

RICE UNIVERSITY

**Spectral characteristics of aperiodic CMV and
Schrödinger operators**

by

Darren C. Ong

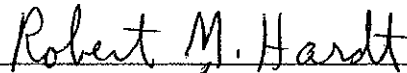
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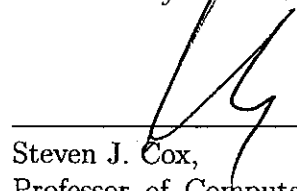
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Abstract

Spectral characteristics of aperiodic CMV and Schrödinger
operators

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This PhD. thesis consists of theorems concerning the spectral theory of CMV and Schrödinger operators that are either almost periodic, derived from low-complexity sequence subshifts, or perturbations of periodic operators. Parts of this thesis are joint work with Milivoje Lukic or Paul Munger.

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Finally, in the tradition of Johann Sebastian Bach, I will affix the words *Soli Deo Gloria* on this text as a testament to the divine hand that is at work in all things.

Dedicated to mummy and daddy.

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Chapter 1

Introduction

1.1 Background

First, some notational conventions. Throughout \mathbb{D} will refer to the open unit disk centered at 0, and \mathbb{N} will refer to the non-negative integers.

1.1.1 The CMV operator

We shall begin with a brief explanation of the CMV operator. A more thorough exposition of the theory can be found in [Sim04]. Let \mathbb{D} be the open unit disk centered at 0. We consider a sequence $\{\alpha(n)\} \in \mathbb{D}$ that runs through n in either \mathbb{Z} or \mathbb{N} . We shall refer to these as *Verblunsky coefficients*, although they are also known as *Schur coefficients* or *reflection coefficients*. We also define $\rho(n)$ as $\sqrt{1 - |\alpha(n)|^2}$.

The *CMV operator*, \mathcal{C} is a unitary linear operator from $\ell^2(\mathbb{N})$ to $\ell^2(\mathbb{N})$ that can be expressed as follows:

$$\mathcal{C} = \begin{pmatrix} \overline{\alpha(0)} & \overline{\alpha(1)}\rho(0) & \rho(1)\rho(0) & 0 & 0 & \dots \\ \rho(0) & -\overline{\alpha(1)}\alpha(0) & -\rho(1)\alpha(0) & 0 & 0 & \dots \\ 0 & \overline{\alpha(2)}\rho(1) & -\overline{\alpha(2)}\alpha(1) & \overline{\alpha(3)}\rho(2) & \rho(3)\rho(2) & \dots \\ 0 & \rho(2)\rho(1) & -\rho(2)\alpha(1) & -\overline{\alpha(3)}\alpha(2) & -\rho(3)\alpha(2) & \dots \\ 0 & 0 & 0 & \overline{\alpha(4)}\rho(3) & -\overline{\alpha(4)}\alpha(3) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \quad (1.1)$$

We also consider the extended CMV operator, which is instead a unitary linear operator from $\ell^2(\mathbb{Z})$ to $\ell^2(\mathbb{Z})$. This operator is denoted as \mathcal{E} , and is expressed as

$$\mathcal{E} = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & -\overline{\alpha(0)}\alpha(-1) & \overline{\alpha(1)}\rho(0) & \rho(1)\rho(0) & 0 & 0 & \dots \\ \dots & -\rho(0)\alpha(-1) & -\overline{\alpha(1)}\alpha(0) & -\rho(1)\alpha(0) & 0 & 0 & \dots \\ \dots & 0 & \overline{\alpha(2)}\rho(1) & -\overline{\alpha(2)}\alpha(1) & \overline{\alpha(3)}\rho(2) & \rho(3)\rho(2) & \dots \\ \dots & 0 & \rho(2)\rho(1) & -\rho(2)\alpha(1) & -\overline{\alpha(3)}\alpha(2) & -\rho(3)\alpha(2) & \dots \\ \dots & 0 & 0 & 0 & \overline{\alpha(4)}\rho(3) & -\overline{\alpha(4)}\alpha(3) & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \quad (1.2)$$

1.1.2 Orthogonal polynomials on the unit circle

The foremost application of the theory of CMV operators is that it allows us to apply spectral theoretic methods to the study of orthogonal polynomials on the unit circle.

We shall sketch the main ideas of this connection. The interested reader might wish

to consult [Sim04] for a fuller explanation.

Let us consider a probability measure μ on $\partial\mathbb{D}$, the unit circle centered at 0. We shall assume that μ is *nontrivial*, that is, μ is not supported on any finite set. We consider the Hilbert space $L^2(\partial\mathbb{D}, d\mu)$ with inner product

$$\langle f, g \rangle = \int_{\partial\mathbb{D}} \overline{f(z)}g(z)d\mu(z).$$

Using the basis $\{1, z, z^2, \dots\}$, we perform a Gram-Schmidt orthonormalization and obtain a sequence of polynomials $\Phi_n(z)$ of degree n . We will usually consider instead the normalized polynomials $\varphi_n = \Phi_n/||\Phi_n||$.

For a polynomial $f_n(z)$ of degree n , let us define

$$f_n^*(z) = z^n \overline{f_n(1/\bar{z})}.$$

The polynomials φ obey a recursion relation known as the *Szegő recurrence*:

$$z\varphi_n(z) = \rho(n)\varphi_{n+1}(z) + \overline{\alpha(n)}\varphi_n^*(z), \tag{1.3}$$

where, again, the $\alpha(n)$ are in \mathbb{D} , and $\rho(n) = \sqrt{1 - |\alpha(n)|^2}$. We also set an initial condition $\alpha(-1) = -1$. Thus starting from a nontrivial probability measure μ on $\partial\mathbb{D}$, we have produced a sequence $\{\alpha(n)\}, n \in \mathbb{N}$ of terms in \mathbb{D} . Let \mathfrak{F} be the map from the set of nontrivial probability measures on $\partial\mathbb{D}$ to $\times_{j=0}^{\infty}\mathbb{D}$ thus described. We then can express a fundamental theorem of the theory of orthogonal polynomials on the unit circle:

Theorem (Verblunsky's Theorem). \mathfrak{F} is a bijection.

There are four proofs of this theorem in [Sim04]. Of particular note is the one expressed as Theorem 4.2.8 of that book, which directly uses the CMV operator. The

key to this application, along with all other applications of the CMV operator to this theory of orthogonal polynomials, is the realization that μ is the spectral measure of the CMV operator whose Verblunsky coefficients are the recurrence coefficients of the Szegő recurrence generated by μ .

We sketch first the relationship between the $\{\alpha(n)\}$ and the CMV operator. Full details are explained in Section 4.2 of [Sim04]. As we mentioned before, the CMV operator on $\ell^2(\mathbb{N})$ is unitarily equivalent to the multiplication by z operator on $L^2(d\mu, \partial\mathbb{D})$. We consider the basis $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$ of $L^2(d\mu, \partial\mathbb{D})$, and perform a Gram-Schmidt orthogonalization to obtain a basis χ . We can calculate that

$$\chi_{2n} = z^{-n}\varphi_{2n}^*, \chi_{2n-1} = z^{-n+1}\varphi_{2n-1}.$$

Furthermore, we can verify that

$$\begin{aligned} \langle \chi_{2n-2}, z\chi_{2n} \rangle &= \rho_{2n-1}\rho_{2n-2}, & \langle \chi_{2n}, z\chi_{2n} \rangle &= -\overline{\alpha_{2n}}\alpha_{2n-1}, \\ \langle \chi_{2n-1}, z\chi_{2n} \rangle &= -\alpha_{2n-2}\rho_{2n-1}, & \langle \chi_{2n+1}, z\chi_{2n} \rangle &= -\alpha_{2n-1}\rho_{2n}, \\ \langle \chi_{2n-2}, z\chi_{2n-1} \rangle &= \overline{\alpha_{2n-1}}\rho_{2n-2}, & \langle \chi_{2n}, z\chi_{2n-1} \rangle &= \overline{\alpha_{2n}}\rho_{2n-1}, \\ \langle \chi_{2n-1}, z\chi_{2n-1} \rangle &= -\overline{\alpha_{2n-1}}\alpha_{2n-2}, & \langle \chi_{2n+1}, z\chi_{2n-1} \rangle &= \rho_{2n}\rho_{2n-1}, \end{aligned}$$

with all other entries zero. This is where we derive the matrix form of the CMV operator.

Let us consider the simplest example, that where μ is normalized Lebesgue measure on the unit circle (that is, $\mu(\theta) = d\theta/2\pi$). In this case, $\{1, z, z^2, \dots\}$ are already orthogonal, and thus our orthogonal polynomials are simply $\varphi_n(z) = z^n$. When we

apply (1.3), we calculate that $\alpha(n) = 0$ for all n , and thus $\rho(n) = 1$ for all n . Our CMV matrix is thus (recall that $\alpha(-1) = -1$

$$\begin{pmatrix} 0 & 0 & 1 & & & & & & \\ 1 & 0 & 0 & & & & & & \\ & 0 & 0 & 0 & 1 & & & & \\ & 1 & 0 & 0 & 0 & & & & \\ & & & 0 & 0 & 0 & 1 & & \\ & & & & 1 & 0 & 0 & 0 & \\ & & & & & \vdots & & & \end{pmatrix}.$$

The Verblunsky coefficients are related to the moments of μ as well in the following way. Given this measure μ on the unit circle, we can define

$$F(z) = \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta).$$

Where $c_n = \int_0^{2\pi} e^{-in\theta} d\mu(\theta)$ is the n th moment of μ , it is straightforward to calculate that

$$F(z) = 1 + 2 \sum_{n=1}^{\infty} c_n z^n.$$

It is easy to verify that $F(z)$ is a Carathéodory function. That is, it is an analytic function on \mathbb{D} that maps to the right half plane, and so that $F(0) = 1$.

We can also define

$$f(z) = \frac{1 - F(z)}{z F(z) + 1}.$$

Here f is a Schur function, that is, it is an analytic function on \mathbb{D} such that $\sup_{z \in \mathbb{D}} |f(z)| \leq 1$.

The connection between F, f and the Verblunsky coefficients of μ proceeds as follows. We define $\gamma_0 = f_0(0)$, a Schur function $f_1(z)$ that satisfies

$$f(z) = \frac{\gamma_0 + zf_1(z)}{1 + \overline{\gamma_0}zf_1(z)}.$$

We can continue by defining $\gamma_n = f_n(0)$ for $n \geq 1$, and defining f_{n+1} inductively using

$$f_n(z) = \frac{\gamma_n + zf_{n+1}(z)}{1 + \overline{\gamma_n}zf_{n+1}(z)}.$$

This process is known as the Schur algorithm. If μ is a nontrivial measure on $\partial\mathbb{D}$, the γ_n will all lie in \mathbb{D} . In that case, we have

Theorem (Geronimus' Theorem). *Let μ be a nontrivial measure on $\partial\mathbb{D}$. Let $\{\alpha(n)\}$, $n \geq 0$ be the sequence of Verblunsky coefficients generated using the Szegő recursion. Let γ_n be the coefficients generated by the Schur function f corresponding to μ . We then have $\gamma_n = \alpha(n)$.*

This gives us a bijection between sequences of Verblunsky coefficients and the functions $\{F, f\}$, and hence also to the moments c_n .

1.1.3 Dynamically generated Verblunsky coefficients

Consider $(\Omega, d\beta)$, a probability measure space and let $\mathcal{T} : \Omega \rightarrow \Omega$ be an invertible measure-preserving ergodic transformation and $f : \Omega \rightarrow \mathbb{D}$ a nonconstant measurable function. We can then generate Verblunsky coefficients associated to $(\Omega, d\beta, \mathcal{T}, f)$ as

random variables on Ω :

$$\alpha_\omega(n) = f(\mathcal{T}^n \omega). \quad (1.4)$$

Thus for each $\omega \in \Omega$ we have a corresponding extended CMV operator \mathcal{E}_ω and its corresponding spectral measures on $\partial\mathbb{D}$. Let us now enumerate a few examples:

Limit-periodic Verblunsky coefficients

As an example, let Ω be a *Cantor group*, that is, a totally disconnected compact Abelian topological group with no isolated points. We say that a map $\mathcal{T} : \Omega \rightarrow \Omega$ is a translation if it is of the form $\mathcal{T}\omega = \omega + a$ for some $a \in \Omega$. The α_ω s generated in this setting are *limit-periodic sequences*, that is, they are uniform limits of periodic sequences. Avila in [Avi09] first expressed limit-periodic sequences in this form to prove theorems concerning limit-periodic Schrödinger operators.

Quasiperiodic shifts

Let Ω be a d -dimensional torus $(\mathbb{R}/\mathbb{Z})^d$, and let $\mathcal{T} : \Omega \rightarrow \Omega$ be a minimal translation. Then if f is continuous α_ω will be a quasiperiodic sequence.

Sequence subshifts

Given a 1-sided infinite sequence u , the *subshift* Ω generated by u is the set of all two-sided sequences all of whose subblocks are also subblocks of u . In these examples, \mathcal{T} is typically a right shift, and $f(\omega)$ only depends on the 0th letter of the word $\omega \in \Omega$.

To define a Sturmian sequence, we choose a rotation angle $\theta \in [0, 1)$, and let a_n denote the coefficients in its continued fraction expansion. That is,

$$\theta = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ddots}}.$$

We also define p_n/q_n to be the n th rational approximation of θ . Then the q_n are integers that satisfy

$$q_{n+1} = a_n q_n + q_{n-1}, \tag{1.5}$$

with initial conditions $q_0 = 0, q_1 = 1$.

Let f be a nonconstant function from $\{0, 1\}$ to \mathbb{D} . We then define our Verblunsky coefficients of our extended CMV matrix $\mathcal{E}_{\theta, \beta}$ by

$$\alpha_{\theta, \beta}(n) = f(\chi_{[1-\theta, 1]}(n\theta + \beta \pmod{1})).$$

An important special case is the Fibonacci subshift, where θ is equal to the golden mean. The CMV operator generated by the Fibonacci subshift has applications in the theory of quantum walks (see [DMY13a],[DMY13b]).

As for codings of rotations, we instead choose an interval $I \subseteq \mathbb{R}/\mathbb{Z}$, and define our Verblunsky coefficients by

$$\alpha_{\theta, \beta}(n) = f(\chi_I(n\theta + \beta \pmod{1})).$$

Another example is that of the period doubling subshift. Consider an alphabet $\mathcal{A} = \{a, b\}$, and the set of words \mathcal{A}^* using letters in \mathcal{A} . Define $S : \mathcal{A}^* \rightarrow \mathcal{A}^*$ by $S(b) = aa, S(a) = ab$ and such that if $w = c_1 \dots c_l$ with $c_j \in \mathcal{A}$, then $S(w) = S(c_1) \dots S(c_l)$. Then we define u to be the invariant limit $u = S^\infty(a)$.

In all these cases, by standard results in ergodic theory (see for instance Theorem 10.5.21 of [Sim04]), the spectrum Σ_θ of $\mathcal{E}_{\theta, \beta}$ is independent of β .

1.1.4 Examples

Here we highlight some examples of CMV operators and their corresponding spectral measures. Section 1.6 of [Sim04] includes several other standard examples.

Constant Verblunsky coefficients

If we let $\alpha(n)$ all be equal to a constant α , The absolutely continuous part of the measure will be supported on the arc from $\theta_{|\alpha|}$ to $2\pi - \theta_{|\alpha|}$, where $\theta_{|\alpha|} = 2 \arcsin(|\alpha|)$. There will also sometimes be a pure point in the arc that isn't on the supporting set of the measure, depending on the choice of α . The $\ell^2(\mathbb{Z})$ CMV operator corresponds to the same measure on the unit circle, except that there will be no pure points. This is a consequence of the Gordon lemma.

Periodic Verblunsky coefficients

This case is covered in detail in Sections 11 of [Sim04]. For both the $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ CMV operator with q -periodic coefficients, we get corresponding spectral measures that consist of q absolutely continuous intervals on the circle. These intervals may intersect in at most one point. In addition, for the $\ell^2(\mathbb{N})$ version of the operator, there may be pure points away from the interiors of these q bands, either one or zero pure points in each gap between a pair of neighboring band interiors. For the $\ell^2(\mathbb{Z})$ version of the CMV operator, these pure points do not exist.

Almost periodic Verblunsky coefficients

Take f to be a nonconstant function from $\{0, 1\}$ to \mathbb{D} , and let $\alpha(n) = f(n\theta + \beta \pmod{1})$ for an irrational number θ . These are known as Verblunsky coefficients generated by quasiperiodic shifts. For both the $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ versions of the CMV operator, for almost every $\beta \in [0, 1)$ the corresponding spectral measures contain no absolutely continuous part, and the singular continuous part consists of a Cantor set. By Theorem 1.2 of this thesis, we also know that the $\ell^2(\mathbb{Z})$ operator has empty point spectrum for almost every β .

1.2 Summary of results

We are in general interested in understanding the relationship between the Verblunsky coefficients $\alpha(n)$ and the corresponding probability measure μ . Most of the theorems listed in this section will possess that flavor.

We will also note here that there is a strong connection between the CMV operator and the Jacobi operator. Indeed, one may view the CMV operator as the unitary analogue of the Jacobi operator on $\ell^2(\mathbb{Z})$ or $\ell^2(\mathbb{N})$. Not a few of the results to follow are motivated by corresponding results in the theory of Jacobi operators or discrete Schrödinger operators.

1.2.1 Gordon lemma

The Gordon lemma was discovered in [Gor76], for the Schrödinger operator. Assuming that the potential sequence is close to periodic in a precise sense, we can use this

lemma to prove absence of pure point spectrum.

We show that we can apply similar ideas to the case where the Verblunsky coefficients are close to periodic in the same way. Using this Gordon lemma for the extended CMV operator, we can rule out point spectra for several classes of CMV operators. Some of the theorems we prove that rely on these methods are as follows.

Theorem 1.1 ([Ong12]). *Let Ω be a Cantor group, $\omega \in \Omega$, and \mathcal{T} be a shift. Let \mathcal{E}_f be the CMV operator corresponding to dynamically generated Verblunsky coefficients generated by $(\Omega, \mathcal{T}, f, \omega)$.*

1. *There exists a dense G_δ set $\mathcal{G} \subseteq C(\Omega, \mathbb{D})$ such that for every $f \in \mathcal{G}$ and $\omega \in \Omega$, the extended CMV operator, \mathcal{E}_f has empty point spectrum.*
2. *There exists a dense G_δ set $\mathcal{R} \subseteq C(\Omega, \mathbb{D})$ such that for every $f \in \mathcal{R}$ and $\omega \in \Omega$, the spectrum of the extended CMV operator, \mathcal{E}_f is a Cantor set.*
3. *There exists a dense set $\mathcal{R}' \subseteq C(\Omega, \mathbb{D})$ such that for every $f \in \mathcal{R}'$ and $\omega \in \Omega$, the spectrum of the extended CMV operator, \mathcal{E} is a Cantor set of positive Lebesgue measure and all spectral measures are purely absolutely continuous.*

Theorem 1.2 ([Ong13a]). *For almost every $\omega \in \Omega$ and T either a minimal shift on $\Omega = \mathbb{T}^d$ or a skew-shift on \mathbb{T}^2 given by $T(\omega_1, \omega_2) = (\omega_1 + 2a, \omega_1 + \omega_2)$, there exists a dense G_δ set of sampling functions in $C(\Omega, \mathbb{D})$ for which the corresponding CMV operator has purely singular continuous spectrum.*

Theorem 1.3 ([Ong13b]). *All CMV operators generated by the period doubling subshift are purely singular continuous.*

Theorem 1.4 ([Ong13b]). *Suppose θ is chosen so that there exist odd a_n for arbitrarily large n . Then the CMV operator corresponding to a Sturmian subshift generated by the rotation number θ is purely singular continuous for Lebesgue almost every phase β .*

Theorem 1.5 ([Ong13b]). *Let I be an interval in \mathbb{R}/\mathbb{Z} and suppose θ is chosen such that $\limsup_{n \rightarrow \infty} a_n \geq 5$ and there exist odd a_n for arbitrarily large n . Then the spectrum of the CMV operator corresponding to a subshift generated by the coding of the rotation θ is purely singular continuous for Lebesgue almost every phase β .*

Theorem 1.6. *Consider a Sturmian subshift, and suppose θ is chosen so that there exists an N such that $a_n = 1$ for all $n > N$ (The Fibonacci subshift satisfies this condition). Then the CMV operator corresponding to a Sturmian subshift generated by the rotation number θ is purely singular continuous for Lebesgue almost every phase β .*

For the following two theorems, we note that the existence of q_n even for arbitrarily large n is equivalent to the existence of a_m odd for arbitrarily large m .

Theorem 1.7. *Consider a Sturmian subshift, and suppose θ is chosen so that there exists an infinite subset $\tilde{\mathbb{N}}$ of \mathbb{N} so that q_n is even for $n \in \tilde{\mathbb{N}}$. Assume further that $\limsup_{n \in \tilde{\mathbb{N}}} a_n \geq 3$. Then the CMV operator corresponding to a Sturmian subshift generated by the rotation number θ is purely singular continuous for Lebesgue almost every phase β .*

Theorem 1.8. *Consider a rotation coding subshift, where I is an interval in \mathbb{R}/\mathbb{Z} . Suppose also that θ is chosen such that there exists an infinite subset $\tilde{\mathbb{N}}$ of \mathbb{N} so that*

q_n is even for $n \in \tilde{\mathbb{N}}$. Assume further that $\limsup_{n \in \tilde{\mathbb{N}}} a_n \geq 5$. Then the spectrum of the CMV operator corresponding to a subshift generated by the coding of the rotation θ is purely singular continuous for Lebesgue almost every phase β .

The analogues of the preceding theorems appear in the Schrödinger setting in [DG11], [BD08], [Dam01],[Kam96].

1.2.2 Power law bounds on solutions.

The results of this subsection are joint work with Paul E. Munger. The goal is to say something about the spectrum of the extended CMV operator \mathcal{E} , using only information about the corresponding one-sided infinite CMV operator \mathcal{C} .

Theorem 1.9 ([MO13]). *Let Σ be a subset of $\partial\mathbb{D}$, and let \mathcal{C} be a CMV operator on $\ell^2(\mathbb{N})$. Suppose there are constants γ_1, γ_2 such that for each $z \in \Sigma$, every normalized solution of $(\mathcal{C}u - zu) = 0$ obeys the estimate*

$$C_1(z)L^{\gamma_1} \leq \|u\|_L \leq C_2(z)L^{\gamma_2}$$

for $L > 0$ sufficiently large. Let $\beta = 2\gamma_1/(\gamma_1 + \gamma_2)$. Then any extension \mathcal{E} to $\ell^2(\mathbb{Z})$ has purely β -continuous spectrum on Σ . Moreover, if $C_1(z)$ and $C_2(z)$ are independent of z , then for any $\varphi \in \ell^2$ of compact support, the spectral measure of (\mathcal{E}, φ) is uniformly β -Hölder continuous on Σ .

This is a CMV analogue for a corresponding result [DKL00] for Schrödinger operators.

We then apply this theorem to CMV operators generated by Sturmian subshifts. This result is actually the second of a trilogy of papers (together with [DFV13] and

[DMY13b]) that together prove dynamical spreading of a certain class of quantum walks.

1.2.3 Decaying oscillatory perturbations

Lukic had produced several theorems concerning the Schrödinger operator with decaying oscillatory potential of the Wigner-von Neumann type. More precisely, in [Luk13a] he considered the following potential function:

$$V(x) = \sum_{k=1}^{\infty} c_k e^{-i\phi_k x} \gamma_k(x), \quad (1.6)$$

with the following conditions:

1. (uniformly bounded variation) $\gamma_k(x) : (0, \infty) \rightarrow \mathbb{C}$ are functions of bounded variation whose variation is bounded uniformly in k , i.e.

$$\sup_k \text{Var}(\gamma_k, (0, \infty)) < \infty; \quad (1.7)$$

2. (uniform L^p condition) for some integer $p \geq 2$,

$$\sup_k \int_0^{\infty} |\gamma_k(x)|^p dx < \infty; \quad (1.8)$$

3. (decay of coefficients) for some $\alpha \in (0, 1/(p-1))$,

$$\sum_{k=1}^{\infty} |c_k|^\alpha < \infty. \quad (1.9)$$

He was able to show that the Schrödinger operator

$$(H_0 u)(x) = -u''(x) + V(x)u(x), \quad (1.10)$$

had absolutely continuous spectrum on $(0, \infty)$. In fact, when the sum in (1.6) is finite, he managed to show that the spectrum of the operator was purely absolutely continuous except for an explicitly determined finite set of points. In the case where the sum in (1.6) is infinite, he instead bounded the Hausdorff dimension of the embedded singular spectrum in $(0, \infty)$. In [Luk11], he also proved version of this theorem in the Jacobi and CMV settings.

In joint work with Milivoje Lukic, we were able to generalize his results to decaying oscillatory perturbations of periodic Schrödinger operators: that is

Theorem 1.10 ([LO13]). *Let the potential V be given by (1.6). Assume that V_0 is periodic, and consider the perturbed periodic Schrödinger operator*

$$(Hu)(x) = -u''(x) + (V(x) + V_0(x))u(x). \quad (1.11)$$

Given the \mathbf{a} in 1.9 we can find a set S of Hausdorff dimension at most $(p-1)\mathbf{a}$, so that on $\text{Int}(\sigma_{ess}(H)) \setminus S$, the spectral measure μ is mutually absolutely continuous with Lebesgue measure, so the absolutely continuous spectrum of H is equal to $\sigma_{ess}(H)$.

Further, in the case where the sum in (1.6) is finite the embedded singular spectrum is pure point with finitely many points in each absolutely continuous band in $\sigma_{ess}(H)$ ($\sigma_{ess}(H)$ is a union of compact intervals, which we refer to as bands), and we can determine the locations of those points explicitly.

We also partially extended this result to the CMV and Jacobi operator settings, by adapting the idea of generalized Prüfer variables of [KRS99]. We end up with a version of Theorem 1.10 for finite frequencies. First, a definition:

Definition 1.11. A sequence $\{\alpha_n\}_{n=N}^\infty$ has generalized bounded variation with the set of phases $A = \{\phi_1, \dots, \phi_L\}$ if it can be expressed as a sum

$$\alpha_n = \sum_{l=1}^L e^{-i\phi_l n} \gamma_n^{(l)}$$

where $L < \infty$ and so that $\gamma^{(l)}$ has bounded variation. The set of sequences with generalized bounded variation with set of phases A will be denoted $GBV(A)$ or $GBV(\phi_1, \dots, \phi_L)$.

We can now state our theorems.

Theorem 1.12. *Consider q -periodic Jacobi coefficients $\{a_n\}, \{b_n\}$, and perturbations a'_n, b'_n of these Jacobi coefficients (i.e. $a_n, a_n + a'_n$ are both positive, and $b_n, b_n + b'_n$ are both real). Let A be a finite set of phases and assume $\{a'_n, b'_n\}$ are in $\ell^p \cap GBV(A)$ for some $p \geq 2$.*

Then the spectral measure of the $\ell^2(\mathbb{N})$ Jacobi operator

$$\begin{pmatrix} b_1 + b'_1 & a_1 + a'_1 & 0 & 0 \\ a_1 + a'_1 & b_2 + b'_2 & a_2 + a'_2 & 0 \\ 0 & a_2 + a'_2 & \ddots & \ddots \\ 0 & 0 & \ddots & \ddots \end{pmatrix}. \quad (1.12)$$

consists of a union of absolutely continuous bands, each of which contains finitely many embedded pure points.

Theorem 1.13. *Consider q -periodic Verblunsky coefficients $\alpha(n)$, $n \in \mathbb{N}$ and a perturbation $\alpha'(n)$ of these Verblunsky coefficients (i.e. $\alpha(n) + \alpha'(n)$ are in \mathbb{D} for every n). Let A be a finite set of phases and assume that $\alpha'(n)$ is in $\ell^p \cap GBV(A)$ for some $p \geq 2$.*

Then the spectral measure of the corresponding CMV operator consists of a union of absolutely continuous bands, each of which contains finitely many pure points.

Chapter 2

Two Gordon lemmas for CMV operators and their applications

In this chapter, we shall prove Theorems 1.1, 1.2, 1.3, 1.6, 1.7 and 1.8. The proofs of all these theorems rely at least partially on an idea initially developed in [Gor76] for the context of Schrödinger operators. A good expository treatment about the use of the Gordon idea in the Schrödinger setting can be found in [Dam00]. We adapt this idea to the CMV context in a few different ways.

The Gordon lemma roughly says that an operator on $\ell^2(\mathbb{Z})$ that is close to periodic in a strong uniform way cannot have eigenvalues. As a simple heuristic as to why this is reasonable, let us consider a q -periodic periodic operator \mathcal{E} on $\ell^2(\mathbb{Z})$. Let us consider a sequence in $\ell^\infty(\mathbb{Z})$ (not necessarily in $\ell^2(\mathbb{Z})$) that solves $\mathcal{E}u = zu$ for some $z \in \partial\mathbb{D}$. It seems reasonable for $|u(n)|$ to be periodic, or for $|u(n)|$ to increase or decrease as n increases. However, for $u(n)$ to be in $\ell^2(\mathbb{Z})$, it would have to decay as $n \rightarrow \infty$ and as $n \rightarrow -\infty$. So there must be some “peak” between the values of

n where u decays at $+\infty$, and where u decays at $-\infty$. But since the operator is symmetric on jumps of size q , it would be odd to see a peak in $u(n)$ that isn't present at $u(n + mq)$ for arbitrary integers m . Thus we can expect that u cannot be in $\ell^2(\mathbb{Z})$ and hence u isn't an eigenvector. We will develop these ideas with more precision below.

2.1 A few versions of the CMV Gordon lemma

In this section, we will state a few versions of the Gordon lemma and provide their proofs. The first subsection uses the transfer matrices discovered by Gesztesy and Zinchenko in [GZ06]. The treatment here is similar to the Gordon lemmas given in [Dam00]. The second subsection is a CMV adaptation of an older version of the Gordon lemma expressed in the Schrödinger setting in [CFKS08]. While this version is less elegant computationally, it is nevertheless useful when the symmetries in the Verblunsky coefficient sequence are less exact.

2.1.1 A Gordon lemma for Gesztesy-Zinchenko transfer matrices

Specify $z \in \partial\mathbb{D}$. For integer n , let us first introduce the transfer matrix T_n described in (2.18) of [GZ06]. It is defined by

$$T_n(z) = \begin{cases} \frac{1}{\rho(n)} \begin{pmatrix} -\overline{\alpha(n)} & z \\ 1/z & -\alpha(n) \end{pmatrix}, & n \text{ odd}, \\ \frac{1}{\rho(n)} \begin{pmatrix} -\alpha(n) & 1 \\ 1 & -\overline{\alpha(n)} \end{pmatrix}, & n \text{ even}. \end{cases} \quad (2.1)$$

Note that the notation in [GZ06] differs slightly from ours. In particular, their $\alpha(n)$ is our $-\overline{\alpha(n)}$.

Let us also define

$$\Theta_n = \begin{pmatrix} \overline{\alpha(n)} & \rho(n) \\ \rho(n) & -\alpha(n) \end{pmatrix}.$$

Where u is a solution to the CMV eigenvalue equation $\mathcal{E}u = zu$, there is a corresponding solution v to the eigenvalue equation $\mathcal{E}^T v = zv$ that satisfies $\mathcal{M}u = zv$, where $\mathcal{M} : \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ is the direct sum of 2×2 matrices of the form Θ_{2k-1} with k running from $-\infty$ to $+\infty$. The $(0,0)$ entry in the matrix representation of \mathcal{M} is $-\alpha(-1)$. The existence of such v is proven in Lemma 2.2 of [GZ06].

The transfer matrices satisfy

$$\begin{pmatrix} u_n(z) \\ v_n(z) \end{pmatrix} = T_n(z) \begin{pmatrix} u_{n-1}(z) \\ v_{n-1}(z) \end{pmatrix}. \quad (2.2)$$

Note that the T_n have determinant -1 . We also define

$$M_n(z) = \begin{cases} T_n(z) \dots T_1(z), & n \geq 1, \\ I, & n = 0, \\ T_{n+1}(z)^{-1} \dots T_0(z)^{-1}, & n \leq -1. \end{cases} \quad (2.3)$$

Proposition 2.1 (Two-block CMV Gordon lemma). *Fix a sequence of Verblunsky coefficients α and fix $z \in \partial\mathbb{D}$. Suppose there is an even whole number sequence $n_k \rightarrow \infty$ and a constant $C \geq 1$ so that for every k*

$$\alpha(j) = \alpha(j + n_k), 1 \leq j \leq n_k, \quad (2.4)$$

and also $|\operatorname{tr} M_{n_k}(z)| \leq C$.

