Convergence Analysis of Discontinuous Galerkin Methods for Poroelasticity Equations

by

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ABSTRACT

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This thesis analyzes a numerical method for solving the poroelasticity equations. The model incorporating the poroelasticity equations in this thesis can be applied in intestinal edema, which is a medical condition referring to the accumulation of excess fluid in the spaces between cells of intestinal wall tissue. The model has a dilatation term and can give a comprehensive prediction of pressure and displacement for intestinal edema. I approximate the pressure, displacement and dilatation by the discontinuous Galerkin method, which includes symmetric, nonsymmetric and incomplete interior penalty Galerkin cases. Moreover, in order to solve for the nonsymmetric case, I introduce an additional penalty term in the scheme. Theoretical convergence error estimates derived in a discrete-in-time setting show the a priori error can be bounded by some constant, which is related to the pressure, displacement, dilatation and the mesh size.
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Chapter 1

Introduction

This thesis provides a theoretical analysis of a numerical method for solving the poroelasticity equations. The model consisting of the poroelasticity equations in this thesis can be applied in intestinal edema, which is a medical condition referring to the accumulation of excess fluid in the space between cells of intestinal wall tissue. I approximate the pressure, displacement and dilatation by the discontinuous Galerkin method, which includes symmetric, nonsymmetric and incomplete interior penalty Galerkin cases. Moreover, in order to solve for the nonsymmetric case, I introduce an additional penalty term in the scheme.

1.1 About poroelasticity

There are many porous materials in the world, such as natural objects: rock, soil, biological tissues, and artifacts: plastic, ceramics. Poroelasticity is the modeling of transport phenomena in porous media. From the 1930s to the 1960s, Maurice Anthony Biot [7, 5, 6] built the foundation of theories of poroelasticity, which is now known as Biot theory. Modeling poroelasticity involves the coupling between a transport law (i.e. Darcy’s law) and a balance law. Darcy’s law describes the fluid flow in the porous media, and the balance law is related to the balance laws of mass and momentum.
These two laws can be coupled by Biot theory. The poroelasticity model can be applied in many areas, such as biomechanics, reservoir engineering, environmental engineering and so on.

Showalter has done many theoretical works on poroelasticity model, such as proving the existence and uniqueness of weak solution of some special models in porous media. For example, in the work [35], Showalter reported an existence, uniqueness and regularity theory for a general initial-boundary-value problem for a system of partial differential equations which describes the Biot consolidation model in poroelasticity, and in the paper [34], he gave a standard weak solution for the coupled system of fluid transport and poro-mechanics. From the works [33, 36, 37, 34] of Showalter et al., we know that the true weak solutions corresponding to the problems of poroelasticity exist and are unique.

Mikelić and Wheeler [22] demonstrated the stability and convergence of two schemes: the undrained split method and the fixed stress split method for the quasi-static Biot system. Additionally, they reported a new method with faster convergence rate than known schemes.

There is a large amount of published work on the numerical computation of the poroelasticity equations. For instance, in the work of Mercer et al. [21], the authors illustrated a numerical scheme in an implicit time stepping routine and approximated the deformation, pressure and flow by finite difference method in a two-dimensional rectangular deformable porous medium. Atalla et al. [2] presented a mixed displace-
ment pressure formulation which was derived from Biot’s poroelasticity equations, and they discussed the numerical implementation of the presented approximation in a finite element code. Pain et al. [26] developed a numerical method for the poroelasticity Biot equations coupled to a simplified form of Maxwell’s equations, and they solved the equations by a mixed finite-element method in space domain and by an implicit unconditionally stable time-stepping method in time domain.

In the work of Korsawe et al. [19], the authors developed two different numerical approaches in space domain for consolidation in porous media, which were Galerkin finite element method (GFEM) and least-squares mixed finite methods (LS-MFEM), and they discretized the time derivative by a first-order finite difference scheme. The comparison of results of the two methods, showed that the GFEM preserved steep pressure gradients but it overestimated effective stress, and the LS-MFEM can obtain the direct approximation of stress but the cost is larger than GFEM.

Ferronato et al. [12] developed an advanced fully coupled 3-D mixed finite element model of the Biot equations. The model can alleviate the pore pressure numerical oscillations at the interface between materials with different permeabilities. They approximated the medium displacement by a family of linear piecewise polynomials and the fluid flow rate by the lowest order Raviart-Thomas mixed space, and numerically integrated the equations in time by a finite difference scheme. The authors implemented their algorithm and showed that the model was successfully experimented within realistic applications of soil consolidation.
In 2008, Tchonkova et al. [39] developed an original mixed least-squares method for solving Biot consolidation problems. They obtained the solution by minimization of a least-squares functional, related to the equations of equilibrium (conservation of mass and momentum). The authors approximated displacements, stresses, fluid pressures and velocities by linear continuous functions and used backward-Euler method in time. The authors tested the method for one- and two-dimensional classical problems in poroelasticity and proved that the rates of convergence for the four unknowns were the same, when the same interpolation spaces were used.

Haga et al. [17] wanted to avoid the non-physical pressure oscillations observed in finite element calculations of Biot's poroelastic equations in low-permeable media. For consistency, they employed a first-order backward finite difference method in time in the whole paper. By comparing the oscillations in low-permeable porous media with the oscillations in low-compressible porous media, they found using stable mixed elements in space can help free the oscillations.

Next I will describe in detail the few papers that contain convergence error estimates of discretization of the Biot's consolidation model in poroelasticity by different numerical methods.

Zenisek [45] proved the existence and uniqueness of the weak solution of an initial-boundary value for Biot's model of consolidation of clay, and introduced some error estimates. In this paper, the author approximated the pressure and displacement by using the simplest finite elements, and they discretized in time by backward-Euler
In the paper [13] of Gaspar et al., the authors presented stability estimates and convergence analysis of finite difference methods for the Biot consolidation model for a saturated homogeneous, isotropic, porous medium composed of an incompressible solid matrix. The authors dealt with the one-dimensional domain case. They approximated the pressure, displacement and effective stress by the finite difference method in continuous time. They carried out some numerical tests in order to illustrate the theoretical results.

Moreover, Phillip et al. [27, 28] formulated a finite element procedure for approximating the coupled fluid and mechanics in Biot’s consolidation model in poroelasticity. In these two papers, they approximated the pressure by a mixed finite element method and the displacements by continuous Galerkin (CG) method. In the paper [27], the authors gave theoretical convergence errors for the continuous in time scheme. In the paper [28], the authors also gave theoretical convergence errors for the fully discrete scheme. They used Taylor expansion in time. The results of numerical experiments in both papers showed very good agreement with the theoretical convergence errors. In the article [29] of Phillip et al., they approximated the flow variable by a mixed finite element method and the displacement by discontinuous Galerkin (DG) method, and the authors gave convergence error estimates for the continuous in time scheme. The authors showed that by the results of the above three papers, although the DG method produced the same theoretical convergence rates as the CG method, the DG
method was more suitable than the CG method on unstructured grids.

This thesis will focus on a poroelasticity model which can be applied in intestinal edema. The next section describes existing models of intestinal edema.

1.2 Intestinal Edema

Intestinal edema is the accumulation of abnormally large amounts of fluid in the space between cells of intestinal wall tissue [15]. This medical condition is commonly seen in patients with gastroschisis, inflammatory bowel disease and cirrhosis, especially the patients recovering from abdominal injury. Intestinal edema can cause other diseases, such as ileus and even death.

In the field of intestinal edema, both significant experimental work [24, 8, 23, 40, 41] and theoretical work about volume regulation, fluid balance and fluid transport in the interstitium [10, 16, 38, 11] have been done. There are some edema models [25, 43] developed for other organs. In the work of Nagashima [25], they developed a two-dimensional model for vasogenic brain edema by using the finite element method. Wiener et al. [43] developed a mathematical model that integrated different processes affect fluid and protein transport in the lung. These models [25, 43] do not consider the changes of the tissue that surrounds the interstitium. Hence, volume changes in these models depend only on current pressure but not on updated tissue stress.

Based on the Biot model for poroelasticity [7, 5], Young et al. [44] constructed a mathematical model of intestinal edema formation, which can overcome the pre-
Figure 1.1: The left picture is a cross-section of the intestine [44]; the right picture is a microscopy-section of the intestine.

vious limitations and correctly capture the pressure and volume changes in the four experimental scenarios presented in [8]. But they did not give any theoretical results.

Why could Young et al. construct the model from the poroelasticity Biot model? First, the intestine is a multi-layered organ of the digestive system. The left picture of Figure 1 shows a circular cross-sectional diagram of the intestinal layers, and the right one* is a microscopy-sectional diagram of the intestinal layers. The fluids can accumulate in the interstitium between tissue cells. In balanced conditions, fluid volume in the interstitium is constant because of a balanced fluid exchange between blood capillaries and lymphatics [11]. When the balance is destroyed and the rate of fluid added is larger than the rate of fluid removed, edema will occur.

Young et al. [44] considered that the blood and lymph capillaries are the fluid sources and modeled the intestinal wall as a poroelasticity medium. Based on the

Biot model [7, 5], their model was developed under the following assumptions:

- The interstitial fluid is incompressible.
- The interface between the fluid and solid phases is impermeable.
- The solid phase at the microscopic level preserves its volume.

The first assumption means the compressibility coefficient of the interstitial fluid is zero.

1.3 My work

In the application, the interstitial fluid is compressible. Therefore, I extend the work of Young et al. [44] and add a compressibility term to the model. My model related to the Biot model for poroelasticity holds under the following two assumptions:

- The interface between the fluid and solid phases is impermeable.
- The solid phase at the microscopic level preserves its volume.

In the following chapter, I will describe the model problem and construct the numerical scheme in space by the DG method [30, 31, 14] and discretize in time by the backward-Euler method. In order to solve for the nonsymmetric case, I introduce an additional penalty term in the DG scheme. Then I will show that the approximated numerical solutions exist and are unique. In chapter 3, I will give some results of the convergence error estimate for symmetric form and nonsymmetric form, respectively.
Chapter 2

Model Problem and Scheme

First Section 2.1 simply introduces the Biot model. Then in Section 2.3, the numerical scheme of the model constructed by the DG method is defined. In order to estimate the numerical error for the nonsymmetric case, an additional penalty term is introduced. Finally Section 2.4 contains a proof of existence and uniqueness of the numerical solutions.

2.1 Model problem

In this section we outline the derivation of the Biot model (see book by Bear et al. [4]). We consider the case of mass transport of a single fluid phase under isothermal conditions, and assume the following:

Assumption 2.1 : There is no mass of the considered fluid phase crossing the fluid-solid interface, i.e., the interface is impermeable.

The fluid’s mass balance can then be written in the form:

\[
\frac{\partial (n \rho_f)}{\partial t} + n \mathbf{V}_f \cdot \nabla \rho_f + \rho_f \nabla \cdot \left( -\frac{\kappa}{\mu_f} \nabla p + n \mathbf{V}_s \right) + \rho_f \phi(p) = 0 \tag{2.1}
\]

where \( n \) is the porosity; \( \rho_f \) is the density of the fluid; \( \mathbf{V}_f \) denotes the velocity of the fluid; \( \mu_f \) is the fluid viscosity; \( \kappa \) is the medium’s permeability; \( p \) is the hydro-
static pressure; $V_s$ is the solid’s velocity; $\phi(p)$ is the pressure dependent source term representing fluid that is added or removed.

The solid’s mass balance can be written as:

$$
\frac{\partial((1 - n)\rho_s)}{\partial t} + \nabla \cdot ((1 - n)\rho_s V_s) = 0 \tag{2.2}
$$

where $\rho_s$ is the density of the solid.

From equations (2.1) and (2.2), we eliminate the term $\nabla \cdot V_s$ and obtain:

$$
-\frac{\kappa}{\mu_f} \Delta p + \frac{n}{\rho_f} \frac{D_f \rho_f}{Dt} + \frac{1}{1 - n} \frac{D_s n}{Dt} - \frac{n}{\rho_s} \frac{D_s \rho_s}{Dt} + \phi(p) = 0 \tag{2.3}
$$

where $D_f(\cdot)/Dt$ and $D_s(\cdot)/Dt$ are the material derivatives with respect to the fluid and the solid, and the definitions are:

$$
\frac{D_f(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + V_f \cdot \nabla (\cdot)
$$

$$
\frac{D_s(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + V_s \cdot \nabla (\cdot)
$$

Assumption 2.2 : The solid phase keeps its volume microscopically.

The assumption 2.2 means $\frac{D_s \rho_s}{Dt} = 0$. Then the equation (2.3) becomes:

$$
-\frac{\kappa}{\mu_f} \Delta p + \frac{n}{\rho_f} \frac{D_f \rho_f}{Dt} + \frac{1}{1 - n} \frac{D_s n}{Dt} + \phi(p) = 0 \tag{2.4}
$$

By assumption 2.2, the equation (2.2) becomes:

$$
\frac{1}{1 - n} \frac{D_s (1 - n)}{Dt} + \nabla \cdot V_s = 0 \tag{2.5}
$$

Combining equations (2.4) and (2.5), we obtain:

$$
-\frac{\kappa}{\mu_f} \Delta p + \frac{n}{\rho_f} \left( \frac{\partial \rho_f}{\partial t} + V_f \cdot \nabla \rho_f \right) + \nabla \cdot V_s + \phi(p) = 0 \tag{2.6}
$$
We denote by $w$ the position of the solid phase. Then we have:

$$V_s = \frac{\partial w}{\partial t}$$

We make an additional assumption:

**Assumption 2.3**: $|\frac{\partial \rho_f}{\partial t}| \gg |\mathbf{V}_f \cdot \nabla \rho_f|$ which, with $\rho_f = \rho_f(p_f)$ and $\frac{\partial \rho_f}{\partial t} = \frac{d\rho_f}{dp} \frac{\partial p}{\partial t} = c \rho_f \frac{\partial p}{\partial t}$ (here $p$ is the pressure and $c$ is the fluid’s coefficient of compressibility), is equivalent to $|\frac{\partial p}{\partial t}| \gg |\mathbf{V}_f \cdot \nabla p|$.

Then equation (2.6) becomes:

$$-\frac{\kappa}{\mu_f} \Delta p + nc \frac{\partial p}{\partial t} + \frac{\partial}{\partial t} (\nabla \cdot w) + \phi(p) = 0 \quad (2.7)$$

From the conservation of momentum, we have:

$$\nabla \cdot (\mu(\nabla w + (\nabla w)^T) + \lambda(\nabla \cdot w)I) - \nabla p = 0 \quad (2.8)$$

where $\mu$ and $\lambda$ are Lamé coefficients of the solid material.

We simplify the above equation and obtain:

$$\mu \Delta w + (\mu + \lambda) \nabla (\nabla \cdot w) - \nabla p = 0 \quad (2.9)$$

Throughout the thesis, $\Omega$ denotes a bounded polygonal domain in $\mathbb{R}^d$ (d=2 or 3) and we consider functions defined on $\Omega \times (0, T)$, where $(0, T)$ is the time interval.

We introduce the dilatation:

$$\varepsilon = \nabla \cdot w \quad (2.10)$$
The source term is a linear function of pressure [44]: \( \phi(p) = -(\alpha p + \nu) \). Combining equations (2.7) and (2.9), we obtain our PDE model:

\[
\begin{align*}
\frac{c}{\partial t} \frac{\partial p}{\partial t} + \frac{\partial \varepsilon}{\partial t} - \kappa \Delta p &= \alpha p + \nu, \text{ in } \Omega \times (0, T) \\
\nabla \cdot \mathbf{w} - \varepsilon &= 0, \text{ in } \Omega \times (0, T) \\
-\mu \Delta \mathbf{w} + \nabla p - (\mu + \lambda) \nabla \varepsilon &= 0, \text{ in } \Omega \times (0, T)
\end{align*}
\] (2.11a-b-c)

We use the following boundary conditions:

\[
\partial \Omega = \Gamma_{pD} \cup \Gamma_{pN} = \Gamma_{wD} \cup \Gamma_{wN}
\]

\[
p = p_D \text{ on } \Gamma_{pD}, \quad \nabla p \cdot \mathbf{n}_{pN} = g_N \text{ on } \Gamma_{pN}
\]

\[
\mathbf{w} = \mathbf{w}_D \text{ on } \Gamma_{wD}, \quad \mu \nabla \mathbf{w} \mathbf{n}_{wN} + (\mu + \lambda) \varepsilon \mathbf{n}_{wN} - \rho \mathbf{n}_{wN} = g_w \text{ on } \Gamma_{wN}
\]

We recall that \( \mu, \lambda \) are the Lamé coefficients of the solid material; \( \kappa \) is the medium’s permeability; \( c \geq 0 \) is a constrained specific storage coefficient; \( \alpha \) is a positive constant; \( \nu \) is some constant.

We set the following initial conditions:

\[
p = p_0, \text{ on } \Omega \times \{0\}
\]

\[
\varepsilon = \varepsilon_0, \text{ on } \Omega \times \{0\}
\]

\[
\mathbf{w} = \mathbf{w}_0, \text{ on } \Omega \times \{0\}
\]
2.2 Notation

Let $E_h = \{E_1, E_2, \cdots, E_{N_h}\}$ be a nondegenerate quauniform subdivision of $\Omega$. In this thesis, the two-dimensional elements $E_i$ ($i = 1, 2, \cdots, N_h$) are triangles and the three-dimensional elements $E_i$ ($i = 1, 2, \cdots, N_h$) are tetrahedrons.

There exists some positive constant $\tau$, such that:

$$\frac{h}{h_i} \leq \tau, \; i = 1, \cdots, N_h$$

where $h_{E_i} = \sup_{x,y \in E_i} \|x - y\|$ and $h = \max\{h_{E_i}, \; i = 1, \cdots, N_h\}$.

Denote the edges (faces for three dimensions) of $E_h$ by $\{e_1, e_2, \cdots, e_{M_h}\}$. Then we have:

$$\forall j \in \{1, \cdots, M_h\}, \; \exists i_j \in \{1, \cdots, N_h\}, \text{ such that } e_j \in \partial E_{i_j}.$$

Let $|E|$ denote the area of $E$ in two dimensions (2D) and the volume of $E$ in three dimensions (3D). Similarly, $|e|$ denotes the length of $e$ in 2D and the area in 3D. There exist two constants $C_1$ and $C_2$ independent of $e$ such that

$$|E| = C_1|e|^2 \text{ in 2D, } |E| = C_1|e|^{3/2} \text{ in 3D}$$

$$h_E = C_2|e| \text{ in 2D, } h_E = C_2|e|^{1/2} \text{ in 3D}.$$  

Especially, for $d$ is the dimension of $\Omega \in \mathbb{R}^d$, we have:

$$\forall e \subset \partial E, \; |e| \leq h_{E}^{d-1} \leq h^{d-1}$$

Let $\Gamma_h$ denote the set of interior edges (or faces in 3D) of the subdivision $E_h$. A unit normal vector $n_e$ is associated with each edge (or faces in 3D) $e$. If $e$ is in on
the boundary $\partial \Omega$, then $\mathbf{n}_e$ is taken to be the unit outward vector normal to $\partial \Omega$. For a function $v$ over $\Omega$, we assume the normal vector $\mathbf{n}_e$ is oriented from $E_1$ to $E_2$ and define the jump and average of $v$ on an edge $e$ shared by two elements $E_1$ and $E_2$:

$$\{v\} = \frac{1}{2}(v|_{E_1}) + \frac{1}{2}(v|_{E_2}), \forall e = \partial E_1 \cap \partial E_2$$

$$[v] = (v|_{E_1}) - (v|_{E_2}), \forall e = \partial E_1 \cap \partial E_2$$

If $e \in \partial \Omega$, then $\{v\} = [v] = (v|_{E_1})$, $\forall e = \partial E_1 \cap \partial \Omega$.

We introduce the Sobolev space for general positive integer $s$:

$$H^s(\Omega) = \{v \in L^2(\Omega) : \forall 0 \leq |\varrho| \leq s, D^\varrho v \in L^2(\Omega)\}$$

Here

$$\varrho = (\varrho_1, \ldots, \varrho_l) \in \mathbb{N}^l, |\varrho| = \sum_{i=1}^{l} \varrho_i, D^\varrho v = \frac{\partial^{|\varrho|} v}{\partial x_1^{\varrho_1} \cdots \partial x_l^{\varrho_l}}$$

The Sobolev norm associated with $H^s(\Omega)$ is:

$$\|v\|_{H^s(\Omega)} = \left( \sum_{0 \leq |\varrho| \leq s} \|D^\varrho v\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}$$

and the semi-norm is:

$$|v|_{H^s(\Omega)} = \left( \sum_{|\varrho| = s} \|D^\varrho v\|_{L^2(\Omega)}^2 \right)^{\frac{1}{2}}$$

satisfying $|v|_{H^s(\Omega)} \leq \|v\|_{H^s(\Omega)}$.

The broken Sobolev space for any real number $s$ is defined as following:

$$H^s(\mathcal{E}_h) = \{v \in L^2(\Omega) : \forall E \in \mathcal{E}_h, v|_E \in H^s(E)\}$$

equipped with the broken Sobolev norm: $\|v\|_{H^s(\mathcal{E}_h)} = \left( \sum_{E \in \mathcal{E}_h} \|v\|_{H^s(E)}^2 \right)^{\frac{1}{2}}$ and the semi-norm: $|v|_{H^s(\mathcal{E}_h)} = \left( \sum_{E \in \mathcal{E}_h} |v|_{H^s(E)}^2 \right)^{\frac{1}{2}}$, satisfying $|v|_{H^s(\mathcal{E}_h)} \leq \|v\|_{H^s(\mathcal{E}_h)}$. 
Let \( N_T \) be a positive integer and let \( \Delta t = T/N_T \) denote the time step. We use the following notation for any function \( f \):

\[
\forall n \geq 0, \ t_n = n\Delta t, \ \forall x \in \Omega, \ f^n(x) = f(t_n)(x) = f(t_n, x)
\]

Denote \( \mathbb{P}_k(E) \) the space of polynomials of degree less than or equal to \( k \):

\[
\mathbb{P}_k(E) = \text{span}\{x_1^{\varrho_1} \cdots x_l^{\varrho_l} : \varrho_1 + \cdots + \varrho_l \leq k, x \in E\}
\]

If \( A \) and \( B \) are matrices in \( \mathbb{R}^{n \times n} \), define: \( A : B = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij} \).

There is an inequality:

\[
\forall x, y \in \mathbb{R}, \frac{1}{2}(x^2 - y^2) \leq \frac{1}{2}(x^2 - y^2 + (x - y)^2) = (x - y)x \quad (2.12)
\]

### 2.3 Numerical scheme

We first define the bilinear forms that are used in our scheme. Then we explain how we derive the forms from the PDE model.

Define three DG bilinear forms:

\[
a_1(p, r) = \sum_{E \in \mathcal{E}_h} \int_{E} \nabla p \cdot \nabla r - \sum_{e \in \mathcal{F}_h} \int_{e} \{\nabla p\} \cdot \mathbf{n}_e[r]
\]

\[
- \sum_{e \in \Gamma_{PD}} \int_{e} \nabla p \cdot \mathbf{n}_e r + \sum_{e \in \Gamma_h \cup \Gamma_{PD}} \frac{\sigma_e}{|e|^\beta} \int_{e} [p][r] + \theta_1 \sum_{e \in \Gamma_h \cup \Gamma_{PD}} \int_{e} \{\nabla r\} \cdot \mathbf{n}_e[p]
\]

\[
b(v, q) = - \sum_{E \in \mathcal{E}_h} \int_{E} q \nabla \cdot v + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \int_{e} \{q\}[v] \cdot \mathbf{n}_e
\]

\[
a_2(u, v) = \sum_{E \in \mathcal{E}_h} \int_{E} \nabla u : \nabla v + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \frac{\sigma_e}{|e|^\beta} \int_{e} [u] \cdot [v]
\]

\[
- \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \int_{e} \{\nabla u\} \mathbf{n}_e \cdot [v] + \theta_2 \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \int_{e} \{\nabla v\} \mathbf{n}_e \cdot [u]
\]
Here, $\beta = 1$ in 2D, $\beta = \frac{1}{2}$ in 3D and $|e|^\beta \leq h$; $\sigma_e \geq 0$ is the penalty parameter. If $\theta_1 = -1$ and $\sigma_e$ is bounded below by a large enough constant, the form of $a_1(\cdot, \cdot)$ is symmetric, and the method is called the symmetric interior penalty Galerkin (SIPG) method [1, 42]; if $\theta_1 = 1$ and $\sigma_e = 1$, the form of $a_1(\cdot, \cdot)$ is nonsymmetric, and the method is called the nonsymmetric interior penalty Galerkin (NIPG) method [31]; if $\theta_1 = 0$ and $\sigma_e$ is bounded below by a large enough constant, the method is called the incomplete interior penalty Galerkin (IIPG) method [9]. A similar terminology applies to $a_2$.

Next we begin constructing the DG scheme for the model (2.11a)-(2.11c).

