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Essays in Cooperative Stability

by

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ABSTRACT

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We define a very general group manipulation idea and the corresponding stability concept of “absence-proofness”. In the first chapter, we analyze this concept in surplus sharing transferable utility games, exchange economies with private endowments and fair division problems. Solutions that are stable in our sense are core selections. We also show that it is weaker than population monotonicity in cooperative games and fair division problems, and very demanding for the allocation problems with private endowments. Particularly, the Walrasian allocation rule is not immune to manipulation. Also, it is the first external stability concept defined for fair division problems. In the second chapter, we work on cooperative stability in cost sharing of a minimum cost spanning tree and give a family of stable solutions that are responsive to the asymmetries in the cost data. Interpreting population monotonicity as a strong stability property, in the third chapter, we study population monotonicity in the fair division of indivisible goods where monetary compensations are allowed. We show that if there are more than three goods no efficient solution satisfies this property. For the two goods case we define hybrid solutions that are efficient and population monotonic.
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Contents

Abstract ii
Acknowledgements iii
Chapter 1. Absence-proofness: A new cooperative stability concept 1
  1.1 Introduction 1
    1.1.1 TU games 2
    1.1.2 Exchange economies with private endowments 4
    1.1.3 Fair division problems 4
  1.2 Absence-proofness in TU surplus sharing games 5
    1.2.1 Absence-proof allocation schemes 6
      1.2.1.1 Comparing the core, AP and PM: A normative approach 10
    1.2.2 Nucleolus on the set of convex games 12
    1.2.3 Bilateral assignment games 13
    1.2.4 General nucleoli 18
      1.2.4.1 The extended nucleolus 19
      1.2.4.2 The $K$-monoclus: An APAS whenever a game admits one 20
      1.2.4.3 Comparing the $K$-monoclus and the extended nucleolus 22
  1.3 Exchange economies with private endowments 24
    1.3.1 On the core, the competitive equilibrium and AP 26
    1.3.2 Housing markets 29
    1.3.3 Classical exchange economies 31
  1.4 Fair division problems 34
    1.4.1 On the AP and PM 36
1.4.2 Perfectly divisible goods with no monetary transfers 39
1.4.3 Models with monetary transfers 42
  1.4.3.1 A single indivisible object 44

1.5 Appendix 51

Chapter 2. Strongly stable and responsive cost sharing solutions for minimum spanning tree problems
  2.1 Introduction 54
  2.2 The setting 58
  2.3 Basic properties and solutions 59
  2.4 Absence-proofness: A strong stability property 62
  2.5 Achieving SRK₁, SRK₂, SCM and AP 65

Chapter 3. Population monotonicity in fair division of multiple indivisible goods 73
  3.1 Introduction 73
  3.2 The setting 75
  3.3 Preferences and basic properties 76
  3.4 Concave TU games and solutions 78
  3.5 Problems with |Ω| ≥ 3 80
  3.6 Problems with |Ω| ≤ 2 83
    3.6.1 The case |Ω| = 2 84
1 Absence-proofness: A New Cooperative Stability Concept

1.1 Introduction

Individuals involved in joint economic activities do so voluntarily; whether they show up at the scene or not depends on what they get (or expect to get) in the activity, and their outside option. If an individual or a subgroup of individuals know or anticipate that he (they) can do better on his (their) own, we expect them not to join. Designing allocation rules that make everyone or every subgroup willing to participate is the motivation for the familiar properties known as individual rationality, and core stability. We may also need to prevent the kind of partial secession discussed first in the context of exchange economies by Postlewaite (1979). Given an allocation rule, an individual may benefit by withholding some of his endowment and consuming it together with his allocation in the “reduced” economy where he provides the rest of his endowment. A rule that makes such moves unprofitable is called withholding-proof.

In the same spirit, we propose a very general group manipulation and the related concept of “absence-proofness” (AP). Given an allocation problem in a society $N$, and a solution well defined for all subsocieties, a group of people $S \subseteq N$ may benefit by leaving a subgroup $T \subseteq S$ “out” of the allocation process. After the allocation takes place in the society $N \setminus T$, agents in $S \setminus T$ may reallocate what they received, plus $T$’s endowments (if they have any) among all of $S$. This reallocation is profitable if it is Pareto superior to what $S$ would get in the society $N$ had $T$ not been left aside. We call the solutions that are immune to this kind of manipulations absence-proof.
Absence-proofness is related to both core stability and withholding-proofness (WP). Indeed, AP implies core stability, as we see by taking \( T = S \). However, like WP, AP requires the knowledge of how the solution works in the problem reduced to \( N \setminus T \). Therefore, while the core is defined for a single allocation problem, absence-proofness is a property of an allocation rule. Note that by taking \( N = S = T \), AP also implies Pareto optimality.

Our work suggests that absence-proofness, which is quite different than the well-known population-monotonicity (PM) property in spirit, has surprisingly close formal implications with PM in fair division problems and TU games. Also, AP is a very demanding property in exchange economies with private endowments.

An important feature of absence-proofness is that it applies to a larger range of models than the core and WP. Both WP and core stability apply to exchange economies or any economy where resources are privately owned to start with. Core stability also works in more abstract problems such as transferable utility (TU) cooperative games. However, both concepts lose their bite for the classical fair division problem where we distribute a commonly owned set of goods. On the other hand, AP has much to say in all these problems. Here, we study absence-proofness in three different problems; surplus sharing TU games\(^1\), economies with private endowments, and fair division problems.

### 1.1.1 TU games

In the TU game \((N,v)\), AP is related to but more complicated than core stability. Agents in \( T \) can generate a surplus of \( v(T) \) if they stay out. This surplus is no more than their total allocation had they appeared if a rule is a core selection (as required by AP). Therefore, for \( S \supseteq T \) to manipulate, the loss of agents in \( T \) due to staying out of allocation process should be compensated by an increase in the total payoff of agents in \( S \setminus T \) in the reduced problem. Thus, PM, which

\(^1\) Our results also apply to the cost sharing TU games with necessary adjustments where some inequalities are inversed. See Section 2.4.
simply requires that no agent gains from the departure of some agents, rules out the possibility of manipulation.

Population-monotonicity has both a normative and a strategic interpretation in TU games. It was first discussed in this context by Sprumont (1990) and Moulin (1990a) (See Section 1.2 for a more detailed literature on PM). If an additional agent extends the cooperative opportunities, this should not harm any existing agents. On the strategic side, if an existing agent loses, he may veto the newcomer. However, analyzing this kind of strategies requires extensive modeling of the coalition formation process. The strategic move we discuss here is simple. The loser just pays the newcomer to stay out. If a rule is absence-proof, no agent $i$ has an incentive to pay the newcomer to make him stay out even if $i$ loses when newcomer appears.