Then z is not an eigenvalue for \mathcal{E} .

Proposition 2.2 (Three-block CMV Gordon lemma). *Fix a sequence of Verblunsky coefficients α and fix $z \in \partial\mathbb{D}$. Suppose there is an even whole number sequence $n_k \rightarrow \infty$ so that for every k*

$$\alpha(j - n_k) = \alpha(j) = \alpha(j + n_k), 1 \leq j \leq n_k. \quad (2.5)$$

Then z is not an eigenvalue for \mathcal{E} .

We will prove both versions simultaneously

Proof of Propositions 2.1 and 2.2. By the Cayley-Hamilton theorem, for any even $n \geq 2$ we have the following matrix equation:

$$M_n(z)^2 - \operatorname{tr}(M_n(z))M_n(z) + I = 0.$$

Note that the conditions we have on the Verblunsky coefficients imply that

$$M_{n_k}(z)^2 = M_{2n_k}(z).$$

Thus we have for any k

$$\begin{pmatrix} u_{2n_k} \\ v_{2n_k} \end{pmatrix} - \operatorname{tr}(M_{n_k}(z)) \begin{pmatrix} u_{n_k} \\ v_{n_k} \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = 0.$$

Let us assume without loss of generality that $|u_0|^2 + |v_0|^2 = 1$. This implies that one of

$$\begin{pmatrix} u_{2n_k} \\ v_{2n_k} \end{pmatrix}, \operatorname{tr}(M_{n_k}(z)) \begin{pmatrix} u_{n_k} \\ v_{n_k} \end{pmatrix},$$

has norm at least $1/2$. Stated another way, we have

$$\max \left(\left\| \begin{pmatrix} u_{2n_k} \\ v_{2n_k} \end{pmatrix} \right\|, \left\| \operatorname{tr}(M_{n_k}(z)) \begin{pmatrix} u_{n_k} \\ v_{n_k} \end{pmatrix} \right\| \right) \geq \frac{1}{2} \min \left(1, \frac{1}{\operatorname{tr}(M_{n_k}(z))} \right).$$

Assume that $\operatorname{tr}(M_{n_k}(z))$ is bounded. Thus the existence of a sequence $\{n_k\}$ so that $\alpha(j) = \alpha(j + n_k)$, $1 \leq j \leq n_k$ implies that the sequence u doesn't decay on the right for any initial values u_0, v_0 . Note that it is not possible for v to be large at $+\infty$ and for u to decay at $+\infty$, due to the fact that $\mathcal{M}u = zv$, and that all the 2×2 blocks that make up \mathcal{M} have determinant 1 and uniformly bounded trace. This all implies that z is not an eigenvalue for the operator \mathcal{E} and so we prove Proposition 2.1. We note here that it actually suffices that $\operatorname{tr}(M_n(z))$ is bounded for n in the sequence $\{n_k\}$.

Now assume instead that $\operatorname{tr}(M_n(z)) > 1$ for $n = m_k$, where the $\{m_k\}$ are a

subsequence of the $\{n_k\}$. Now apply the matrix equation to

$$\begin{pmatrix} u_{-m_k} \\ v_{-m_k} \end{pmatrix},$$

and we obtain

$$\begin{pmatrix} u_{m_k} \\ v_{m_k} \end{pmatrix} - \text{tr}(M_{m_k}(z)) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \begin{pmatrix} u_{-m_k} \\ v_{-m_k} \end{pmatrix} = 0,$$

and given the assumption $|u_0|^2 + |v_0|^2 = 1$, we must have

$$\max \left(\left\| \begin{pmatrix} u_{m_k} \\ v_{m_k} \end{pmatrix} \right\|, \left\| \begin{pmatrix} u_{-m_k} \\ v_{-m_k} \end{pmatrix} \right\| \right) \geq \frac{1}{2}.$$

Again, it is not possible for v to be large at $+\infty$ or $-\infty$ and for u to decay, since $\mathcal{M}u = zv$. Thus the solution u cannot decay at both $\pm\infty$, and we conclude that z is not an eigenvalue for the operator \mathcal{E} . This proves Proposition 2.2. \square

2.1.2 A CMV Gordon lemma á la Cycon, Froese, Kirsch and Simon

We will begin by presenting another sequence of transfer matrices related to \mathcal{E} . Let us first define an ℓ^∞ -eigenvector of \mathcal{E} as an ℓ^∞ vector u that satisfies $zu = \mathcal{E}u$ for some z , but such that u is not necessarily in ℓ^2 . Based on the structure given in (1.1),

we determine that for n odd, an ℓ^∞ -eigenvector u of \mathcal{E} must satisfy

$$\begin{aligned} zu_{n+1} = & \overline{\alpha(n+1)}\rho(n)u_n - \overline{\alpha(n+1)}\alpha(n)u_{n+1} \\ & + \overline{\alpha(n+2)}\rho(n+1)u_{n+2} + \rho(n+2)\rho(n+1)u_{n+3}, \end{aligned} \quad (2.6)$$

$$\begin{aligned} zu_{n+2} = & \rho(n+1)\rho(n)u_n - \rho(n+1)\alpha(n)u_{n+1} \\ & - \overline{\alpha(n+2)}\alpha(n+1)u_{n+2} - \rho(n+2)\alpha(n+1)u_{n+3}. \end{aligned} \quad (2.7)$$

We multiply Equation (2.6) by $\alpha(n+1)$ and Equation (2.7) by $\rho(n+1)$ and then add the two equations to eliminate the u_{n+3} term. This gets us

$$\begin{aligned} z\rho(n+1)u_{n+2} = & (\overline{\alpha(n+1)}\rho(n)\alpha(n+1) + \rho(n+1)^2\rho(n))u_n \\ & - ((\overline{\alpha(n+1)}\alpha(n) + z)\alpha(n+1) + \rho(n+1)^2\alpha(n))u_{n+1}. \end{aligned}$$

Similarly, if we multiply Equation (2.6) by $\overline{\alpha(n+2)}\alpha(n+1) + z$ and Equation (2.7) by $\overline{\alpha(n+2)}\rho(n+1)$, adding the two equations eliminates the u_{n+2} term and we have

$$\begin{aligned} & - z\rho(n+2)\rho(n+1)u_{n+3} \\ = & (\overline{\alpha(n+1)}\rho(n)(\overline{\alpha(n+2)}\alpha(n+1) + z) + \rho(n+1)^2\rho(n)\overline{\alpha(n+2)})u_n \\ & - ((\overline{\alpha(n+1)}\alpha(n) + z)(\overline{\alpha(n+2)}\alpha(n+1) + z) + \rho(n+1)^2\alpha(n)\overline{\alpha(n+2)})u_{n+1}. \end{aligned}$$

These equations give us a two-step transfer matrix

$$A_n = \frac{1}{z\rho(n+1)\rho(n+2)} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

with entries

$$\begin{aligned}
a_{11} &= \rho(n+2)(\overline{\alpha(n+1)}\alpha(n+1)\rho(n) + \rho(n+1)^2\rho(n)), \\
a_{12} &= -\rho(n+2)[(\overline{\alpha(n+1)}\alpha(n) + z)\alpha(n+1) + \rho(n+1)^2\alpha(n)], \\
a_{21} &= -[(\overline{\alpha(n+2)}\alpha(n+1) + z)\overline{\alpha(n+1)}\rho(n) + \rho(n+1)^2\rho(n)\overline{\alpha(n+2)}], \\
a_{22} &= (\overline{\alpha(n+2)}\alpha(n+1) + z)(\overline{\alpha(n+1)}\alpha(n) + z) + \overline{\alpha(n+2)}\rho(n+1)^2\alpha(n).
\end{aligned}$$

This satisfies

$$\begin{pmatrix} u_{n+2} \\ u_{n+3} \end{pmatrix} = A_n \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix}. \quad (2.8)$$

We can compute that $\det A_n = \rho(n)/\rho(n+2)$, which is never zero. Hence A_n is always invertible.

Note 2.3. *It is possible to express the recurrence relation so that the transfer matrices have determinant 1. Let \mathfrak{A}_n be the matrix that results from multiplying the top row of A_n by $\rho(n+2)$ and the left column of A_n by $1/\rho(n)$. Consequently, we can express Equation (2.8) instead as*

$$\begin{pmatrix} \rho(n+2)u_{n+2} \\ u_{n+3} \end{pmatrix} = \mathfrak{A}_n \begin{pmatrix} \rho(n)u_n \\ u_{n+1} \end{pmatrix}, \quad (2.9)$$

where this time the matrix \mathfrak{A}_n has determinant 1. This adjustment is based on a similar trick used to obtain determinant 1 transfer matrices in the OPRL case. We choose not to use this expression for the recurrence relation, since it is not essential for our context that the transfer matrices have determinant 1, and the $\rho(n)$ terms complicate the proof slightly.

Let $\mathbb{D}(r)$ be an open disk of radius $r < 1$ centered at the origin. Note that $a_{11}, a_{12}, a_{21}, a_{22}$ (as well as the fraction $1/z\rho(n+1)\rho(n+2)$) are all continuous functions of $\alpha(n), \alpha(n+1), \alpha(n+2)$. In addition, if we restrict $\alpha(n), \alpha(n+1), \alpha(n+2)$ to $\mathbb{D}(r)$, $a_{11}, a_{12}, a_{21}, a_{22}$ and $1/z\rho(n+1)\rho(n+2)$ are also bounded. There must exist a positive real-valued function $\gamma(k, q, r)$ with $k, q \in \mathbb{Z}_+$ so that if $\{\widetilde{\alpha(i)}\}$ are a sequence of Verblunsky coefficients corresponding to the matrices \widetilde{A}_n , and if $|\widetilde{\alpha(n)} - \alpha(n)|, |\widetilde{\alpha(n+1)} - \alpha(n+1)|, |\widetilde{\alpha(n+2)} - \alpha(n+2)|$ are all less than $\gamma(k, q, r)$, this implies that $\|A_n - \widetilde{A}_n\| < k^{-q}$.

Proposition 2.4. *Let $\alpha(n)$ and $\alpha_k(n)$, $k \in \mathbb{Z}_+$, be two sided-sequences in \mathbb{D}^∞ . Let α_k be periodic with even period q_k . We then know that there must exist $r_k < 1$ so that the sequence $\alpha_k(n)$ lies in $\mathbb{D}(r_k)$, the open disk of radius r_k centered at the origin. If*

$$\sup_{-2q_k+1 \leq n \leq 2q_k+1} |\alpha_k(n) - \alpha(n)| \leq \gamma(k, q_k, r_k),$$

and \mathcal{E} is the extended CMV operator corresponding to Verblunsky coefficients $\alpha(n)$, there is no eigenvector u of \mathcal{E} in $\ell^2(\mathbb{Z})$.

We first introduce the following lemma:

Lemma 2.5. *Let A be an invertible 2×2 matrix, and x a vector of norm 1. Then*

$$\max(\|Ax\|, \|A^2x\|, \|A^{-1}x\|, \|A^{-2}x\|) \geq 1/2.$$

Proof. This proof is in Section 10.2 of [CFKS08]. □

Note 2.6. *This is known as the “four block” version of the Gordon idea. We don’t use the “three block” version of the idea since we don’t want to assume our matrices have determinant 1.*

Proof of Proposition. Let \mathcal{E}_k be the extended CMV operator corresponding to the Verblunsky coefficients $\alpha_k(n)$. Let $w^{(k)}$ be a nonzero ℓ^∞ -eigenvector of \mathcal{E}_k with the same initial condition as the nonzero ℓ^∞ -eigenvector u of \mathcal{E} : i.e. $u_1 = w_1^{(k)}, u_2 = w_2^{(k)}$. Note that u_1, u_2 cannot both be zero, since otherwise the recurrence would imply that u is zero. Define for odd n

$$\phi(n) = \begin{pmatrix} u_n \\ u_{n+1} \end{pmatrix}, \phi_k(n) = \begin{pmatrix} w_n^{(k)} \\ w_{n+1}^{(k)} \end{pmatrix}.$$

and $A_n^{(k)}$, as the transfer matrix of \mathcal{E}_k . Assume for now that the odd number n is positive. In the following calculations, bear in mind that all the suprema only run through odd n , and the sums only run through odd j . Also, let $\Phi = \|\phi(1)\|$.

$$\begin{aligned} & \sup_{1 \leq n \leq 2q_k+1} \|\phi(n) - \phi_k(n)\| \\ & \leq \sup_{1 \leq n \leq 2q_k+1} \|A_{n-2}A_{n-4} \dots A_3A_1 - A_{n-2}^{(k)}A_{n-2}^{(k)} \dots A_3^{(k)}A_1^{(k)}\| \Phi, \\ & = \sup_{1 \leq n \leq 2q_k+1} \left\| \sum_{1 \leq j \leq n-2} A_{n-2} \dots A_j A_{j-2}^{(k)} \dots A_1^{(k)} - A_{n-2} \dots A_{j+2} A_j^{(k)} \dots A_1^{(k)} \right\| \Phi, \\ & \leq \sup_{1 \leq n \leq 2q_k+1} \sum_{1 \leq j \leq n-2} \|A_{n-2} \dots A_{j+2} (A_j - A_j^{(k)}) A_{j-2}^{(k)} \dots A_1^{(k)}\| \Phi, \\ & \leq \sup_{1 \leq n \leq 2q_k+1} \left(\sum_{1 \leq j \leq n-2} \|A_{n-2} \dots A_{j+2}\| \cdot \|A_j - A_j^{(k)}\| \cdot \|A_{j-2}^{(k)} \dots A_1^{(k)}\| \right) \Phi. \end{aligned}$$

The entries of the transfer matrices are all bounded. Note also that since

$\sup_{n \leq 2q_k+1} |\alpha(n) - \alpha_k(n)| \leq \gamma(k, q_k, r_k)$, it is true that $\|A_j - A_j^{(k)}\| \leq k^{-q_k}$. This

means that for some constants C, D we have the estimate

$$\begin{aligned} \sup_{1 \leq n \leq 2q_k+1, n \text{ odd}} \|\phi_k(n) - \phi(n)\| &\leq \sup_{1 \leq n \leq 2q_k+1, n \text{ odd}} C|n|e^{|Dn|}k^{-q_k}, \\ &= C(2q_k + 1)e^{D(2q_k+1)}k^{-q_k}. \end{aligned}$$

If instead n is negative, the preceding argument proceeds completely analogously.

We have now that

$$\max_{a=\pm 1, \pm 2} \|\phi(aq_k + 1) - \phi_k(aq_k + 1)\| \rightarrow 0.$$

as $k \rightarrow \infty$. Let us set $A = A_{q_k-1}^{(k)} \dots A_3^{(k)} A_1^{(k)}$. Note that, due to the fact that the $\{\alpha_i^{(k)}\}$ are q_k -periodic, we then have

$$\begin{aligned} \phi_k(2q_k + 1) &= A^2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \\ \phi_k(q_k + 1) &= A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \\ \phi_k(-q_k + 1) &= A^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \\ \phi_k(-2q_k + 1) &= A^{-2} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \end{aligned}$$

Thus by applying Lemma 2.5 we obtain

$$\max_{a=\pm 1, \pm 2} \|\phi_k(aq_k + 1)\| \geq \frac{1}{2} \|\phi_k(1)\| = \frac{\sqrt{|u_1|^2 + |u_2|^2}}{2}.$$

Then

$$\limsup_{n \text{ odd integer}} \frac{|u_n|^2 + |u_{n+1}|^2}{|u_1|^2 + |u_2|^2} \geq \limsup_{q_k} \frac{\max_{a=\pm 1, \pm 2} \|\phi_k(aq_k + 1)\|^2}{\|\phi(1)\|^2} \geq \frac{1}{4}.$$

□

In particular, a CMV operator associated with Verblunsky coefficients that satisfy the conditions of Proposition 2.4 must have empty point spectrum.

Corollary 2.7. *Consider a two-sided sequence of Verblunsky coefficients $\alpha(n)$ such that there exists a sequence of even positive integers $q_k \rightarrow \infty$ so that*

$$\max_{-q_k+1 \leq n \leq q_k+1} |\alpha(n) - \alpha(n \pm q_k)| \leq \frac{\gamma(k, q_k, r_k)}{4}.$$

Here, r_k is a positive real number less than 1 for which $\alpha(-2q_k + 1), \dots, \alpha(2q_k + 1)$ all lie in a disk of radius r_k centered at the origin. The CMV operator associated with these Verblunsky coefficients has no point spectrum.

Definition 2.8. A two-sided sequence of Verblunsky coefficients $\alpha(n) \in \mathbb{D}$ that satisfies the conditions of Corollary 2.7 is referred to as a *Gordon sequence*.

2.2 CMV operators with limit-periodic Verblunsky coefficients

In this section, we will prove Theorem 1.1. With small changes, the idea of this proof follows closely the proof in [DG11] concerning limit-periodic discrete Schrödinger operators. The difference is in our CMV Gordon lemma, and the fact that the computations in the CMV context tend to be more complicated.

We will rely on the fact that limit-periodic sequences are very well approximated by periodic sequences. The following remarks about periodic sampling functions will prove essential:

Definition 2.9. A sampling function f is k -periodic if $f(T^k\omega) = f(\omega)$ for every $\omega \in \Omega$. The space of periodic sampling functions is denoted P .

We note that Lemma 2.4 in [DG11] applies for our context. This lemma states

Proposition 2.10. *There exists a sequence of Cantor subgroups Ω_k of Ω of finite index p_k such that $\bigcap \Omega_k = \{0\}$. The sampling functions that are defined on Ω/Ω_k are periodic with period p_k . We denote those sampling functions as the space $P_k \subset P$. All periodic sampling functions belong to some P_k .*

2.2.1 Absence of point spectrum

Using the Gordon lemma in Section 2.1.2, we prove generic absence of point spectrum for limit-periodic extended CMV operators.

Proof of part 1 of Theorem 1.1. The set of periodic sampling functions is dense in $C(\Omega, \mathbb{D})$. For every j -periodic sampling function f_j , choose a positive real number $R(f_j) < 1$ so that $\|f_j\| < R(f_j)$. Now for $j \in 2\mathbb{Z}_+, k \in \mathbb{Z}_+$, let

$$\mathcal{G}_{j,k} = \{f \in C(\Omega, \mathbb{D}) : \exists j\text{-periodic } f_j, \text{ such that } \|f\| < R(f_j), \\ \text{and } |f(v) - f_j(v)| < \frac{1}{2} \left(\frac{\gamma(k, jk, R(f_j))}{4} \right), v \in \mathbb{Z}\},$$

Clearly, $\mathcal{G}_{j,k}$ is open. For $k \in \mathbb{Z}_+$, let

$$\mathcal{G}_k = \bigcup_{j=2, j \text{ even}}^{\infty} \mathcal{G}_{j,k}.$$

The set \mathcal{G}_k is open by construction, and dense since it contains all periodic sampling functions (since we may double the period of every odd-periodic sampling function).

Thus

$$\mathcal{G} = \bigcap_{k=1}^{\infty} \mathcal{G}_k.$$

is a dense G_δ set. We claim that for every $f \in \mathcal{G}$ and every $\omega \in \Omega$, the sequence of Verblunsky coefficients given by $\alpha(n) = f(T^n \omega)$ is Gordon.

Let $f \in \mathcal{G}$ and $\omega \in \Omega$ be given. Since $f \in \mathcal{G}_k$ for every k , we can find j_k -periodic f_{j_k} satisfying

$$|f(v) - f_{j_k}(v)| < \frac{1}{2} \left(\frac{\gamma(k, j_k k, R(f_{j_k}))}{4} \right), v \in \mathbb{Z}.$$

Let $q_k = j_k k$, so that $q_k \rightarrow \infty$ as $k \rightarrow \infty$. Then, we have

$$\begin{aligned} & \max_{-q_k+1 \leq n \leq q_k+1} \|\alpha(n) - \alpha(n \pm q_k)\| \\ &= \max_{-q_k+1 \leq n \leq q_k+1} \|f(T^n \omega) - f(T^{n \pm q_k} \omega)\|, \\ &= \max_{-j_k k+1 \leq n \leq j_k k+1} \|f(T^n \omega) - f_{j_k}(T^n \omega) + f_{j_k}(T^{n \pm j_k k} \omega) - f(T^{n \pm j_k k} \omega)\|, \\ &\leq \max_{-j_k k+1 \leq n \leq j_k k+1} \|f(T^n \omega) - f_{j_k}(T^n \omega)\| \\ &\quad + \max_{-j_k k+1 \leq n \leq j_k k+1} \|f(T^{n \pm j_k k} \omega) - f_{j_k}(T^{n \pm j_k k} \omega)\|, \\ &< \frac{\gamma(k, q_k, R(f_{j_k}))}{4}. \end{aligned}$$

We may, of course simply define $r_k = R(f_{j_k})$. It follows that α is a Gordon sequence. □

2.2.2 Cantor spectra

In this section we will prove part 2 of Theorem 1.1. First, we establish a connection between perturbations of the sampling function and perturbations of the corresponding CMV spectra.

Proposition 2.11. *Given two sampling functions f, g with $\|f - g\|_\infty < \epsilon^2/72$ for some $\epsilon > 0$, and $\mathcal{E}_f, \mathcal{E}_g$ denoting the extended CMV operators induced by f, g respectively, we have*

$$\|\mathcal{E}_f - \mathcal{E}_g\| \leq \epsilon.$$

Proof. This follows immediately from Equation (4.3.11) in [Sim04]. Note that Equation (4.3.11) states a result for regular CMV matrices. The proof works perfectly well for our extended CMV matrices. \square

Let us first introduce the following elementary lemma:

Lemma 2.12. *Let \mathcal{E} be a unitary operator. Let z be a point on $\partial\mathbb{D}$, and let d be the distance from z to the spectrum $\Sigma(\mathcal{E})$ of \mathcal{E} (measured in absolute distance, rather than arclength). Then where R_z is the resolvent of \mathcal{E} at z , $\|R_z\| = 1/d$.*

Proof. Let ψ denote an arbitrary normalized vector. Let μ be the spectral measure of \mathcal{E} corresponding to ψ . Since μ is supported by $\Sigma(\mathcal{E})$, by the Spectral Theorem,

$$\|R_z\psi\|^2 = \int_{\Sigma(\mathcal{E})} \frac{1}{|w - z|^2} d\mu(w) \leq \frac{1}{d^2}.$$

Thus we have demonstrated that $\|R_z\| \leq 1/d$. The assertion that $\|R_z\| \geq 1/d$ is given as (2.88) in [Tes00]. \square

Proposition 2.13. *Given two CMV operators $\mathcal{E}, \mathcal{E}'$, if $\|\mathcal{E} - \mathcal{E}'\| < \epsilon$, then if z_0 is in \mathcal{E} 's spectrum, then $|z_0 - z'_0| < \epsilon$ for some z'_0 in \mathcal{E}' 's spectrum.*

Proof. By Lemma 2.16 in [Tes00], if $z_0 \in \partial\mathbb{D}$ and there exists a sequence $\{f_n\} \in \ell^2(\mathbb{Z})$, $f_n \neq 0$ so

$$\frac{\|(\mathcal{E} - z_0)f_n\|}{\|f_n\|} \rightarrow 0, n \rightarrow \infty,$$

then z_0 is in the spectrum of \mathcal{E} . If z_0 is in the boundary of the resolvent set, then the converse also holds.

We apply the converse. So let $\mathcal{E}, \mathcal{E}'$ be two CMV operators such that $\|\mathcal{E} - \mathcal{E}'\| < \epsilon$ for some $\epsilon > 0$. Let z_0 be in $\Sigma(\mathcal{E})$, and let z'_0 be a point in $\Sigma(\mathcal{E}')$ whose distance from z_0 is minimum (z'_0 exists since the spectrum is closed). If $z_0 = z'_0$, we're done. If not, it is clear that z'_0 is on the boundary of the resolvent set of \mathcal{E}' . Hence we know that there exists a sequence $\{f_n\}$ for which

$$\frac{\|(\mathcal{E}' - z'_0)f_n\|}{\|f_n\|} \rightarrow 0.$$

But then we have

$$\frac{\|(\mathcal{E} - z'_0)f_n\|}{\|f_n\|} \leq \frac{\|(\mathcal{E} - \mathcal{E}')f_n\| + \|(\mathcal{E}' - z'_0)f_n\|}{\|f_n\|}.$$

Thus given any $\delta > 0$,

$$e_n = \frac{\|(\mathcal{E} - z'_0)f_n\|}{\|f_n\|} \leq \epsilon + \delta,$$

for sufficiently large n . We know that e_n is a bounded sequence, and thus must have a convergent subsequence. That subsequence must converge to a value not greater

than ϵ . Let $\{e_{k(n)}\}_{n=-\infty}^{\infty}$ be that subsequence, and let it converge to $\gamma \leq \epsilon$ as $n \rightarrow \infty$.

But then we know that

$$\begin{aligned} 1 &= \frac{\|(\mathcal{E} - z'_0)^{-1}(\mathcal{E} - z'_0)f_{k(n)}\|}{\|f_{k(n)}\|}, \\ &\leq \|(\mathcal{E} - z'_0)^{-1}\| \cdot \frac{\|(\mathcal{E} - z'_0)f_{k(n)}\|}{\|f_{k(n)}\|}, \end{aligned}$$

this then implies that

$$\frac{1}{\gamma} \leq \|(\mathcal{E} - z'_0)^{-1}\| = \|\text{Resolvent function of } \mathcal{E} \text{ at } z'_0\|.$$

Thus the distance of z'_0 from the spectrum of \mathcal{E} is at most $\gamma \leq \epsilon$, by Lemma 2.12. \square

Lemma 2.14. *Let \mathcal{E}_f denote the CMV operator induced by the sampling function f .*

Then $\mathcal{R} = \{f \in C(\Omega, \mathbb{D}) : \Sigma(\mathcal{E}_f) \text{ has empty interior}\}$ is a G_δ set.

Proof. This proof is similar in spirit to Lemma 1.1 in [AS81]. For a, b real, we have $e^{ia}, e^{ib} \in \partial\mathbb{D}$ and let us define

$$S_{(a,b)} = \{f \mid f \in C(\Omega, \mathbb{D}), \text{ the resolvent set of } \mathcal{E}_f \text{ intersects the open counter-clockwise arc between } e^{ia} \text{ and } e^{ib}\}.$$

Then

$$\mathcal{R} = \bigcap_{a,b \in 2\pi\mathbb{Q}} S_{(a,b)},$$

so it suffices to show that $S_{(a,b)}$ is open for every choice of $a, b \in 2\pi\mathbb{Q}$. Let $f \in S_{(a,b)}$.

Let B be the open counter-clockwise arc from e^{ia} and e^{ib} . Since the resolvent set

of an operator is always open, there must exist $c \in \partial\mathbb{D}, \delta \in \mathbb{R}_+$ so that the open counterclockwise arc A centered at c with endpoints each δ away from c in absolute distance satisfies

$$A \subseteq B \cap \text{resolvent set of } \mathcal{E}_f.$$

For every $g \in C(\Omega, \mathbb{D})$ with $\|f - g\| < \delta^2/72$, we assert that c is in the resolvent set of \mathcal{E}_g . This is because we have $\|\mathcal{E}_f - c\|^{-1} \leq \delta^{-1}$ by Lemma 2.12, so that

$$1 + (\mathcal{E}_g - \mathcal{E}_f)(\mathcal{E}_f - c)^{-1},$$

is invertible if $\|f - g\| < \delta^2/72$, since Proposition 2.11 would then imply that

$$\|\mathcal{E}_f - \mathcal{E}_g\| < \delta.$$

□

We will now say a few words first about the spectrum when the sampling function f (and hence α) is periodic. This case is very extensively studied in Chapter 11 of [Sim04], but we will state a few results that are of particular relevance to us here.

Given a p -periodic sequence of Verblunsky coefficients with p even, we introduce the discriminant function, $\Delta(z)$, which is defined in (11.1.2) of [Sim04] and whose properties are given by Theorem 11.1.1 of [Sim04]. As a summary, on $\partial\mathbb{D}$, Δ is continuous and takes on real values. According to Theorem 11.1.2 of [Sim04] the essential support of the absolutely continuous spectrum corresponding to the periodic Verblunsky coefficients is given by

$$\{e^{i\theta} \mid -2 \leq \Delta(e^{i\theta}) \leq 2\}.$$

This set comprises of p closed bands on $\partial\mathbb{D}$ on each of which Δ as a function of θ is either strictly increasing or strictly decreasing (Figure 11.2 of [Sim04] provides a typical picture of the graph of Δ). Two adjacent bands intersect not at all or at just one point. If they do not intersect, we say that the gap between two adjacent bands is *open*.

Lemma 2.15. *For a dense open set of periodic sampling functions $f \in P$, all the gaps of $\Sigma(\mathcal{E}_f)$ are open.*

Proof. Theorem 11.13.1 in [Sim04] tells us that a the set of sequences in \mathbb{D}^p which represent the first p terms of a p -periodic sequence corresponding to a spectrum with at least one gap closed is a closed set of measure zero. In particular, if we have a p -periodic Verblunsky sequence whose first p terms is $\alpha_0, \dots, \alpha_{p-1}$, and that sequence corresponds to a spectrum with at least one closed gap, we can find another p -periodic Verblunsky coefficient sequence starting with $\tilde{\alpha}_0, \dots, \tilde{\alpha}_{p-1}$ with $\max_{1 \leq j \leq p-1} |\alpha_j - \tilde{\alpha}_j|$ as small as we like, which corresponds to a spectrum with all gaps open instead.

Take an $\epsilon > 0$. Now consider a p -periodic sampling function $f \in P$ that corresponds to a spectrum with at least one closed gap. By the ω -independence of the spectrum, it suffices to choose an $\omega_0 \in \Omega$, and let $\alpha_j = f(T^j \omega_0)$ for $0 \leq j \leq p-1$. We can find $\tilde{\alpha}_0, \dots, \tilde{\alpha}_{p-1}$ such that $|\alpha_j - \tilde{\alpha}_j| < \epsilon$ so that a p -periodic sequence starting with $\tilde{\alpha}_0, \dots, \tilde{\alpha}_{p-1}$ corresponds to a spectrum with all gaps open.

By Proposition 2.10, we know that p -periodic functions are defined on Ω/Ω_k for some k , where Ω_k is an index p subgroup of Ω . In particular, $\alpha_0, \dots, \alpha_{p-1}$ are the images of f on the p cosets of Ω_k in Ω . We define \tilde{f} by replacing the images of those

p cosets with $\tilde{\alpha}_0, \dots, \tilde{\alpha}_{p-1}$. In this case, \tilde{f} is clearly a continuous p -periodic sampling function on Ω , for which $\|f - \tilde{f}\| < \epsilon$ and \tilde{f} corresponds to a spectrum with all gaps open. \square

Lemma 2.16. *Let $f \in P$ have period p . Then the measure of each band of $\Sigma(\mathcal{E}_f)$ is at most $2\pi/p$.*

Proof. This follows from Theorem 11.1.3 in [Sim04], in particular equations (11.1.18) and (11.1.21). The first part of that Theorem 11.1.3 tells us that the essential spectrum of the spectral measure is equal to the support of the equilibrium measure $d\nu$ on the bands, and together (11.1.18) and (11.1.21) state that the equilibrium measure of any single band is $1/p$.

Now consider the estimate of the equilibrium measure given in Theorem 10.11.21 of [Sim04]. Periodic Verblunsky coefficients are a special case of stochastic Verblunsky coefficients, so this theorem applies. We have then that we can write the equilibrium measure as $d\nu(\theta) = g(\theta)\frac{d\theta}{2\pi}$, with $g(\theta) > 1$ for almost every θ in the support. But this then implies that the normalized Lebesgue measure of any band is at most $1/p$. Hence the Lebesgue measure of any band is at most $2\pi/p$. \square

Proof of part 2 of Theorem 1.1. This proof follows closely the proof of Theorem 4.1 in [DG11].

We need to show that the subset \mathcal{R} of $C(\Omega, \mathbb{D})$ defined in Lemma 2.14 is dense. Since P is dense in $C(\Omega, \mathbb{D})$, we need only show that given $f \in P$ and $\epsilon > 0$, there is an \tilde{f} such that $\|f - \tilde{f}\| < \epsilon$ and $\Sigma(\mathcal{E}_{\tilde{f}})$ is nowhere dense.