We multiply equation (2.11a) by a function $r$ and integrate over $E$, then sum over all elements $E$,

$$
c \sum_{E \in \mathcal{E}_h} \int_E \frac{\partial p}{\partial t} r + \sum_{E \in \mathcal{E}_h} \int_E \frac{\partial \varepsilon}{\partial t} r - \kappa \sum_{e \in \Gamma_h} \int_e \{\nabla p\} \cdot n_e [r] - \kappa \sum_{e \in \Gamma_{PN}} \int_e g_N r
$$

$$
- \kappa \sum_{e \in \Gamma_{PD}} \int_e \nabla \cdot n_e r + \kappa \sum_{E \in \mathcal{E}_h} \int_E \nabla p \cdot \nabla r = \sum_{E \in \mathcal{E}_h} \int_E (\alpha p + \nu) r
$$

With the definition of $a_1$, the above equation becomes:

$$
k a_1(p, r) + c \sum_{E \in \mathcal{E}_h} \int_E \frac{\partial p}{\partial t} r + \sum_{E \in \mathcal{E}_h} \int_E \frac{\partial \varepsilon}{\partial t} r - \sum_{E \in \mathcal{E}_h} \int_E (\alpha p + \nu) r
$$

$$
= \kappa \sum_{e \in \Gamma_{PN}} \int_e g_N r + \kappa \theta_1 \sum_{e \in \Gamma_{PD}} \int_e \nabla r \cdot n_{ePD} + \kappa \sum_{e \in \Gamma_{PD}} \frac{\sigma_e}{|e|^\beta} \int_e p_{DR} r \quad (2.13)
$$

We multiply equation (2.11b) by a function $q$ and integrate over $E$, then sum over all elements $E$,

$$
\sum_{E \in \mathcal{E}_h} \int_E \varepsilon q = \sum_{E \in \mathcal{E}_h} \int_E \nabla \cdot w q
$$
With the definition of $b$, we see that

$$
\sum_{E \in \mathcal{E}_h} \int_E \nabla \cdot \mathbf{w} q = -b(\mathbf{w}, q) + \sum_{e \in \Gamma_{wD}} \int_e q \mathbf{w}_D \cdot \mathbf{n}_e 
$$

Then we obtain:

$$
\sum_{E \in \mathcal{E}_h} \int_E \varepsilon q = -b(\mathbf{w}, q) + \sum_{e \in \Gamma_{wD}} \int_e q \mathbf{w}_D \cdot \mathbf{n}_e 
$$

(2.14)

We multiply equation (2.11c) by a vector function $\mathbf{v}$, and we integrate over $E$. Then we sum over all elements $E$.

$$
-(\mu + \lambda) \sum_{E \in \mathcal{E}_h} \int_E \nabla \varepsilon \cdot \mathbf{v} - \mu \sum_{E \in \mathcal{E}_h} \int_E \Delta \mathbf{w} \cdot \mathbf{v} + \sum_{E \in \mathcal{E}_h} \int_E \nabla p \cdot \mathbf{v} = 0 
$$

With the definition of $a_2$, we obtain:

$$
\mu a_2(\mathbf{w}, \mathbf{v}) + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \frac{\sigma_e}{|e|^{1/\beta}} \int_e [\frac{\partial \mathbf{w}}{\partial t} \cdot [\mathbf{v}] - (\mu + \lambda)b(\mathbf{v}, \varepsilon) + b(\mathbf{v}, p) 
$$

$$
= \mu \sum_{e \in \Gamma_{wD}} \frac{\sigma_e}{|e|^{1/\beta}} \int_e \mathbf{w}_D \cdot \mathbf{v} + \mu \theta_2 \sum_{e \in \Gamma_{wD}} \int_e \nabla n_e \cdot \mathbf{w}_D 
$$

$$
+ \sum_{e \in \Gamma_{wN}} \int_e \mathbf{g}_w \cdot \mathbf{v} + \sum_{e \in \Gamma_{wD}} \frac{\sigma_e}{|e|^{1/\beta}} \int_e \frac{\partial \mathbf{w}_D}{\partial t} \cdot \mathbf{v} 
$$

(2.15)

When $a_2(\cdot, \cdot)$ is symmetric, we choose: $\beta_1 = 1$ in 2D, $\beta_1 = \frac{1}{2}$ in 3D. When $a_2(\cdot, \cdot)$ is nonsymmetric, we choose $\beta_1 \geq 1$ in 2D, $\beta_1 \geq \frac{1}{2}$ in 3D.

Remark 2.1 The penalty item $\sum_{e \in \Gamma_h \cup \Gamma_{wD}} \frac{\sigma_e}{|e|^{1/\beta}} \int_e [\frac{\partial \mathbf{w}}{\partial t} \cdot [\mathbf{v}]$ added to both sides of the equation (2.15), is needed in the error estimation.
Now, we have obtained the following three equations:

\[ \kappa_1 p + c \sum_{E \in \mathcal{E}_h} \int_E \frac{\partial p}{\partial t} r + \sum_{E \in \mathcal{E}_h} \int_E \frac{\partial \varepsilon}{\partial t} r - \sum_{E \in \mathcal{E}_h} \int_E (\alpha p + \nu) r = \kappa \sum_{e \in \Gamma} \int_e g_N r + \kappa \sum_{e \in \Gamma} \int_e \nabla r \cdot \mathbf{n}_D + \kappa \sum_{e \in \Gamma} \frac{\sigma_e}{|e|^3} \int_e p_D r \]  

\[ \sum_{E \in \mathcal{E}_h} \int_E \varepsilon q = -b(w, q) + \sum_{e \in \Gamma_w} \int_e q w_D \cdot \mathbf{n}_e \]  

\[ \mu a_2(w, v) + \sum_{e \in \Gamma_w \backslash \Gamma_{w_D}} \frac{\sigma_e}{|e|^3} \int_e [\frac{\partial w}{\partial t}] \cdot [v] - (\mu + \lambda) b(v, \varepsilon) + b(v, p) = \mu \sum_{e \in \Gamma_w} \int_e w_D \cdot v + \mu \theta_2 \sum_{e \in \Gamma_w} \int_e \nabla v e \cdot w_D \]  

\[ \quad + \sum_{e \in \Gamma_{w_N}} \int_e g_w \cdot v + \sum_{e \in \Gamma_{w_D}} \frac{\sigma_e}{|e|^3} \int_e \frac{\partial w_D}{\partial t} \cdot v \]  

The test functions \( r, q, \) and \( v \) belong to \( Q_k(\mathcal{E}_h), U_r(\mathcal{E}_h) \) and \( V_l(\mathcal{E}_h) \), where \( k \leq r \).

The spaces are defined as below:

\[ Q_k(\mathcal{E}_h) = \{ f \in L^2(\Omega) : f \in P_k(E), \forall E \in \mathcal{E}_h \} \]

\[ V_l(\mathcal{E}_h) = \{ g \in (L^2(\Omega))^d : g \in (P_l(E))^d, \forall E \in \mathcal{E}_h \} \]

\[ U_r(\mathcal{E}_h) = \{ \phi \in L^2(\Omega) : \phi \in P_r(E), \forall E \in \mathcal{E}_h \} \]
Let us define linear forms for the input data:

$$\ell_1(r) = \kappa \sum_{e \in \Gamma_{PN}} \int_e g^{n+1}_N r + \kappa \theta_1 \sum_{e \in \Gamma_{PD}} \int_e \nabla r \cdot n_e p^{n+1} + \kappa \sum_{e \in \Gamma_{PD}} \sigma_e \int_e p^{n+1}_D r \quad (2.17a)$$

$$\ell_2(q) = \sum_{e \in \Gamma_{WD}} \int_e q w^{n+1}_D \cdot n_e \quad (2.17b)$$

$$\ell_3(v) = \mu \sum_{e \in \Gamma_{WD}} \frac{\sigma_e}{|e|^3} \int_e w^{n+1}_D \cdot v + \mu \theta_2 \sum_{e \in \Gamma_{WD}} \int_e \nabla n_e \cdot w^{n+1}_D$$

$$\quad + \sum_{e \in \Gamma_{WN}} \int_e w^{n+1}_D \cdot v + \sum_{e \in \Gamma_{WD}} \frac{\sigma_e}{|e|^3} \int_e w^{n+1}_D - \frac{w^n_D}{\Delta t} \cdot v \quad (2.17c)$$

We fully discretize in time using the backward-Euler method. Our proposed numerical scheme is:

$$\kappa a_1(p^{n+1}_h, r) + c \sum_{E \in \mathcal{E}_h} \int_E p^{n+1}_h - p^n_h r + \sum_{E \in \mathcal{E}_h} \int_E \varepsilon^{n+1}_h - \varepsilon^n_h r$$

$$- \sum_{E \in \mathcal{E}_h} \int_E (\alpha p^n_h + \nu) r = \ell_1(r), \quad \forall r \in Q_k(\mathcal{E}_h) \quad (2.18a)$$

$$\sum_{E \in \mathcal{E}_h} \int_E \varepsilon^{n+1}_h q + b(w^{n+1}_h, q) = \ell_2(q), \quad \forall q \in U_r(\mathcal{E}_h) \quad (2.18b)$$

$$\mu a_2(w^{n+1}_h, v) + \sum_{e \in \Gamma_h \cup \Gamma_{WD}} \frac{\sigma_e}{|e|^3} \int_e w^{n+1}_h - w^n_h = \ell_3(v), \quad \forall v \in V_l(\mathcal{E}_h) \quad (2.18c)$$

where, the unknowns are sequences $(p^n_h)_{n \geq 0}$ of functions in $Q_k(\mathcal{E}_h)$, $(w^n_h)_{n \geq 0}$ of functions in $V_l(\mathcal{E}_h)$, $(\varepsilon^n_h)_{n \geq 0}$ of functions in $U_r(\mathcal{E}_h)$. 
For the initial conditions, we define:

\[(p_0^h, r)_{\mathcal{E}_h} = (\tilde{p}_0, r)_{\mathcal{E}_h}, \forall r \in Q_k(\mathcal{E}_h) \tag{2.19}\]

\[(w_0^h, v)_{\mathcal{E}_h} = (\tilde{w}_0, v)_{\mathcal{E}_h}, \forall v \in V_l(\mathcal{E}_h) \tag{2.20}\]

\[(\varepsilon_0^h, q)_{\mathcal{E}_h} = -b(w_0^h, q) + \sum_{e \in \Gamma_{wD}} (w_D^0 \cdot n_e, q)_e, \forall q \in \mathcal{U}_c(\mathcal{E}_h) \tag{2.21}\]

where \(\tilde{p}_0\) can be chosen to be \(p_0\) if \(p_0\) belongs to the discrete space \(Q_k(\mathcal{E}_h)\), or it can be chosen to be the \(L^2\) projection of \(p_0\); \(\tilde{w}_0\) can be chosen to be \(w_0\) if \(w_0\) belongs to the discrete space \(V_l(\mathcal{E}_h)\), or it can be chosen to be an elliptic projection of \(w_0\) if \(a_2\) is symmetric and \(L^2\) projection of \(w_0\) otherwise. The elliptic projection is defined in Chapter 3.

Here, we define two norms:

For a vector function \(v\):

\[
\|v\|_V = \left( \sum_{E \in \mathcal{E}_h} \|\nabla v\|_{L^2(E)}^2 + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \frac{\sigma_e}{|e|^{\beta}} \|[v]\|_{L^2(e)}^2 \right)^{\frac{1}{2}}
\]

For a scalar function \(r\):

\[
\|r\|_Q = \left( \sum_{E \in \mathcal{E}_h} \|\nabla r\|_{L^2(E)}^2 + \sum_{e \in \Gamma_h \cup \Gamma_{pD}} \frac{\sigma_e}{|e|^{\beta}} \|[r]\|_{L^2(e)}^2 \right)^{\frac{1}{2}}
\]

Lemma 2.1 : There are four constants \(C_1, C_2, C_3, C_4\), such that :

\[a_1(r, r) \geq C_1\|r\|_Q^2, \forall r \in Q_k(\mathcal{E}_h) \tag{2.22}\]

\[a_2(v, v) \geq C_2\|v\|_V^2, \forall v \in V_l(\mathcal{E}_h) \tag{2.23}\]
and

\[ a_1(r_1, r_2) \leq C_5 \| r_1 \|_Q \| r_2 \|_Q, \forall r_1, r_2 \in Q_k(\mathcal{E}_h) \]  \hspace{1cm} (2.24)

\[ a_2(v_1, v_2) \leq C_4 \| v_1 \|_V \| v_2 \|_V, \forall v_1, v_2 \in V_l(\mathcal{E}_h) \] \hspace{1cm} (2.25)

Proof is in [30].

2.4 Existence and uniqueness of numerical solution

Theorem 2.1 (Existence and uniqueness) : There exist sequences \((p^n_h)_{n \geq 0}\) of functions in \(Q_k(\mathcal{E}_h)\), \((w^n_h)_{n \geq 0}\) of functions in \(V_l(\mathcal{E}_h)\), \((\varepsilon^n_h)_{n \geq 0}\) of functions in \(U_r(\mathcal{E}_h)\) satisfying the scheme (2.18). Especially, these sequences are unique.

Proof.

Since in the finite dimensional space, the existence of solution is equivalent to the uniqueness of solution. In the following process, we only show the DG numerical solutions of the model (2.18) are unique.

Usually in order to prove the uniqueness of the solution, we assume there exist two solutions and we show they are equal to each other. In this proof, we simplify the argument by assuming \(p_D = 0, g_N = 0, w_D = 0, g_w = 0, \nu = 0\) as the problem is linear. For the same reason, we assume: \(p_h^n = 0, \varepsilon_h^n = 0\) and \(w_h^n = 0\).

Equations (2.18) become:
\( k a_1(p_h^{n+1}, r) + c \sum_{E \in \mathcal{E}_h} \int_E p_h^{n+1} r + \sum_{E \in \mathcal{E}_h} \int_E \varepsilon_h^{n+1} r = 0 \) \hspace{1cm} (2.26a)

\( b(w_h^{n+1}, q) + \sum_{E \in \mathcal{E}_h} \int_E \varepsilon_h^{n+1} q = 0 \) \hspace{1cm} (2.26b)

\[ \mu a_2(w_h^{n+1}, v) + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \frac{\sigma_e}{|e|^{1/2}} \int_e \left[ \frac{w_h^{n+1}}{\Delta t} \right] \cdot [v] - (\mu + \lambda) b(v, \varepsilon_h^{n+1}) + b(v, p_h^{n+1}) = 0 \] \hspace{1cm} (2.26c)

Equation (2.26b) directly implies:

\[ b\left( \frac{w_h^{n+1}}{\Delta t}, q \right) + \sum_{E \in \mathcal{E}_h} \int_E \frac{\varepsilon_h^{n+1}}{\Delta t} q = 0 \] \hspace{1cm} (2.27)

Choose \( q = \varepsilon_h^{n+1} \), the equation (2.27) becomes:

\[ b\left( \frac{w_h^{n+1}}{\Delta t}, \varepsilon_h^{n+1} \right) + \sum_{E \in \mathcal{E}_h} \int_E \frac{\varepsilon_h^{n+1}}{\Delta t} \varepsilon_h^{n+1} = 0 \] \hspace{1cm} (2.28)

Choose \( q = p_h^{n+1} \), the equation (2.27) becomes:

\[ b\left( \frac{w_h^{n+1}}{\Delta t}, p_h^{n+1} \right) + \sum_{E \in \mathcal{E}_h} \int_E \frac{\varepsilon_h^{n+1}}{\Delta t} p_h^{n+1} = 0 \] \hspace{1cm} (2.29)

Choose \( r = p_h^{n+1} \), the equation (2.26a) becomes:

\[ k a_1(p_h^{n+1}, p_h^{n+1}) + c \sum_{E \in \mathcal{E}_h} \int_E p_h^{n+1} p_h^{n+1} + \sum_{E \in \mathcal{E}_h} \int_E \varepsilon_h^{n+1} p_h^{n+1} = 0 \] \hspace{1cm} (2.30)

Subtracting equation (2.29) from (2.30), we obtain:

\[ k a_1(p_h^{n+1}, p_h^{n+1}) + c \sum_{E \in \mathcal{E}_h} \int_E p_h^{n+1} p_h^{n+1} - b\left( \frac{w_h^{n+1}}{\Delta t}, p_h^{n+1} \right) = 0 \] \hspace{1cm} (2.31)
Choose $v = \frac{w^{n+1}_h}{\Delta t}$, the equation (2.26c) becomes:

$$\mu a_2(w^{n+1}_h, \frac{w^{n+1}_h}{\Delta t}) + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \sigma_e \frac{\| \frac{w^{n+1}_h}{\Delta t} \|_{L^2(e)}}{\| \frac{w^{n+1}_h}{\Delta t} \|_{L^2(e)}} - (\mu + \lambda) b \left( \frac{w^{n+1}_h}{\Delta t}, \varepsilon^{n+1}_h \right) + b \left( \frac{w^{n+1}_h}{\Delta t}, p^{n+1}_h \right) = 0$$

(2.32)

We add equations (2.32) and (2.31):

$$\mu a_2(w^{n+1}_h, \frac{w^{n+1}_h}{\Delta t}) + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \sigma_e \frac{\| \frac{w^{n+1}_h}{\Delta t} \|_{L^2(e)}}{\| \frac{w^{n+1}_h}{\Delta t} \|_{L^2(e)}} - (\mu + \lambda) b \left( \frac{w^{n+1}_h}{\Delta t}, \varepsilon^{n+1}_h \right) + k a_1(p^{n+1}_h, p^{n+1}_h) + c \sum_{E \in \mathcal{E}_h} \int_E \frac{p^{n+1}_h}{\Delta t} p^{n+1}_h = 0$$

(2.33)

Combine (2.28) and (2.33):

$$\mu a_2(w^{n+1}_h, \frac{w^{n+1}_h}{\Delta t}) + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \sigma_e \frac{\| \frac{w^{n+1}_h}{\Delta t} \|_{L^2(e)}}{\| \frac{w^{n+1}_h}{\Delta t} \|_{L^2(e)}} + (\mu + \lambda) \sum_{E \in \mathcal{E}_h} \int_E \frac{\varepsilon^{n+1}_h}{\Delta t} \varepsilon^{n+1}_h + k a_1(p^{n+1}_h, p^{n+1}_h) + c \sum_{E \in \mathcal{E}_h} \int_E \frac{p^{n+1}_h}{\Delta t} p^{n+1}_h = 0$$

(2.34)

By Lemma 2.1, there exist some constants $C_1, C_2$ s.t.:

$$\frac{\mu}{\Delta t} C_1 \| w^{n+1}_h \|_V^2 \leq \mu a_2(w^{n+1}_h, \frac{w^{n+1}_h}{\Delta t})$$

$$k C_2 \| p^{n+1}_h \|_Q^2 \leq k a_1(p^{n+1}_h, p^{n+1}_h)$$

Then inequality (2.34) becomes:

$$\frac{\mu}{\Delta t} C_1 \| w^{n+1}_h \|_V^2 + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \sigma_e \frac{\| \frac{w^{n+1}_h}{\Delta t} \|_{L^2(e)}}{\| \frac{w^{n+1}_h}{\Delta t} \|_{L^2(e)}} + k C_2 \| p^{n+1}_h \|_Q^2 + (\mu + \lambda) \sum_{E \in \mathcal{E}_h} \| \varepsilon^{n+1}_h \|_{L^2(E)}^2 + \frac{c}{\Delta t} \sum_{E \in \mathcal{E}_h} \| p^{n+1}_h \|_{L^2(E)}^2 \leq 0$$

From the above inequality, we can obtain $w^{n+1}_h = 0, p^{n+1}_h = 0, \varepsilon^{n+1}_h = 0$. Therefore, we have shown that the DG solution is unique.
Remark 2.2 : If the equation (2.18a) is:

\[ \kappa a_1(p_{h}^{n+1}, r) + c \sum_{E \in \mathcal{E}_h} \int_E \frac{p_{h}^{n+1} - p_{h}^{n}}{\Delta t} r + \sum_{E \in \mathcal{E}_h} \int_E \frac{\varepsilon_{h}^{n+1} - \varepsilon_{h}^{n}}{\Delta t} r - \sum_{E \in \mathcal{E}_h} \int_E (\alpha p_{h}^{n+1} + \nu) r \]

\[ = \kappa \sum_{e \in \Gamma_{PN}} \int_e g_{N}^{n+1} r + \kappa \theta \sum_{e \in \Gamma_{PD}} \int_e \nabla r \cdot n_e p_{D}^{n+1} + \kappa \sum_{e \in \Gamma_{PD}} \frac{\sigma_e}{|e|^3} \int_e p_{D}^{n+1} r \]

If \( \alpha > 0 \), then we need the condition: \( \frac{c}{2\Delta t} \geq \alpha \), i.e. \( \frac{c}{2\alpha} \geq \Delta t \) in order to get that the DG solution is unique. If \( \alpha \leq 0 \), the DG solution is unique without any condition.
Chapter 3

A Priori Error Estimates

This chapter is mainly focused on a priori error estimates between the numerical solution and the exact solution. Since the proof for the symmetric form and the nonsymmetric form $a_2(\cdot, \cdot)$ is different, this chapter presents the error estimates for the two cases respectively.

3.1 Approximation results

We now state important results that we will use in the proof frequently.

Theorem 3.1 (Trace theorem [20]) : Let $\Omega$ be a bounded domain with polygonal boundary $\partial \Omega$ and outward normal vector $\mathbf{n}$. There exist trace operators $\gamma_0 : H^s(\Omega) \rightarrow H^{s-1/2}(\partial \Omega)$ for $s > 1/2$ and $\gamma_1 : H^s(\Omega) \rightarrow H^{s-3/2}(\partial \Omega)$ for $s > 3/2$ that are extensions of the boundary values and boundary normal derivatives, respectively. The operators $\gamma_j$ are surjective. Furthermore, if $v \in C^1(\bar{\Omega})$, then $\gamma_0 v = v|_{\partial \Omega}$, $\gamma_1 v = \nabla v \cdot \mathbf{n}|_{\partial \Omega}$.
Lemma 3.1 (Trace inequalities [20]) : There is a constant $C$ independent of $h_E$ and $v$ such that for any $v \in H^s(E)$:

\[
\begin{align*}
    s &\geq 1, \forall e \subset \partial E, \|\gamma_0 v\|_{L^2(e)} \leq C h_E^{-1/2} \left( \|v\|_{L^2(E)} + h_E \|\nabla v\|_{L^2(E)} \right) \\
    s &\geq 2, \forall e \subset \partial E, \|\gamma_1 v\|_{L^2(e)} \leq C h_E^{-1/2} \left( \|\nabla v\|_{L^2(E)} + h_E \|\nabla^2 v\|_{L^2(E)} \right)
\end{align*}
\]  

(3.1a)  
(3.1b)

In the space of polynomials in 2D, the trace inequalities become:

\[
\begin{align*}
    \forall v \in P_k(E), \forall e \subset \partial E, \|v\|_{L^2(e)} &\leq C_2 |e|^{-1/2} \|v\|_{L^2(E)} \\
    \forall v \in P_k(E), \forall e \subset \partial E, \|\nabla v \cdot n\|_{L^2(e)} &\leq C_2 |e|^{-1/2} \|\nabla v\|_{L^2(E)}
\end{align*}
\]  

(3.2a)  
(3.2b)

where $C_1, C_2$ are independent of $h_E$.

Theorem 3.2 (Approximation results [3, 32]) : Let $E$ be a triangle or parallelogram in 2D or a tetrahedron or hexahedron in 3D. Let $v \in H^s(E)$ for $s \geq 1$. Let $k \geq 0$ be an integer. There exist a constant $C$ independent of $v$ and $h_E$ and a function $\tilde{v} \in P_k(E)$ such that

\[
\forall 0 \leq q \leq s, \|v - \tilde{v}\|_{H^q(E)} \leq C h_E^{\min(k+1,s)-q} \|v\|_{H^s(E)}
\]  

(3.3)

Theorem 3.3 (Green’s theorem) : Given $E$ a bounded domain and $n_E$ the outward normal vector to $\partial E$, for all $v \in H^2(E)$ and $w \in H^1(E)$

\[
- \int_E w \Delta v = \int_E \nabla v \cdot \nabla w - \int_E \nabla v \cdot n_E w
\]  

(3.4)

The following two inequalities are often used in this thesis.
Cauchy-Schwarz’s inequality:

\[ \forall f, g \in L^2(\Omega), \quad |(f, g)\Omega| \leq \|f\|_{L^2(\Omega)} \|g\|_{L^2(\Omega)} \quad (3.5) \]

Young’s inequality:

\[ \forall \epsilon > 0, \forall a, b \in \mathbb{R}, \quad ab \leq \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2 \quad (3.6) \]

Gronwall’s inequalities are very important for analyzing time-dependent problems. Since we only discuss the problem in discrete time in this thesis, here we present the discrete Gronwall’s inequality [18].

**Lemma 3.2 (Discrete Gronwall’s inequality) :** Let \( \Delta t, B, C > 0 \) and let \( (a_n)_n, (b_n)_n, (c_n)_n, (d_n)_n \) be sequences of nonnegative numbers satisfying

\[ \forall n \geq 0, \quad a_n + \Delta t \sum_{i=0}^{n} b_i \leq B + C \Delta t \sum_{i=0}^{n} a_i + \Delta t \sum_{i=0}^{n} c_i \]

Then, if \( C \Delta t < 1 \),

\[ \forall n \geq 0, \quad a_n + \Delta t \sum_{i=0}^{n} b_i \leq e^{C(n+1)\Delta t} (B + \Delta t \sum_{i=0}^{n} c_i) \quad (3.7) \]

**Lemma 3.3 (Poincare’s inequality) :** There is a constant \( C \) such that

\[ \forall v \in H^1(\Omega), \quad \|v\|_{L^2(\Omega)} \leq C(\|\nabla v\|_{L^2(\Omega)} + |\int_{\partial \Omega} v|) \quad (3.8) \]

Assume that \( \Gamma_D \) is a subset of the boundary \( \partial \Omega \) with \( |\Gamma_D| > 0 \). For the broken Sobolev space \( H^1(\mathcal{E}_h) \), if \( \beta > (d - 1)^{-1} \), there is a constant \( C \) such that:

\[ \forall v \in H^1(\mathcal{E}_h), \quad \|v\|_{L^2(\Omega)} \leq C(\|\nabla v\|_{L^2(\mathcal{E}_h)}^2 + \sum_{e \in \Gamma_h \cup \Gamma_D} \frac{1}{|e|^\beta \|v\|_{L^2(e)}^2})^{\frac{1}{2}} \quad (3.9) \]
where the broken norm is defined as:

\[ \| \nabla r \|_{L^2(E_h)} = \left( \sum_{E \in E_h} \| \nabla r \|_{L^2(E)}^2 \right)^{\frac{1}{2}} \]

Corresponding to Poincare’s inequality, there is an inverse inequality:

**Lemma 3.4 (Inverse inequality)**: Let \( E \) be a bounded domain in \( \mathbb{R}^d \) with diameter \( h_E \). Then, there is a constant \( C \) independent of \( h_E \) such that for any polynomial function \( v \) of degree \( k \) defined on \( E \) we have

\[ \forall 0 \leq j \leq k, \quad \| \nabla^j v \|_{L^2(E)} \leq C h^{-j}_E \| v \|_{L^2(E)} \quad (3.10) \]

The above results are very important for our error analysis. Next we are going to introduce some definitions.

