

An Analytical Method for Analyzing Symmetry-Breaking Bifurcation and Period-Doubling Bifurcation

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Abstract

A new modification of homotopy analysis method (HAM) is proposed in this paper. The auxiliary differential operator is specifically chosen so that more than one secular term must be eliminated. The proposed method can capture asymmetric and period-2 solutions with satisfactory accuracy and hence can be used to predict symmetry-breaking and period-doubling bifurcation points. The variation of accuracy is investigated when different number of frequencies are considered.

Keywords: Symmetry-breaking bifurcation; period-doubling bifurcation; homotopy analysis method; analytical approximation

1. Introduction

Bifurcation refers to a qualitative change in periodic orbits (or equilibrium points), or in their stability attributes, caused by a small variation in system parameters. Bifurcations are important phenomena which exist in the behavior of many nonlinear systems and are closely related to system stability and more complicated behavior such as chaos. If an analytical method cannot be used to investigate bifurcations, then the understanding of nonlinear systems that it brings to us will be insufficient. Although there is a vast literature [1–6] investigating nonlinear systems through analytical methods, few studies attempted to investigate bifurcation phenomena with these analytical methods. Tesi et al. proposed to use a combination of the Loeb criterion, a criterion for numerically determining the stability of a limit cycle, and the first-order harmonic balance approximation to investigate period-doubling bifurcations and showed that this method could be applied in conjunction with a control design to delay or even eliminate a period-doubling cascade to chaos [7]. Bonani and Gilli put forward an approach for investigating limit-cycle bifurcations in nonlinear control systems, which not only detected bifurcations of the limit cycles, but also calculated the Floquet multipliers as the roots of an algebraic equation as well as determined the stability [8]. Berns et al. proposed a quasi-analytical computational scheme for predicting period-doubling bifurcation based one high-order harmonic balance analysis and characteristic multiplier tracking [9]. Luo and Huang analytically predicted period-m solutions of a periodically excited Duffing oscillator with the generalized harmonic balance method [10]; Luo and Huang presented the analytical expressions for period-m flows and chaos of a damped Duffing oscillator subjected to harmonic excitation [11]; Luo and Huang derived the analytical period-1 solution to a Duffing oscillator with harmonic excitation and a twin-well potential and studied the routes of the period-1 motion to chaos [12]; the three papers successfully derived analytical period-m solutions and predicted their stability and bifurcations as well the routes to chaos.

Analytical study on the dynamics of nonlinear systems dates back to early celestial mechanics. In 1830, Poisson first introduced an early form of the perturbation methods which obtained periodic solutions

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based on a power series expansion with respect to a small change of a parameter in the nonlinear system. Lindstedt put forward a variant of the perturbation method without strict mathematical foundation [13]. It was not until 1892 that the mathematical foundation of the perturbation method was established by Poincaré [14]. Standard perturbation methods like the Lindstedt-Poincaré method work very well for celestial bodies where the oscillations are subjected to weak nonlinearity most of the time, but will yield large errors when the investigated system has strong nonlinearity. However, in many disciplines like mechanical engineering, earthquake engineering, and structural engineering, strongly nonlinear dynamic systems are very common. Therefore, numerous studies have attempted to find analytical methods for solving strongly nonlinear differential equations. Burton proposed a modified Lindstedt-Poincaré method which can be applied to systems with strong nonlinearity [15]. The standard method of multi-scale was modified by Burton and Rahman to enable accurate periodic solutions to be acquired for strongly nonlinear oscillators [16]. Cheung et al. introduced a new parameter which can always be kept small regardless of the magnitude of the original system parameters [17]. Liao combined the concept of homotopy in topology with Maclaurin series expansion and developed a method called the homotopy analysis method (HAM) [18]. HAM also works for strongly nonlinear systems [19–21]. Although a number of analytical methods for strongly nonlinear systems have been put forward, they can only be used to obtain periodic solutions of some selected nonlinear problems, which are mostly period-one solutions. Most of the bifurcation phenomena like symmetry-breaking bifurcations and period-doubling bifurcations are beyond the capability of these existing analytical methods.

Nayfeh and Balachandran described a route to chaos, which is now basically the most well-known one [22]. This route to chaos starts with a symmetry-breaking bifurcation and period-doubling bifurcations. Typically a symmetry-breaking bifurcation occurs prior to a period-doubling bifurcation, which is followed by a cascade of period-doubling bifurcations [23, 24]. Finally it reaches the onset of chaos. One of the examples reported in [22] is for the following Duffing oscillator.

$$\ddot{x} + c\dot{x} + x - x^3 = F \cos \Omega t. \quad (1)$$

For the Duffing oscillator represented by Eq. (1), parameters held constant are c at 0.4 and Ω at 0.8. When F is equal to 0.350, one stable symmetric period-one oscillation occurs. Its limit cycle and amplitude of frequency components are presented in Fig. 1(a) and Fig. 1(b), respectively.

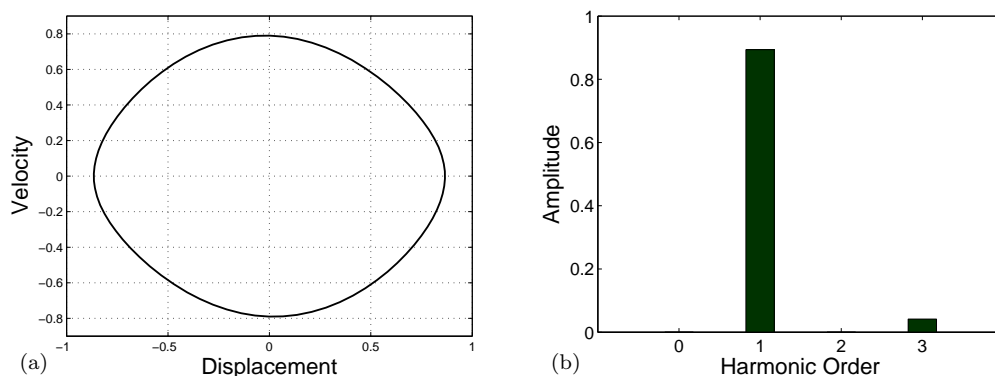


Fig. 1. The limit cycle of a symmetric period-one solution. (a) A symmetric limit cycle; (b) harmonic components.

When F increases to 0.380, the steady-state solution becomes asymmetric as shown in Fig. 2(a), which suggests a symmetry-breaking bifurcation has already occurred. The appearance of a constant term can be easily seen as a zeroth order harmonic in Fig. 2(b).

When the excitation amplitude F further increases to 0.386, a period-2 steady-state solution arises as shown in Fig. 3(a), which suggests a period-doubling bifurcation occurs between $F = 0.380$ and $F = 0.386$. The appearance of a frequency component at $\frac{\Omega}{2}$ can be seen in Fig. 3(b).