Let P be the set of periodic sampling functions, and P_k be the set of periodic

sampling functions of period p_k , as described in Proposition 2.10. Let $f \in P$ and $\epsilon > 0$ be given. We can write $f = \sum_{j=1}^N a_j W_j$, $W_j \in P_j$. We also construct $s_0 = \sum_{i=0}^N a_i^{(0)} W_i$ so that $\|s_0\| < \epsilon^2/72$ and $f_0 = f + s_0$ is p_N -periodic and the corresponding spectrum has all p_N gaps open. This is possible due to Lemma 2.15, since we know that having gaps open is generic behavior.

Suppose that we have chosen s_0, s_1, \dots, s_{k-1} and f_0, f_1, \dots, f_{k-1} . Let A_{k-1} be the minimal gap size of $\Sigma(f_{k-1})$ (we define gap size by absolute distance, rather than distance along the circular arc). We can then define $B_k = \min\{A_0, A_1, \dots, A_{k-1}\}$. Then we pick $s_k = \sum_{i=0}^{N+k} a_i^{(k)} W_i$ (using Lemma 2.15) so that

$$\|s_k\| < \left(\frac{\epsilon}{2^k}\right)^2 / 72, \quad (2.10)$$

$$\|s_k\| < \frac{1}{2^k} \frac{B_k^2}{3^2 \cdot 72}, \quad (2.11)$$

$$f_k = f + \sum_{j=0}^k s_j \text{ has all the gaps of its corresponding spectrum open.} \quad (2.12)$$

So the limit of f_k exists, and we let $\tilde{f} = \lim_{k \rightarrow \infty} f_k$. We have by construction $\|\tilde{f} - f\| < \epsilon^2/54$. We want to show that $\Sigma(\mathcal{E}_{\tilde{f}})$ is nowhere dense, or equivalently that it has a dense complement.

Given $z \in \Sigma(\mathcal{E}_{\tilde{f}})$ and $\tilde{\epsilon} > 0$, for sufficiently large k ,

$$\|\tilde{f} - f_k\| < \left(\frac{\tilde{\epsilon}}{3}\right)^2 / 72, \quad (2.13)$$

$$\frac{2\pi}{p_{N+k}} < \frac{\tilde{\epsilon}}{3}, \quad (2.14)$$

$$\frac{\epsilon}{2^k} < \frac{\tilde{\epsilon}}{3}. \quad (2.15)$$

By (2.13) and Proposition 2.11, we know that $\|\mathcal{E}_{\tilde{f}} - \mathcal{E}_{f_k}\| < \tilde{\epsilon}/3$, and so by Proposition 2.13 there exists $z' \in \Sigma(\mathcal{E}_{f_k})$ such that $|z - z'| < \tilde{\epsilon}/3$. By Lemma 2.16 and (2.14) we

can find \tilde{z} in a gap of $\Sigma(\mathcal{E}_{f_k})$ such that $|z' - \tilde{z}| < \tilde{\epsilon}/3$. We then have a gap of $\Sigma(f_k)$ that contains \tilde{z} as $I_\delta(a)$, which refers to an open interval on the unit circle centered at the point $a \in \partial\mathbb{D}$, whose endpoints are each δ away from a (this distance calculated in terms of absolute value, rather than arclength). By the triangle inequality, we have $2\delta > B_{k+1}$. We then know that, by (2.11),

$$\|\tilde{f} - f_k\| = \left\| \sum_{j=k+1}^{\infty} s_j \right\| < \frac{B_{k+1}^2}{3^2 \cdot 72} \left(\frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \dots \right) < \left(\frac{2\delta}{3} \right)^2 \frac{1}{72}.$$

This implies $\|\mathcal{E}_{\tilde{f}} - \mathcal{E}_{f_k}\| < 2\delta/3$. So we have, by Proposition 2.13 and the triangle inequality that $I_{\frac{\delta}{3}}(a) \cap \Sigma(\mathcal{E}_{\tilde{f}}) = \emptyset$. We claim that there exists $\delta' \in [\delta/3, \delta)$ such that $I_{\delta'}(a) \cap \Sigma(\mathcal{E}_{\tilde{f}}) = \emptyset$ and $|\delta - \delta'| < \epsilon/2^k$. The above demonstrates that we may arrange for $\delta' \geq \delta/3$. Suppose that it is impossible to find such a δ' so that $|\delta - \delta'| < \epsilon/2^k$. Then there will be an x such that $x \in \Sigma(\mathcal{E}_{\tilde{f}})$ and $I_{\frac{\epsilon}{2^k}}(x) \subseteq I_\delta(a)$. This contradicts the fact that $I_{\frac{\epsilon}{2^k}}(x) \cap \Sigma(\mathcal{E}_{f_k}) \neq \emptyset$, since we had

$$\|\tilde{f} - f_k\| = \left\| \sum_{j=k+1}^{\infty} s_j \right\| < \left(\frac{\epsilon}{2^k} \right)^2 / 72,$$

and so $\|\mathcal{E}_{\tilde{f}} - \mathcal{E}_{f_k}\| < \epsilon/2^k$.

Thus we can choose \hat{z} in the gap of $\Sigma(\mathcal{E}_{\tilde{f}})$ that contains $I_{\delta'}(a)$ and so that $|\hat{z} - \tilde{z}| < \epsilon/2^k < \tilde{\epsilon}/3$. The second inequality follows from (2.15). Since we also have $|\tilde{z} - z'| < \tilde{\epsilon}/3$ and $|z' - z| < \tilde{\epsilon}/3$, it follows that $|\hat{z} - z| < \tilde{\epsilon}$. Finally, we have that $\partial\mathbb{D} \setminus \Sigma(\tilde{f})$ is dense and the conclusion immediately follows. \square

2.2.3 Absolutely continuous spectrum

We now prove the third part of Theorem 1.1. The Floquet Theory results required in the proof are described in fuller detail in [Sim04], Sections 11.1 and 11.2.

Let q be an even integer so that the Verblunsky coefficients are q -periodic. That is, if the period p of the Verblunsky coefficients is odd, we take q to be an even multiple. We shall consider the extended CMV matrix \mathcal{E} as an operator acting on $\ell^\infty(\mathbb{Z})$. The operator \mathcal{E} is bounded on $\ell^\infty(\mathbb{Z})$, since every row has only four nonzero terms that are bounded as long as α is bounded away from the boundary of the disk. Let M be the shift operator $(Mu)_m = u_{m+q}$. We then have

$$M\mathcal{E} = \mathcal{E}M.$$

Let $\Theta \in [0, 2\pi)$ and define

$$X_\Theta = \{u \in \ell^\infty(\mathbb{Z}) \mid Mu = e^{i\Theta}u\}.$$

Note that \mathcal{E} takes X_Θ to itself. Also, X_Θ is clearly q -dimensional, since it is determined by any q consecutive coordinates of u . We then define $\mathcal{E}_q(\Theta)$ as the restriction of \mathcal{E} to X_Θ . The matrix $\mathcal{E}_q(\Theta)$ is described explicitly in Figure 11.3, (11.2.6), and (11.2.7) of [Sim04]. Given this machinery, let us now summarize Floquet Theory for OPUC.

Proposition 2.17. (a) *We have $z \in \Sigma(\mathcal{E})$ if and only if $zu = \mathcal{E}u$ for some solution*

$\{u(n)\}$ obeying $Mu = e^{i\Theta}u$ for some $e^{i\Theta} \in \partial\mathbb{D}$. In this case, $\tilde{u} = \langle u(n) \rangle_{n=0}^{q-1}$ is an eigenvector of $\mathcal{E}_q(\Theta)$ corresponding to the eigenvalue z .

(b) *We have*

$$\Sigma(\mathcal{E}) = \bigcup_{\Theta} \Sigma(\mathcal{E}_q(\Theta)).$$

(c) For $\Theta \neq 0, \pi$, we have

$$\det(z - \mathcal{E}_q(\Theta)) = \left(\prod_{j=0}^{q-1} \rho(j) \right) [z^{q/2} [\Delta(z) - (e^{i\Theta} + e^{-i\Theta})]],$$

where $\Delta(z)$ is the discriminant function defined in (11.1.2) of [Sim04]. Also, we know that

$$\Sigma(\mathcal{E}) = \{z : |\Delta(z)| \leq 2\}.$$

The set $\Sigma(\mathcal{E})$ is made of q bands such that on each band, $\Sigma(\mathcal{E})$ is either strictly increasing or strictly decreasing.

(d) If z is on the boundary of some band, then $\Delta(z) = \pm 2$.

We know that there is no point spectrum from Corollary 2.7. The discriminant $\Delta(z)$ is described in Section 11.1, and is explicitly defined in (11.1.2) in [Sim04]. Furthermore, again according to Theorem 11.1.2 of [Sim04] the spectrum of \mathcal{E} consists precisely of the points z on $\partial\mathbb{D}$ for which $|\Delta(z)| \leq 2$.

We shall first put the Fourier transform into a mod q setting, in a similar spirit as that in Section 5.3 of [Sim11]. We define

$$\mathcal{F} : \ell^2(\mathbb{Z}) \rightarrow L^2 \left(\partial\mathbb{D}, \frac{d\Theta}{2\pi}; \mathbb{C}^q \right),$$

the L^2 functions with values in \mathbb{C}^q . The expression $\frac{d\Theta}{2\pi}$ refers to normalized Lebesgue measure on the unit circle. Thus given $n = 0, \dots, q-1$:

$$(\mathcal{F}u)_n(\Theta) = \sum_{l=-\infty}^{\infty} u_{n+lq} e^{-il\Theta}.$$

We define this initially for $u \in \ell^1$ and then extend by using

$$\int_{\partial\mathbb{D}} \|\mathcal{F}u(\Theta)\|^2 \frac{d\Theta}{2\pi} = \sum_n |u_n|^2,$$

since $\{e^{il\Theta}\}_{l=-\infty}^{\infty}$ is a basis for $L^2(\partial\mathbb{D}, \frac{d\Theta}{2\pi})$. The inverse

$$\mathcal{F}^{-1} : L^2\left(\partial\mathbb{D}, \frac{d\Theta}{2\pi}; \mathbb{C}^q\right) \rightarrow \ell^2(\mathbb{Z}),$$

is given by

$$(\mathcal{F}^{-1}f)_{n+lq} = \int e^{il\Theta} f_n(\Theta) \frac{d\Theta}{2\pi},$$

for $l \in \mathbb{Z}$ and $n = 0, \dots, q-1$.

The function $\psi : z \rightarrow [0, \pi]$, defined in (11.2.19) of [Sim04] plays the role of kp in [DG11]. That is, given any $z \in \Sigma(\mathcal{E})$, we must have a solution $u = \phi$ to $\mathcal{E}u = zu$ that satisfies $\phi_{n+q} = e^{i\psi(z)}\phi_n$. We also have an equilibrium measure $d\nu$ (denoted $d\rho$ in [DG11]) on the bands, that is written in the form $d\nu = V(\theta)d\theta$ where

$$V(\theta) = \frac{1}{q\pi} \left| \frac{d\psi(e^{i\theta})}{d\theta} \right|. \quad (2.16)$$

This is (11.2.25) in [Sim04].

Given any $z \in \text{Int}\Sigma(\mathcal{E})$, if we choose $\Theta = \psi(z)$ we know that z is an eigenvalue of $\mathcal{E}_q(\Theta)$. Furthermore, from (11.2.19) in [Sim04] we can define $\psi(z)$ in terms of an arccos of a polynomial, and thus ψ is smooth as a function on a band. In fact from (2.8) it is clear that if z is in the interior of a spectral band there are two linearly independent solutions of $z\phi = \mathcal{E}\phi$. We label them $\phi^+(z), \phi^-(z)$ and assume without loss of generality that

$$\sum_{j=0}^{q-1} |\phi_j^\pm|^2 = 1.$$

For a vector u of finite support, we define

$$Vu^\pm(z) = \frac{q}{2} \sum_{n \in \mathbb{Z}} \overline{\phi_n^\pm(z)} u_n.$$

Lemma 2.18. *The $U = V\mathcal{F}$ operator extends to a unitary map of $\ell^2(\mathbb{Z})$ to $L^2(\Sigma(\mathcal{E}), d\nu(\theta); \mathbb{C}^2)$.*

Proof. Note that $\tilde{\phi}^+ \equiv \{\phi_n^+\}_{n=0}^{l-1}$ is a normalized eigenvector of $\mathcal{E}_q(\Theta)$. Thus if $\lambda_1, \dots, \lambda_q$ are the eigenvalues for a certain $\mathcal{E}_q(\Theta)$, $\{\tilde{\phi}^+(\lambda_j)\}_{j=1}^q$ is an orthonormal basis for \mathbb{C}^q . Hence V is just a unitary change of basis: this is made clear when we compute

$$\begin{aligned} d\nu &= V(\theta)d\theta, \\ &= \frac{1}{q\pi} \left| \frac{d\psi(e^{i\theta})}{d\theta} \right| d\theta, \\ &= \frac{1}{q\pi} \left| \frac{d\Theta}{d\theta} \right| d\theta, \\ &= \frac{2}{q} \frac{d\Theta}{2\pi}. \end{aligned}$$

Furthermore, since \mathcal{F} is unitary as well U must be also. □

We clearly also have

$$[U(\mathcal{E}u)]^\pm(z) = M_z[Uu]^\pm(z),$$

where M_z is the multiplication by z operator in $L^2(\Sigma(\mathcal{E}), d\nu(\theta); \mathbb{C}^2)$.

Lemma 2.19. *Given the q -periodic sequence α and a vector u of finite support, we can write the density function*

$$g_{\alpha,u} = (|(Uu)^+(e^{i\theta})|^2 + |(Uu)^-(e^{i\theta})|^2) \left| \frac{d\nu(\theta)}{d\theta} \right|, \quad (2.17)$$

so that the spectral measure associated with the sequence α and the vector u is

$$d\mu_{\alpha,u} = g_{\alpha,u}(e^{i\theta})d\theta.$$

Proof. Let u be a finitely supported vector, and let $d\mu_u$ be the corresponding spectral measure of \mathcal{E} . Let M_z be the multiplication by z operator on $L^2(\Sigma(\mathcal{E}), d\mu_u)$. If f is a polynomial, we have, clearly

$$\begin{aligned} \langle Uu, M_{f(z)}Uu \rangle_{L^2(d\nu)} &= \langle Uu, Uf(\mathcal{E})u \rangle_{L^2(d\nu)}, \\ &= \langle u, f(\mathcal{E})u \rangle, \\ &= \int_{\Sigma(\mathcal{E})} f(e^{i\theta}) d\mu_u(\theta). \end{aligned}$$

This then clearly holds if we let f be any L^2 function. But we also know

$$\langle Uu, M_{f(z)}Uu \rangle_{L^2(d\nu)} = \int_{\Sigma(\mathcal{E})} f(e^{i\theta}) \|Uu(e^{i\theta})\|^2 d\nu(\theta).$$

□

With these correspondences, the proofs and statements of Lemmas 3.1-3.3 of [DG11] follow verbatim:

Lemma 2.20. *For every $t \in (1, 2)$, there exists a constant $D = D(\|\alpha\|_\infty, q, t)$ such that*

$$\int_{\Sigma(\mathcal{E})} \left| \frac{1}{q\pi} \frac{d\psi}{d\theta}(\theta) \right|^t d\theta \leq D. \quad (2.18)$$

Proof. By (11.2.23) of [Sim04] we have

$$\left| \frac{1}{q\pi} \frac{d\psi}{d\theta}(\theta) \right| = \left| \frac{\Delta'(e^{i\theta})}{2q\pi \sin(\psi)} \right|.$$

Since we can bound $|\Delta'(e^{i\theta})|$ by a $(\|\alpha\|_\infty, q)$ -dependent constant and

$\int_0^\pi (\sin(x))^{1-t} dx < \infty$, we have the following estimates, where C_1, C_2 are constants.

$$\int_{\Sigma(\mathcal{E})} \left| \frac{1}{q\pi} \frac{d\psi}{d\theta}(\theta) \right|^t d\theta \leq C_1 \int_0^\pi \left| \frac{1}{2q\pi \sin(\psi)} \right|^{t-1} d\psi \leq C_2 \int_0^\pi |\sin(\psi)|^{1-t} d\psi,$$

and the last integral may be bounded by a t -dependent constant. \square

Lemma 2.21. *Let $u \in \ell^2(\mathbb{Z})$ have finite support. Then, for every $t \in (1, 2)$, there exists a constant $Q = Q(\|\alpha\|_\infty, q, u, t)$ such that*

$$\int_{\Sigma(\varepsilon)} |g_{\alpha, u}(e^{i\theta})|^t d\theta \leq Q. \quad (2.19)$$

Proof. Since u has a finite support, we can find a constant $M = M(q, u)$ such that

$\| [Uu](e^{i\theta}) \|^2 \leq M$. Thus, by (2.17) we have

$$\begin{aligned} \int_{\Sigma(\varepsilon)} |g_{\alpha, u}(e^{i\theta})|^t d\theta &= \int_{\Sigma(\varepsilon)} \left[(|Uu^+(e^{i\theta})|^2 + |Uu^-(e^{i\theta})|^2) \left| \frac{1}{q\pi} \frac{d\psi}{d\theta}(\theta) \right| \right]^t d\theta, \\ &\leq M^t \int_{\Sigma(\varepsilon)} \left| \frac{1}{q\pi} \frac{d\psi}{d\theta}(\theta) \right|^t d\theta, \\ &\leq M^t D. \end{aligned}$$

with the constant D from the previous lemma. \square

Lemma 2.22. *Let $(X, d\mu)$ be a finite measure space, let $r > 1$ and let $f_n, f \in L^r$ with $\sup_n \|f_n\|_r < \infty$. Suppose that $f_n(x) \rightarrow f(x)$ pointwise almost everywhere. Then, $\|f_n - f\|_p \rightarrow 0$ for every $p < r$.*

Proof. This is [AS81, Lemma 2.6]. \square

Lemma 2.23. *Suppose $u \in \ell^2(\mathbb{Z})$ has finite support and $\alpha^{(n)}, \alpha : \mathbb{Z} \rightarrow \partial\mathbb{D}$ are q -periodic and such that $\|\alpha^{(n)} - \alpha\|_\infty \rightarrow 0$ as $n \rightarrow \infty$. Then, for any $t \in (1, 2)$, we have*

$$\int_{\partial\mathbb{D}} |g_{\alpha^{(n)}, u}(e^{i\theta}) - g_{\alpha, u}(e^{i\theta})|^t d\theta \rightarrow 0,$$

as $n \rightarrow \infty$.

Proof. By Lemmas 2.21 and 2.22 we only need to prove pointwise convergence. Given the explicit identity (2.17), pointwise convergence follows readily from the following two facts: the discriminant of the approximants converges pointwise to the discriminant of the limit and the matrices $\mathcal{E}(e^{i\psi})$ defined in (11.2.4) and Figure 11.3 of [Sim04] associated with the approximants converge pointwise to those associated with the limit and therefore so do the associated eigenvectors. \square

Proof of part 3 of Theorem 1.1. This proof follows closely the proof of Theorem 5.1 of [DG11].

We start from the proof of part 2 of Theorem 1.1, and then refine it. Thus, we will begin once more with an arbitrarily small ball in $C(\Omega, \partial\mathbb{D})$. We then find a point in this ball so that the associated CMV operator has both Cantor spectrum and purely absolutely continuous spectrum. Since we have purely a.c. spectrum, the spectrum cannot be supported on a set with zero Lebesgue measure.

Let us set a $t \in (1, 2)$ and choose $u \in \ell^2(\mathbb{Z})$ so that u has finite support. We repeat the steps in part 2 of Theorem 1.1, except that we choose s_k so that in addition to the conditions in that previous proof, we have

$$\left(\int_{\partial\mathbb{D}} |g_u^{k-1}(e^{i\theta}) - g_u^k(e^{i\theta})|^t d\theta \right)^{\frac{1}{t}} \leq \frac{1}{2^k}, \quad (2.20)$$

where g_u^k is the density of the spectral measure associated with u and the periodic Verblunsky coefficients $n \mapsto f_k(T^n\omega)$. By Lemma 2.23, we can assume the estimate above is uniform in $\omega \in \Omega$. Furthermore, using Lemma 2.21 we can also assume the existence of a constant $Q(u, t) < \infty$ such that $\int_{\mathbb{R}} |g_u^k(e^{i\theta})|^t d\theta \leq Q(u, t)$.

For a fixed $\omega \in \Omega$, let A be a finite union of open sets in $\partial\mathbb{D}$. If P_A^k is the

spectral projection for the Verblunsky coefficients $n \mapsto f_k(T^n\omega)$ and P_A is the spectral projection for the Verblunsky coefficients $n \mapsto \tilde{f}(T^n\omega)$, it follows that $\langle u, P_A u \rangle \leq \limsup_{k \rightarrow \infty} \langle u, P_A^k u \rangle$ since $\|f_k - \tilde{f}\|_\infty \rightarrow 0$. The associated CMV operators converge in norm.

We then have

$$\langle u, P_A u \rangle \leq \limsup_{k \rightarrow \infty} \int_A g_u^k(e^{i\theta}) d\theta \leq Q(u, t) |A|^{\frac{1}{q}},$$

by Hölder's inequality, where $\frac{1}{q} + \frac{1}{t} = 1$ and $|\cdot|$ denotes Lebesgue measure. Thus we know that the spectral measure associated with u and the CMV operator with potential $n \mapsto \tilde{f}(T^n\omega)$ is absolutely continuous. Since this is true for every finitely supported u , the operator must have purely absolutely continuous spectrum, and so we are done. \square

2.3 CMV operators with quasiperiodic Verblunsky coefficients

In this section, we prove a slightly stronger version of Theorem 1.2.

Theorem 2.24. *For almost every $\omega \in \Omega$ and T either a minimal shift on $\Omega = \mathbb{T}^d$ or a skew-shift on \mathbb{T}^2 given by $T(\omega_1, \omega_2) = (\omega_1 + 2\mathbf{a}, \omega_1 + \omega_2)$, there exists a dense G_δ set of sampling functions in $C(\Omega, \mathbb{D})$ for which the corresponding CMV operator has empty point spectrum.*

Also, we can deduce as an immediate corollary, using [Zha12],

Theorem 2.25. *For almost every $\omega \in \Omega$ and T either a minimal shift on $\Omega = \mathbb{T}^d$ or a skew-shift on \mathbb{T}^2 given by $T(\omega_1, \omega_2) = (\omega_1 + 2\mathbf{a}, \omega_1 + \omega_2)$, there exists a dense G_δ set of sampling functions in $C(\Omega, \mathbb{D})$ for which the corresponding CMV operator has purely singular continuous spectrum.*

2.3.1 Absence of point spectrum

We begin with some setup. Any reference to (eTRP), (eMRP), (eGRP) refers to the definitions (TRP), (MRP), (GRP), expressed as Definitions 2,3 and 4 in [BD08] with a small modification. More specifically, we replace the repetition property in those definitions with the following even repetition property.

Definition 2.26 (Even Repetition Property). A sequence $\{\omega_k\}_{k \in [0, j]}$ has the even repetition property if for every $\epsilon > 0, s > 0$ there exists an even integer $q \geq 2$ such that $\text{dist}(\omega_k, \omega_{k+q}) < \epsilon$ for $k = 0, \dots, \lfloor sq \rfloor$.

More explicitly, (Ω, T) satisfies the (eTRP) condition if the set of points in Ω whose forward orbit satisfies the even repetition property is dense in Ω . (Ω, T) satisfies the (eMRP) property if that set is full measure instead of dense, and it satisfies (eGRP) if all forward orbits satisfy the even repetition property.

We shall use Corollary 2.7 to demonstrate absence of point spectrum, by first showing that sequences that obey these repetition properties are Gordon sequences.

Proposition 2.27. *Suppose (Ω, T) is minimal and satisfies (eTRP). Then there exists a dense G_δ subset of \mathcal{F} of $C(\Omega, \mathbb{D})$ such that for every $f \in \mathcal{F}$, there is a residual*

subset $\Omega_f \subseteq \Omega$ with the property that for every $\omega \in \Omega_f$, the sequence of two-sided Verblunsky coefficients defined above is a Gordon sequence.

Proof. The proof is similar to the proof found in [BD08], with some small modifications.

By assumption, there is a point $\omega \in \Omega$ whose forward orbit has the even repetition property. For each $k \in \mathbb{Z}_+$, we consider $\epsilon = 1/k$, $s = 4$ and the associated $q_k = q(\epsilon, s)$ in Definition 2.26. We then have that $q_k \rightarrow \infty$ as $k \rightarrow \infty$. Now take an open ball B_k around ω with radius small enough so that

$$\overline{T^n(B_k)}, 1 \leq n \leq 5q_k$$

are disjoint and, for every $1 \leq j \leq q_k$,

$$\bigcup_{l=0}^4 T^{j+lq_k}(B_k)$$

is contained in some ball of radius 5ϵ . Define

$$\mathcal{C}_k = \{f \in C(\Omega, \mathbb{D}) : f \text{ is constant on each set } \bigcup_{l=0}^4 T^{j+lq_k}(B_k), 1 \leq j \leq q_k\},$$

also define

$$\mathcal{F}_k = \left\{ f \in C(\Omega, \mathbb{D}) : \exists \tilde{f} \in \mathcal{C}_k, \text{ such that } \|f - \tilde{f}\| < \frac{1}{2} \left(\frac{\Gamma(k, q_k, \|f\|)}{4} \right) \right\}.$$

Note that \mathcal{F}_k is an open neighborhood of \mathcal{C}_k , and hence for each m

$$\bigcup_{k \geq m} \mathcal{F}_k$$

is an open and dense subset of $C(\Omega, \mathbb{D})$. This follows since every $f \in C(\Omega, \mathbb{D})$ is uniformly continuous and the diameter of the set $\bigcup_{l=0}^4 T^{j+lq_k}(B_k)$ goes to zero, uniformly

in j as $k \rightarrow \infty$. Thus

$$\mathcal{F} = \bigcap_{m \geq 1} \bigcup_{k \geq m} \mathcal{F}_k$$

is a dense G_δ subset of $C(\Omega, \mathbb{D})$.

Consider some $f \in \mathcal{F}$. Then $f \in \mathcal{F}_{k_l}$ for some sequence $k_l \rightarrow \infty$. Observe that for every $m \geq 1$,

$$\bigcup_{l \geq m} \bigcup_{j=1}^{q_{k_l}} T^{j+q_{k_l}}(B_{k_l})$$

is an open and dense subset of Ω since T is minimal and $q_{k_l} \rightarrow \infty$. Thus

$$\Omega_f = \bigcap_{m \geq 1} \bigcup_{l \geq m} \bigcup_{j=1}^{q_{k_l}} T^{j+q_{k_l}}(B_{k_l})$$

is a dense G_δ subset of Ω .

Given $\omega \in \Omega_f$, ω belongs to $\bigcup_{j=1}^{q_{k_l}} T^{j+q_{k_l}}(B_{k_l})$ for infinitely many l . For each such l , we have by construction that

$$\begin{aligned} \max_{1 \leq j \leq q_{k_l}} |f(T^{j+q_{k_l}}\omega) - f(T^{j+2q_{k_l}}\omega)| &< \frac{\Gamma(k, q_k, \|f\|)}{4}, \\ \max_{1 \leq j \leq q_{k_l}} |f(T^j\omega) - f(T^{j+q_{k_l}}\omega)| &< \frac{\Gamma(k, q_k, \|f\|)}{4}, \\ \max_{1 \leq j \leq q_{k_l}} |f(T^{j-q_{k_l}}\omega) - f(T^j\omega)| &< \frac{\Gamma(k, q_k, \|f\|)}{4}, \end{aligned}$$

and

$$\max_{1 \leq j \leq q_{k_l}} |f(T^{j-2q_{k_l}}\omega) - f(T^{j-q_{k_l}}\omega)| < \frac{\Gamma(k, q_k, \|f\|)}{4}.$$

Thus $\alpha(n) = f(T^n\omega)$ is Gordon. □

Proposition 2.28. *Suppose that (Ω, T, μ) satisfies (eMRP). Then there exists a residual subset \mathcal{F} of $C(\Omega, \mathbb{D})$ such that for every $f \in \mathcal{F}$, there exists a subset $\Omega_f \subseteq \Omega$ of full μ measure with the property that for every $\omega \in \Omega_f$, the corresponding sequence of two-sided Verblunsky coefficients is a Gordon sequence.*

The proof of this proposition is almost exactly the same as that of Theorem 3 of [BD08], except that we define

$$\mathcal{F}_i = \left\{ f \in C(\Omega, \mathbb{D}) : \exists \tilde{f} \in F_i, \text{ such that } \|f - \tilde{f}\| < \frac{1}{2} \left(\frac{\Gamma(k, q_k, \|f\|)}{4} \right) \right\},$$

and make some other minor adjustments.

Proposition 2.29. *Every minimal shift $T\omega = \omega + \mathbf{a}$ on the torus \mathbb{T}^d satisfies (eGRP).*

Proof. Note that the assumption that the condition that (\mathbb{T}^d, T) satisfies (eGRP) is the same thing as saying that (\mathbb{T}^d, T') defined by $T'\omega = \omega + 2\mathbf{a}$ satisfies (GRP). But this is true by Theorem 3 of [BD08]. \square

Definition 2.30. $\mathbf{a} \in \mathbb{T}$ is called *badly approximable* if there exists a constant $c > 0$ such that $\langle \mathbf{a}q \rangle > c/q$ for every $q \in \mathbb{Z} \setminus \{0\}$, where $\langle \dots \rangle$ refers to distance from \mathbb{Z} .

Proposition 2.31. *For a minimal skew-shift $T(\omega_1, \omega_2) = (\omega_1 + 2\mathbf{a}, \omega_1 + \omega_2)$ on the torus \mathbb{T}^2 , the following are equivalent.*

(i) \mathbf{a} is not badly approximable

(ii) (Ω, T) satisfies (eGRP)

(iii) (Ω, T, Leb) satisfies (eMRP)

(iv) (Ω, T) satisfies (eTRP)

Proof. Note that since (eTRP) is a stronger condition than (TRP), and (eGRP) \implies (eMRP) \implies (eTRP), and since Theorem 4 in [BD08] asserts that (TRP) \implies (i), we only need to prove (i) \rightarrow (ii).

Assume that \mathbf{a} is not badly approximable. This means that there is some positive integer sequence $q_k \rightarrow \infty$ such that

$$\lim_{k \rightarrow \infty} q_k \langle \mathbf{a}q_k \rangle = 0.$$

Furthermore, if we double all the q_k s, clearly this limit still equals zero. Thus we may assume all the q_k s are even.

Iterating the skew shift n times, we find

$$T^n(\omega_1, \omega_2) = (\omega_1 + 2n\mathbf{a}, \omega_2 + n\omega_1 + n(n-1)\mathbf{a}),$$

Therefore,

$$T^{n+q}(\omega_1, \omega_2) - T^n(\omega_1, \omega_2) = (2q\mathbf{a}, q\omega_1 + q^2\mathbf{a} + 2nq\mathbf{a} - q\mathbf{a}). \quad (2.21)$$

Let $(\omega_1, \omega_2) \in \mathbb{T}^2$, $\epsilon > 0$ and $r > 0$ be given. We will construct an even sequence $\tilde{q}_k \rightarrow \infty$ so that for $1 \leq n \leq r\tilde{q}_k$,

$$(2\tilde{q}_k\mathbf{a}, \tilde{q}_k\omega_1 + \tilde{q}_k^2\mathbf{a} + 2n\tilde{q}_k\mathbf{a} - \tilde{q}_k\mathbf{a})$$

is of size $O(\epsilon)$. Each \tilde{q}_k will be of the form $m_k q_k$ for some $m_k \in \{1, 2, \dots, \lfloor \epsilon^{-1} \rfloor + 1\}$.

We can see that every term of Equation 2.21 except $\tilde{q}_k\omega_1$ goes to zero as $k \rightarrow \infty$, regardless of the choice of m_k , and hence is less than ϵ for k large enough. To treat the remaining term, we can just choose m_k in the specified ϵ -dependent range so that $\tilde{q}_k\omega_1 = m_k(q_k\omega_1)$ is of size less than ϵ as well. Consequently, the orbit of (ω_1, ω_2) has the repetition property. Since (ω_1, ω_2) was arbitrary, (eGRP) holds. \square

Proof of Proposition 2.24. By Theorem 2.29, we know that minimal shifts satisfy (eGRP) and hence (eMRP) and (eTRP). By Theorem 2.31 we know that skew shifts

with \mathbf{a} not badly approximable satisfy (eMRP). Note that the set of badly approximable \mathbf{a} 's has zero Lebesgue measure ([Khi97], Theorem 29 on p.60). We then apply Propositions 2.27 and 2.28 to conclude our proof. \square

2.4 CMV operators generated by subshifts

For a more comprehensive treatment of the area of CMV operators generated by subshifts, please consult Section 10.5A of [Sim04]. In our section, we use the Gordon lemma to eliminate the possibility of pure point spectrum in several contexts.

