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ON GARABEDIAN'S METHOD OF SOLVING THE WAVE EQUATION

by

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On Garabedian’s Method of Solving the Wave Equation

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Abstract

In this thesis, we shall re-examine and provide as clear an exposition as possible of a method presented by P. R. Garabedian which results in an integral formula representation of a solution to the wave equation. The method involves analytically extending a harmonic function of real arguments along a purely imaginary axis in complex space and establishing the validity of the standard integral formula for harmonic functions as a representation of a solution to the wave equation when one of the arguments is purely imaginary. This is done in the odd dimensional case by integration by parts and an application of the residue theorem, and in the even dimensional case by computing bounds on the integrals.
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INTRODUCTION

In his textbook [3] and in a couple of prior papers [1],[2], Garabedian demonstrates a connection between elliptic and hyperbolic partial differential equations. The most basic of all hyperbolic equations is the wave equation

\[ \frac{\partial^2 v}{\partial t^2} = \sum_{j=1}^{n} \frac{\partial^2 v}{\partial x_j^2}, \]

and the most basic of all elliptic equations is Laplace’s equation

\[ -\frac{\partial^2 v}{\partial t^2} = \sum_{j=1}^{n} \frac{\partial^2 v}{\partial x_j^2}. \]

While standard, traditional procedures for solving the wave equation with given initial conditions, commonly called the Cauchy problem, rarely mention any elliptic partial differential equations, Garabedian chooses to examine connections between the two basic problems mentioned above.

Fundamental solutions of elliptic differential equations lead to integral representations of solutions to Cauchy problems for those equations in bounded domains with smooth boundaries, although one cannot impose independent Cauchy data. Garabedian shows how to use an analytic continuation into the complex domain of a solution to Laplace’s equation to obtain a solution to the wave equation. He then proceeds to apply his method to more general elliptic equations to obtain solutions to other hyperbolic equations. After analytically extending into \( \mathbb{C}^n \), Garabedian examines solutions to the elliptic equations and their integral representations for arguments...
with one of the variables complex and the rest real. Then he shows that the main tools needed for the analysis are contour integration and the residue theorem.

Garabedian anticipated that his techniques would have wide-ranging applications to problems in hyperbolic systems of partial differential equations, fluid dynamics and magnetohydrodynamics [2]. Unfortunately, Garabedian’s ideas and methods seem to have remained dormant for approximately thirty years. Recently mathematicians have rediscovered the method of analytically continuing solutions of one class of differential equations to obtain solutions of another class. Taylor [6] uses an analytic continuation of the Fourier transform of the fundamental solution for the wave equation. The extension of a function of the real variables $y$ and $\xi$ to the half-plane \( \{ y \in \mathbb{C} : \text{Re} \ y \geq 0 \} \) leads to a direct formula for the fundamental solution. Keyfitz and Lopes-Filho [5] utilize Garabedian’s methods to study solutions of $2 \times 2$ systems of conservation laws that change type (hyperbolic to elliptic). They derive real analytic solutions in the elliptic states by analytically extending the systems to ones which are hyperbolic in the complex plane and then finding holomorphic solutions to the systems.

So, what is the connection between the archetypes for hyperbolic and elliptic problems, the wave equation and Laplace’s equation? Using the standard fundamental solution $E(x)$ for the Laplace operator, a function $u$ which is harmonic in a region $\Omega$ with smooth boundary can be represented by the integral formula

$$u(p) = \int_{\partial \Omega} \left[ u \frac{\partial E(x - p)}{\partial \nu} - E(x - p) \frac{\partial u}{\partial \nu} \right] dS,$$
where $\nu$ is the inward pointing normal and $p \in \Omega$. The formula expresses $u$ as an integral function of both its value and the value of its normal derivative on the boundary of $\Omega$. This thesis re-examines and provides as clear an exposition as possible of a method that was presented in [2] and [3] by Garabedian which results in an integral formula representation of a solution to the wave equation for $t > 0$. The starting point for the investigation is the examination of the above integral formula for harmonic functions of an imaginary argument, $u(x_1, \ldots, x_n, it)$. Once the validity of the integral is established, then we can directly substitute true solutions of the wave equation, $v(x_1, \ldots, x_n, t)$, into the integral formula, yielding the desired integral representation of solutions to the wave equation. Thus, the key to obtaining the desired representation is the understanding of the known integral formula when one of the arguments of the integrand is purely imaginary.

In Chapter 1 we introduce an analytic continuation of a harmonic function along a purely imaginary axis in $\mathbb{C}^{n+1}$ and how that continuation affects the choice of the region $\Omega$ and its boundary, which is the surface on which the integration is performed. The surface of integration $\partial \Omega$ must be chosen to encompass the singularities of the integrand. Chapter 2 is an examination of the integral formula for this harmonic function of an imaginary argument in the three space variable case. After a change of coordinates to a cylindrical system, the key step is the calculation of a contour integral by an application of the residue theorem. The two space variable case, examined in Chapter 3, requires a careful choice of initial data and estimates on the limiting values
of the path integrals as the surface is squeezed around the singularities. The solution for more general initial data is addressed in the Appendix. In Chapter 4 we look at the problem in higher odd space dimensions. The contour integral and the use of the residue theorem mirrors that in Chapter 2 for three space dimensions. After an appropriate number of integrations by parts, the solutions are simply derivatives in the time variable of the solution of the three dimensional wave equation obtained in Chapter 2.
Chapter 1: THE WAVE EQUATION

