Bright Soliton Trains of Trapped Bose-Einstein Condensates

U. Al Khawaja,[1] H. T. C. Stoof,[1] R. G. Hulet,[2] K. E. Strecker,[2] and G. B. Partridge[2]
[1]*Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, 3584 CE Utrecht, The Netherlands*
[2]*Department of Physics and Astronomy and Rice Quantum Institute, Rice University, Houston, Texas 77251*
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We variationally determine the dynamics of bright soliton trains composed of harmonically trapped Bose-Einstein condensates with attractive interatomic interactions. In particular, we obtain the interaction potential between two solitons. We also discuss the formation of soliton trains due to the quantum mechanical phase fluctuations of a one-dimensional condensate.

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Introduction.—The existence of solitonic solutions is a very general feature of nonlinear wave equations. In the case of an atomic Bose-Einstein condensate, the macroscopic wave function of the condensate obeys the so-called Gross-Pitaevskii equation, whose nonlinearity is a result of the interatomic interactions. Depending on the repulsive or attractive nature of the interatomic interactions, the Gross-Pitaevskii equation allows for either *dark* or *bright* solitons, respectively. The properties of dark solitons have been extensively studied theoretically [1–8]. They have also been created experimentally in elongated Bose-Einstein condensates [9–11]. Much less is known of bright solitons [12], which have only very recently been created in two experiments with Bose-Einstein condensates of ^{7}Li atoms [13,14].

In the experiment of Strecker *et al.*, soliton trains consisting of up to ten solitons have been observed [14]. Moreover, it was found that, even though the interatomic interactions are attractive, the neighboring solitons repel each other with a force that is dependent on their separation. Here we confirm that the source of this repulsive force is a phase difference of π between two neighboring solitons [15]. Physically, this can be understood from the fact that the antisymmetric nature of the many-soliton wave function prevents the solitons from penetrating each other. Although the phase difference explains the repulsive interactions between the solitons, it presents us with the problem of understanding why the experiment always seems to create soliton trains in which the neighboring solitons have a π phase difference. We also address this interesting question here.

To understand the dynamics of the solitons, we use a variational approach, in which we describe the wave function of the individual solitons as a Gaussian and then make an appropriate linear superposition of these Gaussians to represent the soliton train. Both the width and position of a Gaussian can be varied. However, since we are dealing with solitons, which by definition do not show any dispersion, only the position is allowed to depend on time. Using this trial wave function, we derive equations of motion for the center-of-mass position and the relative distances between the solitons. As expected, the center-of-mass motion decouples completely from the relative motion. The dynamics of the soliton separations compare favorably with the experimentally observed behavior.

Single soliton dynamics.—We first study the equilibrium properties and dynamics of a single soliton. As in the experiments, we consider bright solitons formed from a condensate that is confined by a very elongated axially symmetric harmonic potential, for which the ratio of the radial and the axial trapping frequencies obey $\omega_r/\omega_z \gg 1$. We are, therefore, mostly interested in the motion of the soliton in the axial direction and can represent the wave function of a single soliton by a trial function of the form

$$\psi(\mathbf{x}, t) = A_{\rm sol}(z - \zeta(t), r) \exp\left(i \frac{m}{\hbar} \frac{d\zeta(t)}{dt} z\right), \tag{1}$$

where $\zeta(t)$ represents the soliton's center of mass, and m is the atomic mass. For the amplitude of the soliton, we take the Gaussian

$$A_{\rm sol}(\mathbf{x}) = \sqrt{\frac{N}{\pi^{3/2} q_z q_r^2}} \exp\left(-\frac{z^2}{2q_z^2} - \frac{r^2}{2q_r^2}\right), \tag{2}$$

where q_z and q_r are two variational parameters that determine the width of the soliton in the axial and radial directions, respectively. The total number of atoms in the condensate is denoted by N. Note that this Gaussian *ansatz* is particularly well-suited when the axial width of the soliton is comparable to the axial harmonic oscillator length $\ell_z = \sqrt{\hbar/m\omega_z}$. In the case that $q_z \ll \ell_z$, it would be better to replace the exponential $\exp(-z^2/2q_z^2)$ by $\sqrt{\pi}/[2\cosh(z/q_z)]$ [13,15]. However, we see later that the Gaussian *ansatz* gives physically reasonable results in this case also. Note that we take q_z and q_r as time independent, since we do not want to consider the possible breathing motion of the solitons here [16].