We denote by \((w^n, p^n, \varepsilon^n)\) the exact solution at time \( t_n \), and \((\tilde{P}w^n, P_hp^n, \Pi \varepsilon^n)\) a good approximation of the exact solution to be defined later.

**Definition 3.1 (Elliptic projection)** The function \( \tilde{f} \) is the elliptic projection of \( f \) for a bilinear form \( a \) in space \( H \), if

\[ \forall g \in H, \quad a(f - \tilde{f}, g) = 0 \]

**Definition 3.2 (\( L^2 \) projection)** The function \( \tilde{\phi} \) is the \( L^2 \) projection of \( \phi \) in space \( H \), if

\[ \forall \psi \in H, \quad (\phi - \tilde{\phi}, \psi) = 0 \]

Let \( P_hp^n \) be the \( L^2 \) projection of \( p^n \), and let \( \Pi \varepsilon^n \) be the \( L^2 \) projection of \( \varepsilon^n \). By Theorem 3.2, we can obtain several inequalities for the exact solution and the
approximation.

\[
\|Z^I_{p;n}\|_{L^2(\mathcal{E}_h)} = \|p^n - P_hp^n\|_{L^2(\mathcal{E}_h)} \leq C h^{\min(k+1,s+1)}|p^n|_{H^{s+1}(\mathcal{E}_h)}, \quad \forall n \in \mathbb{N} \quad (3.11a)
\]

\[
\|Z^I_{\varepsilon;n}\|_{L^2(\mathcal{E}_h)} = \|\varepsilon^n - \Pi\varepsilon^n\|_{L^2(\mathcal{E}_h)} \leq C h^{\min(r+1,s)}|\varepsilon^n|_{H^{s}(\mathcal{E}_h)}, \quad \forall n \in \mathbb{Z}^+ \quad (3.11b)
\]

If \(\tilde{P}w^n\) is the elliptic projection of \(w^n\) for \(a_2\) and \(a_2\) is symmetric, we have

\[
\|Z^I_{w;n}\|_{L^2(\mathcal{E}_h)} = \|w^n - \tilde{P}w^n\|_{L^2(\mathcal{E}_h)} \leq C h^{\min(l+1,s+1)}|w^n|_{H^{s+1}(\mathcal{E}_h)}, \quad \forall n \in \mathbb{N} \quad (3.12)
\]

If \(\tilde{P}w^n\) is the elliptic projection of \(w^n\) for \(a_2\) and \(a_2\) is nonsymmetric, we have

\[
\|Z^I_{w;n}\|_{L^2(\mathcal{E}_h)} = \|w^n - \tilde{P}w^n\|_{L^2(\mathcal{E}_h)} \leq C h^{\min(l,s)}|w^n|_{H^{s+1}(\mathcal{E}_h)}, \quad \forall n \in \mathbb{N} \quad (3.13)
\]

If \(\tilde{P}w^n\) is the \(L^2\) projection of \(w^n\), we have

\[
\|Z^I_{w;n}\|_{L^2(\mathcal{E}_h)} = \|w^n - \tilde{P}w^n\|_{L^2(\mathcal{E}_h)} \leq C h^{\min(l+1,s+1)}|w^n|_{H^{s+1}(\mathcal{E}_h)}, \quad \forall n \in \mathbb{N} \quad (3.14)
\]

For any \(n\) in \(\mathbb{N}\), let us define:

\[
Z^A_{w;n} = \tilde{P}w^n - w^n, \quad Z^I_{w;n} = w^n - \tilde{P}w^n
\]

\[
Z^A_{p;n} = P_hp^n - p^n, \quad Z^I_{p;n} = p^n - P_hp^n
\]

\[
Z^A_{\varepsilon;n} = \Pi\varepsilon^n - \varepsilon^n, \quad Z^I_{\varepsilon;n} = \varepsilon^n - \Pi\varepsilon^n
\]

In particular, we remark that we choose \(P_h^0, w_h^0, \varepsilon_h^0\) such that \(Z^A_{p,0} = Z^A_{\varepsilon,0} = 0\) and \(Z^A_{w,0} = 0\).

Then we can decompose the numerical errors:

\[
\begin{align*}
  w^n - w^n_h &= Z^A_{w;n} + Z^I_{w;n} \\
p^n - p^n_h &= Z^A_{p;n} + Z^I_{p;n} \\
\varepsilon^n - \varepsilon^n_h &= Z^A_{\varepsilon;n} + Z^I_{\varepsilon;n}
\end{align*}
\]
For convenience, we make the following definitions:

\[
\|g\|_{L^\infty(L^2)} = \max_{j=0,\ldots,N} \|g(t_j, \mathbf{x})\|_{L^2}, \quad \|g\|_{L^2(H^s)} = (\Delta t \sum_{j=0}^{N} \|g(t_j, \mathbf{x})\|_{L^2})^{\frac{1}{2}}
\]

\[
\|g\|_{L^\infty(t_i,t_j;L^2)} = \sup_{t \in [t_i, t_j]} \|g(t, \mathbf{x})\|_{L^2}
\]

\[
\|g\|_{L^\infty(t_i,t_j;H^s)} = \sup_{t \in [t_i, t_j]} \|g(t, \mathbf{x})\|_{H^s}
\]

\[
\|g\|_{H^k(0,T;H^s)} = \left( \sum_{i=0}^{k} \|\frac{\partial^i g}{\partial t^i}\|_{H^s} \right)^{\frac{1}{2}}
\]

\[
\|r\|_{L^2(E_h)} = \left( \sum_{E \in E_h} \|r\|_{L^2(E)}^2 \right)^{\frac{1}{2}}
\]

We remark that the following relationships hold as a result of Taylor expansion (see the Appendix):

\[
\frac{\varepsilon^{n+1} - \varepsilon^n}{\Delta t} = \varepsilon_t^{n+1} + \Delta t \rho_{\varepsilon;n+1}, \quad \forall x \in \Omega
\]

\[
\frac{p^{n+1} - p^n}{\Delta t} = p_t^{n+1} + \Delta t \rho_{p;n+1}, \quad \forall x \in \Omega
\]

\[
\frac{w^{n+1} - w^n}{\Delta t} = w_t^{n+1} + \Delta t \rho_{w;n+1}, \quad \forall x \in \Omega
\]

where \(\rho_{\varepsilon;n+1}, \rho_{p;n+1}\) and \(\rho_{w;n+1}\) satisfy:

\[
\|\rho_{\varepsilon;n+1}\|_{L^2(\Omega)} \leq C_3 \left\| \frac{\partial^2 \varepsilon}{\partial t^2} \right\|_{L^\infty(t_n, t_{n+1};L^2(\Omega))}
\]

\[
\|\rho_{p;n+1}\|_{L^2(\Omega)} \leq C_4 \left\| \frac{\partial^2 p}{\partial t^2} \right\|_{L^\infty(t_n, t_{n+1};L^2(\Omega))}
\]

\[
\|\rho_{w;n+1}\|_{L^2(\Omega)} \leq C_5 \left\| \frac{\partial^2 w}{\partial t^2} \right\|_{L^\infty(t_n, t_{n+1};L^2(\Omega))}
\]

Here, \(C_3, C_4\) and \(C_5\) are some constants independent of \(\Delta t\) and \(h\).
Lemma 3.5 (Approximation result of $\varepsilon$ for $n = 0$): There is a constant $C$ independent of $h$, such that:

$$\|Z_{\varepsilon,0}\|_{L^2(\mathcal{E}_h)} = \|\varepsilon(0) - \Pi \varepsilon^0\|_{L^2(\mathcal{E}_h)} \leq C(h^{\min(l,s)} - \delta\|w(0)\|_{H^{s+1}(\mathcal{E}_h)} + h^{\min(r+1,s)}\|\varepsilon(0)\|_{H^r(\mathcal{E}_h)})$$

If $\tilde{P} w^0$ is the elliptic projection of $w^0$ for $a_2$ and $a_2$ is nonsymmetric, $\delta = 1$; otherwise, $\delta = 0$.

Proof.

From (2.21), we have:

$$(\varepsilon^0_h, q)_{\mathcal{E}_h} = -b(w^0_h, q) + \sum_{e \in \Gamma_{wp}} (w^0_D \cdot n_e, q)_e, \forall q \in \mathcal{U}_r(\mathcal{E}_h) \quad (3.17)$$

For the exact solution $\varepsilon(0)$, we also have:

$$(\varepsilon(0), q)_{\mathcal{E}_h} = -b(w(0), q) + \sum_{e \in \Gamma_{wp}} (w^0_D \cdot n_e, q)_e, \forall q \in \mathcal{U}_r(\mathcal{E}_h) \quad (3.18)$$

Subtracting (3.17) from (3.18), we obtain:

$$(\varepsilon(0) - \varepsilon^0_h, q)_{\mathcal{E}_h} = -b(w(0) - w^0_h, q) \quad (3.19)$$

Let $I \varepsilon^0 \in \mathcal{U}_r$ be the Lagrange interpolant of $\varepsilon(0)$. There is some constant $C$ such that

$$\|\varepsilon(0) - I \varepsilon^0\|_{L^2(\mathcal{E}_h)} \leq C h^{\min(r+1,s)}|\varepsilon(0)|_{H^r(\mathcal{E}_h)}$$

Choose $q = I \varepsilon^0 - \varepsilon^0_h$, then the above equality (3.19) becomes:

$$(\varepsilon(0) - I \varepsilon^0 + I \varepsilon^0 - \varepsilon^0_h, I \varepsilon^0 - \varepsilon^0_h)_{\mathcal{E}_h} = -b(w(0) - w^0_h, I \varepsilon^0 - \varepsilon^0_h)$$
We rearrange the above equality and obtain:

\[
\|I\varepsilon^0 - \varepsilon_h^0\|_{L^2(\mathcal{E}_h)}^2 = -b(w(0) - w_h^0, I\varepsilon^0 - \varepsilon_h^0) - (\varepsilon(0) - I\varepsilon^0, I\varepsilon^0 - \varepsilon_h^0)_{\mathcal{E}_h} = S_1 + S_2
\]

We analyze \(S_1\) and \(S_2\) separately. By the definition of \(b(\cdot, \cdot)\), we have

\[
S_1 = -b(w(0) - w_h^0, I\varepsilon^0 - \varepsilon_h^0)
\]

\[
= \sum_{E \in \mathcal{E}_h} \int_E (I\varepsilon^0 - \varepsilon_h^0) \nabla \cdot (w(0) - w_h^0) - \sum_{e \in \Gamma \cup \Gamma_w} \int_e \{I\varepsilon^0 - \varepsilon_h^0\}[w(0) - w_h^0] \cdot n_e
\]

Since \(Z_{\omega, b}^A = 0\), which implies \(w_h^0 = \tilde{P}w^0\). We apply Cauchy-Schwarz, Young’s inequalities, trace inequality and approximation result (3.12) and (3.13) to the above equality. Then we obtain:

\[
S_1 \leq \frac{1}{8}\|I\varepsilon^0 - \varepsilon_h^0\|_{L^2(\mathcal{E}_h)}^2 + C_1\|\nabla \cdot (w(0) - w_h^0)\|_{L^2(\mathcal{E}_h)}^2
\]

\[
+ C_2 \sum_{e \in \Gamma \cup \Gamma_w} \|\{I\varepsilon^0 - \varepsilon_h^0\}\|_{L^2(e)}\|\{w(0) - w_h^0\}\|_{L^2(e)}
\]

\[
\leq \frac{1}{8}\|I\varepsilon^0 - \varepsilon_h^0\|_{L^2(\mathcal{E}_h)}^2 + C_1 h^{2\min(l,s)-\delta}\|w(0)\|_{H^{s+1}(\mathcal{E}_h)}^2
\]

\[
+ C_2 \sum_{E \in \mathcal{E}_h} h^{-\frac{1}{2}}\|I\varepsilon^0 - \varepsilon_h^0\|_{L^2(E)} h^{-\frac{1}{2}}\|w(0) - w_h^0\|_{L^2(E)} + h\|\nabla (w(0) - w_h^0)\|_{L^2(E)}
\]

\[
\leq \frac{1}{8}\|I\varepsilon^0 - \varepsilon_h^0\|_{L^2(\mathcal{E}_h)}^2 + C_1 h^{2\min(l,s)-\delta}\|w(0)\|_{H^{s+1}(\mathcal{E}_h)}^2
\]

\[
+ \frac{1}{8}\|I\varepsilon^0 - \varepsilon_h^0\|_{L^2(\mathcal{E}_h)}^2 + C_2 h^{2\min(l,s)-\delta}\|w(0)\|_{H^{s+1}(\mathcal{E}_h)}^2
\]

\[
\leq \frac{1}{4}\|I\varepsilon^0 - \varepsilon_h^0\|_{L^2(\mathcal{E}_h)}^2 + C_3 h^{2\min(l,s)-\delta}\|w(0)\|_{H^{s+1}(\mathcal{E}_h)}^2
\]

(3.21)

where if \(\tilde{P}w^0\) is the elliptic projection of \(w^0\) for \(a_2\) and \(a_2\) is nonsymmetric, \(\delta = 1\); otherwise, \(\delta = 0\).
We apply Cauchy-Schwarz, Young’s inequalities and approximation result to bound \( S_2 \).

\[
S_2 = -(\varepsilon(0) - I\varepsilon^0, I\varepsilon^0 - \varepsilon_h^0)\varepsilon_h
\]

\[
\leq \frac{1}{4} \|I\varepsilon^0 - \varepsilon_h^0\|^2_{L^2(\varepsilon_h^0)} + \|\varepsilon(0) - I\varepsilon^0\|^2_{L^2(\varepsilon_h)}
\]

\[
\leq \frac{1}{4} \|I\varepsilon^0 - \varepsilon_h^0\|^2_{L^2(\varepsilon_h^0)} + C_4 h^{2\min(r+1,s)} \|\varepsilon(0)\|^2_{H^s(\varepsilon_h)} \tag{3.22}
\]

Combining (3.20), (3.21) and (3.22), we obtain:

\[
\|I\varepsilon^0 - \varepsilon_h^0\|^2_{L^2(\varepsilon_h^0)} \leq C_5 (h^{2\min(l,s) - \delta}\|w(0)\|_{H^{s+1}(\varepsilon_h)} + h^{2\min(r+1,s)} \|\varepsilon(0)\|_{H^s(\varepsilon_h)}) \tag{3.23}
\]

Because we know \( Z_{\varepsilon;0}^A = 0 \) which means \( \varepsilon_h^0 = \Pi \varepsilon^0 \), then

\[
\|\varepsilon(0) - \Pi\varepsilon^0\|_{L^2(\varepsilon_h)} \leq \|\varepsilon(0) - I\varepsilon^0\|_{L^2(\varepsilon_h)} + \|I\varepsilon^0 - \Pi\varepsilon^0\|_{L^2(\varepsilon_h)}
\]

\[
\leq C_6 (h^{\min(l,s) - \delta}\|w(0)\|_{H^{s+1}(\varepsilon_h)} + h^{\min(r+1,s)} \|\varepsilon(0)\|_{H^s(\varepsilon_h)}) \tag{3.24}
\]

\[\blacksquare\]

3.2 Error analysis

In this section, we assume the good approximation solution \((\tilde{P}w^{n+1}, P_hp^{n+1}, \Pi\varepsilon^{n+1})\)

belongs to \( V_l(\varepsilon_h) \times Q_k(\varepsilon_h) \times U_r(\varepsilon_h) \) and the exact solution satisfies \( (s \in \mathbb{Z}^+) \):

\[
w \in H^1(0,T; (H^{s+1}(\varepsilon_h))^d)
\]

\[
p \in H^2(0,T; H^{s+1}(\varepsilon_h))
\]

\[
\varepsilon \in H^2(0,T; H^{s}(\varepsilon_h))
\]
We recall that the numerical solution \((w_{n+1}^h, p_{n+1}^h, \varepsilon_{n+1}^h)\) satisfies:

\[
\kappa a_1(p_{n+1}^h, r) + c \sum_{E \in \mathcal{E}_h} \int_E \frac{p_{n+1}^h - p_n^h}{\Delta t} r + \sum_{E \in \mathcal{E}_h} \int_E \frac{\varepsilon_{n+1}^h - \varepsilon_n^h}{\Delta t} r \\
- \sum_{E \in \mathcal{E}_h} \int_E (\alpha p_n^h + \nu)r = \ell_1(r) \quad \forall r \in Q_k(\mathcal{E}_h)
\]

\(3.25a\)

\[
\sum_{E \in \mathcal{E}_h} \int_E \varepsilon_{n+1}^h q + b(w_{n+1}^h, q) = \ell_2(q) \quad \forall q \in \mathcal{U}_r(\mathcal{E}_h)
\]

\(3.25b\)

\[
\mu a_2(w_{n+1}^h, v) + \sum_{e \in \Gamma_h \cup \Gamma_w} \sigma_e \left[ \frac{w_{n+1}^h - w_n^h}{\Delta t} \right] \cdot [v] \\
- (\mu + \lambda)b(v, \varepsilon_{n+1}^h) + b(v, p_{n+1}^h) = \ell_3(v) \quad \forall v \in \mathcal{V}_l(\mathcal{E}_h)
\]

\(3.25c\)

By \((3.15a), (3.15b)\) and \((3.15c)\), we know the exact solution \((w^{n+1}, p^{n+1}, \varepsilon^{n+1})\) satisfying:

\[
\kappa a_1(p_1, r) + c \sum_{E \in \mathcal{E}_h} \int_E \frac{p_{n+1}^h - p_n^h}{\Delta t} r + \sum_{E \in \mathcal{E}_h} \int_E \frac{\varepsilon_{n+1}^h - \varepsilon_n^h}{\Delta t} r - \sum_{E \in \mathcal{E}_h} \int_E (\alpha p_n^h + \nu)r \\
= \ell_1(r) + c\Delta t \sum_{E \in \mathcal{E}_h} \int_E \rho_{p; n+1} r + \Delta t \sum_{E \in \mathcal{E}_h} \int_E \rho_{\varepsilon; n+1} r \quad \forall r \in Q_k(\mathcal{E}_h)
\]

\(3.26a\)

\[
\sum_{E \in \mathcal{E}_h} \int_E \varepsilon_{n+1}^h q + b(w_{n+1}^h, q) = \ell_2(q) \quad \forall q \in \mathcal{U}_r(\mathcal{E}_h)
\]

\(3.26b\)

\[
\mu a_2(w^{n+1}, v) + \sum_{e \in \Gamma_h \cup \Gamma_w} \sigma_e \left[ \frac{w_{n+1}^h - w_n^h}{\Delta t} \right] \cdot [v] \\
- (\mu + \lambda)b(v, \varepsilon_{n+1}^h) + b(v, p_{n+1}^h) = \ell_3(v) \quad \forall v \in \mathcal{V}_l(\mathcal{E}_h)
\]

\(3.26c\)

In the above, we use the fact that the jumps of \(w^n\) and \(w^{n+1}\) are zero in \(L^2(e)\) for all edges \(e\) in \(\Gamma_h\) and something else too.
Subtracting equation (3.25a) from (3.26a), we obtain:

\[
\kappa a_1 \left( Z_{p,n+1}^I + Z_{p,n+1}^A, r \right) + c \sum_{E \in \mathcal{E}_h} \int_E \frac{(Z_{p,n+1}^I + Z_{p,n+1}^A) - (Z_{p,n}^I + Z_{p,n}^A)}{\Delta t} \, \nu_E + \sum_{E \in \mathcal{E}_h} \int_E \frac{(Z_{\varepsilon,n+1}^I + Z_{\varepsilon,n+1}^A) - (Z_{\varepsilon,n}^I + Z_{\varepsilon,n}^A)}{\Delta t} \, \nu_E - \alpha \sum_{E \in \mathcal{E}_h} \int_E (Z_{p,n}^I + Z_{p,n}^A) \, \nu_E \\
= c \Delta t \sum_{E \in \mathcal{E}_h} \int_E \rho_{p,n+1} r + \Delta t \sum_{E \in \mathcal{E}_h} \int_E \rho_{\varepsilon,n+1} r
\]

(3.27)

Subtracting equation (3.25b) from (3.26b), we obtain:

\[
b(Z_{w,n+1}^I + Z_{w,n+1}^A, q) + \sum_{E \in \mathcal{E}_h} \int_E (Z_{\varepsilon,n+1}^I + Z_{\varepsilon,n+1}^A) q = 0
\]

(3.28)

Subtracting equation (3.25c) from (3.26c), we obtain:

\[
\mu a_2 (Z_{w,n+1}^I + Z_{w,n+1}^A, v) - (\mu + \lambda)b(v, Z_{\varepsilon,n+1}^I + Z_{\varepsilon,n+1}^A) + b(v, Z_{p,n+1}^I + Z_{p,n+1}^A) \\
+ \sum_{e \in \Gamma_h \cup \Gamma_{\Omega D}} \frac{\sigma_e}{|e|^{\beta_e}} \left[ \left( \frac{(Z_{w,n+1}^I + Z_{w,n+1}^A) - (Z_{w,n}^I + Z_{w,n}^A)}{\Delta t}, [v] \right)_e \right] = 0
\]

(3.29)

From equation (3.28), we obtain:

\[
b\left( \frac{(Z_{w,n+1}^I + Z_{w,n+1}^A) - (Z_{w,n}^I + Z_{w,n}^A)}{\Delta t}, q \right) \\
+ \sum_{E \in \mathcal{E}_h} \int_E \frac{(Z_{\varepsilon,n+1}^I + Z_{\varepsilon,n+1}^A) - (Z_{\varepsilon,n}^I + Z_{\varepsilon,n}^A)}{\Delta t} \, q = 0
\]

(3.30)

Choosing \( q = Z_{\varepsilon,n+1}^A \) in the equation (3.30), we obtain:

\[
b\left( \frac{(Z_{w,n+1}^I + Z_{w,n+1}^A) - (Z_{w,n}^I + Z_{w,n}^A)}{\Delta t}, Z_{\varepsilon,n+1}^A \right) \\
+ \sum_{E \in \mathcal{E}_h} \left( \frac{(Z_{\varepsilon,n+1}^I + Z_{\varepsilon,n+1}^A) - (Z_{\varepsilon,n}^I + Z_{\varepsilon,n}^A)}{\Delta t}, Z_{\varepsilon,n+1}^A \right)_E = 0
\]

(3.31)

Choosing \( q = Z_{p,n+1}^A \) in the equation (3.30), we obtain:

\[
b\left( \frac{(Z_{w,n+1}^I + Z_{w,n+1}^A) - (Z_{w,n}^I + Z_{w,n}^A)}{\Delta t}, Z_{p,n+1}^A \right) \\
+ \sum_{E \in \mathcal{E}_h} \left( \frac{(Z_{\varepsilon,n+1}^I + Z_{\varepsilon,n+1}^A) - (Z_{\varepsilon,n}^I + Z_{\varepsilon,n}^A)}{\Delta t}, Z_{p,n+1}^A \right)_E = 0
\]

(3.32)
Choosing \( r = Z^A_{p; n+1} \) in the equation (3.27), we obtain:

\[
\kappa a_1(Z^I_{p; n+1} + Z^A_{p; n+1}, Z^A_{p; n+1}) + c \sum_{E \in \mathcal{E}_h} \left( \frac{(Z^I_{p; n+1} + Z^A_{p; n+1}) - (Z^I_{p; n} + Z^A_{p; n})}{\Delta t} \right) Z^A_{p; n+1} E
\]

\[
+ \sum_{E \in \mathcal{E}_h} \left( \frac{(Z^I_{\varepsilon; n+1} + Z^A_{\varepsilon; n+1}) - (Z^I_{\varepsilon; n} + Z^A_{\varepsilon; n})}{\Delta t} \right) Z^A_{p; n+1} E - \alpha \sum_{E \in \mathcal{E}_h} (Z^I_{p; n} + Z^A_{p; n}, Z^A_{p; n+1}) E
\]

\[
= c \Delta t \sum_{E \in \mathcal{E}_h} (\rho_{p; n+1}, Z^A_{p; n+1}) E + \Delta t \sum_{E \in \mathcal{E}_h} (\rho_{\varepsilon; n+1}, Z^A_{p; n+1}) E \quad (3.33)
\]

Subtracting the equation (3.32) from (3.33), we obtain:

\[
\kappa a_1(Z^I_{p; n+1} + Z^A_{p; n+1}, Z^A_{p; n+1}) + c \sum_{E \in \mathcal{E}_h} \left( \frac{(Z^I_{p; n+1} + Z^A_{p; n+1}) - (Z^I_{p; n} + Z^A_{p; n})}{\Delta t} \right) Z^A_{p; n+1} E
\]

\[
- b(\frac{(Z^I_{\varepsilon; n+1} + Z^A_{\varepsilon; n+1}) - (Z^I_{\varepsilon; n} + Z^A_{\varepsilon; n})}{\Delta t} \right) Z^A_{p; n+1} E - \alpha \sum_{E \in \mathcal{E}_h} (Z^I_{p; n} + Z^A_{p; n}, Z^A_{p; n+1}) E
\]

\[
= c \Delta t \sum_{E \in \mathcal{E}_h} (\rho_{p; n+1}, Z^A_{p; n+1}) E + \Delta t \sum_{E \in \mathcal{E}_h} (\rho_{\varepsilon; n+1}, Z^A_{p; n+1}) E \quad (3.34)
\]

