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FORCE AND HEAT GENERATION IN A
CONDUCTING SPHERE IN AN ALTERNATING
MAGNETIC FIELD

by

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ABSTRACT

Force and Heat Generation in a Conducting Sphere in an Alternating Magnetic Field

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The interaction of an electrically conducting sphere with a time varying magnetic field is useful in the study of "containerless" processing methods such as electromagnetic levitation melting. The fundamental quantities of interest in this interaction are the rate of heat generation in the sphere, the Lorentz force and magnetic pressure on it. These quantities depend upon the nature of the current sources that create the magnetic field, and the material properties of the sphere. In this work, the Maxwell equations for the interaction of a sphere with an arbitrary external alternating magnetic field are first formulated. Then, the density of the induced currents in the sphere is found as a function of the external current sources and the material properties of the sphere. The current density is now used to calculate the heat generated in the sphere. Next, a method to calculate the Lorentz force on an electrically conducting sphere placed in an arbitrary sinusoidally varying magnetic field is developed and a formula for the force on the sphere is given. This formula is used to derive the special case of a sphere in an axisymmetric system of circular current loops. Numerical results for the force on a sphere on the axis of a stack of loops are presented as a function of the stack geometry. The results for the heat generation and the Lorentz force obtained in this study are compared with the results obtained by a previously used model (known as the "homogeneous model") which assumes that the external magnetic field is uniform and unidirectional. It is shown that the homogeneous model is a special case of the present model and that it underestimates heat generation significantly, and overestimates the Lorentz force. In addition, as the size of the sphere decreases, the homogeneous model gives erroneous results, approaching an order of magnitude for heat generation in a very
small sphere. Subsequently, a procedure to determine the magnetic pressure distribution on the surface of a levitated liquid metal droplet is developed. The pressure distribution is calculated in terms of the geometry of the coil that creates the field. Finally, the magnetic fields of helical windings that are commonly used in the laboratory for levitation melting are calculated.
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To Amma and Appa

My prime movers
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**Nomenclature**

**Roman symbols:**

- \( a \) = radius of current loop or radius of the helix, m
- \( A \) = magnetic vector potential, Wb m\(^{-1}\)
- \( B \) = magnetic flux density, Wb m\(^{-2}\)
- \( c \) = velocity of light, m s\(^{-1}\), a parameter of the helical winding, Eq. (6.2)
- \( c_n \) = fraction that denotes height of the \( n \)th loop in the stack of loops
- \( C_{n,k,l,m} \) = vector that represents the influence of the external field on the field inside the sphere, Wb m\(^{1.5}\)
- \( d \) = pitch of helical winding, m
- \( ds(r') \) = tangent vector to a curve at point \( r' \) on it, m
- \( D \) = electric flux density, C m\(^{-2}\)
- \( e \) = base of natural logarithms
- \( E(\kappa) \) = complete Elliptic integral of the second kind
- \( E \) = Electric field intensity, V m\(^{-1}\)
- \( E_n(z) \) = Exponential integral function
- \( F \) = Lorentz force, N
- \( F_{n,l,m} \) = line integral for line source, m\(^{-l}\)
- \( g_l(q_n) \) = skin depth function for Lorentz force (this work)
- \( G(q) \) = skin depth function for Lorentz force (Rony, 1964)
- \( h \) = height of coil, m
- \( H_l(q_n) \) = skin depth function for power absorbed
- \( i \) = imaginary number, \( \sqrt{-1} \)
- \( I \) = current, A
- \( I_{l+1/2}(z) \) = half integer order modified Bessel function of the first kind
- \( \text{Im} \) = Imaginary part of a complex variable or function
- \( I_t(R_s,q_n) \) = skin depth function for magnetic pressure
\( I_{n,l,m} \) = source function of the \( n \)th current source, A

\( J_{l+1/2}(z) \) = half integer order Bessel function of the first kind

\( J \) = current density, A m\(^{-2}\)

\( K(\kappa) \) = complete Elliptic integral of the first kind

\( L \) = angular momentum operator

\( N \) = total number of current loops

\( p \) = pressure, N m\(^{-2}\)

\( P \) = power, W

\( P_l^m(u) \) = Associated Legendre Polynomial

\( q \) = ratio of radius to skin depth, \( \left( \frac{R}{\delta} \right) \)

\( r \) = radial coordinate in spherical coordinates, m

\( r \) = position vector of a point in space

\( R \) = specimen radius, m

\( \text{Re} \) = Real part of a complex variable or function

\( t \) = time, s

\( u \) = \( \cos \theta \)

\( \mathbf{u} \) = unit vector

\( U_{n,l,m} \) = real part of \( I_{n,l,m} \), A

\( V_{n,l,m} \) = real part of \( I_{n,l,m} \), A

\( x_{l+1/2,k} \) = \( k \)th real root of \( J_{l-1/2}(x) = 0 \)

\( Y_l^m(u, \phi) \) = spherical harmonic, see Eq. (2.10)

\( Y_{l,m}^e(u, \phi) \) = \( P_l^m(u) \cos \phi \), real part of spherical harmonic

\( Y_{l,m}^o(u, \phi) \) = \( P_l^m(u) \sin \phi \), imaginary part of spherical harmonic

\( Z \) = complex number, cartesian coordinate or axial distance in cylindrical coordinates, m

\( Z_{l',m',m} \) = surface integrals of products of three spherical harmonics
Greek symbols:

\( \alpha \) = angle of cone, rad

\( \gamma \) = ratio of stack height to least radius \((h/a_o)\)

\( \delta \) = skin depth, m, see Eq. (2.22d)

\( \delta_{l,m} \) = Kronecker delta function

\( \varepsilon_o \) = electrical permittivity of free space, \(8.854 \times 10^{-12} \text{ F m}^{-1}\)

\( \phi \) = polar angle in spherical or cylindrical coordinates, rad

\( \kappa \) = argument of Elliptic integrals

\( \mu_o \) = magnetic permeability of free space, \(4\pi \times 10^{-7} \text{ H m}^{-1}\)

\( \rho \) = radial coordinate in cylindrical coordinates, m

\( \sigma \) = electrical conductivity, \((\Omega \text{ m})^{-1}\)

\( \tau \) = surface tension, J

\( \theta \) = azimuthal angle in spherical coordinates, rad

\( \omega \) = angular frequency of magnetic field, rad s\(^{-1}\)

\( \omega^* \) = natural frequency of droplet oscillation, rad s\(^{-1}\)

\( \psi \) = phase difference, rad

\( \Psi_i(z,s) \) = skin depth function (see Appendix F or Eq.(4.12c))

Subscripts:

\( c \) = capping coil or reverse wound turn

\( e \) = external field

\( \text{hom} \) = homogeneous field model

\( j \) = summation index

\( k \) = summation index

\( l \) = summation index

\( m \) = summation index

\( n \) = nth current source, normal component of a vector at a surface
\( o \) = scale for the appropriate quantity

\( s \) = sphere

\( t \) = tangential component of a vector at a surface

\( 1, 2 \) = medium 1, medium 2 respectively

**Superscripts:**

\( m \) = summation index

\( * \) = complex conjugate

\( \overline{\cdot} \) = overbar, denotes scaled quantity

\( ' \) = prime, denotes source quantities as seen in the coordinate system attached to the sphere

\( '' \) = double prime, denotes quantities as seen from a coordinate system attached to the stack or the source

**Other:**

\( \nabla \) = gradient operator

\( \| \| \) = absolute value
1.1 Motivation:

At high temperatures, most materials react with the walls of their containers. This inevitably leads to material contamination and property degradation. Therefore, it becomes difficult to process materials to the required degree of purity and/or measure their properties at high temperatures (Garnier, 1987). The problems addressed by my thesis were created in part, by an ongoing project on the processing of materials at very high temperatures, and a need to know their thermophysical properties, at the Department of Mechanical Engineering and Materials Science at Rice University.

Electromagnetic levitation melting, first patented in 1923 (Muck, 1923) and demonstrated experimentally in 1952 (Okress et al., 1952), offers a solution to the problem of materials processing at high temperatures. A typical electromagnetic levitation device is shown in Fig. 1.1. It consists of a few turns of hollow copper tubing wound over the length of a few centimeters (the levitation coil) and topped by a few reverse wound turns (the capping coil). A large high frequency current (~ 100 to 300 A, 400 kHz) is allowed to flow through the coil and set up an alternating magnetic field. By virtue of its geometry, the coil creates a magnetic field whose region of minimum field strength (known as the 'potential well') lies in the gap between the levitation and the capping coils. When an electrically conducting sample is placed in this gap, eddy currents are induced in the sample and heat it. Also, they interact with the external field and exert a Lorentz force on the sample in a direction opposite to that of gravity. The sample therefore levitates and melts within a hundred seconds or so (Sneyd and Moffatt, 1982). There are several advantages of suspending a molten sample of liquid metal in this fashion. Firstly, it does away with a material crucible and provides a solution to the problem of contamination due to containers. Secondly, it is a valuable diagnostic tool. For example, study of the heat
transfer from the sample reveals information about the thermal diffusivity of the material and its optical properties such as surface emissivity (Bayazitoglu et al., 1990). When a levitated sample melts, it begins to execute shape oscillations. By analyzing the dynamics of the suspended liquid metal droplet, we may obtain the surface tension and viscosity of the liquid metal (Soda et al., 1977, Bayazitoglu and Suryanarayana, 1991, Suryanarayana and Bayazitoglu, 1991). Besides, electromagnetic levitation melting has applications in casting, in the study of gas-metal reactions, etc. (Advanced Materials and Processes, 1991).

Another related problem concerns a method for the production of very fine metal powders where a spray of liquid metal droplets is allowed to fall through the high frequency alternating magnetic field that is created by a long solenoid. Eddy currents are induced in the droplets and they heat them (Bayazitoglu and Cerny, 1992). If the induction heating is sufficient, the droplets reach boiling point and vaporize, and thus decrease in size (Stickel and Bayazitoglu, 1994). When the droplets reach a targeted size, they are cooled and collected. The metal vapor that is generated as a by-product is also collected and nucleated to form fine metal powders.
Both these processes share a common feature, viz., the interaction of an electrically conducting sphere with an alternating magnetic field. The heat generated in the sphere and the Lorentz forces acting on it are the basic quantities that must be known to understand these processes. For example, the heat transfer from the levitated droplet and the rate of vaporization of the falling droplet in the powder production process depend upon the heat generation in the sphere. The motion of the falling droplet is governed by the combined action of gravitational, magnetic, buoyant, and viscous forces. Furthermore, in the case of the levitated liquid metal droplet, its position and shape oscillations are determined by the net Lorentz force on it and the distribution of the magnetic pressure on its surface, respectively. In order to calculate any or all of these quantities, we must analyze the case of an electrically conducting sphere that is placed in an alternating magnetic field.

1.2 Thesis synopsis:

Following this introduction, there are five self-contained chapters. Each of them contains references appropriate to its subject matter. Hence, I have not devoted a separate section for literature survey here.

It is clear that we must know the heat generation in the levitating sample and the falling droplet, the forces exerted on them by the external magnetic fields, and in the case of the levitated droplet, the distribution of the magnetic pressure on its surface. Consequently, I have attempted to answer the following questions in this work: i) What is the nature of the magnetic fields that are created by typical levitation coils, and how do they depend on the coil geometry? ii) What is the heat generated on a spherical sample that is placed in an arbitrary magnetic field? iii) What is the Lorentz force on the sample? iv) What is the distribution of the magnetic pressure on the surface of the molten sample?

All of the works that address the problem of an electrically conducting sphere in an alternating magnetic field can be classified as belonging to either the 'homogeneous' or the 'nonhomogeneous' model. The homogeneous model assumes that the conducting sphere is
placed in a uniform and unidirectional sinusoidally alternating magnetic field and shows that i) the heat generated in the sphere varies as the square of the external field, and ii) that the Lorentz force on it is proportional to the product of the field and its gradient (Rony, 1964). The assumption of the uniform unidirectional external field gives rise to several problems. For example, the magnetic field can hardly be regarded as being uniform or unidirectional over the diameter of the sphere. In fact, the magnetic fields produced by the laboratory levitation coils are not quite axisymmetric (Bayazitoglu and Sathuvalli, 1993). The nonhomogeneous model, on the other hand, assumes that the sphere is placed in an arbitrarily varying magnetic field. However, most of the works that account for the nonuniformity of the field (Zong et al., 1992) are highly geometry specific and are applicable only to axisymmetric systems. A recent analysis of a sphere placed in an arbitrary magnetic field shows that the induced current density and the power absorbed by it can be expressed in terms of certain "source functions" (Lohofer, 1989). These "source functions" are unique functions that depend upon the geometry of the current sources that create the magnetic field and their relative positions with respect to the sphere. In this light, in Chapter 2, the problem of the interaction of a spherical sample with an arbitrary external magnetic field is formulated by using the Maxwell equations and solved in terms of these "source functions."

In Chapter 3, a method to calculate these source functions (for distributed as well as concentrated line sources) is developed and used to calculate them (Bayazitoglu and Sathuvalli, 1994). It is shown that the source functions strongly depend on the geometry of the coil that produces the applied magnetic field. Subsequently, the role of these source functions (and the nonhomogeneity of the external magnetic field) on the heat generated in the sphere is identified.

Chapter 4 addresses the calculation of the Lorentz force that acts on a sphere that is placed in an arbitrary magnetic field. Here, the density of eddy currents that are generated in a diamagnetic sphere placed in a sinusoidally alternating magnetic field is first written.
Then by using multipole expansion, the vector potential of the external magnetic field is expressed in terms of the above mentioned "source functions." The external magnetic field is then calculated by using a gradient formula. An expression for the Lorentz force is finally obtained in terms of these source functions and a "skin depth" function.

Chapter 5 considers the problem of determining the magnetic pressure distribution on the surface of a levitating liquid metal droplet. This quantity is important for the reasons mentioned in section 1.1.

Chapter 6 tackles the problem of calculating the magnetic fields generated by typical levitation coils that are used in the laboratory for experiments. A model known as the "helix model" is developed and it is used to calculate the magnetic fields of levitation coils. These results are also useful in the study of the coils that are used in the powder production process.

The thesis ends with a few appendices. The appendices contain mathematical material that is important to the main text, such as proofs of key equations, limiting and scaling analyses, and such.

Finally, a note about the bibliography. Most of the literature concerning levitation studies has addressed one or more of the following aspects: (i) calculation of levitation forces on a conducting body that has been placed in an alternating magnetic field, and of the power absorbed by it, (ii) stability and flow phenomena in the levitating sample, and iii) experimental studies. Most of the references provided here mostly belong to (i) or (ii) and only a few key references in (iii) are given.
2.1 Introduction:

As mentioned in section 1.2, one of the earliest analyses of a sphere in a magnetic field with particular reference to levitation melting is by Rony (1964). Rony calculates the force on a conducting sphere and the power absorbed by it, when it is placed in a uniform \( z \)-directed alternating magnetic field. Fromm and Jehn (1965) calculate the power absorbed by conducting specimens and the forces on them, when they are placed in the field produced by circular current carrying loops and compare the results with experimental data. However, both of these works assume that the levitating body is a sphere placed in a uniform unidirectional alternating magnetic field. Also, they assume that the external field is created by a stack of coaxial circular current carrying loops. In this thesis, the model that is used by these authors is referred to as the "homogeneous field model."

There are relatively few works that attempt to calculate the effect of the nonhomogeneity of the external magnetic field on the heat generation in the specimen. El-Kaddah and Szekely (1983, 1984) and Zong et al. (1992) numerically calculate the heating in a conducting sphere and axisymmetric bodies placed in axisymmetric magnetic fields in microgravity and earth-bound situations. However, these methods cannot work when axisymmetry is absent, for example, when the sphere is displaced from the axis of symmetry of a coil. A recent analysis of a sphere placed in an arbitrary magnetic field shows that the induced current density and the power absorbed by it can be expressed in terms of certain "source functions" (Lohofer, 1989). These "source functions" are unique functions that depend upon the geometry of the current sources that create the magnetic field and their relative positions with respect to the sphere.

This chapter formulates the governing equations for a sphere that is placed in an arbitrary time varying magnetic field and presents the solution in terms of the "source
functions" of the external magnetic field. The next chapter shows how to calculate these source functions (for both distributed and concentrated line sources) and then use them to find the heat that is generated in the sphere.

2.2 Formulation:

![Diagram of a sphere placed in a magnetic field](image)

Figure 2.1 A sphere placed in a magnetic field.

Consider a diamagnetic sphere of radius $R_s$ that is placed in an external magnetic field $\mathbf{B}_e(r,t)$ as shown in Fig. 2.1. The interaction of this sphere with a time varying magnetic field is described by the Maxwell equations. Faraday's law dictates that a changing magnetic field should induce an electric field $\mathbf{E}_s(r,t)$ inside the sphere. The induced electric field sets up induced eddy currents, which in turn are responsible for an
induced magnetic field. Let $B_s(r,t)$ denote the magnetic field due to the induced eddy currents inside the sphere. Then Faraday's law for this situation can be expressed as

$$\nabla \times E_s(r,t) = -\frac{\partial}{\partial t} \left[ B_s(r,t) + B_e(r,t) \right], \quad |r| \leq R_s. \quad (2.1)$$

The induced eddy currents in the sphere are related to the net magnetic field inside the sphere by Ampere's law which is

$$\nabla \times \left[ B_s(r,t) + B_e(r,t) \right] = \mu_0 J_s(r,t), \quad |r| \leq R_s, \quad (2.2)$$

where $J_s(r,t)$ is the density of the induced eddy currents and $\mu_0$ is the magnetic permeability of free space*. The induced eddy current density and the induced electric field are related by Ohm's law, i.e.,

$$J_s(r,t) = \sigma_s E_s(r,t), \quad |r| \leq R_s, \quad (2.3)$$

where $\sigma_s$ is the electrical conductivity of the sphere. It is mathematically convenient to introduce the magnetic vector potential $A(r,t)$ such that

$$\nabla \times A_{s or e}(r,t) = B_{s or e}(r,t), \quad (2.4a)$$

$$\nabla \cdot A_{s or e}(r,t) = 0. \quad (2.4b)$$

Now assume that the external magnetic field, $B_e(r,t)$ is created by a set of $N$ discrete conductors that carry sinusoidally varying currents, so that

$$B_e(r,t) = \sum_{n=1}^{N} B_n(r) \cos \omega_n t. \quad (2.5a)$$

Also, by virtue of Ampere's law

$$\nabla \times B_e(r,t) = 0, \quad (2.5b)$$

at all points, except on the conductor boundaries. Further, due to Eq. (2.4a), we may handle the external magnetic field via its vector potential. So,

$$A_e(r,t) = \sum_{n=1}^{N} A_n(r) \cos \omega_n t. \quad (2.6)$$

Substitution of Eq. (2.4a) in Eq. (2.1) and subsequent use of Eq. (2.3) yields

---

* In both powder production and levitation melting, the materials used are high conductivity metals. Most metals (with the exception of iron, nickel, and ferrous alloys) belong to a class of materials known as diamagnetic materials. The permeability of diamagnetic materials is approximately equal to the permeability of free space (Smythe, 1989). Further, one of the defining properties of a diamagnetic material is that when it is placed in a nonuniform magnetic field, it tends to move to a region of weaker field strength.
Finally, by using Eqs. (2.5b) and (2.7) successively in Eq. (2.2), then invoking the definition of the magnetic vector potential (Eqs. (2.4)), and the vector identity,
\[ \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \]
the following equation is obtained
\[ \nabla^2 \mathbf{A}_s(r,t) - \mu_0 \sigma_s \frac{\partial}{\partial t} \mathbf{A}_s(r,t) = \mu_0 \sigma_s \frac{\partial}{\partial t} \mathbf{A}_e(r,t), \quad |r| \leq R_s. \quad (2.8a) \]

The equation for the vector potential in the region outside the sphere is obtained by setting \( \sigma_s = 0 \),
\[ \nabla^2 \mathbf{A}_s(r,t) = 0, \quad |r| > R_s. \quad (2.8b) \]

Equations (2.8) are a variant of the classical heat conduction equation.

A note about the assumptions made so far. First of all, we assume that the electrically conducting body (in the case of levitation as well as the powder production processes) can be modeled as a sphere. Further, we assume that the magnetic field inside the sphere has two components- one due to the external field (denoted by the subscript \( e \)) and the other due to the induced currents in the sphere (denoted by the subscript \( s \)). Since Maxwell, equations are linear, the superposition suggested by Eqs. (2.1), (2.2) and (2.6) is valid. Also, the displacement current, which is equal to \( \frac{\partial (\varepsilon_0 \mathbf{E}_s)}{\partial t} \), has been neglected in Eq. (2.2). This is valid since the frequency of the external field is of the order of \( 10^5 \) Hz. At these frequencies, it can be shown that the error caused by the neglect of the displacement current in the calculation of the induced magnetic field is of the order of \( R_s/c \), where \( c \) is the velocity of light. Therefore, the term due to the displacement current has been neglected (Haus and Melcher, 1989). This kind of analysis is known as magnetoquasistatic analysis.

* Here \( \varepsilon_0 \) is the permittivity of free space. Most diamagnetic materials have a permittivity which is equal to that of free space.
2.3 Solution:

Equations (2.8) must be solved subject to the boundary conditions at the surface of the sphere and the external field $B_s(r,t)$. In this case, it can be shown that the vector potential $A(r,t)$ and the gradients of its tangential and normal components must be continuous across the boundary of the sphere (see Appendix A for details). It must be mentioned here that one of the aims of this project is to obtain the actual heat generation in a sphere (the subject of Chapter 3). Therefore, the steps leading to the solution of Eq. (2.8) as given by Lohofer (1989) are presented in some detail in order to ensure readability.

In the coordinate system of Fig. 2.1, the vector potential $A_s(r,t)$ can be expanded in a series of spherical harmonics, as given by

$$A_s(r,u,\phi,t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} r^{-l/2} R_{l,m}(r,t) Y_l^m(u,\phi)$$

(2.9)

where $Y_l^m(u,\phi)$ are spherical harmonics given by

$$Y_l^m(u,\phi) = (-1)^m \frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!} P_l^m(u) e^{im\phi}.$$  

(2.10)

In the above equations, $u = \cos \theta$ and $P_l^m(u)$ are Associated Legendre polynomials of the first kind, and $R_{l,m}(r,t)$ are vectorial coefficients of the spherical harmonics. The Laplacian in spherical coordinates is given by (Arfken, 1985),

$$\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{r^2} \mathbf{L}^2,$$

(2.11a)

where $\mathbf{L}$ is the angular momentum operator defined by

$$\mathbf{L}^2 = -\left\{ \frac{\partial}{\partial u} \left( 1 - u^2 \right) \frac{\partial}{\partial u} + \frac{1}{1 - u^2} \frac{\partial^2}{\partial \phi^2} \right\}.$$  

(2.11b)

Substitute Eqs. (2.11) in Eq. (2.8a), multiply each term by $Y_l^{m*}(u,\phi)$, the complex conjugate of $Y_l^m(u,\phi)$, and integrate each term over the surface of the sphere to get,

$$\left\{ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \mu_o \sigma_s \frac{\partial}{\partial t} \right\} \int_{-1}^{1} \int_{0}^{2\pi} A_s(r,u,\phi,t) Y_l^{m*}(u,\phi) d\phi du$$

$$- \frac{1}{r^2} \int_{-1}^{1} \int_{0}^{2\pi} \mathbf{L}^2 A_s(r,u,\phi,t) Y_l^{m*}(u,\phi) d\phi du$$

$$= \mu_o \sigma_s \frac{\partial}{\partial t} \int_{-1}^{1} \int_{0}^{2\pi} A_e(r,u,\phi,t) Y_l^{m*}(u,\phi) d\phi du.$$  

(2.12)
Now substitute the spherical harmonic expansion for $A_s$ (Eq. (2.9)) in Eq. (2.12), and integrate the left hand side of Eq. (2.12) to get

$$
\left\{ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \mu_0 \sigma_s \frac{\partial}{\partial t} \right\} \left[ r^{-1/2} \mathbf{R}_{l,m}(r,t) \right] = l(l+1) r^{-5/2} \mathbf{R}_{l,m}(r,t) \\
= \mu_0 \sigma_s \frac{\partial}{\partial t} \int_{-1}^{+1} 2\pi \mathbf{A}_s(r,u,\phi,t) Y_l^m(u,\phi) d\phi du .
$$

(2.13)

In arriving at Eq. (2.13), Eqs. (E1) and (E2) of Appendix E have been used. Equation (2.13) finally reduces to

$$
\left\{ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{(l+1/2)^2}{2} - \mu_0 \sigma_s \frac{\partial}{\partial t} \right\} \mathbf{R}_{l,m}(r,t) \\
= \mu_0 \sigma_s \frac{\partial}{\partial t} \int_{-1}^{+1} 2\pi \mathbf{A}_s(r,u,\phi,t) Y_l^m(u,\phi) d\phi du, \quad r \leq R_s .
$$

(2.14)

Equation (2.14) is similar to the 1-D nonhomogeneous heat conduction equation in cylindrical coordinates in the $r$-direction.