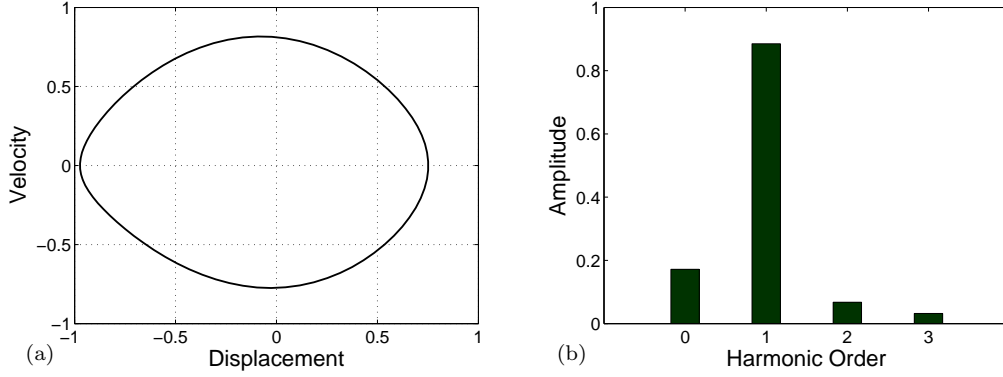


Fig. 2. The limit cycle of an asymmetric period-one solution. (a) An asymmetric limit cycle; (b) harmonic components.

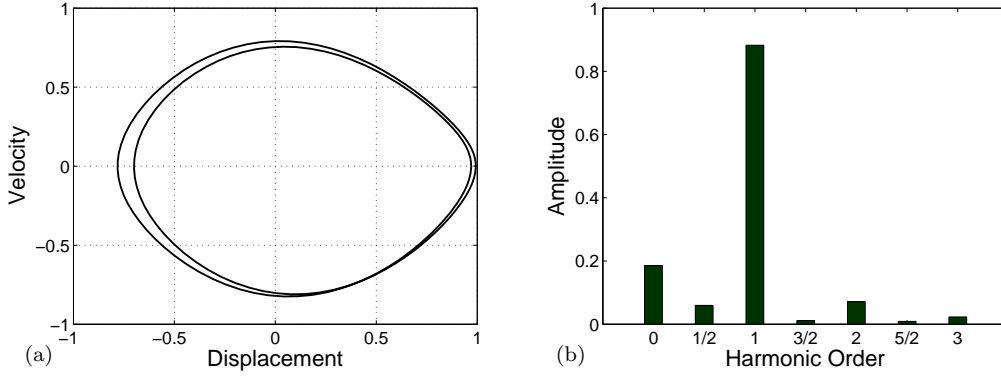


Fig. 3. The limit cycle of an asymmetric period-2 solution. (a) A limit cycle with period doubled; (b) harmonic components.

In the present paper, a new analytical procedure for obtaining approximate solutions for strongly nonlinear problems will be outlined in the framework of HAM. This new procedure will be applied to investigate symmetry-breaking bifurcations and period-doubling bifurcations of a Duffing oscillator.

2. Formulation of multi-frequency HAM

To start the procedure of multi-frequency HAM (MFHAM), an auxiliary linear differential operator conveying information of n (n is an arbitrary positive integer) fundamental frequencies needs to be constructed.

$$\begin{aligned}
 \mathcal{L}_n(x) = & x^{(2n)} + x^{(2n-2)} \sum_r \Omega_r^2 + x^{(2n-4)} \sum_{r \neq j} (\Omega_r^2 \Omega_j^2) \\
 & + x^{(2n-6)} \sum_{r, j, l} (\Omega_r^2 \Omega_j^2 \Omega_l^2) + \cdots + x \prod_r \Omega_r^2.
 \end{aligned} \tag{2}$$

where Ω_r , $r = 1, 2, \dots, n$, are the fundamental frequencies taken into consideration, and the numbers in the superscripted parentheses denote the differential order with respect to time. It is worth mentioning that Ω_r , $r = 1, 2, \dots, n$, can either be known positive real values or positive real variables to be determined in

subsequent secular term elimination steps. Unknown fundamental frequencies are usually used for initial value problems, and in these cases a parameter expansion technique might be required to allow the homotopy differential equations' solutions have particular initial values. The number of equations that the initial value conditions provide always equals the number of unknown fundamental frequencies used plus the number of variables introduced during the parameter expansion process, which makes the solving procedure of MFHAM clear and reliable.

The characteristic polynomial of Eq. (2) is

$$P(\lambda) = \prod_{r=1}^n (\lambda + \Omega_r i)(\lambda - \Omega_r i). \quad (3)$$

Because the considered frequencies multiplied by the imaginary unit i are the roots of the characteristic polynomial in Eq. (3), any sinusoidal terms on the right-hand side of the differential operator $\mathcal{L}_n(x)$ with the considered frequencies will pose a secular term and hence need to be eliminated.

When the solution is expected to contain a constant term, the auxiliary linear operator in Eq. (2) can be revised as follows

$$\begin{aligned} \mathcal{L}_{n+1}(x) = & x^{(2n+1)} + x^{(2n-1)} \sum_r \Omega_r^2 + x^{(2n-3)} \sum_{r \neq j} (\Omega_r^2 \Omega_j^2) \\ & + x^{(2n-5)} \sum_{r,j,l} (\Omega_r^2 \Omega_j^2 \Omega_l^2) + \dots + \dot{x} \prod_r \Omega_r^2. \end{aligned} \quad (4)$$

The multi-frequency auxiliary linear operator in the framework of HAM provides a new thinking of solving nonlinear dynamic systems. For an arbitrary nonlinear dynamic system

$$\mathcal{N}(x) = 0, \quad (5)$$

where \mathcal{N} is a nonlinear integral or differential operator, construct the following homotopy

$$(1 - q)\mathcal{L}_{n+1}(x - g_0(t)) - q\hbar H(t)\mathcal{N}(x) = 0, \quad (6)$$

where $q \in [0, 1]$ is the homotopy-parameter, $g_0(t)$ is the initial solution, \hbar is a convergence-control parameter, and $H(t)$ is an auxiliary function.

The power series solution to Eq. (6) in q is

$$x(t, q) = x_0(t) + qx_1(t) + q^2x_2(t) + \dots, \quad (7)$$

Substituting Eq. (7) into Eq. (6) and separating the terms with identical powers of q , we obtain a infinite number of linear differential equations, the first few of which are as follows.

$$q^0 : \mathcal{L}_{n+1}(x_0 - g_0(t)) = 0, \quad (8)$$

$$q^1 : \mathcal{L}_{n+1}(x_1 - g_0(t)) = \mathcal{L}_{n+1}(x_0 - g_0(t)) + \hbar H(t)\mathcal{N}(x_0), \quad (9)$$

$$q^2 : \mathcal{L}_{n+1}(x_2 - g_0(t)) = \mathcal{L}_{n+1}(x_1 - g_0(t)) + \hbar H(t) \frac{1}{1!} \left. \frac{\partial \mathcal{N}(x)}{\partial q} \right|_{q=0}. \quad (10)$$

Since Eqs. (8), (9), and so forth include linear operator \mathcal{L}_{n+1} on the left-hand side, whose characteristic polynomial has non-repeated roots $\pm\Omega_r i$, $r = 1, 2, \dots, n$, sometimes including a zero root, all the coefficients of the sinusoidal terms with circular frequencies Ω_r , $r = 1, 2, \dots, n$, on the right-hand side must be equal to zero in order to eliminate secular terms in the solutions.

The solution to Eq. (8) is

$$x_0(t) = g_0(t) + A_0 + \sum_{r=1}^n A_r \sin(\Omega_r t + \phi_r), \quad (11)$$

where A_0 , A_r and ϕ_r , $r = 1, 2, 3, \dots, n$, are constants to be determined in subsequent secular term elimination steps.