We wish to find an integral formula representation of the solution to the wave equation in \( n \) space variables

\[
\frac{\partial^2 v}{\partial t^2} = \sum_{j=1}^{n} \frac{\partial^2 v}{\partial x_j^2},
\]

with initial data

\[
v(x_1, \ldots, x_n, 0) = f(x_1, \ldots, x_n),
\]
\[
\frac{\partial v}{\partial t}(x_1, \ldots, x_n, 0) = g(x_1, \ldots, x_n).
\]

In [2] and [3] Garabedian presents a method of obtaining such an integral representation. He utilizes an analytic continuation of a harmonic function in \( \mathbb{R}^{n+1} \) to complex values of \( x_{n+1} \). On this imaginary axis the harmonic function satisfies a wave equation; moreover, this function of an imaginary argument can be substituted into the well-known integral formula for harmonic functions. This integral formula is Garabedian’s prime candidate for a representation of a solution to (1)-(2). The question is then whether this integral formula carries any validity as a solution. Is the integrand infinite anywhere on the surface of integration? Does the integral hold any of the initial data? Can the integral be simplified in any way to make information about the solution and initial data more accessible? We shall examine Garabedian’s method and the integral in detail.

1.1 Laplace’s Equation

To begin, we consider the Laplace equation in \( \mathbb{R}^{n+1} \),

\[
- \Delta u = 0.
\]
Letting $x = (x_1, \ldots, x_n, x_{n+1})$ be represented by $x = (x', x_{n+1})$, the standard fundamental solution for the Laplace operator is

$$E(x) = \frac{1}{(n-1)\omega_{n+1}} |x|^{-(n-1)} = \frac{1}{(n-1)\omega_{n+1}} (|x'|^2 + x_{n+1}^2)^{-(n-1)/2},$$

where $\omega_{n+1}$ is the surface area of the unit sphere in $\mathbb{R}^{n+1}$. Then for a point $p$ in a region $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary,

$$u(p) = \int_{\partial \Omega} \left[ u \frac{\partial E(x - p)}{\partial \nu} - E(x - p) \frac{\partial u}{\partial \nu} \right] dS,$$

where $\nu$ is the inward pointing normal to $\Omega$. Without loss of generality, suppose $0 \in \Omega$. Then $u$ is complex analytic in a small neighborhood of $0 \in C^{n+1}$; in particular, $u$ is analytic for purely imaginary values of $x_{n+1}$ in that neighborhood. Letting $x_{n+1} = it$,

$$-\Delta u = \frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2},$$

since by the chain rule $\frac{\partial u}{\partial x_{n+1}} = \frac{1}{i} \frac{\partial u}{\partial t}$. So, $u(x_1, \ldots, x_n, it)$ satisfies the wave equation.

### 1.2 The Surface of Integration

Now let $P = (0, \ldots, 0, it), t > 0$. Then

$$|x - P|^2 = |x'|^2 + (x_{n+1} - it)^2 = |x'|^2 + x_{n+1}^2 - t^2 - 2ix_{n+1}t.$$

Thus, $|x - P|^2 = 0$ precisely when $x_{n+1} = 0$ and $|x'| = t$, that is, on the $(n-1)$-dimensional sphere $|x'| = t$ in the $n$-space $x_{n+1} = 0$. For equation (4) to remain valid as a representation of the solution of the wave equation, the surface $\partial \Omega$ must stay away from the points where $|x - P|^2 = 0$. Since the integral in (4) is independent of
the surface of integration, $\partial \Omega$ can be a thin, "tubelike" surface which surrounds the $(n - 1)$-sphere where $|x - P|^2 = 0$, and which can be visualized as a contour in the $|x'| - x_{n+1}$ plane.
Chapter 2: THE WAVE EQUATION IN THREE SPACE VARIABLES

To examine the integral in (4) for $x = (x_1, x_2, x_3, x_4)$ and $P = (0, 0, 0, it)$, we will change to cylindrical coordinates:

$$
x_1 = |x'|\sin\theta_1 \cos\theta_2
$$
$$
x_2 = |x'|\sin\theta_1 \sin\theta_2
$$
$$
x_3 = |x'|\cos\theta_1
$$
$$
x_4 = x_4.
$$

Then $|x - P|^2 = 0$ and $\partial \Omega$ can be shown on a $|x'| - x_4$ plane.

The surface of integration $\partial \Omega$ is a path that follows the $|x'|$-axis and circles the point $(0, t)$ on the circle $(|x'| - t)^2 + x_4^2 = \epsilon^2$. We shall let $\epsilon \to 0$ in order to squeeze $\partial \Omega$ around $|x - P|^2 = 0$.

2.1 The Integrand Along the Paths $\alpha$ and $\beta$

We now want to examine the integrand in (4) along the paths $\alpha$ and $\beta$. 
On the path $\alpha$, $x_4 = \epsilon$. So

$$|x - P| = \sqrt{|x'|^2 + (\epsilon - it)^2}$$

$$= \sqrt{\epsilon + i|x'| - it \sqrt{\epsilon - i|x'| - it}};$$

$$\lim_{\epsilon \to 0} |x - P| = \sqrt{i|x'| - it \sqrt{-i|x'| - it}}.$$

On the path $\beta$, $x_4 = -\epsilon$. Then

$$|x - P| = \sqrt{|x'|^2 + (-\epsilon - it)^2}$$

$$= \sqrt{|x'|^2 + (\epsilon + it)^2}$$

$$= \sqrt{\epsilon - i|x'| + it \sqrt{i|x'| + it}};$$

$$\lim_{\epsilon \to 0} |x - P| = \sqrt{-i|x'| + it \sqrt{i|x'| + it}}$$

$$= i \sqrt{i|x'| - it \sqrt{-i|x'| - it}}$$

$$= -\sqrt{i|x'| - it \sqrt{-i|x'| - it}}.$$

So as $\epsilon \to 0$, $|x - P|^2$ is the same on $\alpha$ and $\beta$. Parameterizing the paths by $x_4(s)$ and $|x'|(s)$, we have

