terms’ matrix elements are all zero in this space.

It is not difficult to notice from Figs. 7(a) and 7(c) that there is a discontinuous jump in both the particle numbers and their fluctuations at the critical tunneling rate $t_c$. The jump in $\Delta n_{\text{tot}}$, in particular, indicates that the phase transition from MI to SF may be of first order for even $n_{\text{tot}}$ when $U > V$ and $T^- \neq 0$. To verify this, we fix $U/V = 1.5$, $T^+ = 0$, $T^-/V = -0.05$, $\mu/V = 1.5$, and plot the ground-state energy $E(\phi_1,\phi_2)$ with different $t$ in Fig. 8. When $t$ is very small, $\phi_1 = \phi_2 = 0$ (MI) is the global minimum point [see Fig. 8(a)]. As $t$ increases, additional local minimum points with finite $\phi_1,\phi_2$ (metastable SF) arise, as shown in Fig. 8(b). If we continue to increase $t$, Fig. 8(c) displays that the MI state with $\phi_1,\phi_2 = 0$ becomes a local minimum point (i.e., a metastable MI). Finally, the local minimum point becomes a local maximum point if $t$ is sufficiently large as shown in Fig. 8(d). In conclusion, metastable SF and metastable MI states exist near the boundary of the MI lobes in Fig. 3(e) (between the dashed and solid line) while $n_{\text{tot}}$ is even. This existence of such metastable states is a tell-tale signature that the MI-SF phase transition in this case is of first order. A similar situation can be found in spin-1 bosons which can also host the first-order phase transition [12,13], but they are induced by different effects. In Fig. 8, another feature is also worth our attention. Only
one of the two species is dominant in the SF state, i.e., $|\phi_1| \gg |\phi_2|$ or $|\phi_2| \gg |\phi_1|$, and they are degenerate. This is consistent with Figs. 7(c) and 7(d) which indicate that only one component’s fluctuation is prominent while the other one’s is close to zero. In comparison, both components have significant occupation fluctuations in the SF state when $U < V$, as shown in Figs. 4(c), 4(d), 5(c), and 5(d).

Fig. 9. (Color online) The average particle numbers and their fluctuations obtained by the Gutzwiller method. For all the subplots, we have $V_1 = V_2 = V$, $U = 1.5$, $T^+ = 0.05$, and $T^- = 0$. The chemical potential is taken as $\mu = 1.5$ for (a) and (c); $\mu = 2.5$ for (b) and (d).

Furthermore, if $T^- = 0$ but $T^+ \neq 0$, not only the even occupation’s but also the odd occupation’s MI-SF phase transition will be of first order, as shown in Fig. 9, because all the MI ground states’ degeneracy is lifted in the presence of $T^+$ regardless of whether $n_{\text{tot}}$ is even or odd.

**IV. PERTURBATIVE ANALYSIS**

To gain more physical insights into the phase diagram, we consider the intercomponent interaction (the $U$ term) and the nonlinear coupling (the $T^\pm$ terms) as perturbations and examine how they affect the phase diagram. Furthermore, in order to distinguish the contribution from each of them, we have shown how each of these nonlinear perturbations affects the mean-field phase diagrams in Fig. 10.

In Fig. 10(a), we show the “unperturbed” phase diagram by taking $U = 0$ and $T^\pm = 0$. Here we simply have two uncoupled single-component Bose-Hubbard models. Since we have chosen the intracomponent interaction strength $V_1 = V_2 = V$, the two individual phase diagrams completely overlap with each other. In particular, each MI region is characterized by an equal number of the two polariton modes, and hence only even $N_{\text{tot}}$ MI regions are present. Along the horizontal axis ($t = 0$), the MI region with $N_{\text{tot}} = 2n$ occupies a region $(n - 1)V < \mu < nV$. At the boundary $\mu = nV$, the four Mott states $|n,n\rangle$, $|n,n+1\rangle$, $|n+1,n\rangle$, and $|n+1,n+1\rangle$ are energetically degenerate with energy $-n(n+1)V$.