2.4.1 Floquet Theory for Gesztesy-Zinchenko transfer matrices

As a first step, we have to say a few words about a Floquet theory for the Gesztesy-Zinchenko transfer matrices given as (2.1). Note that a Floquet theory for the CMV operator is already well understood (see Chapter 11.2 of [Sim04]), but using the standard Szegő transfer matrices rather than the Gesztesy-Zinchenko transfer matrices. In some respects, our treatment will be simpler, since the fact that the Gesztesy-Zinchenko transfer matrices have determinant -1 instead of z means that we won't have to keep track of $z^{-1/2}$ terms.

Fix $z \in \partial\mathbb{D}$, even $q \geq 4$, and $\beta \in \partial\mathbb{D}$. Consider the Floquet matrix given by $\mathcal{E}_q(\beta) = \mathcal{L}_q \mathcal{M}_q(\beta)$, where

$$\mathcal{M}_q(\beta) = \begin{pmatrix} -\alpha(q-1) & & & \rho(q-1)\beta^{-1} \\ & \Theta_1 & & \\ & & \ddots & \\ & & & \Theta_{q-3} \\ \rho(q-1)\beta & & & \overline{\alpha(q-1)} \end{pmatrix}, \quad (2.22)$$

and

$$\mathcal{L}_q = \begin{pmatrix} \Theta_0 & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \\ & & & & \Theta_{q-2} \end{pmatrix}. \quad (2.23)$$

Note that $\mathcal{E}_q(\beta)$ is a matrix with four nonzero elements on each row, obtained by taking \mathcal{E} , cutting out the $[0, q-1] \times [0, q-1]$ block, and modifying the first two and last two rows in the following way: The top two rows in \mathcal{E} have one element each cut off in passing to $\mathcal{E}_q(\beta)$. We shift that element right q places and multiply by β^{-1} . In the bottom row, we instead shift left by q places and multiply by β . See Figure 11.3 of [Sim04].

It is then straightforward to verify the following proposition.

Proposition 2.32. $z_0 \in \partial\mathbb{D}$ is an eigenvalue of $\mathcal{E}_q(\beta)$ if and only if β is an eigenvalue of $M_q(z_0)$.

Proof. We simply note that Lemma 2.2 of [GZ06] implies that if the solution u to the eigenvalue equation $\mathcal{E}u = z_0u$ is a Floquet solution with skew angle β , then the

corresponding solution v of $\mathcal{E}^T v = z_0 v$ satisfies $\mathcal{M}u = z_0 v$. It is easy to see that this is equivalent to $\mathcal{M}_q(\beta)\hat{u} = z_0\hat{v}$ for $\hat{u} \in \mathbb{C}^q$ whose terms are $u(1), \dots, u(q)$ and $\hat{v} \in \mathbb{C}^q$ whose terms are $v(1), \dots, v(q)$. But then clearly

$$z_0\beta\hat{v} = \mathcal{M}_q(\beta)\beta\hat{u},$$

and this implies that $\beta v(1) = v(q+1)$. Since we already know that $\beta u(1) = u(q+1)$, and since our matrices $M_q(z_0)$ act on vectors of the form

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix},$$

we can conclude that β will be an eigenvalue of $M_q(z_0)$ if and only if there exists skew-periodic sequences u, v with skew term β , such that $\mathcal{E}u = z_0 u$ (and hence $\mathcal{E}_q\hat{u} = z_0\hat{u}$) and such that v is associated to u in the manner given by Lemma 2.2 of [GZ06]. \square

Note that $M_q(z_0)$ is unimodular. Therefore, if β is an eigenvalue, then β^{-1} must be also. We thus see that z_0 is an eigenvalue of $\mathcal{E}_q(\beta)$ if and only if

$$\Delta(z_0) := \text{Tr}(M_q(z_0)) = \beta + \beta^{-1} = 2 \cos(k), \quad (2.24)$$

where $\beta = e^{ik}$. We may at this point mimic the standard Floquet Theory of CMV operators to assert that the spectrum of a q -periodic CMV operator is precisely the subset of $z \in \partial\mathbb{D}$ when $|\Delta(z)| \leq 2$.

2.4.2 Uniform results for the period doubling subshift

Let us recall our discussion on the period doubling subshift in the introduction. We have an alphabet $\mathcal{A} = \{a, b\}$. We consider the substitution sequence $S(a) = ab$,

$S(b) = aa$, and when we iterate this substitution we obtain a one-sided invariant sequence $u = abaaabab\dots$. Choose a nonconstant function $f : \mathcal{A} \rightarrow \mathbb{D}$ and then consider the associated subshift (Ω, \mathcal{T}, f) . We can define for $\omega \in \Omega$ an extended CMV operator \mathcal{E}_ω whose corresponding Verblunsky coefficients are given by $\alpha(n) = f(\omega_n)$.

By construction, we have the recurrence

$$\begin{aligned} S^n(a) &= S^{n-1}(a)S^{n-1}(b), \\ S^n(b) &= S^{n-1}(a)S^{n-1}(a). \end{aligned} \tag{2.25}$$

Given an n , we define $T_k^a(z)$ for $1 \leq k \leq 2^n$ as $T_k(z)$ taking $\alpha(k) = f(S^n(a)_k)$ (To clarify, $S^n(a)_k$ is the k th letter of the word $S^n(a)$). Similarly, we define $T_k^b(z)$ for $1 \leq k \leq 2^n$ as $T_k(z)$ taking $\alpha(k) = f(S^n(b)_k)$. Following from (2.25) and the discussion on the period doubling sequence earlier in this section, let us define also

$$\begin{aligned} M_{(a),n}(z) &= \prod_{k=2^n}^1 T_k^a(z), \\ M_{(b),n}(z) &= \prod_{k=2^n}^1 T_k^b(z). \end{aligned} \tag{2.26}$$

We will often suppress the dependence on z for notational convenience. We verify using (2.25) that for $n \geq 1$ we have

$$\begin{aligned} M_{(a),n+1} &= \prod_{k=2^{n+1}}^1 T_k^a(z) = \prod_{k=2^n}^1 T_k^b(z) \prod_{k=2^n}^1 T_k^a(z) = M_{(b),n} M_{(a),n}, \\ M_{(b),n+1} &= \prod_{k=2^{n+1}}^1 T_k^b(z) = \prod_{k=2^n}^1 T_k^a(z) \prod_{k=2^n}^1 T_k^a(z) = M_{(a),n} M_{(a),n}. \end{aligned} \tag{2.27}$$

Lemma 2.33. $\operatorname{Tr}(M_{(b),n} M_{(a),n}^{-1}) = \frac{2(\operatorname{Re}(-f(a)\overline{f(b)})+1)}{\sqrt{1-|f(a)|^2}\sqrt{1-|f(b)|^2}} \geq 2$ for $n \geq 1$.

Proof. First, using the fact that $|f(a)|, |f(b)| < 1$, we observe

$$\frac{2(\operatorname{Re}(-f(a)\overline{f(b)}) + 1)}{\sqrt{1 - |f(a)|^2}\sqrt{1 - |f(b)|^2}} \geq \frac{2(1 - |f(a)| \cdot |f(b)|)}{\sqrt{1 - |f(a)|^2}\sqrt{1 - |f(b)|^2}} \geq 2,$$

where the second inequality is a standard exercise in multivariable calculus.

We can compute that the lemma holds for $n = 1$:

$$\begin{aligned} M_{(b),1}M_{(a),1}^{-1} &= T_2^b(z)T_1^b(z)T_1^a(z)^{-1}T_2^a(z)^{-1} \\ &= T_2^b(z)T_2^a(z)^{-1}, \text{ since } T_1^a(z) = T_1^b(z) \\ &= \frac{1}{\sqrt{1 - |f(a)|^2}} \begin{pmatrix} -f(a) & 1 \\ 1 & -\overline{f(a)} \end{pmatrix} \frac{-1}{\sqrt{1 - |f(b)|^2}} \begin{pmatrix} -\overline{f(b)} & -1 \\ -1 & -f(b) \end{pmatrix} \\ &= \frac{1}{\sqrt{1 - |f(a)|^2}} \frac{1}{\sqrt{1 - |f(b)|^2}} \begin{pmatrix} -f(a)\overline{f(b)} + 1 & f(b) - f(a) \\ \overline{f(b)} - \overline{f(a)} & -f(b)\overline{f(a)} + 1 \end{pmatrix} \end{aligned}$$

Thus we have

$$\operatorname{Tr}(M_{(b),1}M_{(a),1}^{-1}) = \frac{2(\operatorname{Re}(-f(a)\overline{f(b)}) + 1)}{\sqrt{1 - |f(a)|^2}\sqrt{1 - |f(b)|^2}}.$$

Note that for $n \geq 1$

$$\begin{aligned} \operatorname{Tr}(M_{(b),n+1}M_{(a),n+1}^{-1}) &= \operatorname{Tr}(M_{(a),n}M_{(a),n}M_{(a),n}^{-1}M_{(b),n}^{-1}) \\ &= \operatorname{Tr}(M_{(a),n}M_{(b),n}^{-1}) \\ &= \operatorname{Tr}(M_{(b),n}M_{(a),n}^{-1}). \end{aligned}$$

The last equality following from the fact that $M_{(a),n}M_{(b),n}^{-1}$ has determinant 1. The lemma then follows from a simple induction argument. \square

Let us define $x_n = \text{Tr}(M_{(a),n})$, $y_n = \text{Tr}(M_{(b),n})$, and $B = \text{Tr}(M_{(b),1}M_{(a),1}^{-1})$. We instead write $x_n(z), y_n(z)$ when we wish to emphasize the dependence on z . We will perform an analysis on the trace map dynamics of M , a central tool in many previous treatments of operators generated by subshifts. See [Pey95] for a survey of the method.

Proposition 2.34. *For $n \geq 1$ the sequences x_n, y_n are real and obey the recursion*

$$x_{n+1} = x_n y_n - B, \tag{2.28}$$

$$y_{n+1} = x_n^2 - 2. \tag{2.29}$$

Proof. Note the following standard identity for unimodular 2×2 matrices M, N (for instance, refer to (17) of [BIST89])

$$\text{tr}(MN) = \text{tr}(M)\text{tr}(N) - \text{tr}(MN^{-1}). \tag{2.30}$$

Using Lemma 2.33, we know that if we set $M = M_{(b),n}, N = M_{(a),n}$ we get the first equation, and if we set $M = N = M_{(a),n}$ we get the second.

We calculate that x_1, y_1 are both real, and as a consequence x_n, y_n is real for all $n \geq 1$. □

Definition 2.35. \mathcal{U} is the set of all points $(x, y) \in \mathbb{R}^2$ such that there is n_0 such that if $x_0 = x, y_0 = y$, for all $n \geq n_0, |x_n| > 2$. We call the interior of \mathcal{U} the set of *unstable points*.

We note here that the interior of \mathcal{U} is certainly nonempty, since by (2.28), (2.29) all points with x and y sufficiently large (say, both larger than B) are in \mathcal{U} . We

will also remark that in the Schrödinger setting, it is known that \mathcal{U} is an open set (Lemma 2 of [BBG91]). That lemma doesn't directly apply in our setting unless $B = 2$. Nevertheless, in our paper the question of whether \mathcal{U} is open or not only affects the proof of Proposition 2.37 below, and as the reader will see it suffices to consider the interior of \mathcal{U} .

Proposition 2.36. *If $(x_0(z), y_0(z))$ is an unstable point, then z is in the resolvent set of the operator \mathcal{E} .*

Proof. In the Schrödinger setting, the proof was given in [Bel90]. Let $z \in \partial\mathbb{D}$, and let us label $\hat{\mathcal{E}}(n)$ as the periodic approximation of \mathcal{E} , generated by Verblunsky coefficients $\hat{\alpha}(j) = \alpha(j')$, where $j' \equiv j \pmod{2^n}$ and $j' \in [0, 2^n)$. In this case, $|x_n(z)| > 2$ means that z is in a spectral gap of $\hat{\mathcal{E}}(n)$. Thus if $(x_N(z), y_N(z))$ is unstable for some N , there is some neighborhood U of z that is in a spectral gap of a sequence $\hat{\mathcal{E}}(t)$ for $t \geq N$ of periodic approximations of \mathcal{E} . We conclude that z is also in a spectral gap of \mathcal{E} . \square

We now establish trace bounds for points in the spectrum.

Proposition 2.37. *If $z \in \partial\mathbb{D}$ is in the spectrum of \mathcal{E} , then for any integer $n \geq 1$ $|x_n(z)| \leq B$ or $|x_{n+1}(z)| \leq B$.*

Proof. Recall that from Lemma 2.33, $B \geq 2$. Assume instead that for some n we have both $|x_n(z)| > B$ and $|x_{n+1}(z)| > B$. By (2.29) we must also have $y_{n+1}(z) > B$. But this implies, by (2.28) that $|x_{n+2}(z)| > B$. Thus it is an easy induction argument to show that $|x_m(z)| > B \geq 2$ for all $m \geq n$. This implies that $(x_0(z), y_0(z)) \in \mathcal{U}$.

Furthermore, let us consider the open subsets D_+, D_- of \mathbb{R}^2 , defined as $D_{\pm} = \{\pm x > B, y > 2\}$. It is clear that if for some n $(x_n, y_n) \in D_+ \cup D_-$ then $(x_0, y_0) \in \mathcal{U}$. Since x_n, y_n are analytic functions of x_0, y_0 for fixed n , if we choose (x_0, y_0) such that $(x_n, y_n) \in D_+ \cup D_-$, then every point in some open neighborhood S of (x_0, y_0) must have their n th images in $D_+ \cup D_-$, and thus every point in S must be unstable. Thus we can show that $(x_0(z), y_0(z))$ is in the interior of \mathcal{U} , and so by Proposition 2.36, we conclude z must be in the resolvent set of \mathcal{E} . \square

Proof of Theorem 1.3. Absolutely continuous spectrum is ruled out in [DL07]. In this section we have already established all the tools necessary in the CMV setting, and so we can now easily replicate the combinatorial analysis in [Dam01] to rule out point spectrum. Specifically, our Propositions 2.37, 2.2 and 2.1 replace Propositions 2.2, 2.4a, and 2.4b respectively in that paper. Since the subshift we consider is identical, what follows is simply an extended case analysis on the given element $\omega \in \Omega$ which works exactly the same way as in [Dam01]. \square

As a direction for future research, it will be interesting to replicate the analysis in Section 3 of [BBG91] where they proved several results about the spectral gaps for the Schrödinger operator with period doubling potential, in addition to proving that the spectrum is uniformly singular continuous. In particular, they were able to determine which gaps were open, and to calculate the rate of growth of the gaps in terms of the absolute value of the potential.

Their results rely on exploiting trace map dynamics for Schrödinger transfer matrices. The analysis in our CMV context will be more complicated, since our trace

map equations (2.28), (2.29) have an additional parameter B , and as a result the nature of the fixed points, et cetera will not be the same. For example, in general it is not clear in our context that \mathcal{U} is an open set, a fact that is true in the Schrödinger setting and an essential first step for many of the theorems in [BBG91].

2.4.3 Almost sure results for subshifts generated by Sturmian sequences and codings of rotations

In this section, we will consider Verblunsky coefficients generated by subshifts of *Sturmian sequences* and *codings of rotations*.

The proofs of Theorems 1.6, 1.7 and 1.8 are a close analogue of the proofs in [Kam96], which we sketch here for the reader's convenience. We first address the preliminaries, Lemmas 1,2,3 and 4 in that paper. Their Lemma 2 is proven in the CMV context as Theorem 10.16.1 in [Sim04] (that is, the fact that the spectrum is a.e independent of $\omega \in \Omega$, and that we have a.e absence of point spectrum). This in turn implies their Lemma 1, which asserts that absence of point spectrum for a positive measure set of β implies absence of point spectrum for a full measure set of β . Their Lemma 3 is simply the Gordon lemma, so we can equivalently use our Proposition 2.2. Finally, their Lemma 4 is a simple calculation. We have to make a small change to that Lemma 4, however. Rather than taking lim sups over $n \in \mathbb{N}$, we only take lim sups over n in $\tilde{\mathbb{N}}$, (recall that $\tilde{\mathbb{N}}$ is the subset of \mathbb{N} so that q_n is even), and we thus obtain instead

$$\limsup_{n \in \tilde{\mathbb{N}}} q_{n+1}/q_n \geq \limsup_{n \in \tilde{\mathbb{N}}} a_n. \quad (2.31)$$

Again, absence of absolutely continuous spectrum was demonstrated in [DL07].

Proof of Theorem 1.8. Where α_β refers to the Verblunsky coefficients generated using the phase β , let us define

$$E(n) := \{\beta | \alpha_\beta(j + q_n) = \alpha_\beta(j - q_n) = \alpha_\beta(j), 1 \leq j \leq q_n\},$$

so $E(n)$ is the set of phases for which Proposition 2.2 is satisfied for q_n . It suffices to show

$$\mu \left(\limsup_{n \in \tilde{\mathbb{N}}} E(n) \right) > 0,$$

where μ is Lebesgue measure on $[0, 1)$. Recall that I is the interval for our rotation coding sequence, and let us denote β_1, β_2 as the two endpoints of I . Then where Φ is the canonical projection from \mathbb{R} to \mathbb{R}/\mathbb{Z} , and $|\cdot|_1$ denotes distance from 0 in \mathbb{R}/\mathbb{Z} , we define sets

$$E_i(n) = \{\beta | \min_{1 \leq j \leq q_n} |\Phi(j\theta) + \beta - \beta_i|_1 > |q_n\theta - p_n|\}, i = 1, 2. \quad (2.32)$$

We can calculate that

$$|(\Phi((j + q_n)\theta) + \beta) - (\Phi(j\theta) + \beta)|_1 = |q_n\theta - p_n|, \quad (2.33)$$

By (2.32) and (2.33) we then have $E_1(n) \cap E_2(n) \subset E(n)$. After some calculations, this gets us

$$\mu(E(n)) \geq 1 - 4 \frac{q_n}{q_{n+1}},$$

and therefore

$$\limsup_{n \in \tilde{\mathbb{N}}} \mu(E(n)) \geq 1 - \frac{4}{\limsup_{n \in \tilde{\mathbb{N}}} \frac{q_{n+1}}{q_n}}.$$

And so with (2.31) and the hypothesis that $\limsup_{n \in \tilde{\mathbb{N}}} a_n \geq 5$, we are done. \square

Proof of Theorem 1.7. We repeat the steps of the previous proof, but this time we set $\beta_1 = 1 - \theta$ and $\beta_2 = 0$. We then perform similar computations to obtain

$$\mu(E(n)) \geq 1 - 2(q_n + 1)|q_n\theta - p_n|, \quad (2.34)$$

and then, using (5) from [Kam96]

$$\mu(E(n)) \geq 1 - \frac{2q_n + 2}{q_{n+1}}.$$

We can then conclude that $\limsup_{n \in \tilde{\mathbb{N}}} \mu(E(n))$ is positive if we apply (2.31) and the hypothesis $\limsup_{n \in \tilde{\mathbb{N}}} a_n \geq 3$. \square

Proof of Theorem 1.6. By (7) of [Kam96],

$$\lim_{n \rightarrow \infty} \frac{q_{n+1}}{q_n} = \frac{2}{\sqrt{5} - 1}.$$

But this certainly implies

$$\limsup_{n \in \tilde{\mathbb{N}}} \frac{q_{n+1}}{q_n} = \frac{2}{\sqrt{5} - 1}.$$

Using this equation to replace (7) (that is, (7) in their paper, not ours) in the proof of Lemma 5 in [Kam96], we can assert that

$$\limsup_{n \in \tilde{\mathbb{N}}} q_n |q_n\theta - p_n| = \frac{1}{\sqrt{5}}.$$

We plug this in to (2.34), and then we are done. \square

As a final remark, we will note that [DKL00] proved singular continuous spectrum for Schrödinger operators for all β and without any restrictions on the a_n 's. Unfortunately, there are at least two obstacles to applying that argument in our setting directly. First, our Gordon lemma for the CMV operator only works for even jumps, and so to apply the argument in [DKL00] directly we would require consecutive q_n to be even, which is impossible. Furthermore, we lack a Floquet theory for solutions of odd-periodic Gesztesy-Zinchenko transfer matrices, which we require to establish trace bounds.

Chapter 3

Power law bounds on solutions of the CMV operator

In this chapter we prove Theorem 1.9. The results of this chapter are joint work with Paul Munger.

In [IRT92], [DL99], and [DKL00] power law bounds on formal eigenvectors are established for the Schrödinger operator with a Sturmian sequence of recursion coefficients. In the last section of this chapter, we apply analogous methods to extended CMV matrices with ω equal to the golden mean.

3.1 A Green's function for the extended CMV operator

In this section, we will explain an essential formula proven in [GZ06]. We require a way to relate the extended CMV operator \mathcal{E} with the two one-sided CMV operators that comprise its two halves. More precisely, if we modify $\alpha(-1) = -1$, then (1.2) becomes the direct sum of operators on $\ell^2([0, \infty) \cap \mathbb{Z})$ and $\ell^2([-1, -\infty) \cap \mathbb{Z})$ of the form (1.1). We label the halves as \mathcal{C}_+ and \mathcal{C}_- respectively. In this section \mathcal{E} refers to the unmodified extended CMV matrix.

First, let us label $F_+(z)$, the Carathéodory function corresponding to \mathcal{C}_+ , and $F_-(z)$, the Carathéodory function corresponding to \mathcal{C}_- . Carathéodory functions are holomorphic maps from \mathbb{D} to the right half plane $\{z | \operatorname{Re} z > 0\}$. We also say a function is anti-Carathéodory when its negative is Carathéodory. The correspondences between a given CMV matrix and its Carathéodory function are explored more fully in Section 1.3 of [Sim04]. Briefly, a Carathéodory function is the CMV analogue of the m -function in the theory of Jacobi matrices, and is connected to the spectral theory of the CMV matrix.

For example, where c_i are the moments of the spectral measure of the one-sided CMV matrix \mathcal{C} , its Carathéodory function F may be expressed as $F(z) = 1 + 2 \sum_{n=1}^{\infty} c_n z^n$. It is also true that $\operatorname{Re} F(re^{i\theta}) d\theta / 2\pi$ converges weakly to the spectral measure of \mathcal{C} as $r \rightarrow 1$. Finally, we note that where μ is the spectral measure of a

CMV matrix, its Carathéodory function is given by the formula

$$F(z) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta).$$

The Green's function (or resolvent function) for \mathcal{E} is computed using formal eigenvalues to \mathcal{C}_\pm and \mathcal{C}_\pm^T .

Lemma 3.1 (Lemma 3.1 in [GZ06]). *Let $z \in \mathbb{C} \setminus (\partial\mathbb{D} \cup \{0\})$, and let M_- be an anti-Carathéodory function in ([GZ06], Lemma 2.20), which is, by (2.139) in [GZ06] related to F_- by*

$$M_-(z) = \frac{\operatorname{Re}(1 - \bar{\alpha}_0) - i\operatorname{Im}(1 + \bar{\alpha}_0)F_-(z)}{i\operatorname{Im}(1 - \bar{\alpha}_0) - \operatorname{Re}(1 + \bar{\alpha}_0)F_-(z)}.$$

Let u_\pm be ℓ^2 solutions to $(\mathcal{C}_\pm - z)u = 0$, and let v_\pm be ℓ^2 solutions to $(\mathcal{C}_\pm^T - z)v = 0$, normalized by

$$\begin{aligned} v_-(z, 0) &= -1 + M_-(z), & v_+(z, 0) &= -1 + F_+(z), \\ u_-(z, 0) &= z + zM_-(z), & u_+(z, 0) &= z + zF_+(z). \end{aligned}$$

We may extend these solutions to solutions of $(\mathcal{E} - z)w = 0$ and $(\mathcal{E}^T - z)w = 0$.

Then the resolvent function $(\mathcal{E} - z)^{-1}(x, y)$ can be expressed as

$$\frac{-1}{2z(F_+(z) - M_-(z))} \begin{cases} u_-(z, x)v_+(z, y) & \text{if } x < y \text{ or } x = y \text{ odd,} \\ u_+(z, x)v_-(z, y) & \text{if } x > y \text{ or } x = y \text{ even,} \end{cases} \quad (3.1)$$

3.2 Applications of the formula

For a $\ell^2(\mathbb{N})$ -vector u , and a positive integer n , we define $\|u\|_n$ as $\sqrt{\sum_{j=0}^n |u(j)|^2}$.

We can also define $\|u\|_k$, for k positive but not an integer so that $\|u\|_k^2$ is a linear interpolation of $\|u\|_n^2$.

Lemma 3.2. *Suppose, for a one-sided CMV matrix \mathcal{C} , that every solution of*

$$\begin{pmatrix} \eta_{2n+2}(z) \\ \eta_{2n+3}(z) \end{pmatrix} = \frac{1}{\rho(n)} \begin{pmatrix} z & -\overline{\alpha(n)} \\ -\alpha(n)z & 1 \end{pmatrix} \begin{pmatrix} \eta_{2n}(z) \\ \eta_{2n+1}(z) \end{pmatrix}, \quad (3.2)$$

with $|\eta_0(z)|^2 + |\eta_1(z)|^2 = 2$ obeys the estimate

$$C_1 L^{\gamma_1} \leq \|\eta(z)\|_L \leq C_2 L^{\gamma_2}, \quad (3.3)$$

for $L > 0$ sufficiently large. Then

$$\sup_{\lambda \in \partial\mathbb{D}} \left| \frac{(1-\lambda) + (1+\lambda)F(rz)}{(1+\lambda) + (1-\lambda)F(rz)} \right| \leq C_3 (1-r)^{\beta-1}, \quad (3.4)$$

where $\beta = 2\gamma_1/(\gamma_1 + \gamma_2)$.

Proof. This is a consequence of the Jitomirskaya-Last inequality for OPUC (see [Sim04] Section 10.8), which says that

$$\frac{\|\psi^\lambda(z)\|_{x(r)}}{\|\varphi^\lambda(z)\|_{x(r)}} \lesssim |F^\lambda(rz)| \lesssim \frac{\|\psi^\lambda(z)\|_{x(r)}}{\|\varphi^\lambda(z)\|_{x(r)}}.$$

Here, F^λ is the Carathéodory function corresponding to the Alexandrov measure μ_λ (refer to Theorem 3.2.14 of [Sim04]). Its first and second kind orthogonal polynomials are φ^λ and ψ^λ . The function $x(r)$ is defined by $(1-r)\|\varphi^\lambda(z)\|_{x(r)}\|\psi^\lambda(z)\|_{x(r)} = \sqrt{2}$.

The required inequality is equivalent to $|F^\lambda(rz)| \lesssim (1-r)^{\beta-1}$. This is true if $\frac{\|\psi^\lambda(z)\|_{x(r)}}{\|\varphi^\lambda(z)\|_{x(r)}} \lesssim (1-r)^{\beta-1}$, by the Jitomirskaya-Last inequality. Because $(\eta_{2n} = \varphi_n^\lambda(z), \eta_{2n+1} = \lambda z^n \overline{\varphi_n(1/\bar{z})})$ and $(\eta_{2n} = \psi_n^\lambda(z), \eta_{2n+1} = -\lambda z^n \overline{\psi_n(1/\bar{z})})$ solve

$$(\eta_{2n+2}(z), \eta_{2n+3}(z))^T = T_n(z)(\eta_{2n}(z), \eta_{2n+1}(z))^T,$$

with initial conditions $(1, \bar{\lambda})$ and $(1, -\bar{\lambda})$, the hypothesis applies to ψ^λ and φ^λ . Therefore,

$$\|\psi^\lambda(z)\|_{x(r)}^\beta \|\varphi^\lambda(z)\|_{x(r)}^{\beta-2} \lesssim x(r)^{\gamma_1(\beta-2) + \gamma_2\beta} \simeq 1.$$

By the definition of $x(r)$, it follows that

$$\|\psi^\lambda(z)\|_{x(r)}^\beta \|\varphi^\lambda(z)\|_{x(r)}^{\beta-2} \lesssim (1-r)^{\beta-1} \|\psi^\lambda(z)\|_{x(r)}^{\beta-1} \|\varphi^\lambda(z)\|_{x(r)}^{\beta-1},$$

which is equivalent to the required inequality. \square

Theorem 3.3. *Given $z \in \Sigma$, suppose that the estimate (3.3) holds. Then, where*

$$G_{kl}(z) = (\delta_k, (\mathcal{E} - z)^{-1} \delta_l),$$

$$|G_{00}(rz) + G_{11}(rz)| \leq C_4(1-r)^{\beta-1},$$

for all $r \in (0.9, 1)$ and C_4 a z and r -independent constant. Consequently, Λ is β -Hölder continuous at z .

In particular, assume that $S \subset \partial\mathbb{D}$ is a Borel set such that there are constants γ_1, γ_2 and, for each $z \in S$, there are constants $C_1(z), C_2(z)$ so that

$$C_1(z)L^{\gamma_1} \leq \|\eta\|_L \leq C_2(z)L^{\gamma_2}$$

for every $z \in S$ and for every solution of (3.2) that is normalized. Then, the restriction of every spectral measure of \mathcal{E} to S is purely $\frac{2\gamma_1}{\gamma_1+\gamma_2}$ -continuous, that is, it gives zero weight to sets of zero $h^{\frac{2\gamma_1}{\gamma_1+\gamma_2}}$ measure.

Proof. The following is a maximum modulus principle argument similar to that in [DKL00]. Fix $z \in \Sigma$ and $r \in (0.9, 1)$. We consider (3.4) and obtain

$$\sup_{\lambda \in \partial\mathbb{D}} \left| \frac{(1-\lambda) + (1+\lambda)F_+(rz)}{(1+\lambda) + (1-\lambda)F_+(rz)} \right| \leq C_3(1-r)^{\beta-1}. \quad (3.5)$$

Since $-M_-(z)$ is a Carathéodory function it maps to the right half plane, and so the expression $(M_-(rz) + 1)/(M_-(rz) - 1)$ has modulus less than 1. Thus by the

maximum modulus principle,

$$\left| \frac{\left(1 - \frac{M_-(rz)+1}{M_-(rz)-1}\right) + \left(1 + \frac{M_-(rz)+1}{M_-(rz)-1}\right) F_+(rz)}{\left(1 + \frac{M_-(rz)+1}{M_-(rz)-1}\right) + \left(1 - \frac{M_-(rz)+1}{M_-(rz)-1}\right) F_+(rz)} \right| \leq C_3(1-r)^{\beta-1}.$$

Now if we simplify the expression on the left, we have

$$\left| \frac{1 - M_-(rz)F_+(rz)}{F_+(rz) - M_-(rz)} \right|.$$

A table on page 181 of [GZ06] computes the values of $u_{\pm}(1)$, $u_{\pm}(0)$, $v_{\pm}(1)$, and $v_{\pm}(0)$. We use $k_0 = -1$ in that table. Note that we index our solutions u, v differently, so in their notation,

$$u_+(n) = q_+(n-1) + F_+ p_+(n-1),$$

$$u_-(n) = q_+(n-1) + M_- p_+(n-1),$$

$$v_+(n) = s_+(n-1) + F_+ r_+(n-1),$$

$$v_-(n) = s_+(n-1) + M_- r_+(n-1).$$

We also label our Verblunsky coefficients differently than they do: their α_n is written as $-\overline{\alpha(n)}$ in our notation. Using these calculations and (3.1), we can write $G_{00} + G_{11}$ as

$$-\frac{(-1 + F_+)(1 + M_-)}{2(F_+ - M_-)} - \frac{[z + \overline{\alpha(0)} + M_-(z - \overline{\alpha(0)})][-1 - \alpha(0)z + F_+(1 - \alpha(0)z)]}{2\rho(0)^2 z (F_+ - M_-)}.$$

For r approaching 1, $|G_{00}(rz) + G_{11}(rz)|$ gets large when $F_+(rz) - M_-(rz)$ is close to zero, or when $F_+(rz)$ and $F_-(rz)$ both go to infinity. In both these cases,

$$|G_{00}(rz) + G_{11}(rz)| \leq C_4 \left| \frac{1 - M_-(rz)F_+(rz)}{F_+(rz) - M_-(rz)} \right|,$$

for an appropriate constant C_4 . It is not difficult to see then, as a consequence

$$|G_{00}(rz) + G_{11}(rz)| \leq C_4(1-r)^{\beta-1}.$$

Let us first note the connection between $G_{00} + G_{11}$ and the Carathéodory function F corresponding to \mathcal{E} and $d\Lambda$. We have by definition

$$F(z) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\Lambda(\theta).$$

Let us also define

$$d\Lambda_r(\theta) = \operatorname{Re} F(re^{i\theta}) \frac{d\theta}{2\pi}.$$

It is well known that $d\Lambda_r$ converges to $d\Lambda$ weakly. We note also that

$$\begin{aligned} F(z) &= \int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\Lambda(\theta) \\ &= 1 + 2z \int \frac{1}{e^{i\theta} - z} d\Lambda(\theta) \\ &= 1 + 2z(G_{00}(z) + G_{11}(z)). \end{aligned}$$

We then deduce that $\Lambda(z)$ is uniformly β -Hölder continuous on Σ . Writing $z = e^{i\Theta}$:

$$\Lambda[e^{i(\Theta-\epsilon)}, e^{i(\Theta+\epsilon)}] = \int_{\Theta-\epsilon}^{\Theta+\epsilon} 1 d\Lambda(\Theta).$$

We note that for sufficiently small ϵ the above is less than

$$2\epsilon (\operatorname{Re} F((1-\epsilon)z) + 1) \leq C\epsilon^\beta,$$

since $\beta \leq 1$. □

Note that if we let C_1, C_2 be z -dependent, the theorem still holds, except that C_3 is also z -dependent. We can then conclude:

Theorem 3.4. *Let Σ be a Borel subset of $\partial\mathbb{D}$, and let \mathcal{C} be a CMV operator on $\ell^2(\mathbb{N})$. Suppose there are constants γ_1, γ_2 such that for each $z \in \Sigma$, every normalized solution of $\eta(z)$ of the transfer matrix recursion (3.2) obeys the estimate*

$$C_1(z)L^{\gamma_1} \leq \|\eta\|_L \leq C_2(z)L^{\gamma_2}$$

for $L > 0$ sufficiently large. Let $\beta = 2\gamma_1/(\gamma_1 + \gamma_2)$. Then any extension \mathcal{E} of \mathcal{C} to $\ell^2(\mathbb{Z})$ has purely β -continuous spectrum on Σ . Moreover, if $C_1(z)$ and $C_2(z)$ are independent of z , then for any $\varphi \in \ell^2$ of compact support, the spectral measure of (\mathcal{E}, φ) is uniformly β -Hölder continuous on Σ .