$$\nu = \frac{(-|x'|, x'_4)}{\sqrt{|x'|^2(s)^2 + x'_4(s)^2}}.$$

Parameterize path $\alpha$ by

$$x_4 = \epsilon, \quad |x'|(s) = (t - \delta)s;$$

$$\frac{dx_4}{ds} = 0, \quad \frac{d|x'|}{ds} = t - \delta.$$

So on $\alpha$, $\nu = (-1, 0)$ and $\frac{\partial}{\partial \nu} = \frac{\partial}{\partial x_4}$. Parameterize path $\beta$ by

$$x_4 = -\epsilon, \quad |x'|(s) = (t - \delta)(1 - s);$$

$$\frac{dx_4}{ds} = 0, \quad \frac{d|x'|}{ds} = -(t - \delta).$$
So on $\beta$, $\nu = (1, 0)$ and $\frac{\partial}{\partial \nu} = \frac{\partial}{\partial x_4}$. The integrand on $\alpha$ is then

$$u \frac{\partial}{\partial \nu} |x - P|^2 - |x - P|^{-2} \frac{\partial u}{\partial \nu}$$

$$= -u \frac{\partial}{\partial x_4} |x - P|^2 + |x - P|^{-2} \frac{\partial u}{\partial x_4},$$

and on $\beta$ is

$$u \frac{\partial}{\partial \nu} |x - P|^2 - |x - P|^{-2} \frac{\partial u}{\partial \nu}$$

$$= u \frac{\partial}{\partial x_4} |x - P|^2 - |x - P|^{-2} \frac{\partial u}{\partial x_4}.$$

So

$$\frac{1}{4\pi^2} \int_{\alpha + \beta} \left( u \frac{\partial}{\partial \nu} |x - P|^2 - |x - P|^{-2} \frac{\partial u}{\partial \nu} \right) dS = 0.$$

### 2.2 A New Path of Integration

Now let $|x'| = r$ and change the path in the $x_4 - r$ plane. $\Omega$ is now the region $|x_4| < \epsilon$, $|r - t| < \delta$. The right side of equation (4) can now be written

$$\frac{1}{4\pi^2} \int_{\partial \Omega} \int_{S^2} \left( u(x) \frac{\partial}{\partial \nu} |x - P|^2 - \frac{1}{|x - P|^2} \frac{\partial u}{\partial \nu} \right) \sqrt{r'(s)^2 + |x'_4(s)|^2} ds dS^2.$$

At each point $(r, x_4) = (a, \kappa)$, we integrate over the sphere $r = a$ in the space $x_4 = \kappa$. 

![Diagram](image-url)
Let \( A(f, y, \rho) = \frac{1}{4\pi} \int_{S(0, 1)} f(y + \rho \sigma) dS(\sigma) \) be the average of \( f \) over the sphere centered at \( y \) with radius \( \rho \). Then the right side of equation (4) becomes

\[
\frac{1}{\pi} \int_{\partial \Omega} \left[ A(u, 0, r) \frac{\partial}{\partial \nu} \frac{1}{|x - P|^2} - \frac{1}{|x - P|^2} \frac{\partial A(u, 0, r)}{\partial \nu} \right] \sqrt{r'(s)^2 + [x'_4(s)]^2} r^2 ds
\]

\[
= \frac{1}{\pi} \int_{\partial \Omega} \left[ \frac{1}{|x - P|^2} \frac{\partial A}{\partial x_4} - A \frac{\partial}{\partial x_4} \left( \frac{1}{|x - P|^2} \right) \right] r^2 ds
\]

\[
- \left[ \frac{1}{|x - P|^2} \frac{\partial A}{\partial r} - A \frac{\partial}{\partial r} \left( \frac{1}{|x - P|^2} \right) \right] r^2 dx_4
\]

\[
= \frac{1}{\pi} \int_{\gamma_1 + \gamma_3} \left[ \frac{r^2}{r^2 + (x_4 - it)^2} \frac{\partial A(u, 0, r)}{\partial x_4} - A(u, 0, r) \frac{\partial}{\partial x_4} \left( \frac{r^2}{r^2 + (x_4 - it)^2} \right) \right] dr
\]

\[
+ \frac{1}{\pi} \int_{\gamma_2 + \gamma_4} \left[ A(u, 0, r) \frac{\partial}{\partial r} \left( \frac{1}{|x - P|^2} \right) - \frac{1}{|x - P|^2} \frac{\partial A(u, 0, r)}{\partial r} \right] r^2 dx_4.
\]

As \( \epsilon \to 0 \), \( \int_{\gamma_2 + \gamma_4} \to 0 \). So we must compute

\[
\lim_{\epsilon \to 0} \frac{1}{\pi} \int_{\gamma_1 + \gamma_3} \left[ \frac{r^2}{r^2 + (x_4 - it)^2} \frac{\partial A(u, 0, r)}{\partial x_4} - A(u, 0, r) \frac{\partial}{\partial x_4} \left( \frac{r^2}{r^2 + (x_4 - it)^2} \right) \right] dr.
\]

### 2.3 An Application of the Residue Theorem

Consider

\[
\int_{\gamma_1 + \gamma_3} \frac{F(r, x_4)}{r^2 + (x_4 - it)^2} dr
\]

\[
= \int_{\gamma_1 + \gamma_3} \frac{F(x_4)}{r^2 + (x_4 - it - i\rho)(x_4 - it + i\rho)} dr.
\]