In Section 1.2, we examine this one-way logical relation between PM and AP in several examples. Sönmez (1993) showed that the nucleolus is not population-monotonic on the set of convex games. In Section 1.2.2, we show that it does not satisfy AP, either. Sprumont (1990) showed that the bilateral assignment games almost never admit a population-monotonic solution. In these games, we have a clear distinction between the two properties. In Section 1.2.3, we give the necessary and sufficient conditions for the existence of AP solutions in $2 \times 2$ bilateral assignment games, and propose some solutions (see Table 1.1, 1.5 and 1.6). Indeed, when they exist, absence-proof allocations constitute a central cube in the set of core allocations (see Figure 1.1). However, these conditions cannot be generalized to cases with more players in each side.

In Section 1.2.1 we provide alternative definitions for the core, AP and PM. These definitions allow us to compare the three properties from a normative viewpoint. Norde and Slikker (2011) designed a set of solutions, which are general nucleoli, and population-monotonic whenever achieving PM is possible. In Section 1.2.4, based on our alternative definitions we reevaluate the nucleolus and design a solution ($\mathcal{K}$-*monoclus*) that is absence-proof, whenever achieving AP is

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2 Players are partitioned into two groups $N_1$ and $N_2$. The characteristic function $\nu$ is defined solely by the surplus generated by the pairs $\nu(i,j)$, $i \in N_1$ and $j \in N_2$, and its superadditive cover.
possible. We also compare this solution with the nucleolus. Interestingly, in a subset of bankruptcy games they coincide.

1.1.2 Exchange economies with private endowments

In Section 1.3, we show that in the context of indivisible goods with monetary transfers, achieving AP is impossible. We illustrate this impossibility in several examples for different problems; Böhm-Bawerk markets, house assignment problems (Shapley and Shubik 1971), and single seller auctions. In housing markets introduced by Shapley and Scarf (1974) where no monetary compensation is available, only a weaker version of AP can be achieved. This version blocks the possibility of a strict improvement in the welfare of all agents in the manipulating coalition. We also show that the core mechanism that is calculated via the famous top trading cycle algorithm uniquely achieves the weak AP. In problems with divisible goods, on the classical domain, Walrasian allocation is not absence-proof. We also show by means of example that the manipulation of the Walrasian allocation is not a rare occurrence. Even in a problem with three agents who have the same Cobb-Douglas preferences, manipulation is possible. Our results coincide with Thomson (2013) where he introduces withdrawal-proofness, a concept related to but weaker than AP, in exchange economies and fair division problems. There, the manipulation argument is exactly the same. However, the manipulating coalition consists of only two agents, i.e. \(|S| = 2\), and \(|T| = 1\).

1.1.3 Fair division problems

In an environment with a fixed common endowment, it is normatively appealing to assume that no one benefits the presence of additional agents. This interpretation of PM was introduced by Thomson (1983). Here, agents have equal claims on the common endowment, and hence, the strategic interpretation of PM loses its bite. However, strategic move in the argument of AP is still intact. If agents in \(T\) renounce their claims, the manipulating coalition \(S \supseteq T\) has only the
allocation of $S \setminus T$ in the reduced problem to redistribute. Suppose a Pareto optimal allocation rule is population-monotonic, and $S$ manipulates it by the absence of $T$. By PM, all the agents in $N \setminus S$ are better off in the reduced problem. Also, the redistribution of what $S \setminus T$ gets in the reduced problem to all the agents in $S$ is Pareto superior to the allocation to $S$ in the society $N$. This contradicts that the allocation in society $N$ is Pareto optimal. Hence, under efficiency, PM implies AP (Theorem 1.1). For a related literature on PM, and the corollaries see Section 1.4.2 and 1.4.3.

Aside from this strong sufficient condition (PM) for AP, we provide a sufficient condition for manipulation (see Proposition 1.15). As a corollary, in Section 1.4.2, we show that the competitive equilibrium with equal incomes solution is not AP in the problem of distributing a perfectly divisible bundle in $\mathbb{R}_+^l$.

The simplicity of the model of allocating a single object with monetary transfers enables us to give a simple characterization of AP rules, which clarifies the difference between PM and AP (see Propositions 1.16 and 1.18). Indeed, it shows how close they are. Thus, by replacing PM with AP, we cannot escape the incompatibility between one of the most desired fairness property “envy-freeness” (Foley 1967) and PM (see Alkan 1994 and Moulin 1990b). However, rules that are AP but not PM exist and we introduce two of them. First one is a serial oligarchy solution. The other lexicographically favors the agents, starting with the agent who has the lowest valuation for the object while respecting two upper bounds (see Proposition 1.20). A weaker version of PM requires that when some agents leave, remaining agents should be affected in the same direction. The second rule we introduce is not even weakly PM.

### 1.2 Absence-proofness in TU Surplus Sharing Games

**Basic notions:** Given a society $N = \{1, 2, \ldots, n\} \in \mathcal{N}$, where $\mathcal{N}$ denotes the set of all finite societies, a characteristic function $v: 2^N \rightarrow \mathbb{R}_+$ describes the value that a group of agents $S \subseteq N$
are able to create on their own, with the convention that \( \nu(\emptyset) = 0 \). Hence, a tuple \((N, \nu)\) defines a surplus sharing TU game.

For a fixed \((N, \nu)\), an allocation \( x = (x_i)_{i \in N} \in \mathbb{R}_+^N \) is a vector s.t. \( \sum_{i \in N} x_i \leq \nu(N) \). \( x \) is efficient if \( \sum_{i \in N} x_i = \nu(N) \), and individually rational if \( x_i \geq \nu(i) \) for all \( i \in N \). The imputation set \( I(N, \nu) \) consists of all the efficient and individually rational allocations. Given \((N, \nu)\) and \( S \subseteq N \), the game \((S, \nu_S)\) is a subgame of \((N, \nu)\) if for all \( T \subseteq S \), \( \nu_S(T) = \nu(T) \). By abuse of notation, we will write \((S, \nu)\) instead of \((S, \nu_S)\) or even sometimes game \( S \) when there is no confusion about the fixed \((N, \nu)\). An allocation scheme \( X(\cdot) \) is a mapping that assigns an efficient allocation to each subgame, i.e. \( X(S) \in \mathbb{R}_+^S \) for all \( S \subseteq N \) and \( \sum_{i \in S} X_i(S) = \nu(S) \).

Let \( \Gamma^N \) be the set of all games admissible for \( N \) and \( \Gamma = \bigcup_{N \in \mathcal{N}} \Gamma^N \). An allocation rule defined on the domain of games \( \mathcal{G} \subseteq \Gamma \) assigns an allocation at each game in \( \mathcal{G} \). We call a domain \( \mathcal{G} \) rich if for any game \((N, \nu)\), all the subgames of that game are in \( \mathcal{G} \), as well. Note that an allocation rule on a rich domain \( \mathcal{G} \) induces a unique allocation scheme for each game in \( \mathcal{G} \) but not vice versa. A game \((N, \nu)\) is convex if for all \( T \subseteq S \subseteq N \) and \( i \notin S \), we have \( \nu(T \cup \{i\}) - \nu(T) \leq \nu(S \cup \{i\}) - \nu(S) \).