For the region outside the sphere, the governing equation, Eq. (2.8b) can be solved by setting $\sigma_s = 0$ in Eq. (2.14), to yield

$$
\left\{ \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \right\} \left[ r^{-1/2} \mathbf{R}_{l,m}(r,t) \right] \\
- l(l+1) r^{-5/2} \mathbf{R}_{l,m}(r,t) = 0, \quad r > R_s .
$$

(2.15)

Equation (2.15) can be solved by assuming that

$$
\mathbf{R}_{l,m}(r,t) = \mathbf{a}_{l,m}(t) f(r) .
$$

(2.16)

The presence of the time dependent vector function $\mathbf{a}_{l,m}(t)$ in the above equation is justified because the field outside the sphere varies in time. Recall that the description of the external field according to Eq. (2.6) assumes sinusoidal variation in time. Straight forward integration of the second order differential equation that results from the substitution of Eq. (2.16) in Eq. (2.15) and the imposition of the condition that $f(r)$ is bounded* as $r \to \infty$ results in

$$
\mathbf{A}_s(r,u,\phi,t) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \mathbf{a}_{l,m}(t) r^{-l(l+1)} Y_l^{m}(u,\phi), \quad r > R_s .
$$

(2.17)

The boundary conditions on the vector potential lead to (see Appendix A)

* The influence of the eddy currents should not be felt at points far away from the sphere.
\[ L_t \left[ r \frac{\partial R_{l,m}(r,t)}{\partial r} + \left( l + \frac{1}{2} \right) R_{l,m}(r,t) \right] = 0. \quad (2.18) \]

Since the vector potential within the sphere is continuous, the solution \( R_{l,m}(r,t) \) (in Eq. (2.14)), for the region \( 0 < r \leq R_s \) can be expanded in a series of fractional order Bessel functions given by (Lohofer, 1989)

\[ R_{l,m}(r,t) = \sum_{k=1}^{\infty} T_{k,l,m}(t) J_{l+1/2,k} \left( x_{l+1/2,k} \frac{r}{R_s} \right), \quad r \leq R_s, \quad (2.19a) \]

where \( J_{l+1/2}(x) \) are Bessel functions, and \( x_{l+1/2,k} \) is the \( k \)th real root of \( J_{l+1/2}(x) = 0 \)

\[ (2.19b) \]

Please see Appendix A for proof of Eq. (2.19b). Substitute Eq. (2.19a) in Eq. (2.14) and multiply each term by \( r J_{l+1/2} \left( x_{l+1/2,k} \frac{r}{R_s} \right) \) and integrate over \( r \) (from \( r = 0 \) to \( r = R_s \)), to yield,

\[ -\frac{x_{l+1/2,k}^2}{R_s^2} T_{k,l,m}(t) - \mu_o \sigma_s \frac{dT_{k,l,m}}{dt} = \]

\[ \frac{2\mu_o \sigma_s}{R_s^2 J_{l+1/2}^2 \left( x_{l+1/2,k} \right)} \frac{d}{dt} \int_0^{R_s} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^{3/2} J_{l+1/2} \left( x_{l+1/2,k} \frac{r}{R_s} \right) Y_{l,m}^* (u, \phi) A_n (r, u, \phi, t) d\phi dw. \]

\[ (2.20) \]

In obtaining Eq. (2.20), the orthogonality of Bessel functions as given by Eq. (F3b) has been used. By using the description of the external magnetic field given in Eq. (2.6), the solution of Eq. (2.20) becomes

\[ T_{k,l,m}(t) = T_{k,l,m}^0 e^{-\frac{x_{l+1/2,k}^2}{R_s^2 J_{l+1/2}^2 \left( x_{l+1/2,k} \right)}} + \sum_{n=1}^{\infty} C_{n,k,l,m} \frac{2q_n^2}{\sqrt{4q_n^4 + x_{l+1/2,k}^4}} \cos(\omega_n t + \psi_{n,k,l}) \quad (2.21) \]

where,

\[ C_{n,k,l,m} = \frac{2}{R_s^2 J_{l+1/2}^2 \left( x_{l+1/2,k} \right)} \int_0^{R_s} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^{3/2} J_{l+1/2} \left( x_{l+1/2,k} \frac{r}{R_s} \right) Y_{l,m}^* (u, \phi) A_n (r, u, \phi) d\phi dw, \]

\[ \psi_{n,k,l} = \cos^{-1} \left( -\frac{2q_n^2}{\sqrt{4q_n^4 + x_{l+1/2,k}^4}} \right), \quad (2.22a) \]
T_{k,l,m}^0 is a vector that depends upon the initial condition. Note that $\delta_n$ is the skin depth and that $q_n$ is the ratio of the sphere radius to the skin depth. Further, note that the vector $C_{n,k,l,m}$ is a volume integral whose integrand is a function of the external magnetic field. This vector represents the influence of the external magnetic field on the induced field inside the sphere.

Typically, the term that represents the influence of the initial condition in Eq. (2.21) can be neglected. For diamagnetic spheres of the order of radius 1 cm, $\mu_o = 4\pi \times 10^{-7} \text{H m}^{-1}$, $\sigma_s \sim 10^7 (\Omega \text{ m})^{-1}$, and $x_{l+1/2,k} \geq \pi$ (Abramowitz and Stegun, 1965). Thus,

$$\frac{x_{l+1/2,k}^2}{R_s^2 \mu_o \sigma_s} > 7853 \text{s}^{-1}.$$  

Hence for periods of time greater than $10^{-4}$ s, this term may be neglected*.

Back substituting Eqs. (2.21) - (2.22) in Eq. (2.19a) and finally in Eq. (2.9), the induced vector potential inside the sphere is obtained as follows:

$$A_z(r,u,\phi,t) = \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{k=1}^{\infty} C_{n,k,l,m} \frac{2q_n^2}{\sqrt{4q_n^4 + x_{l+1/2,k}^4}} r^{-1/2}$$

$$
\cdot J_{l+1/2} \left( \frac{r}{R_s} \right) Y_l^m(u,\phi) \cos(\omega nt + \psi_{n,k,l}), \quad r \leq R_s. 
$$

(2.23)

The solution for the region outside the sphere can be found now. The continuity of the vector potential at the surface of the sphere yields (Eqs. (A12) and (A16) of Appendix A),

$$a_{l,m}(t) = R_{l,m}(R_s,t)R_s^{l+1/2}.$$  

The above equation with Eqs. (2.19a) and (2.17) yields

---

* This is very similar to neglecting the transient term in the solution of the heat conduction equation when the Fourier number is very large. In fact, $(\mu_o \sigma_s)^{-1}$ has the dimensions of a diffusivity coefficient. The exponent of the first term in Eq. (2.21) is thus an electromagnetic analogue of the Fourier number.
\[ A_z(r, u, \phi, t) = \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{k=1}^{\infty} C_{n,k,l,m} \frac{2q_n^2}{\sqrt{4q_n^4 + x_{l+1/2,k}^4}} r^{-1/2} J_{l+1/2}(x_{l+1/2,k}) \]

\[ \cdot \left( \frac{R}{r} \right)^{l+1/2} Y_l^m (u, \phi) \cos(\omega_n t + \varphi_{n,k,l}), \quad r > R_s. \] 

Equation (2.24) predicts that the effect of the induced currents at points far away from the center of the sphere is small, as expected.

The solution for the vector potential due to the induced currents must be used to derive an expression for the eddy current density in the sphere. The calculation of the eddy current density directly leads to the calculation of all the other secondary quantities, viz., the rate of heat generation in the sphere, the Lorentz force on it, and the distribution of magnetic pressure on its surface.
Chapter 3

Heat Generation in the Sphere

3.1 Introduction:

The purpose of this chapter is to calculate these source functions (for both distributed and concentrated line sources) and show that they are strong functions of the geometry of the coil that produces the applied magnetic field. The "source functions" may, in principle be obtained purely from magnetostatic analysis, that is to say that they may calculated by ignoring the contribution of the magnetic field due to induction in the sphere. In other words, these source functions are unique functions of the geometry of the current carrying coils. The source functions are then used to determine the heat generated in the sphere. The role of the nonhomogeneity of the magnetic field on the heat generation is studied. It is shown that the homogeneous model (defined in section 1.2) is a special case of the present model and that it underestimates the heat generation significantly. In addition, as the size of the sphere decreases, the homogeneous model gives erroneous results, approaching an order of magnitude for a very small sphere. A simple equation to correct the power generation for this limiting case is given.

The rate of heat generation $P_s$, in the sphere is given by

$$P_s = \frac{Lt}{T} \int \int \frac{1}{t=0 \text{vol.}} \|J_s(r, t)\|^2 dv dt,$$

$$= \frac{Lt}{T} \int \int \frac{1}{t=0 \text{vol.}} J_s \cdot J_s^* dv dt.$$

Note that $P_s$ denotes the time averaged power integrated over the entire volume of the sphere. In the case of a sinusoidally varying field (as given by Eq. (2.5a), $T$ is time period of variation of the field ($\sim \omega_n / 2 \pi$).
3.2 The vectors $C_{n,k,l,m}$ and $I_{n,l,m}$:

As mentioned in the section 2.1, the vector $C_{n,k,l,m}$ is a volume integral that represents the effect of the external magnetic field on the induced field in the sphere. If the external field is assumed to be created by a set of sinusoidally alternating currents (see Eq. (2.6)), then the vector potential due to these currents may be represented by using the Biot-Savart law as (Smythe, 1989)

$$A_n(r) = \frac{\mu_0}{4\pi} \int_{|r-r'|} J_n(r') dv$$

$$= \frac{\mu_0}{4\pi} \int \int_{S} J_n(r') \left[ \frac{r - r'}{r - r'} \right] r^2 d\phi' du' dr',$$  

where $J_n(r')$ is the current density of the $n$th conductor, and the (volume) integral is carried out over all space outside the sphere. The term $|r - r'|$ in the integrand of Eq. (3.2) may be expanded in spherical harmonics as (Arfken, 1985)

$$\frac{1}{|r - r'|} = \sum_{l=0}^\infty \sum_{m=-l}^l \frac{4\pi}{2l+1} Y_l^m(u,\phi) Y_l^m(u',\phi') \begin{cases} r'^l/r'^{l+1}, & \text{for } r > r' \\ r'^l/r^{l+1}, & \text{for } r' > r. \end{cases}$$

Substitute Eqs. (3.2) and (3.3) in Eq. (2.22a), exchange the orders of integration, and integrate term by term over the surface of the sphere to get (Lohofer, 1989)

$$C_{n,k,l,m} = \frac{2\mu_0 R_s^{l+1/2}}{x_{l+1/2,k}^2} \int_{R_s} \int_{-1}^1 \int_0^{2\pi} J_n(r, u, \phi) r^{-l+1} Y_l^m(u, \phi) d\phi dudr.$$  

(3.4a)

The "source function" $I_{n,l,m}$ is now defined as

$$I_{n,l,m} = R_s^l \int_{R_s} \int_{-1}^1 \int_0^{2\pi} J_n(r, u, \phi) r^{-l+1} Y_l^m(u, \phi) d\phi dudr.$$  

(3.4b)

This leads to the relation,

$$I_{n,l,m} = \frac{x_{l+1/2,k}^2 J_{l+1/2}(x_{l+1/2,k})}{2\mu_0 R_s^{1/2}} C_{n,k,l,m}.$$  

(3.4c)

Note that the vector $I_{n,l,m}$ is independent of the index $k$. This property along with the fact that $I_{n,l,m}$ is purely a function of the external field is useful in the calculation of the power absorbed by the sphere and the force exerted by the magnetic field on it. Equation (3.4c) is significant in that it relates an integral over all space to another over a finite volume.
3.3 The induced current density and power generated in the sphere:

By using Eqs. (2.7) and (2.8a), it can be shown that

$$\nabla^2 \mathbf{A}_s (r, u, \phi, t) = -\mu_0 \mathbf{J}_s (r, u, \phi, t).$$  \hspace{1cm} (3.5)

The eddy current density inside the sphere can then be found by calculating the Laplacian of \( \mathbf{A}_s (r, u, \phi, t) \). The use of Eqs. (2.11), (3.4c), (2.23), and (3.5), yields

$$\mathbf{J}_s (r, u, \phi, t) = \frac{2}{R_s^{3/2}} \sum_{n=1}^{\infty} \sum_{l=0}^{n} \sum_{m=-l}^{l} \sum_{k=1}^{\infty} \frac{I_{n,l,m}}{J_{l+1/2} (x_{l+1/2,k})} \sqrt{4q_n^4 + x_{l+1/2,k}^4}$$

$$r^{-1/2} J_{l+1/2} \left( x_{l+1/2,k} \right) \frac{r}{R_s} Y_l^m (u, \phi) \cos (\omega_n t + \psi_{n,k,l}).$$  \hspace{1cm} (3.6)

The use of Eqs. (3.1) and (3.6) to evaluate the rate of heat generation yields

$$P_s = \frac{1}{\sigma_s R_s} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=1}^{N} \sum_{n'=1}^{N} 4q_n^2 q_{n'}^2 I_{n,l,m} I^*_ {n',l,m} \frac{1}{4q_n^4 + x_{l+1/2,k}^4} \frac{1}{4q_{n'}^4 + x_{l+1/2,k}^4}.$$  \hspace{1cm} (3.7)

The orthogonality of spherical harmonics, Eq. (E1) and Eq. (D6a) to evaluate the \( \text{r-integral} \) have been used to arrive at Eq. (3.7). Equation (3.7) can be recast as (Lohofer, 1989)

$$P_s = \frac{1}{2\sigma_s R_s} \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} H_l (q_n) \left( |I_{n,l,m}|^2 \right) + 2 \sum_{\nu > n} \delta_{\omega_n, \omega_{n'}} \text{Re} \left\{ I_{n,l,m} I^*_{n',l,m} \right\}$$  \hspace{1cm} (3.8)

where

$$H_l (q_n) = \sum_{k=1}^{\infty} \frac{8q_n^4}{4q_n^4 + x_{l+1/2,k}^4} = -\text{Re} \left[ (1 + i)q_n \frac{J_{l+1/2} \left\{ (1 + i)q_n \right\}}{J_{l-1/2} \left\{ (1 + i)q_n \right\}} \right].$$  \hspace{1cm} (3.9a)

thus eliminating the summation over the index \( k \).

The calculation of the power absorbed in a sphere placed in an arbitrary alternating magnetic field according to Eq. (3.8) involves the calculation of two quantities, viz., a "skin depth" function \( H_l (q_n) \) and the vector \( I_{n,l,m} \). The function \( H_l (q_n) \) as given by Eq. (3.9) is a function of the ratio of half integer order Bessel functions with a complex argument, and can be calculated by using the Lentz algorithm (Lentz, 1976). The Lentz algorithm expresses this ratio in terms of continued fractions. The calculation of the vector \( I_{n,l,m} \) is a little more involved and is the subject of section 3.3.
By using some of the results in Appendix F, the skin-depth function $H_i(q_n)$ can be recast as a special case of a more general and useful function, $\Psi_i(z, s)$. Specifically, it can be shown that

$$H_i(q_n) = -4q_n^2 \text{Im}\left[\Psi_i((1+i)q_n, 1)\right],$$

$$= -4q_n^2 \text{Im}\left\{\frac{1}{z} I_{i+1/2}((1+i)q_n)\right\},$$

where $I_{i+1/2}((1+i)q_n)$ denote modified Bessel functions. The function $\Psi_i(z, s)$ is defined ahead in Eq. (4.12c) of Chapter 4.

### 3.4 Calculation of the vector $I_{n,l,m}$

As given by Eq. (3.4b), the vector $I_{n,l,m}$ is an integral of a function of the $n$th external current source. Further, the integration is carried out over all space outside the sphere. The current sources in turn may be discrete current conductors that carry sinusoidal currents. Such concentrated current carrying conductors may be represented mathematically by Dirac delta functions (Van Bladel, 1977). It must be remembered that these current sources are described in a coordinate system that is attached to the center of the sphere. The calculation of the vector $I_{n,l,m}$ directly from Eq. (3.4b) poses problems, because, it changes when the relative position of the sphere with respect to the current sources changes. Closed form integration of Eq. (3.4b) is possible only when there is a high degree of symmetry. For example, the vector $I_{n,l,m}$ may be found for the case of a sphere that lies along the axis of a set of coaxial circular loops. When such symmetry is absent, the description of line current sources in terms of Dirac delta functions is difficult.

Due to Eq. (2.22a) and (3.4c),

$$I_{n,l,m} = \frac{x_{l+1/2,k}^2}{\mu_0 R_1^{1/2} J_{i+1/2}(x_{l+1/2,k})}$$

$$\int_0^\infty \int_{-1}^{+1} 2\pi r^{3/2} J_{i+1/2} \left(x_{l+1/2,k} \frac{r}{R_1}\right) Y_{l+1/2}^m(u, \phi) A_n(r, u, \phi) \, d\phi du dr.$$
It is easier to calculate the vector $\mathbf{I}_{n,l,m}$ by using Eq. (3.10) as compared with Eq. (2.22a). The region of integration in Eq. (3.10) is the finite volume of the sphere. Importantly, the integrand is a function of the vector potential due to the $n$th external current source. Since the vector potential $\mathbf{A}_n(r,u,\phi)$ is continuous inside the sphere, integration does not pose a problem. Equation (3.10) is valid whenever the vector potential due to a current source is known. In particular, when the current source is distributed (say a tightly wound coil which may approximated by cylindrical or conical current sheet, or a conducting annulus that is sometimes used in place of a capping coil), Eq. (3.10) may be used (see Appendix B for illustration). However, in many instances, the sources can be idealized as line currents (say a helical winding or a circular loop). The vector potential of such a source is given by (Smythe, 1989)

$$\mathbf{A}_n(r,u,\phi) = \frac{\mu_0 I_n}{4\pi} \oint \frac{ds_n'}{||\mathbf{r}(r,u,\phi) - \mathbf{r}_n'(r_n',u_n',\phi_n')||}$$

where $\mathbf{r}_n'$ is the position vector of a point on the source, $ds_n'$ is the tangent vector to the source at this point, and the integration is along the path described by the source. Note that quantities with a single prime denote the coordinates of the current source as seen from the center of the sphere. By using Eq. (3.3) to expand $||\mathbf{r} - \mathbf{r}_n||^{-1}$ it can be shown that

$$\mathbf{A}_n(r,u,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{\mu_0 I_n}{2l + 1} Y_{l,l,m}^{m'}(u,\phi) \mathbf{F}_{n,l,m},$$

where

$$\mathbf{F}_{n,l,m} = \oint \frac{(r_n')^{-(l+1)}}{r_{n'}} Y_{l,l,m}^{m'}(u_n',\phi_n') ds_n'.$$

Note that in Eq. (3.12b), the integration is performed over the coordinates of the line source. Substitution of Eq. (3.12a) into Eq. (3.10) and subsequent term by term integration yields, after utilizing the orthogonality of spherical harmonics, (i.e., Eq. (E1)),

$$\mathbf{I}_{n,l,m} = I_n R_{n,l,m} \mathbf{F}_{n,l,m}.$$

The similarity between Eqs. (3.4b) and (3.13) is at once apparent. Further, Eq. (3.12b) implies that $\mathbf{F}_{n,l,m}$ is purely a function of the source geometry.
Figure 3.1 A sphere that is placed off the axis of a circular loop.

Figure 3.2 A sphere on the axis of a stack of circular loops.
In this chapter, the vector \( I_{n,l,m} \) is calculated for circular current loops for arbitrary positions of the sphere with respect to the current loop (see Fig. 3.1). However, let us first consider the axisymmetric case shown in Fig. 3.2. When the center of the sphere lies along the axis of the loop,

\[
 \mathbf{r}'(r_n, u_n, \phi_n) = \mathbf{u}_x r_n (1 - u_n^2)^{1/2} \cos \phi_n + \mathbf{u}_y r_n (1 - u_n^2)^{1/2} \sin \phi_n + u_z r_n u_n,
\]

where \( u_n = \cos \theta_n \) and \( \mathbf{u}_x, \mathbf{u}_y, \) and \( \mathbf{u}_z \) are the unit cartesian vectors. The above equation is derived from the parametric equations for a circle. Also, since \( r_n \) and \( u_n \) do not depend on \( \phi_n \),

\[
ds' = -r_n (1 - u_n^2)^{1/2} \left[ \mathbf{u}_x \sin \phi_n - \mathbf{u}_y \cos \phi_n \right] d\phi_n, 0 \leq \phi_n \leq 2\pi.
\]

Direct substitution of the above equations in Eqs. (3.12b) and (3.13) gives

\[
I_{n,l,m} = \frac{\sqrt{\pi}}{2} \sqrt{\frac{2l+1}{l(l+1)}} R_n^l \left( \frac{R}{r_n} \right) \sin \theta_n p_l'(\cos \theta_n) \left[ -\delta_{m,l}(i \mathbf{u}_x + \mathbf{u}_y) + \delta_{m,-l}(-i \mathbf{u}_x + \mathbf{u}_y) \right]. \tag{3.14}
\]

For the general case shown in Fig. 3.2, straightforward coordinate transformation of the equation of the current loop from the double primed system to the unprimed system (centered on the loop) yields

\[
\mathbf{r}'(r_n, u_n, \phi_n) = \mathbf{u}_x x_n' + \mathbf{u}_y y_n' + \mathbf{u}_z z_n', \tag{3.15a}
\]

\[
ds' = \left( \frac{dx_n'}{d\phi_n'} + \frac{dy_n'}{d\phi_n'} + \frac{dz_n'}{d\phi_n'} \right) d\phi_n', \tag{3.15b}
\]

\[
x_n' = r_n' \left( 1 - u_n'^{1/2} \right) \cos \phi_n' - x_o, \tag{3.15c}
\]

\[
y_n' = r_n' \left( 1 - u_n'^{1/2} \right) \sin \phi_n' - y_o, \tag{3.15d}
\]

\[
z_n' = r_n' u_n' - z_o, \tag{3.15e}
\]

where \((x_o, y_o, z_o)\) is the position of the center of the sphere as seen from the double primed system. Equation (3.15) can be modified by knowing the appropriate parametric equations when the current sources are not circular loops. A similar coordinate transformation must be used to obtain the vector potential of a distributed current source in the case of Eq. (3.10).

* See Appendix B for an alternate derivation.
A program to calculate $I_{n,l,m}$ for distributed sources as well as line currents was developed. Since the functions in the integrand of Eq. (3.10) are high order polynomials, multiple gaussian integration was used. 32 point Gauss-Legendre integration was used to compute the $r$ and $u$ integrals. The $\phi$ integral was computed by computing separate gaussian points and weights, since circular functions were involved (Hildebrand, 1973). If the parametric equations of a line source are all circular functions, it can be shown (Stroud, 1974) that trapezoidal or Simpson’s rules yield sufficient accuracy in the integration of Eq. (3.12b).