Let \( \rho = ir \), \( d\rho = idr \). Then

\[
\int_{\gamma_1} \frac{F(e^{i\theta}, x_4)}{(x_4 - it - \rho)(x_4 - it + \rho)} d\rho.
\]
which by the residue theorem

\[
\frac{1}{i} \cdot 2\pi i \left[ \frac{F\left(\frac{t}{i}, x_4\right)}{x_4 - it - \rho} \right]_{\rho = -x_4 + it}
\]

\[= \frac{2\pi F(ix_4 + t, x_4)}{2(x_4 - it)} = \frac{\pi F(ix_4 + t, x_4)}{x_4 - it}.\]

Then \(\lim_{x_4 \to 0}(7) = \frac{i\pi F(t, 0)}{t}\).

Now

\[
(6) = \frac{1}{\pi} \left[ \frac{i\pi}{t} \cdot t^2 \frac{\partial A(u, 0, t)}{\partial x_4} \right] = \frac{1}{\pi} \cdot i \cdot \frac{\partial}{\partial t} \left[ \frac{i\pi}{t} \left( t^2 A(u, 0, t) \right) \right]
\]

\[= it \frac{\partial A(u, 0, t)}{\partial x_4} + \frac{\partial}{\partial t} (tA(u, 0, t)).\]

(The second term comes from the facts that

1) \(\frac{\partial}{\partial x_4} \left( \frac{r^2}{r^2 + (x_4 - it)^2} \right) = \frac{i\partial}{\partial t} \left( \frac{r^2}{r^2 + (x_4 - it)^2} \right),\)

and 2) \(A(u, 0, r)\) does not depend on \(t\); so that the partial derivative with respect to \(t\) now in the second term of (6) can be moved outside the integral.)

So,

\[
(8) \quad u(x_1, x_2, x_3, it) = it \frac{\partial A}{\partial x_4}(u(x', 0), 0, t) + \frac{\partial}{\partial t} (tA(u(x', 0), 0, t)).
\]
Before examining the integral in (4) for $x = (x_1, x_2, x_3)$, and $P = (0, 0, it)$, we shall once again change coordinates to a cylindrical system:

$$x_1 = |x'| \cos \theta$$
$$x_2 = |x'| \sin \theta$$
$$x_3 = x_3.$$

Then we can exhibit $|x - P|^2 = 0$ and $\partial \Omega$ on a $|x'| - x_3$ plane.

As in the three variable case, $\partial \Omega$ is a path that follows the $|x'|$-axis and circles the point $(0, t)$ on the circle $(|x'| - t)^2 + x_3^2 = \epsilon^2$. We shall let $\epsilon \to 0$ to squeeze $\partial \Omega$ around $|x - P|^2 = 0$.

3.1 The Integrand Along the Paths $\alpha$ and $\beta$

We now want to examine the integrand in (4) along the paths $\alpha$ and $\beta$.

On the path $\alpha$, $x_3 = \epsilon$. So

$$|x - P| = \sqrt{|x'|^2 + (\epsilon - it)^2}$$
$$= \sqrt{\epsilon + i|x'| - it}\sqrt{\epsilon - i|x'| - it};$$
Then
\[
\lim_{\epsilon \to 0} |x - P| = \sqrt{i|x'| - it} \sqrt{-i|x'| - it}.
\]

Call this value $|x - P|_\alpha$.

On the path $\beta$, $x_3 = -\epsilon$. Then
\[
|x - P| = \sqrt{|x'|^2 + (-\epsilon - it)^2} = \sqrt{|x'|^2 + (\epsilon + it)^2} = \sqrt{\epsilon - i|x'| + it\sqrt{\epsilon + i|x'| + it}};
\]
\[
\lim_{\epsilon \to 0} |x - P| = \sqrt{-i|x'| + it\sqrt{i|x'| + it}} = i\sqrt{i|x'| - it\sqrt{-i|x'| - it}} = -\sqrt{i|x'| - it\sqrt{-i|x'| - it}}.
\]

Call this value $|x - P|_\beta$. So as $\epsilon \to 0$, $|x - P|$ changes sign from $\alpha$ to $\beta$. Parameterizing the paths by $x_3(s)$ and $|x'|(s)$, we have
\[
\nu = \frac{(-|x'|', x_3')}{\sqrt{|x'|'(s)^2 + x_3'(s)^2}}.
\]

Parameterize path $\alpha$ by
\[
x_3 = \epsilon, \quad |x'|(s) = (t - \delta)s;
\]
\[
\frac{dx_3}{ds} = 0, \quad \frac{dx_3'}{ds} = t - \delta.
\]

So on $\alpha$, $\nu = (-1, 0)$ and $\frac{\partial}{\partial \nu} = \frac{-\partial}{\partial x_3}$. Parameterize path $\beta$ by
\[
x_3 = -\epsilon, \quad |x'|(s) = (t - \delta)(1 - s);
\]
\[
\frac{dx_3}{ds} = 0, \quad \frac{dx_3'}{ds} = -(t - \delta).
So on $\beta$, $\nu = (1, 0)$ and $\frac{\partial}{\partial \nu} = \frac{\partial}{\partial x_3}$. The integrand on $\alpha$ is then

$$u \frac{\partial}{\partial \nu} |x - P|_\alpha - |x - P|_\alpha \frac{\partial u}{\partial \nu}$$

$$= -u \frac{\partial}{\partial x_3} |x - P|_\alpha + |x - P|_\alpha \frac{\partial u}{\partial x_3},$$

and on $\beta$ is

$$u \frac{\partial}{\partial \nu} |x - P|_\beta - |x - P|_\beta \frac{\partial u}{\partial \nu}$$

$$= u \frac{\partial}{\partial x_3} |x - P|_\beta - |x - P|_\beta \frac{\partial u}{\partial x_3}.$$ 

$$= -u \frac{\partial}{\partial x_3} |x - P|_\alpha + |x - P|_\alpha \frac{\partial u}{\partial x_3}.$$