The core of a game \((N, \nu)\) is the set \( \mathcal{C}(N, \nu) = \{ x \in \mathbb{R}_+^N : \sum_{i \in S} x_i \geq \nu(S), \forall S \subseteq N \} \). A game \((N, \nu)\) is balanced if it has a nonempty core, and totally balanced if all of its subgames are balanced. An allocation scheme is a core selection if it assigns a core allocation to each subgame. Similarly, an allocation rule is a core selection on \( \mathcal{G} \) if it assigns a core allocation to each game in \( \mathcal{G} \).

### 1.2.1 Absence-proof allocation schemes

Core stability is the most fundamental stability property in the context of TU games. If not satisfied, a group of agents \( T \) would stay out and enjoy the surplus of \( \nu(T) \), which is more than their total payoff. This group decision requires three basic assumptions. Agents should know or

---

3 By abuse of notation, we write \( X(S) \) instead of \( X(S, \nu) \), and \( X(1, 2, \ldots, s) \) instead of \( X(\{1, 2, \ldots, s\}) \).
anticipate the outcome, act voluntarily, and surplus generation as well as the redistributions out of
the allocation process is possible. Without any further assumptions, by the help of following
example, we discuss a new outside option that is neither foreseen, nor prevented by the core
stability.

**Example 1.1:** \( N = \{1, \ldots, 5\}, W = \{1,2,3\} \) and \( F = \{4,5\} \); \( v(S) = \min\{|S \cap W|, |S \cap F|\} \), for all \( S \subseteq N \).

There are three workers with the same skill, and two firms are looking for exactly one worker
with that skill. Each firm can employ at most one worker. A worker can create 1 unit of surplus if
hired, and is useless otherwise. Here, the core is simple to describe. For each subgame \( S \), as long
as \( |S \cap W| \neq |S \cap F| \), the scarce type of agents in \( S \) equally share the entire surplus of \( v(S) \).
Hence, the unique core allocations of game \( N \) and subgame \( S = \{4,5\} \) are \( x = (0,0,0,1,1) \),
and \( x' = (1,*,* ,0,0) \), respectively. It is also well known that the game \( (N, v) \) in the example is
totally balanced.

Let \( \mathcal{X}(\cdot) \) be a core selection. Then, \( \mathcal{X}(N) = (0,0,0,1,1) \) and \( \mathcal{X}(1,4,5) = (1,*,* ,0,0) \).
Consider the coalition of workers \( W \). Agent 1 asks the other two to stay at home. Then, the active
agents are 1, 4 and 5. As two firms are competing to hire the same worker, agent 1 gets the entire
surplus of 1 unit in game \( \{1,4,5\} \). Note that any redistribution of 1 unit among the workers is a
Pareto improvement with respect to their allocation in \( \mathcal{X}(N) \).

**Definition 1.1:** Let \((N, v)\) be given, \( \mathcal{X} \) is an absence-proof allocation scheme (APAS) if for
all \( T \subseteq K \subseteq S \subseteq N \),

\[
\sum_{i \in K \setminus T} x_i(S \setminus T) + v(T) \leq \sum_{i \in K} x_i(S)
\]

Here, we enlarge the outside options of the coalition of \( K \). In addition to a total secession of
the coalition (the case \( K = T \) in (1.1)), as prevented by core stability, \( K \) may also leave a strict
subset of it outside the allocation process, and can still benefit. These additional options correspond to the case \( T \subseteq K \) in (1.1).

When we think of \( N \) as a maximal society and the population as a variable, the domain \( G \) of games we would face are \((N, \nu)\) and all of its subgames. Hence, allocation schemes are allocation rules defined on \( G \). Adopting this interpretation, we ask \( \mathcal{X}(\cdot) \) to satisfy the desired property not only in game \((N, \nu)\), but in all the subgames, i.e. for all \( S \subseteq N \).

**Remark 1.1:** An APAS is core selection, and hence efficiency at problem \( N \) even had we not imposed efficiency on \( \mathcal{X}(\cdot) \) by definition. Just set \( K = T \).

In TU game context, allocation schemes are widely used solution objects, especially in the literature on the well-known population-monotonicity property. The same object appears as generalized allocation in Moulin (1990a), and payoff configuration in Thomson (1995). However, use of allocation schemes is more common, following Sprumont (1990) where he defines the *population-monotonic allocation schemes (PMAS)*. An allocation scheme \( \mathcal{X}(\cdot) \) is PMAS if \( \mathcal{X}_i(T) \leq \mathcal{X}_i(S) \) for all \( T \subseteq S \subseteq N, i \in T \). Notice a PMAS \( \mathcal{X}(\cdot) \) is a core selection, as \( \nu(T) = \sum_{i \in T} \mathcal{X}_i(T) \leq \sum_{i \in T} \mathcal{X}_i(S) \) for all \( T \subseteq S \subseteq N \).

The game in Example 1.1 clearly does not admit an APAS. Note that, there, the payoff of worker 1 increases by absence of the other workers. Indeed, this is not a coincidence, but a requirement for the workers to manipulate the allocation scheme.

Sprumont (1990) is the first to discuss the strategic interpretation of PM. He argues that if an agent \( i \) gets more in a subgame, say \( \mathcal{X}_i(N) < \mathcal{X}_i(S) \), then \( i \) will be tempted to form the smaller coalition \( S \) by using his bargaining skills or by any other means. The strategic move we define here is indeed a particular action that \( i \) could take; that is convincing \( N \setminus S \) to stay out by paying them.

**Proposition 1.1:** Given a game \((N, \nu)\), if \( \mathcal{X} \) is a PMAS, it is also an APAS.

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\(^4\) See Sprumont (2008) for a detailed argument on the use of allocation schemes.
Proof: Let \((N, v)\) be a game and \(X\) be a PMAS at this game. Take any \(T \subseteq K \subseteq S \subseteq N\). PM implies \(\Sigma_{i \in K \setminus T} x_i(S \setminus T) \leq \Sigma_{i \in K \setminus T} x_i(S)\), and \(v(T) = \Sigma_{i \in T} x_i(T) \leq \Sigma_{i \in T} x_i(S)\). Then, (1.1) holds. □

Proposition 1.1 provides a strong reason to choose PMAS’s among the core selections. It also tells us a lot about APAS’s. First of all any game that admits a PMAS also admits an APAS. Any game that is a linear combination of monotonic simple games, and only those, admits a PMAS (Sprumont 1990). Norde and Reijnierse (2002) give another characterization by generalizing the idea of vector of balanced weights that is used to characterize the balanced games. Existence of (maybe a similar) a nice and compact characterization of the set of games that admit an APAS is still an open question. We provide a characterization for \(2 \times 2\) bilateral assignment games in Section 1.2.3.

Other corollaries to Proposition 1.1 are as follows: The Shapley value (Shapley 1962) and sequential and monotone Dutta-Ray solutions (Dutta and Ray 1989) are absence-proof on the set of convex games (Sprumont 1990 and Hokari 2002). The proportional allocation scheme is absence-proof on average monotonic games (Izquierdo and Rafels 2001).