Before we proceed, we state and prove the following well-known fact for the reader's convenience:

Lemma 3.5. *For any $n \in \mathbb{Z}$, $\{\delta_{2n}, \delta_{2n+1}\}$ form a spectral basis for \mathcal{E} .*

Proof of lemma. First, let us show that δ_{2n+2} is in the span $S_{2n, 2n+1}$ of

$$\{\mathcal{E}^k \delta_{2n}\}_{k \in \mathbb{Z}} \cup \{\mathcal{E}^k \delta_{2n+1}\}_{k \in \mathbb{Z}}.$$

First, note that we have

$$\begin{aligned} \mathcal{E}\delta_{2n+1} &= \overline{\alpha(2n+1)}\rho(2n)\delta_{2n} - \overline{\alpha(2n+1)}\alpha(2n)\delta_{2n+1} \\ &\quad + \overline{\alpha(2n+2)}\rho(2n+1)\delta_{2n+2} + \rho(2n+2)\rho(2n+1)\delta_{2n+3}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \mathcal{E}\delta_{2n+2} &= \rho(2n+1)\rho(2n)\delta_{2n} - \rho(2n+1)\alpha(2n)\delta_{2n+1} \\ &\quad - \overline{\alpha(2n+2)}\alpha(2n+1)\delta_{2n+2} - \rho(2n+2)\alpha(2n+1)\delta_{2n+3}. \end{aligned} \tag{3.7}$$

This gives us

$$\begin{aligned} \frac{\alpha(2n+1)}{\rho(2n+1)}\mathcal{E}\delta_{2n+1} + \mathcal{E}\delta_{2n+2} &= \left(\frac{|\alpha(2n+1)|^2\rho(2n)}{\rho(2n+1)} + \rho(2n+1)\rho(2n) \right) \delta_{2n} \\ &\quad - \left(\frac{|\alpha(2n+1)|^2\alpha(2n)}{\rho(2n+1)} + \rho(2n+1)\alpha(2n) \right) \delta_{2n+1}, \end{aligned}$$

and we conclude that $S_{2n,2n+1}$ contains $\mathcal{E}\delta_{2n+2}$. Applying \mathcal{E}^{-1} on both sides of the preceding equation shows that it also contains δ_{2n+2} .

By considering the expressions for $\mathcal{E}\delta_{2n-1}$ and $\mathcal{E}\delta_{2n}$ instead, we can similarly show that $\mathcal{E}\delta_{2n-1}$, and hence δ_{2n-1} lies in $S_{2n,2n+1}$.

Now let us demonstrate that δ_{2n+3} is in $S_{2n,2n+1}$. We consider (3.6) and (3.7) once more, and this time by eliminating the $\delta_{2n}, \delta_{2n+1}$ terms we get

This gives us

$$\begin{aligned} \mathcal{E}\delta_{2n+1} - \frac{\overline{\alpha(2n+1)}}{\rho(2n+1)}\mathcal{E}\delta_{2n+2} \\ &= \left(\overline{\alpha(2n+2)}\rho(2n+1) + \frac{|\alpha(2n+1)|^2\alpha(2n+1)}{\rho(2n+1)} \right) \delta_{2n+2} \\ &\quad + \left(\rho(2n+2)\rho(2n+1) + \frac{|\alpha(2n+1)|^2\rho(2n+2)}{\rho(2n+1)} \right) \delta_{2n+3}, \end{aligned}$$

and this demonstrates that δ_{2n+3} lies in $S_{2n+1,2n+2}$, and hence $S_{2n,2n+1}$.

We can similarly show that δ_{2n-2} lies in $S_{2n-1,2n}$ and hence $S_{2n,2n+1}$, by using the expressions for $\mathcal{E}\delta_{2n-1}$ and $\mathcal{E}\delta_{2n}$ and then eliminating the $\delta_{2n}, \delta_{2n+1}$ terms.

We have now shown that $S_{2n,2n+1}$ contains $\{\delta_{2n-2}, \delta_{2n-1}, \delta_{2n+2}, \delta_{2n+3}\}$. A simple induction argument now tells us that $S_{2n,2n+1} = \ell^2(\mathbb{Z})$. \square

Remark It is easy to see that $\{\delta_{2n-1}, \delta_{2n}\}$ for any n also form a spectral basis.

Proof of Theorem 3.4. From the lemma and its proof, we see that given a $\phi \in \ell^2(\mathbb{Z})$

with support on $\{-N, \dots, N+1\}$, there must exist polynomials P_0, P_1 of degree not exceeding N such that $P_0(\mathcal{E})\delta_0 + P_1(\mathcal{E})\delta_1 = \phi$. This implies that the spectral measure for ϕ is bounded by $q(z)d\Lambda(z)$ for some polynomially bounded function $q(z)$. If C_1, C_2 are independent of z , then, by the corollary $d\Lambda$ is uniformly β -Hölder continuous, and this implies that $qd\Lambda$ is also uniformly β -Hölder continuous. In the case that C_1, C_2 depend on z , we know that $d\Lambda$ is β -continuous. Given any $\phi \in \ell^2$, its spectral measure is dominated by Λ and so must be β -continuous as well. \square

3.3 An application to Sturmian Verblunsky coefficients

Theorem 3.6. *Given a sequence $\{A_n : \mathbb{T} \rightarrow \text{SL}(2, \mathbb{C})\}_{n=0}^\infty$, let us write $M_k(z) = \prod_{n=k}^0 A_n(z)$. Suppose there are sequences a_n and q_n of natural numbers related by $q_{n+1} = a_{n+1}q_n + q_{n-1}$, such that $M_{q_{n+1}}(z) = M_{q_{n-1}}(z)M_{q_n}(z)^{a_{n+1}}$. Let $x_n(z) = \text{tr}M_{q_n}(z)$ and $z_n(z) = \text{tr}M_{q_{n-1}}(z)M_{q_n}(z)$, and put $I(z) = x_{n-1}^2 + x_n^2 + z_n^2 - x_{n-1}x_nz_n$.*

Suppose that:

1. *The function I is independent of n .*
2. *The sequence a_n is of bounded density: $d = \limsup \frac{1}{N} \sum_{n=1}^N a_n$ is finite.*
3. *There is a compact set $\Sigma \subset \mathbb{T}$ and a constant K such that $z \in \Sigma$ iff $|x_n(z)| \leq K$ or $|z_n(z)| \leq K$ for all n .*

Then for all $z \in \Sigma$, there exist $\gamma_2(z)$ and $C(z)$ independent of n such that

$$\|M_n(z)\| \leq C(z)n^{\gamma_2(z)}.$$

These conditions are sufficient to apply the argument in [IRT92]. One obtains

$$C(z) = L^{4d}, \quad \gamma_2(z) = 4d \log_2 L,$$

where

$$\begin{aligned} L = & \max(4 \max(2, \sup |x_n|, \sup |z_n|), 4\|M_1\|, 4\|M_0\|, 4\|M_0M_1\|) \\ & \times (4 + 2 \max(2, \sup |x_n|, \sup |z_n|)). \end{aligned}$$

The method used in [DKL00] can be applied to show that for some C' ,

$$\|\xi\|_L \leq C'(z)n^{\gamma(z)}$$

for any solution $\xi(z)$ to the transfer matrix recursion. Compactness of Σ and continuity of C, γ yield a z -independent bound by taking the maximum.

Theorem 3.7. *Let q_n be the convergents of the continued fraction $[a_1, a_2, a_3, \dots]$.*

Relaxing the bounded density hypothesis to require only that q_n be bounded above by a geometric sequence, such a sequence of maps into $\mathrm{SL}(2, \mathbb{C})$ satisfies, for all $z \in \Sigma$,

$$\|\xi(z)\|_L \geq C_2(z)L^{\gamma_1(z)}$$

for some $C(z), \gamma_1(z)$, and for L large enough.

Proof. The method used in [DKL00] applies without any significant changes. It only deals with model-independent properties of the transfer matrices.

□

Claim 3.8. *In both cases, an extended CMV matrix with Verblunsky coefficients that have a Fibonacci sequence as a suffix furnishes an example of such a sequence of maps.*

Proof. Let $\{T_n(z)\}_{n=0}^\infty$ be the sequence of n -step transfer matrices corresponding to the quasiperiodic CMV operator \mathcal{E} . Then $\det(T_n(z)) = z^n$, so that $M_n(z) := T_n(z)/z^{n/2}$ is in $\mathrm{SL}(2, \mathbb{C})$. It is well known (see [Sim04] 12.8) that the family T_n obeys a substitution rule of the necessary type; so does the family M_n . Because the spectrum of \mathcal{E} is contained in \mathbb{T} , M_n and T_n always have the same operator norms. Finally, that the traces $x_n(z)$ obey the required bound is proved in [Sim04] 12.8.

The method in [DKL00] provides a simple expression for γ_1 . Put $q_n \leq B^n$, and let $C(\alpha, \beta) := \max\{\max_{|z|=1} 2 + \sqrt{8 + I(z)}, \frac{4}{\sqrt{1-|\alpha|^2}\sqrt{1-|\beta|^2}}\}$. Then

$$\gamma_1 = \frac{\log\left(1 + \frac{1}{4C(\alpha, \beta)^2}\right)}{16 \log B}.$$

The constant $C(\alpha, \beta)$ occurs because it bounds $|x_n(z)|$ for $z \in \Sigma$.

□

Corollary 3.9. *With the notation and assumptions above, the spectral measure of such a CMV operator is uniformly β -Hölder continuous for $\beta = \frac{2\gamma_1}{\gamma_1 + \gamma_2}$.*

3.4 An application to quantum walks

In this section, we will explain the how the results in the previous sections apply to the theory of quantum walks. In the second subsection, we will provide a brief explanation of quantum walks and the role of the CMV matrix, as well as a brief summary of the

results in [DMY13a] and [DMY13b] concerning the Fibonacci CMV operator and the corresponding quantum walk. In the first subsection, we will summarize results in [DFV13] about the relationship between Hölder continuity of spectral measures and dynamical spreading of the corresponding quantum walk.

3.4.1 Dynamical spreading and unitary operators

Here we summarize the relevant results of [DFV13], which are unitary analogues of the Guarneri-Combes-Last and Guarneri-Schulz-Baldes dynamical spreading theorems (we refer the interested reader to [DT10] for a history of the problem).

We let \mathcal{H} be a complex separable Hilbert space, U a unitary operator on \mathcal{H} and $\psi \in \mathcal{H}$ be a vector normalized so $\|\psi\| = 1$. We wish to consider the time evolution, $\psi(k) = U^k\psi$. We let $\{\varphi_n\}$ be an orthonormal basis (assume for our purposes that n is indexed in \mathbb{Z}). To describe the spreading of ψ with respect to this basis, we define

$$a_\psi(n, k) = |\langle \varphi_n, \psi(k) \rangle|^2.$$

This can be thought of as the probability that ψ is at state φ_n at time k . Let us consider the Cesàro time-averaged probabilities

$$\tilde{a}_\psi(n, K) = \frac{1}{K} \sum_{k=0}^{K-1} a_\psi(n, k).$$

We then consider the moments of the position operator, given by

$$|X|_\psi^p(K) = \sum_n (|n|^p + 1) \tilde{a}_\psi(n, K).$$

We would like to compare the growth of $|X|_\psi^p(K)$ to polynomial growth of the form $K^{p\beta}$ for a suitable exponent β . Thus we define

$$\beta_\psi(p) = \liminf_{K \rightarrow \infty} \frac{\log(|X|_\psi^p(K))}{p \log K}.$$

We can then state the theorem

Theorem 3.10 ([DFV13], Corollary 3.13). *Let μ be a finite Borel measure on $\partial\mathbb{D}$.*

The upper Hausdorff dimension of μ is given by

$$\dim_H^+(\mu) = \inf\{\dim_H(S) : S \subset \partial\mathbb{D} \text{ measurable, } \mu(S) = \mu(\partial\mathbb{D})\}.$$

(Loosely speaking, $\dim_H^+(\mu)$ is the smallest dimension of a set that supports μ).

Let μ_ψ^U be the spectral measure of the operator U . We then have

$$\beta_\psi(p) \geq \dim_H^+(\mu_\psi^U).$$

Naturally, combined with our Theorem 3.4, we can assert dynamical spreading for unitary operators whose eigensolution obey certain power-law bounds.

3.4.2 Quantum walks and CMV operators

We shall now explore the connection between CMV operators and quantum walks, first discovered in [CGMV10]. We consider a Hilbert space $\mathcal{H} = \mathbb{Z} \otimes \mathbb{C}^2$ with basis vectors of the form $|n\rangle \otimes |\uparrow\rangle$, $|n\rangle \otimes |\downarrow\rangle$ for $n \in \mathbb{Z}$. We will refer to elements of \mathcal{H} as *states*. The $|n\rangle$ part is called the *site*, and the $|\uparrow\rangle$ part is called the *spin*.

We are given

$$C_n = \begin{pmatrix} c_n^{11} & c_n^{12} \\ c_n^{21} & c_n^{22} \end{pmatrix} \in U(2),$$

We will refer to the C_n as *coins*. In some settings the coins will depend on a time parameter, but for our purposes we will assume time-independence. We will assume a time transition from t to $t + 1$ on the basis vectors as follows:

$$\begin{aligned} |n\rangle \otimes |\uparrow\rangle &\rightarrow c_n^{11}|n+1\rangle \otimes |\uparrow\rangle + c_n^{21}|n-1\rangle \otimes |\downarrow\rangle, \\ |n\rangle \otimes |\downarrow\rangle &\rightarrow c_n^{12}|n+1\rangle \otimes |\uparrow\rangle + c_n^{22}|n-1\rangle \otimes |\downarrow\rangle. \end{aligned}$$

This extends by linearity to a transition operator $U : \mathcal{H} \rightarrow \mathcal{H}$. The key observation of [CGMV10] is that using a suitable change of basis, U can be rewritten explicitly in the form (1.2).

We will now discuss the special case of the Fibonacci quantum walk as considered in [DMY13b]. Let us recall subshift dynamical systems. That is, given an alphabet $\mathcal{A} = \{a, b\}$, we consider a word $u \in \mathcal{A}^{\mathbb{N}}$. The *hull* Ω_S generated by this substitution is the set of all two-sided sequences all of whose subblocks are also subblocks of u . The *subshift dynamical system* (Ω_S, \mathcal{T}) consists of the hull, the left shift \mathcal{T} on Ω_S , and the product topology on Ω_S . For each $\omega \in \Omega_S$, we can then generate a CMV operator \mathcal{E}_ω by taking $\alpha_\omega(n) = f(\omega_n)$, where ω_n refers to the n th letter of ω , and $f : \mathcal{A} \rightarrow \mathbb{D}$ is a nonconstant function.

We define a substitution as a semigroup homomorphism $S : \mathcal{A}^* \rightarrow \mathcal{A}^*$, where \mathcal{A}^*

is the semigroup of words generated by \mathcal{A} under the concatenation operation. We define the *Fibonacci substitution* by setting $S(a) = ab, S(b) = a$ and extending the definition of S by concatenation. Thus $S(a), S^2(a), S^3(a), \dots$ all start with a and we have a limit $S^\infty(a) \in \mathcal{A}^{\mathbb{Z}^+}$. If we extend S by concatenation to act on infinite words, $S^\infty(a)$ is invariant under the action of S . We take this $S^\infty(a)$ as our sequence u , which we call the *Fibonacci word*.

For two distinct $\theta_a, \theta_b \in (-\pi/2, \pi/2)$, we define two coins

$$C_a = \begin{pmatrix} \cos \theta_a & -\sin \theta_a \\ \sin \theta_a & \cos \theta_a \end{pmatrix}, C_b = \begin{pmatrix} \cos \theta_b & -\sin \theta_b \\ \sin \theta_b & \cos \theta_b \end{pmatrix}.$$

Then we choose an element $\omega \in \Omega_S$ and place these coins on our quantum walk system according to elements of the sequence. In other words, $C_n = C_a$ if the n th element of w is a , and $C_n = C_b$ otherwise. Let us write \mathcal{E}_ω, ψ as the CMV operator corresponding to this quantum walk with initial vector ψ , and $\mu_{\mathcal{E}_\omega, \psi}$ the corresponding spectral measure.

In [DMY13b], the authors apply our Theorems 3.4, 3.6 and 3.7, Claim 3.8 as well as Theorem 3.10 to prove

Theorem 3.11 ([DMY13b], Theorem 2.1). *Define:*

1. $I(z) = \operatorname{Re}(z)^2(\sec^2 \theta_a + \sec^2 \theta_b) + (\operatorname{Re}(z^2) \sec \theta_a \sec \theta_b - \tan \theta_a \tan \theta_b)^2 - 2(\operatorname{Re}(z)^2 \sec^2 \theta_a \sec^2 \theta_b (\operatorname{Re}(z^2) - \sin \theta_a \sin \theta_b)) - 1;$
2. $C(z) = \max\{2 + \sqrt{8 + I(z)}, (\sec \theta_a)^{-1}, (\sec \theta_b)^{-1}\};$
3. $\gamma_1(z) = \frac{\log\left(1 + \frac{1}{4C(z)^2}\right)}{16 \log \phi},$ where ϕ is the golden mean;

4. $\gamma_2(z) = 4 \log_2 K(z)$, where K is a z -dependent constant;

5.
$$\beta(z) = \frac{2\gamma_1(z)}{\gamma_1(z) + 2\gamma_2(z) + 1}.$$

Then, for all ψ, ω, p as above, we have

$$\tilde{\beta}_{\omega, \psi}^{\pm}(p) \geq \max \{ \beta(z) : z \in \text{supp } \mu_{\mathcal{E}_{\omega}, \psi} \},$$

where $\mu_{\mathcal{E}_{\omega}, \psi}$ denotes the spectral measure associated with the unitary operator \mathcal{E}_{ω} and the state ψ .

Chapter 4

Perturbations of periodic operators using generalized Prüfer variables

Results in this chapter are joint work with Milivoje Lukic. We prove Theorems 1.10, 1.12, and 1.13.

4.1 Decaying oscillatory perturbations of periodic Schrödinger operators

Let us first discuss solutions of the unperturbed Schrödinger equation

$$H_0\varphi := -\varphi'' + V_0(x)\varphi = E\varphi, \tag{4.1}$$

where φ is the Floquet solution. We express φ as $p(x)e^{ikx}$, where k is the quasimomentum. We also write $\varpi(x) = \text{Arg } p(x)$. When we wish to emphasize the dependence of k on E , we will write k_E . It is known that the essential spectrum of a periodic

Schrödinger operator consists of a union of absolutely continuous closed intervals (often referred to as “bands”). Any two of those bands can intersect at most at a point. Additionally, by Weyl’s theorem, $\sigma_{ess}(H) = \sigma_{ess}(H_0)$.

Let us define the set S so that

$$S = \{E \in \text{Int}(\sigma_{ess}(H)) \mid \text{not all solutions of (1.11) are bounded}\}.$$

Due to standard results in spectral theory ([GP87],[Beh91],[Sto92]), we know that μ is mutually absolutely continuous with Lebesgue measure on $\text{Int}(\sigma_{ess}(H)) \setminus S$.

Our theorem for the infinite frequency setting is Theorem 1.10. Our theorem for the finite frequency case (i.e. all but finitely many of the c_l s are zero) is given as

Theorem 4.1. *Considering V chosen so that (1.6) is a finite sum,*

$$S \subseteq \left\{ E \in \text{Int}(\sigma_{ess}(H)) \mid \pm 2k_E \in \bigcup_{l=1}^{p-1} A \oplus \dots \oplus A \pmod{2\pi} \text{ (sum of } l \text{ } A\text{'s mod } 2\pi) \right\}, \quad (4.2)$$

where A is the set of all ϕ_l s. This implies that each band of $\sigma_{ess}(H)$ has at most finitely many embedded pure points, that H has no singular continuous spectrum, and that the absolutely continuous spectrum of H is equal to $\sigma_{ess}(H)$.

Conversely, we can also say

Theorem 4.2. *Fix p and the size of $A = \{\phi_l\}$. For any choice of A away from an algebraic set of codimension 1, if we fix an energy E and a quasimomentum k_E in the RHS of (4.2), among all the $V_0 \in L^1_{loc}((0, 1))$ for which k_E corresponds to E , there is an open and dense set of V_0 such that there is a choice of V for which the spectrum of H has an embedded pure point at E .*

These results are an extension of [Luk13b] and [Luk13a], which study the case when our V_0 is identically zero. We may also see these results as extensions of [KN07], which addresses the problem in L^2 for a less general V , using different methods.

The next three sections will explain the proofs of Theorems 4.1 and 1.10. In the last section we will discuss a converse problem, that is, the existence of embedded eigenvalues in the case where (1.6) is a finite sum. This section will culminate in a proof of Theorem 4.2.

4.1.1 Preliminary Lemmas

We wish to control solutions of the perturbed Schrödinger operator (1.11) by comparing them to solutions of the unperturbed operator (4.1), so we will use modified Prüfer variables which were defined in [KRS99] for this purpose.

We first fix E and fix a φ . We will need to choose φ so that it is linearly independent of $\bar{\varphi}$: this is possible since we restrict our attention to E in the interior of σ_{ess} . We define $\rho(x) \in \mathbb{C}$ as in (21) of [KRS99]:

$$\begin{pmatrix} u'(x) \\ u(x) \end{pmatrix} = \text{Im} \left[\rho(x) \begin{pmatrix} \varphi'(x) \\ \varphi(x) \end{pmatrix} \right].$$

We also define

$$R(x) = |\rho(x)| \quad (4.3)$$

$$\eta(x) = \text{Arg}(\rho(x)) \quad (4.4)$$

$$\theta(x) = kx + \varpi(x) + \eta(x). \quad (4.5)$$

We choose η so that $\eta(0) \in (-\pi, \pi]$ and η is continuous. Our Prüfer variables are going to be R and η .

Let ω be the Wronskian of $\varphi, \bar{\varphi}$. Note that the Wronskian is real, nonzero and independent of x . By Theorem 2.3 of [KRS99] we have that

$$[\ln R(x)]' + i\eta'(x) = \frac{\rho'(x)}{\rho(x)} = \frac{2|\varphi(x)|^2}{\omega} V(x) \sin(kx + \varpi(x) + \eta(x)) e^{-ikx - i\varpi(x) - i\eta(x)},$$

which we then rewrite as

$$-\eta'(x) + i[\ln(R(x))]' = \frac{|\varphi(x)|^2}{\omega} V(x) (1 - e^{-2i(kx + \varpi(x) + \eta(x))}).$$

In particular, we can write

$$[\ln R(x)]' = \text{Im} \left(\frac{|\varphi(x)|^2}{\omega} V(x) e^{2ikx + 2i\varpi(x) + 2i\eta(x)} \right), \quad (4.6)$$

$$\eta'(x) = \frac{|\varphi(x)|^2}{\omega} V(x) \left(-1 + \frac{1}{2} \left(e^{2ikx + 2i\varpi(x) + 2i\eta(x)} + e^{-2ikx - 2i\varpi(x) - 2i\eta(x)} \right) \right). \quad (4.7)$$

Note that our choice of $V(x)$ given in (1.6) means that $[\ln(R(x))]'$ can be written as a sum of terms which are products of periodic and decaying factors. Our immediate goal is to perform a sequence of manipulations that look like integration by parts so as to integrate the periodic factors of $[\ln(R(x))]'$ and differentiate the decaying factors, an approach in the spirit of [Luk13b] and [Luk13a]. First, we need a technical lemma to choose the appropriate antiderivative for the periodic factors.

Proposition 4.3. *Let $\Phi(\alpha; x)$ be continuous and 1-periodic in x . Then there exists a continuous 1- periodic function $\tilde{\Phi}_\alpha(x)$ such that*

$$(i\tilde{\Phi}_\alpha(x)e^{i\alpha x})' = (1 - e^{i\alpha})\Phi(\alpha; x)e^{i\alpha x}.$$

Furthermore, if $\alpha \not\equiv 0 \pmod{2\pi}$, this function is unique.

Proof. Let $Q_\alpha(x)$ be the antiderivative of $\Phi(\alpha; x)e^{i\alpha x}$ such that $Q_\alpha(0) = 0$. We then define

$$\tilde{\Phi}_\alpha(x) = -iQ_\alpha(x)e^{-i\alpha x}(1 - e^{i\alpha}) + iQ_\alpha(1)e^{-i\alpha x}. \quad (4.8)$$

We first note that $(i\tilde{\Phi}_\alpha(x)e^{i\alpha x})' = (1 - e^{i\alpha})\Phi(\alpha; x)e^{i\alpha x}$. It remains to show that $\tilde{\Phi}_\alpha(x)$ is 1-periodic.

We observe that since $i\tilde{\Phi}_\alpha(x)e^{i\alpha x}$ is an antiderivative of $(1 - e^{i\alpha})\Phi(\alpha; x)e^{i\alpha x}$, and since

$$\tilde{\Phi}_\alpha(1) = -iQ_\alpha(1)e^{-i\alpha}(1 - e^{i\alpha}) + iQ_\alpha(1)e^{-i\alpha} = iQ_\alpha(1) = \tilde{\Phi}_\alpha(0),$$

we must then have

$$\begin{aligned}
i\tilde{\Phi}_\alpha(x)e^{i\alpha x} - i\tilde{\Phi}_\alpha(0) &= (1 - e^{i\alpha}) \int_0^x \Phi(\alpha; t)e^{i\alpha t} dt \\
&= (e^{-i\alpha} - 1) \int_0^x \Phi(\alpha; t+1)e^{i\alpha(t+1)} dt \quad (\text{due to 1-periodicity of } \Phi) \\
&= (e^{-i\alpha} - 1) \int_1^{x+1} \Phi(\alpha; t)e^{i\alpha t} dt \\
&= e^{-i\alpha} \left(i\tilde{\Phi}_\alpha(x+1)e^{i\alpha(x+1)} - i\tilde{\Phi}_\alpha(1)e^{i\alpha} \right) \\
i\tilde{\Phi}_\alpha(x)e^{i\alpha x} &= i\tilde{\Phi}_\alpha(x+1)e^{i\alpha x} - i\tilde{\Phi}_\alpha(1) + i\tilde{\Phi}_\alpha(0) \\
\tilde{\Phi}_\alpha(x) &= \tilde{\Phi}_\alpha(x+1).
\end{aligned}$$

We now demonstrate uniqueness when $\alpha \not\equiv 0 \pmod{2\pi}$. Consider a 1-periodic continuous function $\Psi(x)$ such that $(\Psi(x)e^{i\alpha x})' = (1 - e^{i\alpha})\Phi(\alpha; x)e^{i\alpha x}$. We then know that for some constant C ,

$$\Psi(x)e^{i\alpha x} = \tilde{\Phi}_\alpha(x)e^{i\alpha x} + C$$

we can then write

$$\Psi(x) = \tilde{\Phi}_\alpha(x) + Ce^{-i\alpha x}.$$

Observe that $e^{-i\alpha x}$ is not 1-periodic since α cannot be a multiple of 2π , by hypothesis. Thus it must be true that $C = 0$, and so $\Psi(x) = \tilde{\Phi}_\alpha(x)$. \square

Remark It is easy to see that if $\alpha \equiv 0 \pmod{2\pi}$, we may define $\tilde{\Phi}_\alpha(x)$ to be an arbitrary constant. The following lemma demonstrates that there is a natural choice for this arbitrary constant.

Lemma 4.4. *Let n be an integer. Assume that the function $\Phi(\alpha; x)$ is continuous and 1-periodic in x , and converges uniformly as $\alpha \rightarrow 2\pi n$ for x on \mathbb{R} . Then $Q_{2\pi n}(1) =$*

$\lim_{\alpha \rightarrow 2\pi n} Q_\alpha(1)$ exists, and $\tilde{\Phi}_\alpha(x)$ converges to $iQ_{2\pi n}(1)$ as $\alpha \rightarrow 2\pi n$ uniformly for x on \mathbb{R}

Proof. Let us define, as in Proposition 4.3, $Q_\alpha(x)$ as the antiderivative of $\Phi(\alpha; x)e^{i\alpha x}$ with $Q_\alpha(2\pi n) = 0$. Then, due to the fact that $\Phi(\alpha; x), e^{i\alpha x}$ uniformly converge as $\alpha \rightarrow 2\pi n$ for x in compact subsets of \mathbb{R} ,

$$\begin{aligned} Q_\alpha(x) &= \int_0^x \Phi(\alpha; t)e^{i\alpha t} dt \\ \lim_{\alpha \rightarrow 2\pi n} \int_0^x \Phi(\alpha; t)e^{i\alpha t} dt &= \int_0^x \lim_{\alpha \rightarrow 2\pi n} \Phi(\alpha; t)e^{i\alpha t} dt \\ &= \int_0^x \Phi(2\pi n; t) dt \\ &= Q_{2\pi n}(x). \end{aligned}$$

Thus indeed $\lim_{\alpha \rightarrow 2\pi n} Q_\alpha(x)$ converges to $Q_{2\pi n}(x)$. Furthermore, since

$$\int_{x_0}^{x_1} \Phi(\alpha; t)e^{i\alpha t} dt,$$

is bounded for $x_0 < x_1$ and x_1 bounded, the convergence is uniform on compact subsets of x . From (4.8), we then find that $\tilde{\Phi}_\alpha(x)$ converges uniformly on compact subsets of \mathbb{R} . Since $\tilde{\Phi}_\alpha(x)$ is 1-periodic, it follows that it in fact converges uniformly on \mathbb{R} .

□

As a consequence, defining $\tilde{\Phi}_{2\pi n}(x)$ as $iQ_{2\pi n}(1)$ ensures that $\tilde{\Phi}_\alpha(x)$ is uniformly continuous at $\alpha \equiv 0 \pmod{2\pi}$.

Let us define $C_{per}(0, \infty)$ as the space of continuous 1-periodic functions on $(0, \infty)$ with the uniform norm. Then the preceding proposition allows us to make the following definition:

Definition 4.5. For $K \in \mathbb{Z}, \alpha \in \mathbb{R}$, $\lambda_{\alpha, K}$ is a linear operator from $C_{per}(0, \infty)$ to itself so that $\lambda_{\alpha, K}$ takes each $\Phi(\alpha; x) \in C_{per}(0, \infty)$ to the corresponding

$$\tilde{\Phi}_\alpha(x)e^{-2Ki\varpi(x)},$$

where $\tilde{\Phi}_\alpha$ is as defined in (4.8).