For purposes of analysis and design, it is useful to nondimensionalize the expressions for the power absorbed by the sphere, in terms of the coil geometry. Choosing the least radius of the stack $a_o$ as the length scale, and $I_o$ as the current scale, it can be shown that

$$\bar{P}_s = \frac{1}{2\bar{R}_s} \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} H_i(q_n) \left[ \frac{I_n^2 |\tilde{I}_{n,l,m}|^2}{2} + \frac{2 \sum_{n' > n} \bar{I}_{n,n'} \Re \{ \tilde{I}_{n,l,m}, \tilde{I}^{*}_{n',l,m} \} }{2} \right]$$

where

$$\bar{P}_s = P_s / P_o, \quad P_o = \frac{I_o^2}{\sigma_o a_o}, \quad \bar{R}_s = R_s / a_o, \quad \bar{I}_n = I_n / I_o,$$

$$\bar{I}_{n,l,m} = \frac{x_{l+1/2,k}^2}{J_{l+1/2} \left( x_{l+1/2,k} R_s \right)^{5/2}} \int_0^\infty \int_0^{2\pi} \frac{r^{3/2} J_{l+1/2} \left( x_{l+1/2,k} R_s \right)}{r} Y_l^{m'}(\bar{u}, \bar{\phi}) \bar{A}_n(\bar{r}, \bar{u}, \bar{\phi}) d\bar{\phi} d\bar{u} d\bar{r}, \text{ dist. source,}$$

$$= \bar{R}_s \left\{ \frac{1}{\bar{I}_n} \right\}^{(l+1)} Y_l^{m'}(\bar{u}_n', \bar{\phi}_n') dS_n', \text{ line source,}$$

$$= \bar{I}_{n,l,m} / I_n,$$

(3.17)

and $\bar{r} = r / a_o$ and $\bar{A}_n(\bar{r}, \bar{u}, \bar{\phi}) = (\mu_o I_o)^{-1} A_n(r, u, \phi)$. This completes the set of equations required for the nonhomogeneous model. It now remains to write down the equations of the homogeneous model.
3.5 The homogeneous field model:

The power absorbed by an electrically conducting sphere placed in a uniform and unidirectional alternating magnetic field \( \mathbf{B}(r,t) = B \cos \omega t \) can be shown to be (Rony, 1964)

\[
P_{\text{hom}} = \frac{3\pi R^2}{\sigma \mu_o} H(q)(\mathbf{B}, \mathbf{B})
\]

where

\[
H(q) = q \frac{\sinh 2q + \sin 2q}{\cosh 2q - \cos 2q} - 1
\]

(3.18b)

When the external magnetic field is created by a stack of \( N \) coaxial circular loops (see Fig. 3.2) the magnetic field has only axial and radial components, i.e.,

\[
\mathbf{B}(\rho, z) = \mathbf{B}_\rho(\rho, z) + \mathbf{B}_z(\rho, z),
\]

\[
= \mathbf{u}_\rho \sum_{n=1}^{N} \mathbf{B}_{\rho,n}(\rho, z) + \mathbf{u}_z \sum_{n=1}^{N} \mathbf{B}_{z,n}(\rho, z).
\]

If all loops carry current of the same frequency, it can be shown that the magnetic field components in circular cylindrical coordinates as given by Eqs. (6.7) can be scaled as,

\[
\overline{\mathbf{B}}_{\rho,n}(\overline{\rho}, \overline{z}) = \frac{\overline{z} - \gamma c_n}{\overline{\rho} \sqrt{\left(1 + \gamma c_n + \overline{\rho}\right)^2 + \left(\overline{z} - \gamma c_n\right)^2}}
\]

\[
\cdot \left[ -K(\kappa_n) + \frac{(1 + \gamma c_n \tan \alpha)^2 + \overline{\rho}^2 + (\overline{z} - \gamma c_n)^2}{(1 + \gamma c_n \tan \alpha - \overline{\rho})^2 + (\overline{z} - \gamma c_n)^2} E(\kappa_n) \right]
\]

(3.19a)

\[
\overline{\mathbf{B}}_{z,n}(\overline{\rho}, \overline{z}) = \frac{1}{\sqrt{\left(1 + \gamma c_n + \overline{\rho}\right)^2 + \left(\overline{z} - \gamma c_n\right)^2}}
\]

\[
\cdot \left[ K(\kappa_n) + \frac{(1 + \gamma c_n \tan \alpha)^2 - \overline{\rho}^2 - (\overline{z} - \gamma c_n)^2}{(1 + \gamma c_n \tan \alpha - \overline{\rho})^2 + (\overline{z} - \gamma c_n)^2} E(\kappa_n) \right]
\]

(3.19b)

where \( \overline{\mathbf{B}}_{\rho,n} = B_{\rho,n}/B_o \), \( \overline{\mathbf{B}}_{z,n} = B_{z,n}/B_o \), \( B_o = \mu_o I_o/2\pi a_o \), \( \overline{\rho} = \rho/a_o \), \( \overline{z} = z/a_o \), \( \gamma = h/a_o \) and \( c_n = (n-1)/(N-1) \). \( K \) and \( E \) are Elliptic integrals of the first and second kind respectively. Their argument is defined by,

\[
\kappa_n = \frac{4\overline{\rho}(1 + \gamma c_n \tan \alpha)}{\sqrt{(1 + \gamma c_n \tan \alpha + \overline{\rho})^2 + (\overline{z} - \gamma c_n)^2}}.
\]
For the special case of a single loop (i.e., \(N = 1\)), put \(\gamma = 0\) in Eqs. (3.19). Using these equations, the power absorbed by the sphere according to Eqs. (3.18) can be scaled as

\[
\bar{P}_{\text{hom}} = \frac{P_{\text{hom}}}{P_o} = \frac{3R}{4\pi} H(q) \left[ \left( \sum_{n=1}^{N} \vec{B}_{P,n}(\vec{r},z) \right)^2 + \left( \sum_{n=1}^{N} \vec{B}_{z,n}(\vec{r},z) \right)^2 \right].
\]  

(3.20)

3.6 Results and discussion:

As pointed out earlier, the calculation of the power absorbed requires the calculation of the vector \(\vec{I}_{n,l,m}\). Programs to calculate the volume and line integrals that determine this vector were developed. Further, a program to find \(H_i(q_n)\) by using the Lentz (1976) algorithm was developed and used. The power absorbed was then calculated according to Eq. (3.16). These results were compared with the results obtained from the homogeneous model equations, i.e. Eq. (3.20).

Figure 3.3 shows the power absorbed along the axis of a right circular stack with five (\(N = 5\)) coaxial circular loops for two different sample sizes. The top half of Fig. 3.3 refers to a sphere whose radius is one-tenth of the coil radius, i.e., the sphere is very small compared to the size of the coil. When such a sphere lies along the axis of a circular stack of loops, the variation of the magnetic field across the diameter is never more than 4 to 5%, and the magnetic field can be assumed to be essentially uniform and directed along the coil axis*. This situation corresponds roughly to the "homogeneous model" for levitation, and one expects that the homogeneous and the nonhomogeneous models should yield identical results. This fact is confirmed in the top half of Fig. 3.3. The bottom half of Fig. 3.3 refers to a spherical sample whose radius is nine-tenths of the coil radius. Typically, the radius of the levitated sample is anywhere between 50 to 70% of the coil radius (Zong et al., 1992; Li, 1993). The ratios in Fig. 3.3 have been chosen to illustrate the extremities of the situation. It is clear, that when the sample radius is not a negligible fraction of the coil radius, the homogeneous model vastly underestimates the power absorbed. Figure 3.4

* That the radial component of the magnetic field vanishes on the axis of a loop can be checked by setting \(\rho = 0\), either in Eq. (3.19a) or Eq. (6.7a).
Figure 3.3 The power absorbed by a sphere for the geometry of Fig. 3.2.

Figure 3.4 Difference between the homogeneous and nonhomogeneous models for the geometry of Fig. 3.2.
shows the percentage difference in the generated power as predicted by the homogeneous
and the nonhomogeneous models along the axis of a right circular stack of coaxial loops as
a function of the sample size (for different values of the height to radius ratio, \( \gamma \) of the
stack). This figure shows that power absorption in the sphere as predicted by the
homogeneous model ceases to be reliable for sample to coil radius ratios in excess of 0.3.
At this point, the homogeneous model underestimates the power generated by 10% or
more.

The calculation of the heat generated in the sphere when it does not lie along the
axis of the coil is of interest in actual practical situations. In earthbound levitation
experiments, the sample oscillates about the axis. The off-axis displacements are usually
significant. In the coils described by El-Kaddah and Szekely (1983) and Zong et al.
(1992), the maximum sample displacement can be as much as 78% and 90% of its diameter
respectively. Therefore, it is important to evaluate the power absorbed at off-axis locations
of the coil. However, at off-axis locations, the problem loses its axisymmetry and the
calculation of the vector \( \mathbf{I}_{n,l,m} \) is more involved. Further, it is clear from Eq. (3.16) that
when the number of turns is greater than one, the calculations become lengthy. Therefore,
only a few illustrative cases for a single loop coil are presented.

Table 3.1 lists the values of \( |\mathbf{I}_{n,l,m}|^2 \) for the case of a sphere placed off the axis of a
circular loop. The radius of the sphere is 50% of the radius of the loop. The values of \( \mathbf{I}_{n,l,m} \)
decrease with increasing values of \( l \) and \( m \). This behavior is expected since the contribution
to the total power by the higher modes should be negligible. The contribution to the total
heat generated by a given mode is \( [H_1(q_n)/(2\pi R_s)]|\mathbf{I}_{n,l,m}|^2 \). It is clear that \( \mathbf{I}_{n,l,m} \) decreases
rapidly with increase in \( l \). When the sphere is large and the magnetic field is highly
nonuniform, an oscillatory convergence may be expected. Substitution of the values shown
in Table 3.1 into the formulae for the power absorption shows that the homogeneous model
underestimates the power by nearly 20% in this case. Table 3.2 lists the values of \( |\mathbf{I}_{n,l,m}|^2 \)
for the same relative positions of the sphere and the loop. However, the sphere is very
Table 3.1

\[
\begin{align*}
\bar{x}_o = \bar{y}_o &= 0.1, \quad \bar{z}_o = 2.5, \quad \bar{R}_x = 0.5, \quad q_1 = 43.0737 \\
\overline{B}_p(\bar{x}_o, \bar{y}_o, \bar{z}_o) &= 1.17e - 3, \quad \overline{B}_z(\bar{x}_o, \bar{y}_o, \bar{z}_o) = 1.56e - 1 \\
\overline{A}_1(\bar{x}_o, \bar{y}_o, \bar{z}_o) &= \mathbf{u}_s 1.805e - 3
\end{align*}
\]

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\(\overline{P}_{hom} = 0.1222, \quad \overline{P}_t = 0.1517\)

% difference is 19.43
Table 3.2

$\bar{x}_o = \bar{y}_o = 0.1, \; \bar{z}_o = 2.5, \; R_z = 0.01, \; q_1 = 0.8615$

$\bar{B}_p(\bar{x}_o, \bar{y}_o, \bar{z}_o) = 1.17 \times 10^{-3}, \; \bar{B}_z(\bar{x}_o, \bar{y}_o, \bar{z}_o) = 1.56 \times 10^{-1}$

$\bar{A}_1(\bar{x}_o, \bar{y}_o, \bar{z}_o) = u_\phi 1.805 \times 10^{-3}$

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$H_i(q_1) = 0.542 \; 0.048 \; 0.012 \; 0.004 \; 0.002 \; 0.001$

$\bar{P}_{\text{hom}} = 2.789 \times 10^{-6}, \; \bar{P}_z = 1.112 \times 10^{-3}$

three orders of magnitude difference
small compared (1% of the loop radius) to the radius of the loop. Once again, the contribution to the power absorbed by the higher modes is negligible. The significant contribution is by the modes \( l = 0, m = 0 \) and \( l = 1, m = \pm 1 \). By using the values given in Table 3.2, it is seen that the difference in power absorption in the sphere as predicted by the homogeneous and the nonhomogeneous models is more than an order of magnitude. This is an unexpected result. As in the on-axis case, it would seem reasonable to expect that in the limiting case of a fairly small sphere (as compared with the size of the coil), the nonhomogeneous and homogeneous models should yield identical results. But it is not so. Table 3.2 shows that the major contribution to the power absorbed is from the \( l = 0, m = 0 \) mode. All other higher order modes are negligible. This result can be explained as follows.

In the limiting case of a very small sphere, the vector potential of the magnetic field can be assumed to be constant over the volume of the sphere. Then Eq. (3.10) may be rewritten as

\[
I_{n,l,m} = \frac{x_{l+1/2,k}^2}{\mu_0 R_s^{5/2}} A_n(r_o, u_o, \phi_o) \frac{r}{R_s} \int_1^{r_o} \left( \frac{x_{l+1/2,k}}{R_s} \right) Y_{l+1/2}^m(u, \phi) d\phi dudr,
\]

where \( A_n(r_o, u_o, \phi_o) \) is the vector potential of the external field at the center of the loop. The integral in the above equation vanishes for all values of \( l \) and \( m \) except \( l = m = 0 \), since

\[
\int_1^{r_o} Y_l^m(u, \phi) d\phi du = \sqrt{4\pi} \delta_{m,0} \delta_{l,0}.
\]

The \( r \)-integral for this case is solved by using Eq. (D6a) to yield

\[
I_{n,0,0} = \frac{2\sqrt{\pi}}{\mu_0} A_n(r_o, u_o, \phi_o).
\]

This result can also be derived by setting \( l = m = 0 \) in Eq. (3.13).

The vector potential for a circular loop can be obtained from any standard reference such as Smythe (1989). When the value of the vector potential due to the loop at the point specified in Tables 3.1 and 3.2 is substituted in Eq. (3.21) the value shown in Table 3.2 is obtained. Table 3.2 also confirms that the vector \( I_{n,l,m} \) is indeed near zero or negligible when \( l \neq 0, m \neq 0 \). Further, Eq. (3.21) indicates that \( I_{n,0,0} \) is independent of the sample size, a fact that is a consequence of Eq. (3.13). Tables 3.1 and 3.2 confirm this. This is a
very important result, since it implies that all previous calculations of the heating generated in a sphere, especially in the limiting case of a small off-axis sphere are in error corresponding to the magnitude of \( I_{n,0,0} \) as given by Eq. (3.21). Physically, this result can be explained by noting that at off-axis positions, the absence of circular symmetry manifests itself in the non-zero \( I_{n,0,0} \) term. The "homogeneous model" does not account for this asymmetry. Rather, the assumption of a uniform \( z \)-directed field inherently introduces symmetry in the model and incorrectly estimates the heat generation. The absence of this discrepancy in the on-axis case can be explained by noting that the field due to a circular loop on its axis is purely axial (and that \( A \) vanishes identically). A scaling analysis of Eq. (3.8) for the limiting case of a small sphere is presented in Appendix B. The discussion in this appendix further illustrates the difference between the homogeneous and the nonhomogeneous power calculations.

The vector potential due to a uniform \( z \)-directed field can be shown to be (Smythe, 1989)

\[
A_n = u_y \frac{1}{2} Br \sin \theta.
\]  

(3.22)

Upon substituting Eq. (3.22) in Eq. (3.10), it can be shown that \( I_{n,l,m} \) exists only when \( l = 1 \), and \( m = \pm 1 \), and that it is given by

\[
I_{n,1,1} = \sqrt{\frac{3\pi}{2}} \left( iu_x + u_y \right) \left( \frac{BR_x}{\mu_o} \right),
\]

(3.23a)

\[
I_{n,1,-1} = -I_{n,1,1}^*.
\]

(3.23b)

By substituting Eq. (3.23) in Eq. (3.16), the homogenous model power equation, Eq. (3.18a) can be recovered. Since fractional order Bessel functions may be expressed in terms of circular functions, it can be verified that the skin depth function in Eq. (3.18b) for the homogeneous model as developed by Rony (1964) can be recovered by putting \( l = 1 \) in Eq. (3.9).

Tables 3.3 and 3.4 indicate the values of \( |I_{n,l,m}|^2 \) for a very large and a very small sphere respectively when the center of the sphere is very slightly displaced from the center.
of the loop. Substituting the values given in Table 3.3 in the formula for power (using the homogeneous as well as the nonhomogeneous models) shows that the homogeneous model underestimates the power absorbed by nearly 36%! This agrees with the numbers shown for the on-axis case in Fig. 3.4. The value of $|I_{n,1,1}|^2$ in Table 3.3 checks very well with its

Table 3.3

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$P_{hom} = 162.3373$, $P_s = 253.3112$

% difference is 35.9

value obtained by using Eq. (3.23a) ($|I_{n,1,1}|^2 = 1.909$). This confirms the fact that when the field is purely $z$-directed, the homogeneous field model predicts only the $l = 1, m = 1$ mode. However, the numbers in Table 3.3 imply that there is a significant contribution by the $l = 3, m = 1$ mode ($|I_{1,3,1}|^2 = 1.097$), comparable to the contribution by the $l = 1, m = 1$ mode. When the sphere is concentric with the loop, by using Eq. (3.14), it can be shown
Table 3.4

\[ x_0 = y_0 = 0.01, z_0 = 0.005, R_0 = 0.01, q_l = 0.8615 \]

\[ B_p(x_0, y_0, z_0) = 3.33e - 4, B_z(x_0, y_0, z_0) = 3.14195 \]

\[ A_1(x_0, y_0, z_0) = 0.534e - 3 \]

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<td>0.001</td>
</tr>
</tbody>
</table>

\[ \bar{P}_{\text{hom}} = 1.1312e - 3, \quad \bar{P}_s = 5.3883e - 3 \]

% difference is 79.00

That only odd values of \( l \) contribute to the power absorbed. When the sphere is large, the significant contributions are by the \( l = 1 \) and \( l = 3 \) modes. Table 3.3 bears this out for very slight displacements from the center, and serves to validate these calculations. Finally, Table 3.4 describes the same situation for a very small sphere. Further, it is interesting to note that \( I_{1,0,0} \) is independent of the sphere size (compare Tables 3.3 and 3.4) and that its value matches with the value predicted by Eq. (3.21).

Finally, it must be mentioned here that Lohofer (1993) presents a power series type solution for \( I_{n,l,m} \) at slightly off-axis locations. Equation (54) of this reference reads

\[ I_{n,l,m} = \sum_{\lambda=0}^{\lambda} \sum_{\mu=-\lambda}^{\lambda} \frac{4\pi}{2\lambda + 1} r_s^4 Y_\lambda^\mu(u_s, \phi_s) a_{\lambda,\mu} \hat{I}_{n,\lambda,l+m} a_{l,m} a_{\lambda,\mu}, \quad (3.24a) \]
where

\[
\hat{I}_{n,l,m} = R_l \int_{R} \int_{-1}^{1} \int_{0}^{2\pi} J_n(r,u,\phi) r^{-l} l^m Y^m_l(u,\phi) d\phi dudr,
\]

(3.24b)

\[
a_{l,m} = (-1)^{l+m} \frac{4\pi}{2l+1} (l+m)!(l-m)!
\]

(3.24c)

Here \((r_s,u_s,\phi_s)\) is the location of the center of the sphere and \(\hat{J}_n(r,u,\phi)\) is the current density of the source as seen from the double primed coordinate system (see Fig. 3.2). The similarity between Eqs. (3.24b) and (3.4b) implies that \(\hat{I}_{n,l,m}\) is the source function as seen from a coordinate system attached to the source. While Eq. (3.24) provides a solution for the source function at an arbitrary position of the sphere it still leaves the problem of finding \(\hat{I}_{n,l,m}\) unresolved. For an arbitrary current source, calculation of the integral in Eq. (3.24b) poses the problems mentioned earlier in connection with the calculation of \(I_{n,l,m}\).

Further, the nature of the (power series) solution limits the region in which Eq. (3.24) is valid. On the other hand, the method presented in this section is valid for all positions of the sphere with respect to a given set of sources, and involves the calculation of a simple line or volume integral. From a computational point of view, the choice is between using the series in Eq. (3.24a) and the calculation of the volume integral in Eq. (3.24b) or directly using Eq. (3.10), or Eqs. (3.12b) and (3.13). We believe that our method is faster and more general, especially when the source geometries are complicated (for example a helical winding with elliptical cross section, or the case of a distributed source). For the case shown in Tables 3.3 and 3.4, Eq. (3.24) yields a value of \(I_{l,0,0}\) that matches with the method shown in this work. However it fails as expected, when applied to the situation shown in Tables 3.1 and 3.2. For further details regarding the applicability of Eq. (3.24) refer Lohofer (1993).

In the case of the very small sphere the vector potential over its volume can be assumed to be constant, and by using Eqs. (3.8) and (3.21) the power generated due to the external magnetic field created by a single current source (i.e., \(N = 1\)), can be shown to be (see Appendix B)
\[ P_s = \frac{1}{2\pi R_s} H_0(q) |I_{0,0}|^2 \tag{3.25a} \]

where
\[ H_0(q) = q \frac{\sinh 2q - \sin 2q}{\cos 2q + \cosh 2q} \tag{3.25b} \]
\[ I_{0,0} = \frac{2\sqrt{\pi}}{\mu_0} A(r) \tag{3.25c} \]

Here \( r \) is the position vector of the center of the sphere with respect to the current source. \( A(r) \) is the vector potential of the magnetic field at the center of the sphere. Equation (3.25b) represents a special case of the skin depth function, and can be obtained by putting \( l = 0 \) in Eq. (3.9).

### 3.7 Conclusions:

From the preceding analysis, it can be concluded that:

i) for a sphere placed in the magnetic field of a circular loop(s), the power absorbed is strongly dependent on the relative size of the sphere and its position with respect to the coil.

ii) The nonhomogeneous model for the power absorbed by the sphere accounts for the variation of the magnetic field across the sphere and provides much better results than the homogeneous model. The homogeneous model is in error by at least 15% in typical laboratory situations.

iii) In the limiting case of a very small sphere that is off the axis of a loop, the homogeneous model yields incorrect results. The present analysis reveals that a term that corresponds to the \( l = 0, m = 0 \) mode is the leading term that contributes to the heat generation in the sphere. This result holds whenever the sphere is very small as compared to the scale of the field and the vector potential of the field is not zero.
4.1 Introduction:

The aim of this chapter is to derive an expression for the Lorentz force that acts on an electrically conducting sphere that is placed in the alternating magnetic field. First, the density of eddy currents that are generated in the diamagnetic sphere due to the sinusoidally alternating magnetic field as given by Eq. (3.6) is written. Then, by using multipole expansion, the vector potential of the external magnetic field is expressed in terms of the previously defined "source functions" $I_{n,l,m}$ of the external current sources. The external magnetic field is calculated by using a gradient formula. The expression for the instantaneous Lorentz force per unit volume is subsequently integrated over the volume of the sphere to obtain the net time averaged force on the sphere in terms of these source functions and a "skin depth" function. This method can also be used with little modification to find the magnetic pressure on the surface of a liquid metal droplet in an arbitrary magnetic field, as indicated in the next chapter. Approximations for the skin depth function for use in practical situations are presented. The general formula for the net Lorentz force is used to derive the force on a sphere placed in an axisymmetric magnetic field that is created by a stack of circular coaxial loops (Fig. 3.2) and obtain some numerical results in terms of the stack geometry. The homogeneous model is shown to be a special case of the nonhomogeneous model and the difference in the predictions of the Lorentz force by the two models is estimated.

4.2 Analysis:

The net time averaged Lorentz force acting on the sphere is,

$$F_s = \lim_{T \to \infty} \frac{L_t}{T} \int_0^T \int_0^{2\pi} \int_0^{R_s} \text{Re}[J_s(r,t)] \times \text{Re}[B_e(r,t)] r^2 d\phi \, du \, dr \, dt.$$  \hfill (4.1)
The form of Eq. (4.1) suggests that it is convenient to express the external magnetic field also in terms of the source function $I_{n,l,m}$, since the induced eddy current density inside the sphere has already been expressed in terms of it (according to Eq. (3.6)).

The vector potential of the external magnetic field can be expressed in terms of the external current densities by Eq. (3.2). The denominator of the integrand in Eq. (3.2) can be written as a multipole expansion in spherical harmonics according to Eq. (3.3). Substituting Eq. (3.3) in Eq. (3.2) and considering only the case $r < r'$, yields

$$A_n(r,u,\phi) = \mu_o \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} Y_l^m(u,\phi) r^l \left[ \int \int \int J_n(r',u',\phi') r'^{-l-1} Y_l^m(u',\phi') d\phi' du' dr' \right].$$

Assuming that $J_n(r,u,\phi)$ is real for all $n$, the above equation in conjunction with Eq. (3.4b) becomes

$$A_n(r,u,\phi) = \mu_o \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{R_{n}^{-l}}{2l+1} r^l Y_l^m(u,\phi) I_{n,l,m}^*.$$ (4.2a)

The vector potential of the external magnetic field is then given by using Eq. (2.6),

$$A_e(r,u,\phi,t) = \mu_o \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{R_{n}^{-l}}{2l+1} r^l Y_l^m(u,\phi) I_{n,l,m}^* \cos(\omega_n t).$$ (4.2b)

The magnetic flux density of the external field can be then found by taking the curl of Eq. (4.2b). The calculation of the external magnetic field in this fashion requires the use of the general gradient formula for functions of the type $f(r)Y_l^m(u,\phi)$ (Rose, 1955). However, this involves the calculation of Clebsch-Gordan coefficients (used in the description of angular momenta in quantum mechanics). For the purposes of the present calculation, it is advantageous (from a purely algebraic point of view) to rewrite the standard spherical harmonics $Y_l^m(u,\phi)$ (see Eq. (2.10)), in a slightly modified form. Following the notation of Morse and Feshbach (1965), we let

$$Y_{l,m}^e(u,\phi) = P_l^m(u) \cos m\phi,$$ (4.3a)

$$Y_{l,m}^\phi(u,\phi) = P_l^m(u) \sin m\phi.$$ (4.3b)

---

* Only the case $r < r'$ is considered, since it corresponds to the case when the center of the sphere lies within the radius of the coil.
so that $Y_{l,m}^r$ and $Y_{l,m}^0$ are the real and imaginary parts of the standard spherical harmonic $Y_{l,m}^n(u, \phi)$, to a multiplicative factor.

Now let the complex vector ("source function")
\[ I_{n,l,m} = U_{n,l,m} + iV_{n,l,m}, \tag{4.4} \]
where both $U_{n,l,m}$ and $V_{n,l,m}$ are real. Then Eqs. (3.6) and (4.4) can be simplified to
\[ \text{Re}\left[ J_s(r,u,\phi,t) \right] = \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ f_{l,k,m}^{\text{ext}}(r,u,\phi)M_{n,k,l,m} - f_{l,k,m}^0(r,u,\phi)N_{n,k,l,m} \right] \cos(\omega_n t + \psi_{n,k,l,m}) \tag{4.5a} \]
where
\[ f_{l,k,m}^{\text{ext}}(r,u,\phi) = r^{-1/2} J_{l+1/2}\left( x_{l+1/2,k} \frac{r}{R_s} \right) Y_{l,m}^n(u, \phi), \tag{4.5b} \]
\[ M_{n,k,l,m} = E_{n,k,l,m} U_{n,l,m} \]
\[ = E_{n,k,l,m} \text{Re}[I_{n,l,m}], \tag{4.5c} \]
\[ N_{n,k,l,m} = E_{n,k,l,m} V_{n,l,m} \]
\[ = E_{n,k,l,m} \text{Im}[I_{n,l,m}], \tag{4.5d} \]
and
\[ E_{n,k,l,m} = (-1)^m \frac{2}{R_s^{3/2}} J_{l+1/2}\left( x_{l+1/2,k} \right) \frac{2q_n^2}{\sqrt{4q_n^4 + x_{l+1/2,k}^4}} \frac{2l+1 (l-m)!}{4 \pi (l+m)!}. \tag{4.5e} \]

Equation (4.5) ensures that the induced current density is expressed solely in terms of known real functions of the external currents. It now remains to express the external magnetic field similarly.