Choosing \( v = \frac{Z^A_{w; n+1} - Z^A_{w; n}}{\Delta t} \), the equation (3.29) becomes:

\[
\mu a_2 (Z^I_{w; n+1} + Z^A_{w; n+1}, \frac{Z^A_{w; n+1} - Z^A_{w; n}}{\Delta t}) - (\mu + \lambda) b(\frac{Z^A_{w; n+1} - Z^A_{w; n}}{\Delta t}, Z^I_{\varepsilon; n+1} + Z^A_{\varepsilon; n+1})
\]

\[
+ b(\frac{Z^A_{w; n+1} - Z^A_{w; n}}{\Delta t}, Z^I_{p; n+1} + Z^A_{p; n+1})
\]

\[
+ \sum_{e \in \Gamma_h + \Gamma_{wD}} \frac{\sigma_e}{|\epsilon|} \beta (\frac{(Z^I_{w; n+1} + Z^A_{w; n+1}) - (Z^I_{w; n} + Z^A_{w; n})}{\Delta t})_e + \frac{Z^A_{w; n+1} - Z^A_{w; n}}{\Delta t} e
\]

\[
= 0 \quad (3.35)
\]
We add equations (3.35) and (3.34):

\[
\begin{align*}
\kappa_1 & (Z^I_{p,n+1} + Z^A_{p,n+1}, Z^A_{p,n+1}) + c \sum_{E \in \mathcal{E}_h} \frac{(Z^I_{p,n+1} + Z^A_{p,n+1}) - (Z^I_{p,n} + Z^A_{p,n})}{\Delta t} Z^A_{p,n+1}E \\
+ & \mu_2 (Z^I_{w,n+1} + Z^A_{w,n+1}, \frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t}) - (\mu + \lambda)b\left(\frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t}, Z^I_{\epsilon,n+1} + Z^A_{\epsilon,n+1}\right) \\
+ & b\left(\frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t}, Z^I_{p,n+1}\right) - b\left(\frac{Z^I_{w,n+1} - Z^I_{w,n}}{\Delta t}, Z^A_{w,n}\right) - \alpha \sum_{E \in \mathcal{E}_h} (Z^I_{p,n} + Z^A_{p,n}, Z^A_{p,n+1})E \\
+ & \sum_{e \in \Gamma_h \cup \Gamma_p} \frac{\sigma_e}{|e|^{\frac{3}{2}}} \left(\frac{(Z^I_{w,n+1} + Z^A_{w,n+1}) - (Z^I_{w,n} + Z^A_{w,n})}{\Delta t}, \frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t}\right)E \\
= & c\Delta t \sum_{E \in \mathcal{E}_h} (\rho_{p,n+1}, Z^A_{p,n+1})E + \Delta t \sum_{E \in \mathcal{E}_h} (\rho_{\epsilon,n+1}, Z^A_{p,n+1})E 
\end{align*}
\]

(3.36)

Combining equations (3.31) and (3.36), we obtain:

\[
\begin{align*}
\kappa_1 & (Z^I_{p,n+1} + Z^A_{p,n+1}, Z^A_{p,n+1}) + c \sum_{E \in \mathcal{E}_h} \frac{(Z^I_{p,n+1} + Z^A_{p,n+1}) - (Z^I_{p,n} + Z^A_{p,n})}{\Delta t} Z^A_{p,n+1}E \\
+ & \mu_2 (Z^I_{w,n+1} + Z^A_{w,n+1}, \frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t}) - (\mu + \lambda)b\left(\frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t}, Z^I_{\epsilon,n+1}\right) \\
+ & (\mu + \lambda) \sum_{E \in \mathcal{E}_h} \frac{(Z^I_{\epsilon,n+1} + Z^A_{\epsilon,n+1}) - (Z^I_{\epsilon,n} + Z^A_{\epsilon,n})}{\Delta t} Z^A_{\epsilon,n+1}E \\
+ & b\left(\frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t}, Z^I_{p,n+1}\right) - b\left(\frac{Z^I_{w,n+1} - Z^I_{w,n}}{\Delta t}, Z^A_{w,n}\right) - \alpha \sum_{E \in \mathcal{E}_h} (Z^I_{p,n} + Z^A_{p,n}, Z^A_{p,n+1})E \\
+ & \sum_{e \in \Gamma_h \cup \Gamma_p} \frac{\sigma_e}{|e|^{\frac{3}{2}}} \left(\frac{(Z^I_{w,n+1} + Z^A_{w,n+1}) - (Z^I_{w,n} + Z^A_{w,n})}{\Delta t}, \frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t}\right)E \\
= & c\Delta t \sum_{E \in \mathcal{E}_h} (\rho_{p,n+1}, Z^A_{p,n+1})E + \Delta t \sum_{E \in \mathcal{E}_h} (\rho_{\epsilon,n+1}, Z^A_{p,n+1})E
\end{align*}
\]

(3.37)
We rearrange the items of equation (3.37) and obtain an error equation:

\[
\begin{align*}
\kappa a_1(Z_{pn+1}^A, Z_{pn+1}^A) + c \sum_{E \in \mathcal{E}_h} \left( \frac{Z_{pn+1}^A - Z_{pn}^A}{\Delta t}, Z_{pn+1}^A \right)_E \\
+ \mu a_2(Z_{wn+1}^A, \frac{Z_{wn+1}^A - Z_{wn}^A}{\Delta t}) + (\mu + \lambda) \sum_{E \in \mathcal{E}_h} \left( \frac{Z_{\varepsilon,n+1}^A - Z_{\varepsilon,n}^A}{\Delta t}, Z_{\varepsilon,n+1}^A \right)_E
\end{align*}
\]

\[= - \kappa a_1(Z_{pn+1}^I, Z_{pn+1}^A) - c \sum_{E \in \mathcal{E}_h} \left( \frac{Z_{pn+1}^I - Z_{pn}^A}{\Delta t}, Z_{pn+1}^A \right)_E
\]

\[- \mu a_2(Z_{wn+1}^I, \frac{Z_{wn+1}^A - Z_{wn}^A}{\Delta t}) + (\mu + \lambda) b\left( \frac{Z_{wn+1}^A - Z_{wn}^A}{\Delta t}, Z_{\varepsilon,n+1}^I \right) - (\mu + \lambda) \sum_{E \in \mathcal{E}_h} \left( \frac{Z_{\varepsilon,n+1}^A - Z_{\varepsilon,n}^I}{\Delta t}, Z_{\varepsilon,n+1}^A \right)_E
\]

\[+ b\left( \frac{Z_{wn+1}^I - Z_{wn}^I}{\Delta t}, Z_{pn+1}^A \right) - b\left( \frac{Z_{wn+1}^A - Z_{wn}^A}{\Delta t}, Z_{pn+1}^A \right)
\]

\[+ c \Delta t \sum_{E \in \mathcal{E}_h} (\rho_{pn+1}, Z_{pn+1}^A)_E + \Delta t \sum_{E \in \mathcal{E}_h} (\rho_{\varepsilon,n+1}, Z_{\varepsilon,n+1}^A)_E
\]

\[- \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \frac{\sigma_e}{|e|} \left( \frac{Z_{wn+1}^I - Z_{wn}^I}{\Delta t}, \frac{Z_{wn+1}^A - Z_{wn}^A}{\Delta t} \right)_e
\]

\[+ \alpha \sum_{E \in \mathcal{E}_h} (Z_{pn+1}^I, Z_{pn+1}^A)_E + \alpha \sum_{E \in \mathcal{E}_h} (Z_{pn}^A, Z_{pn+1}^A)_E \]  \hspace{1cm} (3.38)

### 3.2.1 Symmetric form

In this section we discuss the case where the form \(a_2(\cdot, \cdot)\) is symmetric and we derive an error estimate for this case.

**Theorem 3.4 (Error estimate for symmetric form)**: We assume the exact solution \(w\) belongs to \(H^1(0,T;(H^{s+1}(\mathcal{E}_h))^d)\), \(p\) belongs to \(H^2(0,T;H^{s+1}(\mathcal{E}_h))\) and \(\varepsilon\) belongs to \(H^2(0,T;H^s(\mathcal{E}_h))\). If \(\theta_2 = -1\), i.e. \(a_2(\cdot, \cdot)\) is symmetric, and sequences \((p^n_h)_{n \geq 0}\) of
functions in $Q_h(\mathcal{E}_h)$, $(\mathbf{w}_n^h)_{n \geq 0}$ of functions in $\mathcal{V}_l(\mathcal{E}_h)$, $(\varepsilon_n^h)_{n \geq 0}$ of functions in $\mathcal{U}_r(\mathcal{E}_h)$ satisfy the model (2.18), then we have

\[
\begin{align*}
&\left\| \mathbf{Z}^A_{w:m} \right\|_V^2 + c \left\| \mathbf{Z}^A_{p:m} \right\|_{L^2(\mathcal{E}_h)}^2 + \left\| \mathbf{Z}^A_{\varepsilon:m} \right\|_{L^2(\mathcal{E}_h)}^2 + \Delta t \sum_{n=1}^{m} \left\| \mathbf{Z}^A_{p:n} \right\|_Q^2 \\
&+ \sum_{n=1}^{m} \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \sigma_e \frac{\left\| \mathbf{Z}^A_{w:n} - \mathbf{Z}^A_{w:n-1} \right\|_{L^2(e)}}{\Delta t}^2 \\
&\leq \tilde{C}_1 h^{2\min(k,s)}(\|p\|_{L^2(H^{s+1}(\mathcal{E}_h))}^2 + T h^2 \|\frac{\partial p}{\partial t}\|_{L^\infty(0,T;H^{s+1}(\mathcal{E}_h))}^2) \\
&+ \tilde{C}_2 h^{2\min(l,s)}(\|\mathbf{w}(0)\|_{H^{s+1}(\mathcal{E}_h)}^2 + T h^2 \|\frac{\partial \mathbf{w}}{\partial t}\|_{L^\infty(0,T;H^{s+1}(\mathcal{E}_h))}^2) \\
&+ \tilde{C}_3 h^{2\min(r+1,s)}(\|\varepsilon\|_{L^2(H^{s+1}(\mathcal{E}_h))}^2 + T h^2 \|\frac{\partial \varepsilon}{\partial t}\|_{L^\infty(0,T;H^{s+1}(\mathcal{E}_h))}^2 + \|\varepsilon(0)\|_{H^{s}(\mathcal{E}_h)}^2) \\
&+ \tilde{C}_4 \Delta t^2 (\|\frac{\partial^2 p}{\partial t^2}\|_{L^2(L^\infty(t_n,t_{n+1};L^2(\Omega)))}^2 + \|\frac{\partial^2 \varepsilon}{\partial t^2}\|_{L^2(L^\infty(t_n,t_{n+1};L^2(\Omega)))}^2)
\end{align*}
\]

where $k, l$ and $r$ are the degrees of the polynomial spaces; $s$ is the degree of the Sobolev space; $\tilde{C}_1$, $\tilde{C}_2$, $\tilde{C}_3$ and $\tilde{C}_4$ are constants independent of $h$.

Proof.

Let us write the left-hand side of (3.38) as

\[
\begin{align*}
&\kappa_1 (\mathbf{Z}^A_{p:n+1}, \mathbf{Z}^A_{p:n+1}) + c \sum_{E \in \mathcal{E}_h} (\frac{\mathbf{Z}^A_{p:n+1} - \mathbf{Z}^A_{p:n}}{\Delta t}, \mathbf{Z}^A_{p:n+1})_E \\
&+ \mu a_2 (\mathbf{Z}^A_{w:n+1}, \frac{\mathbf{Z}^A_{w:n+1} - \mathbf{Z}^A_{w:n}}{\Delta t}) + (\mu + \lambda) \sum_{E \in \mathcal{E}_h} (\frac{\mathbf{Z}^A_{\varepsilon:n+1} - \mathbf{Z}^A_{\varepsilon:n}}{\Delta t}, \mathbf{Z}^A_{\varepsilon:n+1})_E \\
&+ \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \sigma_e \frac{\left\| \mathbf{Z}^A_{w:n+1} - \mathbf{Z}^A_{w:n} \right\|_{L^2(e)}}{\Delta t}^2 \\
:= R_1 + R_2 + R_3 + R_4 + R_5
\end{align*}
\]
By Lemma 2.1, we have:

$$R_1 = \kappa a_1(Z_{p,n+1}^A, Z_{p,n+1}^A) \geq C_1 \kappa \|Z_{p,n+1}^A\|^2_Q$$

From the inequality (2.12), we derive:

$$R_2 = c \sum_{E \in \mathcal{E}_h} \left( \frac{Z_{p,n+1}^A - Z_{p,n}^A}{\Delta t} \right)_E \geq \frac{c}{2\Delta t} (\|Z_{p,n+1}^A\|^2_{L^2(\mathcal{E}_h)} - \|Z_{p,n}^A\|^2_{L^2(\mathcal{E}_h)})$$

Similarly, from the inequality (2.12), we have:

$$R_4 = (\mu + \lambda) \sum_{E \in \mathcal{E}_h} \left( \frac{Z_{\w,n+1}^A - Z_{\w,n}^A}{\Delta t} \right)_E \geq \frac{\mu + \lambda}{2\Delta t} (\|Z_{\w,n+1}^A\|^2_{L^2(\mathcal{E}_h)} - \|Z_{\w,n}^A\|^2_{L^2(\mathcal{E}_h)})$$

Finally, we have:

$$R_5 = \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \sigma_e \left|\left|\frac{Z_{w,n+1}^A - Z_{w,n}^A}{\Delta t}\right|\right|_e$$

Then, we conclude:

$$R_1 + R_2 + R_3 + R_4 + R_5$$

$$\geq C_1 \kappa \|Z_{p,n+1}^A\|^2_Q + \frac{\mu}{\Delta t} a_2(Z_{w,n+1}^A, Z_{w,n+1}^A - Z_{w,n}^A) + \frac{c}{2\Delta t} (\|Z_{p,n+1}^A\|^2_{L^2(\mathcal{E}_h)} - \|Z_{p,n}^A\|^2_{L^2(\mathcal{E}_h)}) + \frac{\mu + \lambda}{2\Delta t} (\|Z_{\w,n+1}^A\|^2_{L^2(\mathcal{E}_h)} - \|Z_{\w,n}^A\|^2_{L^2(\mathcal{E}_h)}) + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \sigma_e \left|\left|\frac{Z_{w,n+1}^A - Z_{w,n}^A}{\Delta t}\right|\right|_e$$

Let

$$\tilde{C}_5 = \min(C_1 \kappa, c, \mu + \lambda, 1), \text{ if } c \neq 0$$

$$\tilde{C}_5 = \min(C_1 \kappa, \mu + \lambda, 1), \text{ if } c = 0$$
Then from the error equation (3.38), we get:

\[
\begin{align*}
&\|Z_{p:n+1}^A\|^2_\mathcal{Q} + \frac{\mu}{\Delta t} a_2(Z_{w:n+1}^A, Z_{p:n+1}^A - Z_{w:n}^A) \\
&+ \frac{c}{2\Delta t} \left( \|Z_{p:n+1}^A\|^2_{L^2(\mathcal{E}_h)} - \|Z_{p:n}^A\|^2_{L^2(\mathcal{E}_h)} \right) \\
&+ \frac{1}{2\Delta t} \left( \|Z_{\varepsilon,n+1}^A\|^2_{L^2(\mathcal{E}_h)} - \|Z_{\varepsilon,n}^A\|^2_{L^2(\mathcal{E}_h)} \right) \\
&+ \sum_{e \in \Gamma_h \cup \Gamma_D} \sigma_e \left( \left[ \frac{Z_{w:n+1}^I - Z_{w:n}^I}{\Delta t} \right], \left[ \frac{Z_{p:n+1}^I - Z_{p:n}^I}{\Delta t} \right] \right)_{e} \\
&\leq \frac{1}{C_5} \left( - \kappa a_1(Z_{p:n+1}^I, Z_{p:n+1}^A) - c \sum_{E \in \mathcal{E}_h} (\frac{Z_{p:n+1}^I - Z_{p:n}^I}{\Delta t}, Z_{p:n+1}^A)_{E} \\
&- \mu a_2(Z_{w:n+1}^A, Z_{w:n+1}^A - Z_{w:n}^A) + (\mu + \lambda) b \left( \frac{Z_{w:n+1}^A - Z_{w:n}^A}{\Delta t}, Z_{p:n+1}^A \right) \\
&- (\mu + \lambda) b \left( \frac{Z_{w:n+1}^A - Z_{w:n}^A}{\Delta t}, Z_{\varepsilon,n+1}^A \right) - (\mu + \lambda) \sum_{E \in \mathcal{E}_h} (\frac{Z_{\varepsilon,n+1}^I - Z_{\varepsilon,n}^I}{\Delta t}, Z_{\varepsilon,n+1}^A)_{E} \\
&+ b \left( \frac{Z_{w:n+1}^I - Z_{w:n}^I}{\Delta t}, Z_{p:n+1}^A \right) - b \left( \frac{Z_{w:n+1}^A - Z_{w:n}^A}{\Delta t}, Z_{p:n+1}^I \right) \\
&+ c \Delta t \sum_{E \in \mathcal{E}_h} (\rho_{p:n+1}, Z_{p:n+1}^A)_{E} + \Delta t \sum_{E \in \mathcal{E}_h} (\rho_{\varepsilon,n+1}, Z_{\varepsilon,n+1}^A)_{E} \\
&- \sum_{e \in \Gamma_h \cup \Gamma_D} \sigma_e \left( \left[ \frac{Z_{w:n+1}^I - Z_{w:n}^I}{\Delta t} \right], \left[ \frac{Z_{w:n+1}^A - Z_{w:n}^A}{\Delta t} \right] \right)_{e} \\
&+ \alpha \sum_{E \in \mathcal{E}_h} (Z_{p:n}^I, Z_{p:n+1}^A) + \alpha \sum_{E \in \mathcal{E}_h} (Z_{p:n}^A, Z_{p:n+1}^A) \\
:= & T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 + T_9 + T_{10} + T_{11} + T_{12} + T_{13} \quad (3.40)
\end{align*}
\]

Now, we estimate each term $T_i$, for $i = 1, \cdots, 13$. 
By the definition of $a_1(\cdot, \cdot)$, we know:

$$T_1 = - \frac{\kappa}{C_5} a_1(Z_{p,n+1}^I, Z_{p,n+1}^A)$$

$$= - \frac{\kappa}{C_5} \left( \sum_{E \in \mathcal{E}_h} \int_E \nabla Z_{p,n+1}^I \cdot \nabla Z_{p,n+1}^A - \sum_{e \in \Gamma_h \cup \Gamma_{pD}} \int_e \{ \nabla Z_{p,n+1}^I \} \cdot n_e [Z_{p,n+1}^A] ight)$$

$$+ \sum_{e \in \Gamma_h \cup \Gamma_{pD}} \frac{\sigma_e}{|e|^2} \int_e [Z_{p,n+1}^I] [Z_{p,n+1}^A] + \theta_1 \sum_{e \in \Gamma_h \cup \Gamma_{pD}} \int_e \{ \nabla Z_{p,n+1}^A \} \cdot n_e [Z_{p,n+1}^I])$$

$$= - \frac{\kappa}{C_5} (T_{101} + T_{102} + T_{103} + T_{104})$$

We need to estimate the terms $T_{101}, T_{102}, T_{103}, T_{104}$ separately.

To bound the term $T_{101}$, Cauchy-Schwarz’s inequality (3.5), Young’s inequality (3.6) and the approximation result (3.11a) are applied.

$$T_{101} = \sum_{E \in \mathcal{E}_h} \int_E \nabla Z_{p,n+1}^I \cdot \nabla Z_{p,n+1}^A$$

$$\leq \sum_{E \in \mathcal{E}_h} \| \nabla Z_{p,n+1}^I \|_{L^2(E)} \| \nabla Z_{p,n+1}^A \|_{L^2(E)}$$

$$\leq \tilde{C_5} \sum_{E \in \mathcal{E}_h} \| \nabla Z_{p,n+1}^A \|_{L^2(E)}^2 + C_6 \| \nabla Z_{p,n+1}^I \|_{L^2(\Omega)}^2$$

$$= \frac{\tilde{C_5}}{24\kappa} \| \nabla Z_{p,n+1}^A \|_{L^2(\mathcal{E}_h)}^2 + C_6 \| \nabla Z_{p,n+1}^I \|_{L^2(\Omega)}^2$$

$$\leq \frac{\tilde{C_5}}{24\kappa} \| \nabla Z_{p,n+1}^A \|_{L^2(\mathcal{E}_h)}^2 + C_6 h^{2\min(k,s)} \| p^{n+1} \|_{H^{s+1}(\mathcal{E}_h)}^2$$

Here in the last inequality, the constant $C_6$ is different from the previous one. But we still use the same notation for simplicity. In the sequel, we will also use generic constants $C_i$ for $i \geq 7$. 
We bound $T_{102}$ by Cauchy-Schwarz’s inequality (3.5), Young’s inequality (3.6), the trace inequality (3.1b) and the approximation result (3.11a).

\[ T_{102} = - \sum_{e \in \Gamma_D \cup T_pD} \int_e \{ \nabla Z_{p,n+1}^I \} \cdot n_e [Z_{p,n+1}^A] \]

\[ \leq \sum_{e \in \Gamma_D \cup T_pD} \| [Z_{p,n+1}^A] \|_{L^2(e)} \| \{ \nabla Z_{p,n+1}^I \} \cdot n_e \|_{L^2(e)} \]

\[ \leq \frac{\bar{C}_5}{24k} \sum_{e \in \Gamma_D \cup T_pD} \frac{\sigma_e}{|e|^\beta} \| [Z_{p,n+1}^A] \|_{L^2(e)}^2 + C_7 \sum_{e \in \Gamma_D \cup T_pD} h \| \{ \nabla Z_{p,n+1}^I \} \cdot n_e \|_{L^2(e)}^2 \]

\[ \leq \frac{\bar{C}_5}{24k} \sum_{e \in \Gamma_D \cup T_pD} \frac{\sigma_e}{|e|^\beta} \| [Z_{p,n+1}^A] \|_{L^2(e)}^2 \]

\[ + C_7 \sum_{E \in \mathcal{E}_h} h^{-1} \| \nabla Z_{p,n+1}^I \|_{L^2(E)}^2 + h^2 \| \nabla^2 Z_{p,n+1}^I \|_{L^2(E)}^2 \]

\[ \leq \frac{\bar{C}_5}{24k} \sum_{e \in \Gamma_D \cup T_pD} \frac{\sigma_e}{|e|^\beta} \| [Z_{p,n+1}^A] \|_{L^2(e)}^2 + C_7 h^{2 \min(k,s)} \| P^{n+1} \|_{H^{s+1}(\mathcal{E}_h)}^2 \]

Also, $T_{103}$ is bounded by Cauchy-Schwarz’s inequality, Young’s inequality, the trace inequality (3.1a) and the approximation result (3.11a).

\[ T_{103} = \sum_{e \in \Gamma_D \cup T_pD} \frac{\sigma_e}{|e|^\beta} \int_e [Z_{p,n+1}^I] [Z_{p,n+1}^A] \]

\[ \leq \sum_{e \in \Gamma_D \cup T_pD} \frac{\sigma_e}{|e|^\beta} \| [Z_{p,n+1}^I] \|_{L^2(e)} \| [Z_{p,n+1}^A] \|_{L^2(e)} \]

\[ \leq \frac{\bar{C}_5}{24k} \sum_{e \in \Gamma_D \cup T_pD} \frac{\sigma_e}{|e|^\beta} \| [Z_{p,n+1}^A] \|_{L^2(e)}^2 + \frac{C_8}{h} \sum_{e \in \Gamma_D \cup T_pD} \| [Z_{p,n+1}^I] \|_{L^2(e)}^2 \]

\[ \leq \frac{\bar{C}_5}{24k} \sum_{e \in \Gamma_D \cup T_pD} \frac{\sigma_e}{|e|^\beta} \| [Z_{p,n+1}^A] \|_{L^2(e)}^2 \]

\[ + C_8 \sum_{E \in \mathcal{E}_h} h^{-1} h^{-1} \| Z_{p,n+1}^I \|_{L^2(E)}^2 + h^2 \| \nabla Z_{p,n+1}^I \|_{L^2(E)}^2 \]

\[ \leq \frac{\bar{C}_5}{24k} \sum_{e \in \Gamma_D \cup T_pD} \frac{\sigma_e}{|e|^\beta} \| [Z_{p,n+1}^A] \|_{L^2(e)}^2 + C_8 h^{2 \min(k,s)} \| P^{n+1} \|_{H^{s+1}(\mathcal{E}_h)}^2 \]
Finally, \( T_{104} \) is bounded by Cauchy-Schwarz’s inequality, Young’s inequality, the trace inequalities (3.1a), (3.2a), inverse inequality (3.10) and the approximation result (3.11a).

\[
T_{104} = \theta_1 \sum_{e \in \Gamma_h \cup \Gamma_{pD}} \int_e \{ \nabla Z^A_{p;n+1} \} \cdot n_e \{ Z^I_{p;n+1} \}
\]

\[
\leq C_9 \sum_{e \in \Gamma_h \cup \Gamma_{pD}} \| \{ \nabla Z^A_{p;n+1} \} \cdot n_e \|_{L^2(e)} \| [Z^I_{p;n+1}] \|_{L^2(e)}
\]

\[
\leq C_9 \sum_{E \in \mathcal{E}_h} h^{-1/2} \| \nabla Z^A_{p;n+1} \|_{L^2(E)} h^{-1/2} (\| Z^I_{p;n+1} \|_{L^2(E)} + h \| \nabla Z^I_{p;n+1} \|_{L^2(E)})
\]

\[
\leq \frac{C_5}{24h} \| \nabla Z^A_{p;n+1} \|_{L^2(\mathcal{E}_h)}^2 + C_{10} h^{2 \min(k,s)} \| p^{n+1} \|_{H^{s+1}(\Omega)}^2
\]

Combining the above four inequalities, we obtain:

\[
T_1 = - \frac{k}{C_5} (T_{101} + T_{102} + T_{103} + T_{104})
\]

\[
\leq \frac{1}{12} \| \nabla Z^A_{p;n+1} \|_{L^2(\mathcal{E}_h)}^2 + C_{11} h^{2 \min(k,s)} \| p^{n+1} \|_{H^{s+1}(\Omega)}^2
\]

\[
+ \frac{1}{12} \sum_{e \in \Gamma_h \cup \Gamma_{pD}} \frac{\sigma_e}{|e|} \| [Z^A_{p;n+1}] \|_{L^2(e)}^2
\]

\[
\leq \frac{1}{12} \| Z^A_{p;n+1} \|_Q^2 + C_{11} h^{2 \min(k,s)} \| p^{n+1} \|_{H^{s+1}(\mathcal{E}_h)}^2
\]

(3.41)
To bound the term $T_2$, Cauchy-Schwarz’s inequality, Young’s inequality can be employed. Hence,

$$T_2 = -\frac{c}{C_5} \sum_{E \in \mathcal{E}_h} \left( \frac{Z_{p:n+1} - Z_{p:n}}{\Delta t}, Z_A^{p:n+1} \right)_E$$

$$\leq \frac{c}{C_5} \sum_{E \in \mathcal{E}_h} \left\| Z_A^{p:n+1} \right\|_{L^2(E)} \left\| \frac{Z_{p:n+1} - Z_{p:n}}{\Delta t} \right\|_{L^2(E)}$$

$$\leq \frac{c}{12} \sum_{E \in \mathcal{E}_h} \left\| Z_A^{p:n+1} \right\|_{L^2(E)}^2 + C_{12}c \sum_{E \in \mathcal{E}_h} \left\| \frac{Z_{p:n+1} - Z_{p:n}}{\Delta t} \right\|_{L^2(E)}^2$$

$$= \frac{c}{12} \left\| Z_A^{p:n+1} \right\|_{L^2(\mathcal{E}_h)}^2 + C_{12}c \left\| \frac{Z_{p:n+1} - Z_{p:n}}{\Delta t} \right\|_{L^2(\mathcal{E}_h)}^2$$

Next we use a Taylor series and approximation result (3.11a) for the above inequality, and we obtain:

$$T_2 \leq \frac{c}{12} \left\| Z_A^{p:n+1} \right\|_{L^2(\mathcal{E}_h)}^2 + C_{12}c \left\| \frac{\partial Z_I^p}{\partial t} \right|_{t=(n+\delta_1)\Delta t} \right\|_{L^2(\mathcal{E}_h)}^2$$

$$\leq \frac{c}{12} \left\| Z_A^{p:n+1} \right\|_{L^2(\mathcal{E}_h)}^2 + C_{12}c \Delta t^{2\min(k+1,s+1)} \left\| \frac{\partial p}{\partial t} \right|_{t=(n+\delta_1)\Delta t} \right\|_{H^{s+1}(\mathcal{E}_h)}^2$$

(3.42)

where $\delta_1$ is some constant in $[0, 1]$.