By Remark 1.1, AP solutions exist only if the game is totally balanced. We know by Sönmez (1993) that the nucleolus (Schmeidler 1969) is not PM on convex games. In Section 1.2.2, we show that it is not even AP. Sprumont (1990) and Innara (1993) proved that the Shapley value is a core selection on the class of average (quasi) convex games\(^5\) while Sprumont also showed that the Shapley value is not a PMAS on this domain.

Open Question 1: Is the Shapley value AP on the set of average (quasi) convex games?

It is important to keep in mind that the formulation in (1.1) is critical and not appropriate for all problems that are represented by a TU game. TU games are commonly simplified

\(^5\) A game \((N, v)\) is quasi-convex if for all \(S \subseteq T \subseteq N\), we have \(\Sigma_{i \in S} (v(S) - v(S \setminus i)) \leq \Sigma_{i \in S} (v(T) - v(T \setminus i))\)
representations of the feasible utility space in allocation problems with private or common endowments, and quasilinear preferences that allow monetary transfers. In a fair division problem, for all $T \subseteq N$, $v(T')$ represents the monetary equivalent of the common endowment for agents in $T$ when they are the only claimants. However, when the set of claimants is $N \not\supseteq T$, coalition $T$ can generate no surplus if they renounce their claim and leave the scene. Inequality (1.1) does not represent this situation. The argument is more subtle for the case of private endowments. Suppose we are reallocating privately owned indivisible goods, say cars, and $T$ stays out with the cars they own. $K \setminus T$ brings its allocation after the allocation process in the society $N \setminus T$. If an agent from $T$ has a utility higher than everyone else for a car that $K \setminus T$ brought in, he gets it in the Pareto improving reallocation. Then, the monetary equivalent of $K$’s “new” total endowment (cars that $K \setminus T$ got in the allocation process plus the cars that agents in $T$ own) for agents in $K$ is more than $\sum_{i \in K \setminus T} x_i (N \setminus T) + v(T)$. So, the results in this section are valid for TU games where the allocation is in actual money terms.

As an end note for the formulation (1.1), we may have asked for a weaker property which ensures no manipulation only at the grand game $(N, v)$, and hence (1.1) to hold only for $S = N$. In that case, for any balanced game we can write a “weakly” absence-proof allocation scheme. It is an easy exercise to see that (1.1) holds for the case $S = N$ for the following “proportional” allocation scheme: Fix an arbitrary $y \in C(N, v)$. Define for all $i \in N$, $x_i (N) = y_i$; for all $S \subseteq N$, $i \in S$, $x_i (S) = y_i (v(S)/y(S))$ if $y(S) > 0$, and $x_i (S) = 0$ if $y(S) = 0$.

Applying the proportional allocation scheme to the game in Example 1.1, we have $X(N) = (0, 0, 0, 1, 1)$, and $X(1, 4, 5) = (0, *, *, 0.5, 0.5)$. Then, if for some reason workers 2 and 3 are not available anymore, it does not assigns a core allocation to the game $\{1, 4, 5\}$.

1.2.1.1 Comparing the core, AP and PM: A normative approach

Besides being a stability property, core can be conceived as a normative property as well. Let $(N, v)$ be a totally balanced game and $X$ be a core selection. Consider two disjoint subsocieties $S$,
$S'$ and the subgame $S \cup S'$. Note that for any totally balanced game, we can write $v(S \cup S') = v(S) + v(S') + g(S, S')$ with $g(S, S') \geq 0$. Both subsocieties guarantee a payoff of $v(S)$ and $v(S')$, respectively at $\mathcal{X}$ in subgame $S \cup S'$. Then, the remaining $g(S, S')$ is distributed among the agents in $S \cup S'$. So, if we interpret $g(S, S')$ as the value created by merger of two societies, core dictates that none of the two societies should gain in total more than that amount when they merge. The next proposition relates this normative interpretation to absence-proofness, and population-monotonicity.

**Proposition 1.2:** Let $(N, v)$ be a game, $\mathcal{X}$ be an allocation scheme and $S, S' \subseteq N$ such that $S \cap S' = \emptyset$.

(i) $\mathcal{X}$ is a core selection if and only if we have,

$$\sum_{i \in S}(\mathcal{X}_i(S \cup S') - \mathcal{X}_i(S)) \leq g(S, S') \quad (1.2)$$

(ii) $\mathcal{X}$ is an APAS if and only if for all $K \subseteq S$ we have,

$$\sum_{i \in K}(\mathcal{X}_i(S \cup S') - \mathcal{X}_i(S)) \leq g(S, S') \quad (1.3)$$

(iii) $\mathcal{X}$ is a PMAS if and only if for all $K \subseteq S \cup S'$ we have,

$$\sum_{i \in (K \cap S)}(\mathcal{X}_i(S \cup S') - \mathcal{X}_i(S)) + \sum_{i \in (K \cap S')}(\mathcal{X}_i(S \cup S') - \mathcal{X}_i(S')) \leq g(S, S') \quad (1.4)$$

**Proof:** We only prove (ii). and omit the trivial arguments for (i). and (iii).

**Necessity.** Let $\mathcal{X}$ be an APAS at $(N, v)$. Take any $S, S' \subseteq N$ s.t. $S \cap S' = \emptyset$, and any $K \subseteq S$. (1.1) implies that $\sum_{i \in S \setminus K} \mathcal{X}_i(S) + v(S') \leq \sum_{i \in (S \setminus K) \cup S'} \mathcal{X}_i(S \cup S')$. Then, $v(S) - \sum_{i \in K} \mathcal{X}_i(S) + v(S') \leq v(S \cup S') - \sum_{i \in K} \mathcal{X}_i(S \cup S')$. Therefore, (1.3) is satisfied.

**Sufficiency.** Let $(N, v)$ be a game, $\mathcal{X}$ be an allocation scheme at $(N, v)$, and (1.3) hold. Take any $T \subseteq K \subseteq S \subseteq N$. Then, $(S \setminus K) \subseteq (S \setminus T)$, and by (1.3), $\sum_{i \in S \setminus K} \mathcal{X}_i(S) - \sum_{i \in S \setminus T} \mathcal{X}_i(S \setminus T) \leq$
Thus, \( v(S) - v(S \setminus T) - v(T) = g(S \setminus T, T) \). Therefore, \( \sum_{i \in S \setminus K} x_i(S \setminus T) + v(T) \leq v(S) - \sum_{i \in S \setminus K} x_i(S) \). Therefore, \( \sum_{i \in K \setminus T} x_i(S \setminus T) + v(T) \leq \sum_{i \in K} x_i(S) \), and hence (1.1) holds.

By Proposition 1.2, we can read absence-proofness as follows: When two societies merge, no coalition from one of these societies gets an extra surplus that is more than the value created \((g(S, S'))\) by this merger, while population-monotonicity does not allow any coalitions from the joint society to gain more than this value.