In Fig. 10(b), we show the phase diagram in the presence of a small intercomponent interaction $U \ll V$. As one can see, the $U$ term induces MI regions with odd $n_{\text{tot}} = 2n + 1$, which for small $U$ occurs near the boundaries between two even MI regions at $\mu = nV$. At $t = 0$, these odd MI regions are represented by two degenerate Fock states $|n,n+1\rangle$ and $|n+1,n\rangle$, and occupy a region with $\mu \in (\mu_x, \mu_x^{-})$. The values of $\mu_x$ and $\mu_x^{-}$ can be readily obtained from $E(n,n,\mu_x) = E(n,n+1,\mu_x^{-})$ and $E(n,n+1,\mu_x) = E(n+1,n+1,\mu_x^{-})$, from which we obtain the width of the odd MI region in the $\mu$ axis as $\mu_x^{-} - \mu_x = U$, which is exactly the excitation gap $\Delta_{2n+1}$ of the MI state. The above analysis provides a more quantitative argument for the even-odd effect mentioned earlier. Furthermore, since the tunneling energy is roughly proportional to $\sqrt{n(n+1)}$, the MI regimes with occupation number $2n+1$ become smaller as $n$ grows. Our mean-field numerical results, as shown in the inset of Fig. 10(b), are in complete agreement with this analysis.

In Fig. 10(c), we show the phase diagram in the presence of the nonlinear coupling $T^+$ term, while keeping $U = 0$ and $T^- = 0$. In this case, in the absence of tunneling, the conservation of the number of polaritons in each component is broken, while the total polariton number $n_{\text{tot}}$ for each cavity remains conserved for Hamiltonian $H_0$. We also observe the appearance of odd MI regions except for $n_{\text{tot}} = 1$. Furthermore, the odd MI regions grow in size as $n_{\text{tot}}$ increases, and is in stark contrast with the previous case. These properties can be intuitively understood as follows. Consider an odd MI region with $n_{\text{tot}} = 2n + 1$ at $t = 0$, which is characterized by two degenerate Fock states $|n,n+1\rangle$ and $|n+1,n\rangle$ in the absence of $T^+$. When a small $T^+$ is turned on, a direct coupling between these two states is induced with the corresponding matrix element given by $2n(n + 1)T^+$. To first order in $T^+$, the ground-state energy is shifted down by this amount. The presence of the $T^+$ term will also lower the energies of even MI states, but the energy shift is quadratic in $T^+$ as the unperturbed
even MI states are nondegenerate. This explains the appearance of odd MI regions with \( n_{\text{tot}} = 2n + 1 \) for \( n > 0 \). There is no MI region for \( N_{\text{tot}} = 1 \), because the transition-matrix element between Fock states \( |0,1\rangle \) and \( |1,0\rangle \) vanishes. In addition, it is not difficult to show that the width of the odd MI region on the \( \mu \) axis is \( O(4n(n+1)T^*) \), which grows when \( n \) increases, in good agreement with the numerical results.

Actually, whether \( U \) is zero or not, we can always use the above picture, i.e., first-order energy shift to the state with degeneracy and second-order energy shift to the state without degeneracy, to understand the effect of \( T^* \). And it also applies to the situation even \( U > V \). As mentioned previously, the ground states with odd occupations are doubly degenerate when \( U > V \), \( T^- \neq 0 \) at \( t = 0 \), and \( T^+ \) has nonvanishing matrix elements between them which can induce a first-order energy shift. At the same time, the energy shift is second order in the even occupation case. On the other hand, the tunneling energy grows as \( T^+ \) increases. Therefore, the even-odd effect disappears in Fig. 3(f).

Finally, let us consider the effect of the \( T^- \) term. In the case \( U = 0 \) and \( T^- = 0 \), the \( T^- \) term alone would not induce odd MI regions. This is due to the fact that, unlike the \( T^+ \) term, this term does not induce a direct coupling between \( |n,n+1\rangle \) and \( |n+1,n\rangle \) states, i.e., the corresponding matrix element vanishes. Hence a small \( T^- \) does not have any noticeable effects on the phase diagram. However, the \( T^- \) term can be regarded as a nonlinear coupling between the two components of the polariton, whose effective sign depends on the population difference. As a result, in the superfluid region, when the relative population \( n_1 - n_2 \) changes sign, the relative phase between the two order parameters \( \phi_1 \) and \( \phi_2 \) will change from zero to \( \pi \). As this effect is already present in the single-cavity system which we investigated in detail in Ref. [21], we do not provide a detailed discussion here. Instead, we just present the phase diagram for the superfluid region in Fig. 11, in which 0SF and \( \pi \)SF are superfluid phases with relative phase between \( \phi_1 \) and \( \phi_2 \) being zero and \( \pi \), respectively. The transition between 0SF and \( \pi \)SF is of first order.