Recall $\tilde{P}w^{n+1}$ is the elliptic projection of the true solution $w^{n+1}$ for $a_2(\cdot, \cdot)$, which means:

$$T_3 = -\frac{\mu}{C_5} a_2(Z_{w:n+1}, Z_A^{w:n+1} - Z_{w:n}) = 0$$

(3.43)

Assume $\Pi \varepsilon^{n+1}$ is the $L^2$ projection of the true solution in space $\mathcal{U}_{l-1}$, which means:

$$\forall n \in \mathbb{N}, \forall q \in \mathcal{U}_r(\mathcal{E}_h), (\varepsilon^{n+1} - \Pi \varepsilon^{n+1}, q)_{\mathcal{E}_h} = 0$$
By the definition of $b(\cdot, \cdot)$, we obtain for the term $T_4$:

$$
T_4 = \frac{(\mu + \lambda)}{C_5} b(\frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t}, Z^I_{\varepsilon,n+1})
$$

$$
= \frac{(\mu + \lambda)}{C_5} \left( - \sum_{E \in \mathcal{E}_h} \int_E Z^I_{\varepsilon,n+1} \nabla \cdot \frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t} + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \int_e \{Z^I_{\varepsilon,n+1}\} \{\frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t}\} \cdot \mathbf{n}_e \right)
$$

$$
= \frac{(\mu + \lambda)}{C_5} (T_{401} + T_{402})
$$

Since $Z^A_{w,n}$ and $Z^A_{w,n+1}$ belong to $\mathcal{V}_t$, then we have:

$$
\nabla \cdot \frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t} \in \mathcal{U}_{t-1}
$$

Therefore, since $\Pi \varepsilon^{n+1}$ is the $L^2$ projection of $\varepsilon^{n+1}$ for $n \in \mathbb{N}$, the term $T_{401}$ vanishes:

$$
T_{401} = - \sum_{E \in \mathcal{E}_h} \int_E Z^I_{\varepsilon,n+1} \nabla \cdot \frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t} = 0
$$

For the term $T_{402}$, we apply Young’s inequality and have:

$$
T_{402} = \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \int_e \{Z^I_{\varepsilon,n+1}\} \{\frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t}\} \cdot \mathbf{n}_e
$$

$$
\leq \sum_{e \in \Gamma_h \cup \Gamma_{wD}} |e|^{\frac{\beta_1}{2}} \|\{Z^I_{\varepsilon,n+1}\}\|_{L^2(e)} \|\{\frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t}\}\|_{L^2(e)} |e|^{\frac{\beta_1}{2}}
$$

$$
\leq \frac{C_5}{12(\mu + \lambda)} \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \sigma_e |e|^{\beta_1} \|\{\frac{Z^A_{w,n+1} - Z^A_{w,n}}{\Delta t}\}\|_{L^2(e)} \|\mathbf{n}_e\|^2_{L^2(e)}
$$

$$
+ C_{14} \sum_{e \in \Gamma_h \cup \Gamma_{wD}} h \|\{Z^I_{\varepsilon,n+1}\}\|_{L^2(e)}^2
$$

We recall $\beta_1 \geq 1$ in 2D and $\beta_1 \geq \frac{1}{2}$ in 3D.
Then using the trace inequality (3.1a) employed over each edge-element pair and approximation result (3.11b), we can find the following bound for $T_{402}$:

\[
T_{402} \leq \frac{\tilde{C}_5}{12(\mu + \lambda)} \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \frac{\sigma_e}{|e|^{\beta_1}} \left\| \frac{Z_{\bar{w};n+1}^A - Z_{\bar{w};n}^A}{\Delta t} \right\|_{L^2(e)}^2 \\
+ C_{14} \sum_{E \in \mathcal{E}_h} \frac{\sigma_e}{|e|^{\beta_1}} \left\| \frac{Z_{\bar{w};n+1}^A - Z_{\bar{w};n}^A}{\Delta t} \right\|_{L^2(e)}^2
\]

\[
\leq \frac{\tilde{C}_5}{12(\mu + \lambda)} \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \frac{\sigma_e}{|e|^{\beta_1}} \left\| \frac{Z_{\bar{w};n+1}^A - Z_{\bar{w};n}^A}{\Delta t} \right\|_{L^2(e)}^2 + C_{14} h^{2 \text{min}(r+1,s)} \| \varepsilon_{n+1} \|_{H^s(\mathcal{E}_h)}^2
\]

Combining the above two results, we get:

\[
T_4 \leq \frac{1}{12} \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \frac{\sigma_e}{|e|^{\beta_1}} \left\| \frac{Z_{\bar{w};n+1}^A - Z_{\bar{w};n}^A}{\Delta t} \right\|_{L^2(e)}^2 + C_{14} h^{2 \text{min}(r+1,s)} \| \varepsilon_{n+1} \|_{H^s(\mathcal{E}_h)}^2 \quad (3.44)
\]

By the definition of $b(\cdot, \cdot)$, we expand the term $T_5$:

\[
T_5 = -\frac{(\mu + \lambda)}{\tilde{C}_5} b\left(\frac{Z_{\bar{w};n+1}^I - Z_{\bar{w};n}^I}{\Delta t}, Z_{\varepsilon;n+1}^I\right)
\]

\[
= -\frac{(\mu + \lambda)}{\tilde{C}_5} \left( - \sum_{E \in \mathcal{E}_h} \int_E Z_{\varepsilon;n+1}^I \nabla \cdot \frac{Z_{\bar{w};n+1}^I - Z_{\bar{w};n}^I}{\Delta t} \right)
\]

\[
+ \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \int_e \left\{ Z_{\varepsilon;n+1}^I \right\} \cdot \left( \frac{Z_{\bar{w};n+1}^I - Z_{\bar{w};n}^I}{\Delta t} \cdot \mathbf{n}_e \right)
\]

\[
= -\frac{(\mu + \lambda)}{\tilde{C}_5} (T_{501} + T_{502})
\]

We estimate $T_{501}$ and $T_{502}$ separately.
First we apply Cauchy-Schwarz’s, Young’s inequalities for $T_{501}$ and obtain:

$$T_{501} = - \sum_{E \in \mathcal{E}_h} \int_E Z^A_{\varepsilon;n+1} \nabla \cdot \frac{Z^I_{w;n+1} - Z^I_{w;n}}{\Delta t}$$

$$\leq \sum_{E \in \mathcal{E}_h} \|Z^A_{\varepsilon;n+1}\|_{L^2(E)} \|\nabla \cdot \frac{Z^I_{w;n+1} - Z^I_{w;n}}{\Delta t}\|_{L^2(E)}$$

$$\leq C_{16} \sum_{E \in \mathcal{E}_h} \|Z^A_{\varepsilon;n+1}\|_{L^2(E)} \|\nabla \frac{Z^I_{w;n+1} - Z^I_{w;n}}{\Delta t}\|_{L^2(E)}$$

$$\leq \frac{\tilde{C}_5}{24(\mu + \lambda)} \sum_{E \in \mathcal{E}_h} \|Z^A_{\varepsilon;n+1}\|_{L^2(E)}^2 + C_{16} \sum_{E \in \mathcal{E}_h} \|\nabla \frac{Z^I_{w;n+1} - Z^I_{w;n}}{\Delta t}\|_{L^2(E)}^2$$

Next we use a Taylor series and the approximation result (3.11b) for the above inequality to have:

$$T_{501} \leq \frac{\tilde{C}_5}{24(\mu + \lambda)} \|Z^A_{\varepsilon;n+1}\|_{L^2(E_h)}^2 + C_{16} \|\nabla \frac{Z^I_{w;n+1} - Z^I_{w;n}}{\Delta t}\|_{L^2(E_h)}^2$$

where $\delta_2$ is some constant in $[0, 1]$.

The bound for the term $T_{502}$ uses the Cauchy-Schwarz’s, Young’s inequalities, Taylor series, the trace inequality and the approximation result (3.12).

$$T_{502} = \sum_{e \in \Gamma_h \cup \Gamma_{wd}} \int_e \left\{ Z^A_{\varepsilon;n+1} \right\} \left[ \frac{Z^I_{w;n+1} - Z^I_{w;n}}{\Delta t} \right] \cdot \mathbf{n}_e$$

$$\leq C_{17} \sum_{e \in \Gamma_h \cup \Gamma_{wd}} h^\frac{1}{2} \left\{ Z^A_{\varepsilon;n+1} \right\} \left[ \left\| \frac{\partial Z^I_{w}}{\partial t} \right\|_{L^2(E)} \right] \left\| \frac{\partial Z^I_{w}}{\partial t} \right\|_{L^2(E)} h^{-\frac{1}{2}}$$

$$\leq C_{17} \sum_{E \in \mathcal{E}_h} \|Z^A_{\varepsilon;n+1}\|_{L^2(E)} h^{-1} \left( \left\| \frac{\partial Z^I_{w}}{\partial t} \right\|_{L^2(E)} + h \left\| \nabla \frac{\partial Z^I_{w}}{\partial t} \right\|_{L^2(E)} \right)$$

$$\leq \frac{\tilde{C}_5}{24(\mu + \lambda)} \|Z^A_{\varepsilon;n+1}\|_{L^2(E_h)}^2 + C_{18} h^2 \min(l,s) \left\| \frac{\partial \mathbf{w}}{\partial t} \right\|_{L^2(E_h)}^2$$

where $\delta_3$ is a constant in $[0, 1]$. 
Thus combining the bounds above, we have:

$$T_5 \leq \frac{1}{12} \|Z_{\varepsilon;n+1}^A\|_{L^2(E_h)}^2 + C_{19} h^{2\min(t,s)} \|\frac{\partial w}{\partial t}|_{t=(n+\delta_4)\Delta t}\|_{H^{s+1}(E_h)}^2$$  (3.45)

where $\delta_4$ is a constant in $[0, 1]$.

To bound the term $T_6$, for $n > 0$, the Cauchy-Schwarz’s, Young’s inequalities, the trace inequality and the approximation result (3.11b) are employed.

$$T_6 = -\frac{(\mu + \lambda)}{C_5} \sum_{E \in E_h} \left( \frac{Z_{\varepsilon;n+1}^I - Z_{\varepsilon;n}^I}{\Delta t}, Z_{\varepsilon;n+1}^A \right)_E$$

$$\leq \frac{(\mu + \lambda)}{C_5} \sum_{E \in E_h} \|Z_{\varepsilon;n+1}^A\|_{L^2(E)} \left( \frac{Z_{\varepsilon;n+1}^I - Z_{\varepsilon;n}^I}{\Delta t} \right)_{L^2(E)}$$

$$\leq \frac{1}{12} \sum_{E \in E_h} \|Z_{\varepsilon;n+1}^A\|_{L^2(E)}^2 + C_{20} \sum_{E \in E_h} \left( \frac{Z_{\varepsilon;n+1}^I - Z_{\varepsilon;n}^I}{\Delta t} \right)_{L^2(E)}$$

$$= \frac{1}{12} \|Z_{\varepsilon;n+1}^A\|_{L^2(E_h)}^2 + C_{20} \|\frac{\partial Z_{\varepsilon}^I}{\partial t}|_{t=(n+\delta_5)\Delta t}\|_{L^2(\Omega)}^2$$

$$\leq \frac{1}{12} \|Z_{\varepsilon;n+1}^A\|_{L^2(E_h)}^2 + C_{20} h^{2\min(s+1,t)} \|\frac{\partial \varepsilon}{\partial t}|_{t=(n+\delta_5)\Delta t}\|_{H^{s+1}(E_h)}^2$$  (3.46)

where $\delta_5$ is some constant in $[0, 1]$.

For $n = 0$, from the above inequality, we know:

$$T_6 \leq \frac{1}{12} \sum_{E \in E_h} \|Z_{\varepsilon;1}^A\|_{L^2(E)}^2 + C_{20} \sum_{E \in E_h} \left( \frac{Z_{\varepsilon;1}^I - Z_{\varepsilon;0}^I}{\Delta t} \right)_{L^2(E)}$$  (3.47)

Let $Z_{\varepsilon;0}^I = \varepsilon(0) - I\varepsilon^0$, where $I\varepsilon^0$ is the $L^2$ projection of $\varepsilon(0)$. By the fact $\Pi\varepsilon^0 = \varepsilon_h^0$, then we have:

$$\sum_{E \in E_h} \left( \frac{Z_{\varepsilon;1}^I - Z_{\varepsilon;0}^I}{\Delta t} \right)_{L^2(E)} \leq \sum_{E \in E_h} \left( \frac{Z_{\varepsilon;1}^I - Z_{\varepsilon;0}^I}{\Delta t} \right)_{L^2(E)} + \sum_{E \in E_h} \frac{1}{\Delta t} \|I\varepsilon^0 - \Pi\varepsilon^0\|_{L^2(E)}^2$$

$$= \sum_{E \in E_h} \left( \frac{Z_{\varepsilon;1}^I - Z_{\varepsilon;0}^I}{\Delta t} \right)_{L^2(E)} + \sum_{E \in E_h} \frac{1}{\Delta t} \|I\varepsilon^0 - \varepsilon_h^0\|_{L^2(E)}^2$$
By the triangle inequality, the approximation result (3.11b) and Lemma 3.5, the above inequality becomes:

\[
\sum_{E \in \mathcal{E}_h} \left\| \frac{Z_{\varepsilon,1}^I - Z_{\varepsilon,0}^I}{\Delta t} \right\|_{L^2(E)}^2 \leq \tilde{C}_{20} h^{2 \min(r+1,s)} \left\| \frac{\partial \varepsilon}{\partial t} \right\|_{H^s(\mathcal{E}_h)}^2 \\
+ \frac{C_0}{\Delta t} (h^{2 \min(l,s)} \|w(0)\|_{H^{s+1}(\mathcal{E}_h)}^2 + h^{2 \min(r+1,s)} \|\varepsilon(0)\|_{H^s(\mathcal{E}_h)}^2)
\]

(3.48)

where \(\delta_5\) is some constant in \([0, 1]\) and \(C_0\) is some constant independent of \(h\).

Combining (3.47) and (3.48), when \(n = 0\) we have a bound for \(T_6\):

\[
T_6 \leq \frac{1}{12} \sum_{E \in \mathcal{E}_h} \left\| Z_{\varepsilon,1}^I \right\|_{L^2(E)}^2 + \frac{C_0}{\Delta t} (h^{2 \min(l,s)} \|w(0)\|_{H^{s+1}(\mathcal{E}_h)}^2 + h^{2 \min(r+1,s)} \|\varepsilon(0)\|_{H^s(\mathcal{E}_h)}^2)
\]

(3.49)

The bound for the term \(T_7\) is obtained in a similar way as for the bound of \(T_5\) and applied Poincare’s inequality.

\[
T_7 = \frac{1}{C_5} b \left( \frac{Z_{w:n+1}^A - Z_{w:n}^A}{\Delta t}, Z_{p:n+1}^A \right)
\]

\[
\leq \frac{1}{12} \left\| Z_{p:n+1}^A \right\|_Q^2 + C_{21} h^{2 \min(l,s)} \left\| \frac{\partial w}{\partial t} \right\|_{t=(n+\delta_6)\Delta t}^2 H^{s+1}(\mathcal{E}_h)
\]

(3.50)

where \(\delta_6\) is a constant in \([0, 1]\).

The bound for the term \(T_8\) is obtained in a similar way as for the bound of \(T_4\).

\[
T_8 = -\frac{1}{C_5} b \left( \frac{Z_{w:n+1}^A - Z_{w:n}^A}{\Delta t}, Z_{p:n+1}^A \right)
\]

\[
\leq \frac{1}{12} \sum_{e \in \Gamma_h \cup \Gamma_{w_d}} \frac{\sigma_e}{|e|^\gamma} \left\| \frac{Z_{w:n+1}^A - Z_{w:n}^A}{\Delta t} \right\|_{L^2(e)}^2 + C_{22} h^{2 \min(k+1,s+1)} \|p_{n+1}\|_{H^{s+1}(\mathcal{E}_h)}^2
\]

(3.51)
The term $T_9$ is bounded by Cauchy-Schwarz’s inequality, Young’s inequality and the bound (3.16b).

$$T_9 = \frac{c \Delta t}{C_5} \sum_{E \in \mathcal{E}_h} (\rho_{p;n+1}, Z_{p;n+1}^A)_E$$

$$\leq \frac{c}{12} \| Z_{p;n+1}^A \|_{L^2(\mathcal{E}_h)}^2 + C_{23} c \Delta t^2 \| \rho_{p;n+1} \|_{L^2(\Omega)}^2$$

$$\leq \frac{c}{12} \| Z_{p;n+1}^A \|_{L^2(\mathcal{E}_h)}^2 + C_{23} c \Delta t^2 \| \frac{\partial^2 p}{\partial t^2} \|_{L^\infty(t_n,t_{n+1};L^2(\Omega))}^2 \quad (3.52)$$

Similarly, the term $T_{10}$ is bounded by Cauchy-Schwarz’s inequality, Young’s inequality, Poincare’s inequality and the bound (3.16a).

$$T_{10} = \frac{\Delta t}{C_5} \sum_{E \in \mathcal{E}_h} (\rho_{\varepsilon;n+1}, Z_{p;n+1}^A)_E$$

$$\leq \frac{1}{12} \| Z_{p;n+1}^A \|_{Q}^2 + C_{23} \Delta t^2 \| \rho_{\varepsilon;n+1} \|_{L^2(\Omega)}^2$$

$$\leq \frac{1}{12} \| Z_{p;n+1}^A \|_{Q}^2 + C_{24} \Delta t^2 \| \frac{\partial^2 \varepsilon}{\partial t^2} \|_{L^\infty(t_n,t_{n+1};L^2(\Omega))}^2 \quad (3.53)$$
For the term \( T_{11} \), we use Cauchy-Schwarz's inequality, Young's inequality, inverse inequality (3.10), the approximation result (3.12) and the fact that we choose \( \beta_1 = \beta \).

\[
T_{11} = \frac{1}{C_5} \sum_{e \in \Gamma_h} \frac{\sigma_e}{|e|^{\beta_1}} \left( \frac{Z_{w:n+1}^I - Z_{w:n}^I}{\Delta t}, \frac{Z_{w:n+1}^A - Z_{w:n}^A}{\Delta t} \right)_e
\]

\[
\leq \frac{1}{12} \sum_{e \in \Gamma_h} \frac{\sigma_e}{|e|^{\beta_1}} \left\| \frac{Z_{w:n+1}^A - Z_{w:n}^A}{\Delta t} \right\|_{L^2(e)}^2 + C_{25} \sum_{e \in \Gamma_h} h^{-1} \left\| \frac{Z_{w:n+1}^I - Z_{w:n}^I}{\Delta t} \right\|_{L^2(e)}^2
\]

\[
+ C_{25} \sum_{E \in \mathcal{E}_h} h^{-2} \left\| \frac{\partial W}{\partial t}_{|t=(n+\delta_7)\Delta t} \right\|_{L^2(E)}^2 + h^2 \left\| \nabla \frac{\partial W}{\partial t}_{|t=(n+\delta_7)\Delta t} \right\|_{L^2(E)}^2
\]

\[
\leq \frac{1}{12} \sum_{e \in \Gamma_h} \frac{\sigma_e}{|e|^{\beta_1}} \left\| \frac{Z_{w:n+1}^A - Z_{w:n}^A}{\Delta t} \right\|_{L^2(e)}^2 + C_{25} h^{2 \min(k,s+1)} \left\| \frac{\partial W}{\partial t}_{|t=(n+\delta_7)\Delta t} \right\|_{L^2(E)}^2
\]

(3.54)

where \( \delta_7 \) is a constant in \([0, 1]\).

We bound \( T_{12} \) by Cauchy-Schwarz's inequality, Young's inequality, Poincare's inequality and the approximation result (3.11a):

\[
T_{12} = \frac{\alpha}{C_5} \sum_{E \in \mathcal{E}_h} (Z_{p:n}, Z_{p:n+1})_E \leq C_{26} \left\| Z_{p:n}^I \right\|_{L^2(E)}^2 + \frac{1}{12} \left\| Z_{p:n+1}^A \right\|_{Q}^2
\]

\[
\leq C_{26} h^{2 \min(k+1,s+1)} \left\| p^{n+1} \right\|_{H^{s+1}(E_h)}^2 + \frac{1}{12} \left\| Z_{p:n+1}^A \right\|_{Q}^2
\]

(3.55)

Case 1: \( \alpha > 0 \) and \( c > 0 \)

We bound \( T_{13} \) by Cauchy-Schwarz's inequality, Young's inequality and Poincare's inequality:

\[
T_{13} = \frac{\alpha}{C_5} \sum_{E \in \mathcal{E}_h} (Z_{p:n}, Z_{p:n+1})_E \leq C_{27} \left\| Z_{p:n}^A \right\|_{L^2(E_h)}^2 + \frac{1}{12} \left\| Z_{p:n+1}^A \right\|_{Q}^2
\]

(3.56)
Therefore, combining (3.41), (3.42), (3.43), (3.44), (3.45), (3.46), (3.49), (3.50), (3.51), (3.52), (3.53), (3.54), (3.55) and (3.56), we obtain:

\[
T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 + T_9 + T_{10} + T_{11} + T_{12} + T_{13} \\
\leq \frac{5}{12} \| Z_{p,n+1}^A \|^2_Q + \frac{2c}{12} \| Z_{p,n+1}^p \|_{L^2(\mathcal{E}_h)}^2 + \frac{2}{12} \| Z_{\varepsilon,n+1}^A \|^2_{L^2(\mathcal{E}_h)} + C_{27} \| Z_{p,n}^A \|^2_{L^2(\mathcal{E}_h)} \\
+ \frac{3}{12} \sum_{e \in \Gamma_h} \frac{\sigma_e}{|e|^2} \| \left( \frac{Z_{u,n+1}^A}{\Delta t} - \frac{Z_{u,n}^A}{\Delta t} \right) \|^2_{L^2(\mathcal{e})} \\
+ \varpi_n \frac{C_0}{\Delta t} (h^{2\min(l,s)} \| w(0) \|_{H^{s+1}(\mathcal{E}_h)} + h^{2\min(r+1,s)} \| \varepsilon(0) \|_{H^s(\mathcal{E}_h)}) \\
+ C_{11} h^{2\min(k,s)} \| p^{n+1} \|_{H^{s+1}(\mathcal{E}_h)} + C_{12} h^{2\min(k+1,s+1)} \| \frac{\partial p}{\partial t} \|_{t=(n+\delta_t)\Delta t} \|_{H^{s+1}(\mathcal{E}_h)}^2 \\
+ C_{15} h^{2\min(r+1,s)} \| \varepsilon^{n+1} \|_{H^s(\mathcal{E}_h)} + C_{19} h^{2\min(l,s)} \| \frac{\partial w}{\partial t} \|_{t=(n+\delta_t)\Delta t} \|_{H^{s+1}(\mathcal{E}_h)}^2 \\
+ C_{20} h^{2\min(r+1,s)} \| \frac{\partial \varepsilon}{\partial t} \|_{t=(n+\delta_t)\Delta t} \|_{H^s(\mathcal{E}_h)}^2 + C_{21} h^{2\min(l,s)} \| \frac{\partial w}{\partial t} \|_{t=(n+\delta_t)\Delta t} \|_{H^{s+1}(\mathcal{E}_h)}^2 \\
+ C_{23} \Delta t^2 \| \frac{\partial p}{\partial t} \|_{L^2(t_n,t_{n+1};L^2(\Omega))}^2 + C_{24} \Delta t^2 \| \frac{\partial \varepsilon}{\partial t} \|_{L^2(t_n,t_{n+1};L^2(\Omega))}^2 \\
+ C_{25} h^{2\min(l,s)} \| \frac{\partial w}{\partial t} \|_{t=(n+\delta_t)\Delta t} \|_{H^{s+1}(\mathcal{E}_h)}^2 + (C_{22} + C_{26}) h^{2\min(k+1,s+1)} \| p^{n+1} \|_{H^{s+1}(\mathcal{E}_h)}^2
\]

where \( \varpi_n \) is 1 when \( n = 0 \) and 0 when \( n \geq 1 \).