Substitution of Eqs. (4.3) and (4.4) in Eq. (4.2b) results in
\[ \text{Re}\left[ A_t(r,u,\phi,t) \right] = \mu_o \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ (r^{l+1} Y_{l,m}^r(u, \phi))S_{n,l,m} - (r^{l+1} Y_{l,m}^0(u, \phi))T_{n,l,m} \right] \cos(\omega_n t) \tag{4.6a} \]
where
\[ S_{n,l,m} = D_{l,m} U_{n,l,m} \]
\[ = D_{l,m} \text{Re}[I_{n,l,m}], \tag{4.6b} \]
\[ T_{n,l,m} = D_{l,m} V_{n,l,m} \]
\[ = D_{l,m} \text{Im}[I_{n,l,m}], \tag{4.6c} \]
and
\[ D_{l,m} = (-1)^m \frac{R^{-l}}{2l+1} \sqrt{\frac{2l+1(l-m)!}{4\pi(l+m)!}} \]  

Since \( \nabla \times \mathbf{I}_{n,l,m} = 0 \), the definition of the magnetic vector potential yields,

\[ \text{Re}[\mathbf{B}_e(r,u,\phi,t)] = \mu_0 \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \nabla \left( r^{l} Y_{l,m}(u,\phi) \right) \times \mathbf{S}_{n,l,m} - \nabla \left( r^{l} Y_{l,m}^0(u,\phi) \right) \times \mathbf{T}_{n,l,m} \right] \cos(\omega \cdot t). \]

It is easy to check that Eq. (4.7) satisfies the zero divergence condition,

\[ \nabla \cdot \mathbf{B}_e(r,u,\phi,t) = 0, \]

as required by the Maxwell equations.

### 4.3 The Lorentz force:

The Lorentz force on the sphere can now be found from Eqs. (4.1), (4.5) and (4.7) by direct substitution of the expressions for the current density and the external magnetic field. This substitution yields,

\[ \mathbf{F}_s = \mu_0 \sum_{n,l,m}^{N} \sum_{n',l',m'}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \left\{ \mathbf{M}_{n,k,l,m} f_{l,k,m}(r,u,\phi) - \mathbf{N}_{n,k,l,m} f_{l,k,m}^0(r,u,\phi) \right\} \times \left\{ \nabla \left( r^{l} Y_{l,m}(u,\phi) \right) \times \mathbf{S}_{n,l,m} - \nabla \left( r^{l} Y_{l,m}^0(u,\phi) \right) \times \mathbf{T}_{n,l,m} \right\} \right] r^2 d\phi dr du \]

\[ \cdot \left[ \frac{L}{T} \int_{0}^{T} \cos(\omega \cdot t + \psi_{n,k,l}) \cos(\omega n \cdot t) dt \right], \]

where

\[ \sum_{n,l,m} \rightarrow \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l}. \]

Since the fields are all harmonic, the time dependent integral in Eq. (4.8) reduces to

\[ \frac{\omega_n}{2\pi} \int_{0}^{2\pi} \cos(\omega_n \cdot t + \psi_{n,k,l}) \cos(\omega_n \cdot t) dt = \frac{1}{2} \cos \psi_{n,k,l} \delta_{\omega_n,\omega_n}, \]

The cross product in the volume integral can be expanded to yield the following terms:
\[ X_1 = \sum_{n,l,m} \sum_{n',l',m'} \sum_{k=1}^{\infty} \cos \psi_{n,k,l} \delta_{\omega_n,\omega_{n'}} \int_0^{R+12\pi} \int_0^1 M_{n,k,l,m} f_{l,k,m}^e (r,u,\phi) \times \left[ \nabla \left( r Y_{l,m}^e (u,\phi) \right) \times S_{n',l',m'} \right] \, r^2 \, d\phi \, dr, \]  
(4.10a)

\[ X_{II} = -\sum_{n,l,m} \sum_{n',l',m'} \sum_{k=1}^{\infty} \cos \psi_{n,k,l} \delta_{\omega_n,\omega_{n'}} \int_0^{R+12\pi} \int_0^1 N_{n,k,l,m} f_{l,k,m}^0 (r,u,\phi) \times \left[ \nabla \left( r Y_{l,m}^0 (u,\phi) \right) \times T_{n',l',m'} \right] \, r^2 \, d\phi \, dr, \]  
(4.10b)

\[ X_{III} = -\sum_{n,l,m} \sum_{n',l',m'} \sum_{k=1}^{\infty} \cos \psi_{n,k,l} \delta_{\omega_n,\omega_{n'}} \int_0^{R+12\pi} \int_0^1 N_{n,k,l,m} f_{l,k,m}^0 (r,u,\phi) \times \left[ \nabla \left( r Y_{l,m}^0 (u,\phi) \right) \times S_{n',l',m'} \right] \, r^2 \, d\phi \, dr, \]  
(4.10c)

\[ X_{IV} = \sum_{n,l,m} \sum_{n',l',m'} \sum_{k=1}^{\infty} \cos \psi_{n,k,l} \delta_{\omega_n,\omega_{n'}} \int_0^{R+12\pi} \int_0^1 N_{n,k,l,m} f_{l,k,m}^0 (r,u,\phi) \times \left[ \nabla \left( r Y_{l,m}^0 (u,\phi) \right) \times T_{n',l',m'} \right] \, r^2 \, d\phi \, dr, \]  
(4.10d)

so that the net time averaged Lorentz force on the sphere is

\[ F_s = \frac{1}{2} \mu_o [X_1 + X_{II} + X_{III} + X_{IV}], \]  
(4.11)

It now remains to evaluate each of the terms in Eq. (4.11), which in turn involves the calculation of the sums of the integrals in Eq. (4.10). Fortunately, the four vector integrals in Eq. (4.10) share a common integral. The calculation of a typical term in Eq. (4.10) is shown in Appendix D. The evaluation of the integrals in Eq. (4.10) calls for the calculation of the gradient of functions of the type \( r Y_{l,m}^e (u,\phi) \). These are evaluated in Appendix E.

The final expression for the net time averaged Lorentz force on the sphere is found by substituting the expressions for the terms in Eq. (4.11) from Eqs. (D10), (D11), (D12), and (D13) of Appendix D. Thus,

\[ F_s = \frac{1}{2} \mu_o \sum_{n=1}^{N} \sum_{l=0}^{N} \sum_{m=-l}^{l} \sum_{i=1}^{1} \frac{1}{2} \cos \psi_{n,k,l} \delta_{\omega_n,\omega_{n'}} \left[ (1 + \delta_{m,0}) U_{n,l,m} \times \left( u_x \times F_{n,l,m}^1 + u_y \times Q_{n,l,m}^2 + u_z \times K_{n,l,m}^1 \right) \right] + \left[ (1 - \delta_{m,0}) V_{n,l,m} \times \left( u_x \times Q_{n,l,m}^1 - u_y \times F_{n,l,m}^2 + u_z \times K_{n,l,m}^2 \right) \right], \]  
(4.12a)

where the skin depth function \( g_i (q_n) \) is given by
\[ g_t(q_n) = \sum_{k=1}^{\infty} \frac{-1}{x_{l+1/2,k}} \frac{4q_n^4}{4q_n^4 + x_{l+1/2,k}^4} \]

= \text{Re} \left[ \Psi_t(z,1) - \frac{1}{2(2l+1)} \right], \quad (4.12b)

and for real values of \( s \)

\[ \Psi_t(z,s) = \frac{1}{2z} \frac{I_{l+1/2}(zs)}{I_{l-1/2}(z)}, \quad (4.12c) \]

\[ z = (1 + i)q_n. \quad (4.12d) \]

Also,

\[ F_{n,l,m}^1 = \beta_1 U_{n,l+1,m-1} - \beta_2 U_{n,l+1,m+1}, \quad (4.12d) \]

\[ F_{n,l,m}^2 = \beta_1 U_{n,l+1,m-1} + \beta_2 U_{n,l+1,m+1}, \quad (4.12e) \]

\[ Q_{n,l,m}^1 = \beta_1 V_{n,l+1,m-1} - \beta_2 V_{n,l+1,m+1}, \quad (4.12f) \]

\[ Q_{n,l,m}^2 = \beta_1 V_{n,l+1,m-1} + \beta_2 V_{n,l+1,m+1}, \quad (4.12g) \]

\[ K_{n,l,m}^1 = \beta_3 U_{n,l+1,m}, \quad (4.12h) \]

\[ K_{n,l,m}^2 = \beta_3 V_{n,l+1,m}. \quad (4.12i) \]

The \( \beta \) coefficients are defined as

\[ \beta_1 = \sqrt{(l-m+2)(l-m+1)}, \]

\[ \beta_2 = \sqrt{(l+m+2)(l+m+1)}, \]

and

\[ \beta_3 = 2\sqrt{(l+m+1)(l-m+1)}. \]

Along with Eqs. (3.4b) and (4.4), Eq. (4.12) gives the net time averaged Lorentz force acting on a sphere placed in an alternating magnetic field. Equation (4.12) in conjunction with (3.4b) implies that the Lorentz force is proportional to the \((2l+1)\)th power of the sphere radius. Here, we must point out that Lohofer (1993) presents an expression for the force and torque on the sphere in terms of the same source functions, but a different skin depth function. The skin depth function as given in Lohofer (1993) makes use of ordinary fractional order Bessel functions as opposed to modified Bessel functions that are used in Eq. (4.12b).
Appendix F presents a proof of Eq. (4.12b). The skin depth function $g_l(q_n)$ for the Lorentz force is analogous to the function $H_l(q_n)$ which appears in the expression for the power absorbed in the sphere placed in an alternating magnetic field, viz. Eq. (3.9). The function $g_l(q_n)$ depicts the frequency dependence of every mode in Eq. (4.12a) (see Fig. 4.1). Since modified Bessel functions can be expressed in terms of circular functions (Abramowitz and Stegun, 1965) the special case of $l = 1$ may be obtained explicitly. For $l = 1$,

$$g_1(q_n) = \frac{1}{4q_n} \frac{\sinh 2q_n - \sin 2q_n}{\cosh 2q_n - \cos 2q_n} - \frac{1}{6}.$$  

(4.13a)
A comparison with the corresponding skin depth function derived by Rony (1964) (see Eq. (4.20b)) reveals that

$$G(q_n) = -6g_1(q_n).$$

(4.13b)

This implies that the homogeneous model corresponds to the \( l = 1 \) mode in the general expression for the force.

Since modified Bessel functions can be expressed in terms of ordinary Bessel functions, the Lentz algorithm (Lentz, 1976) may be used to evaluate \( g_1(q_n) \) for arbitrary values of \( q_n \). However, in most practical levitation situations the ratio of the specimen radius to skin depth, \( q_n \), is of the order of 40 to 50. In powder production applications, this ratio is seldom greater than 2. It is therefore worthwhile to examine the behavior of the skin depth function \( g_1(q_n) \), for large and small limiting values of \( q_n \). The high frequency limit (or large \( q_n \)) can be shown to be (see Appendix F)

$$\lim_{q \to \infty} [g_1(q)] = \frac{-1}{2(2l+1)}.$$  

(4.14a)

This is physically reasonable since it implies that increasing the frequency indefinitely does not correspondingly increase the force (see Fig. 4.1). On the other hand, when \( q_n \) is small, it can be shown (see Appendix F) that

$$\lim_{q_n \to 0} [g_1(q_n)] = \frac{4q_n^4}{(2l+1)^3(2l+3)(2l+5)}.$$  

(4.14b)

Since, \( I_{l+1/2}(z) = I_{l-1/2}(z) \) for large values of \( z \), the skin depth function may be approximated by,

$$g_1(q_n) \approx \frac{1}{4q_n} - \frac{1}{2(2l+1)}.$$  

(4.14c)

for intermediate values of \( q_n > 2 \). For \( l = 1 \), Eqs. (4.14b) and (4.14c) give

$$-6g_1(q_n) \approx 1 - \frac{3}{2q_n}$$  

(4.14d)

and

$$-6 \lim_{q_n \to 0} [g_1(q_n)] = 0.025397q_n^4$$  

(4.14e)
respectively. These are the limiting values for the skin depth function obtained by using the homogeneous model of Rony (Eqs. (12) and (15) of Rony (1964)). The relations in Eqs. (4.14) obviate the need for the Lentz (1976) algorithm, which at the minimum requires the use of a computer program to evaluate the ratios of fractional order Bessel functions.

Finally, we note that the function $\Psi_I(z,s)$ as given by Eq. (4.12c) represents an important class of functions in the study of the interaction of a sphere with a magnetic field. These functions take the form of the ratios of modified Bessel functions because the fields are harmonic in time and because the eigenfunctions of Eq. (2.8) in the $r$ direction (in spherical coordinates) are modified Bessel functions. When $s$ is replaced by $(r/R_s)$ in Eq. (4.12c), the function $\Psi_I(z,s)$ can be used to obtain a compact expression for the distribution of the current density inside the sphere (see Chapter 5 and Appendix F for details). By taking the imaginary part of $\Psi_I(z,1)$ we recover the skin depth function $H_I(q_n)$ in the expression for the power absorbed (recall Eq. (3.9b)).

4.4 The axisymmetric case:

Figure 3.2 shows a sphere placed along the axis of a conical stack of coaxial loops. The vector source function $I_{n,l,m}$ for this arrangement has been shown to be given by Eq. (3.14), i.e.,

$$I_{n,l,m} = \frac{\sqrt{\pi}}{2} I_{n,l} \left[ -\delta_{m,1} \left( iu_x + u_y \right) + \delta_{m,-1} \left( -iu_x + u_y \right) \right]$$

(4.15a)

where

$$I_{n,l} = I_n \sqrt{ \frac{(2l+1)}{[l(l+1)](r/R_s)^2} } \sin \theta_n P_l^m(\cos \theta_n),$$

(4.15b)

and $I_n$ is the current flowing in the $n$th loop. The form of Eq. (4.4) allows the following:

$$U_{n,l,m} = u_y I_{n,l} \frac{\sqrt{\pi}}{2}(-\delta_{m,1} + \delta_{m,-1}),$$

(4.16a)

$$V_{n,l,m} = -u_x I_{n,l} \frac{\sqrt{\pi}}{2}(\delta_{m,1} + \delta_{m,-1}).$$

(4.16b)

It is now merely a matter of substituting Eqs. (4.16) into Eqs. (4.12) and plodding through the various steps. Passing lightly over the finer algebraic details, the following points are
noted: (i) only two terms that denote the transverse (i.e., \( x \) and \( y \)) components survive the triple vector products in Eq. (4.12a), (ii) however, these terms do not survive the summation over the index \( m \) due to the presence of factors such as \((-\delta_{m,1} + \delta_{m,-1})(\pm \delta_{m-1,1} + \delta_{m+1,-1})\), and (iii) the two terms that survive the cross products and the summation over \( m \) are along the \( z \)-axis. The net Lorentz force is directed along the positive axis of the stack, and is given by

\[
\mathbf{F}_s = u_c \pi \mu_0 \sum_n \sum_{n'} \sum_{l=1}^{\infty} g_l(q_n) \delta_{\omega_n, \omega_{n'}} \frac{2l+1}{l+1} I_n I_{n'} \sin \theta_n \sin \theta_{n'} P^l(\cos \theta_n) P^{l+1}_{l}(\cos \theta_{n'}) \frac{R_{n'}}{r_{n'}^l r_{n'}^{l+1}}.
\]

(4.17)

Brisley and Thornton (1963) obtain an identical relation, albeit in a slightly different notation by calculating the force exerted on the individual loops by the eddy current field outside the sphere. Also, note that the factor \( \delta_{\omega_n, \omega_{n'}} \) in Eq. (4.17) ensures that there is no time averaged interaction between currents at different frequencies. This fact is also significant in the determination of the pressure distribution on the surface of the sphere.

It is convenient to nondimensionalize Eq. (4.17) by choosing the least radius of the stack of loops as the length scale \( a_0 = (a_1) \) and \( I_o \) as the current scale. With reference to Fig. 3.2, Eq. (4.17) becomes,

\[
\frac{\mathbf{F}_s}{\mu_0 I_o^2} = u_c \pi \sum_n \sum_{n'} \sum_{l=1}^{\infty} g_l(q_n) \frac{2l+1}{l+1} \bar{I}_n \bar{I}_{n'} \sin \theta_n \sin \theta_{n'} P^l(\cos \theta_n) P^{l+1}_{l}(\cos \theta_{n'}) \frac{\bar{R}^{2l+1}}{r_{n'}^l r_{n'}^{l+1}},
\]

(4.18a)

where

\[
\bar{r}_n = \sqrt{(\bar{z}_o - \gamma c_n)^2 + (\gamma c_n \tan \alpha + 1)^2},
\]

(4.18b)

\[
\tan \theta_n = \frac{\gamma c_n \tan \alpha + 1}{\bar{z}_o - \gamma c_n},
\]

(4.18c)

and \( \bar{I}_n = I_n/I_o, \bar{r}_n = r_n/r_o, c_n = (n-1)/(N-1), \bar{z}_o = z_o/a_o, \gamma = h/a_o, \bar{R}_s = R_s/a_o \). Note that \( \bar{z}_o \) is the scaled height of the center of the sphere from the center of the bottom loop in Fig. 3.2. For the special case of a single loop (i.e., \( N = 1 \)), put \( \gamma = 0 \) and \( n = 1 \) in Eq. (4.18) to obtain
Equations (4.18) and (4.19a) are reasonable, since they dictate that the force on the sphere should vanish at points far removed from the coil. Further, it is easy to check that the expression for the force in Eq. (4.19a) is an odd function of the position of the sphere with respect to the center of the loop, i.e.,

\[ F_s|_{\theta_1} = -F_s|_{\pi - \theta_1}. \]  

(4.19b)

From a physical standpoint, this is expected since a diamagnetic body always tends to move to a region of weaker field strength.

### 4.5 The Lorentz force according to the homogeneous model:

As mentioned earlier, the homogeneous model proceeds by assuming that the sphere is placed in a uniform and unidirectional external magnetic field. Let this magnetic field be given by \( B_e = \mathbf{B} \cos \omega t = u_x \mathbf{B} \cos \omega t \). Based on this assumption Rony (1964) shows that the Lorentz force on the sphere is given by

\[ F_s = -\frac{2\pi R^3}{\mu_o} G(q)(\mathbf{B} \cdot \nabla) \mathbf{B}, \]  

(4.20a)

where

\[ G(q) = 1 - \frac{3}{2q} \frac{\sinh(2q) + \sin(2q)}{\cosh(2q) - \cos(2q)}. \]  

(4.20b)

In order to calculate the Lorentz force on the sphere for this configuration of the field by using the present method, the appropriate source function \( I_{n,l,m} \) must be found. From Eqs. (3.23),

\[ I_{n,l,m} = \sqrt{\frac{3\pi}{2}} \left( \frac{B_o R_x}{\mu_o} \right) \delta_{l,1} \left[ (iu_x + u_y) \delta_{m,1} - (-iu_x + u_y) \delta_{m,-1} \right] \]  

(4.21a)

which when coupled with Eq. (4.4) gives

\[ U_{n,l,m} = u_x \sqrt{\frac{3\pi}{2}} \left( \frac{B_o R_x}{\mu_o} \right) \delta_{l,1} \left( \delta_{m,1} - \delta_{m,-1} \right) \]  

(4.22a)
\[ \mathbf{V}_{n,l,m} = u_y \sqrt{\frac{3\pi}{2}} \left( \frac{B_r R}{\mu_0} \right) \delta_{l,1} \left( \delta_{m,1} + \delta_{m,-1} \right) \]  

(4.22b)

These equations may be substituted in the general force equation, Eq. (4.12). The principal steps are essentially similar to the axisymmetric case of the previous section, except that all the terms in the force expression vanish. None of them survives summation over the index \( l \) due to the presence of the factor \( \delta_{l,1} \delta_{l+1,1} \). Therefore, the present method when applied to the sphere in the homogeneous unidirectional field implies that the Lorentz force is zero. Clearly, this result does not agree with Eq. (4.20a). It must however be anticipated, since paradoxically enough Eq. (4.20a) predicts that the Lorentz force vanishes in the absence of a field gradient. In other words, the homogeneous model begins by assuming a homogeneous field and derives a nonhomogeneous field as a precondition for a non-zero Lorentz force! This is not surprising since the homogeneous model calculates the force on the sphere by assuming it to be a small dipole that is placed in a nonhomogeneous field.* From a physical point of view, this result may be interpreted as follows. Since diamagnetic bodies tend to move to regions of weaker field strength in a nonhomogeneous field, they might be expected to not move at all in a field that is uniform (i.e. a zero gradient field).

In section (ii) of Appendix C, it is shown that for a very small sphere placed on the axis of a circular loop, the homogeneous model overestimates the Lorentz force by as much as 33\%. This assertion must be tested experimentally. Fromm and Jehn (1965) measure the forces on small copper spheres placed in the field of a circular loop. However, the radius of the smallest sphere that they use is only a fourth of the radius of the circular loop. The result of Appendix C is very likely valid for the limiting case of a small sphere. It suggests that the forces on smaller spheres (a tenth of the loop radius and smaller) might be overestimated by Eqs. (4.20). These results will be of interest to workers in the area of powder production.

* The force on a dipole of moment \( \mathbf{M} \) due to a magnetic field \( \mathbf{B} \) is \( (\mathbf{M}, \nabla)\mathbf{B} \).
It is worthwhile to consider the limiting case of a sphere with a very small radius, i.e., $R_s \to 0$. The source function for such a sphere placed in an arbitrary magnetic field is given by Eq. (3.21) as

$$I_{n,l,m} = \frac{2\sqrt{\pi}}{\mu_0} A_n(r_0, u_0, \phi_0) \delta_{l,0} \delta_{m,0}. \quad (4.23)$$

where $(r_0, u_0, \phi_0)$ are the coordinates of the center of the sphere. Since the source function vanishes identically for all values of $l = m \neq 0$, Eq. (4.12) indicates a net zero Lorentz force on the sphere. This is an expected result since a magnetic field cannot exert a force on a zero-radius sphere. Also, in the case of an extremely small sphere, the magnetic field is essentially uniform over its volume. The sphere is not large enough to sense the nonhomogeneity of the magnetic field and hence does not experience a force.

### 4.6 Results and discussion:

The results in the paper by Brisley and Thornton (1963) provide a benchmark to compare the results obtained here. They present an order of magnitude calculation for a 0.5 cm radius copper sphere ($\sigma_s = 2.5 \times 10^7 (\Omega \text{ m})^{-1}$) placed 1 cm from the center of a loop of radius 1 cm. The loop carries a current of 1320 A at 400 kHz. This current is sufficient to balance the weight of the sphere (45.72 mN). Using these values in Eq. (4.19a) yields a value of 46.2 mN.

Figure 4.2 shows the variation of the magnitude of the Lorentz force along the axis of a single loop for three different values of the radius to skin depth ratio. The figure indicates that the force is a very weak function of $q$, for large values. Also, note that the case of $q = 31.4$ corresponds to the case shown in Fig. 4 of Brisley and Thornton (1963). The difference in the scaled values along the force axis arises from the present choice of rationalized MKS units. When the values of the forces shown in Fig. 4 are multiplied by a conversion factor of $4\pi$ (to convert from rationalized MKS to electromagnetic units), the curve reported by Brisley and Thornton (1963) is recovered exactly. Recall that Eq. (4.20)
Figure 4.2 Force on a sphere on the axis of a single loop.

predicts that the Lorentz force on a sphere varies as the product of the field and the field gradient. The magnetic field due to a circular loop has only an axial component on its axis, and is given by Eq. (C8) of Appendix C. The maximum of the product of the function in Eq. (C1) and its derivative occurs at $z_0 = 0.378$, which is close to the point at which the force shown in Fig. 4.2 peaks. Figure 4.2 also implies that the force does not vary appreciably with the frequency. For example, when the skin depth parameter increases by a factor of 2 (i.e. the frequency increases by a factor 4), there is no appreciable change in the force on the sphere. This is due to the asymptotic behavior of the skin depth function $g_1(q_s)$ at high frequency (as borne out by Fig. 4.1).

The top half of Fig. 4.3 shows the variation of the force along the axis of a right circular stack with two and five loops. The effect of the individual loops is clear. There are as many peaks as there are loops. In the case of only two loops, the loops are situated at
Figure 4.3 Force on a sphere for the geometry of Fig. 3.2.