The perturbation calculation presented above allows us to better understand the phase diagram shown in Fig. 3. In particular, we can now explain why the interspecies interaction gives rise to the even-odd effect, and how the nonlinear coupling term \( T^+ \) weakens the even-odd effect.

V. EXACT DIAGNOSTICATION

So far we have investigated a homogeneous system of an interconnected cavity in one dimension using a mean-field approach. In this section, the exact diagonalization method is used to study this model. To make the calculation manageable, we consider a finite number, \( N_c \) of cavities with \( N_p \) total particles. The whole Hilbert space is spanned by the Fock state basis \( |\psi_i\rangle = |n_1^{\alpha_1},n_1^{\alpha_2},\ldots,n_1^{\alpha_N},n_2^{\gamma_1},n_2^{\gamma_2},\ldots,n_2^{\gamma_N}\rangle \) where \( n_\alpha^k \) denotes the number of polaritons in component \( \alpha \) in the \( k \)th cavity, and they are constrained as \( n_\alpha^k \geq 0 \) and \( \sum_{k=1}^{N_c}(n_1^k + n_2^k) = N_p \). The dimension of the Hilbert space is therefore \( D = N_p^{N_c} \).

We write the Hamiltonian into a matrix form under this Fock state basis using the periodic boundary condition, and obtain the ground state \( \langle G \rangle \) of this large sparse matrix through exact diagonalization. In Fig. 12, we plot the total number fluctuation per cavity which is defined as

\[
\Delta n = \langle G |n^k \rangle^2 \langle G \rangle - \langle G |n^k \rangle \langle G \rangle^2,
\]

where \( n^k = n_1^k + n_2^k \) is the total number operator for the \( k \)th cavity. Under the periodic boundary condition, this quantity is independent of the cavity index \( k \). We vary \( N_p \) and \( N_c \) to some extent while restricting their ratio \( N_p/N_c \) (i.e., the number of polaritons per cavity) to be 1, 2, or 3. As can be seen, the behavior of the number fluctuation is sensitive to \( N_p/N_c \). The parameters of Fig. 12(a) are the same as those used in Fig. 10(c). Here one can see that in the limit \( t \rightarrow 0 \) the number fluctuation for systems with \( N_p/N_c = 1 \) remains finite, which indicates the lack of MI region for \( N_{\text{tot}} = 1 \). By contrast, in the same limit, the number fluctuations for \( N_p/N_c = 2 \) and 3 vanish and more specifically \( \Delta n \) for \( N_p/N_c = 3 \) tends to zero with a much steeper slope. We thus expect to see a large MI.
region for $N_{\text{tot}} = 2$ and a small MI region for $N_{\text{tot}} = 3$. All these are fully consistent with the mean-field phase diagram shown in Fig. 10(c). The parameters used for Fig. 10(b) are the same as those used for Fig. 10(b). Here $\Delta n$ for all three ratios or $N_p/N_c$ vanish in the limit $t \to 0$. The slopes of $\Delta n$ near this limit indicate that two small MI regions for $N_{\text{tot}} = 1$ and 3 and a large MI region for $N_{\text{tot}} = 2$ in the mean-field limit are expected, which is again consistent with the results obtained earlier.

VI. SUMMARY

In summary, we have presented a scheme to realize a two-component BHM with nonlinear intercomponent coupling in a system of cavity polaritons. We mapped out the phase diagram showing the boundaries between the MI phase and the SF phase. Using several different approaches—the mean-field decoupling method, the Gutzwiller method, the perturbation calculation, and the exact diagonalization—we show how the interspecies interaction and the nonlinear coupling terms affect the phase diagram, and particularly how they induce the first-order MI-SF phase transition and give rise to or weaken the even-odd effect. Additionally, the competition between the nonlinear coupling strengths $T^+$ and $T^-$ can drive a first-order quantum phase transition within the SF regime that changes the relative condensate phase of the two polariton components. Through our study, we have obtained a clear understanding about this two-component BHM. In the future, we could also realize a two-component BHM that breaks the time-reversal symmetry [16,17] by manipulating the external fields’ relative phase. This model may host more exotic $p$-wave superfluid phases [15].

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