Lemma 4.6. *With $\|\dots\|$ denoting the operator norm, $\|\lambda_{\alpha, K}\| \leq 2$.*

Proof. Let $\Phi(\alpha; x) \in C_{per}(0, \infty)$. Note that since $|(\lambda_{\alpha, K}\Phi)(x)|$ is 1-periodic and continuous, it must have a maximum in $[0, 1)$. Let $Q_\alpha(x)$ once again be the antiderivative of $\Phi(\alpha; x)e^{i\alpha(x)}$ such that $Q_\alpha(0) = 0$. It is clear that $|Q_\alpha(x)| \leq \|\Phi\|$ for any $x \in [0, 1)$. In fact, it is also clear that $|Q_\alpha(1) - Q_\alpha(x)| \leq \|\Phi\|$. When we rewrite (4.8) as

$$\tilde{\Phi}_\alpha(x) = ie^{i\alpha}Q_\alpha(x)e^{-i\alpha x} + i(Q_\alpha(1) - Q_\alpha(x))e^{-i\alpha x},$$

it becomes clear that $\|\tilde{\Phi}_\alpha\| \leq 2$. □

Since the variations of the γ_i are uniformly bounded, it is possible to define

$$\tau = \sup_l \text{Var}(\gamma_l, (0, \infty)) < \infty.$$

Lemma 4.7. *We let $J, K \in \mathbb{Z}$ with $J \geq 1$ and $0 \leq K \leq J$, and $0 \leq a < b < \infty$. We also define $\Gamma(x) = \gamma_{m_1}(x) \dots \gamma_{m_J}(x)$, and we let ϕ be some phase in $[0, 2\pi)$. Then*

where Φ is some continuous 1-periodic function,

$$2\|\lambda_{2Kk-\phi,K}\Phi\|\tau^J \geq \left| \int_a^b \left((1 - e^{i(2Kk-\phi)})e^{2Ki(\eta(x)+kx)}e^{-i\phi x}\Gamma(x)\Phi(x) - 2Ke^{2Ki(\eta(x)+kx+\varpi(x))}e^{-i\phi x}\Gamma(x)\frac{d\eta(x)}{dx}\lambda_{2Kk-\phi,K}\Phi(x) \right) dx \right|. \quad (4.9)$$

Proof. The proof is identical to the proof of Lemma 2.1 of [Luk13a], except that we redefine $\psi(x)$ in that proof to be equal to

$$e^{2Ki\eta(x)} \cdot ie^{i(2Kkx+2K\varpi(x)-\phi x)}\lambda_{2Kk-\phi,K}\Phi(x).$$

Keep in mind that the derivative of $ie^{i(2Kkx+2K\varpi(x)-\phi x)}\lambda_{2Kk-\phi,K}(\Phi(x))$ is

$$(1 - e^{i(2Kk-\phi)})e^{(2Kk-\phi)ix}\Phi(x), \text{ by the definition of } \lambda_{2Kk-\phi,K}. \quad \square$$

4.1.2 A recursion relation

For integers J, K with $J \geq 1$ and $0 \leq K \leq J$, let us now define functions $f_{J,K}, g_{J,K}$ as follows. They are functions of $1 + J$ variables, x, ϕ_1, \dots, ϕ_J , and they also depend implicitly on E .

For convenience, we define $\Phi_0(x) = |\varphi(x)|^2/\omega$. We first set

$$f_{1,0}(x; \phi_1) = 0, f_{1,1}(x; \phi_1) = \Phi_0(x). \quad (4.10)$$

We then define

$$g_{J,K}(x; \{\phi_j\}_{j=1}^J) = \frac{2K}{1 - e^{i(2Kk-\sum_{j=1}^J \phi_j)}} \lambda_{2Kk-\sum_{j=1}^J \phi_j, K} [e^{2iK\varpi(x)} f_{J,K}(x; \{\phi_j\}_{j=1}^J)], \quad (4.11)$$

and for $J \geq 2$,

$$f_{J,K}(x; \{\phi_j\}_{j=1}^J) = \sum_{l=K-1}^{K+1} \sum_{\sigma \in S_J} \frac{\Phi_0(x)}{J!} w_{K-l} g_{J-1,l}(x; \{\phi_{\sigma(j)}\}_{j=1}^{J-1}). \quad (4.12)$$

Here S_J denotes the symmetric group in J elements and, motivated by (4.7), we define the constant function

$$w_a(x; \phi) = \begin{cases} -1 & a = 0, \\ \frac{1}{2} & a = \pm 1, \\ 0 & |a| \geq 2. \end{cases} \quad (4.13)$$

We use the symmetric product defined as Definition 2.1 of [Luk13a], with some slight modifications:

Definition 4.8. For a function p_I of I variables and a function q_J of J variables, their symmetric product is a function $p_I \odot q_J$ of $1 + I + J$ variables defined by

$$(p_I \odot q_J)(x; \{\phi_i\}_{i=1}^{I+J}) = \frac{1}{(I+J)!} \sum_{\sigma \in S_{I+J}} p_I(x; \{\phi_{\sigma(i)}\}_{i=1}^I) q_J(x; \{\phi_{\sigma(i)}\}_{i=I+1}^{I+J}).$$

Where δ refers to the Kronecker delta, we can express $f_{J,K}$ as

$$f_{J,K} = \delta_{J-1} \Phi_0(x) + \sum_{a=-1}^1 \Phi_0(x) w_a \odot g_{J-1, K+a}. \quad (4.14)$$

Lemma 4.9. For $0 \leq K \leq I$ and $0 < l < I$,

$$f_{I,K} = \frac{1}{2} \sum_{j=0}^I f_{j,l} \odot g_{I-j, K-l}. \quad (4.15)$$

$$g_{I,K} = \frac{1}{2} \sum_{j=0}^I g_{j,l} \odot g_{I-j, K-l}. \quad (4.16)$$

Proof. Let us assume for now that the $2Kk - \sum \phi$ terms are never congruent to 0 mod 2π . We prove both (4.15) and (4.16) at the same time, inductively. The

statement is vacuously true for $I \leq 1$. Let us assume now that it holds for $I - 1$. We use (4.14) and notice

$$\sum_{t=0}^I f_{t,l} \odot g_{I-t,K-l} = \sum_{t=0}^I (\delta_{t-1} \Phi_0(x) + \sum_{a=-1}^1 \Phi_0(x) w_a \odot g_{t-1,l+a}) \odot g_{I-t,K-l}.$$

Using the inductive assumption, we may apply (4.16) to the $g \odot g$ terms, unless $l + a \leq 0$. But $l + a \leq 0$ holds only for $l = 1, a = -1$, and in this exceptional case $g_{t-1,l+a} = 0$. Thus,

$$\begin{aligned} \sum_{t=0}^I f_{t,l} \odot g_{I-t,K-l} &= \sum_{t=0}^I \delta_{t-1} \Phi_0(x) \odot g_{I-t,K-l} \\ &\quad + \sum_{a=-1}^1 \sum_{t=0}^I \Phi_0(x) w_a \odot g_{t-1,l+a} \odot g_{I-t,K-l} \\ &= \delta_{l-1} \Phi_0(x) \odot g_{I-1,K-1} \\ &\quad + \sum_{a=-1}^1 \sum_{t=0}^I \Phi_0(x) w_a \odot 2(g_{I-1,K+a} - \delta_{a+1} \delta_{l-1} g_{I-1,K-1}) \\ &\quad \text{(because } t = 1, l > 1 \text{ implies } f_{t,l} = 0) \\ &= \delta_{l-1} \Phi_0(x) \odot g_{I-1,K-1} \\ &\quad + 2f_{I,K} - \delta_{I-1} \Phi_0(x) - 2w_{-1} \Phi_0(x) \odot \delta_{l-1} g_{I-1,K-1} \\ &= 2f_{I,K}. \end{aligned}$$

The last equality is due to the fact that we are assuming $I > 1$. It remains to prove (4.16).

We can calculate that for any $\sigma \in S_I$, using the product rule and (4.11)

$$\begin{aligned} & \frac{d}{dx} \left(\frac{ie^{i(2K(kx+\varpi(x))-\sum_{j=1}^I \phi_j x)}}{2} \sum_{t=0}^I g_{t,l}(\{x; \phi_{\sigma(j)}\}_{j=1}^t) g_{I-t,K-l}(\{x; \phi_{\sigma(j)}\}_{j=t+1}^I) \right) \\ &= \frac{d}{dx} \left(\frac{i}{2} \sum_{t=0}^I \left[e^{i(2l(kx+\varpi(x))-\sum_{j=1}^t \phi_{\sigma(j)} x)} g_{t,l}(\{x; \phi_{\sigma(j)}\}_{j=1}^t) \right. \right. \\ & \quad \left. \left. \times \left[e^{i(2(K-l)(kx+\varpi(x))-\sum_{j=t+1}^I \phi_{\sigma(j)} x)} g_{I-t,K-l}(x; \{\phi_{\sigma(j)}\}_{j=t+1}^I) \right] \right] \right). \end{aligned}$$

Recalling (4.11), Proposition 4.3 and Definition 4.5 and applying the product rule,

we find that this expression is equal to

$$\begin{aligned} & \left(\frac{1}{2} \sum_{t=0}^I \left[2le^{i(2l(kx+\varpi(x))-\sum_{j=1}^t \phi_{\sigma(j)} x)} f_{t,l}(\{x; \phi_{\sigma(j)}\}_{j=1}^t) \right] \right. \\ & \quad \times \left[e^{i(2(K-l)(kx+\varpi(x))-\sum_{j=t+1}^I \phi_{\sigma(j)} x)} g_{I-t,K-l}(x; \{\phi_{\sigma(j)}\}_{j=t+1}^I) \right] \Big) \\ & + \left(\frac{1}{2} \sum_{t=0}^I \left[e^{i(2l(kx+\varpi(x))-\sum_{j=1}^t \phi_{\sigma(j)} x)} g_{t,l}(x; \{\phi_{\sigma(j)}\}_{j=1}^t) \right] \right. \\ & \quad \left. \times \left[2(K-l)e^{i(2(K-l)(kx+\varpi(x))-\sum_{j=t+1}^I \phi_{\sigma(j)} x)} f_{I-t,K-l}(x; \{\phi_{\sigma(j)}\}_{j=t+1}^I) \right] \right). \end{aligned}$$

Collecting terms, we find that this in turn can be written as

$$\begin{aligned} & \frac{e^{i(2K(kx+\varpi(x))-\sum_{j=1}^I \phi_j x)}}{2} \left(\sum_{t=0}^I 2l f_{t,l}(x; \{\phi_{\sigma(j)}\}_{j=1}^t) g_{I-t,K-l}(x; \{\phi_{\sigma(j)}\}_{j=t+1}^I) \right. \\ & \quad \left. + 2(K-l) g_{t,l}(x; \{\phi_{\sigma(j)}\}_{j=1}^t) f_{I-t,K-l}(x; \{\phi_{\sigma(j)}\}_{j=t+1}^I) \right). \end{aligned}$$

Once we average across all permutations $\sigma \in S_I$, we arrive at

$$\begin{aligned}
& \left(\sum_{t=1}^I \frac{i e^{i(2K(kx+\varpi(x))-\sum_{j=1}^I \phi_j x)}}{2} g_{t,l} \odot g_{I-t,K-l} \right)' \\
&= 2l \frac{e^{i(2K(kx+\varpi(x))-\sum_{j=1}^I \phi_j x)}}{2} \sum_{t=1}^I f_{t,l} \odot g_{I-t,K-l} \\
&\quad + 2(K-l) \frac{e^{i(2K(kx+\varpi(x))-\sum_{j=1}^I \phi_j x)}}{2} \sum_{t=1}^I f_{t,K-l} \odot g_{I-t,l} \\
&= 2K e^{i(2K(kx+\varpi(x))-\sum_{j=1}^I \phi_j x)} f_{I,K}.
\end{aligned}$$

Since $\frac{1}{2}g_{t,l} \odot g_{I-t,K-l}$ is 1-periodic in x , we conclude by uniqueness in Proposition 4.3 that it is equal to $g_{I,K}$.

Now in the case where some $2Kk - \sum \phi \equiv 0 \pmod{2\pi}$, we may apply Lemma 4.4 to assert that f, g are continuous at $\alpha \equiv 0 \pmod{2\pi}$ and thus since the equalities (4.15),(4.16) hold for every $2Kk - \sum \phi$ in a neighborhood of $0 \pmod{2\pi}$, they must be true for $2Kk - \sum \phi \equiv 0 \pmod{2\pi}$ as well. \square

Let us define functions h_j of $1+j$ variables recursively by $h_0(x) = 1$ and

$$\begin{aligned}
& h_J(x; \phi_1, \dots, \phi_J) \\
&= \frac{1}{|1 - e^{i(2k-\phi_1-\dots-\phi_J)}|} \sum_{j=0}^{J-1} h_j(x; \phi_1, \dots, \phi_j) h_{J-j-1}(x; \phi_{j+1}, \dots, \phi_{J-1}).
\end{aligned}$$

Next, we recall that by Lemma 4.6, $\|\lambda_\alpha\| \leq 2$.

Lemma 4.10. *Where $\|\Phi_0\|$ refers to the maximum of the periodic continuous function $|\Phi_0(x)|$, the function $g_{J,1}$ can be bounded in terms of h_J in the following manner:*

$$|g_{J,1}(x; \{\phi_j\}_{j=1}^J)| \leq \frac{2(2\|\Phi_0\|)^J}{J!} \sum_{\sigma \in S_J} h_J(x; \{\phi_{\sigma(j)}\}_{j=1}^J).$$

Proof. We prove this by induction. First we note that

$$|g_{1,1}(x; \phi_1)| = \left| 2 \frac{\lambda_{2k-\phi_1} e^{2i\varpi(x)} f_{1,1}(x; \phi_1)}{1 - e^{i(2k-\phi_1)}} \right| \leq 4\|\Phi_0\| h_1(x; \phi_1).$$

For $J \geq 2$, we deduce, using (4.12), (4.14) and the inductive hypothesis,

$$\begin{aligned} & |g_{J,1}(x; \{\phi_j\}_{j=1}^J)| \\ &= \left| 2 \frac{\lambda_{2k-\sum_{j=1}^J \phi_j} e^{2i\varpi(x)} f_{J,1}}{1 - e^{i(2k-\sum_{j=1}^J \phi_j)}} \right| \\ &= \left| \frac{2}{1 - e^{i(2k-\sum_{j=1}^J \phi_j)}} \right. \\ &\quad \left. \times \lambda_{2k-\sum_{j=1}^J \phi_j} \left(e^{2i\varpi(x)} \Phi_0(x) \left[w_0 \odot g_{J-1,1} + \frac{1}{2} w_1 \odot \sum_{j=1}^{J-2} g_{j,1} \odot g_{J-j-1,1} \right] \right) \right| \\ &\leq \frac{4\|\Phi_0\|}{|1 - e^{i(2k-\sum_{j=1}^J \phi_j)}|} \left| \left(w_0 \odot g_{J-1,1} + \frac{1}{2} w_1 \odot \sum_{j=1}^{J-2} g_{j,1} \odot g_{J-j-1,1} \right) \right| \\ &\leq \frac{2(2\|\Phi_0\|)^J}{|1 - e^{i(2k-\sum_{j=1}^J \phi_j)}|} \left| \left(2w_0 \odot h_{J-1,1} + 2w_1 \odot \sum_{j=1}^{J-2} h_{j,1} \odot h_{J-j-1,1} \right) \right| \\ &\leq \frac{2(2\|\Phi_0\|)^J}{|1 - e^{i(2k-\sum_{j=1}^J \phi_j)}|} \left(2|w_0| \odot h_{J-1,1} + 2w_1 \odot \sum_{j=1}^{J-2} h_{j,1} \odot h_{J-j-1,1} \right) \\ &= \frac{2(2\|\Phi_0\|)^J}{J!} \sum_{\sigma \in S_J} h_J(x; \{\phi_{\sigma(j)}\}_{j=1}^J). \end{aligned}$$

□

We are now prepared to prove the following lemma:

Lemma 4.11. *Let $E \in (0, \infty)$, and let V be given by (1.6) so that the following conditions are satisfied:*

(i) $\sum_{l=1}^{\infty} |c_l| < \infty$

(ii) for $j = 1, \dots, p-1$, and $1 \leq K \leq j$,

$$\sum_{l_1, \dots, l_j=1}^{\infty} |c_{l_1} \dots c_{l_j} h_j(x; \phi_{l_1}, \dots, \phi_{l_j})| < \infty. \quad (4.17)$$

then all the solutions of (1.11) are bounded.

Before we prove this lemma, we first define

$$\mathcal{S}_{J,K}(x) = \sum_{m_1, \dots, m_J}^{\infty} f_{J,K}(\phi_{m_1}, \dots, \phi_{m_J}) \beta_{m_1}(x) \dots \beta_{m_J}(x) e^{2iK[kx + \varpi(x) + \eta(x)]}, \quad (4.18)$$

where $\beta_l(x) = c_l e^{-i\phi_l x} \gamma_l(x)$.

We can then rewrite (4.6) as

$$\ln R(b) - \ln R(a) = \operatorname{Im} \int_a^b \mathcal{S}_{1,1}(x) dx. \quad (4.19)$$

We also define

$$E_{J,K} = \sum_{m_1, \dots, m_J=1}^{\infty} |c_{m_1} \dots c_{m_J}| \cdot \|g_{J,K}(\phi_{m_1}, \dots, \phi_{m_J})\|, \quad (4.20)$$

where $\|g\|$ refers to the maximum of the continuous periodic function $|g|$.

Lemma 4.12. *Assume the hypotheses of Lemma 4.11. For $J = 1, \dots, p-1$,*

$$\left| \int_a^b \left(\sum_{K=1}^J \mathcal{S}_{J,K} - \sum_{K=0}^{J+1} \mathcal{S}_{J+1,K} \right) dx \right| \leq \sum_{K=1}^J \frac{E_{J,K} \tau^J}{K} \quad (4.21)$$

Proof. We apply Lemma 4.7, setting $\Phi = e^{2iK\varpi(x)} f_{J,K}$, with $\Gamma(x) = \gamma_{m_1} \dots \gamma_{m_J}$ and

$\phi = \phi_{m_1} + \dots + \phi_{m_J}$. Noting that the assumption (4.17) implies $2Kk - \phi \not\equiv 0 \pmod{2\pi}$,

we obtain

$$\frac{\|g_{J,K}\|}{K} \tau^J \geq \left| \int_a^b \left(e^{2Ki(\eta(x) + kx + \varpi(x))} e^{-i\phi x} \Gamma(x) f_{J,K} \right. \right. \\ \left. \left. - e^{2Ki(\eta(x) + kx + \varpi(x))} e^{-i\phi x} \Gamma(x) \frac{d\eta(x)}{dx} g_{J,K} \right) dx \right|. \quad (4.22)$$

We then expand $d\eta/dx$ using (4.7), apply (4.12), multiply by $c_{m_1} \dots c_{m_J}$, sum in m_1, \dots, m_J from 1 to ∞ , and sum in K from 1 to J to prove the lemma. \square

Let us define

$$\mathbf{m} = \sup_l \|\gamma_l(x)\|_p. \quad (4.23)$$

We know it is finite by assumptions we placed on V .

Lemma 4.13. *Assume the hypotheses of Lemma 4.11. $\mathcal{S}_{J,K}(x)$ is absolutely convergent when $1 \leq K \leq J \leq p$, and if in addition $J \geq 2$ then*

$$\begin{aligned} & \sum_{m_1, \dots, m_J=1}^{\infty} |f_{J,K}(\phi_{m_1}, \dots, \phi_{m_J}) \beta_{m_1}(x) \dots \beta_{m_J}(x)| \\ & \leq |\Phi_0(x)| \sum_{a=-1}^1 |w_a| E_{J-1, K+a} \sum_{l=1}^{\infty} |c_l| \tau^J. \end{aligned} \quad (4.24)$$

Furthermore, if $J = p$, we have that also

$$\begin{aligned} & \int_0^{\infty} \sum_{m_1, \dots, m_J=1}^{\infty} |f_{J,K}(\phi_{m_1}, \dots, \phi_{m_J}) \beta_{m_1}(x) \dots \beta_{m_J}(x)| dx \\ & \leq |\Phi_0(x)| \sum_{a=-1}^1 |w_a| E_{p-1, K+a} \sum_{l=1}^{\infty} |c_l| \mathbf{m}(x)^J. \end{aligned} \quad (4.25)$$

Proof. From (4.12) we have

$$\begin{aligned} & |f_{J,K}(\phi_{m_1}, \dots, \phi_{m_J})| \\ & \leq |\Phi_0(x)| \sum_{a=-1}^1 \sum_{\sigma \in S_J} |w_a| \cdot |g_{J-1, K+a}(\phi_{m_{\sigma(1)}}, \dots, \phi_{m_{\sigma(J-1)}})| \end{aligned}$$

We then multiply by

$$|\beta_{m_1}(x) \dots \beta_{m_J}(x)| \leq |c_{m_1} \dots c_{m_J}| \tau^J.$$

Summing in m_1, \dots, m_J completes the proof of (4.24).

For $J = p$, we multiply instead by

$$\int_0^\infty |\beta_{m_1}(x) \dots \beta_{m_J}(x)| dx \leq |c_{m_1} \dots c_{m_J}| \mathbf{m}^J,$$

to get (4.25). □

Proof of Lemma 4.11. We sum (4.21) in $J = 1, \dots, p-1$, to obtain

$$\left| \int_a^b \left(\mathcal{S}_{1,1}(x) - \sum_{K=1}^p \mathcal{S}_{p,K}(x) - \sum_{j=2}^p \mathcal{S}_{j,0}(x) \right) dx \right| \leq \sum_{j=1}^{p-1} \sum_{l=1}^j \frac{1}{l} E_{j,l} \tau^j. \quad (4.26)$$

Note that the RHS converges due to the assumption (4.17) together with Lemma 4.10. By using Lemma 4.13 for $J = p$, integrating in x and summing in K ,

$$\left| \sum_{K=1}^p \int_a^b \mathcal{S}_{p,K}(x) dx \right| \leq \|\Phi_0\| \sum_{r=0}^{p-1} E_{p-1,r} \sum_{l=1}^{\infty} |c_l| \mathbf{m}^p.$$

We have now that

$$\begin{aligned} |\ln R(b) - \ln R(a)| - \tilde{B}(b) &\leq \sum_{j=1}^{p-1} \sum_{l=1}^j \frac{1}{l} E_{j,l} \tau^j \\ &\quad + \|\Phi_0\| \sum_{r=0}^{p-1} E_{p-1,r} \sum_{l=1}^{\infty} |c_l| \mathbf{m}^p. \end{aligned}$$

where $\tilde{B}(b)$, a bound on the $\sum \mathcal{S}_{j,0}$ term, is independent of our choices of R and η .

In other words, we can write

$$\ln R(b) - \ln R(a) + i(\eta(b) - \eta(a)) = A(b) + B(b),$$

where $A(b)$ converges as $b \rightarrow \infty$, and $B(b)$ is independent of η and R . Consider now two solutions of (1.11), u_1, u_2 with the corresponding $(R_1, \eta_1), (R_2, \eta_2)$. We then also

have

$$\ln R_1(b) - \ln R_1(a) + i(\eta_1(b) - \eta_1(a)) = A_1(b) + B_1(b)$$

$$\ln R_2(b) - \ln R_2(a) + i(\eta_2(b) - \eta_2(a)) = A_2(b) + B_2(b)$$

We note that $B_1 = B_2$. Therefore subtracting the first equation from the second we obtain

$$\ln R_2(b) + i\eta_2(b) - \ln R_1(b) - i\eta_1(b) = A_2(b) - A_1(b).$$

In particular, both sides of this equation converge when $b \rightarrow \infty$, and so we must know that

$$\ln \frac{R_2(b)}{R_1(b)}, \eta_1(b) - \eta_2(b),$$

both converge as $b \rightarrow \infty$.

However, we know that the Wronskian of u_1, u_2 does not depend on b . We may express the Wronskian as

$$\omega = R_1(b)R_2(b) \sin |\eta_1(b) - \eta_2(b)|$$

But we know that $(\eta_1(b) - \eta_2(b))$ converges as $b \rightarrow \infty$, and so in response to a choice of u_2 it is possible to choose solution u_1 so that $\lim_{b \rightarrow \infty} \sin |\eta_1(b) - \eta_2(b)| = \epsilon$ for some $\epsilon > 0$. This is because convergence of $\eta_1(b) - \eta_2(b)$ is at a rate independent of initial conditions. Thus for sufficiently large b , we can choose u_1 by, say, the initial condition $\eta_1(b) = \eta_2(b) + \pi/2$, and this would guarantee the limit is nonzero. But then we have

$$\lim_{b \rightarrow \infty} \ln(R_2(b)^2) = -\ln((\epsilon)/\omega) + \lim_{b \rightarrow \infty} \operatorname{Re}(A_2(b) - A_1(b)).$$

Thus $R_2(b)$ converges, and hence we have proven our lemma. \square

4.1.3 Proofs of theorems

Lemma 4.14. *Assume that (1.9) holds. Then for a positive integer j , the set of k for which the condition (4.17) fails has Hausdorff dimension at most $j\mathbf{a}$.*

Proof. The proof is similar to that of Lemma 4.2 in [Luk13a], even though our h is defined slightly differently. The most significant difference is that our singularities are at $2k - \sum \phi = 2\pi n$ rather than just at 0, so each choice of $\sum \phi$ generates infinitely many singularities rather than just one. We adjust the proof by restricting the measure ν in Lemma 4.1 of [Luk13a] to be a finite uniformly β -Hölder continuous measure on $[-\pi, \pi]$. \square

Proof of Theorem 1.10. Note that by standard results in Floquet Theory (cf. [Wei03]) the quasimomentum k is monotone and analytic on bands of the ac spectrum of the unperturbed operator. Thus the theorem follows immediately from Lemmas 4.11 and 4.14, and the fact that monotone analytic maps preserve Hausdorff dimension. \square

Proof of Theorem 4.1. It is clear that for finite frequencies, the points of our S are the only ones which might not satisfy the small divisor condition (4.17). \square

4.1.4 Existence of embedded eigenvalues

We already know that the set S described in Theorem 4.1 is optimal, since they are optimal for $V_0 = 0$, by [Krü12] and [Luk13b]. In this section we wish to demonstrate

examples of point spectrum even when the background potential V_0 is not identically zero.

The proofs in this section will be similar to the proofs in Section 6 of [Luk13b], except that f in our proofs are periodic in x instead of constant in x . The following lemma will thus prove useful.

Lemma 4.15. *Let $P(x)$ be a C^1 1-periodic function on \mathbb{R}_+ , and let \mathcal{P} be its mean. Let $q(x)$ be a C^1 function of bounded variation on \mathbb{R}_+ such that $q'(x) \in L^1(\mathbb{R}_+)$ and $\lim_{x \rightarrow \infty} q(x) = 0$. Then $\int_0^\infty (P(x) - \mathcal{P})q(x)dx$ is finite.*

Proof. This follows from integration by parts. Let $A(x)$ be an antiderivative of $P(x) - \mathcal{P}$ and notice that $A(x)$ is periodic. We then calculate

$$\int_0^\infty (P(x) - \mathcal{P})q(x)dx = A(x)q(x) \Big|_0^\infty - \int_0^\infty A(x)q'(x)dx,$$

and observe that the term on the RHS is finite. □

Lemma 4.16. *Let $R(x), \eta(x)$ be the Prüfer variables corresponding to some solution of (1.11). Assume that*

$$\frac{d}{dx} \log R(x) \sim -\frac{B(x)}{x^{(p-1)\gamma}}, \tag{4.27}$$

for some periodic C^1 function $B(x)$ with positive mean, and the limit

$$\eta_\infty = \lim_{x \rightarrow \infty} \eta(x)$$

exists. Then for some $A > 0$,

$$u(x) = Af(x)e^{i[kx + \eta_\infty]}(1 + o(1)), x \rightarrow \infty.$$

where denoting \mathcal{B} as the mean of $B(x)$ (remember that \mathcal{B} is positive),

$$f(x) = \begin{cases} x^{-\mathcal{B}} & \gamma = \frac{1}{p-1} \\ \exp\left(-\frac{\mathcal{B}}{1-(p-1)\gamma}x^{1-(p-1)\gamma}\right) & \gamma \in \left(\frac{1}{p}, \frac{1}{p-1}\right) \end{cases}$$

These asymptotics imply the existence of an L^2 solution of (1.11) if $\gamma \in \left(\frac{1}{p}, \frac{1}{p-1}\right)$, and hence an eigenvalue.

Proof. This follows immediately from Lemma 6.1 in [Luk13b] and our Lemma 4.15. □

Theorem 4.17. *Consider*

$$V(x) = \sum_{l=1}^K L_k \frac{1}{x^\gamma} \cos(\alpha_l x + \xi_l(x)) + \beta_0(x), x \geq x_0 \quad (4.28)$$

where

$$\gamma \in \left(\frac{1}{p}, \frac{1}{p-1}\right],$$

$L_k > 0$, and

$$\beta_0(x) \in C^1, \frac{d}{dx}(\beta_0(x)) = O(x^{-p\gamma}), \beta_0(x) = O(x^{-\gamma}), x \rightarrow \infty. \quad (4.29)$$

The functions $\xi_l(x) \in C^1$ have the property that

$$\xi_l'(x) = O(x^{-(p-1)\gamma}), x \rightarrow \infty.$$

If $\beta_0(x)$ has bounded variation, this ensures that (4.28) has generalized bounded variation with phases

$$\{0, \pm\alpha_1, \dots, \pm\alpha_K\}.$$

Thus V is L^p . Our values for ϕ_1, ϕ_2, \dots are then drawn from $\{0, \pm\alpha_1, \dots, \pm\alpha_K\}$.

Consider a value of k for which $2k \equiv \phi_{j_1} + \dots + \phi_{j_{p-1}} \pmod{2\pi}$, such that $2k$ cannot be written similarly as a sum of fewer phases. If the 1-periodic function $f_{p-1,1}(x; \phi_{j_1}, \dots, \phi_{j_{p-1}})e^{2i\varpi(x)}$ does not have mean 0, then there are choices of β_0 and ξ_l so that the operator H given by (1.11) has point spectrum at all energies E with the given quasimomentum k .

We will remark that the methods of the previous section make clear that the converse is true, i.e. if $f_{p-1,1}(x; \phi_{j_1}, \dots, \phi_{j_{p-1}})$ has mean zero, then there is no point spectrum at the specified value of the quasimomentum.

Proof of Theorem 4.17. This proof will follow closely the proof of Theorem 1.2 of [Luk13b].

We start from (4.6) and apply the iterative algorithm in the previous section. Recall that the algorithm could not deal with the term

$$f_{p-1,1}(\phi_{j_1}, \dots, \phi_{j_{p-1}})\beta_{j_1}(x) \dots \beta_{j_{p-1}}(x)e^{2ikx+2i\varpi(x)+2i\eta(x)},$$

and that instead we had to bound it separately in the form of Lemma 4.10. Thus if we denote the number of distinct permutations of (j_1, \dots, j_{p-1}) by C_1 , we obtain

$$\frac{d}{dx} \log R(x) \sim \operatorname{Im} \left(\frac{\Lambda(x)}{x^{(p-1)\gamma}} e^{i\xi(x)+2i\eta(x)} \right), \quad (4.30)$$

where

$$\Lambda(x) = C_1 f_{p-1,1}(x; \phi_{j_1}, \dots, \phi_{j_p}) e^{2i\varpi(x)} L_{j_1} \dots L_{j_p},$$

and

$$\xi(x) = \xi_{j_1}(x) + \dots + \xi_{j_{p-1}}(x).$$

Conversely, once we have an appropriate $\xi(x)$, we can construct $\xi_j(x)$ by taking $\xi_j(x) = c_j \xi(x)$, where the c_j are real numbers such that $c_{j_1} + \dots + c_{j_{p-1}} = 1$.

We now need to show that $\eta(x)$ has a limit as $x \rightarrow \infty$. We apply (4.7) and see that

$$\begin{aligned} \frac{d\eta}{dx} &= \Phi_0(x)V(x)\operatorname{Re}(-1 + e^{2ikx+2i\varpi(x)+2i\eta(x)}) \\ &\sim \operatorname{Re}\left(\Omega(x) + \frac{\Lambda(x)}{x^{(p-1)\gamma}}e^{i\xi(x)+2i\eta(x)}\right), \end{aligned}$$

with

$$\Omega(x) = \sum_{I=1}^{p-1} \sum_{\phi_{j_1}+\dots+\phi_{j_I} \in 2\pi\mathbb{Z}} f_{I,0}(x; \phi_{j_1}, \dots, \phi_{j_I}) \beta_{j_1}(x) \dots \beta_{j_I}(x).$$

Let us replace every function f with its mean in the definition of $\Omega(x)$, to obtain $\hat{\Omega}(x)$. Observe that by Lemma 6.2 of [Luk13b], there is a choice of $\beta_0(x)$ such that $\int_0^\infty \hat{\Omega}(x)dx$ is finite. By applying Lemma 4.15, we can see that $\int_0^\infty \Omega(x)dx$ is finite as well.