Substituting Eq. (11) into Eq. (9) and eliminating the secular terms, A_0 , A_r and ϕ_r , $r = 1, 2, 3, \dots, n$, will be decided and x_1 is thus obtained.

In the similar manner, x_r , $r = 1, 2, 3, \dots, n$, can be derived one by one. The solution to the original nonlinear problem Eq. (5) is

$$x(t) = \lim_{q \rightarrow 1} x(t, q) = x_0(t) + x_1(t) + x_2(t) + \dots \quad (12)$$

When one uses the auxiliary differential operator concerning only one single frequency, $\mathcal{L}_2(x) = \ddot{x} + \Omega^2 x$, in MFHAM to solve Eq. (1), the proposed MFHAM simplifies to the standard HAM. Since the standard HAM has been well studied, the first-order HAM approximation to Eq. (1) is directly given as follows.

$$x(t) = A_{c0} \cos \Omega t + A_{s0} \sin \Omega t + \frac{\hbar}{32\Omega^2} [(A_{c0}^3 - 3A_{c0}A_{s0}^2) \cos 3\Omega t + (3A_{s0}A_{c0}^2 - A_{s0}^3) \sin 3\Omega t], \quad (13)$$

where A_{c0} and A_{s0} are governed by

$$F + (\Omega^2 - 1)A_{c0} + \frac{3}{4}A_{c0}^3 - c\Omega A_{s0} + \frac{3}{4}A_{c0}A_{s0}^2 = 0, \quad (14)$$

$$(\Omega^2 - 1)A_{s0} + \frac{3}{4}A_{s0}^3 + c\Omega A_{c0} + \frac{3}{4}A_{s0}A_{c0}^2 = 0. \quad (15)$$

When one uses the auxiliary differential operator, $\mathcal{L}_{2s}(x) = \mathcal{L}_2(x)/\Omega^2 = [\ddot{x} + \Omega^2 x]/\Omega^2$, the first-order approximation to Eq. (1) becomes

$$x(t) = A_{c0} \cos \Omega t + A_{s0} \sin \Omega t + \hbar [(A_{c0}^3 - 3A_{c0}A_{s0}^2) \cos 3\Omega t + (3A_{s0}A_{c0}^2 - A_{s0}^3) \sin 3\Omega t], \quad (16)$$

where A_{c0} and A_{s0} are also governed by Eq. (14) and Eq. (15).

With $c = 0.4$ and $F = 0.1$ in Eq. (1), the frequency response curves by Eq. (13) and Eq. (16) are compared with the one obtained through numerical simulation in Fig. 4. As can be seen in Fig. 4, the frequency response curve obtained with $\mathcal{L}_2(x)$ has large difference with the numerical one at small frequencies, while the frequency response curve with $\mathcal{L}_{2s}(x)$ agrees satisfactorily with the numerical one in all frequency ranges. Therefore, scaling of the auxiliary differential operator increase the convergence range and applicability of HAM.

Based on a direct comparison of Eq. (13) and Eq. (16), this improvement on convergence range stems from the fact that the driven frequency's influence on the amplitudes of the high harmonic terms is eliminated. In view of this, the auxiliary differential operator used in MFHAM should also be scaled to maintain MFHAM's accuracy for all frequencies. Like $\mathcal{L}_{n+1}(x)$ in Eq. (4) should be reconstructed as follows.

$$\begin{aligned} \mathcal{L}_{n+1}(x) = & \frac{1}{\Omega^{2n+1}} x^{(2n+1)} + x^{(2n-1)} \sum_r \Omega_r^2 + x^{(2n-3)} \sum_{r \neq j} (\Omega_r^2 \Omega_j^2) \\ & + x^{(2n-5)} \sum_{r, j, l} (\Omega_r^2 \Omega_j^2 \Omega_l^2) + \dots + \dot{x} \prod_r \Omega_r^2. \end{aligned} \quad (17)$$

After being divided by Ω to the order the same as the highest differential order of $\mathcal{L}_{n+1}(x)$, the auxiliary differential operator improves MFHAM's accuracy at small frequencies.

3. Investigation of Symmetry-Breaking Bifurcation Using MFHAM

An arbitrary Duffing oscillator

$$\hat{m} \frac{d^2 \hat{x}}{d\hat{t}^2} + \hat{c} \frac{d\hat{x}}{d\hat{t}} + \hat{k} \hat{x} + \hat{\alpha} \hat{x}^3 = \hat{F} \cos \hat{\Omega} \hat{t}, \quad (18)$$

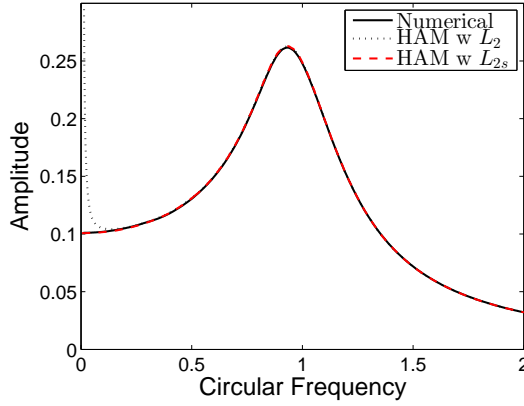


Fig. 4. Frequency response curves obtained with $\mathcal{L}_2(x)$ and $\mathcal{L}_{2s}(x)$ compared to the numerical frequency response curve.

can be transformed with the following linear scaling of time and space

$$\hat{t} = t \sqrt{\frac{\hat{m}}{|\hat{k}|}}, \quad (19)$$

$$\hat{x} = x \sqrt{\left| \frac{\hat{k}}{\hat{\alpha}} \right|}, \quad (20)$$

into

$$\ddot{\hat{x}} + \hat{c}\dot{\hat{x}} + \text{sgn}(\hat{k})\hat{x} + \text{sgn}(\hat{\alpha})\hat{x}^3 = F \cos \Omega t, \quad (21)$$

where the over dot denotes differentiation with respect to the nondimensional time t , sgn is the sign function, and

$$c = \frac{\hat{c}}{\sqrt{\hat{m}|\hat{k}|}}, \quad (22)$$

$$F = \frac{\sqrt{|\hat{\alpha}|}}{|\hat{k}|\sqrt{|\hat{k}|}} \hat{F}, \quad (23)$$

$$\Omega = \hat{\Omega} \sqrt{\frac{\hat{m}}{|\hat{k}|}}. \quad (24)$$

When $\hat{k} > 0$ and $\hat{\alpha} < 0$, [Eq. \(21\)](#) becomes [Eq. \(1\)](#). Since linear scaling of time and space does not change the substantial dynamics of the Duffing oscillator, [Eq. \(1\)](#) is a very widely representative non-dimensionalized Duffing system.