The equilibrium widths of the soliton can be calculated by minimizing the Gross-Pitaevskii energy functional

$$E[\psi^*, \psi] = \int d\mathbf{x}\left[\frac{\hbar^2}{2m}|\nabla\psi(\mathbf{x}, t)|^2 + V^{\rm ext}(\mathbf{x})|\psi(\mathbf{x}, t)|^2 + \frac{1}{2} T^{2B} |\psi(\mathbf{x}, t)|^4\right], \tag{3}$$

with respect to q_z and q_r [17]. The external potential obeys $V^{\rm ext}(\mathbf{x}) = m(\omega_z^2 z^2 + \omega_r^2 r^2)/2$ and the interatomic interaction strength $T^{2B} = 4\pi a\hbar^2/m$ is proportional to the negative s-wave scattering length a. Assuming that q_r has been found in the above manner, the physics becomes effectively one dimensional. The one-dimensional energy for $\psi(z, t) \equiv \int dr 2\pi r \psi(\mathbf{x}, t)$ is again given by Eq. (3) but now we must use $T^{2B} = 4\pi\kappa\hbar^2/m$ with $\kappa = a/2\pi q_r^2$ [18]. It should be kept in mind, however, that the one-dimensional theory can be applied only for a soliton containing less than $N_{\max} = \mathcal{O}(\ell_r/|a|)$ atoms, where $\ell_r = \sqrt{\hbar/m\omega_r}$ is the radial harmonic oscillator length. Above this number of atoms the soliton will collapse [19–21].

Using this effective one-dimensional picture, the equation of motion for $\zeta(t)$ is finally derived by substituting the Gaussian *ansatz* $\psi(z, t)$ into the action $S[\psi^*, \psi] = \int dt(\int dz i\psi^* \partial\psi/\partial t - E[\psi^*, \psi])$. The resulting equation $d^2\zeta(t)/dt^2 = -\omega_z^2\zeta(t)$ shows that the center of mass of the soliton oscillates sinusoidially with the trap frequency. This is, of course, an exact result that follows from the Kohn theorem [22].

Two soliton dynamics.—Our trial wave functions for two solitons with a phase difference of 0 or π are, respectively, the symmetric and antisymmetric combinations of the two Gaussian wave functions of a single soliton. In detail, we use

$$\psi_\pm(z, t) = \frac{1}{\sqrt{N_\pm(t)}}\left[A_{\rm sol}(z - z_1(t))\exp\left(i\frac{m}{\hbar}\frac{dz_1(t)}{dt}(z - \zeta(t))\right) \pm A_{\rm sol}(z - z_2(t))\exp\left(i\frac{m}{\hbar}\frac{dz_2(t)}{dt}(z - \zeta(t))\right)\right]. \tag{4}$$

Here $z_1(t)$ and $z_2(t)$ denote the positions of the two solitons and $\zeta(t) = (z_1(t) + z_2(t))/2$ is their center of mass. For simplicity we consider here only the case of a phase difference of 0 or π between the solitons. In addition, we take the number of atoms in each soliton, and therefore also their widths, equal to each other. The generalization to an arbitrary phase and an arbitrary number of atoms is straightforward but somewhat tedious. Fortunately, the above assumptions turn out to be reasonable when we compare our results with experiments, and it allows us to bring out the dynamics of the relative separation most clearly. Finally, for the rest of this section, we scale length to $\ell_z = \sqrt{\hbar/m\omega_z}$, time to $1/\omega_z$, and energy to $\hbar\omega_z$. The normalization constant $N_\pm(t)$ is then given by

$$N_\pm(t) = 2\left(1 \pm \exp\left[-\frac{\eta^2(t)}{4q_z^2} - \frac{q_z^2}{4}\left(\frac{d\eta(t)}{dt}\right)^2\right]\right), \tag{5}$$

where $\eta(t) = z_1(t) - z_2(t)$ is the distance between the two solitons. Note that N is still the total number of atoms in the condensate, so a single soliton contains only $N/2$ atoms.