**1.2.2 Nucleolus on the set of convex games**

The class of convex games has been a special area of interest in the literature on PMAS for two reasons: they constitute a rich domain, and usually have a large set of core allocations. The literature on PM mainly focused on allocation schemes that are based on applying allocation rules (or sometimes referred to as value operators) to all subgames of a game in a given rich domain. Sönmez (1993) showed that the “extended” nucleolus is not PM on convex games in general. However, on a particular subset of (dual of) these games, known as airport games, it is PM.

The lexicographic ordering of \( \mathbb{R}^l \) is denoted by \( \succeq_L \); that is \( x \succeq_L y \) for \( x, y \in \mathbb{R}^l \) if \( x = y \) or there is \( t \in \{1, ..., l\} \) such that \( x_t = y_t, \) for all \( t' < t \) and \( x_t > y_t. \)

Now, let \((N, v)\) be such that \( I(N, v) \) is nonempty. For each allocation \( x \in I(N, v) \) define the excess of coalition \( S \subseteq N \) as \( e(x; S) = \sum_{i \in S} x_i - v(S) \). Let \( e(x) \in \mathbb{R}^{2^N} \) have the excesses of allocation \( x \) ordered increasingly. Then, the unique allocation \( \mu(N, v) \) such that for each \( x \in I(N, v), e(\mu(N, v)) \succeq_L e(x) \) is called the nucleolus of the game. Let \( \mathcal{M} \) denote the extended nucleolus, i.e. given any game \((N, v), \mathcal{M}(S) = \mu(S, v) \) for all \( S \subseteq N. \)

**Proposition 1.3:** The extended nucleolus \( \mathcal{M} \) is not AP on the set of convex games.

**Proof:** Consider the following games \((N, v), (N', v')\); \( N = \{1, ..., 6\}, N' = N \cup \{7\}: \)
Note that \((N, v)\) is a subgame of \((N', v')\) and both games are convex. Also, for all \(i, j \in K \equiv \{1, 2, 3, 4, 5\}\) and \(S\) s.t. \(i, j \notin S\) we have \(v(S \cup i) = v(S \cup j)\), and \(v'(S \cup i) = v'(S \cup j)\). It is easy to check that here nucleolus treats these agents equally, and we have \(\mu_i(N) = y\) for all \(i \in K\) and \(\mu_6(N) = 12 - 5y\). Then, the minimum excess is maximized for \(S = \{i\}\) with \(i \in K\) and \(y = 1.5\). Hence we have:

\[
\mu_i(N) = 1.5 \text{ for all } i \in K \text{ and } \mu_6(N) = 4.5
\]

In game \((N', v')\), the minimum excess is trivially maximized for \(S = \{7\}\) and \(S' = N' \setminus \{7\}\). Then, we have \(\mu_7(N') = 0.2\) and \(\sum_{i \in N} \mu_i(N') = 12.2\). The argument for distributing 12.2 to agents in \(N\) is similar to the one in game \((N, v)\). Hence, we have:

\[
\mu_i(N') = 1.6 \text{ for all } i \in K; \mu_6(N') = 4.2 \text{ and } \mu_7(N') = 0.2
\]

Then, by absence of agent 7 at game \((N', v')\), agents 6 and 7 would enjoy a total of 4.5 instead of 4.4. □

1.2.3 Bilateral assignment games

The society \(N\) consists of two disjoint type of agents \(A\) and \(B\), i.e. \(N = A \cup B\), and \(A \cap B = \emptyset\). No coalition consisting of agents only from \(A\) or \(B\) can create a surplus. A generic pair \((a_i, b_j)\) \(\in A \times B\) can create \(v(i, j) \geq 0\). A coalition \(S\) containing several agents of each type generates the surplus \(v(S)\) by forming pairs \((a_i, b_j)\) efficiently and summing up the corresponding \(v(i, j)\)’s, i.e. for \(S = A' \cup B'\) s.t. \(A' \subseteq A\) and \(B' \subseteq B\), \(v(S)\) is the maximal sum \(\sum v(i, j)\) when we assign agents of \(A'\) to those of \(B'\). We call an assignment of \(A'\) to \(B'\) optimal if it generates \(v(S)\), and we say
that a pair is optimal in $S$ if it appears in some optimal assignment that generates $v(S)$. We will represent a bilateral assignment game by the matrix $v(A \times B)$.

Regarding the economic environments that would induce this type of games, we can think of $A$ as the set of potential workers with similar skills, and $B$ as the firms that needs only one worker with this specific skill. We can also think of $A$ as a set of flute players, and $B$ as a set of piano players, who are seeking performance positions for duos.

One nice feature of these games is that they are totally balanced. However, as we show in Proposition 1.5, a PMAS is almost never admissible for this class of games. Although, there are games which admit an APAS, there are severe limitations on the surplus opportunities. In games with two agents on each side we are able to identify the necessary and sufficient conditions for the existence of an APAS. Unfortunately, there is no simple procedure to generate an APAS. Moreover, these conditions would not be nicely generalized to the games with $n \geq 5$.

**Lemma 1.1:** Shapley and Shubik (1971). Given a bilateral assignment game $((A; B), v)$, $x(N)$ is a core allocation if and only if for all $i \in A$, $j \in B$ we have $x_i(N) + x_j(N) \geq v(i, j)$; with equality if $(a_i, b_j)$ is an optimal pair in $N$. An agent who is not in any optimal pair gets 0.

**Lemma 1.2:** Let $((A; B), v)$ be an assignment game, and $(a_i, b_j)$ be an optimal pair at subgame $(S, v)$. Then, for any APAS $X$ we have, $X_i(S) = X_i(i, j)$, and $X_j(S) = X_j(i, j)$.

**Proof:** Let everything be as in the statement of the lemma. By Lemma 1.1, we have $X_i(S) + X_j(S) = X_i(i, j) + X_j(i, j) = v(i, j)$. Also, as $(a_i, b_j)$ is an optimal pair at $(S, v)$, $v(i, j) + v(S\{i, j\}) = v(S)$. Let wlog $X_i(S) < X_j(i, j)$. Then, as $X_i(i, j) + v(S\{i, j\}) > X_i(S) + \sum_{k \in S \setminus \{i, j\}} X_k(S)$, the coalition $S\{j\}$ is better off by leaving $S\{i, j\}$ out of the game $S$. □

The idea in Lemma 1.2 is crucial for the characterization of APAS’s in 3-person games, and the conditions for the existence of an APAS in 4-person $(2 \times 2)$ games. As the size of the society grows, in case $(a_i, b_j)$ is an optimal pair in game $N$, it is possibly optimal for many other
subgames of $N$. Lemma 1.2 indicates that absence-proofness become more restrictive with the size of $N$.

A 3-person ($1 \times 2$) game is defined by two numbers and a 4-person ($2 \times 2$) game is defined by four numbers. To get rid of notational complexity we write these numbers in an increasing manner i.e., $u_1 \leq u_2$ for 3-person games and $u_1 \leq u_2 \leq u_3 \leq u_4$ for 4-person games.