We then have

$$\frac{d\eta}{dx} \sim \frac{\Lambda(x)}{x^{(p-1)\gamma}}. \quad (4.31)$$

Firstly, since we assumed $f_{p-1,1}(x)$ and hence $\Lambda(x)$ has nonzero mean, it must be true that $\operatorname{Im}(\Lambda(x)e^{it})$ has positive mean for some real t . We denote $\psi(x) = \xi(x) + 2\eta(x)$.

Lemma 4.18. *Let $R(x), \eta(x)$ be Prüfer variables corresponding to some solution of (1.11). Assume that (4.30) and (4.31) hold. Then we may pick $\xi(x)$ with $\xi'(x) \in$*

$O(x^{-(p-1)\gamma})$ such that $\lim_{x \rightarrow \infty} \psi(x) = t$.

Proof. The proof is identical to that of Lemma 6.3 of [Luk13b], except that we replace the constant Λ with the periodic function $\Lambda(x)$, and Lemma 4.1 with Lemma 4.7. \square

With that choice of $\xi(x)$,

$$\begin{aligned} \frac{d}{dx} \log R(x) &\sim \operatorname{Im} \left(\frac{\Lambda(x)}{x^{(p-1)\gamma}} e^{i\psi_\infty} + \frac{\Lambda(x)}{x^{(p-1)\gamma}} (e^{i\psi(x)} - e^{i\psi_\infty}) \right) \\ &\sim \operatorname{Im} \left(\frac{\Lambda(x) e^{it}}{x^{(p-1)\gamma}} + O(x^{-p\gamma}) \right) \\ &\sim \operatorname{Im} \left(\frac{\Lambda(x) e^{it}}{x^{(p-1)\gamma}} \right), \end{aligned}$$

and then we simply apply Lemma 4.16 to complete our proof. \square

We now need to determine how often the condition that

$$f_{p-1,1}(x; \phi_{j_1}, \dots, \phi_{j_{p-1}}) e^{2i\varpi(x)}$$

has nonzero mean is satisfied. We will show that this condition is satisfied for a non-trivial class of periodic functions $\varphi(x) e^{-ikx}$. We will start with a suggestive example. For notational convenience, let us adjust the order of the phases so we can rewrite $\phi_{j_1}, \phi_{j_2}, \phi_{j_{p-1}}$ as $\phi_1, \phi_2, \dots, \phi_{p-1}$.

Proposition 4.19. *Assume that the Floquet solution φ is given as Ce^{ikx} for some positive C (i.e., $\Phi_0(x) = \frac{C}{\omega}$ and $\varpi(x) = 0$). Then for a choice of phases $\{\alpha_i\}$ away from an algebraic set of codimension 1, for every $1 \leq l \leq j \leq p-1$*

$$f_{j,l}(x; \phi_1, \dots, \phi_j) e^{2il\varpi(x)},$$

is a nonzero constant in x (the constant depends on j, l).

Proof. Let us assume that $2Kk - \sum_{t=1}^l \phi_{\sigma(t)} \equiv 0 \pmod{2\pi}$ does not hold for any $l \leq j$, any choice of phases, any permutation σ of $p - 1$ elements, and any $K < p$. We can make this assumption since it is a codimension 1 condition on the $\{\alpha_i\}$.

We may calculate that for a constant C , $e^{2il\varpi(x)} \lambda_{\alpha,l} C = -\frac{C}{\alpha}$. Thus applying (4.12), we discover that

$$f_{j,l}(x; \phi_1, \dots, \phi_j) e^{2il\varpi(x)}$$

is a rational function in the variables k, ϕ_1, \dots, ϕ_j , with denominator terms of the form $2Kk - \sum \phi$. Note that for large enough k all the terms are strictly positive (the quasimomentum k only takes values in a π -interval, but in the context of this proof we are viewing it as a variable in \mathbb{R}). But then it is an easy induction argument that

$$(-1)^{l+1} f_{j,l}(x; \phi_1, \dots, \phi_j) e^{2il\varpi(x)}$$

is strictly positive, using (4.11) and (4.12). This demonstrates that it is a nontrivial rational function, and therefore is only zero on a set of ϕ_j s of codimension 1. \square

Lemma 4.20. *Assume that the 1-periodic function $\varphi(x)e^{-ikx}$ has finite Fourier expansion*

$$\sum_{n=-N}^N \hat{\varphi}(n) e^{2\pi i n x}.$$

Assume that the phases $\{\alpha_i\}$ are chosen away from the codimension 1 algebraic set described in Proposition 4.19. Then if the Fourier coefficients $\hat{\varphi}(n)$ are chosen away from another algebraic set of codimension 1, the corresponding

$$f_{j,l}(x; \phi_1, \dots, \phi_j) e^{2il\varpi(x)},$$

have nonzero mean for all $1 \leq l \leq j \leq p - 1$.

Proof. As a first step, we have to understand how $e^{2il\varpi(x)}\lambda_{\alpha,l}$ acts on finite Fourier sums. So let us consider $\Phi(x) = \sum_{n=-N}^N \hat{\Phi}(n)e^{2\pi inx}$. For the reader's convenience, we will take this somewhat tedious calculation step by step, using the proof and notation of Proposition 4.3 as a guide.

First, we need to determine the value of $Q_\alpha(x)$, that is, the antiderivative of $\Phi(x)e^{i\alpha x}$ with $Q_\alpha(0) = 0$. We find that

$$\begin{aligned} Q_\alpha(x) &= \int \sum_{n=-N}^N \hat{\Phi}(n)e^{(2\pi n+\alpha)ix} dx \\ &= C + \sum_{n=-N}^N \hat{\Phi}(n) \frac{e^{(2\pi n+\alpha)ix}}{(2\pi n + \alpha)i}. \end{aligned}$$

Using $Q(0) = 0$, it is easy to calculate the value of C . We then obtain, finally

$$Q_\alpha(x) = \sum_{n=-N}^N \hat{\Phi}(n) \frac{e^{(2\pi n+\alpha)ix} - 1}{(2\pi n + \alpha)i}.$$

Then, by (4.8), we have

$$\begin{aligned} \tilde{\Phi}_\alpha(x) &= - \left(\sum_{n=-N}^N \hat{\Phi}(n) \frac{e^{(2\pi n+\alpha)ix} - 1}{(2\pi n + \alpha)} \right) e^{-i\alpha x} (1 - e^{i\alpha}) \\ &\quad + \left(\sum_{n=-N}^N \hat{\Phi}(n) \frac{e^{(2\pi n+\alpha)i} - 1}{(2\pi n + \alpha)} \right) e^{-i\alpha x} \\ &= - \left(\sum_{n=-N}^N \hat{\Phi}(n) \frac{e^{2\pi nix} - e^{-i\alpha x}}{(2\pi n + \alpha)} \right) (1 - e^{i\alpha}) \\ &\quad + \left(\sum_{n=-N}^N \hat{\Phi}(n) \frac{e^{i\alpha(1-x)} - e^{-i\alpha x}}{(2\pi n + \alpha)} \right) \\ &= \sum_{n=-N}^N - \frac{\hat{\Phi}(n)(1 - e^{i\alpha})}{(2\pi n + \alpha)} e^{2\pi nix}. \end{aligned}$$

and thus by Definition 4.5, we know that

$$\Phi(x) \rightarrow e^{2il\varpi(x)} \lambda_{\alpha,l} \frac{\Phi(x)}{1 - e^{i\alpha}}$$

modifies the Fourier coefficients so that $\hat{\Phi}(n) \rightarrow -\frac{\hat{\Phi}(n)}{2\pi n + \alpha}$.

Now using (4.11),(4.12), and the fact that

$$\Phi_0(x) = \frac{\sum_{n=-N}^N \hat{\varphi}(n)e^{2\pi inx} \sum_{n=-N}^N \overline{\hat{\varphi}(-n)}e^{2\pi inx}}{\omega},$$

we deduce that

$$f_{j,l}(x; \phi_1, \dots, \phi_j)e^{2il\varpi(x)},$$

is a finite Fourier sum, whose coefficients are all polynomials in $\{\hat{\varphi}(n)\} \cup \{\overline{\hat{\varphi}(n)}\}$.

In particular, the zeroth Fourier coefficient, which gives us the mean of $f_{j,l}$, is a polynomial in $\{\hat{\varphi}(n)\} \cup \{\overline{\hat{\varphi}(n)}\}$. But we know it is not identically zero, since by Proposition 4.19 the mean is nonzero when $\hat{\varphi}(0) = \overline{\hat{\varphi}(0)} = C$ a positive constant and all the other $\hat{\varphi}(n), \overline{\hat{\varphi}(n)}$ are zero. Thus the mean can be zero only on a codimension 1 set of $\{\hat{\varphi}(n)\} \cup \{\overline{\hat{\varphi}(n)}\}$. \square

Proposition 4.21. *Assume that the phases $\{\alpha_i\}$ are chosen away from the codimension 1 algebraic set described in Proposition 4.19. Then for a dense open set of $V_0(x)$ in (4.1) in the $L^1(0,1)$ -topology, the corresponding*

$$f_{j,l}(x; \phi_1, \dots, \phi_j)e^{2il\varpi(x)},$$

have nonzero mean for all $1 \leq l \leq j \leq p - 1$.

Proof. Let us for notational convenience set $\psi(x) = \varphi(x)e^{-ikx}$. First note that trigonometric polynomials are dense in the space of 1-periodic functions under the $W^{2,1}((0,1))$ topology. We further claim that the set of trigonometric polynomials in Lemma 4.20 (i.e., missing an algebraic codimension 1 set) is still dense in the space of 1-periodic functions. This is obvious, since for any n we can apply an arbitrarily

small trigonometric polynomial (in the $W^{2,1}((0, 1))$ sense) perturbation of degree n to a trigonometric polynomial in that algebraic codimension 1 set so that the perturbed polynomial is not in the codimension 1 set.

Also, the condition that the

$$f_{j,l}(x; \phi_1, \dots, \phi_j) e^{2il\varpi(x)},$$

have nonzero mean for all $1 \leq l \leq j \leq p-1$ is clearly an open condition in $W^{2,1}((0, 1))$ -norm of $\psi(x)$, since the expressions are sums of antiderivatives of ψ . Thus this condition is an open and dense condition in the space of 1-periodic functions $\psi(x)$ under the $W^{2,1}((0, 1))$ topology.

Furthermore, if we write (4.1) in terms of ψ and k , we get

$$V_0(x) - E = \frac{\psi''(x)}{\psi(x)} + 2ik \frac{\psi'(x)}{\psi(x)} - k^2.$$

Recall that since we assumed $\varphi, \bar{\varphi}$ are linearly independent, it must be true that $\psi(x)$ is nonzero for all x . So, noting that the quasimomentum k depends continuously on V_0 , it is clear that an open and dense set in ψ corresponds to an open and dense set in V_0 (using the $L^1(0, 1)$ topology).

□

Theorem 4.2 is an immediate corollary of this proposition.

4.2 Decaying oscillatory perturbations of periodic Jacobi and CMV operators

In this section, we prove Theorems 1.12 and 1.13. The key to these two theorems is in adapting the generalized Prüfer variables of [KRS99] to the Jacobi and Szegő recurrence settings.

4.2.1 Generalized Prüfer variables for the Jacobi recursion

Consider $x \in \mathbb{R}$, $\{a_n\}$ a sequence of positive real numbers, $\{a'_n\}$ a sequence of real numbers such that $a_n + a'_n$ is positive, and $\{b_n\}, \{b'_n\}$ two sequences of real numbers. For $n \geq 1$ let $\varphi(n), u(n)$ be two sequences that satisfy

$$x\varphi(n) = a_{n+1}\varphi(n+1) + b_{n+1}\varphi(n) + a_n\varphi(n-1), \quad (4.32)$$

and

$$xu(n) = (a_{n+1} + a'_{n+1})u(n+1) + (b_{n+1} + b'_{n+1})u(n) + (a_n + a'_n)u(n-1), \quad (4.33)$$

respectively.

The transfer matrix $T_0(x; n)$ is defined as

$$T_0(x; n) = \prod_{j=n}^1 \frac{1}{a_j} \begin{pmatrix} x - b_{j+1} & -a_j^2 \\ 1 & 0 \end{pmatrix},$$

so

$$T_0(x; n) \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a_{n+1}\varphi(n+1) \\ \varphi(n) \end{pmatrix},$$

if φ obeys $a_1\varphi(1) = a, \varphi(0) = b$. Also, we make the assumption

$$K = \sup_{n>0} \|T_0(x; n)\| < \infty. \quad (4.34)$$

Let us define some variations on the Wronskian. For two sequences f, g , we have

$$W_{0,0}(f, g) = a_{n+1}f(n)g(n+1) - a_{n+1}f(n+1)g(n),$$

$$W_{a',a'}(f, g) = (a_{n+1} + a'_{n+1})f(n)g(n+1) - (a_{n+1} + a'_{n+1})f(n+1)g(n),$$

$$W_{0,a'}(f, g) = (a_{n+1} + a'_{n+1})f(n)g(n+1) - a_{n+1}f(n+1)g(n).$$

The motivation behind the notation $W_{*1,*2}(f, g)$ is that we usually choose f to be a solution to the transfer matrix equation with the a_n perturbed by $*_1$, and g to be a solution to the transfer matrix equation with the a_n perturbed by $*_2$.

If we assume

$$a_{n+1}f(n+1) + a_n f(n-1) = (x - b_{n+1})f(n),$$

and

$$(a_{n+1} + a'_{n+1})g(n+1) + (a_n + a'_n)g(n-1) = (x - b_{n+1} - b'_{n+1})g(n),$$

then

$$\begin{aligned} W_{0,a'}(f, g)(n) - W_{0,a'}(f, g)(n-1) &= -b'_{n+1}f(n)g(n) \\ &\quad - a'_n(f(n)g(n-1) + f(n-1)g(n)). \end{aligned} \quad (4.35)$$

Consider $\varphi, \bar{\varphi}$, two complex-valued solutions of (4.32). We then have

$$W_{0,0}(\bar{\varphi}, \varphi)(n) = 2ia_{n+1}\text{Im}(\overline{\varphi(n)}\varphi(n+1)) = i\omega, \quad (4.36)$$

for some real nonzero constant ω . Let us define $\gamma(n)$ by

$$\varphi(n) = |\varphi(n)|e^{i\gamma(n)}. \quad (4.37)$$

By the constancy of the Wronskian, this implies

$$2|\varphi(n)| \cdot |\varphi(n+1)|a_{n+1} \sin(\gamma(n+1) - \gamma(n)) = \omega. \quad (4.38)$$

Given a complex reference solution φ to (4.32) and a real valued solution u to (4.33), we define $Z(n)$ as follows:

$$\begin{pmatrix} (a_n + a'_n)u(n) \\ u(n-1) \end{pmatrix} = \frac{1}{2i} \left(Z(n) \begin{pmatrix} a_n \varphi(n) \\ \varphi(n-1) \end{pmatrix} - \overline{Z(n)} \begin{pmatrix} a_n \overline{\varphi(n)} \\ \overline{\varphi(n-1)} \end{pmatrix} \right) \quad (4.39)$$

$$= \text{Im} \left[Z(n) \begin{pmatrix} a_n \varphi(n) \\ \varphi(n-1) \end{pmatrix} \right]. \quad (4.40)$$

Define $R(n) > 0$ and $\eta(n) \in \mathbb{R}$ by

$$Z(n) = R(n)e^{i\eta(n)}.$$

We can use Wronskians to invert (4.39) to get

$$Z(n) = \frac{2}{\omega} W_{0,a'}(\overline{\varphi}, u)(n-1), \quad (4.41)$$

which is the same as (41) in [KRS99].

Theorem 4.22.

$$\begin{aligned}
\frac{Z(n+1)}{Z(n)} = & 1 - \frac{i}{\omega} \frac{a_n}{a_n + a'_n} b'_{n+1} |\varphi(n)|^2 (e^{-2i(\eta(n)+\gamma(n))} - 1) \\
& + \frac{i}{\omega} a'_n |\varphi(n-1)| \cdot |\varphi(n)| e^{i(\gamma(n)-\gamma(n-1))} \\
& - \frac{i}{\omega} a'_n |\varphi(n-1)| \cdot |\varphi(n)| e^{-2i\theta} e^{i(\gamma(n-1)-\gamma(n))} \\
& + \frac{i}{\omega} \frac{a_n}{a_n + a'_n} a'_n (1 - e^{-2i(\eta(n)+\gamma(n))}) |\varphi(n-1)| \cdot |\varphi(n)| e^{-i(\gamma(n-1)-\gamma(n))}.
\end{aligned}$$

Proof. For notational convenience, let us define

$$\theta(n) = \eta(n) + \gamma(n).$$

We first note that

$$\begin{aligned}
u(n) &= \frac{a_n}{a_n + a'_n} R(n) |\varphi(n)| \sin(\theta(n)), \\
u(n-1) &= R(n) |\varphi(n-1)| \sin(\theta(n) - \gamma(n) + \gamma(n-1)).
\end{aligned}$$

We have by (4.41), (4.35), (4.33), (4.32),

$$\begin{aligned}
Z(n+1) - Z(n) &= -\frac{2}{\omega} (b'_{n+1} u(n) \overline{\varphi(n)} + a'_n \overline{\varphi(n)} u(n-1) + a'_n \overline{\varphi(n-1)} u(n)) \\
&= -\frac{2}{\omega} \frac{a_n}{a_n + a'_n} b'_{n+1} R(n) \sin(\theta(n)) |\varphi(n)|^2 e^{-i\gamma(n)} \\
&\quad - \frac{2}{\omega} a'_n R(n) \sin(\theta(n) - \gamma(n) + \gamma(n-1)) |\varphi(n)| \cdot |\varphi(n-1)| e^{-i\gamma(n)} \\
&\quad - \frac{2}{\omega} \frac{a_n}{a_n + a'_n} a'_n R(n) \sin(\theta(n)) |\varphi(n-1)| \cdot |\varphi(n)| e^{-i\gamma(n-1)}. \quad (4.42)
\end{aligned}$$

From the definitions of R, θ , we then have

$$\begin{aligned}
& Z(n+1) - Z(n) \\
&= -\frac{2}{\omega} \frac{a_n}{a_n + a'_n} b'_{n+1} |\varphi(n)|^2 Z(n) \sin(\theta(n)) e^{-i\theta(n)} \\
&\quad - \frac{2}{\omega} a'_n |\varphi(n-1)| \cdot |\varphi(n)| Z(n) \sin(\theta(n) - \gamma(n) + \gamma(n-1)) e^{-i\theta(n)} \\
&\quad - \frac{2}{\omega} \frac{a_n}{a_n + a'_n} a'_n Z(n) \sin(\theta(n)) |\varphi(n+1)| \cdot |\varphi(n)| e^{-i(\theta(n) + \gamma(n) - \gamma(n+1))}.
\end{aligned}$$

which implies

$$\begin{aligned}
\frac{Z(n+1)}{Z(n)} &= 1 - \frac{2}{\omega} \frac{a_n}{a_n + a'_n} b'_{n+1} |\varphi(n)|^2 \sin(\theta(n)) e^{-i\theta(n)} \\
&\quad - \frac{2}{\omega} a'_n |\varphi(n-1)| \cdot |\varphi(n)| \sin(\theta(n) + \gamma(n-1) - \gamma(n)) e^{-i\theta(n)} \\
&\quad - \frac{2}{\omega} \frac{a_n}{a_n + a'_n} a'_n \sin(\theta(n)) |\varphi(n-1)| \cdot |\varphi(n)| e^{-i(\theta(n) + \gamma(n-1) - \gamma(n))}.
\end{aligned}$$

We rewrite this slightly by expanding out the $e^{-i\theta}$ terms to obtain

$$\begin{aligned}
& \frac{Z(n+1)}{Z(n)} \\
&= 1 - \frac{2}{\omega} \frac{a_n}{a_n + a'_n} b'_{n+1} |\varphi(n)|^2 \sin(\theta(n)) [\cos(\theta(n)) - i \sin(\theta(n))] \\
&\quad - \frac{2}{\omega} a'_n |\varphi(n-1)| \cdot |\varphi(n)| \sin(\theta(n) + \gamma(n-1) - \gamma(n)) [\cos(\theta(n)) - i \sin(\theta(n))] \\
&\quad - \frac{2}{\omega} \frac{a_n}{a_n + a'_n} a'_n \sin(\theta(n)) |\varphi(n-1)| \cdot |\varphi(n)| [\cos(\theta(n)) - i \sin(\theta(n))] e^{-i(\gamma(n-1) - \gamma(n))},
\end{aligned}$$

and using the double angle formulae, this turns out to become

$$\begin{aligned}
& \frac{Z(n+1)}{Z(n)} \\
= & 1 - \frac{1}{\omega} \frac{a_n}{a_n + a'_n} b'_{n+1} |\varphi(n)|^2 \sin(2\theta(n)) \\
& + \frac{i}{\omega} \frac{a_n}{a_n + a'_n} b'_{n+1} |\varphi(n)|^2 (1 - \cos(2\theta(n))) \\
& - \frac{2}{\omega} a'_n |\varphi(n-1)| \cdot |\varphi(n)| \cdot [\sin(\theta(n)) \cos(\gamma(n-1) - \gamma(n)) \\
& + \cos(\theta(n)) \sin(\gamma(n-1) - \gamma(n))] [\cos(\theta(n)) - i \sin(\theta(n))] \\
& - \frac{1}{\omega} \frac{a_n}{a_n + a'_n} a'_n \sin(2\theta(n)) |\varphi(n-1)| \cdot |\varphi(n)| e^{-i(\gamma(n-1) - \gamma(n))} \\
& + \frac{i}{\omega} \frac{a_n}{a_n + a'_n} a'_n (1 - \cos(2\theta(n))) |\varphi(n-1)| \cdot |\varphi(n)| e^{-i(\gamma(n-1) - \gamma(n))},
\end{aligned}$$

and so

$$\begin{aligned}
& \frac{Z(n+1)}{Z(n)} \\
= & 1 - \frac{1}{\omega} \frac{a_n}{a_n + a'_n} b'_{n+1} |\varphi(n)|^2 \sin(2\theta(n)) \\
& + \frac{i}{\omega} \frac{a_n}{a_n + a'_n} b'_{n+1} |\varphi(n)|^2 (1 - \cos(2\theta(n))) \\
& - \frac{1}{\omega} a'_n |\varphi(n-1)| \cdot |\varphi(n)| \cdot \sin(2\theta(n)) \cos(\gamma(n) - \gamma(n-1)) \\
& - \frac{1}{\omega} a'_n |\varphi(n-1)| \cdot |\varphi(n)| \cdot (1 + \cos(2\theta(n))) \sin(\gamma(n) - \gamma(n-1)) \\
& + \frac{i}{\omega} a'_n |\varphi(n-1)| \cdot |\varphi(n)| \cdot (1 - \cos(2\theta(n))) \cos(\gamma(n) - \gamma(n-1)) \\
& + \frac{i}{\omega} a'_n |\varphi(n-1)| \cdot |\varphi(n)| \cdot \sin(2\theta(n)) \sin(\gamma(n) - \gamma(n-1)) \\
& - \frac{1}{\omega} \frac{a_n}{a_n + a'_n} a'_n \sin(2\theta(n)) |\varphi(n-1)| \cdot |\varphi(n)| e^{-i(\gamma(n-1) - \gamma(n))} \\
& + \frac{i}{\omega} \frac{a_n}{a_n + a'_n} a'_n (1 - \cos(2\theta(n))) |\varphi(n-1)| \cdot |\varphi(n)| e^{-i(\gamma(n-1) - \gamma(n))},
\end{aligned}$$

furthermore,

$$\begin{aligned}
& \frac{Z(n+1)}{Z(n)} \\
&= 1 - \frac{i}{\omega} \frac{a_n}{a_n + a'_n} b'_{n+1} |\varphi(n)|^2 (e^{-2i\theta(n)} - 1) \\
&\quad - \frac{i}{\omega} a'_n |\varphi(n-1)| \cdot |\varphi(n)| \cdot (e^{-2i\theta(n)} - 1) \cos(\gamma(n) - \gamma(n-1)) \\
&\quad - \frac{1}{\omega} a'_n |\varphi(n-1)| \cdot |\varphi(n)| (1 + e^{-2i\theta(n)}) \sin(\gamma(n) - \gamma(n-1)) \\
&\quad + \frac{i}{\omega} \frac{a_n}{a_n + a'_n} a'_n (1 - e^{-2i\theta(n)}) |\varphi(n-1)| \cdot |\varphi(n)| e^{-i(\gamma(n-1) - \gamma(n))},
\end{aligned}$$

which completes the proof. \square

This allows us to make a first application of these Prüfer variables.

Corollary 4.23. *Assume that $\inf_n a_n > 0$ and that $a', b' \in \ell^1$. If x is such that solutions of the unperturbed Jacobi matrix are bounded, then solutions of the perturbed Jacobi matrix are bounded.*

Proof. Under the above assumptions, $\lim_{n \rightarrow \infty} \frac{a_n}{a_n + a'_n} = 1$ so

$$M = \sup_n \frac{a_n}{a_n + a'_n}$$

is finite. It follows from Theorem 4.22 that

$$\left| \frac{Z(n+1)}{Z(n)} - 1 \right| \leq \frac{\|\varphi\|_\infty^2}{|\omega|} (2M|b'_{n+1}| + 2|a'_n| + 2M|a'_n|). \quad (4.43)$$

Since the right-hand side of (4.43) is ℓ^1 , it follows that $Z(n)$ converges as $n \rightarrow \infty$ and, in particular, $R(n)$ is bounded in n . This implies that $u(n)$ is bounded in n and completes the proof. \square

Next, we prove a lemma which will be necessary in the applications which follow.

Lemma 4.24. Fix x and a solution φ of (4.32). For different solutions u of (4.33), denote the corresponding Prüfer variables by Z_u, R_u, η_u . If there exists a sequence $A(n)$ independent of u such that the series

$$\sum_{n=1}^{\infty} \left(\log \frac{Z_u(n+1)}{Z_u(n)} - A(n) \right) \quad (4.44)$$

converges uniformly in solution u , then $R_u(n)$ converges as $n \rightarrow \infty$ for any solution u .

Proof. Let us consider two solutions $u_1(n), u_2(n)$ of (4.33). Subtracting (4.44) for the two solutions, we conclude that the series

$$\sum_{n=1}^{\infty} \left(\log \frac{Z_{u_1}(n+1)}{Z_{u_1}(n)} - \log \frac{Z_{u_2}(n+1)}{Z_{u_2}(n)} \right)$$

is convergent. In particular, taking real and imaginary parts, we see that the sequences

$$\log \frac{R_{u_1}(n)}{R_{u_2}(n)}, \quad \eta_{u_1}(n) - \eta_{u_2}(n)$$

converge as $n \rightarrow \infty$.

By uniform convergence, there is an n_0 such that for all solutions u ,

$$\left| \sum_{n=n_0+1}^{\infty} \left(\log \frac{Z_u(n+1)}{Z_u(n)} - A(n) \right) \right| \leq \frac{\pi}{8}.$$

Taking imaginary parts and subtracting this for u_1, u_2 ,

$$\left| \sum_{n=n_0+1}^{\infty} ((\eta_{u_1}(n+1) - \eta_{u_1}(n)) - (\eta_{u_2}(n+1) - \eta_{u_2}(n))) \right| \leq \frac{\pi}{4}.$$

Thus,

$$\left| \lim_{n \rightarrow \infty} (\eta_{u_1}(n) - \eta_{u_2}(n)) - (\eta_{u_1}(n_0) - \eta_{u_2}(n_0)) \right| \leq \frac{\pi}{4}.$$

In particular, if we were to pick the solution u_1 arbitrarily and pick the solution u_2 so that

$$Z_{u_2}(n_0) = iZ_{u_1}(n_0),$$

then we would have $\eta_{u_2}(n_0) - \eta_{u_1}(n_0) \in \frac{\pi}{2} + 2\pi\mathbb{Z}$ so

$$\lim_{n \rightarrow \infty} (\eta_{u_2}(n) - \eta_{u_1}(n)) \in \left(\frac{\pi}{4}, \frac{3\pi}{4} \right) + 2\pi\mathbb{Z}.$$

We consider now the Wronskian of u_1 and u_2 .

$$\begin{aligned} & W_{a', a'}(u_1, u_2)(n-1) \\ &= (a_n + a'_n)[u_1(n-1)u_2(n) - u_1(n)u_2(n-1)] \\ &= a_n |\varphi(n)\varphi(n-1)| R_{u_2}(n) R_{u_1}(n) [\sin(\eta_{u_2}(n) + \gamma(n)) \sin(\eta_{u_1}(n) + \gamma(n-1)) \\ &\quad - \sin(\eta_{u_1}(n) + \gamma(n)) \sin(\eta_{u_2}(n) + \gamma(n-1))] \\ &= a_n |\varphi(n)\varphi(n-1)| R_{u_2}(n) R_{u_1}(n) \sin(\eta_{u_1}(n) - \eta_{u_2}(n)) \sin(\gamma(n) - \gamma(n-1)) \\ &= \frac{1}{2} R_{u_2}(n) R_{u_1}(n) \sin(\eta_{u_1}(n) - \eta_{u_2}(n)) W(\varphi, \varphi^*) \end{aligned}$$

The Wronskian is nonzero and independent of n , but we know from the above that the quantities

$$\frac{R_{u_1}(n)}{R_{u_2}(n)}, \quad \sin(\eta_{u_1}(n) - \eta_{u_2}(n))$$

have nonzero limits as $n \rightarrow \infty$. From our final formula for the Wronskian, it then follows that $R_{u_1}^2(n)$, and then $R_{u_1}(n)$ has a nonzero limit as $n \rightarrow \infty$. \square

4.2.2 Decaying oscillatory perturbations of periodic Jacobi matrices

Let us assume the a_n, b_n are periodic of period q and that a'_n, b'_n are decaying oscillatory sequences each of which satisfies Definition 1.11.

Let us fix E in the interior of the spectrum of the periodic Jacobi matrix $J(a, b)$, and let $\varphi(n)$ be a Floquet solution at E . Then we can write $\gamma(n) = \varpi(n) + kn$ where k is the quasimomentum and $|\varphi(n)|$ and $\varpi(n)$ are q -periodic. We can thus rewrite the equation again:

$$\begin{aligned}
& \frac{Z(n+1)}{Z(n)} \\
= & 1 - \frac{i}{\omega} \frac{a_n}{a_n + a'_n} b'_{n+1} |\varphi(n)|^2 (e^{-2i\eta(n)} e^{-2i\varpi(n) - 2ikn} - 1) \\
& + \frac{i}{\omega} a'_n |\varphi(n-1)| \cdot |\varphi(n)| e^{i(\varpi(n) - \varpi(n-1) + k)} \\
& - \frac{i}{\omega} a'_n |\varphi(n-1)| \cdot |\varphi(n)| e^{-2i\eta(n)} e^{-2i\varpi(n) - 2ikn} e^{-i(\varpi(n) - \varpi(n-1) + k)} \\
& + \frac{i}{\omega} \frac{a_n}{a_n + a'_n} a'_n (1 - e^{-2i\eta(n)} e^{-2i\varpi(n) - 2ikn}) |\varphi(n-1)| \cdot |\varphi(n)| e^{i(\varpi(n) - \varpi(n-1) + k)}, \quad (4.45)
\end{aligned}$$

Using the geometric series

$$\frac{a_n}{a_n + a'_n} = \frac{1}{1 + a'_n/a_n} = 1 - \frac{a'_n}{a_n} + \left(\frac{a'_n}{a_n}\right)^2 - \dots$$

we can write

$$\frac{Z(n+1)}{Z(n)} = 1 + P(n) + Q(n),$$

where $P(n)$ collects all terms with at most $p-1$ factors of a'_n and b'_{n+1} ,

$$P(n) = \sum_{\substack{K, L \geq 0 \\ 0 < K+L < p}} \sum_{M=0}^1 \xi_{K, L, M}(n) a_n'^K b'_{n+1}{}^L e^{-2iM\eta}$$

and $\xi_{K,L,M}(n)$ are q -periodic sequences. The remainder $Q(n)$ collects all terms with p or more factors of a'_n and b'_{n+1} , so $Q(n) \in \ell^1$.