Since

\[
\| \frac{\partial w}{\partial t} \|_{L^\infty(t_n,t_{n+1};H^{s+1}(\mathcal{E}_h))} = \max_{t \in [t_n,t_{n+1}]} \| \frac{\partial w}{\partial t} \|_{H^{s+1}(\mathcal{E}_h)}(\mathcal{E}_h),
\]

\[
\| \frac{\partial p}{\partial t} \|_{L^\infty(t_n,t_{n+1};H^{s+1}(\mathcal{E}_h))} = \max_{t \in [t_n,t_{n+1}]} \| \frac{\partial p}{\partial t} \|_{H^{s+1}(\mathcal{E}_h)}(\mathcal{E}_h),
\]

\[
\| \frac{\partial \varepsilon}{\partial t} \|_{L^\infty(t_n,t_{n+1};H^s(\mathcal{E}_h))} = \max_{t \in [t_n,t_{n+1}]} \| \frac{\partial \varepsilon}{\partial t} \|_{H^s(\mathcal{E}_h)}(\mathcal{E}_h),
\]

we simplify the above inequality and obtain:
\[
\sum_{i=1}^{13} T_i \leq \frac{5}{12} \|Z^{A}_{p,n+1}\|_Q^2 + \frac{C}{6} \|Z^{A}_{p,n+1}\|_{L^2(\mathcal{E}_h)}^2 + \frac{1}{6} \|Z^{A}_{\varepsilon,n+1}\|_{L^2(\mathcal{E}_h)}^2 \\
+ C_{27} \|Z^{A}_{p,n}\|_{L^2(\mathcal{E}_h)}^2 + \frac{1}{4} \sum_{e \in \Gamma_h} \sigma_e \|\frac{Z^{A}_{w,n+1} - Z^{A}_{w,n}}{\Delta t}\|_{L^2(e)}^2 \\
+ C_{28} \left( h^2 \min(k,s) \|p^{n+1}\|_{H^{s+1}(\mathcal{E}_h)}^{2} + h^2 \min(k+1,s+1) \|\frac{\partial p}{\partial t}\|_{L^\infty(t_n,t_{n+1};H^{s+1}(\mathcal{E}_h))}^{2} \right) \\
+ h^2 \min(r+1,s) \|\varepsilon^{n+1}\|_{H^{s}(\mathcal{E}_h)}^{2} + h^2 \min(t,s) \|\frac{\partial w}{\partial t}\|_{L^\infty(t_n,t_{n+1};H^{s+1}(\mathcal{E}_h))}^{2} \\
+ h^2 \min(r+1,s) \|\frac{\partial \varepsilon}{\partial t}\|_{L^\infty(t_n,t_{n+1};H^{s}(\mathcal{E}_h))}^{2} \right) \\
+ \omega_n \frac{C_0}{\Delta t} \left( h^2 \min(t,s) \|w(0)\|_{H^{s+1}(\mathcal{E}_h)}^{2} + h^2 \min(t+1,s) \|\varepsilon(0)\|_{H^{s}(\mathcal{E}_h)}^{2} \right) \\
+ C_{29} \Delta t^2 \left( \|\frac{\partial^{2} p}{\partial t^{2}}\|_{L^\infty(t_n,t_{n+1};L^2(\Omega))}^{2} + \|\frac{\partial^{2} \varepsilon}{\partial t^{2}}\|_{L^\infty(t_n,t_{n+1};L^2(\Omega))}^{2} \right) (3.57)
\]

Combining (3.40) and (3.57), we obtain:

\[
\|Z^{A}_{p,n+1}\|_Q^2 + \frac{1}{\Delta t} a_2 \left( Z^{A}_{w,n+1}, Z^{A}_{\varepsilon,n+1} \right) - Z^{A}_{w,n} \\
+ \frac{C}{2\Delta t} \left( \|Z^{A}_{p,n+1}\|_{L^2(\mathcal{E}_h)}^{2} - \|Z^{A}_{p,n}\|_{L^2(\mathcal{E}_h)}^{2} \right) + \frac{1}{2\Delta t} \left( \|Z^{A}_{\varepsilon,n+1}\|_{L^2(\mathcal{E}_h)}^{2} - \|Z^{A}_{\varepsilon,n}\|_{L^2(\mathcal{E}_h)}^{2} \right) \\
+ \sum_{e \in \Gamma_h \cup \Gamma_{w,n}} \sigma_e \|\frac{Z^{A}_{w,n+1} - Z^{A}_{w,n}}{\Delta t}\|_{L^2(e)}^{2} \\
\leq C_{30} \|Z^{A}_{p,n+1}\|_{L^2(\mathcal{E}_h)}^{2} + C_{31} \|Z^{A}_{\varepsilon,n+1}\|_{L^2(\mathcal{E}_h)}^{2} + C_{32} \|Z^{A}_{p,n}\|_{L^2(\mathcal{E}_h)}^{2} \\
+ C_{33} \left( h^2 \min(k,s) \|p^{n+1}\|_{H^{s+1}(\mathcal{E}_h)}^{2} + h^2 \min(k+1,s+1) \|\frac{\partial p}{\partial t}\|_{L^\infty(t_n,t_{n+1};H^{s+1}(\mathcal{E}_h))}^{2} \right) \\
+ h^2 \min(r+1,s) \|\varepsilon^{n+1}\|_{H^{s}(\mathcal{E}_h)}^{2} + h^2 \min(t,s) \|\frac{\partial w}{\partial t}\|_{L^\infty(t_n,t_{n+1};H^{s+1}(\mathcal{E}_h))}^{2} \\
+ h^2 \min(r+1,s) \|\frac{\partial \varepsilon}{\partial t}\|_{L^\infty(t_n,t_{n+1};H^{s}(\mathcal{E}_h))}^{2} \right) \\
+ \omega_n \frac{C_0}{\Delta t} \left( h^2 \min(t,s) \|w(0)\|_{H^{s+1}(\mathcal{E}_h)}^{2} + h^2 \min(t+1,s) \|\varepsilon(0)\|_{H^{s}(\mathcal{E}_h)}^{2} \right) \\
+ C_{34} \Delta t^2 \left( \|\frac{\partial^{2} p}{\partial t^{2}}\|_{L^\infty(t_n,t_{n+1};L^2(\Omega))}^{2} + \|\frac{\partial^{2} \varepsilon}{\partial t^{2}}\|_{L^\infty(t_n,t_{n+1};L^2(\Omega))}^{2} \right) (3.58)
\]
Since \( a_2(\cdot, \cdot) \) is symmetric, i.e. \( \theta_2 = -1 \) and \( \beta_1 = \beta \) and from the inequality (2.12), we obtain:

\[
a_2(Z_{w_{n+1}}^A, Z_{w_{n+1}}^A) - a_2(Z_{w_{n}}^A, Z_{w_{n}}^A) \geq \frac{1}{2\Delta t} (a_2(Z_{w_{n+1}}^A, Z_{w_{n}}^A) - a_2(Z_{w_{n}}^A, Z_{w_{n}}^A))
\]

Therefore (3.58) becomes:

\[
\frac{||Z_{p,n+1}^A||_{L^2(E_h)}^2}{Q} + \frac{1}{2\Delta t} (a_2(Z_{w_{n+1}}^A, Z_{w_{n+1}}^A) - a_2(Z_{w_{n}}^A, Z_{w_{n}}^A)) + \frac{c}{2\Delta t} (||Z_{p,n+1}^A||_{L^2(E_h)}^2 - ||Z_{p,n}^A||_{L^2(E_h)}^2) + \frac{1}{2\Delta t} (||Z_{\varepsilon,n+1}^A||_{L^2(E_h)}^2 - ||Z_{\varepsilon,n}^A||_{L^2(E_h)}^2) + \sum_{e \in \Gamma_h \cup \Gamma_w} \frac{\sigma_e}{|e|\beta_1} \frac{||Z_{w_{n+1}}^A - Z_{w_{n}}^A||_{L^2(e)}}{\Delta t} \leq C_{30} ||Z_{p,n+1}^A||_{L^2(E_h)}^2 + C_{31} ||Z_{\varepsilon,n+1}^A||_{L^2(E_h)}^2 + C_{32} ||Z_{p,n}^A||_{L^2(E_h)}^2
\]

\[
+ C_{33} \left( h^{2 \min(k,s)} \right) \left( \varepsilon^{n+1} \right)^{2 \min(k+1,s+1)} \left( \frac{\partial p}{\partial t} \right)^2 \left( t_n, t_{n+1}; H^s(E_h) \right) + h^{2 \min(k,s)} \left( \varepsilon^{n+1} \right)^{2 \min(k+1,s+1)} \left( \frac{\partial w}{\partial t} \right)^2 \left( t_n, t_{n+1}; H^s(E_h) \right) + h^{2 \min(r+1,s)} \left( \varepsilon^{n+1} \right)^{2 \min(r+1,s)} \left( \frac{\partial \varepsilon}{\partial t} \right)^2 \left( t_n, t_{n+1}; H^s(E_h) \right) + \frac{C_0}{\Delta t} \left( h^{2 \min(l,s)} \right) \left( \varepsilon(0) \right)^2 \left( \frac{\partial w}{\partial t} \right)^2 \left( t_n, t_{n+1}; H^s(E_h) \right) + C_{34} \Delta t^2 \left( \frac{\partial^2 p}{\partial t^2} \right)^2 \left( t_n, t_{n+1}; L^2(\Omega) \right) + \left( \frac{\partial^2 \varepsilon}{\partial t^2} \right)^2 \left( t_n, t_{n+1}; L^2(\Omega) \right)
\] (3.59)
We multiply the inequality (3.59) by $2\Delta t$ and sum over $n$ from 0 to $m - 1$. Then we obtain:

$$
\Delta t \sum_{n=1}^{m} \|Z_{p; n}^A\|_Q^2 + (a_2(Z_{w; n}^A, Z_{w; n}^A) - a_2(Z_{w; 0}^A, Z_{w; 0}^A))
$$

$$
+ c(\|Z_{p; n}^A\|_{L^2(\xi_h)}^2 - \|Z_{p; 0}\|_{L^2(\xi_h)}^2) + (\|Z_{\varepsilon; n}^A\|_{L^2(\xi_h)}^2 - \|Z_{\varepsilon; 0}\|_{L^2(\xi_h)}^2)
$$

$$
+ \sum_{n=1}^{m} \sum_{e \in \Gamma_h \cup \Gamma_w \cup \Gamma_D} \left[ e |\varepsilon | \right] \left[ \| \frac{Z_{w; n}^A - Z_{w; n-1}^A}{\Delta t} \|_{L^2(e)} \right]^2
$$

$$
\leq C_{35} \Delta t \sum_{n=0}^{m} \|Z_{p; n}^A\|_{L^2(\xi_h)}^2 + C_{36} \Delta t \sum_{n=0}^{m} \|Z_{\varepsilon; n}^A\|_{L^2(\xi_h)}^2
$$

$$
+ C_{37} \Delta t \sum_{n=0}^{m} \left( h^{2 \min(k, s)} \| p^{n+1} \|_{H^{s+1}(\xi_h)}^2 + h^{2 \min(r+1, s)} \| \varepsilon^{n+1} \|_{H^r(\xi_h)}^2 \right)
$$

$$
+ C_{37} T h^{2 \min(k, s+1)} \left\| \frac{\partial p}{\partial t} \right\|_{L^\infty(0, T; H^{s+1}(\xi_h))}^2 + C_{37} T h^{2 \min(l, s)} \left\| \frac{\partial w}{\partial t} \right\|_{L^\infty(0, T; H^{s+1}(\xi_h))}^2
$$

$$
+ C_{37} T h^{2 \min(r+1, s)} \left\| \frac{\partial \varepsilon}{\partial t} \right\|_{L^\infty(0, T; H^{r}(\xi_h))}^2
$$

$$
+ C_0 (h^{2 \min(l, s)} \| w(0) \|_{H^{s+1}(\xi_h)}^2 + h^{2 \min(r+1, s)} \| \varepsilon(0) \|_{H^r(\xi_h)}^2)
$$

$$
+ C_{38} \Delta t^2 \Delta t \sum_{n=0}^{m} \left( \| \frac{\partial^2 p}{\partial t^2} \|_{L^\infty(t_n, t_n+1; L^2(\Omega))}^2 + \| \frac{\partial^2 \varepsilon}{\partial t^2} \|_{L^\infty(t_n, t_n+1; L^2(\Omega))}^2 \right)
$$

$$
\text{(3.60)}
$$

From Lemma 2.1, we have:

$$
a_2(Z_{w; n}^A, Z_{w; n}^A) \geq C_1 \|Z_{w; n}^A\|_V^2
$$

$$
a_2(Z_{w; 0}^A, Z_{w; 0}^A) \leq C_{31} \|Z_{w; 0}^A\|_V^2
$$

By definition, we have: $\bar{p} w^0 = w_h^0$, $P_h p^0 = p_h^0$, $\Pi \varepsilon^0 = \varepsilon_h^0$. This implies: $Z_{w; 0}^A = 0$, $Z_{p; 0}^A = 0$, $Z_{\varepsilon; 0}^A = 0$.  

\[Z_{w; 0}^A = 0, \quad Z_{p; 0}^A = 0, \quad Z_{\varepsilon; 0}^A = 0.\]
Then divided by the smaller number between 1 and $C_1$, the above inequality (3.60) becomes:

$$
\| Z_{w;m}^A \|_V^2 + c \| Z_{p;n}^A \|_{L^2(\mathcal{E}_h)}^2 + \| Z_{\varepsilon;m}^A \|_{L^2(\mathcal{E}_h)}^2 + \Delta t \sum_{n=1}^m \| Z_{p;n}^A \|_Q^2 \\
+ \sum_{n=1}^m \sum_{e \in \Gamma_h \cup \Gamma_{w,D}} \sigma_e \left\| \frac{Z_{w;n}^A - Z_{w;n-1}^A}{\Delta t} \right\|_{L^2(e)}^2 \\
\leq C_{35} \Delta t \sum_{n=1}^m \| Z_{p;n}^A \|_{L^2(\mathcal{E}_h)}^2 + C_{36} \Delta t \sum_{n=1}^m \| Z_{\varepsilon;n}^A \|_{L^2(\mathcal{E}_h)}^2 \\
+ C_{37} \Delta t \sum_{n=0}^m \left( h^{2 \min(k,s)} \| P^{n+1} \|_{H^{s+1}(\mathcal{E}_h)}^2 + h^{2 \min(r+1,s)} \| \varepsilon^{n+1} \|_{H^{s}(\mathcal{E}_h)}^2 \right) \\
+ C_{37} T h^{2 \min(k+1,s+1)} \| \frac{\partial p}{\partial t} \|_{L^\infty(0,T;\mathcal{E}_h)}^2 + C_{37} T h^{2 \min(l,s)} \| \frac{\partial w}{\partial t} \|_{L^\infty(0,T;\mathcal{E}_h)}^2 \\
+ C_{37} T h^{2 \min(r+1,s)} \| \frac{\partial \varepsilon}{\partial t} \|_{L^\infty(0,T;\mathcal{E}_h)}^2 \\
+ C_{0} \left( h^{2 \min(l,s)} \| \varepsilon(0) \|_{H^{s+1}(\mathcal{E}_h)}^2 + h^{2 \min(r+1,s)} \| \varepsilon(0) \|_{H^{s}(\mathcal{E}_h)}^2 \right) \\
+ C_{38} \Delta t^2 \left( \| \frac{\partial^2 p}{\partial t^2} \|_{L^2(\mathcal{T})}^2 + \| \frac{\partial^2 \varepsilon}{\partial t^2} \|_{L^2(\mathcal{T})}^2 \right) \quad(3.61)
$$
By Gronwall’s inequality, we obtain:

\[
\begin{align*}
\|Z_{w;m}^A\|^2_V + c\|Z_{p;m}^A\|^2_{L^2(\varepsilon_h)} + \|Z_{\varepsilon;m}^A\|^2_{L^2(\varepsilon_h)} + \Delta t \sum_{n=1}^{m} \|Z_{p;n}^A\|^2_Q & \\
+ \Delta t \sum_{n=1}^{m} \sum_{\varepsilon \in \Gamma_h \cup \Gamma_{w,D}} \frac{\sigma_{\varepsilon}}{|\varepsilon|^{\beta t}} \left\| \frac{Z_{w;n}^A - Z_{w;n-1}^A}{\Delta t} \right\|^2_{L^2(\varepsilon)} & \\
\leq C^{(m+1)} \Delta t \left( \Delta t \sum_{n=0}^{m} \left( h^{2 \min(k,s)} \|p^{n+1}\|_{H^{s+1}(\varepsilon_h)}^2 + h^{2 \min(r+1,s)} \|\varepsilon^{n+1}\|_{H^s(\varepsilon_h)}^2 \right) \\
+ Th^{2 \min(k+1,s+1)} \|\partial_p \varepsilon\|_{L^2((0,T;H^{s+1}(\varepsilon_h))}^2 + Th^{2 \min(l,s)} \|\partial_w \varepsilon\|_{L^2((0,T;H^{s+1}(\varepsilon_h))}^2 \\
+ Th^{2 \min(r+1,s)} \|\partial_p \varepsilon\|_{L^2((0,T;H^{s}(\varepsilon_h))}^2 \right)
\end{align*}
\]

we obtain:

\[
\begin{align*}
\|Z_{w;m}^A\|^2_V + c\|Z_{p;m}^A\|^2_{L^2(\varepsilon_h)} + \|Z_{\varepsilon;m}^A\|^2_{L^2(\varepsilon_h)} + \Delta t \sum_{n=1}^{m} \|Z_{p;n}^A\|^2_Q & \\
+ \Delta t \sum_{n=1}^{m} \sum_{\varepsilon \in \Gamma_h \cup \Gamma_{w,D}} \frac{\sigma_{\varepsilon}}{|\varepsilon|^{\beta t}} \left\| \frac{Z_{w;n}^A - Z_{w;n-1}^A}{\Delta t} \right\|^2_{L^2(\varepsilon)} & \\
\leq C^{(m+1)} \Delta t \left( h^{2 \min(k,s)} \|p\|_{L^2(H^{s+1}(\varepsilon_h))}^2 + h^{2 \min(r+1,s)} \|\varepsilon\|_{L^2(H^s(\varepsilon_h))}^2 \right) \\
+ Th^{2 \min(k+1,s+1)} \|\partial_p \varepsilon\|_{L^2((0,T;H^{s+1}(\varepsilon_h))}^2 + Th^{2 \min(l,s)} \|\partial_w \varepsilon\|_{L^2((0,T;H^{s+1}(\varepsilon_h))}^2 \\
+ Th^{2 \min(r+1,s)} \|\partial_p \varepsilon\|_{L^2((0,T;H^{s}(\varepsilon_h))}^2 \right)
\end{align*}
\]

Case 2: \(\alpha = 0\)
If \( \alpha = 0 \), then

\[
T_{13} = 0 \quad (3.64)
\]

If \( c = 0 \), combining (3.40), (3.41), (3.42), (3.43), (3.44), (3.45), (3.46), (3.49), (3.50), (3.51), (3.52), (3.53), (3.54), (3.55) and (3.64), we obtain:

\[
\begin{align*}
&\|Z_{p;n+1}\|_Q^2 + \frac{1}{\Delta t} a_2(Z_{w;n+1}, Z_{w;n+1} - Z_{w;n}) \\
&+ \frac{1}{2\Delta t} (\|Z_{p;n+1}\|_{L^2(\mathcal{E}_h)}^2 - \|Z_{c;n}\|_{L^2(\mathcal{E}_h)}^2) + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \sigma_e \|\frac{Z_{w;n+1} - Z_{w;n}}{\Delta t}\|_{L^2(e)}^2 \\
\leq & \frac{1}{3} \|Z_{p;n+1}\|_Q^2 + \frac{1}{6} \|Z_{c;n+1}\|_{L^2(\mathcal{E}_h)}^2 + \frac{1}{4} \sum_{e \in \Gamma_h} \sigma_e \|\frac{Z_{w;n+1} - Z_{w;n}}{\Delta t}\|_{L^2(e)}^2 \\
&+ C_{28} \left( h^{2 \min(k,s)} \|p_{n+1}\|_{H^{s+1}(\mathcal{E}_h)}^2 + h^{2 \min(k+1,s+1)} \|\frac{\partial p}{\partial t}\|_{L^\infty(t_n,t_{n+1};H^{s+1}(\mathcal{E}_h))}^2 \right) \\
&+ h^{2 \min(r+1,s)} \|\varepsilon_{n+1}\|_{H^s(\mathcal{E}_h)}^2 + h^{2 \min(l,s)} \|\frac{\partial w}{\partial t}\|_{L^\infty(t_n,t_{n+1};H^{s+1}(\mathcal{E}_h))}^2 \right) \\
&+ h^{2 \min(r+1,s)} \|\frac{\partial \varepsilon}{\partial t}\|_{L^\infty(t_n,t_{n+1};H^{s}(\mathcal{E}_h))}^2 \\
&+ \varpi_n \frac{C_0}{\Delta t} \left( h^{2 \min(l,s)} \|w(0)\|_{H^{s+1}(\mathcal{E}_h)}^2 + h^{2 \min(r+1,s)} \|\varepsilon(0)\|_{H^s(\mathcal{E}_h)}^2 \right) \\
&+ C_{29} \Delta t^2 \left( \|\frac{\partial^2 p}{\partial t^2}\|_{L^\infty(t_n,t_{n+1};L^2(\Omega))}^2 + \|\frac{\partial^2 \varepsilon}{\partial t^2}\|_{L^\infty(t_n,t_{n+1};L^2(\Omega))}^2 \right) \quad (3.65)
\end{align*}
\]
We apply the similar steps of case 1 to the inequality (3.65) and obtain:

$$
\begin{align*}
\|Z^A_{w;m}\|^2_V + \|Z^A_{e;m}\|^2_{L^2(\mathcal{E}_h)} + \Delta t \sum_{n=1}^{m} \|Z^A_{p;n}\|^2_Q \\
+ \Delta t \sum_{n=1}^{m} \sum_{e\in \Gamma_h \cup \Gamma_w} \frac{\sigma_e}{c^{\beta_1}} \left\| \frac{Z^A_{w;n} - Z^A_{w;n-1}}{\Delta t} \right\|^2_{L^2(e)} \\
\leq e^{C(m+1)} \Delta t \left( h^{2\min(k,s)} \|p\|^2_{H^{s+1}(\mathcal{E}_h)} + h^{2\min(r+1,s)} \|\varepsilon\|^2_{H^s(\mathcal{E}_h)} \\
+ Th^{2\min(k+1,s+1)} \left\| \frac{\partial p}{\partial t} \right\|^2_{L^2(0,T;H^{s+1}(\mathcal{E}_h))} + Th^{2\min(l,s)} \left\| \frac{\partial w}{\partial t} \right\|^2_{L^2(0,T;H^s(\mathcal{E}_h))} \\
+ C_0 (h^{2\min(l,s)} \|w(0)\|^2_{H^s(\mathcal{E}_h)} + h^{2\min(r+1,s)} \|\varepsilon(0)\|^2_{H^s(\mathcal{E}_h)}) \\
+ \Delta t^2 \left( \left\| \frac{\partial^2 p}{\partial t^2} \right\|^2_{L^2(0,T;H^s(\mathcal{E}_h))} + \left\| \frac{\partial^2 \varepsilon}{\partial t^2} \right\|^2_{L^2(0,T;H^s(\mathcal{E}_h))} \right) \right) \\
\end{align*}
$$

(3.66)

If \( c \neq 0 \), combining (3.40), (3.41), (3.42), (3.43), (3.44), (3.45), (3.46), (3.49), (3.50), (3.51), (3.52), (3.53), (3.54), (3.55) and (3.64), similarly, we obtain:

$$
\begin{align*}
\|Z^A_{w;m}\|^2_V + c \|Z^A_{e;m}\|^2_{L^2(\mathcal{E}_h)} + \|Z^A_{e;m}\|^2_{L^2(\mathcal{E}_h)} + \Delta t \sum_{n=1}^{m} \|Z^A_{p;n}\|^2_Q \\
+ \Delta t \sum_{n=1}^{m} \sum_{e\in \Gamma_h \cup \Gamma_w} \frac{\sigma_e}{c^{\beta_1}} \left\| \frac{Z^A_{w;n} - Z^A_{w;n-1}}{\Delta t} \right\|^2_{L^2(e)} \\
\leq e^{C(m+1)} \Delta t \left( h^{2\min(k,s)} \|p\|^2_{H^{s+1}(\mathcal{E}_h)} + h^{2\min(r+1,s)} \|\varepsilon\|^2_{H^s(\mathcal{E}_h)} \\
+ Th^{2\min(k+1,s+1)} \left\| \frac{\partial p}{\partial t} \right\|^2_{L^2(0,T;H^{s+1}(\mathcal{E}_h))} + Th^{2\min(l,s)} \left\| \frac{\partial w}{\partial t} \right\|^2_{L^2(0,T;H^s(\mathcal{E}_h))} \\
+ T h^{2\min(r+1,s)} \left\| \frac{\partial \varepsilon}{\partial t} \right\|^2_{L^2(0,T;H^s(\mathcal{E}_h))} \\
+ C_0 (h^{2\min(l,s)} \|w(0)\|^2_{H^s(\mathcal{E}_h)} + h^{2\min(r+1,s)} \|\varepsilon(0)\|^2_{H^s(\mathcal{E}_h)}) \\
+ \Delta t^2 \left( \left\| \frac{\partial^2 p}{\partial t^2} \right\|^2_{L^2(0,T;H^s(\mathcal{E}_h))} + \left\| \frac{\partial^2 \varepsilon}{\partial t^2} \right\|^2_{L^2(0,T;H^s(\mathcal{E}_h))} \right) \right) \\
\end{align*}
$$

(3.67)

From the inequalities (3.63), (3.66) and (3.67), we prove the theorem is true. \( \blacksquare \)
3.2.2 Nonsymmetric form

In this section we discuss the case: $a_2(\cdot, \cdot)$ is nonsymmetric and there is an error estimate for $a_2(\cdot, \cdot)$ has nonsymmetric form.