So

$$\frac{1}{4\pi} \int_{\alpha + \beta} \left( u \frac{\partial}{\partial \nu} |x - P| - |x - P| \frac{\partial u}{\partial \nu} \right) dS = \frac{1}{4\pi} \int_{\alpha + \beta} 2 \left( u \frac{\partial}{\partial \nu} |x - P|_\alpha - |x - P|_\alpha \frac{\partial u}{\partial \nu} \right) dS.$$ 

### 3.2 A New Path of Integration

Now let $|x'| = r$ and change the path in the $x_3 - r$ plane. $\Omega$ is now the region $|x_3| < \epsilon$, $0 \leq r \leq t + c$. The right side of equation (4) can now be written

$$\frac{1}{4\pi} \int_{\partial \Omega} \int_{S^1} \left[ u(x) \frac{\partial}{\partial \nu} \frac{1}{|x - P|} - \frac{1}{|x - P|} \frac{\partial u}{\partial \nu} \right] \sqrt{r'(s)^2 + [x_3'(s)]^2} ds \right) r dS^1.$$ 

At each point $(r, x_3) = (a, \kappa)$, we integrate over the circle $r = a$ in the plane $x_3 = \kappa$. 

![Diagram](image)
Let \(A(f, y, \rho) = \frac{1}{2\pi} \int_{S(0,1)} f(y+\rho \sigma) dS(\sigma)\) be the average of \(f\) over the circle centered at \(y\) with radius \(\rho\). Then the right side of equation (4) becomes

\[
\frac{1}{2} \int_{\partial \Gamma} \left[ A(u, 0, r) \frac{\partial}{\partial u} \frac{1}{|x-P|} - \frac{1}{|x-P|} \frac{\partial A(u, 0, r)}{\partial v} \right] \left( \sqrt{r'(s)^2 + |x_3(s)|^2} r ds \right)
\]

\[
= \frac{1}{2} \int_{\partial \Gamma} \left( \frac{1}{|x-P|} \frac{\partial A}{\partial x_3} - A \frac{\partial}{\partial x_3} \left( \frac{1}{|x-P|} \right) \right) r dr - \frac{1}{2} \int_{\gamma_1+\gamma_3} \left( \frac{1}{|x-P|} \frac{\partial A}{\partial r} - A \frac{\partial}{\partial r} \left( \frac{1}{|x-P|} \right) \right) r dx_3
\]

As \(\epsilon \to 0\), \(\int_{\gamma_2+\gamma_4} \to 0\). Denoting \(|x-P|\) on \(\gamma_1\) above \(r = t\) by \(|x-P|_1\) and on \(\gamma_3\) above \(r = t\) by \(|x-P|_3\),

\[
|x-P|_1 = \sqrt{|x'|^2 - t^2 + \epsilon^2 - 2it\epsilon};
\]

\[
\lim_{\epsilon \to 0} |x-P|_1 = \sqrt{|x'|^2 - t^2};
\]

\[
|x-P|_3 = \sqrt{|x'|^2 - t^2 + \epsilon^2 + 2it\epsilon};
\]

\[
\lim_{\epsilon \to 0} |x-P|_3 = \sqrt{|x'|^2 - t^2}.
\]

So on the pieces of \(\gamma_1\) and \(\gamma_3\) where \(t < r \leq t + c\), the integral goes to 0 with \(\epsilon\). So we are left to compute

\[
\lim \lim_{\delta \to 0} \int_0^{t+\delta} \left[ -A \frac{\partial}{\partial x_3} \left( \frac{r}{|x-P|} \right) \right] dr
\]

\[
= \lim \lim_{\delta \to 0} \int_0^{(t+\delta)} \frac{r}{|x-P|} \frac{\partial A}{\partial x_3} dr - \lim \lim_{\delta \to 0} \int_0^{(t+\delta)} A \frac{\partial}{\partial x_3} \left( \frac{r}{|x-P|} \right) dr.
\]
3.3 The Limits of the Integrals

We shall now examine the first integral in (9).

\[
\lim_{\delta \to 0} \lim_{\epsilon \to 0} \int_0^{t+\delta} \frac{r}{|x-P|} \frac{\partial A}{\partial x_3} \, dr
\]

\[
= \lim_{\delta \to 0} \int_0^{t+\delta} \frac{r}{\sqrt{ir - it\sqrt{-ir - it}} \frac{\partial A}{\partial x_3} \, dr
\]

\[
= \lim_{\delta \to 0} i \int_0^{t+\delta} \frac{r}{\sqrt{t^2 - r^2}} \frac{\partial A}{\partial x_3} \, dr.
\]

\[
= i \int_0^t \frac{r}{\sqrt{t^2 - r^2}} \frac{\partial A}{\partial x_3} \, dr.
\]

To examine the second integral in (9), we seem to need to restrict ourselves to the initial condition \( f(x_1, x_2) = 0 \) in (2). Then \( A(u, 0, r) = O(\epsilon) \), and

\[
\frac{\partial}{\partial x_3} \left( \frac{1}{|x-P|} \right) = \frac{-(x_3 - it)}{(r^2 + (x_3 - it)^2)^{\frac{3}{2}}}
\]