For symmetry-breaking bifurcations of the forced and damped Duffing oscillator governed by [Eq. \(1\)](#), a constant term will appear in the periodic solution. Considering this, it is necessary to take into account four frequencies, 0 , Ω , 2Ω and 3Ω . Thus the characteristic polynomial of the auxiliary differential operator becomes

$$P(\lambda) = \lambda \prod_{r=1}^3 (\lambda + r\Omega i)(\lambda - r\Omega i), \quad (25)$$

and consequently, the following auxiliary differential operator is used in the the proposed procedure.

$$\mathcal{L}_4(x) = x^{(7)} + 14\Omega^2 x^{(5)} + 49\Omega^4 x^{(3)} + 36\Omega^6 \dot{x}. \quad (26)$$

The following homotopy is constructed.

$$\mathcal{H}(x, q) = \mathcal{L}_4(x) - q [\mathcal{L}_4(x) + \hbar H(t)(\ddot{x} + c\dot{x} + x - x^3 - F \cos \Omega t)] = 0, \quad (27)$$

where $q \in [0, 1]$ is the homotopy parameter.

Assuming the solution of Eq. (27) can be expanded as

$$x = x_0 + qx_1 + q^2x_2 + \dots \quad (28)$$

Substituting Eq. (28) into Eq. (27), then collecting the terms with the identical powers of q , the following is obtained

$$q^0 : \mathcal{L}_4(x_0) = 0, \quad (29)$$

$$q^1 : \mathcal{L}_4(x_1) = \mathcal{L}_4(x_0) + \hbar H(t)(\ddot{x}_0 + c\dot{x}_0 + x_0 - x_0^3 - F \cos \Omega t). \quad (30)$$

The solution to Eq. (29) when considering the four frequencies is

$$x_0 = A_0 + B_0 \cos(\Omega t + \phi_1) + C_0 \cos(2\Omega t + \phi_2) + D_0 \cos(3\Omega t + \phi_3), \quad (31)$$

where $A_0, B_0, \phi_1, C_0, \phi_2, D_0$ and ϕ_3 are to be determined.

Substitution of Eq. (31) and Eq. (29) into Eq. (30) gives

$$\begin{aligned} \mathcal{L}_4(x_1) = & -\hbar H(t) \left\{ \Omega^2 [B_0 \cos(\Omega t + \phi_1) + 4C_0 \cos(2\Omega t + \phi_2) \right. \\ & + 9D_0 \cos(3\Omega t + \phi_3)] - [B_0 \cos(\Omega t + \phi_1) + C_0 \cos(2\Omega t + \phi_2) \\ & + A_0 + D_0 \cos(3\Omega t + \phi_3)] + F \cos \Omega t + c\Omega [B_0 \sin(\Omega t + \phi_1) \\ & + 2C_0 \sin(2\Omega t + \phi_2) + 3D_0 \sin(3\Omega t + \phi_3)] + [B_0 \cos(\Omega t + \phi_1) \\ & \left. + A_0 + C_0 \cos(2\Omega t + \phi_2) + D_0 \cos(3\Omega t + \phi_3)]^3 \right\}, \quad (32) \end{aligned}$$

where a full expansion of Eq. (32) is provided in Eq. A.1 in the appendix.

In order to eliminate secular terms in $x_1(t)$, the following equations must be satisfied.

$$\begin{aligned} -4A_0 + 4A_0^3 + 6A_0B_0^2 + 6A_0C_0^2 + 6A_0D_0^2 + 3B_0^2C_0 \cos(2\phi_1 - \phi_2) \\ + 6B_0C_0D_0 \cos(\phi_1 - \phi_3 + \phi_2) = 0, \quad (33) \end{aligned}$$

$$\begin{aligned} 4F \cos \phi_1 + 3B_0^2D_0 \cos(\phi_3 - 3\phi_1) + 12A_0B_0C_0 \cos(\phi_2 - 2\phi_1) \\ + 12A_0C_0D_0 \cos(\phi_3 - \phi_1 - \phi_2) + 3D_0C_0^2 \cos(2\phi_2 - \phi_1 - \phi_3) \\ + [-4(1 - \Omega^2)B_0 + 12A_0^2B_0 + 3B_0^3 + 6B_0D_0^2 + 6B_0C_0^2] = 0, \quad (34) \end{aligned}$$

$$\begin{aligned} 12A_0B_0C_0 \sin(\phi_2 - 2\phi_1) - 4F \sin \phi_1 + 3D_0C_0^2 \sin(2\phi_2 - \phi_1 - \phi_3) \\ - 4c\Omega B_0 + 12A_0C_0D_0 \sin(\phi_3 - \phi_1 - \phi_2) + 3B_0^2D_0 \sin(\phi_3 - 3\phi_1) = 0, \quad (35) \end{aligned}$$

$$\begin{aligned} 6A_0B_0^2 \cos(2\phi_1 - \phi_2) + 12A_0B_0D_0 \cos(\phi_1 - \phi_3 + \phi_2) \\ + [-4(1 - 4\Omega^2)C_0 + 12A_0^2C_0 + 6B_0^2C_0 + 6D_0^2C_0 + 3C_0^3] \\ + 6B_0C_0D_0 \cos(2\phi_2 - \phi_1 - \phi_3) = 0, \quad (36) \end{aligned}$$

$$\begin{aligned} -6A_0B_0^2 \sin(2\phi_1 - \phi_2) + 12A_0B_0D_0 \sin(\phi_1 - \phi_3 + \phi_2) \\ + 8c\Omega C_0 + 6B_0C_0D_0 \sin(2\phi_2 - \phi_1 - \phi_3) = 0, \quad (37) \end{aligned}$$

$$\begin{aligned} B_0^3 \cos(\phi_3 - 3\phi_1) + 12A_0B_0C_0 \cos(\phi_1 - \phi_3 + \phi_2) \\ + [-4(1 - 9\Omega^2)D_0 + 12A_0^2D_0 + 6B_0^2D_0 + 3D_0^3 + 6D_0C_0^2] \\ + 3B_0C_0^2 \cos(2\phi_2 - \phi_1 - \phi_3) = 0, \quad (38) \end{aligned}$$

$$\begin{aligned} B_0^3 \sin(\phi_3 - 3\phi_1) + 12c\Omega D_0 - 12A_0B_0C_0 \sin(\phi_1 - \phi_3 + \phi_2) \\ - 3B_0C_0^2 \sin(2\phi_2 - \phi_1 - \phi_3) = 0. \quad (39) \end{aligned}$$

When Eqs. (33)-(39) are satisfied, Eq. (32) is reduced and with $H(t) = 1$, the particular solution is calculated and presented as Eq. (40). The full expression for Eq. (32) after secular term elimination is provided in Eq. A.2 in the appendix.