**Proposition 1.4:** Let $((A;B), v)$ be a generic 3-person game as shown in the matrix. Then, $X$ is an APAS if and only if it is efficient at each $S$, $X_1(N) = X_1(1,2) \geq u_1$, $X_2(N) = X_2(1,2)$, $X_3(N) = 0$ and $X_2(N) \geq X_3(1,3)$.

Here in 3-person games, the only difference between an APAS and a PMAS is that a PMAS gives nothing to agent 3 in games $N$ and $\{1,3\}$, while an APAS admits a positive payoff to agent 3 in game $\{1,3\}$. For the sake of completeness we now duplicate the result, Proposition 2 in Sprumont (1990).

**Proposition 1.5:** Let $((A;B), v)$ be a bilateral assignment game with $|A|, |B| \geq 2$ such that for some $i, i' \in A$ and $j, j' \in B$, we have $v(k, l) > 0$ for $k \in \{i, i'\}$, and $l \in \{j, j'\}$. Then, game $((A;B), v)$ does not admit a PMAS.

**Proof:** Let $((A;B), v)$ be as in the premise of the proposition with $\{i, i'\} = \{1,2\}$, $\{j, j'\} = \{3,4\}$ and wlog $v(1,3) = u_4$. Suppose for a contradiction that $X$ is a PMAS. Then, $X$ is a core selection, and by Lemma 1.1 we have, $X_1(1,2,3) = X_3(1,3,4) = 0$. Then, PM implies $X_1(1,3) = X_3(1,3) = 0$ and this contradicts that $X$ is efficient. □

**Proposition 1.6:** Let $((A;B), v)$ be a game s.t. $|A| = |B| = 2$. There are three generic configurations of the value matrix and the conditions stated for each generic case are necessary and sufficient for the existence of an APAS.
Case 1: $u_3, u_4$ are diagonal

\[
\begin{array}{ccc}
 b_3 & b_4 \\
 a_1 & u_1 & u_4 \\
a_2 & u_1 & u_2 \\
\end{array} \Rightarrow (i) \ u_3 \geq u_1 + u_2 \quad (ii) \ u_3 + u_4 \geq 3u_2
\]

Case 2.1: $u_3, u_4$ are inline

\[
\begin{array}{ccc}
 b_3 & b_4 \\
 a_1 & u_1 & u_4 \\
a_2 & u_2 & u_4 \\
\end{array} \Rightarrow (i) \ u_4 \geq u_2 + u_3 \quad (ii) \ u_2 + u_3 \geq 3u_1
\]

Case 2.2: $u_3, u_4$ are inline

\[
\begin{array}{ccc}
 b_3 & b_4 \\
 a_1 & u_1 & u_4 \\
a_2 & u_2 & u_3 \\
\end{array} \Rightarrow (i) \ u_3 \geq u_1 + u_2 \quad (ii) \ u_4 \geq u_2 + u_3
\]

Note that the game in Example 1.1 is a particular bilateral assignment game where all entries are 1. Any $2 \times 2$ subgame violates all the conditions above. In Appendix, we show that the allocation scheme in Table 1.1 is an APAS for Case 1. We also propose two different APAS’s for the other cases, and show that the conditions in each case are necessary.

When all entries are strictly positive, for Case 1 and Case 2.2, efficiency induces a unique optimal assignment. For Case 2.1, in case $u_1 > 0$, absence-proofness requires a unique optimal assignment where agents who generate the highest payoff of $u_4$ are matched.

<table>
<thead>
<tr>
<th>S</th>
<th>$X_1(S)$</th>
<th>$X_2(S)$</th>
<th>$X_3(S)$</th>
<th>$X_4(S)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>${1, 2, 3, 4}$</td>
<td>$u_4 - u_2$</td>
<td>$u_2$</td>
<td>$u_3 - u_2$</td>
<td>$u_2$</td>
</tr>
<tr>
<td>${1, 2, 3}$</td>
<td>0</td>
<td>$u_2$</td>
<td>$u_3 - u_2$</td>
<td>*</td>
</tr>
<tr>
<td>${1, 2, 4}$</td>
<td>$u_4 - u_2$</td>
<td>0</td>
<td>*</td>
<td>$u_2$</td>
</tr>
<tr>
<td>${1, 3, 4}$</td>
<td>$u_4 - u_2$</td>
<td>*</td>
<td>0</td>
<td>$u_2$</td>
</tr>
<tr>
<td>${2, 3, 4}$</td>
<td>*</td>
<td>$u_2$</td>
<td>$u_3 - u_2$</td>
<td>0</td>
</tr>
<tr>
<td>${1, 3}$</td>
<td>$u_4 / 2$</td>
<td>*</td>
<td>$u_4 / 2$</td>
<td>*</td>
</tr>
<tr>
<td>${1, 4}$</td>
<td>$u_4 - u_2$</td>
<td>*</td>
<td>*</td>
<td>$u_2$</td>
</tr>
<tr>
<td>${2, 3}$</td>
<td>*</td>
<td>$u_2$</td>
<td>$u_3 - u_2$</td>
<td>*</td>
</tr>
<tr>
<td>${2, 4}$</td>
<td>*</td>
<td>$u_2 - X_4(2, 4)$</td>
<td>*</td>
<td>$\min{u_3 - u_2, u_2 / 2}$</td>
</tr>
</tbody>
</table>

Table 1.1

Even in a $2 \times 2$ game, AP rules out most of the core allocations available at $(N, v)$. Consider the game $(N, v) = ((A; B), v)$ in Case 1 of Proposition 1.6. Let $(\alpha_1, \alpha_2, \beta_3, \beta_4)$ be an efficient allocation. By Lemma 1.1, core is described by the following inequalities:

\[
\alpha_1 + \beta_4 = u_4, \ \alpha_2 + \beta_3 = u_3, \ \alpha_1 + \beta_3 \geq u_1, \ \alpha_2 + \beta_4 \geq u_2
\]
We can rewrite these inequalities as follows:

\[ \alpha_1 + (u_3 - u_1) \geq \alpha_2, \alpha_2 + (u_4 - u_2) \geq \alpha_1, u_4 \geq \alpha_1, u_3 \geq \alpha_2 \]

The dotted area in Figure 1.1 shows the set of all core allocations. By Lemma 1.2 agents 1 and 4 gets \((\alpha_1, \beta_4)\) in games \(\{1,2,4\}\), and \(\{1,3,4\}\). Then, applying Proposition 1.4 to these games, we have the first inequality below. By a similar argument for agents \(\{2,3\}\), and games \(\{1,2,3\}\) and \(\{2,3,4\}\), we have the second inequality. So, AP induces the following additional restrictions on this allocation:

\[ u_4 - u_2 \geq \alpha_1 \geq u_1, \; u_3 - u_1 \geq \alpha_2 \geq u_2 \]

The dashed area in Figure 1.1 shows the set of core allocations that an APAS may assign to game \((N,v)\). This rules out the two famous (extreme) solutions, A-optimal and B-optimal allocations. Here, the A-optimal allocation is \(\alpha_1 = u_4, \alpha_2 = u_3\) and conversely B-optimal allocation is \(\alpha_1 = \alpha_2 = 0\). The allocation scheme in Table 1.1 corresponds to the southeast corner of the dashed square below. We can think of the northeast and the southwest corners of
that square as the restricted A-optimal and B-optimal allocations, respectively. Unfortunately, we may not be able to write an APAS that assigns one of those two allocations at \((N,v)\) regardless of the surplus opportunities, even though they satisfy both conditions in Case 1. We omit the exhaustive argument.