Using this trial wave function, the Lagrangian per atom takes the form

$$\frac{L[\zeta, \eta]}{N} = \frac{1}{2}\left[\left(\frac{d\zeta}{dt}\right)^2 - \zeta^2\right] + \frac{1}{8}\left[\left(\frac{d\eta}{dt}\right)^2 - \eta^2\right] - \frac{1}{4}\left[\frac{1}{q_z^2} + q_z^2 + V[\eta]\right], \tag{6}$$

where the effective potential $V[\eta]$ is given by

$$V[\eta] = \mp\frac{1}{N_\pm}[(1 + q_z^{-4})\eta^2 + (q_z^4 - 3)\dot{\eta}^2]\exp\left(-\frac{\eta^2 + q_z^4\dot{\eta}^2}{4q_z^2}\right) + \frac{8\sqrt{2\pi}N\kappa}{N_\pm^2 q_z} \times\left[1 + \exp\left(-\frac{\eta^2 + q_z^4\dot{\eta}^2}{2q_z^2}\right) + 2\exp\left(-\frac{\eta^2}{2q_z^2}\right) \pm 4\exp\left(-\frac{3\eta^2 + q_z^4\dot{\eta}^2}{8q_z^2}\right)\cos\left(\frac{\eta\dot{\eta}}{4}\right)\right], \tag{7}$$

and $\dot{\eta} = d\eta/dt$. It is clear from this Lagrangian that the center-of-mass motion and the relative motion are completely decoupled. Furthermore, the equation of motion for the center of mass is simply $d^2\zeta(t)/dt^2 = -\zeta(t)$, which corresponds again to the Kohn mode that oscillates at the frequency of the trap. On the other hand, the equation of motion for the relative motion is rather complex due to the coupling between the two solitons that is determined by the velocity-dependent potential $V[\eta]$.

In Fig. 1, we compare the exact numerical solution of the relative motion with experimental data. The parameters of the experiment and the theory are the same as those of Ref. [14], and we take the antisymmetric combination of Gaussians. Very good agreement is obtained with both the experimentally observed amplitude of oscillation of the relative coordinate and its frequency, which is approximately $2\omega_z$. In particular, we see that the two solitons never cross each other. This result, and the nonsinusoidal nature of $\eta(t)$, are both due to the hard-core nature of the interaction when the solitons start to overlap.

To show this, we neglect the velocity dependence of $V[\eta]$, assume that $q_z \ll 1$, and take the limit of large distances between the two solitons, i.e., $\eta \gg q_z$. With these approximations, we obtain

$$V[\eta] = \frac{2\sqrt{2\pi}N\kappa}{q_z} \mp \left(\frac{4\sqrt{2\pi}N\kappa}{q_z} + \frac{\eta^2}{2q_z^4}\right)\exp\left(-\frac{\eta^2}{4q_z^2}\right). \tag{8}$$

The second term in the right-hand side represents the interaction energy $V(\eta)$ between the two solitons. For the antisymmetric choice of the two-soliton wave function,

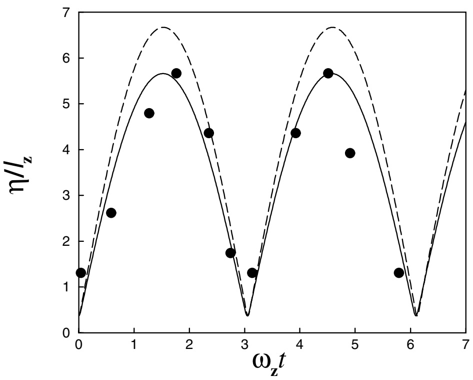

FIG. 1. The relative motion of two solitons. There are 5000 atoms in each soliton and $a = -3a_0$, with a_0 the Bohr radius. The solid curve is the solution of the exact equation of motion for $\eta(t)$. The dashed curve is the solution of Eq. (9) with $V(\eta) = (\eta^2/2q_z^4)\exp(-\eta^2/4q_z^2)$. The points are experimental data obtained using the same apparatus described in Ref. [14]. In the experiment more than two solitons were created. We focused on two large and adjacent ones.

this potential energy is positive, which implies that the two solitons will indeed repel each other.