$z_o / h = 0$ and $z_o / h = 1$. Then the field in between the two loops is relatively gradient free, and thus the force is close to zero. As in the case of the single loop, the peak occurs at a point just above the top loop. When the number of turns is increased to five (5), there is significant variation of the field over the length of the stack, and this is reflected in the force. In fact, when the number of loops per unit length is large, the variations die out completely, and the force profile becomes smooth within the coil (Bayazitoglu and Sathuvalli, 1993). The bottom half of Fig. 4.3 shows the same results for a conical stack of $150$ semiangle. Due to the increasing diameter of the loops with height, the field and the
field gradient become smaller with increasing distance from the bottom of the stack. Consequently, the effect of the individual loops is not seen. The effect of a non-zero semiangle is to reduce the average magnitude of the force.

4.7 Conclusions:

A method of calculating the Lorentz forces on a sphere has been presented. The method relies on expressing the induced eddy current density and the magnetic flux density of the external field in terms of certain "source functions" of the current sources that create the field. Once the "source functions" for a given configuration, are known purely from magnetostatic analysis, the force on the sphere may be evaluated.
5.1 Introduction:

As we noted earlier, the technique of electromagnetic levitation melting has been used for the measurement of thermophysical properties of high temperature liquid metals and reactive materials.

When an electrically conducting sample is allowed to melt during levitation, it melts after a short time (~100 s). The dynamics of the levitated droplet are governed by the balance between the hydrostatic, viscous, surface tension, and electromagnetic forces. At very high frequencies, the induced eddy currents are confined to the 'skin depth' and the Lorentz force does not act on the bulk of the fluid. Instead, it manifests as an effective magnetic pressure on the surface of the droplet. The details of how such an external pressure affects the dynamics of the droplets are given by Suryanarayana and Bayazitoglu (1991). In addition, they provide the 'inverse' theory to determine the surface tension from the frequency of the droplet oscillations.

In theory, the surface tension of the liquid may be found by knowing the frequency of the droplet oscillations (Lamb, 1945). However, this theory assumes that the droplet is spherical and inviscid. It predicts that the surface tension $\tau$ is related to the natural frequency $\omega_i^*$, by

$$\omega_i^* = \frac{\tau(l-1)(l+2)}{\rho R_s^3}$$

(5.1)

where $l$ denotes the mode of oscillation. Equation (5.1) implies that there is only one peak in the frequency spectrum of the droplet. However, experimental studies have revealed otherwise (Trinh, 1988, Krishnan et al., 1988, Egry et al., 1992). Suryanarayana and Bayazitoglu (1991) explain the occurrence of these peaks by assuming that the droplet is slightly deformed. First, they calculate the extent of this deformation in terms of an externally acting force (for example an acoustic or magnetic pressure) and then relate it to
the modal frequencies of oscillation of the droplet. The natural inviscid frequency given by Eq. (5.1) is the average of the modal frequencies. The surface tension of the liquid can be then found. Therefore, it is necessary to determine the distribution of the magnetic pressure on the surface of the droplet.

Secondly, it is well-known that at very high frequencies, due to the 'skin depth' effect, the local magnetic pressure on the surface of the droplet is almost entirely tangential to it. This gives birth to the problem of the "magnetic hole" at the bottom-most point of the levitating droplet, where the hydrostatic pressure is balanced only by the surface tension of the liquid. By knowing the magnetic pressure distribution on the droplet surface in terms of the "source functions" of the external sources, a criterion to design a "good coil (or set of coils)" which does (do) not have the "magnetic hole" could be generated. It is hoped that the pressure distribution on the surface of the sphere as calculated here will be useful in solving both these problems.

While there are several works that consider the problem of a body of liquid metal in a time dependent magnetic field, almost none of them can be used to calculate the pressure distribution on a spherical droplet in terms of the external magnetic. Sneyd and Moffatt (1982) solve the magnetohydrodynamic problem of a circular torus and obtain the pressure distribution at a cross section of the torus. Cummings and Blackburn (1992), in their study of the oscillations of a magnetically levitated droplet, obtain the pressure distribution by assuming a 'linear' magnetic field. However, their analysis involves several questionable assumptions (Egry et al., 1992, 1994) such as the assumption of a linear magnetic field, etc.

In this light, this chapter extends the method developed in Chapter 4 to determine the distribution of the magnetic pressure on the surface of a spherical levitated liquid metal droplet.
5.2. The magnetic pressure on the surface of a levitated droplet:

The instantaneous Lorentz force per unit volume on an electrically conducting sphere placed in an alternating magnetic field is given by

\[ d\mathbf{F}_s(r,u,\phi,t) = \mathbf{J}_s(r,u,\phi,t) \times \mathbf{B}_e(r,u,\phi,t). \] (5.2a)

Here, the instantaneous Lorentz force per unit volume as given by Eq. (5.2a), is the magnetic pressure gradient \( \nabla p_{mag} \), in the liquid droplet. So,

\[ \nabla p_{mag}(r,u,\phi,t) = \mathbf{J}_s(r,u,\phi,t) \times \mathbf{B}_e(r,u,\phi,t). \] (5.2b)

Therefore, the pressure distribution (actually, the difference between the pressure inside and outside) at the surface of the sphere is given by

\[ \Delta p_{mag}(u,\phi,t) = \int_{r=0}^{R} \nabla p_{mag}(r,u,\phi,t).dr. \] (5.2c)

The dynamic response of the droplet is constrained by its surface tension. The droplet cannot respond to the variations of the pressure on a time scale \( \sim \omega_n^{-1} \). Instead, it responds to the time averaged magnetic pressure. Hence, it is useful to consider only time averaged quantities. The time averaged pressure distribution on the sphere surface, over the period \( T \) is

\[ \Delta \bar{p}_{mag}(u,\phi) = \frac{1}{T} \int_{t=0}^{T} \int_{r=0}^{R} \left\{ \mathbf{J}_s(r,u,\phi,t) \times \mathbf{B}_e(r,u,\phi,t) \right\}.dr.dt. \] (5.2d)

As in the case of the net Lorentz force, once the induced current density \( \mathbf{J}_s \) and the external magnetic field are known, the pressure distribution can be found from Eq. (5.2d).

When the sphere is placed in an arbitrary but sinusoidally alternating magnetic field that is created by \( N \) current sources whose current densities are given by \( \mathbf{J}_n(r,u,\phi) \cos \omega_n t, n = 1,2,...,N \), the induced current density \( \mathbf{J}_s \) is given by Eq. (3.6). The magnetic field \( \mathbf{B}_e \) which is obtained by taking the curl of Eq. (4.2b) is given by

\[ \mathbf{B}_e(r,u,\phi,t) = \mu_o \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{R_s^{-l}}{2l+1} \nabla \left\{ r^l Y_l^m(\phi) \right\} \times \mathbf{I}_{n,l,m}^* \cos(\omega_n t). \] (5.3)

Equations (3.6) and (5.3) can, in principle, be used to find \( \Delta \bar{p}_{mag} \) on the surface of the sphere via Eq. (5.2d). However, the presence of the terms with subscript \( k \) does not
allow closed form integration of Eq. (5.2d). Therefore, we sum the series in $k$ in Eq. (3.6) (as shown in Appendix F, Eqs. (F1) through (F6)) and rewrite the current density as

$$J_s(r,u,\phi,t) = \frac{-4}{R_s^{3/2}} \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} q_n^2 \Psi_i(z, r/R_s) e^{i(\omega_n t + \pi/2)} \text{Re} \left[ Y_l^m(u, \phi) \right],$$

where for real values of $s$,

$$\Psi_i(z, s) = \frac{1}{2z} \frac{I_{i-1/2}(zs)}{I_{i+1/2}(z)},$$

$$z = (1+i)q_n.$$

A further simplification of Eqs. (5.4) results from the fact that in the case of electromagnetic levitation, the frequencies involved are very high (~100 kHz). Consequently, the skin depth is small (~$10^{-4}$ m for diamagnetic metals) and $q_n$ can usually be considered very large (~50 for a 0.5 cm radius copper sphere). The modified Bessel functions can therefore approximated by their asymptotic values (for large arguments) by using (Abramowitz and Stegun, 1965)

$$\lim_{z \to \infty} I_{i\pm 1/2}(z) = \frac{e^z}{\sqrt{2\pi z}}$$

to yield the following approximation for the current density induced in a sphere by a high frequency magnetic field,

$$J_s(r,u,\phi,t) \approx \frac{-\sqrt{2}}{R_s} \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} q_n I_{n,l,m} f_n(r,t) Y_l^m(u, \phi),$$

where

$$f_n(r,t) = \frac{1}{r} e^{-q_n \left( \frac{r}{R_s} \right)} \cos \left[ q_n \left( \frac{1 - \frac{r}{R_s}}{2} \right) + \alpha_n t - \frac{\pi}{4} \right].$$

Note that Eqs. (5.5) predict an exponentially decaying current density towards the center of the sphere, which is physically reasonable since the induced eddy currents are confined to the surface of the conductor (i.e., a few skin depths) at high frequencies. If the source function $I_{n,l,m}$ is known, the current density in this limit can be found. For example, for the case of a sphere that is placed in a uniform unidirectional external magnetic field, say $B_e(r,u,\phi,t) = u_z B \cos \omega t$ where $u_z$ is the unit cartesian vector along the $z$-axis, the
source function $I_{n,l,m}$ is given by Eqs. (3.23). By substituting Eqs. (3.23) in Eq. (5.5a), the current density in the sphere placed in such a field can be shown to be

$$J_s(r,\theta,\phi,t) = u_0 \frac{B_0 \sin \theta}{\sqrt{2}} \frac{1}{r} \frac{1}{\mu_0} e^{-q \left( \frac{1-r}{R} \right)} \left[ -i q \left( \frac{1-r}{R} \right) - \alpha \right] ^{-3} \left[ -i q \left( \frac{1-r}{R} \right) - \alpha \right] ^{-\frac{3\pi}{4}}. \tag{5.6}$$

Equation (5.6) matches exactly with Eq. (9.22) of Van Bladel (1964) who obtains this quantity by using a different approach. Thus, Eq. (5.6) serves as a check for the validity for Eqs. (5.5).

It now remains to find the pressure distribution from Eqs. (5.5), (5.3) and (5.2d). The typical electromagnetic levitator may be approximately modeled by a sphere placed along the axis of a stack of coaxial loops, when the pitch of the coil winding is small (see Chapter 6), as shown in Fig. 3.2. The source function for this case is given by Eq. (3.14). Substituting Eq. (3.14) in Eq. (5.5a), the induced eddy current density in the sphere for the axisymmetric situation (of Fig. 3.2) is obtained as

$$J_s(r,\theta,\phi,t) = u_x J_{s,x}(r,\theta,\phi,t) + u_y J_{s,y}(r,\theta,\phi,t) \tag{5.7a}$$

where

$$J_{s,x}(r,\theta,\phi,t) = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} q_n \Theta_{n,l} f_n(r,t) P_l^1(u) \sin \phi, \tag{5.7b}$$

$$J_{s,y}(r,\theta,\phi,t) = \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} q_n \Theta_{n,l} f_n(r,t) P_l^1(u) \cos \phi, \tag{5.7c}$$

$$\Theta_{n,l} = \frac{1}{\sqrt{2} R} \frac{2l+1}{l+1} I_n \left( \frac{R}{r_n} \right) ^l \sin \theta_n P_l^1(\cos \theta_n), \tag{5.7d}$$

and $f_n(r,t)$ is defined in Eq. (5.5b). The magnetic field $B_e(r,\theta,\phi,t)$ for the configuration in Fig. 3.2 can be found either by using Eq. (3.14) in Eq. (5.3) (Sathuvalli and Bayazitoglu, 1994) or as given in Smythe (1989). The external magnetic field, with reference to Fig. 3.2, finally reduces to

$$B_e(r,\theta,\phi,t) = u_x B_{e,x} + u_y B_{e,y} + u_z B_{e,z} \tag{5.8a}$$

where

$$B_{e,x} = -\sum_{n=1}^{N} \sum_{l=1}^{\infty} Q_{n,l} r_{l-1} P_{l-1}^1(u) \cos \phi \cos \omega_n t, \tag{5.8b}$$
\begin{align*}
B_{e,y} &= - \sum_{n=1}^{N} \sum_{l=1}^{\infty} Q_{n,l} r^{l-1} P_{l-1}^{1}(u) \sin \phi \cos \omega_n t, \\
B_{e,z} &= \sum_{n=1}^{N} \sum_{l=1}^{m} Q_{n,l} r^{l-1} P_{l-1}^{1}(u) \cos \omega_n t, \\
Q_{n,t} &= \frac{\mu_0 I_n}{2} \sin \theta_n \frac{1}{l} r_n' P_{l}^{1}(\cos \theta_n).
\end{align*}

Eq. (5.2d) may be shown to reduce to

\[ \Delta \bar{p}_{mag}(u, \phi) = -\frac{1}{2} \sum_{n=1}^{N} \sum_{n'=1}^{N} \sum_{l=1}^{\infty} \frac{\delta_{\omega_n, \omega_{n'}}}{l} \Theta_{n,l} Q_{n,l} q_n(R_s, q_n) \]
\[ \left[ l P_{l}^{1}(u) P_{l-1}^{1}(u) \sin \theta + P_{l}^{1}(u) P_{l}^{1}(u) \cos \theta \right], \]

where

\[ I_n(R_s, q_n) = \int_{r=0}^{R_s} r^{l-2} e^{-q_n \left( \frac{1-r}{R_s} \right)} \cos \left[ q_n \left( 1 - \frac{r}{R_s} \right) - \frac{\pi}{4} \right] dr. \]

The function \( I_n(R_s, q_n) \) may be imagined to be the "skin depth" function for the pressure distribution, in analogy with the functions \( H_1(q_n) \) and \( g_1(q_n) \) for the power and force respectively. Note that Eq. (5.10) is the result of time averaging (as indicated in Eq. (5.2d)) over the time period \( T = \omega_n/2\pi \). The expression in the square parentheses in the expression for \( \Delta \bar{p}_{mag}(u, \phi) \) can be simplified by using the recursion relations for associated Legendre functions. This simplification leads to

\[ \Delta \bar{p}_{mag}(u, \phi) = -\frac{\mu_0}{8} \sum_{n=1}^{N} \sum_{n'=1}^{N} \sum_{l=1}^{\infty} \frac{\delta_{\omega_n, \omega_{n'}}}{l} \frac{1}{l(l+1)} \left[ \sqrt{2} q_n I_n(R_s, q_n) \right] \]
\[ L_n I_n \sin \theta_n \sin \theta_n P_{l}^{1}(\cos \theta_n) P_{l}^{1}(\cos \theta_n) \frac{R_s^{l-1}}{r_n'^{l-1} r_n'^{l-1}} P_{l}^{1}(u) P_{l}^{1}(u). \]

The term \( \sqrt{2} q_n I_n(R_s, q_n) \) is evaluated in Appendix F (Eqs. (F16) through (F19)). It is shown that when \( q_n \) is large,

\[ q_n \to \infty \left\{ \sqrt{2} q_n I_n(R_s, q_n) \right\} = R_s^{l-1}, \quad l = 1, 2, \ldots. \]
Together with Eq. (5.11b), Eq. (5.11a) implies that the pressure distribution is a) axisymmetric (as expected), b) purely a function of the relative position of the sphere with respect to the external coil, and c) independent of the frequency at large frequencies* and given by

\[
\Delta P_{\text{mag}}(u, \phi) = -\frac{\mu_0}{8} \sum_{n=1}^{N} \sum_{n'=1}^{N} \sum_{l=1}^{\infty} \sum_{l'=1}^{\infty} \delta_{\omega_n, \omega_{n'}} \frac{1}{l} \frac{2l+1}{l(l+1)} I_n I_n' \sin \theta_n \sin \theta_{n'} P_l^1(\cos \theta_n) P_{l'}^1(\cos \theta_{n'}) \frac{R_{n'}^{l-1} R_{n}^{l'-1}}{r_{n'}^{l} r_{n}^{l'}} P_l^1(u) P_{l'}^1(u).
\]

(5.12a)

As in the case of the force, there is no time averaged interaction of the currents at different frequencies. The above equation can be scaled by using the least radius of the stack of loops, \(a_o (= a_i)\) as the length scale and \(I_o\) as the current scale. With reference to Fig. 3.2, Eq. (5.12a) becomes

\[
\Delta P_{\text{mag}}(u, \phi) = -\left(\frac{\mu_0 I_o^2}{a_o^2}\right) \frac{1}{8} \sum_{n=1}^{N} \sum_{n'=1}^{N} \sum_{l=1}^{\infty} \sum_{l'=1}^{\infty} \delta_{\omega_n, \omega_{n'}} \frac{1}{l} \frac{2l+1}{l(l+1)} \bar{I}_n \bar{I}_n' \sin \theta_n \sin \theta_{n'} P_l^1(\cos \theta_n) P_{l'}^1(\cos \theta_{n'}) \frac{\bar{R}_{n'}^{l-1} \bar{R}_{n}^{l'-1}}{\bar{r}_{n'}^{l} \bar{r}_{n}^{l'}} P_l^1(u) P_{l'}^1(u),
\]

(5.12b)

where

\[
\bar{r}_n = \sqrt{(\bar{z}_o - \gamma c_n)^2 + (\gamma c_n \tan \alpha + 1)^2},
\]

(5.12c)

\[
\tan \theta_n = \frac{\gamma c_n \tan \alpha + 1}{\bar{z}_o - \gamma c_n},
\]

(5.12d)

and \(\bar{I}_n = I_n/I_o\), \(\bar{r}_n = r_n/r_o\), \(c_n = (n-1)/(N-1)\), \(\bar{z}_o = z_o/a_o\), \(\gamma = h/a_o\), \(\bar{R}_n = R_n/a_o\). Figure 5.1 shows the distribution of the pressure on the surface of a sphere placed along the axis of a stack of loops shown in Fig. 3.2. The pressure profiles are axisymmetric about the \(\theta = 0\) axis of the sphere. The curve to the left of the \(\theta = 0\) axis shows the pressure distribution along the coil in the absence of a reverse wound turn (the capping coil). The

* Physically, we might expect this independence with respect to frequency because of the skin depth phenomenon. Also, mathematically, the behavior of \(g_1(q_s)\) at large values of \(q_s\) forebodes this independence.
Figure 5.1 Pressure on the surface of a sphere on the axis of the stack of loops of Fig. 3.2.

The curve to the right of the $\theta = 0$ axis shows the corresponding distribution when the stack of loops is topped by a reverse wound turn carrying a current in the opposite direction. The reverse turn is assumed to be concentric with the stack, has a radius $\bar{a}_c = a_c/a_o = 1.5$ and is situated at a height $\bar{h}_c = h_c/a_o = 1.2\gamma$ above the center of the bottom most loop in the stack. It carries a scaled current $\bar{I}_c = I_c/I_o = -1.0$. The figure confirms that pressure at the top and bottom of the sphere vanishes, as suggested by Eq. (5.12b), since

$$P_l^m(\pm 1) = 0, \ l = 1, 2, ..., \ m \neq 0.$$ 

Also, the reverse turn tends to the bulge in the lower hemisphere of the sample, thus indicating a greater net upward force on the sample.
It is interesting to note that the corresponding pressure for the case of a sphere in a homogeneous field can be obtained by using Eq. (5.6) for the current density and following a similar procedure. The pressure distribution is then,

\[ \Delta p_{\text{mag}}(u, \phi) = -\frac{3B^2\sin^2 \theta}{2\mu_o} \left\{ \sqrt{2qI_1(R_s, q)} \right\}. \]  

(5.13a)

At high frequencies, by virtue of Eq. (5.11b), Eq. (5.13a) becomes

\[ \Delta p_{\text{mag}}(u, \phi) = -\frac{3B^2\sin^2 \theta}{2\mu_o}. \]  

(5.13b)

Note that the pressure distribution in Eqs. (5.13) is independent of the polar angle \( \phi \) implying axisymmetry. According to this analysis, the unit normal to the surface of the sphere is positive outward. Therefore, the negative sign in the expressions (Eqs. (5.12) and (5.13)) for the pressure on the sphere indicates that the magnetic pressure acts towards the center of the sphere, which is a physically reasonable implication, if the sphere must levitate. Finally, the form of Eq. (5.13) suggests a symmetric profile for the pressure distribution, which in turn suggests a net zero force on the sphere. This is consistent with the result of section 4.5.

In the case of a sphere placed in a arbitrary field, \( \Delta p_{\text{mag}} \) can be found by a similar method. However, it must be remembered that the quantities \( J_s \) and \( B_e \) are real physical quantities. Therefore, only the real parts of Eqs. (5.5a) and (5.3) must be used in evaluating \( \Delta p_{\text{mag}} \) (as given by Eq. (5.2c). The actual algebraic details for the case of the sphere in an arbitrary field are uninstructive and only the final steps in the derivation are shown here.

Rewrite the spherical harmonics \( Y_l^m(u, \phi) \) in terms of their real and imaginary parts as

\[ Y_l^m(u, \phi) = (-1)^m \sqrt{\frac{2l+1(l-m)!}{4\pi (l+m)!}} \left[ Y_l^m(u, \phi) + iY_l^0(u, \phi) \right] \]  

(5.14)

where \( Y_l^r \) and \( Y_l^0 \) are defined in Eqs. (4.3). By using Eqs. (5.14) and (4.4) in Eq. (5.5a), we obtain
\[
\text{Re}\left[J_s(r,u,\phi,t)\right] = \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e_{i,m} f_{n}(r,t) \left[U_{n,l,m} Y_{l,m}^e(u,\phi) - V_{n,l,m} Y_{l,m}^0(u,\phi)\right]
\] (5.15a)

where
\[
e_{i,m} = (-1)^{m+1} \frac{\sqrt{2}}{R_s} \frac{\sqrt{2l+1} (l-m)!}{4\pi (l+m)!}.
\] (5.15b)

The expression for the real part of the magnetic field is given as before by Eq. (4.7). The substitution of Eqs. (5.15) and (4.7) in Eq. (5.2c) gives,
\[
\Delta \overline{p}_{\text{mag}}(u,\phi) = \frac{\mu_0}{2\sqrt{2}} \sum_{n,l,m} \delta_{\omega_e, \omega_e} e_{i,m} D_{l,m} \left\{\sqrt{2} q_n I_l(R_s, q_n) \right\}
\]
\[
\left[\mathbf{u}_r \{\mathbf{x}_1 - \mathbf{x}_n - \mathbf{x}_{III} + \mathbf{x}_{IV}\} \right]
\] (5.16a)

where
\[
\sum_{n,l,m} \rightarrow \sum_{n=1}^{N} \sum_{l=0}^{\infty} \sum_{m=-l}^{l}
\]

and
\[
\mathbf{x}_1 = U_{n,l,m} Y_{l,m}^e(u,\phi) \times \left\{\nabla \left(r Y_{l,m}^e(u,\phi)\right) \times \mathbf{U}_{n',l',m'}\right\},
\] (5.16b)
\[
\mathbf{x}_n = V_{n,l,m} Y_{l,m}^0(u,\phi) \times \left\{\nabla \left(r Y_{l,m}^0(u,\phi)\right) \times \mathbf{U}_{n',l',m'}\right\},
\] (5.16c)
\[
\mathbf{x}_{III} = U_{n,l,m} Y_{l,m}^e(u,\phi) \times \left\{\nabla \left(r Y_{l,m}^0(u,\phi)\right) \times \mathbf{V}_{n',l',m'}\right\},
\] (5.16d)
\[
\mathbf{x}_{IV} = V_{n,l,m} Y_{l,m}^0(u,\phi) \times \left\{\nabla \left(r Y_{l,m}^0(u,\phi)\right) \times \mathbf{V}_{n',l',m'}\right\}.
\] (5.16e)

Equations (5.16) require that we know the gradients of the functions of the type \(\nabla (r Y_{l,m}^{e,0}(u,\phi))\) and these are calculated Appendix E. Equation (5.16a) expresses the pressure distribution on the surface of the sphere, when it is placed in an arbitrary magnetic field. Once the source functions for the external field are found (by the methods given Chapter 3), it is merely a matter of substituting them in Eq. (5.16a) in order to obtain the pressure distribution.