Note that the boundaries of the square in Figure 1.1 are derived by Proposition 1.4, while the conditions in Proposition 1.6 ensure that this set is nonempty. The following example shows that even if all the \(2 \times 2\) subgames of a game satisfy the conditions in Proposition 1.6, that game may not admit an APAS. The game in the example is only a \(2 \times 3\) game, and this gives us a hint of how much more restrictive absence-proofness becomes as the society expands.

**Example 1.2:**

<table>
<thead>
<tr>
<th></th>
<th>(b_3)</th>
<th>(b_4)</th>
<th>(b_5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_1)</td>
<td>3</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>(a_2)</td>
<td>2</td>
<td>8</td>
<td>3</td>
</tr>
</tbody>
</table>

Note that, all three of the \(2 \times 2\) subgames fall into a different case of the Proposition 1.6. Suppose \(X\) is an APAS at the game \((N,v)\). The optimal assignment with the surplus of 11 is \(((a_1,b_3),(a_2,b_4))\). By Proposition 1.4, we have \(X_3(1,2,3) = X_3(1,3) \geq 2\), and \(X_4(1,2,4) = X_4(2,4) \geq 5\). By Lemma 1.2, we have \(X_3(N) \geq 2\), and \(X_4(N) \geq 5\). Thus, \(X_1(N) + X_2(N) + X_5(N) \leq 4\). Now, let \(T = \{2,5\}\) and consider the game \(N \setminus T = \{1,3,4\}\). By Proposition 1.4, \(X_4(1,3,4) \geq 3\). Then, as \(v(T) = 3\), \(\{1,2,5\}\) would be better off by absence of \(T\) at \((N,v)\). Thus, this game does not admit an APAS.

**1.2.4 General nucleoli**

In the literature on PMAS, the primary issue has been to figure out if well-known allocation rules satisfy population-monotonicity on certain classes of games. Norde, and Slikker (2011) departed this trend and focused on constructing PMAS’s for an arbitrary game \((N,v)\). They introduced the solution *monoclus* (a general nucleolus, see Maschler et al 1992) and some variations, which are population-monotonic whenever a PMAS exists. Among those, we inspire from \(\mathcal{T}\)-monoclus and define \(\mathcal{K}\)-monoclus, which is absence-proof whenever an APAS exists.
Given a game \((N, v)\) and an allocation scheme \(X\), for any triple \((i, S, T)\) with \(i \in S \subseteq T \subseteq N\), the \textit{monotonicity} of \(X\) with respect to \((i, S, T)\) is defined as \(\text{mon}(X, (i, S, T)) = X_i(T) - X_i(S)\). Note that an allocation scheme is population-monotonic if and only if all the monotonicities are nonnegative.

Now, define the set \(\mathcal{T} = \{(i, S, T): i \in S \subseteq T \subseteq N\}\) and let \(\theta(X) \in \mathbb{R}^{|\mathcal{T}|}\) have all the monotonicities of \(X\) corresponding to elements in \(\mathcal{T}\) as its coordinates, in a weakly increasing order. Let \(\mathbb{X}(I(N, v))\) denote the set of all allocation schemes at game \((N, v)\) that yields efficient and individually rational allocations at each subgame i.e., \(X(S, v) \in I(S, v)\) for all \(S \subseteq N\) for all \(X \in \mathbb{X}(I(N, v))\). Then, \(\mathcal{T}\)-\textit{monoclus} is defined as follows:

\[
\mathcal{M}^\mathcal{T}(N, v) = \{X \in \mathbb{X}(I(N, v)) : \theta(X) \trianglerighteq \theta(Y) \text{ for all } Y \in \mathbb{X}(I(N, v))\} \quad \text{(1.5)}
\]

\(\mathcal{M}^\mathcal{T}(N, v)\) is nonempty and consist of only one element at each game with a nonempty imputation set \(I(N, v)\) as the set \(\{\theta(X) \in \mathbb{R}^{|\mathcal{T}|} : X \in \mathbb{X}(I(N, v))\}\) is convex and compact. Note that if a game admits an allocation scheme \(X\) with a nonnegative \(\theta(X)\), and hence admits a PMAS, \(\theta \left(\mathcal{M}^\mathcal{T}(N, v)\right)\) is also nonnegative, and \(\mathcal{M}^\mathcal{T}(N, v)\) is a PMAS.

\subsection{1.2.4.1 The extended nucleolus}

By Proposition 1.2, we say an allocation scheme is a core selection if and only if we have \(g(P, Q \setminus P) - \sum_{i \in P} (X_i(Q) - X_i(P)) \geq 0\) for all \(P \subseteq Q \subseteq N\). We can rewrite the inequality as follows:

\[
v(Q) - v(P) - v(Q \setminus P) - \sum_{i \in P} X_i(Q) + v(P) \geq 0 \iff \sum_{i \in Q \setminus P} X_i(Q) - v(Q \setminus P) \geq 0.
\]

Renaming \(Q \setminus P = S\) and \(Q = T\), we have \(\sum_{i \in S} X_i(T) - v(S) \geq 0\) for all \(S \subseteq T \subseteq N\). Now, define \(\sum mon(X, (S, T)) = \sum_{i \in S} X_i(T) - v(S)\), and the set \(S = \{(S, T) : S \subseteq T \subseteq N\}\). Note that \(\sum mon(X, (S, T)) = \sum_{i \in S} \text{mon}(X, (i, S, T)) = e(X(T); S)\) (as defined in Section 1.2.2). Let
\( \epsilon(\mathcal{X}) \in \mathbb{R}^{\lvert S \rvert} \) be the vector that has all the \( \Sigma \)-monotonicities of \( \mathcal{X} \) corresponding to elements in \( S \) as its coordinates in a weakly increasing order. Then, \( S \)-monoclus is as follows:

\[
\mathcal{M}^S(N, v) = \left\{ \mathcal{X} \in \mathcal{X}(I(N, v)) : \epsilon(\mathcal{X}) \succ_L \epsilon(Y) \text{ for all } Y \in \mathcal{X}(I(N, v)) \right\}
\]

(1.6)

As the set \( \mathcal{X} = \left\{ \epsilon(\mathcal{X}) \in \mathbb{R}^{\lvert S \rvert} : \mathcal{X} \in \mathcal{X}(I(N, v)) \right\} \) is convex and compact, \( \mathcal{M}^S(N, v) \) that lexicographically maximizes \( \epsilon(\mathcal{X}) \) within \( \epsilon \) exists and is unique at each game with a nonempty imputation set \( I(N, v) \).