$$\begin{aligned}
x_1 = & -\hbar \left\{ -\frac{A_0 B_0 D_0}{1680 \Omega^7} \sin(4\Omega t + \phi_1 + \phi_3) - \frac{3B_0^2 D_0}{53760 \Omega^7} \sin(5\Omega t + 2\phi_1 + \phi_3) \right. \\
& - \frac{A_0 D_0^2}{120960 \Omega^7} \sin(6\Omega t + 2\phi_3) - \frac{B_0 D_0^2}{53760 \Omega^7} \sin(5\Omega t - \phi_1 + 2\phi_3) \\
& - \frac{B_0 D_0^2}{806400 \Omega^7} \sin(7\Omega t + \phi_1 + 2\phi_3) - \frac{D_0^3}{15966720 \Omega^7} \sin(9\Omega t + 3\phi_3) \\
& - \frac{B_0^2 C_0}{6720 \Omega^7} \sin(4\Omega t + 2\phi_1 + \phi_2) - \frac{A_0 C_0 D_0}{13440 \Omega^7} \sin(5\Omega t + \phi_2 + \phi_3) \\
& - \frac{C_0^3}{725760 \Omega^7} \sin(6\Omega t + 3\phi_2) - \frac{B_0 C_0 D_0}{3360 \Omega^7} \sin(4\Omega t - \phi_1 + \phi_2 + \phi_3) \\
& - \frac{B_0 C_0 D_0}{120960 \Omega^7} \sin(6\Omega t + \phi_1 + \phi_2 + \phi_3) - \frac{D_0^2 C_0}{6720 \Omega^7} \sin(4\Omega t + 2\phi_3 - \phi_2) \\
& - \frac{D_0^2 C_0}{2217600 \Omega^7} \sin(8\Omega t + 2\phi_3 + \phi_2) - \frac{B_0 C_0^2}{53760 \Omega^7} \sin(5\Omega t + \phi_1 + 2\phi_2) \\
& \left. - \frac{A_0 C_0^2}{3360 \Omega^7} \sin(4\Omega t + 2\phi_2) - \frac{D_0 C_0^2}{806400 \Omega^7} \sin(7\Omega t + \phi_3 + 2\phi_2) \right\}, \tag{40}
\end{aligned}$$

where for a particular set of system parameters A_0 , B_0 , C_0 , D_0 , ϕ_1 , ϕ_2 , and ϕ_3 can be given by Eqs. (33)-(39).

Thus, the first-order approximation of x is

$$\begin{aligned}
x &= \lim_{q \rightarrow 1} (x_0 + qx_1) \\
&= x_0 + x_1. \tag{41}
\end{aligned}$$

Given $c = 0.4$, $F = 0.380$, $\Omega = 0.8$, $\hbar = -1$, the limit cycles and harmonic components from the first-order MFHAM-approximation and simulation with bifurcation software AUTO, which is based on the pseudo-arclength continuation method, are compared in Fig. 5(a). The limit cycles and harmonic components from both methods agree satisfactorily with each other. The lack of harmonic terms with frequency higher than 9Ω in the 1st order MFHAM-approximation accounts for the small difference between the two limit cycles.

Equations (33)-(39) yield several sets of real solutions, one of them is with $A_0 > 0$ and $C_0 > 0$, and corresponds to the asymmetric limit cycle in Fig. 5(a). Another set of real solutions from Eqs. (33)-(39) is with $A_0 = C_0 = 0$, which means it yields a symmetric solution. Comparison of the numerical results and MFHAM results of the symmetric solution at $F = 0.380$ is shown in Fig. 6. It is easy to see from Fig. 6(a) and Fig. 6(b) that the proposed method also captures the unstable symmetric solution with satisfactory accuracy after the symmetry-breaking bifurcation occurs.

Figure 7 depicts variation of A_0 and C_0 with excitation amplitude F in the MFHAM analysis. When F is smaller than 0.360, A_0 and C_0 only have real solutions which are both zero. When F increases beyond 0.360, nonzero real solutions for both A_0 and C_0 appear, which suggests that a symmetry-breaking bifurcation occurs at $F = 0.360$. The symmetry-breaking point detected by AUTO lies at $F = 0.362449$. The relative error is 0.68%. Therefore, the proposed method can provide a satisfactory estimation of the symmetry-breaking bifurcation point.

Since the MFHAM-approximation turns out to be accurate enough, it is safe to conclude that Eqs. (33)-(39), which are a set of high-order algebraic equations, provide a legitimate approximation to the relationship between the harmonic components of the asymmetric solution and the system parameters.

It should be noted when both A_0 and C_0 are zero, all the sinusoidal terms in Eq. (41) with circular frequencies at even multiples of Ω will become zero. Hence, the first-order MFHAM-approximation suggests

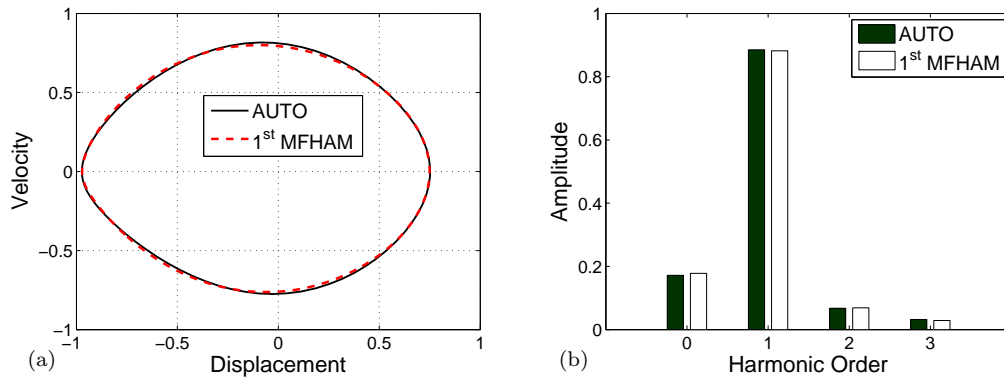


Fig. 5. Comparison of limit cycles concerning symmetry-breaking bifurcation. (a) Asymmetric limit cycles with $F = 0.380$; (b) harmonic components $F = 0.380$.

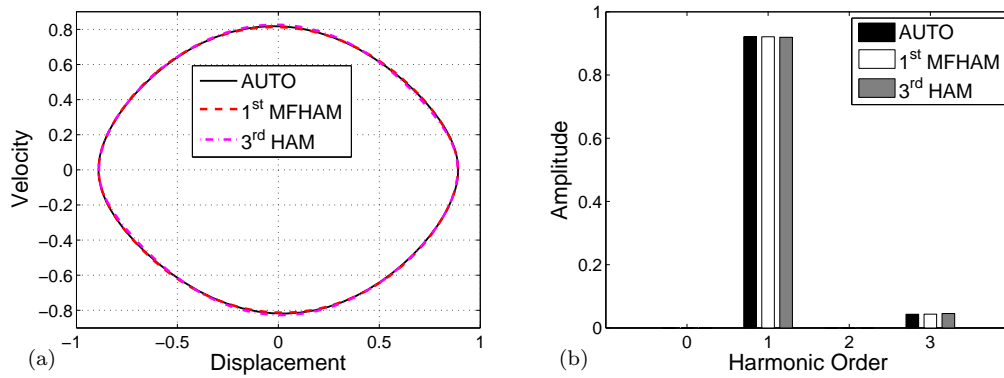


Fig. 6. Comparison of symmetric periodic solutions obtained by AUTO, HAM and MFHAM. (a) Symmetric limit cycles with $F = 0.380$; (b) harmonic components with $F = 0.380$

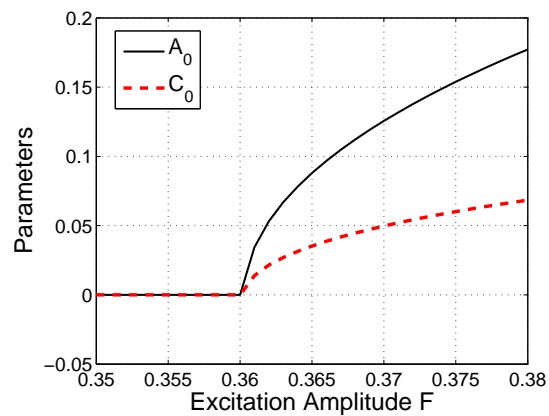


Fig. 7. Variation of A_0 and C_0 versus F .