Then

\[
\left| \int_0^{t+\delta} A \frac{\partial}{\partial x_3} \left( \frac{r}{|x-P|} \right) \, dr \right|
\]

\[
\leq C \epsilon \int_0^{t+\delta} \left( |r^2 + \epsilon^2 - t^2 - 2\epsilon it| \frac{3}{2} (\epsilon - it) \right) |r| \, dr
\]

\[
\leq C \epsilon \int_0^{t+\delta} \left( |r^2 + \epsilon^2 - t^2| + 2\epsilon t \right)^{\frac{3}{2}} \, r \, dr
\]

\[
\leq C \epsilon \int_{-\infty}^{\infty} \left( |\rho + \epsilon^2 - t^2| + 2\epsilon t \right)^{\frac{3}{2}} \, d\rho
\]

(by the change of variables \( r^2 = \rho \))

\[
= C \epsilon \int_{-\infty}^{\infty} (|\rho| + 2\epsilon t)^{\frac{3}{2}} \, d\rho
\]

\[
= C \epsilon \int_0^{\infty} (|\rho| + 2\epsilon t)^{\frac{3}{2}} \, d\rho
\]

\[
= C \epsilon \left( \rho + 2\epsilon t \right)^{\frac{1}{2}} \bigg|_0^{\infty}.
\]
So the absolute value of the second integral in (9) is bounded by

\[
\frac{C\epsilon}{\sqrt{2\epsilon t}} = C\sqrt{\epsilon},
\]

That integral then goes to 0 with \(\epsilon\).

Thus,

\[
(10) \quad u(x_1, x_2, it) = i \int_0^t \frac{r}{\sqrt{t^2 - r^2}} \partial_A(u(x', 0), 0, r)dr.
\]
Chapter 4: HIGHER DIMENSIONS

4.1 The Surface of Integration

We wish to examine the integral in (4) for \( x = (x_1, \ldots, x_n, x_{n+1}) \) and \( P = (0, \ldots, it) \) when \( n > 3 \) is odd. Our cylindrical coordinate system consists of \( |x'|, x_{n+1}, \) and \( \theta_1, \ldots, \theta_{n-1} \).

We integrate on a surface represented in the \( |x'| - x_{n+1} \) plane by a path which follows the \( |x'| \)-axis and circles the point \((0, t)\) on a circle of radius \( \epsilon \). We shall let \( \epsilon \to 0 \) to squeeze \( \partial \Omega \) around \( |x - P|^2 = 0 \).

![Diagram](image)

As in the three-variable case, integrating along the paths \( \alpha \) and \( \beta \) yields 0. So, let \( |x'| = r \) and change the path of integration. \( \Omega \) will be the region \( |x_{n+1}| < \epsilon, |r - t| < \delta \).

The right side of equation (4) can now be written

\[
\frac{1}{\sigma_{n+1}} \int_{\partial \Omega} \int_{S^{n-1}} \left[ u(x) \frac{\partial}{\partial \nu} \frac{1}{|x - P|^{n-1}} - \frac{1}{|x - P|^{n-1}} \frac{\partial u}{\partial \nu} \right] \sqrt{r'(s)^2 + [x'_{n+1}(s)]^2} r^{n-1} dS^{n-1},
\]

where \( \sigma_{n+1} = (n - 1)\omega_{n+1} \). At each point \((r, x_{n+1}) = (a, \kappa)\), we integrate over the \((n-1)\)-sphere \( r = a \) in the space \( x_{n+1} = \kappa \).
Let $I(f, y, \rho) = \int_{S(0,1)} f(y + \rho \sigma) dS(\sigma)$ be the integral of $f$ over the $(n-1)$-sphere centered at $y$ with radius $\rho$. Then the right side of equation (4) becomes

$$
\frac{1}{\sigma_{n+1}} \int_{\partial \Omega} \left[ I(u, 0, r) \frac{\partial}{\partial \nu} \frac{1}{|x - P|^{n-1}} - I(u, 0, r) \frac{\partial}{\partial r} \left( \frac{1}{|x - P|^{n-1}} \right) \right] \sqrt{r'(s)^2 + [x'_{n+1}(s)]^2} r^{n-1} ds
$$

$$
= \frac{1}{\sigma_{n+1}} \int_{\partial \Omega} \left[ \frac{1}{|x - P|^{n-1}} \frac{\partial I}{\partial x_{n+1}} - I \frac{\partial}{\partial r} \left( \frac{1}{|x - P|^{n-1}} \right) \right] r^{n-1} dr
$$

$$
- \frac{1}{\sigma_{n+1}} \int_{\partial \Omega} \left[ \frac{1}{|x - P|^{n-1}} \frac{\partial I}{\partial r} - I \frac{\partial}{\partial r} \left( \frac{1}{|x - P|^{n-1}} \right) \right] r^{n-1} dx_{n+1}
$$

$$
= \frac{1}{\sigma_{n+1}} \int_{\gamma_1 + \gamma_3} \frac{r^{n-1}}{(r^2 + (x_{n+1} + it)^2)^{n-1/2}} \frac{\partial I(u, 0, r)}{\partial x_{n+1}} dr
$$

$$
- I(u, 0, r) \frac{\partial}{\partial x_{n+1}} \left( \frac{r^{n-1}}{(r^2 + (x_{n+1} - it)^2)^{n-1/2}} \right) |dr|
$$

$$
+ \frac{1}{\sigma_{n+1}} \int_{\gamma_2 + \gamma_4} \left[ I(u, 0, r) \frac{\partial}{\partial \nu} \left( \frac{1}{|x - P|^{n-1}} \right) - I(u, 0, r) \frac{\partial}{\partial r} \left( \frac{1}{|x - P|^{n-1}} \right) \right] r^{n-1} dx_{n+1}.
$$