In this approximation, the equation of motion of $\eta(t)$ takes the form of Newton's equation

$$\frac{d^2\eta(t)}{dt^2} = -\eta(t) - \frac{dV(\eta(t))}{d\eta}. \qquad (9)$$

Thus, the problem reduces to that of a particle moving in the potential $\eta^2/2 + V(\eta)$, which has two minima located symmetrically around $\eta = 0$. Depending on the initial conditions, the two solitons can be trapped in these minima. In other words, the separation between them oscillates and the solitons never cross each other, as verified by the exact solution. To investigate the importance of the interatomic interactions, we show in Fig. 1 with the dashed curve the solution to Eq. (9) when the potential contains only the contribution from the kinetic energy of the atoms. A solution of Eq. (9) with the full potential $V(\eta)$ from Eq. (8) is almost identical to the exact solution. The smaller amplitude of oscillation in the relative motion is, therefore, a clear signature of the reduction of the repulsive force between the solitons due to the interatomic interactions. We have also analytically calculated the interaction energy for two solitons with an amplitude proportional to $\sqrt{\pi}/[2\cosh(z/q_z)]$ and solved the corresponding equation of motion. It leads to only minor corrections that are due to the fact that the potential $V(\eta)$ is now somewhat more repulsive and only falls off as $(4\eta/q_z^3)\exp(-\eta/q_z)$ for large separations.

Soliton train formation.—In the experiment of Strecker *et al.* [14], soliton trains are formed by making use of a so-called Feshbach resonance [23]. In summary, the procedure consists of first making a stable condensate with a relatively large positive scattering length and then switching to a small negative value. An important clue for understanding the mechanism for the creation of the soliton train is the observation that all the solitons repel each other and, thus, have a phase difference equal to π. This suggests that, by changing the sign of the scattering length, the phase modes of the condensate become unstable. Roughly speaking, the most important wavelength of the unstable phase modes must then be comparable to the separation between solitons in the train.

To make this physical idea quantitative, we consider a homogeneous condensate with a density $n = N/L$, where L is the (Thomas-Fermi) length of the condensate. Denoting the phase of the condensate wave function $\psi(z,t)$ by $\chi(z,t)$, it is easy to show from the action $S[\psi^*,\psi]$ that $\chi_k(t) \equiv \int dz e^{-ikz}\chi(z,t)$ corresponds to a (complex) harmonic oscillator for every wave vector $k > 0$. Physically, this implies that the quantum mechanics of every phase mode is analogous to the quantum mechanics of a fictitious particle in a two-dimensional harmonic oscillator potential. The mass of this fictitious particle is $m_k = 4\epsilon_k n/\omega_k^2 L$, and the frequency of the harmonic oscillator potential is $\omega_k = \sqrt{\epsilon_k^2 + 2nT^{2B}\epsilon_k}/\hbar$, with $\epsilon_k = \hbar^2k^2/2m$. Initially, the scattering length is positive and all these frequencies are positive. Assuming that the condensate has come to equilibrium, the quantum fluctuations of $\chi_k(t)$ are then fully determined at zero temperature by the ground-state wave function $(1/\pi\ell_k^2)\exp(-|\chi_k|^2/2\ell_k^2)$, with $\ell_k = \sqrt{\hbar/m_k\omega_k}$. In particular, we have that $\langle\chi(z,t)\rangle = 0$.