Table 1: The amplitudes and phase angles of the complementary solutions by standard HAM

Unknowns	Solutions
(B_0, φ_0)	(0.913, 2.264)
(B_1, φ_1)	(0.019, 1.386)
(B_2, φ_2)	(0.012, 0.216)

that when the solution is symmetric, only sinusoidal terms with odd multiples of Ω exist in the solution, which agrees with the results in Fig. 1(b).

4. The Effect of the Number of Considered Frequencies

It is true that the first order standard HAM can not capture the asymmetric solution or predict a symmetry-breaking bifurcation since its initial guess does not include a constant term, but it is still possible for higher order HAM approximations. What follows is to investigate this possibility. Following the standard HAM, a homotopy should be constructed like

$$(1 - q)(\ddot{x} + \Omega^2 x) = q [\ddot{x} + \Omega^2 x + \hbar(\ddot{x} + c\dot{x} + x - x^3 - F \cos \Omega t)]. \quad (42)$$

Substituting Eq. (28) into Eq. (42), then collecting the terms with the identical powers of q , one has

$$q^0 : \ddot{x}_0 + \Omega^2 x_0 = 0, \quad (43)$$

$$q^1 : \ddot{x}_1 + \Omega^2 x_1 = \hbar(\ddot{x}_0 + c\dot{x}_0 + x_0 - x_0^3 - F \cos \Omega t), \quad (44)$$

$$q^2 : \ddot{x}_2 + \Omega^2 x_2 = \ddot{x}_1 + \Omega^2 x_1 + \hbar(\ddot{x}_1 + c\dot{x}_1 + x_1 - 3x_0^2 x_1), \quad (45)$$

$$q^3 : \ddot{x}_3 + \Omega^2 x_3 = \ddot{x}_2 + \Omega^2 x_2 + \hbar(\ddot{x}_2 + c\dot{x}_2 + x_2 - 3x_0^2 x_2 - 3x_1^2 x_0). \quad (46)$$

Assuming the solution to Eq. (43) is

$$x_0 = B_0 \cos(\Omega t + \varphi_0), \quad (47)$$

and the complementary solution for x_1 and x_2 are $B_1 \cos(\Omega t + \varphi_1)$ and $B_2 \cos(\Omega t + \varphi_2)$, respectively, where $B_0 \geq 0$, $B_1 \geq 0$, $B_2 \geq 0$, and $\varphi_0, \varphi_1, \varphi_2 \in [0, 2\pi)$. Following the standard HAM procedure, we find that after three steps of secular term elimination, only one solution set exists for the same set of parameter values, i.e. $c = 0.4$, $F = 0.380$, $\Omega = 0.8$, and $\hbar = -1$. The obtained values for $B_0, B_1, B_2, \varphi_0, \varphi_1, \varphi_2$ are listed in Table 1.

It is easily seen from Table 1 that the amplitude of successive solution terms decreases very fast as the order gets higher, and beyond the third order, we still can not obtain more than one set of amplitudes and phases for the solution terms. Since there exists only one set of solutions for $B_0, B_1, B_2, \varphi_0, \varphi_1$, and φ_2 , we can only obtain one symmetric third-order HAM approximate solution, which agrees well with the numerical result as depicted in Fig. 6. Hence, one can not expect standard HAM with a second-order auxiliary differential operator to capture asymmetric solutions or analyze symmetry-breaking bifurcations.

After MFHAM is chosen, it is still unknown as to how many frequencies must be taken into consideration. The excitation frequency Ω will be retained naturally; the zero frequency, which corresponds to the constant term, is indispensable since we are looking at asymmetric solutions and symmetry-breaking bifurcations. In Table 2, we present the results when we consider different harmonic orders for the same set of parameter values, i.e. $c = 0.4$, $F = 0.380$, $\Omega = 0.8$, and $\hbar = -1$.

Table 2 illustrates that only after considering at least four frequencies do we obtain a real solution with A_0 close to 0.17, which is the constant term in the asymmetric solution as seen in Fig. 5(b). This means that we need at least four frequencies to capture the asymmetric solution for the given parameters. All the choices with at least Ω considered can be used to obtain the symmetric period-one solution with satisfactory accuracy.

Table 2: Solutions when different number of frequencies are considered

Studied Freq.	Unknowns	Real Solutions
0, Ω	(A_0, B_0, ϕ_1)	(0.303, 0.778, 2.427), (0, 0.913, -2.264), (-0.983, 0.147, 3.017)
0, Ω , 2Ω	$(A_0, B_0, \phi_1, C_0, \phi_2)$	(0, 0.913, -2.264, 0, -), (-0.983, 0.149, 3.268, 0.007, 3.838)
0, Ω , 3Ω	$(A_0, B_0, \phi_1, D_0, \phi_3)$	(0.310, 0.776, 2.424, 0.019, 3.972), (0, 0.921, -2.240, 0.032, 3.422), (0.984, 0.147, 3.017, 0.0001, 5.786)
0, Ω , 2Ω , 3Ω	$(A_0, B_0, \phi_1, C_0, \phi_2, D_0, \phi_3)$	(0.177, 0.883, 4.014, 0.068, 5.102, 0.024, 2.737), (0, 0.921, -2.240, 0, -, 0.031, 3.422), (-0.983, 0.149, 3.268, 0.007, 3.838, 0.0003, 0.396)

Table 3: Accuracy with different number of frequencies considered

No.	Studied Freq.	Err
1	Ω (1st-order HAM)	0.0480
2	Ω (3rd-order HAM)	0.0111
3	0, Ω	0.0882
4	0, Ω , 2Ω	0.0760
5	0, Ω , 3Ω	0.0028
6	0, Ω , 2Ω , 3Ω	0.0028

The errors of standard HAM and MFHAM approximations of the same symmetric limit cycle in Fig. 6(a) with different frequencies considered are investigated. The parameters used are $c = 0.4$, $F = 0.38$ and $\Omega = 0.8$. The results are listed in Table 3. The error measures the difference between the approximations and the numerical result obtained by AUTO and are calculated as

$$Err = \sqrt{\int_0^{2\pi/\Omega} (x(t) - x^*(t))^2 dt}, \quad (48)$$

where $x(t)$ denotes an analytical approximation and $x^*(t)$ denotes the numerical symmetric periodic solution from AUTO.

The overall trend in Table 3 is that considering more frequencies results in better accuracy. It should be noted that No. 3 and No. 4 choices are less accurate than No. 1, because for this symmetric solution which does not contain a constant term or a harmonic term with frequency of 2Ω , considering 0 or 2Ω makes the particular solution of $x_1(t)$ distorted despite that B_0 and ϕ_1 obtained under these three choices are the same. But still, No. 4 is more precise than No. 3 because one more frequency is taken into account for No. 4. Once the correct frequency, like 3Ω , with which a harmonic term exists in the symmetric solution, is taken into consideration, the first-order MFHAM approximation for No. 5 or No. 6 provides much better accuracy than does the third-order standard HAM.