**Theorem 3.5 (Error estimate for nonsymmetric form)**: Assume the exact solution $w$ belongs to $H^1(0,T; (H^{s+1}(\mathcal{E}_h))^d)$, $p$ belongs to $H^2(0,T; H^{s+1}(\mathcal{E}_h))$ and $\varepsilon$ belongs to $H^2(0,T; H^{s}(\mathcal{E}_h))$. If $\theta_2 = 1$, i.e. $a_2(\cdot, \cdot)$ is nonsymmetric, and sequences $(p^n_h)_{n \geq 0}$ of functions in $Q_k(E_h)$, $(w^n_h)_{n \geq 0}$ of functions in $V_l(E_h)$, $(\varepsilon^n_h)_{n \geq 0}$ of functions in $U_r(E_h)$ satisfy the model (2.18). Then

$$\|Z^A_{w; m}\|_V^2 + c\|Z^A_{p; m}\|_{L^2(\mathcal{E}_h)}^2 + \|Z^A_{w; m}\|_{L^2(\mathcal{E}_h)}^2 + \Delta t \sum_{n=1}^m \|Z^A_{p; n}\|_Q^2$$

$$+ \Delta t \sum_{n=1}^m \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \frac{\sigma_e}{|e|^{\beta_1}} \left\| \frac{Z^A_{w; n} - Z^A_{w; n-1}}{\Delta t} \right\|^2_{L^2(e)} \leq \tilde{C}_6 h^{2 \min(k,s)} \left( \|p\|_{L^2(0,T; H^{s+1}(\mathcal{E}_h))}^2 + Th^2 \left\| \frac{\partial p}{\partial t} \right\|^2_{L^\infty(0,T; H^{s+1}(\mathcal{E}_h))} \right)$$

$$+ \tilde{C}_7 h^{2 \min(l,s)} \left( \|w(0)\|_{H^{s+1}(\mathcal{E}_h)}^2 + \|w^m\|_{H^{s+1}(\mathcal{E}_h)}^2 + Th^2 \left\| \frac{\partial w}{\partial t} \right\|^2_{L^\infty(0,T; H^{s+1}(\mathcal{E}_h))} \right)$$

$$+ \tilde{C}_8 h^{2 \min(r+1,s)} \left( \|\varepsilon(0)\|_{H^{s}(\mathcal{E}_h)}^2 + \|\varepsilon\|_{L^\infty(0,T; H^{s+1}(\mathcal{E}_h))}^2 + \|\varepsilon\|_{H^{s}(\mathcal{E}_h)}^2 \right)$$

$$+ \tilde{C}_9 \Delta t^2 \left( \left\| \frac{\partial^2 p}{\partial t^2} \right\|^2_{L^\infty(t_n,t_{n+1}; L^2(\Omega))} + \left\| \frac{\partial^2 \varepsilon}{\partial t^2} \right\|^2_{L^\infty(t_n,t_{n+1}; L^2(\Omega))} \right)$$

where $k, l$ are the degrees of the polynomial spaces; $s$ is the degree of the Sobolev space; $\tilde{C}_6, \tilde{C}_7, \tilde{C}_8$ and $\tilde{C}_9$ are constants independent of $h$.

**Proof.**

Note: in this section we suppose $\hat{P}w^{n+1}$ is not the elliptic projection of the true solution $w^{n+1}$ for $a_2$. Then we know $T_3 \neq 0$. 


The idea is to decompose $a_2$ into a symmetric form and a remainder. We define:

$$\tilde{a}_2(u, v) = \sum_{E \in \mathcal{E}_h} \int_E \nabla u : \nabla v + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \frac{\sigma_e}{|e|^2} \int_e [u] \cdot [v]$$

and we write

$$a_2(u, v) = \tilde{a}_2(u, v) - \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \int_e \{\nabla u\} n_e \cdot [v] + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \int_e \{\nabla v\} n_e \cdot [u]$$

From (3.38) and (3.40), we have:

$$R_1 + R_2 + \mu \tilde{a}_2(Z_{w;n+1}^A, \frac{Z_{w;n+1}^A - Z_{w;n}^A}{\Delta t}) + R_4 + R_5$$

$$= \tilde{C}_5(T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 + T_9 + T_{10} + T_{11} + T_{12} + T_{13})$$

$$+ \mu \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \int_e \{\nabla Z_{w;n+1}^A\} n_e \cdot [\frac{Z_{w;n+1}^A - Z_{w;n}^A}{\Delta t}]$$

$$- \mu \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \int_e \{\nabla Z_{w;n+1}^A - Z_{w;n}^A\} n_e \cdot [Z_{w;n+1}^A]$$

(3.68)

where

$$\tilde{a}_2(Z_{w;n+1}^A, \frac{Z_{w;n+1}^A - Z_{w;n}^A}{\Delta t}) \geq \frac{1}{2\Delta t} (\|Z_{w;n+1}^A\|^2_V - \|Z_{w;n}^A\|^2_V)$$

(3.69)

Here, the bounds for the terms $R_i, i = 1, 2, 4, 5, T_j, j = 1, \cdots, 13$ are the same as before. From the previous steps, we only estimate $T_{11}, T_{14}$ and $T_{15}$, where

$$T_{14} = \mu \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \int_e \{\nabla Z_{w;n+1}^A\} n_e \cdot [\frac{Z_{w;n+1}^A - Z_{w;n}^A}{\Delta t}]$$

$$T_{15} = \mu \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \int_e \{\nabla Z_{w;n+1}^A - Z_{w;n}^A\} n_e \cdot [Z_{w;n+1}^A]$$
To bound $T_{11}$, the Cauchy-Schwarz, Young's inequalities, the trace inequality and
the approximation result (3.13) are used.

$$T_{11} = \sum_{e \in \Gamma_h} \sigma_e \left( \frac{Z_{w;n+1}^I - Z_{w;n}^I}{\Delta t}, \frac{Z_{w;n+1}^A - Z_{w;n}^A}{\Delta t} \right)_e$$

$$\leq C_{39} \sum_{e \in \Gamma_h \cup \Gamma^D} \sigma_e \| \frac{Z_{w;n+1}^A - Z_{w;n}^A}{\Delta t} \|_{L^2(e)}^2$$

$$+ C_{40} |e|^{-\beta_1} h^{2 \min(l+1/2,s+1/2)} \| \frac{\partial w}{\partial t} \|_{L^2 \{ t=(n+\delta_9)\Delta t \}^2}$$

where $\delta_9$ is a constant in $[0, 1]$.

In fact, we can set $\beta_1 = \beta$, then $\frac{|e|^{\beta_1}}{h} = 1$. The above inequality becomes:

$$T_{11} \leq C_{39} \sum_{e \in \Gamma_h \cup \Gamma^D} \sigma_e \left( \frac{Z_{w;n+1}^A - Z_{w;n}^A}{\Delta t} \right)_e$$

$$+ C_{40} \| \frac{\partial w}{\partial t} \|_{L^2 \{ t=(n+\delta_9)\Delta t \}^2}$$

(3.70)

The bound for the term $T_{14}$ is found by applying the Cauchy-Schwarz, Young's
inequalities, and the trace inequality.

$$T_{14} = \mu \sum_{e \in \Gamma_h \cup \Gamma^D} \int e \left\{ \nabla Z_{w;n+1}^A \right\} n_e \cdot \left( \frac{Z_{w;n+1}^A - Z_{w;n}^A}{\Delta t} \right)$$

$$\leq C_{41} \sum_{e \in \Gamma_h \cup \Gamma^D} \sigma_e \left( \frac{Z_{w;n+1}^A - Z_{w;n}^A}{\Delta t} \right)_e$$

$$+ C_{42} \sum_{e \in \Gamma_h \cup \Gamma^D} \| \nabla \frac{Z_{w;n+1}^A}{h} \|_{L^2(e)}^2$$

By the fact $\frac{|e|^{\beta_1}}{h} = 1$, the above inequality becomes:

$$T_{14} \leq C_{41} \sum_{e \in \Gamma_h \cup \Gamma^D} \sigma_e \left( \frac{Z_{w;n+1}^A - Z_{w;n}^A}{\Delta t} \right)_e$$

$$+ C_{42} \| \nabla Z_{w;n+1}^A \|_{L^2(e)}^2$$

(3.71)
From the previous steps, we know:

\[ R_1 + R_2 + R_3 + R_4 + R_5 \]
\[ \geq \|Z_{p,n+1}^A\|_Q^2 + \frac{1}{2\Delta t}(\|Z_{w,n+1}^A\|_V^2 - \|Z_{w,n}^A\|_V^2) \]
\[ + \frac{c}{2\Delta t}(\|Z_{p,n+1}^A\|_{L^2(\mathcal{E}_h)}^2 - \|Z_{p,n}^A\|_{L^2(\mathcal{E}_h)}^2) + \frac{1}{2\Delta t}(\|Z_{\varepsilon,n+1}^A\|_{L^2(\mathcal{E}_h)}^2 - \|Z_{\varepsilon,n}^A\|_{L^2(\mathcal{E}_h)}^2) \]
\[ + \sum_{e \in \Gamma_h \cup \Gamma_w} \frac{\sigma_e}{|e|} \|\frac{Z_{w,n+1}^A - Z_{w,n}^A}{\Delta t}\|_\epsilon^2 \]

(3.72)

Similar to symmetric form, we discuss two cases: \( \alpha > 0 \) and \( \alpha = 0 \).

Case 1: \([\alpha > 0 \text{ and } c > 0]\)

Let \( C_{39} = C_{41} = \frac{1}{12} \), then:

\[ \tilde{C}_5(T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + T_7 + T_8 + T_9 + T_{10} + T_{11} + T_{12} + T_{13}) \]
\[ + T_{14} + T_{15} \]
\[ \leq \frac{5}{12} \|Z_{p,n+1}^A\|_Q^2 + \frac{c}{6} \|Z_{p,n+1}^A\|_{L^2(\mathcal{E}_h)}^2 + \frac{1}{6} \|Z_{\varepsilon,n+1}^A\|_{L^2(\mathcal{E}_h)}^2 + C_{27} \|Z_{p,n}^A\|_{L^2(\mathcal{E}_h)}^2 \]
\[ + C_{28} (2^{\min(k,s)} \|p^{n+1}\|_{H^{k+1}(\mathcal{E}_h)}^2 + 2^{\min(k+1,s+1)} \|\frac{\partial p}{\partial t}\|_{L^\infty(t_n,t_{n+1};H^{k+1}(\mathcal{E}_h))}^2) \]
\[ + h^{2\min(r+1,s)} \|e^{n+1}\|_{H^r(\mathcal{E}_h)}^2 + h^{2\min(l,s)} \|\frac{\partial w}{\partial t}\|_{L^\infty(t_n,t_{n+1};H^{l+1}(\mathcal{E}_h))}^2 \]
\[ + h^{2\min(r+1,s)} \|\frac{\partial Z_{w,n}^A}{\partial t}\|_{L^\infty(t_n,t_{n+1};H^{s+1}(\mathcal{E}_h))}^2 \]
\[ + \tilde{\varepsilon}_n \frac{C_0}{\Delta t} (h^{2\min(l,s)} \|w(0)\|_{H^{l+1}(\mathcal{E}_h)}^2 + h^{2\min(r+1,s)} \|\varepsilon(0)\|_{H^r(\mathcal{E}_h)}^2) \]
\[ + C_{29} \Delta t^2 (\|\frac{\partial^2 p}{\partial t^2}\|_{L^\infty(t_n,t_{n+1};L^2(\Omega))}^2 + \|\frac{\partial^2 \varepsilon}{\partial t^2}\|_{L^\infty(t_n,t_{n+1};L^2(\Omega))}^2) \]
\[ + \frac{5}{12} \sum_{e \in \Gamma_h} \frac{\sigma_e}{|e|} \|\frac{Z_{w,n+1}^A - Z_{w,n}^A}{\Delta t}\|_{L^2(e)}^2 + C_{40} h^{2\min(l,s)-1} \|\frac{\partial w}{\partial t}\|_{t=(n+\delta_0)\Delta t}^2 H^{l+1}(\Omega) \]
\[ + C_{42} \|Z_{w,n+1}^A\|_V^2 + \tilde{C}_5 T_3 + T_{15} \]

(3.73)

where \( \tilde{\varepsilon}_n \) is 1 if \( n = 0 \) and 0 if \( n \geq 1 \).
Then from (3.68), (3.72) and (3.73), we obtain:

\[
\begin{align*}
& \| Z_{A,p}^{n+1} \|_Q^2 + \frac{1}{\Delta t} (\| Z_{A,w}^{n+1} \|_V^2 - \| Z_{A,w}^n \|_V^2) \\
& + \frac{c}{\Delta t} (\| Z_{A,p}^{n+1} \|_{L^2(\mathcal{E}_h)}^2 - \| Z_{A,p}^n \|_{L^2(\mathcal{E}_h)}^2) + \frac{1}{\Delta t} (\| Z_{A,\varepsilon}^{n+1} \|_{L^2(\mathcal{E}_h)}^2 - \| Z_{A,\varepsilon}^n \|_{L^2(\mathcal{E}_h)}^2) \\
& + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \sigma_e \| [Z_{A,w}^{n+1} - Z_{A,w}^n] \|_{L^2(e)}^2 \\
\leq & \ C_{46} c \| Z_{A,p}^{n+1} \|_{L^2(\mathcal{E}_h)}^2 + C_{47} \| Z_{A,\varepsilon}^{n+1} \|_{L^2(\mathcal{E}_h)}^2 + C_{48} \| Z_{A,p}^n \|_{L^2(\mathcal{E}_h)}^2 + C_{49} \| Z_{A,w}^{n+1} \|_V^2 \\
& + C_{50} \left( h^2 \min(k,s) \| p^{n+1} \|_{H^{s+1}(\mathcal{E}_h)} + h^2 \min(k+1,s+1) \| \frac{\partial p}{\partial t} \|_{L^\infty(t_n,t_{n+1};H^{s+1}(\mathcal{E}_h))} \\
& + h^2 \min(r+1,s) \| \varepsilon^{n+1} \|_{H^{s}(\mathcal{E}_h)} + h^2 \min(l,s) \| \frac{\partial w}{\partial t} \|_{L^\infty(t_n,t_{n+1};H^{s+1}(\mathcal{E}_h))} \\
& + h^2 \min(r+1,s) \| \frac{\partial \varepsilon}{\partial t} \|_{L^\infty(t_n,t_{n+1};H^{s+1}(\mathcal{E}_h))} \right) + 2 \tilde{C}_5 T_3 + 2 T_{15} \\
& + \omega_n \frac{C_0}{\Delta t} \left( h^2 \min(l,s) \| w(0) \|_{H^{s+1}(\mathcal{E}_h)} + h^2 \min(r+1,s) \| \varepsilon(0) \|_{H^{s}(\mathcal{E}_h)} \right) \\
& + C_{52} \Delta t^2 \left( \| \frac{\partial^2 p}{\partial t^2} \|_{L^\infty(t_n,t_{n+1};L^2(\Omega))} + \| \frac{\partial^2 \varepsilon}{\partial t^2} \|_{L^\infty(t_n,t_{n+1};L^2(\Omega))} \right) \quad (3.74)
\end{align*}
\]

The above inequality (3.74) times $\Delta t$ and sums $n$ from 0 to $m - 1$ and we have:

\[
\tilde{P}w^0 = w_h^0, \quad P_h p^0 = p_h^0, \quad \Pi \varepsilon^0 = \varepsilon_h^0,
\]

which implies: $Z_{A,w}^0 = 0$, $Z_{A,p}^0 = 0$, $Z_{A,\varepsilon}^0 = 0$. Then
we obtain:

\[ \| Z_{w;0}^A \|_V^2 + c \| Z_{p;0}^A \|_{L^2(\mathcal{E}_h)}^2 + \| Z_{\varepsilon;0}^A \|_{L^2(\mathcal{E}_h)}^2 + \Delta t \sum_{n=1}^m \| Z_{p;n}^A \|_Q^2 \]

\[ + \Delta t \sum_{n=1}^m \sum_{e \in \Gamma_k \cup \Gamma_{\omega D}} \frac{\sigma_e}{\epsilon|e|^{p_1}} \| \frac{Z_{w;n}^A - Z_{w;n-1}^A}{\Delta t} \|_{L^2(e)}^2 \]

\[ \leq C_{46} \Delta t \sum_{n=1}^m \| Z_{p;n}^A \|_{L^2(\mathcal{E}_h)}^2 + C_{47} \Delta t \sum_{n=1}^m \| Z_{\varepsilon;n}^A \|_{L^2(\mathcal{E}_h)}^2 + C_{49} \Delta t \sum_{n=1}^m \| Z_{w;n}^A \|_V^2 \]

\[ + C_{53} \Delta t \sum_{n=0}^m \left( h^{2 \min(k,s)} \| P_{n+1} \|_{H^{s+1}(\mathcal{E}_h)}^2 + h^{2 \min(r+1,s)} \| \varepsilon_{n+1} \|_{H^s(\mathcal{E}_h)}^2 \right) \]

\[ + C_{53} \Delta t \sum_{n=0}^m \left( h^{2 \min(k,s+1)} \| \frac{\partial p}{\partial t} \|_{L^\infty(0,T;H^{s+1}(\mathcal{E}_h))}^2 + C_{53} \Delta t \sum_{n=0}^m \left( h^{2 \min(l,s)} \| \frac{\partial w}{\partial t} \|_{L^\infty(0,T;H^s(\mathcal{E}_h))}^2 \right) \]

\[ + C_{54} \Delta t \sum_{n=0}^m \left( \| \frac{\partial^2 p}{\partial t^2} \|_{L^\infty(t_n,t_{n+1};L^2(\Omega))}^2 + \| \frac{\partial^2 \varepsilon}{\partial t^2} \|_{L^\infty(t_n,t_{n+1};L^2(\Omega))} \right) \]

\[ + 2 C_5 \Delta t \sum_{n=0}^{m-1} T_3 + 2 \Delta t \sum_{n=0}^{m-1} T_{15} \quad (3.75) \]

From the definition of \( T_3 \), we have:

\[ 2 C_5 \Delta t \sum_{n=0}^{m-1} T_3 = -2 \mu \Delta t \sum_{n=0}^{m-1} a_2(Z_{w;n+1}^I, \frac{Z_{w;n+1}^A - Z_{w;n}^A}{\Delta t}) \]

\[ = -2 \mu a_2(Z_{w;m}^I, Z_{w;m}^A) + 2 \mu \Delta t \sum_{n=0}^{m-1} a_2(\frac{Z_{w;n+1}^I - Z_{w;n}^I}{\Delta t}, Z_{w;n}^A) \]

\[ + 2 \mu a_2(Z_{w;1}^I - Z_{w;0}^I, Z_{w;0}^A) \quad (3.76) \]

Since \( Z_{w;0}^A = 0 \), then we know:

\[ 2 \mu a_2(Z_{w;1}^I - Z_{w;0}^I, Z_{w;0}) = 0 \quad (3.77) \]
By Lemma 2.1 and approximation results, we obtain:

\[-2\mu a_2(Z_{w;m}^I, Z_{w;m}^A) \leq \frac{1}{4}\|Z_{w;m}^A\|^2_V + C_{51}h^{2\min(l,s)}\|w^m\|^2_{H^{s+1}(\Omega)}\]  

(3.78)

By Lemma 2.1, Cauchy-Schwarz, Young’s inequalities and approximation results, we have:

\[2\mu \Delta t \sum_{n=1}^{m-1} a_2\left(\frac{Z_{w:n+1}^I - Z_{w:n}^I}{\Delta t}, Z_{w:n}^A\right) \leq C_{52}\Delta t \sum_{n=1}^{m-1} \|Z_{w:n}^A\|^2_V + C_{55}\Delta t \sum_{n=1}^{m-1} h^{2\min(l,s)}\|\frac{\partial w}{\partial t}\|_{t=(n+\delta_{10})\Delta t}^2_{H^{s+1}(\Omega)}\]  

(3.79)

where \(\delta_{10}\) is a constant in \([0, 1]\).

Combining (3.77), (3.78) and (3.79), the equality (3.76) becomes:

\[2\tilde{C}_5 \Delta t \sum_{n=0}^{m-1} T_3 \leq \frac{1}{4}\|Z_{w;m}^A\|^2_V + C_{51}h^{2\min(l,s)}\|w^m\|^2_{H^{s+1}(\Omega)} + C_{52}\Delta t \sum_{n=1}^{m-1} \|Z_{w:n}^A\|^2_V + C_{55}\Delta t h^{2\min(l,s)}\|\frac{\partial w}{\partial t}\|_{L^\infty(0,T;H^{s+1}(\mathcal{E}_h))}^2\]  

(3.80)

From the definition of \(T_{15}\), we have:

\[2\tilde{C}_5 \Delta t \sum_{n=0}^{m-1} T_{15} = 2\mu \Delta t \sum_{n=0}^{m-1} \sum_{e \in \Gamma_h \cup \Gamma_{wD}} (\{\nabla Z_{w:n+1}^A - Z_{w:n}^A\}_e, [Z_{w:n+1}^A])_e + 2\mu \sum_{e \in \Gamma_h \cup \Gamma_{wD}} (\{\nabla Z_{w:0}^A\}_e, [Z_{w:1}^A])_e + 2\mu \sum_{e \in \Gamma_h \cup \Gamma_{wD}} (\{\nabla Z_{w:0}^A\}_e, [Z_{w:1}^A])_e\]  

(3.81)

Since \(Z_{w:0}^A = 0\), then we know:

\[2\mu \sum_{e \in \Gamma_h \cup \Gamma_{wD}} (\{\nabla Z_{w:0}^A\}_e, [Z_{w:1}^A])_e = 0\]  

(3.82)
By Cauchy-Schwarz, Young’s inequalities and the trace inequality, we obtain:

$$2\mu \sum_{e \in \Gamma_h \cup \Gamma_{wD}} (\{\nabla Z_{w;m}^A\}n_e, [Z_{w;m}^A])_e$$

$$\leq \frac{1}{4} \sum_{E \in \mathcal{E}_h} \|\nabla Z_{w;m}^A\|^2_{L^2(E)} + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \frac{4\mu^2 C_t^2}{|e|^\beta} \|Z_{w;m}^A\|^2_{L^2(e)} \tag{3.83}$$

where $C_t$ is the constant from Trace inequality.