5.3 The frequencies of oscillation due to the magnetic pressure:

The frequencies of oscillation in mode \(l\) of an electromagnetically levitated droplet have been shown to be (Suryanarayana, 1991, Suryanarayana and Bayazitoglu, 1991)
\[
\omega_{l,m}^2 = \omega_i^2 + \frac{1}{\rho R_i^2} \sum_{l'=-\infty}^{\infty} \sum_{m'=-l'}^{l'} p_{l',m'} \left( \frac{3l(l-1)(l+2)}{(l'-1)(l'+2)} + \frac{2ll'(l'+1)}{(l'-1)(l'+2)} \right) \tilde{Z}_{l,l'}^{m,0,m}, \tag{5.17}
\]

where
\[
\tilde{Z}_{l',l,l'}^{m',m,m} = \int_{-1}^{1} \int_{0}^{2\pi} Y_{l'}^{m'}(u,\phi) Y_l^m(u,\phi) Y_l^m(u,\phi) d\phi du \tag{5.18}
\]

and \( p_{l',m'} \) are the coefficients in the spherical harmonic expansion of the electromagnetic pressure on the surface of the droplet, and \( \omega_i^* \) is the inviscid Rayleigh frequency given in Eq. (5.1). These coefficients are defined by assuming that the magnetic pressure on the droplet can be expanded in a series of spherical harmonics as
\[
\Delta \tilde{p}_{\text{mag}}(u,\phi) = \sum_{l'=0}^{\infty} \sum_{m'=-l'}^{l'} p_{l',m'} Y_{l'}^m(u,\phi). \tag{5.19a}
\]

By using the orthogonality of spherical harmonics, the coefficients \( p_{l',m'} \) may be shown to be
\[
p_{l',m'} = \int_{-1}^{1} \Delta \tilde{p}_{\text{mag}}(u,\phi) Y_{l'}^{m'}(u,\phi) d\phi du. \tag{5.19b}
\]

For the axisymmetric case of a sphere lying on the axis of a stack of loops (Fig. 3.2), we may substitute for \( \Delta \tilde{p}_{\text{mag}}(u,\phi) \) from Eq. (5.12) in Eq. (5.19b), and obtain the coefficients \( p_{l',m'} \). This leads to
\[
p_{l',m'} = \frac{-H_o}{8} \sum_{n=1}^{N} \sum_{n'=1}^{N} \sum_{l=1}^{\infty} \sum_{l'=1}^{\infty} \delta_{n,n'} \omega_{n,\omega_{n'}} \frac{1}{l(l+1)} \frac{2l+1}{l} I_n I_{n'} \sin \theta_n \sin \theta_{n'} P_l^1(\cos \theta_n) P_l^1(\cos \theta_{n'}) \frac{R_s^{l-1} R_s^{l'-1}}{R_i R_i'} \int_{-1}^{1} \int_{0}^{2\pi} P_l^1(u) P_l^1(u) Y_{l'}^{m'}(u,\phi) d\phi du.
\]

The integral on the right hand side of the above equation can be shown to exist only when \( m' = 0 \) (see Appendix G). By using Eq. (G4) of Appendix G, the above expression for \( p_{l',m'} \) reduces to

* The summation in Eq. (5.17) does not include \( l' = 0 \) and \( l' = 1 \). The condition \( l' = 0 \) describes a change in the radius of the droplet. Therefore, for an incompressible fluid, \( l' = 0 \) is not allowed. The \( l' = 1 \) mode represents the translation of the droplet from its position of equilibrium, and therefore does not correspond to a shape oscillation of the sphere. Since, Eq. (5.17) represents the frequencies of shape oscillations, the exclusion of the \( l' = 0 \) and \( l' = 1 \) is valid (Cummings and Blackburn, 1991).
where

\[ p_{r',0} = \left( \frac{\mu_o I_o^2}{a_o^2} \right) \frac{\pi}{2} \sum_{l=1}^{\infty} \sum_{l'=1}^{\infty} h_{l,l'} \sqrt{\frac{2l+1\ell+1}{2l'+1l+1}} Z_{l',l}^{0,-1,1} \]  

(5.21a)

The variables on the right hand side of the above equation are defined in Eqs. (5.12c) and (5.12d). Note that the coefficients \( p_{r',m'} \), as given by Eqs. (5.21) are purely functions of the relative position of the sphere with respect to the coil and the coil geometry. Once the coil geometry is known, the \( p_{r',m'} \) can be found and the frequencies determined from Eqs. (5.17).

The formulae for the \( p_{r',m'} \) are infinite series in \( l \) and \( l' \). For numerical evaluation, the series must be truncated at some point. It can be seen that the functions denoted by \( h_l \) decrease very rapidly with increasing \( l \). Therefore, it is sufficient to consider only terms in \( l \) and \( l' \leq 3 \). With this in mind it can be shown that

\[ p_{0,0} = \left( \frac{\mu_o I_o^2}{a_o^2} \right) \frac{\pi}{2} 0.282095 \left( \tilde{h}_1^2 + \tilde{h}_2^2 + \tilde{h}_3^2 \right), \]

\[ p_{1,0} = \left( \frac{\mu_o I_o^2}{a_o^2} \right) \frac{\pi}{2} (0.437626 \tilde{h}_1 \tilde{h}_2 + 0.467332 \tilde{h}_2 \tilde{h}_3), \]

\[ p_{2,0} = \left( \frac{\mu_o I_o^2}{a_o^2} \right) \frac{\pi}{2} \left( 0.126157 \tilde{h}_1^2 - 0.090119 \tilde{h}_2^2 - 0.126157 \tilde{h}_3^2 - 0.405803 \tilde{h}_h \right), \]

\[ p_{3,0} = \left( \frac{\mu_o I_o^2}{a_o^2} \right) \frac{\pi}{2} \left( 0.286493 \tilde{h}_h \tilde{h}_2 - 0.118977 \tilde{h}_2 \tilde{h}_3 \right), \]

\[ p_{4,0} = \left( \frac{\mu_o I_o^2}{a_o^2} \right) \frac{\pi}{2} \left( 0.30247 \tilde{h}_1 \tilde{h}_2 + 0.161197 \tilde{h}_2^2 - 0.025645 \tilde{h}_3 \right), \]

\[ p_{5,0} = \left( \frac{\mu_o I_o^2}{a_o^2} \right) \frac{\pi}{2} 0.338965 \tilde{h}_h \tilde{h}_3, \]

\[ p_{6,0} = \left( \frac{\mu_o I_o^2}{a_o^2} \right) \frac{\pi}{2} 0.177861 \tilde{h}_3^2. \]

It is now only a matter of substituting the values of these coefficients in order to obtain the frequencies of oscillation from Eq. (5.17). These frequencies can be determined purely in terms of the coefficients \( h_l \), which are functions of the coil geometry alone. In other,
words, given the coil geometry, it is possible to interpret the spectrum of the levitating droplet in terms of externally measurable quantities, and thus obtain its surface tension.

5.4 Conclusions:

In conclusion, a method and an expression for the magnetic pressure distribution has been presented in terms of the geometry of the coil that produces the magnetic field. The usefulness of this expression in interpreting actual experimental data and then determining surface tension of liquid metals remains to be tested. Importantly, this method assumes that the droplet continues to remain spherical after it melts. This is probably a severe assumption, especially in earth bound situations, where the sample is known to assume a shape resembling a pear. The pressure distribution for a perturbed (aspherical) droplet must be determined, for a better analysis of the droplet dynamics.
Chapter 6

The Magnetic Fields of Conical Helices

6.1 Introduction:

In levitation experiments, a suitable excitation coil is very important in order to levitate and stably contain the molten mass of liquid metal. Often, the sample may not levitate at all, or it may exhibit lateral drift and eventually move out of the region of the field and therefore out of the levitator. In order to design a coil that can contain a given mass of molten metal, it is necessary to have an idea of the nature of the field generated by the coil.

The purpose of this chapter is to calculate the magnetic field of an excitation coil used in typical levitation applications. In spite of the number of works that are available about levitation studies, few of them address the calculation of the field produced by the currents flowing in a typical winding. The available works either model the coil as a coaxial stack of loops (Brisley and Thornton, 1963, Fromm and Jehn, 1965, Zong et al., 1992) or only calculate the fields along the axis of symmetry of the winding (Holmes, 1978, Smythe, 1989). The stack model assumes that a coil can be represented by a set of coaxial current carrying circular loops. In this model, due to circular symmetry, the radial component of the magnetic field vanishes identically along the coil axis. However, experiments have indicated that there is a significant radial component along the axis of the excitation coil, sometimes large enough to cause transverse ejection of the sample (from poorly wound coils). This transverse component of the field vanishes only when the coil is closely wound and when there are a large number of turns per unit length of the coil. Ordinarily, the levitation coil consists of a few turns (3 to 5) of copper tubing wound over a few centimeters, plus a reverse wound capping coil (see Fig. 1.1). In such a case, approximating the levitation coil as a stack of coaxial circular loops may not be satisfactory.

In this chapter, the excitation coil is modeled as a conical helix. The helix model gives expressions that can be easily integrated by using an integration routine. First, the
expressions for the fields for a conical helix wound on the surface of a cone with a circular cross section, in cartesian coordinates are developed and the results for the magnetic field due to the current flowing in it are shown. These results are compared with the results for the equivalent stack model. It is shown that the helix model is more realistic for small laboratory coils. Further, the helix model is validated by showing that the helix and stack models give nearly identical results for closely wound coils.

6.2 Calculation of the magnetic field:

Smythe (1989) calculates the field on the axis of a circular helix and Holmes (1978) obtains the equations for the field on the axis of a conical helix. We extend a similar approach to a helix wound on the surface of a cone of circular cross section as shown in Fig. 6.1. The parametric equations of this helix are

\[ x' = a(\theta)\cos \phi \]  
\[ y' = a(\theta)\sin \phi \]  
\[ z' = c\phi \]

(6.1a)  
(6.1b)  
(6.1c)

\( \phi \) is the polar angle measured in the \( x-y \) plane and \( a(\phi) \) is the radius at any cross section of the cone. It is defined as

\[ a(\phi) = a_o(1 + \beta\phi) \]  

(6.2a)

where

\[ \beta = \frac{d\sin \alpha}{2\pi a_o} \]  
\[ c = \frac{d\cos \alpha}{2\pi} \]

(6.2b)  
(6.2c)

\( (a_o,0,0) \) is the starting point of the helix and corresponds to \( \phi = 0 \). Here, \( \alpha \) is the semiangle of the cone on which the helix is wound and \( d \) is the pitch of the winding. When \( \phi \) increases by \( 2\pi \) radians, the radius \( a(\phi) \) increases by \( d\sin \alpha \) and the helix advances along the \( z \)-axis by \( d\cos \alpha \), so that for a winding of \( N \) turns,

\[ Nd\cos \alpha = h \]

(6.2d)
Figure 6.1 The geometry of the conical helix.

When $\alpha = 0$, Eqs. (6.1) and (6.2) describe a right circular helix. Setting $d = 0$ corresponds to a plane circular loop.

Let a current $I_o$ flow though the winding. From Biot-Savart law the magnetic induction, $B(r)$, at a point $r(x,y,z)$, is

$$B(r) = \frac{\mu_o I_o}{4\pi} \oint_{\text{coil}} \frac{ds' \times (r - r')}{|r - r'|^3}$$

(6.3)

where $ds'$ is a tangent vector to the curve at a point $r'$ on it. In cartesian coordinates,
and \( dx', dy', \) and \( dz' \) are found by differentiating Eqs. (6.1). Also,
\[
\mathbf{r} - \mathbf{r}' = u_x(x - x') + u_y(y - y') + u_z(z - z')
\]
(6.5)
The \( x \)-component of the magnetic field becomes
\[
B_x(x, y, z) = \frac{\mu_0 I_o}{4\pi} \int_{\phi=0}^{2\pi} \frac{[ds' \times (\mathbf{r} - \mathbf{r}')]_x}{|\mathbf{r} - \mathbf{r}'|^3}
\]
(6.6)
where \( N \) is the number of turns in the winding. Analogous expressions for \( B_y \) and \( B_z \) can be similarly obtained. The integrals in Eqs. (6.6) have closed form solutions only for a few special cases and in general have to be numerically integrated.

As mentioned earlier, previous models for the excitation coil have modeled the coil as a stack of coaxial loops. Formulas to calculate the magnetic induction due to a circular loop of radius \( a \) carrying a current \( I_o \) are readily available (Smythe, 1989). In cylindrical coordinates, the field due to a loop of radius \( a \), is
\[
B_\rho(\rho, \phi, z) = \frac{\mu_0 I_o}{2\pi} \frac{z}{\rho \sqrt{(a + \rho)^2 + z^2}} \left[ -K(\kappa) + \frac{a^2 + \rho^2 + z^2}{(a - \rho)^2 + z^2} E(\kappa) \right]
\]
(6.7a)
\[
B_\phi(\rho, \phi, z) = 0
\]
(6.7b)
\[
B_z(\rho, \phi, z) = \frac{\mu_0 I_o}{2\pi} \frac{1}{\sqrt{(a + \rho)^2 + z^2}} \left[ K(\kappa) + \frac{a^2 - \rho^2 - z^2}{(a - \rho)^2 + z^2} E(\kappa) \right]
\]
(6.7c)
where
\[
\kappa^2 = \frac{4ap}{(a + \rho)^2 + z^2}
\]
(6.7d)
and \( K(\kappa) \) and \( E(\kappa) \) are complete elliptic integrals of the first and second kind respectively. In Eq. (6.7), the coordinates are measured in a cylindrical system centered on the loop.

### 6.3 Results and discussion:

A Fortran code using Simpson's rule for integrating the expressions for \( \mathbf{B} \) (Eq. (6.3)) for a conical helix was used. Also, a code to evaluate the field due to an equivalent set of loops (by using Eq. 6.7 and superposition) was used to compare the results for the
helix. The equivalent stack of loops for a conical helix of \( N \) turns and height \( h \) consists of \( N \) coaxial circular loops placed at equal distances from each other over the same height.

The results were computed for a helix of height 5 cm and starting radius \( a_0 \) of 1 cm with varying number of turns. The semiangle of the cone was varied from \( 0^\circ \) to \( 75^\circ \) in steps of \( 15^\circ \). These results were compared with the corresponding results for an equivalent conical stack of loops for a given height and semiangle. A current of 1 A was assumed to flow through the windings.

Figure 6.2 shows that there is an appreciable \( x \)-component of \( \mathbf{B} \) when the number of turns is small and approaches zero as the number of turns increases. (In Fig. 6.2 and the rest of the figures, the coil is assumed to lie between the planes \( z = 0 \), and \( z = 5 \) cm.) From Fig. 6.3 we see that the \( y \)-component of \( \mathbf{B} \) along the coil axis reduces at a slower rate than the \( x \)-component when the number of turns per unit length is increased. The \( y \)-component never quite vanishes, and there is an appreciable \( y \)-component even when the number of turns is 100.

Figure 6.4 shows the variation of \( B_x \) along the axis of a helix wound on a cone with different semiangles for different numbers of turns. For a given value of \( \alpha \), the \( x \)-component of \( \mathbf{B} \) is very large for a small number of turns and diminishes as the number of turns is increased. When the number of turns is very large (\( N = 100 \)), \( B_x \) vanishes. When the number of turns is small, (less than 10), the number of cycles of variations in \( B_x \) equals the number of turns. These variations occur over a distance whose order of magnitude corresponds to the pitch of the winding. The behavior of the field of the coil approaches that of a spiral \( \alpha = 90^\circ \) in the limit of increasing semiangle of the cone. The bottom segment of Fig. 6.4 shows the results for an equivalent spiral. In computing the results for the equivalent spiral, it has been assumed that it contains the same number of turns as the helix, and the difference between the outer and inner radii of the spiral is equal to the height of the helix. At this point it must be pointed out that the \( x \)-component of \( \mathbf{B} \) along the axis of an equivalent stack of loops is identically zero, since the radial component of \( \mathbf{B} \) vanishes.
Figure 6.2 $B_x$ along the axis of a right circular helix.

Figure 6.3 $B_y$ along the axis of a right circular helix.
Figure 6.4 $B_z$ along the axis of a conical helix.
Figure 6.5 $B_y$ along the axis of a conical helix.
along the axis of a circular loop due to circular symmetry. Figure 6.5 shows the variation of $B_y$ along the axis of a helix with different number of turns, for different values of $\alpha$. Like in the case of $B_x$, the value of $B_y$ is significant when the number of turns is small (for $N = 2$, $B_x$ and $B_y$ are of the same order of magnitude). $B_y$ reduces in magnitude with increase in the number of turns, but never quite vanishes. $B_y$ reaches a limiting value as $N$ becomes very large. $B_y$ too, exhibits cyclic variation along the axis. Observations similar to those made for $B_x$ can be made for $B_y$ with the exception that $B_y$ does not vanish even when the number of turns is very large. The three lower segments of Fig. 6.5 show the variation of $B_y$ for larger values of $\alpha$.

The behavior of $B_y$ can be qualitatively explained by considering an order of magnitude argument. For a single turn of a helical winding (see Fig. 6.1), it can be shown that to first order in the parameters $\beta$ and $c$, the magnitudes of the $x$ and $y$ components at a point on the helix axis is given by (Holmes, 1978)

\begin{align*}
B_x(O, 0, z) &= 0 \\
B_y(O, 0, z) &= \frac{a_x^2}{2r_z^3} \left[ \beta \left( \frac{z - n \pi c}{a_x} \right) \left( \frac{3a_x^2 - r_z^2}{r_z^2} \right) - c \left( \frac{3(z - n \pi c)^2 - r_z^2}{r_z^2} \right) \right] 
\end{align*}

where

\begin{align*}
a_x &= a_0(1 + \beta \pi) \\
r_z^2 &= \left[ a_x^2 + (z - a_0 \pi c)^2 \right]
\end{align*}

Equations (6.8) are valid for a single turn with $z$ measured from the center of the single turn. We can obtain the order of contribution due to all the turns in the winding by taking the sum over $N$ turns in Eq. (6.8c). With reference to Fig. 6.1, this leads to

\begin{align*}
B_y(O, 0, z) &= \mu \sum_{n=1}^{N} \frac{a_x^2}{2r_{z,n}^3} \left[ \beta \left( \frac{z_n - n \pi c}{a_x} \right) \left( \frac{3a_x^2 - r_{z,n}^2}{r_{z,n}^2} \right) - c \left( \frac{3(z_n - n \pi c)^2 - r_{z,n}^2}{r_{z,n}^2} \right) \right],
\end{align*}

where

\begin{align*}
z_n &= z - \frac{1}{2} (2n - 1) d \cos \alpha, \\
r_z^2 &= \left[ a_x^2 + (z_n - a_0 \pi c)^2 \right].
\end{align*}
Equation (6.9b) accounts for the fact that \( z \) is measured from the center of the \( n \)th turn.

We can rewrite Eq. (6.9) only in terms of the height \( h \) of the winding and the total number of turns \( N \) (by using Eqs. (6.2)), and this leads to

\[
B_y(0,0,z) = \mu_0 (\text{term I} + \text{term II})
\]

where

\[
\text{term I:} \sum_{n=1}^{N} \frac{c_2}{N} \left[ 1 + \left( \frac{c_1 - n\gamma}{N} \right)^2 \right]^{-3/2} \left( \frac{c_1 - n\gamma}{N} \right) \frac{2 - \left( \frac{c_1 - n\gamma}{N} \right)^2}{1 + \left( \frac{c_1 - n\gamma}{N} \right)^2}
\]

\[
\text{term II:} \sum_{n=1}^{N} \frac{c_2}{N} \left[ 1 + \left( \frac{c_1 - n\gamma}{N} \right)^2 \right] \left[ 1 - 2 \left( \frac{c_1 - n\gamma}{N} \right)^2 \right] \frac{1}{1 + \left( \frac{c_1 - n\gamma}{N} \right)^2}
\]

and \( c_1 = z/a_o \), \( c_2 = h/(4\pi a_o^2) \), and \( \gamma = h/a_o \). Therefore, in the limit of very large \( N \), \( B_y \) would seem to vary approximately as the reciprocal of \( N \), unlike \( B_x \) which vanishes.

Further, the results in Table 6.1 shows a comparison of the values obtained by using Eqs. (6.10c) and the program that numerically integrates the field equations shown in Eqs. (6.1) through (6.6).

Table 6.1 indicates that for small values of the semiangle \( \alpha \) and pitch \( d \), the order of magnitude estimate for \( B_y \) as given by Eq. (6.9) is reasonably good and that the transverse component of \( B \) along the axis of symmetry varies as the pitch of the winding.
For all the cases: \( a_o = 1 \text{ cm}, \ h = 5 \text{ cm} \)

Coordinates of the point where \( B_y \) is computed: \( x = 0, \ y = 0, \ z = 1 \text{ cm} \)

### \( \alpha = 0^0 \)

<table>
<thead>
<tr>
<th>Number of turns, ( N )</th>
<th>( B_y ) (Wb / m²)</th>
<th>( B_y ) (Wb / m²)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Order of magnitude</td>
<td>Actual value</td>
</tr>
<tr>
<td>10</td>
<td>4.120 x 10⁻⁶</td>
<td>3.42 x 10⁻⁶</td>
</tr>
<tr>
<td>100</td>
<td>4.106463 x 10⁻⁶</td>
<td>4.106463 x 10⁻⁶</td>
</tr>
<tr>
<td>1000</td>
<td>4.106208 x 10⁻⁶</td>
<td>4.106217 x 10⁻⁶</td>
</tr>
</tbody>
</table>

### \( \alpha = 15^0 \)

| 10                      | 4.684413 x 10⁻⁶    | 4.546321 x 10⁻⁶   |
| 100                     | 4.629622 x 10⁻⁶    | 4.476701 x 10⁻⁶   |
| 1000                    | 4.629046 x 10⁻⁶    | 4.476021 x 10⁻⁶   |

### \( \alpha = 75^0 \)

| 10                      | 9.602775 x 10⁻⁶    | 3.330131 x 10⁻⁶   |
| 100                     | 9.586346 x 10⁻⁶    | 3.634213 x 10⁻⁶   |
| 1000                    | 9.586183 x 10⁻⁶    | 3.632927 x 10⁻⁶   |

For the sake of comparison, Figs. 6.6 and 6.7 show the variation of the \( z \) component of the magnetic field along the axis of a right circular helix and an equivalent stack respectively. The \( z \)-component of \( B \) along the axis of the stack is highly non-uniform for \( N = 2 \) and highly uniform (within the length of the stack of loops) for \( N = 10 \). The field behavior for large \( N \) is identical for the helix and stack models. However, for \( N = 4 \) turns, there is a significant difference in the behavior of the \( z \)-component of \( B \) for the stack and helix models - almost 20% in the center of the coil and -50% near the top and bottom of the coil. Unlike, the \( x \) and \( y \) components of \( B \) which show variations (along the
Figure 6.6 $B_z$ along the axis of the right circular axis.

Figure 6.7 $B_z$ along the axis of the right circular stack.
axis of the coil) over a length that corresponds to the pitch, the $z$ component of the coil does not show any variation. This is because, the current density vector of the current in the coil has a component directed along the positive $z$ axis throughout the coil.

Spiral windings have been occasionally used as capping coils in levitation experiments. Figure 6.8 shows the behavior of the fields due to a spiral winding, which may be treated as a special case of the conical helix. The figure shows the variation of the three components of $\mathbf{B}$ along the $x$-axis at $z=1\text{ cm}$ above the plane of the spiral. The variations in $B_x$ as expected, occur over a length corresponding to the pitch of the spiral. Also, the average magnitude of $B_x$ increases with the number of turns. As the number of turns in the spiral is increased, the peaks merge and the variation of $B_x$ is smooth. The $y$-component of $\mathbf{B}$ exhibits similar behavior, but reaches a limiting value for large $N$. The $z$-component increases monotonically with the number of turns as in the case of the right circular helix.

The capping coil in a levitation device is crucial for the creation of the "potential well" where the sample is electromagnetically contained. The capping coil was therefore modeled as a reverse wound turn of the helix by setting

\begin{align*}
y' &= -a(\theta)\sin \theta, \\
\beta &= -\frac{d \sin \alpha}{2\pi a_o},
\end{align*}

in Eqs. (6.1). Figure 6.9 shows the magnetic field along the axis of a levitation plus capping coil system. The levitation coil is a 5 turn right circular helix ($\alpha = 0$) with a starting radius of 1 cm and height of 5 cm. The capping coil consists of 1 turn wound over 0.5 cm, with radius 1 cm. The two coils are coaxial, separated by a distance of 0.5 cm. Figure 6.9 illustrates the general effect of the capping coil. As expected, the effect of the capping coil is confined to the gap between the levitation and the capping coils. The capping coil causes steeper field gradients in this region, thus causing greater Lorentz forces. Elsewhere, the capping coil has little effect.
Figure 6.8 The components of $\mathbf{B}$ along the $x$-axis for a spiral in a plane that is 1 cm above it.
Figure 6.9 The effect of the capping coil on the fields of a levitation coil.
Figure 6.10 The difference between the helix and stack models for a levitation plus capping coil system.
Finally, Fig. 6.10 shows the differences between the magnetic field values obtained by using the helix and stack models for a conical levitation coil along a line parallel to the coil axis. The levitation coil has a starting radius of 1 cm, height of 5 cm, 5 turns, and semiangle of 25°. The capping coil consists of a single reverse turn of radius 3.3 cm and height 0.5 cm. The capping coil is situated 0.5 cm above the levitation coil. The off axis line is parallel to the z-axis and is situated at $x = y = 0.072$ cm. The figure shows that there is a significant difference between the values predicted by the stack and the helix models. More results and details of the fields of conical helices and their gradients are presented in by Bayazitoglu and Sathuvalli (1993).