5. Investigation of Period-Doubling Bifurcation Using MFHAM

For period-doubling bifurcations of the dynamic system represented by Eq. (1), a constant term will remain in the periodic solution. Seven frequencies, 0, $\frac{\Omega}{2}$, Ω , $\frac{3\Omega}{2}$, 2Ω , $\frac{5\Omega}{2}$ and 3Ω are taken into consideration.

Hence the characteristic polynomial of the auxiliary differential operator

$$P(\lambda) = \lambda \prod_{r=1}^6 \left(\lambda + \frac{r\Omega}{2}i \right) \left(\lambda - \frac{r\Omega}{2}i \right), \quad (49)$$

and we shall adopt the following corresponding auxiliary differential operator in the proposed procedure.

$$\begin{aligned} \mathcal{L}_7(x) = & x^{(13)} + \frac{91\Omega^2}{4}x^{(11)} + \frac{3003\Omega^4}{16}x^{(9)} + \frac{44473\Omega^6}{64}x^{(7)} \\ & + \frac{37037\Omega^8}{32}x^{(5)} + \frac{48321\Omega^{10}}{64}x^{(3)} + \frac{2025\Omega^{12}}{16}\dot{x}. \end{aligned} \quad (50)$$

We construct the following homotopy.

$$\mathcal{L}_7(x) = q \left[\mathcal{L}_7(x) + \hbar(\ddot{x} + c\dot{x} + x - x^3 - F \cos \Omega t) \right], \quad (51)$$

where $q \in [0, 1]$ is the homotopy parameter.

Assuming the solution of Eq. (51) can be expanded as

$$x = x_0 + qx_1 + q^2x_2 + \dots \quad (52)$$

Substitute Eq. (52) into Eq. (51), and then collect the terms with the identical powers of q , the following is obtained.

$$q^0 : \mathcal{L}_7(x_0) = 0; \quad (53)$$

$$q^1 : \mathcal{L}_7(x_1) = \mathcal{L}_7(x_0) + \hbar(\ddot{x}_0 + c\dot{x}_0 + x_0 - x_0^3 - F \cos \Omega t). \quad (54)$$

The solution to Eq. (53) is

$$\begin{aligned} x_0 = & A_0 + F_0 \cos\left(\frac{\Omega}{2}t + \phi_0\right) + B_0 \cos(\Omega t + \phi_1) + G_0 \cos\left(\frac{3\Omega}{2}t + \phi_4\right) \\ & + C_0 \cos(2\Omega t + \phi_2) + H_0 \cos\left(\frac{5\Omega}{2}t + \phi_5\right) + D_0 \cos(3\Omega t + \phi_3), \end{aligned} \quad (55)$$

where $A_0, B_0, C_0, D_0, F_0, G_0, H_0$ and $\phi_r, r = 0, 1, \dots, 5$, are to be determined.

After substituting Eq. (55) and Eq. (53) into Eq. (54), eliminating secular terms in $x_1(t)$ yields thirteen equations, and with these thirteen equations, we can obtain real solutions for the thirteen unknown parameters in Eq. (55). If the resultant MFHAM-approximation is accurate enough, then these thirteen high-order equations can be used to approximate the relationship between the harmonic components of the period-2 solution and the system parameters.

Therefore, following the same procedure as presented in section 3 the right-hand side of Eq. (54) will become entirely known, and consequently $x_1(t)$ can be easily obtained by solving Eq. (54). Then the first-order MFHAM-approximation of x can be given by

$$\begin{aligned} x &= \lim_{q \rightarrow 1} (x_0 + qx_1) \\ &= x_0 + x_1. \end{aligned} \quad (56)$$

Given $c = 0.4$, $F = 0.386$, $\Omega = 0.8$, and $\hbar = -1$ the limit cycles and harmonic components from simulation and the 1st order MFHAM-approximation are compared in Fig. 8, which agree satisfactorily with each other. The small difference between the two limit cycles is due to lack of harmonic terms with frequency higher than 9Ω in the 1st order MFHAM-approximation.

Figure 9 illustrates variation of F_0, G_0 and H_0 with excitation amplitude F in the MFHAM analysis. When F is smaller than 0.3801, F_0, G_0 and H_0 are all zero. When F becomes larger than 0.3801, F_0, G_0 and H_0 start to have nonzero solutions, which suggests that a period-doubling bifurcation occurs at $F = 0.3801$. The period-doubling point detected by AUTO lies at $F = 3.82540$. The relative error is 0.61%. Therefore, the proposed method can provide a satisfactory estimation of the period-doubling bifurcation point.

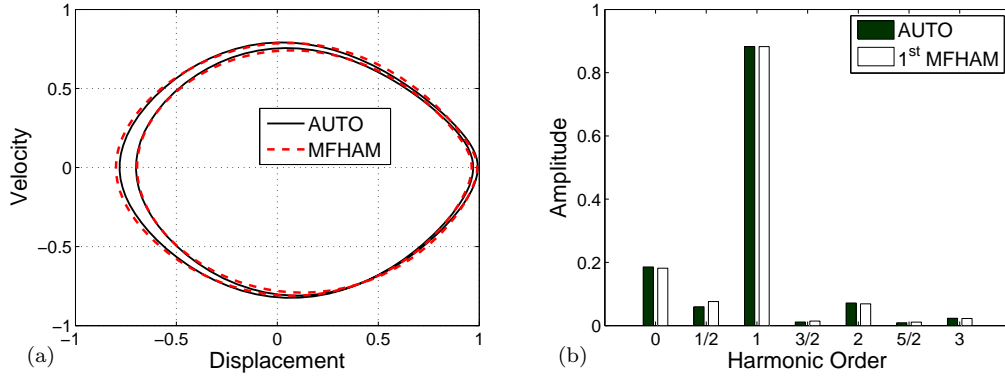


Fig. 8. Comparison of limit cycles after a period-doubling bifurcation. (a) Limit cycles with $F = 0.386$; (b) harmonic components with $F = 0.386$.

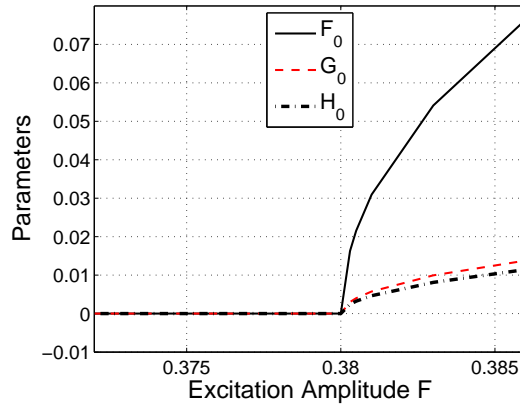


Fig. 9. Variation of F_0 , G_0 and H_0 versus F .

Only the driven frequency is considered by the standard HAM, while the example in this section shows that the frequency component at a frequency lower than the driven frequency, which is the key to analyze period- n solutions, can be computed by the proposed method with satisfactory accuracy. The accuracy of the proposed MFHAM for period-doubling bifurcation can also be improved by increasing the number of considered frequencies. Following the similar procedure, period-4 solutions, period-8 solutions, and so forth, can be captured by the proposed method as long as a big enough number of frequencies are taken into consideration.