By Cauchy-Schwarz, Young’s inequalities and the trace inequality, we have:

$$2\mu \Delta t \sum_{n=1}^{m-1} \sum_{e \in \Gamma_h \cup \Gamma_{wD}} (\{\nabla Z_{w;n+1}^A\}n_e, [\frac{Z_{w;n+1}^A - Z_{w;n-1}^A}{\Delta t}])_e$$

$$\leq \frac{1}{4} \Delta t \sum_{n=1}^{m} \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \frac{\sigma_e}{|e|^{\beta_1}} \|\frac{Z_{w;n}^A - Z_{w;n-1}^A}{\Delta t}\|^2_{L^2(e)} + C_{56} \Delta t \sum_{n=1}^{m} \|Z_{w;n}^A\|^2_{V} \tag{3.84}$$

Combining (3.82), (3.83) and (3.84), the equality (3.81) becomes:

$$2\Delta t \sum_{n=0}^{m-1} T_{15} \leq \frac{1}{4} \Delta t \sum_{n=1}^{m} \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \frac{\sigma_e}{|e|^{\beta_1}} \|\frac{Z_{w;n}^A - Z_{w;n-1}^A}{\Delta t}\|^2_{L^2(e)} + C_{56} \Delta t \sum_{n=1}^{m} \|Z_{w;n}^A\|^2_{V}$$

$$+ \frac{1}{4} \sum_{E \in \mathcal{E}_h} \|\nabla Z_{w;m}^A\|^2_{L^2(E)} + \sum_{e \in \Gamma_h \cup \Gamma_{wD}} \frac{4\mu^2 C_t^2}{|e|^\beta} \|Z_{w;m}^A\|^2_{L^2(e)} \tag{3.85}$$
From (3.80) and (3.85), the inequality (3.75) becomes:

\[
\frac{3}{4} \| Z^{A}_{w;m} \|_{V}^{2} + c \| Z^{A}_{p;n} \|_{L^{2}(\mathcal{E}_{h})}^{2} + \| Z^{A}_{e;m} \|_{L^{2}(\mathcal{E}_{h})}^{2} + \Delta t \sum_{n=1}^{m} \| Z^{A}_{p;n} \|_{Q}^{2} \\
+ \frac{3}{4} \Delta t \sum_{n=1}^{m} \sum_{e \in \Gamma_{h} \cup \Gamma_{wD}} \frac{\sigma_{e}}{|e|^{\beta_{1}}} \| \frac{Z^{A}_{w;n} - Z^{A}_{w;n-1}}{\Delta t} \|^2_{L^{2}(e)} \\
\leq C_{46} \Delta t \sum_{n=1}^{m} \| Z^{A}_{p;n} \|_{L^{2}(\mathcal{E}_{h})}^{2} + C_{47} \Delta t \sum_{n=1}^{m} \| Z^{A}_{e;n} \|_{L^{2}(\mathcal{E}_{h})}^{2} + C_{49} \Delta t \sum_{n=1}^{m} \| Z^{A}_{w;n} \|_{V}^{2} \\
+ C_{53} \Delta t \sum_{n=0}^{m} \left( h^{2 \min(k,s)} \| p^{n+1} \|_{H^{s+1}(\mathcal{E}_{h})} + h^{2 \min(r+1,s)} \| \varepsilon^{n+1} \|_{H^{s}(\mathcal{E}_{h})} \right) \\
+ C_{53} Th^{2 \min(k+1,s+1)} \left( \| \partial p \|_{L^{\infty}(0,T;H^{s+1}(\mathcal{E}_{h}))} + C_{53} Th^{2 \min(l,s)-1} \| \partial \varepsilon \|_{L^{\infty}(0,T;H^{s}(\mathcal{E}_{h}))} \right) \\
+ C_{53} Th^{2 \min(r+1,s)} \left( \| \partial \varepsilon \|_{L^{\infty}(0,T;H^{s}(\mathcal{E}_{h}))} \right) \\
+ C_{51} \left( h^{2 \min(l,s)} \| w(0) \|_{H^{s+1}(\mathcal{E}_{h})} + h^{2 \min(l,s)} \| \varepsilon^{n+1} \|_{H^{s}(\mathcal{E}_{h})} \right) \\
+ C_{54} \Delta t^{2} \Delta t \sum_{n=0}^{m} \left( \| \partial p \|_{L^{\infty}(t_{n},t_{n+1};L^{2}(\Omega))} + \| \partial \varepsilon \|_{L^{\infty}(t_{n},t_{n+1};L^{2}(\Omega))} \right) \\
+ \frac{1}{4} \sum_{E \in \mathcal{E}_{h}} \| \nabla Z^{A}_{w;m} \|_{L^{2}(E)}^{2} + \sum_{e \in \Gamma_{h} \cup \Gamma_{wD}} \frac{4 \mu^{2} C_{t}^{2}}{|e|^{\beta}} \| Z^{A}_{w;m} \|_{L^{2}(e)}^{2} \\
\tag{3.86}
\]

Since

\[
\frac{3}{4} \| Z^{A}_{w;m} \|_{V}^{2} - \frac{1}{4} \sum_{E \in \mathcal{E}_{h}} \| \nabla Z^{A}_{w;m} \|_{L^{2}(E)}^{2} - \sum_{e \in \Gamma_{h} \cup \Gamma_{wD}} \frac{4 \mu^{2} C_{t}^{2}}{|e|^{\beta}} \| Z^{A}_{w;m} \|_{L^{2}(e)}^{2} \\
= \frac{1}{2} \sum_{E \in \mathcal{E}_{h}} \| \nabla Z^{A}_{w;m} \|_{L^{2}(E)}^{2} + \sum_{e \in \Gamma_{h} \cup \Gamma_{wD}} \left( \frac{3}{4} \sigma_{e} - 4 \mu^{2} C_{t}^{2} \right) \frac{1}{|e|^{\beta}} \| Z^{A}_{w;m} \|_{L^{2}(e)}^{2} \\
\]

Suppose \( \sigma_{e} \) is big enough satisfying

\[
\frac{3}{4} \sigma_{e} - 4 \mu^{2} C_{t}^{2} > 0.
\]

Then there exists some constant \( \tilde{C} \) such that:

\[
\frac{3}{4} \| Z^{A}_{w;m} \|_{V}^{2} - \frac{1}{4} \sum_{E \in \mathcal{E}_{h}} \| \nabla Z^{A}_{w;m} \|_{L^{2}(E)}^{2} - \sum_{e \in \Gamma_{h} \cup \Gamma_{wD}} \frac{4 \mu^{2} C_{t}^{2}}{|e|^{\beta}} \| Z^{A}_{w;m} \|_{L^{2}(e)}^{2} \geq \tilde{C} \| Z^{A}_{w;m} \|_{V}^{2}
\]
The inequality (3.86) becomes:

\[
\|Z_{w;m}^A\|^2_V + c\|Z_{p;m}^A\|^2_{L^2(\mathcal{E}_h)} + \|Z_{e;m}^A\|^2_{L^2(\mathcal{E}_h)} + \Delta t \sum_{n=1}^{m} \|Z_{p:n}^A\|^2_Q
\]

\[
+ \Delta t \sum_{n=1}^{m} \sum_{e \in \Gamma_h \cup \Gamma_{w,D}} \frac{\sigma_e}{|e|^{\beta_1}} \|\frac{Z_{w:n}^A - Z_{w:n-1}^A}{\Delta t}\|^2_{L^2(e)}
\]

\[
\leq C_{46} \Delta t \sum_{n=1}^{m} \|Z_{p;n}^A\|^2_{L^2(\mathcal{E}_h)} + C_{47} \Delta t \sum_{n=1}^{m} \|Z_{e;n}^A\|^2_{L^2(\mathcal{E}_h)} + C_{49} \Delta t \sum_{n=1}^{m} \|Z_{w:n}^A\|^2_V
\]

\[
+ C_{53} \Delta t \sum_{n=0}^{m} \left( h^{2 \min(k,s)} \|\beta^{n+1}\|^2_{H^{s+1}(\mathcal{E}_h)} + h^{2 \min(r+1,s)} \|\varepsilon^{n+1}\|^2_{H^{s}(\mathcal{E}_h)} \right)
\]

\[
+ C_{53} \|w(0)\|^2_{H^{s+1}(\mathcal{E}_h)} + h^{2 \min(l,s)} \|\w|\|^2_{H^{s+1}(\mathcal{E}_h)} + h^{2 \min(r+1,s)} \|\varepsilon(0)\|^2_{H^{s}(\mathcal{E}_h)}
\]

\[
+ C_{54} \Delta t^2 \Delta t \sum_{n=0}^{m} \left( \|\frac{\partial^2 p}{\partial t^2}\|^2_{L^\infty(t_n,t_{n+1};L^2(\Omega))} + \|\frac{\partial^2 \varepsilon}{\partial t^2}\|^2_{L^\infty(t_n,t_{n+1};L^2(\Omega))} \right)
\]

(3.87)

By Gronwall’s inequality, we have:

\[
\|Z_{w;m}^A\|^2_V + c\|Z_{p;m}^A\|^2_{L^2(\mathcal{E}_h)} + \|Z_{e;m}^A\|^2_{L^2(\mathcal{E}_h)} + \Delta t \sum_{n=1}^{m} \|Z_{p:n}^A\|^2_Q
\]

\[
+ \Delta t \sum_{n=1}^{m} \sum_{e \in \Gamma_h \cup \Gamma_{w,D}} \frac{\sigma_e}{|e|^{\beta_1}} \|\frac{Z_{w:n}^A - Z_{w:n-1}^A}{\Delta t}\|^2_{L^2(e)}
\]

\[
\leq C^{(m+1)} \Delta t \left( \sum_{n=0}^{m} \left( h^{2 \min(k,s)} \|\beta^{n+1}\|^2_{H^{s+1}(\mathcal{E}_h)} + h^{2 \min(r+1,s)} \|\varepsilon^{n+1}\|^2_{H^{s}(\mathcal{E}_h)} \right)
\]

\[
+ \|w(0)\|^2_{H^{s+1}(\mathcal{E}_h)} + h^{2 \min(l,s)} \|\w|\|^2_{H^{s+1}(\mathcal{E}_h)} + h^{2 \min(r+1,s)} \|\varepsilon(0)\|^2_{H^{s}(\mathcal{E}_h)}
\]

\[
+ \Delta t^2 \left( \|\frac{\partial^2 p}{\partial t^2}\|^2_{L^\infty(t_n,t_{n+1};L^2(\Omega))} + \|\frac{\partial^2 \varepsilon}{\partial t^2}\|^2_{L^\infty(t_n,t_{n+1};L^2(\Omega))} \right)
\]

(3.88)
Therefore, we obtain:

$$\|Z_{w;m}^A\|_V^2 + c\|Z_{e;m}^A\|_{L^2(\mathcal{E}_h)}^2 + \|Z_{e;m}^A\|_{L^2(\mathcal{E}_h)}^2 + \Delta t \sum_{n=1}^{m} \|Z_{p;n}^A\|_Q^2$$

$$+ \Delta t \sum_{n=1}^{m} \sum_{\epsilon \in \Gamma_h \cup \Gamma_w} \frac{\sigma_{\epsilon}}{|\epsilon|^3} \left\| \frac{Z_{w;n}^A - Z_{w;n-1}^A}{\Delta t} \right\|_{L^2(\epsilon)}^2$$

$$\leq e^{C(m+1)\Delta t} \left( h^{2 \min(k,s)} \|p\|_{L^2(\mathcal{H}_{s+1}(\mathcal{E}_h))}^2 + h^{2 \min(r+1,s)} \|\varepsilon\|_{L^2(\mathcal{H}^s(\mathcal{E}_h))}^2 \right)$$

$$+ Th^{2 \min(k+1,s+1)} \|\frac{\partial p}{\partial t}\|_{L^\infty(0,T;\mathcal{H}_{s+1}(\mathcal{E}_h))}^2 + Th^{2 \min(t,s)} \|\frac{\partial w}{\partial t}\|_{L^\infty(0,T;\mathcal{H}^s(\mathcal{E}_h))}^2$$

$$+ Th^{2 \min(r+1,s)} \|\frac{\partial \varepsilon}{\partial t}\|_{L^\infty(0,T;\mathcal{H}^s(\mathcal{E}_h))}^2$$

$$+ h^{2 \min(t,s)} \left( \|w(0)\|_{\mathcal{H}_{s+1}(\mathcal{E}_h)}^2 + \|w_m\|_{\mathcal{H}_{s+1}(\mathcal{E}_h)}^2 \right) + h^{2 \min(r+1,s)} \|\varepsilon(0)\|_{\mathcal{H}^s(\mathcal{E}_h)}^2$$

$$+ \Delta t^2 \left( \|\frac{\partial^2 p}{\partial t^2}\|_{L^2(\mathcal{H}^{s+1}(\mathcal{E}_h))}^2 + \|\frac{\partial^2 \varepsilon}{\partial t^2}\|_{L^2(\mathcal{H}^s(\mathcal{E}_h))}^2 \right)$$

(3.89)

Case 2: $[\alpha = 0]$

From the case 2 of symmetric form and the case 1 of nonsymmetric form, similarly, we obtain:

$$\|Z_{w;m}^A\|_V^2 + \|Z_{e;m}^A\|_{L^2(\mathcal{E}_h)}^2 + \Delta t \sum_{n=1}^{m} \|Z_{p;n}^A\|_Q^2$$

$$+ \Delta t \sum_{n=1}^{m} \sum_{\epsilon \in \Gamma_h \cup \Gamma_w} \frac{\sigma_{\epsilon}}{|\epsilon|^3} \left\| \frac{Z_{w;n}^A - Z_{w;n-1}^A}{\Delta t} \right\|_{L^2(\epsilon)}^2$$

$$\leq e^{C(m+1)\Delta t} \left( h^{2 \min(k,s)} \|p\|_{L^2(\mathcal{H}_{s+1}(\mathcal{E}_h))}^2 + h^{2 \min(r+1,s)} \|\varepsilon\|_{L^2(\mathcal{H}^s(\mathcal{E}_h))}^2 \right)$$

$$+ Th^{2 \min(k+1,s+1)} \|\frac{\partial p}{\partial t}\|_{L^\infty(0,T;\mathcal{H}_{s+1}(\mathcal{E}_h))}^2 + Th^{2 \min(t,s)} \|\frac{\partial w}{\partial t}\|_{L^\infty(0,T;\mathcal{H}^s(\mathcal{E}_h))}^2$$

$$+ Th^{2 \min(r+1,s)} \|\frac{\partial \varepsilon}{\partial t}\|_{L^\infty(0,T;\mathcal{H}^s(\mathcal{E}_h))}^2$$

$$+ h^{2 \min(t,s)} \left( \|w(0)\|_{\mathcal{H}_{s+1}(\mathcal{E}_h)}^2 + \|w_m\|_{\mathcal{H}_{s+1}(\mathcal{E}_h)}^2 \right) + h^{2 \min(r+1,s)} \|\varepsilon(0)\|_{\mathcal{H}^s(\mathcal{E}_h)}^2$$

$$+ \Delta t^2 \left( \|\frac{\partial^2 p}{\partial t^2}\|_{L^2(\mathcal{H}^{s+1}(\mathcal{E}_h))}^2 + \|\frac{\partial^2 \varepsilon}{\partial t^2}\|_{L^2(\mathcal{H}^s(\mathcal{E}_h))}^2 \right)$$

(3.90)
From the inequalities (3.89) and (3.90), we prove the theorem is true.

3.3 A priori error estimate

Combining Theorem 3.4 and Theorem 3.5, we have an a priori error estimate for the numerical solution:

**Theorem 3.6 (A priori error estimate for DG scheme)**: Assume the exact solution \( w \) belongs to \( H^1(0,T;(H^{s+1}(E_h))^d) \), \( p \) belongs to \( H^2(0,T;H^{s+1}(E_h)) \) and \( \epsilon \) belongs to \( H^2(0,T;H^s(E_h)) \). Sequences \((p^n_h)_{n\geq 0}\) of functions in \( Q_k(E_h) \), \((w^n_h)_{n\geq 0}\) of functions in \( V_l(E_h) \), \((\epsilon^n_h)_{n\geq 0}\) of functions in \( U_r(E_h) \) satisfy the model (2.18). Then

\[
\|w^m - w_h^m\|^2_{V} + c\|p^m - p_h^m\|^2_{L^2(E_h)} + \|\epsilon^m - \epsilon_h^m\|^2_{L^2(E_h)} + \Delta t \sum_{n=1}^{m} \|p^n - p_h^n\|_Q^2 \\
\leq \tilde{C}_{10} h^{2 \min(k,s)} (\|p\|^2_{L^2(0,T;H^{s+1}(E_h))} + Th^2 \|\frac{\partial p}{\partial t}\|^2_{L^\infty(0,T;H^{s+1}(E_h))}) \\
+ \tilde{C}_{11} h^{2 \min(l,s)} (\|w\|^2_{L^\infty(0,T;H^{s+1}(E_h))} + Th^2 \|\frac{\partial w}{\partial t}\|^2_{L^\infty(0,T;H^{s+1}(E_h))}) \\
+ \tilde{C}_{12} h^{2 \min(r+1,s)} (\|\epsilon\|^2_{L^2(0,T;H^s(E_h))} + Th^2 \|\frac{\partial \epsilon}{\partial t}\|^2_{L^\infty(0,T;H^{s+1}(E_h))} + \|\epsilon(0)\|^2_{H^s(E_h)}) \\
+ \tilde{C}_{13} \Delta t^2 (\|\frac{\partial^2 p}{\partial t^2}\|^2_{L^2(0,T;H^2(\Omega)))} + \|\frac{\partial^2 \epsilon}{\partial t^2}\|^2_{L^2(0,T;H^2(\Omega)))})
\]

where \( k, l \) are the degrees of the polynomial spaces; \( s \) is the degree of the Sobolev space; \( \tilde{C}_{10}, \tilde{C}_{11}, \tilde{C}_{12} \) and \( \tilde{C}_{13} \) are constants independent of \( h \).

**Proof.**

When \( a_2(\cdot, \cdot) \) is symmetric, by approximation results (3.12), (3.11a) and (3.11b), we have:
From Theorem 3.4, the above inequality (3.91) becomes:

\[
\Delta t \sum_{n=1}^{m} \left( \|p^n - p_h^n\|_Q^2 + \|w^n - w_h^n\|_V^2 + \|p_m^n - P_h^n\|_{L^2(\mathcal{E}_h)} + \|\varepsilon^m - \varepsilon_h^m\|_{L^2(\mathcal{E}_h)} \right) \\
= \Delta t \sum_{n=1}^{m} \left( \|Z_{p:c}^A + Z_{I:p:n}^I\|_Q^2 + \|Z_{w:m}^A + Z_{I:w:m}^I\|_V^2 \right) \\
+ \|Z_{p:c}^A + Z_{I:p:n}^I\|_{L^2(\mathcal{E}_h)} + \|Z_{I:w:m}^A + Z_{I:w:m}^I\|_{L^2(\mathcal{E}_h)} \\
\leq C \left( \Delta t \sum_{n=1}^{m} \|Z_{p:c}^A\|_Q^2 + \|Z_{w:m}^A\|_V^2 + \|Z_{p:c}^A\|_{L^2(\mathcal{E}_h)} + \|Z_{I:w:m}^A\|_{L^2(\mathcal{E}_h)} \right) \\
+ \Delta t \sum_{n=1}^{m} \left( \|Z_{I:p:n}^I\|_Q^2 + \|Z_{I:w:m}^I\|_V^2 + \|Z_{I:p:n}^I\|_{L^2(\mathcal{E}_h)} + \|Z_{I:w:m}^I\|_{L^2(\mathcal{E}_h)} \right) \\
\leq C \left( \Delta t \sum_{n=1}^{m} \|Z_{p:c}^A\|_Q^2 + \|Z_{w:m}^A\|_V^2 + \|Z_{p:c}^A\|_{L^2(\mathcal{E}_h)} + \|Z_{I:w:m}^A\|_{L^2(\mathcal{E}_h)} \right) \\
+ \Delta t \sum_{n=1}^{m} h^{2\min(k,s)} \|p_m^n\|_{H^{s+1}(\mathcal{E}_h)}^2 + h^{2\min(l,s)} \|w^m\|_{H^{s+1}(\mathcal{E}_h)}^2 \]

\begin{equation}
= \Delta t \sum_{n=1}^{m} \left( \|p^n - p_h^n\|_Q^2 + \|w^n - w_h^n\|_V^2 + \|p_m^n - P_h^n\|_{L^2(\mathcal{E}_h)} + \|\varepsilon^m - \varepsilon_h^m\|_{L^2(\mathcal{E}_h)} \right) \\
\leq \tilde{C}_1 h^{2\min(k,s)} \left( \|p\|_{L^2(H^{s+1}(\mathcal{E}_h))}^2 + Th^2 \left\| \frac{\partial p}{\partial t} \right\|_{L^2(0,T;H^{s+1}(\mathcal{E}_h))}^2 \right) \\
+ \tilde{C}_2 h^{2\min(l,s)} \left( \|w\|_{L^2(0,T;H^{s+1}(\mathcal{E}_h))}^2 + Th^2 \left\| \frac{\partial w}{\partial t} \right\|_{L^2(0,T;H^{s+1}(\mathcal{E}_h))}^2 \right) \\
+ \tilde{C}_3 h^{2\min(r+1,s)} \left( \|\varepsilon\|_{L^2(H^{s}(\mathcal{E}_h))}^2 + Th^2 \left\| \frac{\partial \varepsilon}{\partial t} \right\|_{L^2(0,T;H^{s+1}(\mathcal{E}_h))}^2 + \|\varepsilon(0)\|_{H^s(\mathcal{E}_h)}^2 \right) \\
+ \tilde{C}_4 \Delta t^2 \left( \left\| \frac{\partial^2 p}{\partial t^2} \right\|_{L^2(\Omega)}^2 + \left\| \frac{\partial^2 \varepsilon}{\partial t^2} \right\|_{L^2(\Omega)}^2 \right) \tag{3.92}
\end{equation}
When $a_2(\cdot, \cdot)$ is nonsymmetric, similarly from Theorem 3.5, we have:

$$
\Delta t \sum_{n=1}^{m} \| p^n - p_h^n \|^2_Q + \| w^m - w_h^m \|^2_V + \| p^m - p_h^m \|_{L^2(\mathcal{E}_h)}^2 + \| \varepsilon^m - \varepsilon_h^m \|_{L^2(\mathcal{E}_h)}^2 \\
\leq \tilde{C}_6 h^{2 \min(k, s)} (\| p \|^2_{L^2(H^{s+1}(\mathcal{E}_h))} + T h^2 \| \frac{\partial p}{\partial t} \|^2_{L^\infty(0, T ; H^{s+1}(\mathcal{E}_h))}) \\
+ \tilde{C}_7 h^{2 \min(l, s)} (\| w \|^2_{L^\infty(0, T ; H^{s+1}(\mathcal{E}_h))} + T h^2 \| \frac{\partial w}{\partial t} \|^2_{L^\infty(0, T ; H^{s+1}(\mathcal{E}_h))}) \\
+ \tilde{C}_8 h^{2 \min(r+1, s)} (\| \varepsilon \|^2_{L^2(H^{s}(\mathcal{E}_h))} + T h^2 \| \frac{\partial \varepsilon}{\partial t} \|^2_{L^\infty(0, T ; H^{s+1}(\mathcal{E}_h))} + \| \varepsilon(0) \|^2_{H^s(\mathcal{E}_h)}) \\
+ \tilde{C}_9 \Delta t^2 (\| \frac{\partial^2 p}{\partial t^2} \|^2_{L^2((t_n, t_{n+1} ; L^2(\Omega)))} + \| \frac{\partial^2 \varepsilon}{\partial t^2} \|^2_{L^2((t_n, t_{n+1} ; L^2(\Omega)))}) \tag{3.93}
$$

From the above two inequalities, we obtain the result.
Chapter 4

Conclusion and Future Work

In this thesis, discontinuous Galerkin method in space domain and backward-Euler method in time domain are applied to solve the poroelasticity equations which are based on Biot’s model and can be applied to study the formation of intestinal edema.

In Chapter 2, we first introduce the model which include a dilatation term. Then we present discontinuous Galerkin scheme for the poroelasticity equations on a bounded polygonal domain in $\mathbb{R}^d \ (d = 2 \ or \ 3)$, and fully discretize the DG scheme in time domain by backward-Euler method. At the end of this chapter, we prove the existence and uniqueness of the numerical solution to the scheme.

In Chapter 3, since the error analysis for the symmetric form and the nonsymmetric form is different, we present the error estimates for the two cases separately. Then we combine two results of two different cases to get an a priori error estimate.

For future work, we propose to implement the method and to confirm the theoretical rates numerically.
Appendix A

Taylor Expansion

Lemma A.1 If
\[ \frac{\varepsilon^{n+1}(x) - \varepsilon^n(x)}{\Delta t} = \varepsilon_t^{n+1}(x) + \Delta t \rho_{\varepsilon; n+1}, \ \forall x \in \Omega \]
then there is a constant C such that:
\[ \| \rho_{\varepsilon; n+1} \|_{L^2(\Omega)} \leq C \| \frac{\partial^2 \varepsilon}{\partial t^2} \|_{L^\infty(t_n, t_{n+1}; L^2(\Omega))} \]

Proof.

By Taylor expansion, we know:
\[ \varepsilon^n = \varepsilon^{n+1} + (t_n - t_{n+1})\varepsilon_{t}^{n+1} + \frac{(t_n - t_{n+1})^2}{2} \varepsilon_{tt}(t_n^*) \]
where \( t_n^* \in [t_n, t_{n+1}] \). Then we obtain:
\[ \frac{\varepsilon^{n+1} - \varepsilon^n}{\Delta t} = \varepsilon_t^{n+1} + \Delta t \varepsilon_{tt}(t_n^*) \]
which means \( \rho_{\varepsilon; n+1} = \varepsilon_{tt}(t_n^*)/2 \). Then
\[ \| \rho_{\varepsilon; n+1} \|_{L^2(\Omega)} = \| \varepsilon_{tt}(t_n^*)/2 \|_{L^2(\Omega)} \]
\[ \leq \frac{1}{2} \| \frac{\partial^2 \varepsilon}{\partial t^2} \|_{L^\infty(t_n, t_{n+1}; L^2(\Omega))} \]

Lemma A.2 If
\[ \frac{p^{n+1}(x) - p^n(x)}{\Delta t} = p_t^{n+1}(x) + \Delta t \rho_{p;n+1}, \ \forall x \in \Omega \]
then there is a constant C such that:
\[ \| \rho_{p;n+1} \|_{L^2(\Omega)} \leq C \| \frac{\partial^2 p}{\partial t^2} \|_{L^\infty(t_n, t_{n+1}; L^2(\Omega))} \]
Lemma A.3 If

\[
\frac{w^{n+1}(x) - w^n(x)}{\Delta t} = w^{n+1}_t(x) + \Delta t \rho_{w^{n+1}}, \quad \forall x \in \Omega
\]

then there is a constant $C$ such that:

\[
\|\rho_{w^{n+1}}\|_{L^2(\Omega)} \leq C\|\frac{\partial^2 w}{\partial t^2}\|_{L^\infty(t_n, t_{n+1}; L^2(\Omega))}
\]
Bibliography