6.4 Conclusions:

The fields due to the current flowing in a conical helix have been calculated. The influence of pitch and semiangle of the cone on the behavior of the field have been studied. The results were compared with the results obtained by using the stack model of the winding. It is shown that the helix model yields more realistic values of the field. Also, the results obtained by the two models are shown to match in the limit of vanishing pitch of the helix. Since, the Lorentz force on a diamagnetic specimen in a nonuniform can be shown to vary (to leading order) as the product of the field and its gradient the preceding analysis suggests that modeling a levitation coil as a stack of loops might introduce significant errors in the analysis. Also, the capping coil is modeled as a reverse wound turn. It is shown that the effect of the capping coil (as expected) is to increase the field gradients in the region between the levitation and the capping coils.
References


Muck, O., 1923, German Patent No. 422004.


Appendix A

Boundary Conditions for Diamagnetic Bodies

In the solution of eddy current problems, the governing Maxwell equations are often cast in terms of the vector potential defined in Eq. (2.4). The resulting governing equation is similar to the classical diffusion or the heat conduction equation. The appropriate boundary conditions on the vector potential $A$ follow from the boundary conditions that govern the behavior of the electric ($E$) and magnetic ($B$) fields at the interface that separates two media. The boundary conditions at one such interface are as follows:

\begin{align*}
\mathbf{u}_n \cdot (\mathbf{D}_2 - \mathbf{D}_1) &= \rho_{surf}, \quad (A1) \\
\mathbf{u}_n \times (\mathbf{E}_2 - \mathbf{E}_1) &= 0, \quad (A2) \\
\mathbf{u}_n \cdot (\mathbf{B}_2 - \mathbf{B}_1) &= 0, \quad (A3) \\
\mathbf{u}_n \times (\mathbf{H}_2 - \mathbf{H}_1) &= \mathbf{J}_{surf}. \quad (A4)
\end{align*}

where $\rho_{surf}$ and $\mathbf{J}_{surf}$ are the surface charge density and current density at the medium interface. Also, $\mathbf{u}_n$ is the unit outward drawn normal to the surface and $\mathbf{D}$ is the electric flux density vector defined as

\begin{equation}
\mathbf{D} = \varepsilon \mathbf{E} \quad (A5)
\end{equation}

where $\varepsilon$ is the electric permittivity of the medium, and $\mathbf{H}$ is the magnetic field intensity vector defined as

\begin{equation}
\mathbf{B} = \mu \mathbf{H}. \quad (A6)
\end{equation}

Combining Eq. (2.4a) with Faraday's law and Eq. (A5) yields

\begin{equation}
\mathbf{D} = -\varepsilon \frac{\partial \mathbf{A}}{\partial t}. \quad (A7)
\end{equation}

Equation (2.4) in combination with Faraday's and Ohm's laws gives

\begin{equation}
\mathbf{J} = \sigma \frac{\partial \mathbf{A}}{\partial t}. \quad (A8)
\end{equation}

In the case of the electrically conducting sphere placed in the alternating magnetic field, there are no surface charges or currents. Then, by Eqs. (A7) and (A1),
\[ \frac{\varepsilon_1}{\varepsilon_2} = \frac{\partial A_{2n}}{\partial t} = \frac{\partial A_{1n}}{\partial t} \]

where the subscript \( n \) denotes the component of the vector potential along the outward drawn normal to the medium interface. If all the fields are assumed to be harmonic, then \( \frac{\partial}{\partial t} \) may be replaced by \( j\omega \) and hence,

\[ \frac{\varepsilon_1}{\varepsilon_2} = \frac{A_{2n}}{A_{1n}}. \]  \hspace{1cm} (A9)

For materials like copper, aluminum (which are used in levitation experiments), the permittivity is approximately that of free space or air, \( \varepsilon_o \). So at the interface of a good electrical conductor and air, Eq. (A9) becomes

\[ A_{1n} = A_{2n}, \]  \hspace{1cm} (A10)

i.e., the normal component of the vector potential is continuous across the interface between a good conductor and air. A similar analysis of Eq. (A2) yields

\[ A_{1t} = A_{2t}. \]  \hspace{1cm} (A11)

The subscript \( t \) denotes the tangential component of the vector potential at the interface. Equation (A11) indicates that the tangential component of the vector potential is continuous across the interface. Equations (A10) and (A11) together imply that

\[ \mathbf{A}_1 = \mathbf{A}_2 \]  \hspace{1cm} (A12)

thus stating that the vector potential is continuous across the boundary between a good electrical conductor and air or free space.

Using the definition of the vector potential, Eq. (2.4) in Eq. (A3), gives

\[ \mathbf{u}_n \cdot (\nabla \times \mathbf{A}_2) = \mathbf{u}_n \cdot (\nabla \times \mathbf{A}_1). \]  \hspace{1cm} (A13)

By the use of the identity

\[ \nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \]

and Eq. (A12), Eq. (A13) can be shown to yield

\[ \nabla A_{1t} = \nabla A_{2t}. \]  \hspace{1cm} (A14)
Finally, in Eq. (A4), putting $J_{surf} = 0$, $\mu_1 = \mu_2 = \mu_o$ and expressing all quantities in terms of $A$ yields,

$$u_n \times [\nabla \times (A_2 - A_1)] = 0.$$  

Invoke the vector identity,

$$\nabla (a \cdot b) = a \times (\nabla \times b) + b \times (\nabla \times a) + (b \cdot \nabla) a + (a \cdot \nabla) b$$

and then put $a = A_2 - A_1$ and $b = u_n$. This leads to

$$u_n \times [\nabla \times (A_2 - A_1)] = \nabla [u_n \cdot (A_2 - A_1)] - (u_n \cdot \nabla) (A_2 - A_1)$$

$$- (A_2 - A_1) \times (\nabla \times u_n) - [(A_2 - A_1) \cdot \nabla] u_n.$$  

In the above relation, terms 2, 3 and 4 on the right hand side vanish by virtue of Eq. (A12), thus leading to

$$\nabla A_{2n} = \nabla A_{1n}. \quad (A15)$$

Equations (A14) and (A15) indicate that the vector potential is differentiable across the interface between a good conductor and free space.

Equation (2.18) can be now proved. Let $A_{so} (r, u, \phi, t)$ and $A_{si} (r, u, \phi, t)$ denote the vector potentials due to the induced eddy currents outside and inside the sphere respectively. Then from Eq. (2.9), (2.17), and (A12),

$$R_{i,m}(R_s, t) = a_{i,m}(t) R_{s}^{(i+1/2)} \quad \quad (A16)$$

Also, Eq. (A15) implies

$$\left. \frac{\partial A_{so}}{\partial r} \right|_{r = R_s} = \left. \frac{\partial A_{si}}{\partial r} \right|_{r = R_s}.$$  

Substituting Eqs. (2.9) and (2.17) in the above equality, and then using Eq. (A16) yields Eq. (2.18). Substitute Eq. (2.19a) in Eq. (2.18) to get

$$r \to R_s \left[ \frac{d}{dr} J_{i+1/2} \left( \frac{r}{R_s} \right) \right] + J_{i+1/2} \left( \frac{r}{R_s} \right) \right] = 0$$

and then make use of the relation (Abramowitz and Stegun, 1964)

$$\frac{d}{dz} J_v(z) = J_{v-1}(z) - \frac{\nu}{z} J_v(z). \quad (A17)$$

This leads to the Eq. (2.19b).
Appendix B

The Source Function for a Stack of Loops
and a Finite Cylindrical Current Sheet

In this appendix, Eq. (3.10) is used to calculate the source functions for two common configurations of current sources- i) the (discrete) axisymmetric conical stack of current carrying circular loops shown in Fig. 3.2, and ii) the (continuous) right circular cylindrical current sheet.

i) The stack of loops:

The vector potential of a circular loop of current (with reference to Fig. 3.2) as seen from the center of the sphere is given by (Smythe, 1989)

\[ A_n(r, u, \phi) = u_{\phi} \frac{\mu_0 I}{2} \sum_{j=1}^{\infty} \frac{\sin \theta_n}{j(j+1)} \left( \frac{r}{r_n} \right)^j P_j(u) P_j(\cos \theta_n) \]  

(B1)

In terms of unit cartesian vectors, it is given by

\[ u_{\phi} = -\sin \phi \ u_x + \cos \phi \ u_y \]  

(B2)

Substitute the expression for the vector potential of the circular loop in Eq. (3.10). Then the volume integral on the right hand side of Eq. (3.10) becomes

\[ \int_0^R \int_{-1}^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^{3/2} J_{l+1/2}(x_{l+1/2,k} \frac{r}{R}) P^m_j(u)(\cos m\phi - i \sin m\phi). \]

\[ \cdot \left(-\sin \phi \ u_x + \cos \phi \ u_y\right) \sum_{j=1}^{\infty} \frac{\sin \theta_n}{j(j+1)} \left( \frac{r}{r_n} \right)^j P_j(u) P_j(\cos \theta_n) d\phi du dr. \]

The \( \phi \) integral vanishes for all values of \( m \), except \( m = \pm 1 \), since

\[ \int_0^{2\pi} (\cos m\phi - i \sin m\phi)(-\sin \phi \ u_x + \cos \phi \ u_y) d\phi = \begin{cases} \pi(\i u_x + u_y), & m = 1 \\ \pi(-\i u_x + u_y), & m = -1 \\ 0, & \text{otherwise}. \end{cases} \]

The \( u \) integral can be found by using Eq. (E17) to give

\[ \int_{-1}^{+1} P_j^l(u) P_j^l(u) du = \frac{2l(l+1)}{2l+1} \delta_{l,j}. \]
The $r$ integral can be found by using Eq. (D6b). Together, these expressions yield Eq. (3.14)

**ii) The right circular cylindrical sheet:**

In Fig. 3.1, set $\alpha = 0$ and then imagine that the number of current loops per unit length is increased indefinitely, so that the gap between successive current loops vanishes in the limit. The configuration corresponds to a very tightly wound hollow cylindrical current coil or the "cylindrical sheet." The vector potential of this sheet of current is (Smythe, 1989)

$$\mathbf{A}(r,\alpha,\phi) = \mathbf{u}_o \sum_{l=0}^{\infty} C_l r^{2l+1} P_{2l+1}^1(u) \tag{B3a}$$

where

$$C_l = \begin{cases} \frac{-1}{2} \mu_o i_o \cos \beta, & l = 0, \\ \mu_o i_o \left( \frac{h}{2} \right)^{2l} \left[ 2l(2l+1)(2l+2) \right]^{-1} \sin \beta \ P_{2l}^1(\cos \beta), & l = 1,2,3,..., \end{cases} \tag{B3b}$$

$$\tan \beta = \frac{2h}{a_o} \tag{B3c}$$

and $i_o$ is the current per unit length in the sheet. Upon substituting Eq. (B3) in Eq. (3.10), we obtain

$$I_{n,l,m} = \frac{x_{l+1/2,k}^2}{\mu_o R_s^2 J_{l+1/2}^0(x_{l+1/2,k})} \int_{0}^{2\pi} \int_{0}^{1} \mathbf{u}_o(\phi) C_l r^{2l+5/2} J_{l+1/2} \left( \frac{r}{R_s} \right) d\phi \ du \ dr \ P_{2l+1}^1(u)Y_l^{m*}(u) \tag{B4a}$$

After the use of Eqs. (B2) and (2.10), the $\phi$ integral in the above equation is

$$\delta_{m,1}(iu_x + u_y) + \delta_{m,-1}(-iu_x + u_y). \tag{B4a}$$

The $u$ integral is identical to the previous case and implies that for non-zero values of $I_{n,l,m}$, $m = \pm 1$ and $l = (l-1)/2$. With this in mind, the $r$ integral can be shown to be

$$\frac{R_s^l}{x_{l+1/2,k}^2} (2l+1) J_{l+1/2} \left( x_{l+1/2,k} \right) \tag{B4b}$$
Putting everything together, we finally obtain the source function for the cylindrical current sheet as

$$I_{n,l,m} = \frac{2}{\mu_0} C_{l-1} \sqrt{\frac{2l+1}{4\pi}} R_y \left[ -\delta_{m,1} (iv_x + u_y) + \delta_{m,-1} (-iv_x + u_y) \right] \left( 1 - \delta_{l,0} \right), \quad l = 1, 2, 3, \ldots \quad (B5)$$

Note the similarity of Eq. (B4) with Eq. (3.14). The source functions for both these configurations vanish for $m = \pm 1$ in keeping with the axisymmetry of the situation. Also, note that Eq. (B5) is valid when the center of the sphere is at the center of axis of the current sheet. If the center of the sphere is at any other point, the source function must be re-calculated.
Appendix C

The Limiting Case of a Very Small Sphere:
Power Absorbed and Lorentz Force

This appendix is an analysis of the power and force equations for the sphere in when radius of the sphere is i) much smaller than the length scale of the magnetic field, and ii) is comparable to the skin depth.

**Power:**

Consider a sphere placed in the field of a single current (\( N = 1 \)) source. By dropping the subscript \( n \) and using Eq. (2.22c), Eq. (3.8) can be written as

\[
P_s = \frac{1}{2\sigma_s\delta} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \frac{H_l(q)}{q} \right] I_{l,m} \cdot I_{l,m}.
\]

Then for constant radian frequency (i.e. constant \( \delta \)),

\[
R_s \to 0 \left[ P_s \right] = \lim_{q \to 0} \left[ \frac{1}{2\sigma_s\delta} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \frac{H_l(q)}{q} \right] I_{l,m} \right].
\]

From Eq. (3.9)

\[
q \to 0 \left( \frac{H_l(q)}{q} \right) = 8q^3 \sum_{k=1}^{\infty} x_{l+1/2,k}^{-4}.
\]

From Eq. (F18) of Appendix F

\[
R_s \to 0 \left[ P_s \right] = \frac{1}{2\sigma_s\delta} \lim_{q \to 0} \left[ \frac{H_0(q)}{q} \right] I_{0,0}^2 + \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{4q^3}{(2l + 3)(2l + 1)^2} I_{l,m}^2.
\]

Since \( q = 0 \) when \( R_s = 0 \), Eq. (C3) gives the physically reasonable result \( P_s = 0 \) for a sphere of zero radius. When \( R_s \) is small but not zero, Eq. (3.21) implies that \( I_{0,0} \) is the only term that contributes to \( P_s \) when the vector potential of the field is finite at the location of the sphere. On the other hand, the homogeneous model predicts only the term that corresponds to \( I_{1,1} \), irrespective of the geometry of the external field.

By using Taylor expansion for the circular and hyperbolic functions in Eq. (3.25b), it can be shown that
Neglecting all terms in \( l > 1 \), for small \( R_s \), Eq. (C3) is (after making use of Eq. (3.23b))

\[
P_s \simeq \frac{2q^3}{\sigma_s \delta} \left[ \frac{1}{3} |I_{0,0}|^2 + \frac{2}{45} |I_{1,1}|^2 \right].
\] (C4)

The corresponding power absorbed according to the homogeneous model is

\[
P_{\text{hom}} = \frac{2q^3}{\sigma_s \delta} \left[ \frac{1}{3} |I_{0,0}|^2 + \frac{2}{45} |I_{1,1}|^2 \right].
\] (C5)

The use of Eqs. (3.21) and (3.23a) in Eqs. (C4) and (C5) leads to

\[
\frac{P_s}{P_{\text{hom}}} = \frac{15}{2} \frac{|I_{0,0}|^2}{|I_{1,1}|^2} + 1,
\] (C6)

\[
= 20 \frac{|A(r_o, u_o, \phi_o)|^2}{B^2 R_s^2} + 1.
\]

If \( A(r_o, u_o, \phi_o) \sim A_o \) then by Eq. (2.4a) \( B \sim A_o/a_o \). This implies that when \( R_s << 1 \) and the vector potential of the magnetic field at the center of the sphere is not zero

\[
\frac{P_s}{P_{\text{hom}}} \sim \frac{20}{R_s^2}.
\] (C7)

This explains why the homogeneous and nonhomogeneous models yield identical results when the sphere lies along the axis of the stack of loops. Along the axis of a stack of loops, the vector potential vanishes identically thus making \( I_{0,0} \equiv 0 \). At off axis points, there is a non-zero vector potential that creates a finite \( I_{0,0} \). This term is comparable to \( I_{1,1} \) (and sometimes even greater than \( I_{1,1} \) since \( I_{1,1} \approx R_s \)), thus leading to Eq. (C7). It must be noted that Eq. (C7) is merely an estimate of the error inherent in the homogeneous model and therefore must be treated only as a justification for the use of the nonhomogeneous model and term given in Eq. (3.25b) for a small sphere.

**Lorentz force:**

The magnetic field on the axis of a circular loop is purely \( z \)-directed and is found by setting \( \rho = 0 \) in Eq. (6.7c),

\[
\frac{H_0(q)}{q} \to 0 \frac{4q^3}{3}.
\]
\[ B(z_o) = \mu_0 \mu_o \frac{1}{2} \frac{a^2}{(a^2 + z_o^2)}, \tag{C8} \]

where \( z_o \) is the axial distance from the center of the loop of radius \( a \). Using superposition and Eqs. (4.20), the homogeneous model yields the following expression for the Lorentz force* on the sphere along the axis of the stack of loops shown in Fig. 3.2

\[ \frac{F_s}{\mu_o I_o^{\text{hom}}} = \pi R^3 \sum_n \sum_{n'} G(q) \frac{t_n^2 s_{n'}}{(t_n^2 + s_n^2)^{3/2} (t_{n'}^2 + s_{n'}^2)^{3/2}}, \tag{C9} \]

where \( t_n = 1 + \gamma c_n \) and \( s_n = (\gamma - \gamma c_n^2) \), \( \gamma = h/a_o \) and \( c_n = (n-1)/(N-1) \) For the special case of a single loop, the force on the sphere at a height \( z_o \) above the center of the loop becomes

\[ \frac{F_s}{\mu_o I_o^{\text{hom}}} = \pi R^3 G(q) \frac{a_1^2 z_o}{(a_1^2 + z_o^2)^4}. \tag{C10} \]

If the sphere is small compared to the radius of the loop, then all terms of order \( l \geq 2 \) in Eq. (4.17) may be dropped. Considering only the \( l = 1 \) term and expressing all functions in terms of the dimensions in Fig. 3.2, Eq. (4.17a) can be written as

\[ \frac{F_s}{\mu_o I_o^{\text{non-hom}}} = \frac{9}{2} \pi R^3 g_1(q) \frac{a_1^4 z_o}{(a_1^2 + z_o^2)^4}. \tag{C11} \]

By Eq. (4.13b), it follows that

\[ \left[ \frac{F_s}{\mu_o I_o^{\text{non-hom}}} - \frac{F_s}{\mu_o I_o^{\text{hom}}} \right] \left[ \frac{F_s}{\mu_o I_o^{\text{non-hom}}} \right]^{-1} = -\frac{1}{3}. \tag{C12} \]

* Assuming that all the loops carry current at the same frequency.
Appendix D

Calculation of a Typical Term in Eq. (4.11)

The substitution of the gradient formula, Eqs. (4.5c) and (4.6b), into Eq. (4.10a), leads to

\[ X_r = \sum_{j=x,y,z} X_{1,j}, \quad (D1) \]

where

\[ X_{1,j} = \sum_{n,l,m} \sum_{n',l',m'} \sum_{k=1}^{\infty} \cos \psi_{n,k,l} \delta_{\omega_n,\omega_k} E_{n,k,l,m} D_{r,m} U_{n,l,m} \times (u_j \times U_{n',l',m'}) \Lambda_{l,k,m,j}^{r,e}, \quad (D2) \]

and

\[ \Lambda_{l',k,m,j}^{p,q} = \int_0^{+12\pi} \int_0^{+12\pi} f_{l',k,m} (u,\phi) \frac{\partial}{\partial j} \left( r^p Y_{l,m}^q (u,\phi) \right) r^2 d\phi dudr, \quad j = x, y, z, \quad p, q = e, 0. \quad (D3a) \]

The calculation of the various terms of (4.10a) finally involve the calculation of triple integrals of the type shown in Eq. (D3a). As an illustrative case, consider the case \( j = x, p = q = e \). Equation (D3a) then becomes

\[ \Lambda_{l,k,m}^{r,e} = \frac{R_{l+1/2}^{12\pi}}{2} \int_0^{+1/2} J_{l+1/2} \left( x_{l+1/2,k} r R_s \right) \frac{\partial}{\partial x} \left( r^e Y_{l,m}^e (u,\phi) \right) r^2 d\phi dudr \]

\[ = \frac{1}{2} \left[ \left\{ \left( l + m' \right) \left( l + m' - 1 \right) \int_0^{+1/2} J_{l+1/2} \left( x_{l+1/2,k} r R_s \right) Y_{l,m}^e (u,\phi) Y_{l-1,m}^e r^2 d\phi dudr \right\} \right]. \quad (D3b) \]

The quantities of the type \( \frac{\partial}{\partial j} \left( Y_{l,m}^{r,e,0} \right) \) may be found by looking up Eqs. (E12) and (E13).

The orthogonality relation, Eq. (E14) implies that \( l = l + 1, m' = m \pm 1 \) for non-zero values of the above integral. This enables the evaluation of the \( r \)–integral in both the terms above to yield

\[ \int_0^{R_s} r^{l+3/2} J_{l+1/2} \left( x_{l+1/2,k} r R_s \right) dr = \frac{R_s^{l+5/2}}{x_{l+1/2,k}^2} (2l + 1) J_{l+1/2,k} \left( x_{l+1/2,k} \right) \quad (D4) \]
Finally,

\[
\Lambda_{l,m,x} = \left[ -\frac{R^{l+5/2}}{x_{l+1/2,k}^2} J_{l+1/2} \left( x_{l+1/2,k} \right) \frac{(l+m)!}{(l-m)!} \pi (1 + \delta_{m,0}) \delta_{l,l-1} \left\{ \delta_{m,m+1} - (l + m + 2)(l + m + 1) \delta_{m,m-1} \right\} \right]
\]  
\[(D5)\]

For real values of \( \nu \), the following relations are useful in evaluating (Abramowitz and Stegun, 1965) the above equation,

\[
\int_0^\infty \frac{z^{\nu+1} J_\nu(z)}{z^{\nu+1}} dz = \frac{2\nu}{z} J_\nu(z).
\]
\[(D6a)\]

\[
J_{\nu+1}(z) + J_{\nu-1}(z) = \frac{2\nu}{z} J_\nu(z).
\]
\[(D6b)\]

After some straightforward algebra, the term \( X_{l,x} \) can now be found from Eqs. (D2), (D5), (4.5e), and (4.6d), as

\[
X_{l,x} = \sum_n \sum_{n'} \sum_{l=0}^{2} \sum_{m=-l}^{l} \left\{ \frac{1}{2} \sum_{k=1}^{\infty} \frac{4q_n^4}{4q_n^4 + x_{l+1/2,k}^2} \right\} \delta_{a_n,a_{n'}} \left( 1 + \delta_{m,0} \right) \sqrt{\frac{2l+1}{2l+3}} \left[ U_{n,l,m} \times \left\{ \sqrt{(l-m+2)(l-m+1)} U_{n',l+1,m-1} - \right\} \right].
\]
\[(D7)\]

In Eq. (D5), the sums over the indices \( l' \) and \( m' \) vanish due to the presence of the Kronecker delta functions \( \delta_{l,l-1} \) and \( \delta_{m,m\pm 1} \). Note that the expression for \( \cos \psi_{n,k,l} \) from Eq. (2.22b) has been used in arriving at (D7).