As $\epsilon \to 0$, $\int_{\gamma_2 + \gamma_4} \to 0$. 
4.2 An Integration by Parts

So we are left with

\[
\lim_{\epsilon \to 0} \frac{1}{\sigma_{n+1}} \int_{\Omega_{n+3}} \frac{r^{n-1}}{(r^2 + (x_{n+1} - it)^2)^{\frac{n-1}{2}}} \frac{\partial I(u, 0, r)}{\partial x_{n+1}}
\]

\[-I(u, 0, r) \frac{\partial}{\partial x_{n+1}} \left( \frac{r^{n-1}}{|r^2 + (x_{n+1} - it)^2|^{\frac{n-1}{2}}} \right) \, dr \]

\[= \lim_{\epsilon \to 0} \frac{1}{\sigma_{n+1}} \int_{\Omega_{n+3}} \frac{r}{(r^2 + (x_{n+1} - it)^2)^{\frac{n-1}{2}}} \frac{\partial I}{\partial x_{n+1}}
\]

\[-I \frac{\partial}{\partial x_{n+1}} \left( \frac{r}{|r^2 + (x_{n+1} - it)^2|^{\frac{n-1}{2}}} \right) \, r^{n-2} \, dr \]

(11) \[= \lim_{\epsilon \to 0} \frac{1}{(n-3)\sigma_{n+1}} \int_{\Omega_{n+3}} \left( -\frac{\partial}{\partial r} \left( \frac{1}{|r^2 + (x_{n+1} - it)^2|^{\frac{n-3}{2}}} \right) \right) \frac{\partial I(u, 0, r)}{\partial x_{n+1}}
\]

\[+ I(u, 0, r) \frac{\partial}{\partial x_{n+1}} \frac{\partial}{\partial r} \left( \frac{1}{|r^2 + (x_{n+1} - it)^2|^{\frac{n-3}{2}}} \right) r^{n-2} \, dr \]

because

\[
\frac{r}{|x - P|^{n-1}} = -\frac{1}{n-3} \frac{\partial}{\partial r} \left( \frac{1}{|r^2 + (x_{n+1} - it)^2|^{\frac{n-3}{2}}} \right).
\]

Performing an integration by parts, equation (11) is equal to

\[
\lim_{\epsilon \to 0} \frac{1}{(n-3)\sigma_{n+1}} \int_{\Omega_{n+3}} \frac{1}{\partial r} \left( \frac{r^{n-2}}{|r^2 + (x_{n+1} - it)^2|^{\frac{n-3}{2}}} \right) \frac{\partial}{\partial x_{n+1}} \left( \frac{1}{|r^2 + (x_{n+1} - it)^2|^{\frac{n-3}{2}}} \right)
\]

\[- \frac{\partial I}{\partial x_{n+1}} \left( \frac{r^{n-2}}{|r^2 + (x_{n+1} - it)^2|^{\frac{n-3}{2}}} \right) \, \partial \gamma_{1+\partial \gamma_{3}}
\]

\[+ \lim_{\epsilon \to 0} \frac{1}{(n-3)\sigma_{n+1}} \int_{\Omega_{n+3}} \frac{1}{\partial r} \left( \frac{1}{|r^2 + (x_{n+1} - it)^2|^{\frac{n-3}{2}}} \right) \frac{\partial}{\partial x_{n+1}} \left( \frac{r^{n-2}}{|r^2 + (x_{n+1} - it)^2|^{\frac{n-3}{2}}} \right)
\]

\[- \frac{\partial}{\partial x_{n+1}} \left( \frac{1}{|r^2 + (x_{n+1} - it)^2|^{\frac{n-3}{2}}} \right) \frac{\partial}{\partial r} \left( r^{n-2} \right) \, dr \).\]
The integrated terms to be evaluated at the boundaries of $\gamma_1$ and $\gamma_3$ stay away from $|x - P| = 0$. So, as $\epsilon \to 0$, those terms sum to zero. Then, after $\frac{n-5}{2}$ more integrations by parts, equation (11) is equal to

\[
\lim_{\epsilon \to 0} \frac{1}{2 \cdot 4 \cdots (n-3) \sigma_{n+1}} \int_{\gamma_1 + \gamma_3} \left[ \frac{r}{r^2 + (x_{n+1} - it)^2} \left( \frac{\partial}{r \partial r} \right)^{\frac{n-3}{2}} \left( r^{n-2} \frac{\partial I}{\partial x_{n+1}} \right) \right. \\
\left. - \frac{\partial}{\partial x_{n+1}} \left( \frac{r}{r^2 + (x_{n+1} - it)^2} \right) \left( \frac{\partial}{r \partial r} \right)^{\frac{n-3}{2}} \left( r^{n-2} I \right) \right] dr.
\]

Now using the residue calculation from the three variable case (pp.10-11),

\[
(12) = \frac{1}{2 \cdot 4 \cdots (n-3) \sigma_{n+1}} \left[ i \pi \left( \frac{\partial}{t \partial t} \right)^{\frac{n-3}{2}} \left( t^{n-2} \frac{\partial I}{\partial x_{n+1}} \right) + \pi \frac{\partial}{\partial t} \left( i \partial t \right)^{\frac{n-3}{2}} \left( t^{n-2} I \right) \right].
\]