Note that we have

$$\frac{R(n+1)}{R(n)} = \left| \frac{Z(n+1)}{Z(n)} \right|, \quad (4.46)$$

$$e^{2i(\eta(n+1)-\eta(n))} = \frac{Z(n+1)}{Z(n)} \div \frac{\overline{Z(n+1)}}{\overline{Z(n)}}. \quad (4.47)$$

For some set of phases A , let us assume in this section that the a'_n, b'_n are all in $GBV(A)$ and in $\ell^2(\mathbb{N})$.

Lemma 4.25. *Given a q -periodic sequence $f(n)$ and a real number κ such that $\kappa q \not\equiv 0 \pmod{2\pi}$, there exists a q -periodic sequence $g(n)$ such that*

$$e^{i\kappa n} g(n) - e^{i\kappa(n-1)} g(n-1) = e^{i\kappa n} f(n) \quad (4.48)$$

and

$$\|g\|_\infty \leq \frac{q\|f\|_\infty}{|e^{i\kappa q} - 1|}.$$

Proof. We define

$$g(0) = \frac{\sum_{j=1}^q e^{i\kappa j} f(j)}{e^{i\kappa q} - 1},$$

$$g(n) = e^{-i\kappa n} \left(g(0) + \sum_{j=1}^n e^{i\kappa j} f(j) \right), \quad 1 \leq n \leq q-1,$$

and then extend g periodically. We can verify that this g satisfies the conditions of the lemma. □

Lemma 4.26. *Let $m \in \mathbb{Z}$ and $\phi \in [0, 2\pi)$, with m and ϕ not both equal to 0. Let $B \subset \mathbb{R}$ be a finite set and $f : \mathbb{R} \setminus (B + 2\pi\mathbb{Z}) \rightarrow \mathbb{C}$ be a q -periodic continuous function such that $g(k; n) = f(k; n)/(e^{-i(2mk-\phi)} - 1)$ is also continuous on $\mathbb{R} \setminus (B + 2\pi\mathbb{Z})$, with removable singularities allowed.*

If γ_n has bounded variation and $\gamma_n \rightarrow 0$, then for a phase ϕ and positive constants C, D ,

$$\left| \sum_{j=1}^n e^{-i(2mk-\phi)j} f(k; j) e^{-2im\eta(j)} \gamma_j - (e^{-2im\eta(j)} - e^{-2im\eta(j+1)}) e^{-i(2mk-\phi)j} g(k; j) \gamma_j \right|$$

$$\leq C \sup_j |g(k; j)| \cdot \sup_j |\gamma_j| + D \sup_j |g(k; j)|.$$

Proof. We use $\kappa = 2mk - \phi$ in (4.48). By the discrete product rule, we have

$$\begin{aligned} & e^{-2im\eta(n+1)} e^{-i(2mk-\phi)n} g(k; n) - e^{-2im\eta(n)} e^{-i(2mk-\phi)(n-1)} g(k; n-1) \\ &= (e^{-i(2mk-\phi)n} g(k; n) - e^{-i(2mk-\phi)(n-1)} g(k; n-1)) \\ & \quad \times \left(e^{-2im\eta(1)} + \sum_{j=1}^{n-1} (e^{-2im\eta(j+1)} - e^{-2im\eta(j)}) \right) \\ & \quad + (e^{-2im\eta(n+1)} - e^{-2im\eta(n)}) \\ & \quad \times \left(g(k; 0) + \sum_{j=1}^n (e^{-i(2mk-\phi)j} g(k; j) - e^{-i(2mk-\phi)(j-1)} g(k; j-1)) \right) \\ &= e^{-i(2mk-\phi)n} f(k; n) e^{-2im\eta(n)} \\ & \quad + (e^{-2im\eta(n+1)} - e^{-2im\eta(n)}) e^{-i(2mk-\phi)n} g(k; n). \end{aligned}$$

and so

$$\begin{aligned}
& e^{-2im\eta(n+1)} e^{-i(2mk-\phi)n} g(k; n) \gamma_{n+1} \\
& - e^{-2im\eta(1)} g(k; 0) \gamma_1 \\
& = \sum_{j=1}^n e^{-2im\eta(j+1)} e^{-i(2mk-\phi)j} g(k; j) \gamma_{j+1} - e^{-2im\eta(j)} e^{-i(2mk-\phi)(j-1)} g(k; j-1) \gamma_j \\
& = \sum_{j=1}^n e^{-2im\eta(j+1)} e^{-i(2mk-\phi)j} g(k; j) \gamma_j - e^{-2im\eta(j)} e^{-i(2mk-\phi)(j-1)} g(k; j-1) \gamma_j \\
& \quad + \sum_{j=1}^n (\gamma_{j+1} - \gamma_j) e^{-2im\eta(j+1)} e^{-i(2mk-\phi)j} g(k; j) \\
& = \sum_{j=1}^n e^{-i(2mk-\phi)j} f(k; j) e^{-2im\eta(j)} \gamma_j + (e^{-2im\eta(j+1)} - e^{-2im\eta(j)}) e^{-i(2mk-\phi)j} g(k; j) \gamma_j \\
& \quad + \sum_{j=1}^n (\gamma_{j+1} - \gamma_j) e^{-2im\eta(j+1)} e^{-i(2mk-\phi)j} g(k; j).
\end{aligned}$$

□

We rewrite (4.45) as

$$\frac{Z(n+1)}{Z(n)} = 1 - X(n) - Y(n) e^{-2i(\eta(n)+kn)}.$$

Observe that

$$\frac{a_n}{a_n + a'_n} = \frac{1}{1 + a'_n/a_n} = 1 - \frac{a'_n}{a_n} + \left(\frac{a'_n}{a_n}\right)^2 - \dots$$

Thus we can see that both $X(n)$ and $Y(n)$ are sums of terms that are products of a'_n s and b'_{n+1} s with some periodic factor.

In fact, ignoring all ℓ^p terms, we can express

$$B(n) \sim \operatorname{Re} \sum_{K,L,M \geq 0, K+L < p} \xi_{K,L,M}(n) a'_n{}^K b'_{n+1}{}^L e^{-i(2Mk-\phi)},$$

where $\xi_{K,L,M}(n)$ is some continuous q -periodic function.

Let us define now, by expanding (4.47),

$$e^{2im(\eta(n+1)-\eta(n))} - 1 = P_{m,l}(a'_n, b'_{n+1}, e^{2i(kn+\eta(n))}) + O(|a'_n|^l, |b'_{n+1}|^l),$$

where

$$P_{m,l}(a'_n, b'_{n+1}, e^{2i(kn+\eta(n))}) = \sum_{-l < j < l} K_j(n) e^{2ij(kn+\eta(n))}.$$

Here $K_j(n)$ are each products of at most $l-1$ copies of a'_n or b'_{n+1} .

Let us assume that a'_n, b'_{n+1} can be written as

$$a'_n = \sum_{l=1}^L e^{-i\phi_l n} \gamma_n^{(a;l)},$$

$$b'_{n+1} = \sum_{l=1}^L e^{-i\phi_l n} \gamma_n^{(b;l)},$$

where $\gamma_n^{(a;l)}, \gamma_n^{(b;l)}$, are both of bounded variation.

We note that $B(n)$ can then be expressed as sums of terms of the form

$$\operatorname{Re} \left(f(n) e^{-2iM(\eta(n)+kn)} \prod_{j=1}^K e^{i\phi_{l_{K,j}} n} \overline{\gamma_n^{(a;l_{K,j})}} \prod_{j=1}^L e^{i\phi_{l_{L,j}} n} \overline{\gamma_n^{(b;l_{L,j})}} \right).$$

This term will occur in all possible permutations of $l_{K,1}, \dots, l_{K,K}, l_{L,1}, \dots, l_{L,L}$. We average those terms, and then we write the resulting $f(n)$ in the form

$$f_{K,L,M}(n; \phi_{l_{K,1}} \cdots \phi_{l_{K,K}}; \phi_{l_{L,1}} \cdots \phi_{l_{L,L}}).$$

Let us define $\chi(x) = 1/(e^{-ix} - 1)$. We write the corresponding $g(n)$ as constructed by Lemma 4.26 as

$$g_{K,L,M} = \chi \left(2Mk - \sum_{j=1}^K \phi_{K,j} - \sum_{j=1}^L \phi_{L,j} \right) f_{K,L,M}.$$

Also, if K, L , or M is negative, we define $f_{K,L,M} = g_{K,L,M} = 0$.

We define a symmetrized product as follows.

Definition 4.27. For a function $p_{K,L}$ of $1 + K + L$ variables and a function $q_{K',L'}$ of $1 + I' + J' + K' + L'$ variables, we define their symmetric product $p_{K,L} \odot q_{K',L'}$ of $1 + K + L + K' + L'$ variables as

$$\begin{aligned} & (p_{K,L} \odot q_{K',L'})(n; \{y_i\}_{i=1}^{K+K'}; \{z_i\}_{i=1}^{L+L'}) \\ &= \frac{p_{K,L}(n; \{y_{\sigma(K;i)}\}_{i=1}^K; \{z_{\sigma(L;i)}\}_{i=1}^L)}{(K + K')!(L + L')!} \\ & \quad \times q_{K',L'}(n; \{y_{\sigma(K;i)}\}_{i=K+1}^{K'}; \{z_{\sigma(L;i)}\}_{i=L+1}^{L'}). \end{aligned}$$

The sum runs through $\sigma(K; \cdot) \in S_{K+K'}$, and $\sigma(L; \cdot) \in S_{L+L'}$, where S_N is the symmetric group of N elements.

We can then write

$$f_{K,L,M}(n) = \xi_{K,L,M}(n) + \sum \omega_{K,\alpha,\beta,\delta}(n) \odot g_{K-\alpha,L-\beta,M-\delta}(n),$$

where $\omega_{K,\alpha,\beta,\delta}(n)$ is q -periodic and the sum runs through $\alpha, \beta \geq 0$ and $-l < \delta < l$, with α, β not both zero.

Thus by repeated application of Lemma 4.26 we can show that $B(n)$ converges.

Note that to use the Lemma we have to assume that

$$2Mk - \sum_{j=1}^K \phi_{K,j} - \sum_{j=1}^L \phi_{L,j},$$

is nonzero. Thus we may express our theorem as follows.

Theorem 4.28. *Consider q -periodic Jacobi coefficients $\{a_n\}, \{b_n\}$, and perturbations a'_n, b'_n of these Jacobi coefficients (i.e. $a_n, a_n + a'_n$ are both positive, and $b_n, b_n + b'_n$ are both real). Let A be a finite set of phases and assume $\{a'_n, b'_n\}$ are in $\ell^p \cap GBV(A)$ for some $p \geq 2$.*

Let us define A_p as n -sums of elements in A or $-A$ (possibly repeated) where n ranges from 1 to $p - 1$. Let $S = \{k | 2lk \in A_p, l = \pm 1, \pm 2, \dots, \pm(p - 1)\}$, where k is the quasimomentum corresponding to Floquet solutions of the unperturbed Jacobi recursion. Then where $\mu_{\mathbb{R}}$ is the probability measure corresponding to the $\{a_n\}, \{b_n\}$ and $\tilde{\mu}_{\mathbb{R}}$ is the probability measure corresponding to $\{a_n + a'_n\}, \{b_n + b'_n\}$, $\text{supp}\tilde{\mu}_{\mathbb{R},s} \cap \text{supp}\mu_{\mathbb{R},ess}$ only has points with quasimomenta in S .

This theorem is a stronger version of Theorem 1.12.

4.2.3 Generalized Prüfer variables for the Szegő recursion

Let $z \in \partial\mathbb{D}$. For a sequence of Verblunsky coefficients $\{\alpha\}$, and $\rho(n)$ defined as $\sqrt{1 - |\alpha(n)|^2}$, consider the Szegő recursion written in terms of transfer matrices:

that is given

$$A(n) = \frac{1}{\rho(n)} \begin{pmatrix} z & -\overline{\alpha(n)} \\ -\alpha(n)z & 1 \end{pmatrix}, \quad (4.49)$$

we write the recursion as

$$v(n+1) = z^{-1/2} A(n) v(n) \quad (4.50)$$

Where the $v(n)$ are in \mathbb{C}^2 .

Consider now complex sequences $\{\alpha'(n)\}, \{\rho'(n)\}$, such that $\alpha(n) + \alpha'(n) \in \mathbb{D}$, and

$$\rho'(n) = \sqrt{1 - |\alpha(n) + \alpha'(n)|^2} - \rho(n). \quad (4.51)$$

Consider also matrices A'_n defined as

$$A'(n) = \frac{1}{\rho(n) + \rho'(n)} \begin{pmatrix} z & -\overline{\alpha(n) + \alpha'(n)} \\ -(\alpha(n) + \alpha'(n))z & 1 \end{pmatrix}, \quad (4.52)$$

Choosing $\kappa \in \mathbb{C}$, we wish to compare these solutions v with solutions u of

$$u(n+1) \begin{pmatrix} \kappa & 0 \\ 0 & \bar{\kappa} \end{pmatrix} = z^{-1/2} A'(n) u(n) \begin{pmatrix} \kappa & 0 \\ 0 & \bar{\kappa} \end{pmatrix}. \quad (4.53)$$

We will assume the initial conditions

$$u(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Note that the standard Szégo recursion doesn't have the $z^{-1/2}$ term. The term is added here so that the matrix recursion will have determinant 1 rather than z . We

will denote the solutions of the “standard” matrix recursions (i.e., without the $z^{-1/2}$ term) as v_{st}, u_{st} respectively. Note that $u_{st}(n) = z^{n/2}u(n)$, $v_{st}(n) = z^{n/2}v(n)$.

Let us define an antilinear operator C by

$$C \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \overline{w_2} \\ \overline{w_1} \end{pmatrix}. \quad (4.54)$$

Let us write $v^*(n) = Cv(n)$. Notice that $Cz^{-1/2}A(n)C = z^{-1/2}A(n)$, so if $v(n)$ is a solution for (4.50), so is $v^*(n)$. Similarly, defining $u^*(n) = Cu(n)$, if $u(n)$ is a solution for (4.53) then so is $u^*(n)$.

Given f, g two sequences in \mathbb{C}^2 , we define their Wronskian as

$$W(f, g)(n) = f_2(n)g_1(n) - g_2(n)f_1(n). \quad (4.55)$$

Let us assume f, g are both solutions of a Szégo transfer matrix where $\alpha_f(n), \alpha_g(n)$ are the Verblunsky coefficients corresponding to f, g , and ρ_f, ρ_g also correspond respectively to f, g :

We then have

$$\begin{aligned}
& W(f, g)(n) - W(f, g)(n-1) \\
&= f_2(n)g_1(n) - g_2(n)f_1(n) \\
&\quad - (f_2(n-1)g_1(n-1) - g_2(n-1)f_1(n-1)) \\
&= (\rho_f(n-1)z^{-1/2}f_2(n-1) - \alpha_f(n-1)f_1(n))g_1(n) \\
&\quad - (\rho_g(n-1)z^{-1/2}g_2(n-1) - \alpha_g(n-1)g_1(n))f_1(n) \\
&\quad - f_2(n-1)(\rho_g(n-1)z^{-1/2}g_1(n) + z^{-1}\overline{\alpha_g(n-1)g_2(n-1)}) \\
&\quad + g_2(n-1)(\rho_f(n-1)z^{-1/2}f_1(n) + z^{-1}\overline{\alpha_f(n-1)f_2(n-1)}) \\
&= (\rho_f(n-1) - \rho_g(n-1))z^{-1/2}(f_2(n-1)g_1(n) + g_2(n-1)f_1(n)) \\
&\quad - (\alpha_f(n-1) - \alpha_g(n-1))f_1(n)g_1(n) \\
&\quad + (\overline{\alpha_f(n-1)} - \overline{\alpha_g(n-1)})z^{-1}f_2(n-1)g_2(n-1). \tag{4.56}
\end{aligned}$$

Our first order of business is to verify that constancy of the Wronskian holds.

Proposition 4.29. *For $(f_1, f_2)^T, (g_1, g_2)^T$ two solutions of the Szégo recursion, and n a positive integer, $W(f, g)(n) = W(f, g)(n-1)$.*

Proof. Let us write

$$M(n) = \begin{pmatrix} g_1(n) & f_1(n) \\ g_2(n) & f_2(n) \end{pmatrix}.$$

We can then write

$$M(n) = z^{-n/2} A_{n-1} A_{n-2} \dots A_0 M(0).$$

Taking determinants of both sides, we find that

$$f_2(n)g_1(n) - g_2(n)f_1(n) = \det M(0),$$

which proves our proposition. \square

Let us assume that $v(0), v^*(0)$ are linearly independent. Then there exist complex numbers $Z(n), \mathfrak{s}(n)$ such that

$$u(n) = Z(n)v(n) + \mathfrak{s}(n)v^*(n). \quad (4.57)$$

Applying $z^n C$ to both sides of this equation we get

$$u^*(n) = \overline{Z(n)}v^*(n) + \overline{\mathfrak{s}(n)}v(n).$$

This implies that $\mathfrak{s}(n) = \overline{Z(n)}$.

Let us define $\gamma_1(n), \gamma_2(n), R(n), \eta(n)$ as

$$Z(n) = R(n)e^{i\eta(n)}, \quad (4.58)$$

$$v_1(n) = |v_1(n)|e^{i\gamma_1(n)}. \quad (4.59)$$

$$v_2(n) = |v_2(n)|e^{i\gamma_2(n)}. \quad (4.60)$$

Let us write ω as the Wronskian of v and v^* . Note that ω will be a nonzero real constant, due to the assumption that v, v^* are linearly independent.

We can thus write

$$\omega = W(v, v^*) = v_2(n)v_1^*(n) - v_2^*(n)v_1(n) = |v_2(n)|^2 - |v_1(n)|^2. \quad (4.61)$$

Recall that this expression is actually n -independent.

From (4.57) we can write

$$Z(n) = \frac{W(u, v^*)(n)}{\omega}$$

Using (4.56)

$$\begin{aligned} & Z(n+1) - Z(n) \\ &= \frac{W(u, v^*)(n+1) - W(u, v^*)(n)}{\omega} \end{aligned} \quad (4.62)$$

$$\begin{aligned} &= + \frac{((\rho(n) + \rho'(n) - \rho(n))z^{-1/2}(u_2(n)v_1^*(n+1) + v_2^*(n)u_1(n+1)))}{\omega} \\ &\quad - \frac{((\alpha(n) + \alpha'(n)) - \alpha(n))v_1^*(n+1)u_1(n+1)}{\omega} \\ &\quad + \frac{(\overline{(\alpha(n) + \alpha'(n))z^{-1} - \alpha(n)})v_2^*(n)u_2(n)}{\omega}. \\ &= + \frac{\rho'(n)z^{-1/2}(u_2(n)v_1^*(n+1) + v_2^*(n)u_1(n+1))}{\omega} \\ &\quad - \frac{\alpha'(n)v_1^*(n+1)u_1(n+1)}{\omega} + \frac{\overline{\alpha'(n)z^{-1}v_2^*(n)u_2(n)}}{\omega}. \end{aligned} \quad (4.63)$$

Note that we can write, from (4.57),

$$\begin{aligned} u_1(n) &= Z(n)v_1(n) + \overline{Z(n)}v_1^*(n) \\ &= R(n)e^{in(n)}|v_1(n)|e^{i\gamma_1(n)} + R(n)e^{-in(n)}|v_2(n)|e^{-i\gamma_2(n)}. \end{aligned} \quad (4.64)$$

Similarly,

$$\begin{aligned} u_2(n) &= Z(n)v_2(n) + \overline{Z(n)}v_2^*(n) \\ &= R(n)e^{in(n)}|v_2(n)|e^{i\gamma_2(n)} + R(n)e^{-in(n)}|v_1(n)|e^{-i\gamma_1(n)}. \end{aligned} \quad (4.65)$$

Theorem 4.30. *We have*

$$\begin{aligned}
& \frac{R(n+1)e^{i\eta(n+1)}}{R(n)e^{i\eta(n)}} \\
&= [\omega + (\rho'(n)z^{-1/2}(|v_2(n)|e^{i\gamma_2(n)} + e^{-2i\eta(n)}|v_1(n)|e^{-i\gamma_1(n)})|v_2(n+1)|e^{-i\gamma_2(n+1)}) \\
&\quad + \overline{\alpha'(n)}z^{-1}|v_1(n)|e^{-i\gamma_1(n)}(|v_2(n)|e^{i\gamma_2(n)} + e^{-2i\eta(n)}|v_1(n)|e^{-i\gamma_1(n)})] \\
&\quad \div [\omega - \rho'(n)z^{-1/2}(|v_1(n)|e^{-i\gamma_1(n)}|v_1(n+1)|e^{i\gamma_1(n+1)}) \\
&\quad - \rho'(n)z^{-1/2}(|v_1(n)|e^{-i\gamma_1(n)}e^{-2i\eta(n+1)}|v_2(n+1)|e^{-i\gamma_2(n+1)}) \\
&\quad + \alpha'(n)|v_2(n+1)|e^{-i\gamma_2(n+1)}|v_1(n+1)|e^{i\gamma_1(n+1)} \\
&\quad + \alpha'(n)|v_2(n+1)|e^{-i\gamma_2(n+1)}e^{-2i\eta(n+1)}|v_2(n+1)|e^{-i\gamma_2(n+1)}].
\end{aligned}$$

Proof. First, we observe that (4.63), (4.64), (4.65) together give us

$$\begin{aligned}
& R(n+1)e^{i\eta(n+1)} - R(n)e^{i\eta(n)} \\
&= + \frac{\rho'(n)z^{-1/2}(R(n)e^{i\eta(n)}|v_2(n)|e^{i\gamma_2(n)} + R(n)e^{-i\eta(n)}|v_1(n)|e^{-i\gamma_1(n)})v_1^*(n+1)}{\omega} \\
&\quad + \frac{\rho'(n)z^{-1/2}(v_2^*(n)R(n+1)e^{i\eta(n+1)}|v_1(n+1)|e^{i\gamma_1(n+1)})}{\omega} \\
&\quad + \frac{\rho'(n)z^{-1/2}(v_2^*(n)R(n+1)e^{-i\eta(n+1)}|v_2(n+1)|e^{-i\gamma_2(n+1)})}{\omega} \\
&\quad - \frac{\alpha'(n)v_1^*(n+1)R(n+1)e^{i\eta(n+1)}|v_1(n+1)|e^{i\gamma_1(n+1)}}{\omega} \\
&\quad - \frac{\alpha'(n)v_1^*(n+1)R(n+1)e^{-i\eta(n+1)}|v_2(n+1)|e^{-i\gamma_2(n+1)}}{\omega} \\
&\quad + \frac{\overline{\alpha'(n)}z^{-1}v_2^*(n)(R(n)e^{i\eta(n)}|v_2(n)|e^{i\gamma_2(n)} + R(n)e^{-i\eta(n)}|v_1(n)|e^{-i\gamma_1(n)})}{\omega}
\end{aligned}$$

this implies

$$\begin{aligned}
& \frac{R(n+1)e^{i\eta(n+1)}}{R(n)e^{i\eta(n)}} \\
&= [\omega + (\rho'(n)z^{-1/2}(|v_2(n)|e^{i\gamma_2(n)} + e^{-2i\eta(n)}|v_1(n)|e^{-i\gamma_1(n)})v_1^*(n+1)) \\
& \quad + \overline{\alpha'(n)}z^{-1}v_2^*(n)(|v_2(n)|e^{i\gamma_2(n)} + e^{-2i\eta(n)}|v_1(n)|e^{-i\gamma_1(n)})] \\
& \div [\omega - \rho'(n)z^{-1/2}(v_2^*(n)|v_1(n+1)|e^{i\gamma_1(n+1)}) \\
& \quad - \rho'(n)z^{-1/2}(v_2^*(n)e^{-2i\eta(n+1)}|v_2(n+1)|e^{-i\gamma_2(n+1)}) \\
& \quad + \alpha'(n)v_1^*(n+1)|v_1(n+1)|e^{i\gamma_1(n+1)} \\
& \quad + \alpha'(n)v_1^*(n+1)e^{-2i\eta(n+1)}|v_2(n+1)|e^{-i\gamma_2(n+1)}].
\end{aligned}$$

and now we consider for two solutions $u(1; n), u(2; n)$ corresponding to two different κ s, the Wronskian $W(u(1; \cdot), u(2; \cdot))(n)$. For R_1, η_1, Z_1 corresponding to $u(1; n)$ and R_2, η_2, Z_2 corresponding to $u(2; n)$. We can then write

$$\begin{aligned}
& W(u(1; \cdot), u(2; \cdot))(n) \\
&= (R_1(n)e^{i\eta_1(n)}|v_2(n)|e^{i\gamma_2(n)} + R_1(n)e^{-i\eta_1(n)}|v_1(n)|z^n e^{-i\gamma_1(n)}) \\
& \quad \times (R_2(n)e^{i\eta_2(n)}|v_1(n)|e^{i\gamma_1(n)} + R_2(n)e^{-i\eta_2(n)}|v_2(n)|z^n e^{-i\gamma_2(n)}) \\
& \quad - (R_2(n)e^{i\eta_2(n)}|v_2(n)|e^{i\gamma_2(n)} + R_2(n)e^{-i\eta_2(n)}|v_1(n)|z^n e^{-i\gamma_1(n)}) \\
& \quad \times (R_1(n)e^{i\eta_1(n)}|v_1(n)|e^{i\gamma_1(n)} + R_1(n)e^{-i\eta_1(n)}|v_2(n)|z^n e^{-i\gamma_2(n)}) \\
&= 2R_1(n)R_2(n)\text{Im}(|v_2(n)|^2 e^{i(\eta_1(n)-\eta_2(n))} + |v_1(n)|^2 e^{i(\eta_2(n)-\eta_1(n))}) \\
&= 2R_1(n)R_2(n)(|v_2(n)|^2 - |v_1(n)|^2) \sin(\eta_1(n) - \eta_2(n)) \\
&= 2R_1(n)R_2(n)W(v, v^*)(n) \sin(\eta_1(n) - \eta_2(n))
\end{aligned}$$

□

4.2.4 Decaying oscillatory perturbations of periodic CMV operators

We have, once again

$$\begin{aligned}\frac{R(n+1)}{R(n)} &= \left| \frac{Z(n+1)}{Z(n)} \right|, \\ e^{2i(\eta(n+1)-\eta(n))} &= \frac{Z(n+1)}{Z(n)} \div \frac{\overline{Z(n+1)}}{\overline{Z(n)}}.\end{aligned}$$

We then get (ignoring terms that are ℓ^2)

$$\begin{aligned}\frac{R(n+1)}{R(n)} &\sim \operatorname{Re} \left[1 + \frac{2}{\omega} (\rho'(n) z^{-1/2} (|v_2(n)| e^{i\gamma_2(n)} + \frac{2}{\omega} e^{-2i\eta(n)} |v_1(n)| e^{-i\gamma_1(n)}) v_1^*(n+1)) \right. \\ &\quad \left. + \frac{2}{\omega} \overline{\alpha'(n)} z^{-1} v_2^*(n) (|v_2(n)| e^{i\gamma_2(n)} + \frac{2}{\omega} e^{-2i\eta(n)} |v_1(n)| e^{-i\gamma_1(n)}) \right] \\ &\quad \div \operatorname{Re} \left[1 - \frac{2}{\omega} \rho'(n) z^{-1/2} (v_2^*(n) |v_1(n+1)| e^{i\gamma_1(n+1)} \right. \\ &\quad \left. - \frac{2}{\omega} \rho'(n) z^{-1/2} (v_2^*(n) e^{-2i\eta(n+1)} |v_2(n+1)| e^{-i\gamma_2(n+1)}) \right. \\ &\quad \left. + \frac{2}{\omega} \alpha'(n) v_1^*(n+1) |v_1(n+1)| e^{i\gamma_1(n+1)} \right. \\ &\quad \left. + \frac{2}{\omega} \alpha'(n) v_1^*(n+1) e^{-2i\eta(n+1)} |v_2(n+1)| e^{-i\gamma_2(n+1)} \right],\end{aligned}$$

and so, again ignoring ℓ^2 terms,

$$\begin{aligned}
& \ln \left(\frac{R(n+p)}{R(n)} \right) \\
& \sim \sum_{r=n}^{n+p-1} \operatorname{Re} \left[\frac{2}{\omega} (\rho'(r) z^{-1/2} (|v_2(r)| e^{i\gamma_2(r)} \right. \\
& \quad + \frac{2}{\omega} e^{-2i\eta(r)} |v_1(r)| e^{-i\gamma_1(r)}) |v_2(r+1)| e^{-i\gamma_2(r+1)} \\
& \quad + \frac{2}{\omega} \overline{\alpha'(r)} z^{-1} |v_1(r)| e^{-i\gamma_1(r)} (|v_2(r)| e^{i\gamma_2(r)} + \frac{2}{\omega} e^{-2i\eta(r)} |v_1(r)| e^{-i\gamma_1(r)})] \\
& \quad - \operatorname{Re} \left[-\frac{2}{\omega} \rho'(r) z^{-1/2} (|v_1(r)| e^{-i\gamma_1(r)} |v_1(r+1)| e^{i\gamma_1(r+1)}) \right. \\
& \quad - \frac{2}{\omega} \rho'(r) z^{-1/2} (|v_1(r)| e^{-i\gamma_1(r)} e^{-2i\eta(r+1)} |v_2(r+1)| e^{-i\gamma_2(r+1)}) \\
& \quad + \frac{2}{\omega} \alpha'(n) |v_2(r+1)| e^{-i\gamma_2(r+1)} |v_1(r+1)| e^{i\gamma_1(r+1)} \\
& \quad \left. + \frac{2}{\omega} \alpha'(n) |v_2(r+1)| e^{-i\gamma_2(r+1)} e^{-2i\eta(r+1)} |v_2(r+1)| e^{-i\gamma_2(r+1)} \right].
\end{aligned}$$

Furthermore, note that

$$\begin{aligned}
& \rho'(n) \\
& = -\rho(n) + \sqrt{1 - |\alpha(n)|^2 - |\alpha'(n)|^2 - 2\operatorname{Re}(\alpha(n)\alpha'(n))} \\
& = -\rho(n) \\
& \quad + \rho(n) \left(1 - \frac{1}{2} \frac{|\alpha'(n)|^2 + 2\operatorname{Re}(\alpha(n)\alpha'(n))}{(1 - |\alpha(n)|^2)} - \frac{1}{8} \frac{(|\alpha'(n)|^2 + 2\operatorname{Re}(\alpha(n)\alpha'(n)))^2}{(1 - |\alpha(n)|^2)^2} - \dots \right) \\
& = \rho(n) \left(-\frac{1}{2} \frac{|\alpha'(n)|^2 + 2\operatorname{Re}(\alpha(n)\alpha'(n))}{(1 - |\alpha(n)|^2)} - \frac{1}{8} \frac{(|\alpha'(n)|^2 + 2\operatorname{Re}(\alpha(n)\alpha'(n)))^2}{(1 - |\alpha(n)|^2)^2} - \dots \right).
\end{aligned}$$

As a consequence, we may once again write $\ln(R(n+p)/R(n))$ as a sum of terms that are either q -periodic and independent of κ , or a product of q -periodic and $GBV(A)$ terms. Then by repeating the techniques of the previous section, we can show

Theorem 4.31. *Consider q -periodic Verblunsky coefficients $\{\alpha(n)\}$, and perturbations $\alpha'(n)$ of these Jacobi coefficients (i.e. $\alpha(n), \alpha(n) + \alpha'(n)$ are both in \mathbb{D}). Let A be a finite set of phases and assume all the $\{\alpha(n)\}$ are in $\ell^p \cap GBV(A)$ for $p \geq 2$. Let us define A_p as n -sums of elements in A or $-A$ (possibly repeated) where n ranges from 1 to $p - 1$. Let $S = \{k|2lk \in A_p, l = \pm 1, \pm 2, \dots, \pm(p - 1)\}$, where k is the quasimomentum corresponding to Floquet solutions of the unperturbed Szegő recursion. Then where $\mu_{\mathbb{D}}$ is the probability measure corresponding to $\{\alpha(n)\}$ and $\mu'_{\mathbb{D}}$ is the probability measure corresponding to $\{\alpha(n) + \alpha'(n)\}$, $\text{supp}\tilde{\mu}_{\mathbb{D},s} \cap \text{supp}\mu_{\mathbb{D},ess}$ only has points with quasimomenta in A .*

This theorem is a stronger version of Theorem 1.13.

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