6. Conclusions

A new modification of the homotopy analysis method is proposed such that more than one secular term corresponding to different harmonic orders must be eliminated. This so-called multi-frequency homotopy analysis method can be used to capture asymmetric solutions, period-2 solutions and deducibly periodic solutions of even higher period. Thus, the proposed method can be used to investigate symmetry-breaking bifurcation and period-doubling bifurcation. The proposed method has been applied to two well-known bifurcation examples. Its ability to predict symmetry-breaking and period-doubling bifurcations has been verified, and the accuracy is satisfactory. Symmetric, asymmetric, period-one and period-2 solutions can be

obtained by the proposed method simultaneously for a same set of parameters. A set of high-order algebraic equations is required to approximate the relationship between the harmonic components of an asymmetric solution or a period-2 solution and the system parameters. The higher the order of the subharmonic solution, the higher the necessary number of equations and their orders.

It has been shown that the standard HAM can not capture asymmetric or period-2 solutions no matter how high the approximation order is, while the proposed method can with a first-order approximation. Given that the harmonic components with the considered frequencies exist in the periodic solution of interest, increasing the number of considered frequencies in the proposed method generally will improve the accuracy of the approximation and enhance the ability to detect periodic solutions of higher period.

7. Acknowledgments

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Appendix A. Several Long Expressions

A full expansion of Eq. (32) before secular term elimination is presented in Eq. A.1. The full expression for Eq. (32) after secular term elimination is provided in Eq. A.2.

$$\begin{aligned}
\mathcal{L}_4(x_1) = & -\hbar H(t) \left\{ -4A_0^3 - 6A_0B_0^2 - 6A_0D_0^2 - 3B_0^2C_0 \cos(2\phi_1 - \phi_2) + 4A_0 \right. \\
& - 6A_0C_0^2 - 4F \cos \Omega t - 6B_0C_0D_0 \cos(\phi_1 - \phi_3 + \phi_2) - 4c\Omega B_0 \sin(\Omega t + \phi_1) \\
& - 3B_0^2D_0 \cos(\Omega t - 2\phi_1 + \phi_3) - 12A_0B_0C_0 \cos(\Omega t - \phi_1 + \phi_2) \\
& - 12A_0C_0D_0 \cos(\Omega t + \phi_3 - \phi_2) - 3D_0C_0^2 \cos(\Omega t - \phi_3 + 2\phi_2) \\
& + (4B_0 - 4\Omega^2 B_0 - 12A_0^2B_0 - 3B_0^3 - 6B_0D_0^2 - 6B_0C_0^2) \cos(\Omega t + \phi_1) \\
& - 6A_0B_0^2 \cos(2\Omega t + 2\phi_1) - 12A_0B_0D_0 \cos(2\Omega t - \phi_1 + \phi_3) \\
& + (4C_0 - 16\Omega^2 C_0 - 12A_0^2C_0 - 6B_0^2C_0 - 6D_0^2C_0 - 3C_0^3) \cos(2\Omega t + \phi_2) \\
& - 12A_0B_0C_0 \cos(3\Omega t + \phi_1 + \phi_2) - 6B_0C_0D_0 \cos(2\Omega t + \phi_1 + \phi_3 - \phi_2) \\
& + (4D_0 - 36\Omega^2 D_0 - 12A_0^2D_0 - 6B_0^2D_0 - 3D_0^3 - 6D_0C_0^2) \cos(3\Omega t + \phi_3) \\
& - B_0^3 \cos(3\Omega t + 3\phi_1) - 12c\Omega D_0 \sin(3\Omega t + \phi_3) - 8c\Omega C_0 \sin(2\Omega t + \phi_2) \\
& - 3B_0C_0^2 \cos(3\Omega t - \phi_1 + 2\phi_2) + 3A_0B_0D_0 \cos(4\Omega t + \phi_1 + \phi_3) \\
& + \frac{3}{4}B_0^2D_0 \cos(5\Omega t + 2\phi_1 + \phi_3) + \frac{3}{2}A_0D_0^2 \cos(6\Omega t + 2\phi_3) \\
& + \frac{3}{4}B_0D_0^2 \cos(5\Omega t - \phi_1 + 2\phi_3) + \frac{3}{4}B_0D_0^2 \cos(7\Omega t + \phi_1 + 2\phi_3) \\
& + \frac{1}{4}D_0^3 \cos(9\Omega t + 3\phi_3) + \frac{3}{4}B_0^2C_0 \cos(4\Omega t + 2\phi_1 + \phi_2) \\
& + 3A_0C_0D_0 \cos(5\Omega t + \phi_2 + \phi_3) + \frac{1}{4}C_0^3 \cos(6\Omega t + 3\phi_2) \\
& + \frac{3}{2}B_0C_0D_0 \cos(4\Omega t - \phi_1 + \phi_2 + \phi_3) + \frac{3}{2}B_0C_0D_0 \cos(6\Omega t + \phi_1 + \phi_2 + \phi_3) \\
& + \frac{3}{4}D_0^2C_0 \cos(4\Omega t + 2\phi_3 - \phi_2) + \frac{3}{4}D_0^2C_0 \cos(8\Omega t + 2\phi_3 + \phi_2) \\
& + \frac{3}{2}A_0C_0^2 \cos(4\Omega t + 2\phi_2) + \frac{3}{4}B_0C_0^2 \cos(5\Omega t + \phi_1 + 2\phi_2) \\
& \left. + \frac{3}{4}D_0C_0^2 \cos(7\Omega t + \phi_3 + 2\phi_2) \right\}. \tag{A.1}
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_4(x_1) = & -\hbar H(t) \left\{ 3A_0 B_0 D_0 \cos(4\Omega t + \phi_1 + \phi_3) + \frac{3}{4} B_0^2 D_0 \cos(5\Omega t + 2\phi_1 + \phi_3) \right. \\
& + \frac{1}{4} C_0^3 \cos(6\Omega t + 3\phi_2) + \frac{3}{4} B_0 D_0^2 \cos(5\Omega t - \phi_1 + 2\phi_3) \\
& + \frac{3}{4} B_0 D_0^2 \cos(7\Omega t + \phi_1 + 2\phi_3) + \frac{1}{4} D_0^3 \cos(9\Omega t + 3\phi_3) \\
& + \frac{3}{4} B_0^2 C_0 \cos(4\Omega t + 2\phi_1 + \phi_2) + 3A_0 C_0 D_0 \cos(5\Omega t + \phi_2 + \phi_3) \\
& + \frac{3}{2} B_0 C_0 D_0 \cos(4\Omega t - \phi_1 + \phi_2 + \phi_3) + \frac{3}{2} B_0 C_0 D_0 \cos(6\Omega t + \phi_1 + \phi_2 + \phi_3) \\
& + \frac{3}{2} A_0 D_0^2 \cos(6\Omega t + 2\phi_3) + \frac{3}{4} D_0^2 C_0 \cos(4\Omega t + 2\phi_3 - \phi_2) \\
& + \frac{3}{4} D_0^2 C_0 \cos(8\Omega t + 2\phi_3 + \phi_2) + \frac{3}{2} A_0 C_0^2 \cos(4\Omega t + 2\phi_2) \\
& \left. + \frac{3}{4} + B_0 C_0^2 \cos(5\Omega t + \phi_1 + 2\phi_2) + \frac{3}{4} D_0 C_0^2 \cos(7\Omega t + \phi_3 + 2\phi_2) \right\}. \tag{A.2}
\end{aligned}$$

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