So, for $n > 3$ odd,

\[
(13) \ u(x', it) = \frac{1}{K \sigma_{n+1}} \left[ i \pi \left( \frac{\partial}{t \partial t} \right)^{\frac{n-3}{2}} \left( t^{n-2} \frac{\partial I}{\partial x_{n+1}} \right) + \pi \frac{\partial}{\partial t} \left( i \partial t \right)^{\frac{n-3}{2}} \left( t^{n-2} I \right) \right],
\]

with $K = 2 \cdot 4 \cdots (n-3)$. 
CONCLUSION: Solution to the Wave Equation in All Dimensions

The solution to (1)-(2) in one space variable is essentially trivial and can be found in introductory textbooks on partial differential equations (see [4] for example). We remark that applying Garabedian's method to the one space variable case presents a different set of obstacles because the fundamental solution to the Laplacian in two dimensions involves a logarithmic function of $|x|$.

For all the other odd space dimensions ($n \geq 3$) and for the even space dimension $n = 2$, we have demonstrated how to find integral representations of solutions to (1)-(2) ($f = 0$ for $n = 2$) that are harmonic functions $u$ of an imaginary argument $x_{n+1} = it$. The standard method of descent can be applied to obtain solutions in the higher even space dimensions. This method can also be found in partial differential equations textbooks ([3],[4]). In the Appendix we show how to find the solution for $n = 2$ space variables with initial data $f \neq 0$.

Finally, to interpret the equations (8), (10) and (13) as representing solutions to the Cauchy problem (1)-(2), we note that the spherical averages and their derivatives in the $(n + 1)^{st}$ variable evaluated at $x_{n+1} = 0$ are independent of whether $x_{n+1}$ is regarded as purely real or purely imaginary. Therefore, we can substitute actual solutions $v$ of the wave equation into those integral representations,

$$v(x_1, ..., x_n, t) = u(x_1, ..., x_n, it).$$
APPENDIX

In section 3.3, the analysis of the integrals in the two space variable case, we restricted ourselves to the initial condition \( f(x_1, x_2) = 0 \) in (2). We want to now take care of arbitrary Cauchy data \( f \) and \( g \). We shall demonstrate the method for any dimension, and show the explicit integral representation in the case of two space variables.

Suppose \( v_1 \) solves the Cauchy problem for the wave equation with initial value of \( v_1 \) identically zero and initial values of \((v_1)_t\) equal to the function \( g \):

\[
\frac{\partial^2 v_1}{\partial t^2} = \sum_{j=1}^{n} \frac{\partial^2 v_1}{\partial x_j^2},
\]

(14)

\[
v_1(x_1, \ldots, x_n, 0) = 0,
\]

(15)

\[
\frac{\partial v_1}{\partial t}(x_1, \ldots, x_n, 0) = g(x_1, \ldots, x_n),
\]

and suppose \( v_2 \) solves the Cauchy problem for the wave equation with initial value of \( v_2 \) identically zero and initial values of \((v_2)_t\) equal to the function \( f \):

\[
\frac{\partial^2 v_2}{\partial t^2} = \sum_{j=1}^{n} \frac{\partial^2 v_2}{\partial x_j^2},
\]

(16)

\[
v_2(x_1, \ldots, x_n, 0) = 0,
\]

(17)

\[
\frac{\partial v_2}{\partial t}(x_1, \ldots, x_n, 0) = f(x_1, \ldots, x_n)
\]

We would like to find a solution \( v \) to (1)-(2). To begin we shall find a solution \( v'_2 \) to the Cauchy problem for the wave equation with initial values of \( v'_2 \) equal to the above
function $f$ and initial value of $(v'_2)_t$ identically zero:

\[ (18) \quad \frac{\partial^2 v'_2}{\partial t^2} = \sum_{j=1}^{n} \frac{\partial^2 v'_2}{\partial x_j^2}, \]

\[ (19) \quad v'_2(x_1, \ldots, x_n, 0) = f(x_1, \ldots, x_n), \]

\[ \frac{\partial v'_2}{\partial t}(x_1, \ldots, x_n, 0) = 0. \]

To obtain a solution $v'_2$ to (18)-(19), we can differentiate the solution $v_2$ to (16)-(17) with respect to the time variable,

\[ v'_2 = \frac{\partial v_2}{\partial t}. \]

Now to solve the wave equation with arbitrary Cauchy data (1)-(2), simply add the solution $v_1$ to (14)-(15) to the solution $v'_2$ of (18)-(19):

\[ v = v_1 + v'_2 \]

\[ = v_1 + \frac{\partial v_2}{\partial t}. \]

So in the two space variable case,

\[ v = v_1 + \frac{\partial v_2}{\partial t}, \]

where $v_1(x_1, x_2, t)$ and $v_2(x_1, x_2, t)$ are obtained by substitution into the integral representations (10) for harmonic functions $u_1(x_1, x_2, it)$ and $u_2(x_1, x_2, it)$ of imaginary arguments which solve (14)-(15) and (16)-(17) respectively:

\[ u_1(x_1, x_2, it) = i \int_0^t \frac{r}{\sqrt{t^2 - r^2}} \frac{\partial A(u_1, 0, r)}{\partial x_3} dr; \]

\[ \frac{\partial u_2}{\partial t}(x_1, x_2, it) = \frac{\partial}{\partial t} \left( i \int_0^t \frac{r}{\sqrt{t^2 - r^2}} \frac{\partial A(u_2, 0, r)}{\partial x_3} dr \right). \]
BIBLIOGRAPHY


(Chapter 2, Section III, Subsection 5)