The other two terms in Eq. (D1) can be found by following a similar set of steps. This yields

\[
X_{l,y} = 0,
\]
\[(D8)\]

and

\[
X_{l,z} = \sum_n \sum_{n'} \sum_{l=0}^{2} \sum_{m=-l}^{l} \left\{ \frac{1}{2} \sum_{k=1}^{\infty} \frac{4q_n^4}{4q_n^4 + x_{l+1/2,k}^2} \right\} \delta_{a_n,a_{n'}} \left( 1 + \delta_{m,0} \right) \sqrt{\frac{2l+1}{2l+3}} \left[ U_{n,l,m} \times \left\{ \sqrt{(l+m+2)(l-m+1)} U_{n',l+1,m-1} \right\} \right].
\]
\[(D9)\]

Finally, Eqs. (D1), (D7), (D8) and (D9) result in the following expression for the first term in Eq. (4.11),
\[ X_1 = \sum_n \sum_{n'} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2} g_l(q_n) \delta_{\alpha_n, \alpha_{n'}} \left(1 + \delta_{m,0}\right) \sqrt{\frac{2l+1}{2l+3}} \]

\[ U_{n,l,m} \times \begin{cases} u_x \times \left( \sqrt{(l-m+2)(l-m+1)} U'_{n',l+1,m-1} \right) \left( \sqrt{l+m+2}(l+m+1) U'_{n',l+1,m+1} \right) \\ u_z \times \left( 2\sqrt{(l+m+1)(l-m+1)} U'_{n',l+1,m} \right) \end{cases} \]

The remaining three terms in Eq. (4.11) can be obtained by a similar procedure. For the sake of brevity, only the final expression for each of these terms is given below:

\[ X_\Pi = \sum_n \sum_{n'} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2} g_l(q_n) \delta_{\alpha_n, \alpha_{n'}} \left(1 + \delta_{m,0}\right) \sqrt{\frac{2l+1}{2l+3}} \]

\[ V_{n,l,m} \times \begin{cases} u_x \times \left( \sqrt{(l-m+2)(l-m+1)} U'_{n',l+1,m-1} \right) \left( \sqrt{l+m+2}(l+m+1) U'_{n',l+1,m+1} \right) \\ u_y \times \left( \sqrt{(l-m+2)(l-m+1)} U'_{n',l+1,m-1} \right) \left( \sqrt{l+m+2}(l+m+1) U'_{n',l+1,m+1} \right) \end{cases} \]

\[ X_\Pi = \sum_n \sum_{n'} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2} g_l(q_n) \delta_{\alpha_n, \alpha_{n'}} \left(1 - \delta_{m,0}\right) \sqrt{\frac{2l+1}{2l+3}} \]

\[ V_{n,l,m} \times \begin{cases} u_x \times \left( \sqrt{(l-m+2)(l-m+1)} U'_{n',l+1,m-1} \right) \left( \sqrt{l+m+2}(l+m+1) U'_{n',l+1,m+1} \right) \\ u_y \times \left( \sqrt{(l-m+2)(l-m+1)} U'_{n',l+1,m-1} \right) \left( \sqrt{l+m+2}(l+m+1) U'_{n',l+1,m+1} \right) \end{cases} \]

\[ X_{IV} = \sum_n \sum_{n'} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2} g_l(q_n) \delta_{\alpha_n, \alpha_{n'}} \left(1 - \delta_{m,0}\right) \sqrt{\frac{2l+1}{2l+3}} \]

\[ V_{n,l,m} \times \begin{cases} u_x \times \left( \sqrt{(l-m+2)(l-m+1)} U'_{n',l+1,m-1} \right) \left( \sqrt{l+m+2}(l+m+1) U'_{n',l+1,m+1} \right) \\ u_y \times \left( \sqrt{(l-m+2)(l-m+1)} U'_{n',l+1,m-1} \right) \left( \sqrt{l+m+2}(l+m+1) U'_{n',l+1,m+1} \right) \end{cases} \]
The Gradient Formulae

Spherical harmonics, defined in Eq. (2.10) obey the following properties:

i) the orthogonal property
\[ \int_{-1}^{1} \int_{0}^{2\pi} Y_l^m(u, \phi) Y_{l'}^{m'}(u, \phi) \, d\phi \, du = \delta_{l,l'} \delta_{m,m'} , \]  
(E1)

ii) the operator \( \mathbf{L} \) operates on \( Y_l^m(u, \phi) \) as
\[ \mathbf{L}^2 Y_l^m(u, \phi) = l(l+1)Y_l^m(u, \phi) . \]  
(E2)

iii) and the complex conjugate is given by
\[ Y_l^m(u, \phi)^* = (-1)^m Y_l^{-m}(u, \phi) . \]  
(E3)

In evaluating the volume integrals in Eq. (4.10), it is convenient to employ a cartesian system of axes, since the unit vectors in this system are space independent throughout the region of integration (i.e., the volume of the sphere). Hence, it is easier to obtain the gradient formula in cartesian components. The definition of the gradient gives
\[ \nabla [ r^l P_l^m(u)e^{im\phi} ] = \sum_{j=x,y,z} \mathbf{u}_j \frac{\partial}{\partial j} [ r^l P_l^m(u)e^{im\phi} ] . \]  
(E4)

The partial derivatives on the right hand side of Eq. (E4) are evaluated by knowing that (see page 361 of Bateman, 1932)
\[ \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) [ r^l P_l^m(u)e^{im\phi} ] = - r^{l-1} P_{l-1}^{m+1}(u)e^{i(m+1)\phi} , \]  
(E5)

\[ \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) [ r^l P_l^m(u)e^{im\phi} ] = (l+m)(l+m-1) r^{l-1} P_{l-1}^{m-1}(u)e^{i(m-1)\phi} , \]  
(E6)

and
\[ \frac{\partial}{\partial z} [ r^l P_l^m(u)e^{im\phi} ] = (l+m) r^{l-1} P_{l-1}^{m-1}(u)e^{i(m-1)\phi} . \]  
(E7)

Using these relations, it can be shown that
\[ \nabla [ r^l Y_l^m(u, \phi) ] = \frac{1}{\sqrt{2}} \sqrt{\frac{2l+1}{2l-1} r^{l-1}} \begin{bmatrix} s_{-1} \sqrt{(l-m)(l-m-1)} Y_{l-1}^{m+1}(u, \phi) \\ + \sqrt{2} s_0 \sqrt{(l-m)(l+m)} Y_{l-1}^{m}(u, \phi) \\ - s_{+1} \sqrt{(l+m)(l+m-1)} Y_{l-1}^{-m}(u, \phi) \end{bmatrix} , \]  
(E8)

where \( s_j, j = -1,0,+1 \), are known as the spherical basis vectors. They are given by
The gradient of \( r'Y_{l}^{m'}(u, \phi) \) by using Eq. (E3) and knowing that
\[
P_{l}^{m}(u) = (-1)^{m} \frac{(l - m)!}{(l + m)!} P_{l}^{m}(u).
\] (E10)

From this, it can be shown that
\[
\nabla \left[ r'Y_{l}^{m'}(u, \phi) \right] = -\frac{1}{\sqrt{2}} \sqrt{\frac{2l+1}{2l-1}} r^{l-1} \begin{bmatrix}
  s_{z} & \sqrt{(l + m)(l + m - 1)} Y_{l-1}^{m'}
  \\
  -\sqrt{2} s_{0} & \sqrt{(l - m)(l + m)} Y_{l-1}^{m'}
  \\
  + s_{z} & \sqrt{(l - m)(l - m - 1)} Y_{l-1}^{m'
} \end{bmatrix}.
\] (E11)

Following through with some tedious but straightforward algebra yields the following gradient formulae for the surface harmonic defined in Eq. (4.3),
\[
\nabla \left[ r'Y_{l,m}(u, \phi) \right] = \frac{1}{2} r^{l-1} \begin{bmatrix}
  u_{x} \left\{ -Y_{l-1,m+1}^{e} + (l + m)(l + m - 1) Y_{l-1,m-1}^{e} \right\} \\
  u_{y} \left\{ Y_{l-1,m+1}^{e} + (l + m)(l + m - 1) Y_{l-1,m-1}^{e} \right\} \\
  u_{z} \left\{ 2(l + m) Y_{l-1,m}^{e} \right\} \\
  u_{x} \left\{ -\left( l - m \right) \right\} \cdot Y_{l-1,m+1}^{e} + (l + m)(l + m - 1) Y_{l-1,m-1}^{e} \right\} \\
  u_{y} \left\{ Y_{l-1,m+1}^{e} + (l + m)(l + m - 1) Y_{l-1,m-1}^{e} \right\} \\
  u_{z} \left\{ 2(l + m) Y_{l-1,m}^{e} \right\} \\
  u_{x} \left\{ -Y_{l-1,m+1}^{o} + (l + m)(l + m - 1) Y_{l-1,m-1}^{o} \right\} \\
  u_{y} \left\{ Y_{l-1,m+1}^{o} + (l + m)(l + m - 1) Y_{l-1,m-1}^{o} \right\} \\
  u_{z} \left\{ 2(l + m) Y_{l-1,m}^{o} \right\}
\end{bmatrix}
\] (E12)

Finally, the surface harmonic functions as defined in Eq. (4.3) can be shown to obey the following orthogonal properties. Unlike the standard spherical harmonics, these functions are not orthonormal. By direct substitution of the definitions for the functions \( Y_{l,m}^{e}(u, \phi) \) and \( Y_{l,m}^{o}(u, \phi) \), the following can be verified:
\[
\int_{-1}^{1} \int_{-\pi}^{\pi} Y_{l,m}^{e}(u, \phi) Y_{l,m}^{e}(u, \phi) d\phi du = \frac{2}{2l+1} \frac{(l + m)!}{(l - m)!} \pi \left( 1 + \delta_{m,0} \right) \delta_{m,m'} \delta_{l,l'},
\] (E14)
\[
\int_{-1}^{1} \int_{-\pi}^{\pi} Y_{l,m}^{o}(u, \phi) Y_{l,m}^{o}(u, \phi) d\phi du = \frac{2}{2l+1} \frac{(l + m)!}{(l - m)!} \pi \left( 1 - \delta_{m,0} \right) \delta_{m,m'} \delta_{l,l'},
\] (E15)
\[
\int_{-1}^{1} \int_{-\pi}^{\pi} Y_{l,m}^{e}(u, \phi) Y_{l,m}^{o}(u, \phi) d\phi du = 0,
\] (E16)

where the orthogonality of associated Legendre polynomials (Arfken, 1985), i.e.,
\[
\int_{-1}^{+1} P_i^m(u) P_r^m(u) du = \frac{2}{2l + 1} \frac{(l+m)!}{(l-m)!} \delta_{l,r}
\]

(E17)

has been used.
Appendix F

The Skin Depth Dependent Functions

The \( k \) - dependent terms in Eq. (3.6) (after using Eq. (2.22b)) are

\[
\sum_{k=1}^{\infty} \frac{J_{l+1/2} \left( x_{l+1/2,k} \frac{r}{R} \right)}{J_{l+1/2} (x_{l+1/2,k})} \frac{2q_n^2}{\sqrt{4q_n^4 + x_{l+1/2,k}^4}} \cos(\omega_n t + \psi_{n,k,t}),
\]

\[(F1)\]

The above sum is evaluated by expressing the modified Bessel function \( I_{l+1/2}(zt) \) in a Fourier-Bessel series as (Tolstov, 1962)

\[
I_{l+1/2}(zs) = \sum_{k=1}^{\infty} \lambda_{l+1/2,k}(z) J_{l+1/2} (x_{l+1/2,k}s), 0 \leq s \leq 1, l = 0, 1, 2, \ldots,
\]

\[(F2)\]

where \( z \) may be complex, \( \lambda_{l+1/2,k} \) are yet undetermined coefficients, and \( x_{l+1/2,k} \) are defined in Eq. (2.19b). Multiplying both sides of (F2) by \( sJ_{l+1/2} (x_{l+1/2,k}s) \) and integrating from \( s = 0 \) to \( s = 1 \), gives

\[
\lambda_{l+1/2,k} = \frac{2}{\int J_{l+1/2} (x_{l+1/2,k})} \int_{s=0}^{1} sJ_{l+1/2} (x_{l+1/2,k}s) I_{l+1/2}(zs) ds,
\]

\[(F3a)\]

due to the orthogonality of the Bessel functions, i.e.,

\[
\int_{s=0}^{1} sJ_{l+1/2} (x_{l+1/2,k}s) J_{l+1/2} (x_{l+1/2,k}s') ds = \frac{1}{2} J_{l+1/2} (x_{l+1/2,k}) \delta_{k,k'}.
\]

\[(F3b)\]

The integral in (F3a) can be evaluated to yield (Eq. (11.3.29) of Abramovitz and Stegun, 1965)

\[
\int_{s=0}^{1} sJ_{l+1/2} (x_{l+1/2,k}s) I_{l+1/2}(zs) dt = e^{-\frac{\pi}{2}(l-1)} \frac{J_{l+1/2} (x_{l+1/2,k}) J_{l-1/2}(iz)}{z^2 + x_{l+1/2,k}^2}
\]

and so

\[
\lambda_{l+1/2,k} = \frac{2z}{\int J_{l+1/2} (x_{l+1/2,k})} \frac{1}{z^2 + x_{l+1/2,k}^2} I_{l-1/2}(z).
\]

\[(F4a)\]

Therefore,
\[ I_{l+1/2}(z) = \sum_{k=1}^{\infty} \frac{2z}{z^2 + x_{l+1/2,k}^2} \frac{J_{l+1/2}(x_{l+1/2,k}s)}{J_{l+1/2}(x_{l+1/2,k})} I_{l-1/2}(z). \]  
(\text{F4b})

For \( z = (1 + i)q_n \), the above equation gives
\[ \Psi_1(z, s) = \frac{1}{2z} \frac{I_{l+1/2}(z)}{I_{l-1/2}(z)} \sum_{k=1}^{\infty} \frac{J_{l+1/2}(x_{l+1/2,k}s)}{J_{l+1/2}(x_{l+1/2,k})} \left[ -\frac{2i\gamma q_n^2 + x_{l+1/2,k}^2}{4q_n^4 + x_{l+1/2,k}^2} \right]. \]  
(\text{F5})

Finally, putting \( s = r/R_s \) leads to
\[ \text{Re}\left[ \Psi(z, t)e^{i(\omega t + \pi/2)} \right] = -\sum_{k=1}^{\infty} \frac{J_{l+1/2}(x_{l+1/2,k}r/R_s)}{J_{l+1/2}(x_{l+1/2,k})} \left[ \frac{2q_n^2 \cos \omega t + x_{l+1/2,k}^2 \sin \omega t}{4q_n^4 + x_{l+1/2,k}^2} \right]. \]  
(\text{F6})

Together, Eqs. (F1), (F6) and (2.22b) establish Eqs. (5.1a) and (5.1b). Finally, for \( z = (1 + i)q_n \) and \( s = 1 \), equating real parts throughout Eq. (F5) gives
\[ \sum_{k=1}^{\infty} \frac{x_{l+1/2,k}^2}{4q_n^4 + x_{l+1/2,k}^2} = \text{Re}\left[ \frac{1}{2z} \frac{I_{l+1/2}(z)}{I_{l-1/2}(z)} \right] = \text{Re}\left[ \Psi_1(z, 1) \right] \]  
(\text{F7a})

For the same values of \( z \) and \( s \), equating the imaginary parts in Eq. (F5), leads to Eq. (3.9b).

To prove Eq. (4.12b), consider the function \( s^{l+1/2} \). Since it is continuous in the interval \( 0 \leq s \leq 1 \), it can be expanded in a Fourier-Bessel series as
\[ s^{l+1/2} = \sum_{k=1}^{\infty} \lambda_{l+1/2,k} J_{l+1/2}(x_{l+1/2,k}s). \]  
(\text{F8})

As before, the undetermined coefficients \( \lambda_{l+1/2,k} \) are determined by making use of the orthogonality of Bessel functions. Thus,
\[ \lambda_{l+1/2,k} = \frac{2(2l + 1)}{x_{l+1/2,k}^2 J_{l+1/2}(x_{l+1/2,k})}, \]  
(\text{F9a})

and
\[ s^{l+1/2} = 2(2l + 1) \sum_{k=1}^{\infty} \frac{J_{l+1/2}(x_{l+1/2,k}s)}{x_{l+1/2,k}^2 J_{l+1/2}(x_{l+1/2,k})}. \]  
(\text{F9b})

For \( s = 1 \),
\[ \sum_{k=1}^{\infty} \frac{1}{x_{l+1/2,k}^2} = \frac{1}{2(2l + 1)}. \]  
(\text{F10})
Finally, the subtraction of Eq. (F10) from Eq. (F7) gives the desired result, i.e., (4.12b).

The limiting case in Eq. (4.14a) can be proved by rewriting Eq. (4.12b) as
\[ g_i(q_n) = \sum_{k=1}^{\infty} \frac{1}{x_{l+1/2,k}^2} \frac{-1}{1 + \left( \frac{x_{l+1/2,k}^4}{4q_n^4} \right)} \]  

By virtue of Eq. (F10), Eq. (4.14a) follows. Similarly, in the limit of very small \( q_n \), Eq. (4.12b) can be recast as
\[ Lt \left[ g_i(q_n) \right] = 4q_n^4 \sum_{k=1}^{\infty} \frac{-1}{x_{l+1/2,k}^6}. \]  

Multiply Eq. (F9b) by \( s^{l+3/2} \) and integrate from \( s = 0 \) to \( s = s \). The use of Eq. (D6a) leads to
\[ s^{l+3/2} = 2(l+1)(2l+3) \sum_{k=1}^{\infty} \frac{J_{l+3/2}(x_{l+1/2,k}s)}{x_{l+1/2,k}^3 J_{l+1/2}(x_{l+1/2,k})}. \]

Multiply the above equation once again, by \( s^{l+5/2} \) and integrate from \( s = 0 \) to \( s = s \). This leads to
\[ s^{l+5/2} = 2(l+1)(2l+3)(2l+5) \sum_{k=1}^{\infty} \frac{1}{x_{l+1/2,k}^6} \frac{J_{l+5/2}(x_{l+1/2,k}s)}{J_{l+1/2}(x_{l+1/2,k})}. \]  

Putting \( s = 1 \) in Eq. (F13), and then using the recursion relation Eq. (A6b) in conjunction with Eq. (2.19b), gives
\[ 1 = 2(l+1)(2l+3)(2l+5) \sum_{k=1}^{\infty} \left[ \frac{(2l+1)(2l+3)}{x_{l+1/2,k}^4} - \frac{1}{x_{l+1/2,k}^4} \right]. \]  

Squaring both sides of Eq. (F8) and then multiplying throughout by \( s \), and finally integrating from \( s = 0 \) to \( s = 1 \) (Parseval's theorem) yields,
\[ \int_0^1 z^{2l+2}dz = \sum_{k=1}^{\infty} \sum_{k'=1}^{\infty} \lambda_{l+1/2,k} \lambda_{l+1/2,k'} \int_0^1 z J_{l+1/2}(x_{l+1/2,k}z) J_{l+1/2}(x_{l+1/2,k'}z)dz. \]

By using the orthogonality of Bessel functions and then Eq. (F9a), results in
\[ \sum_{k=1}^{\infty} \frac{1}{x_{l+1/2,k}^4} = \frac{1}{2(2l+3)(2l+1)^2}. \]  

Together, Eqs. (F15) and (F14) give Eq. (4.14b).

Finally, we consider the skin depth function obtained during the determination of the pressure distribution in Chapter 5. Equation (5.10) can be rewritten as
where \( t \) is a dummy variable \((= r / R_s)\). For \( l > 2 \), the integral in Eq. (F16) is given by (Abramowitz and Stegun, 1965)

\[
\int_{t=0}^{1} t^{l-2} e^{z} t^{l} dt = \frac{e^{z}}{(z)^{l-1}} \left[ \sum_{k=0}^{l-3} (-1)^{k} (l-2)^{k} P_{k}(z^{*})^{l-k-2} \right] + (-1)^{l-2} (l-2)!
\]

where for any two integers \( m \) and \( n \),

\[
\binom{m}{n} = \frac{m!}{(m-n)!}.
\]

The substitution of Eq. (F17) in Eq. (F16) gives

\[
I_{l}(R_s, q_n) = R_{s}^{l-1} \Re \left[ e^{-\pi/4} \left\{ \sum_{k=0}^{l-3} (-1)^{k} (l-2)^{k} P_{k}(z^{*})^{(l-k+1)} \right\} + (-1)^{l-2} (l-2)! \left( 1 - e^{-z} \right) (z^{*})^{-l+1} \right].
\]

In Eq. (F18) every term on the right hand side contains a term in \( q_n \) whose order is \( \leq -1 \). Therefore, in the limiting case of very large \( q_n \), all terms except the one containing \( q_n^{-1} \) vanish in the expression for \( q_n I_{l}(R_s, q_n) \). Thus, in this limit, Eq. (F18) together with Eq. (F16) gives

\[
\frac{L_{l} t}{q_n} \rightarrow \infty \left\{ q_n I_{l}(R_s, q_n) \right\} = \frac{R_{s}^{l-1}}{\sqrt{2}}, \quad l > 2.
\]

The case for \( l = 2 \) is trivial and it can be shown that

\[
I_{2}(R_s, q_n) = \frac{R_{s}}{\sqrt{2} q_n} \left( 1 - e^{-q_{n}} \right)
\]

which implies that in the large frequency limit, \( q_n I_{2}(R_s, q_n) \rightarrow \left( \frac{R_{s}}{\sqrt{2}} \right) \).

When \( l = 1 \), the integral on the right hand side of Eq. (F16) becomes

\[
\int_{t=0}^{1} e^{z} t^{l} dt = -E_{1}(z^{*})
\]

where \( E_{1}(z^{*}) \) is the exponential integral function.
where $E_n(z)$ is known as the exponential integral and defined as

$$E_n(z) = \int_1^\infty \frac{e^{-zt}}{t^n} \, dt,$$

for $n = 0, 1, 2, \ldots$ and $\text{Re} \, z > 0$. These functions are tabulated in mathematical handbooks such as Abramowitz and Stegun (1965). The function $E_1(z)$ is not defined at $z = 0$ and therefore, the evaluation of the integral in Eq. (F16) for $l = 1$ becomes difficult. However, the physical nature of the problem suggests a way out. Due to the skin depth effect, we are only interested in the region of the sphere that is close to the surface. Therefore, we may write

$$\int_{\varepsilon} e^{\varepsilon t} \, dt = \int_{\varepsilon}^1 e^{\varepsilon t} \, dt$$

where $0 < \varepsilon < 1$. Therefore, we finally obtain

$$\sqrt{2} q_n I_1(R_s, q_n) = \text{Re} \left[ (1-i)q_n e^{-(1-i)q_n} \int_{1}^{\varepsilon} e^{\varepsilon t} \, dt \right],$$

$$= \text{Re} \left[ (1-i)q_n e^{-(1-i)q_n} \left\{ -E_1(-(1-i)q_n) + E_1(-(1-i)q_n, \varepsilon) \right\} \right].$$

For $\varepsilon = 0.6$, this function is plotted as a function of $q_n$ in Fig. F1. The graph shows that when $q_n$ is large, the function $\sqrt{2} I_1(R_s, q_n) \to 1$ which establishes Eq. (5.11b).

![Figure F1](image-url)
Appendix G

The Integral $Z_{r',t',l',l}^{m',m,m}$

Let
\[ y_{r',t',l}^{m',m',m'} = \frac{1}{2\pi} \int_{-1}^{1} Y_{r'}^{m'*}(u,\phi)P_{r'}^{m'}(u)P_{l'}^{m}(u) \, d\phi \, du. \]  
\hspace{1cm} (G1)

Then the integral on the right hand side of Eq. (G1) is
\[ y_{r',t',l}^{m',1,1} = (-1)^{m'} \sqrt{\frac{2r'+1}{4\pi}} \frac{(l+m)!}{(l+m)!} \int_{-1}^{1} [P_{r'}^{m'}(u)e^{-im'\phi}] P_{l'}^{1}(u)P_{l}^{1} \, d\phi \, du \]  
\hspace{1cm} (G2)

where we have made use of the definition for the spherical harmonic given in Eq. (2.10). The presence of the term $e^{-im\phi}$ in the $\phi$ integral ensures that the integral in Eq. (G2) vanishes unless $m'=0$ and we are only required to find $\int_{-1}^{1} P_{r'}(u)P_{r'}^{m'}(u)P_{l}^{m}(u) \, du$.

Now consider the integral $Z_{r',t',l',l}^{m',m,m}$ as defined by Eq. (5.18). By making use of the definition of spherical harmonics (Eq. (2.10)) and using Eq. (E10) and then putting $m'=0$, $m'=-1$, and $m=1$, it can be shown that
\[ \int_{-1}^{1} P_{r'}(u)P_{r'}^{m'}(u)P_{l}^{m}(u) \, du = -4 \frac{\pi l' (l+1)(l'+1)}{(2l'+1)(2l+1)(2l'+1)} Z_{r',t',l'}^{0,-1,1} \]  
\hspace{1cm} (G3)

and hence
\[ y_{r',t',l}^{0,1,1} = -4\pi \frac{l+1}{2l+1} Z_{r',t',l}^{0,-1,1} \]  
\hspace{1cm} (G4)

The integrals $Z_{r',t',l',l}^{m',m,m}$ exist only for certain special values of the indices. They occur frequently in quantum mechanics and are computed by using 3-j symbols (Jones, 1985). For the numerical details of calculating these integrals please refer Jones (1985). In general, $Z_{r',t',l',l}^{m',m,m}$ is
\[ Z_{r',t',l',l}^{m',m,m} = \frac{1}{2\pi} \int_{-1}^{1} Y_{r'}^{m'*}(u,\phi)Y_{r'}^{m'}(u,\phi)Y_{l}^{m}(u,\phi) \, d\phi \, du \]  
\hspace{1cm} = (-1)^{m'} \frac{(2l'+1)(2l+1)(2l+1)}{4\pi} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} l' & l & l \\ -m' & m' & m \end{pmatrix} \left[ \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \right]

where $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ denotes the 3-j symbol. This symbols exists only when the following conditions are all satisfied,
\[ m_1 + m_2 + m_3 = 0, \]
\[ j_1 + j_2 - j_3 \geq 0, \]
\[ j_1 - j_2 + j_3 \geq 0, \]
\[ -j_1 + j_2 + j_3 \geq 0, \]
\[ j_1 + j_2 + j_3 \text{ is integral.} \]

These conditions, by extension determine the existence of \( Z_{i_1, i_2, i_3}^{m_1, m_2, m_3} \). The formulae for the 3-j symbols are rather lengthy and given in Jones (1985). They are therefore not repeated here.