However, if at $t = 0$ we suddenly change the scattering length to a negative value, the modes with $k < k_{\max} = \sqrt{16\pi|\kappa|n}$ become unstable because the spring constant $m_k\omega_k^2$ of these harmonic oscillators becomes negative and equal to $-m_k\Omega_k^2$. Solving the quantum mechanics of a particle in an inverted harmonic oscillator potential, we can then show that, for $t \gg 1/\Omega_k$,

$$\langle|\chi_k(t)|^2\rangle \simeq \frac{\hbar}{4m_k\omega_k}\left(1 + \frac{\omega_k^2}{\Omega_k^2}\right)e^{2\Omega_k t}. \qquad (10)$$

As in the case of a spontaneously broken symmetry, the latter result can physically be understood by saying that quantum mechanical fluctuations *imprint* the condensate wave function with the phase

$$\langle\chi(z,t)\rangle \simeq \int_0^{k_{\max}} \frac{dk}{2\pi}\cos(kz)\sqrt{\frac{\hbar\omega_k L}{4\epsilon_k n}\left(1 + \frac{\omega_k^2}{\Omega_k^2}\right)}e^{\Omega_k t}. \qquad (11)$$

Having obtained this result, we are now able to simulate the experiments. We first determine the ground-state wave function of a condensate with a positive scattering length for the experimental parameters of interest. We then imprint this wave function with the phase given in Eq. (11) and let the resulting wave function evolve under the appropriate Gross-Pitaevskii equation with a negative scattering length. We note here that, due to numerical limitations, we have used an initial number of 10^4 atoms which is an order of magnitude less than that in the

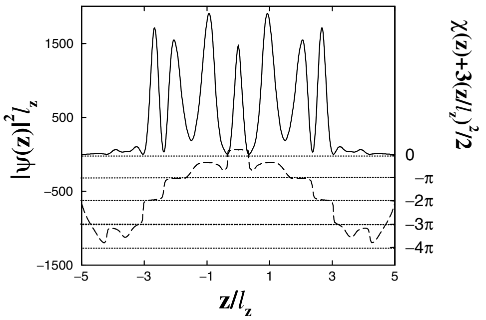

FIG. 2. Soliton train formation. The solid curve is the density and the dashed curve is the phase of the condensate. Initially, we start with an equilibrium condensate profile for 10^4 atoms with a scattering length of $200a_0$. The scattering length is then instantaneously changed to $a = -3a_0$. The trap parameters are obtained from Strecker *et al.* [14].

experiment. In this case, we find that after an evolution time of $t \simeq 1.8/\omega_z$, seven solitons are formed, together with phonon excitations. The solitons are clearly visible as plateaus in the phase, after subtraction of the parabolic behavior of the phase due to the overall shrinking of the condensate. The phase difference between two adjacent solitons is indeed approximately equal to π. Some deviations from π are expected because the solitons are moving relative to each other. These results are shown in Fig. 2, where we plot both the density and the phase of the condensate wave function. Note that our calculation includes no relaxation processes due to the presence of a thermal cloud [24]. Including these stabilizes the soliton train by damping the phonon excitations and the soliton motion.

Conclusions.—In this Letter, we have considered the formation and dynamics of bright soliton trains in a trapped Bose-Einstein condensate with attractive interactions. In particular, we have shown that two solitons with a phase difference of π repel each other. The resulting dynamics compares favorably with experiment. In the experiment, always more than two solitons were created, while here we have focused on only two. In principle, a generalization to more than two solitons is straightforward. In this case, we just sum in the Lagrangian of Eq. (6) over all the positions $z_i(t)$ of the bright solitons.

We have also shown that soliton trains can be dynamically generated by quantum mechanical phase fluctuations of the condensate, due to the modulational instability [25] that exists in a Bose-Einstein condensate with attractive interactions. This mechanism gives an intuitive explanation why adjacent solitons have a π phase difference. It also predicts that the number of solitons that is formed after a rapid, i.e., fast compared to $1/\omega_z$, sign change in the scattering length is equal to the ratio of the initial condensate size and the wavelength of the most unstable phase mode. However, this prediction does not take dissipation into account. Because solitons are not topological objects, dissipation can lead to a considerably different number of solitons as it drives the gas to equilibrium. In the experiment, the number of solitons in the train is about half of the dissipationless prediction [14]. This observation suggests that dissipation processes are indeed present. A detailed comparison with experiment thus requires knowledge of the properties of the thermal cloud, which are not well known experimentally. It was also observed that the number of solitons in the train is, over a large range, independent of the exponential time constant with which the scattering length is changed. This is as expected as long as dissipation plays an important role and the sign change in the scattering length remains nonadiabatic.

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