Note that \( \epsilon(\mathcal{X}) \) consist of the excess vectors \( e(\mathcal{X}(T); S) \) associated with the allocation \( \mathcal{X}(T) \) induced by \( \mathcal{X} \) at each subgame \( T \). The difference between the \( S \)-monoclus and the extended nucleolus is that the latter lexicographically maximizes the ordered vector of excesses associated with each subgame separately, while the former does the maximization at once for all subgames. However, both procedures yield the same outcome.

**Proposition 1.7:** Given any \( (N, v) \), \( S \)-monoclus \( (\mathcal{M}^S(\cdot)) \) is the extended nucleolus \( (\mathcal{M}(\cdot)) \).

**Proof:** Suppose for some \( \mathcal{M}^S(T) \neq \mathcal{M}(T) \) for some \( T \subseteq N \). Consider the allocation scheme:

\[ \mathcal{X}(T') = \mathcal{M}^S(T') \text{ for all } T' \neq T \text{ and } \mathcal{X}(T) = \mu(T) \text{ where } \mu(T) \text{ is the nucleolus of subgame } (T, v). \]

Note that for any \( T' \neq T \) and \( S \subseteq T' \), \( e(\mathcal{X}(T'); S) = e(\mathcal{M}^S(T'); S) \). Then, by definition of the nucleolus we have \( \epsilon(\mathcal{X}) \succ_L \epsilon(\mathcal{M}^S) \). \( \square \)

At a first glance, defining \( S \)-monoclus may seem artificial and unnecessary. However, its construction, and its equivalence to the extended nucleolus are important to understand the general nucleolus we define in the next subsection, and to compare it with \( \mathcal{M} \).

### 1.2.4.2 The \( \mathcal{K} \)-monoclus: An APAS whenever a game admits one

By Proposition 1.2, we say an allocation scheme is absence-proof if and only if we have \( g(P, Q \setminus P) - \sum_{i \in L} (x_i(Q) - x_i(P)) \geq 0 \) for all \( L \subseteq P \subset Q \subseteq N \). We can rewrite the inequality as follows:
\[ v(Q) - v(P) - v(Q \setminus P) - \sum_{i \in L} x_i(Q) + \sum_{i \in \mathcal{L}} x_i(P) \geq 0 \]

\[ \iff \sum_{i \in Q \setminus P} x_i(Q) + \sum_{i \in P \setminus \mathcal{L}} x_i(Q) - \sum_{i \in P \setminus \mathcal{L}} x_i(P) - v(Q \setminus P) \geq 0 \]

\[ \iff e(x(Q); Q \setminus P) + \sum_{i \in P \setminus \mathcal{L}} x_i(Q) - \sum_{i \in P \setminus \mathcal{L}} x_i(P) \geq 0 \]

Renaming \( Q = T, Q \setminus P = S, \) and \( P \setminus \mathcal{L} = K, \) we have \( x(\cdot) \) is absence proof if and only if \( e(x(T); S) + e(x(T); K) - e(x(T \setminus S); K) \geq 0 \) for all \( S \subseteq T \subseteq N, (T \setminus S) \neq K \in 2^{T \setminus S}. \) Now, let \( \mathcal{K} \equiv \{(K,S,T); S \subseteq T \subseteq N, (T \setminus S) \neq K \in 2^{T \setminus S}\}. \) Then, the \( \Sigma \)-monotonicities of \( x \) corresponding to elements in \( \mathcal{K} \) are defined as follows:

\[ \Sigma \text{mon}(x,(K,S,T)) = e(x(T); S) + e(x(T); K) - e(x(T \setminus S); K) \quad (1.7) \]

Indeed, the manipulation argument is clear when a \( \Sigma \)-monotonicity is negative. If \( S \) leaves the game \( T \) and produces \( v(S) \) outside the allocation procedure, their loss is \( e(x(T); S). \) Also, the loss of \( K \) due to the absence of \( S \) is \( e(x(T); K) - e(x(T \setminus S); K). \) If the total loss of agents in \( S \cup K \) is negative, i.e., \( \Sigma \text{mon}(x,(K,S,T)) < 0, \) then they would manipulate by absence of \( S \) at \( T \).

Let \( \kappa(x) \in \mathbb{R}^{||\mathcal{K}|} \) be the vector that has all the \( \Sigma \)-monotonicities of \( x \) corresponding to elements in \( \mathcal{K} \) as its coordinates in a weakly increasing order. Then, \( \mathcal{K} \)-monoclus is defined as follows:

\[ \mathcal{M}^\mathcal{K}(N,v) = \{ \mathcal{X} \in \mathcal{X}(I(N,v)): \kappa(\mathcal{X}) \equiv_L \kappa(Y) \text{ for all } Y \in \mathcal{X}(I(N,v)) \} \quad (1.8) \]

Note that the set \( \kappa \equiv \{\kappa(x) \in \mathbb{R}^{||\mathcal{K}|}; x \in \mathcal{X}(I(N,v))\} \) is convex and compact. Then, the following proposition follows immediately by construction of the \( \mathcal{K} \)-monoclus.

**Proposition 1.8:** \( \mathcal{M}^\mathcal{K}(N,v) \) exist and unique at each game with a nonempty imputation set \( I(N,v). \) Moreover, if a game \( (N,v) \) admits an APAS, then \( \mathcal{M}^\mathcal{K}(N,v) \) is an APAS.
1.2.4.3 Comparing the $\mathcal{K}$-monocus and the extended nucleolus

Note that $\Sigma_{\text{mon}}(\mathcal{X}, (\emptyset, S, T)) = \Sigma_{\text{mon}}(\mathcal{X}, (S, T))$. Hence, for any allocation scheme $\mathcal{X}$, each component of the vector $\mathcal{e}(\mathcal{X})$ is also a component of $\kappa(\mathcal{X})$. However, $\kappa(\mathcal{X})$ has extra components corresponding to the cases $K \neq \emptyset$. When the size of the society grows, the number of these extra components in $\kappa(\mathcal{X})$ grows rapidly, and that makes $\mathcal{K}$-monocus much harder to calculate compared to the extended nucleolus in general. However, if the number of agents is sufficiently small and/or the game is highly symmetric across partitions of the agents, calculation is feasible. Note that for any 2-person game $\mathcal{e}(\mathcal{X}) = \kappa(\mathcal{X})$, and hence $\mathcal{M}^{\mathcal{K}} = \mathcal{M}$.

Here, we will analyze the differences and similarities between these two solutions in two sets of games. In the 3-person bilateral assignment games they always differ. In a certain subset of pessimistic bankruptcy games (see Example 1.4) both solutions coincide.

**Example 1.3:** Let $\mathcal{G}$ be the set of 3-person bilateral games as in Section 1.2.3. For any $(N, \nu) \in \mathcal{G}$, wlog $\nu(N) = \nu(1,2) = u_2$, $\nu(1,3) = u_1$, and $\nu(S) = 0$ otherwise, with $u_2 \geq u_1 > 0$.

The extended nucleolus of the game above is simple to calculate and is as follows:

$$\mathcal{M}(N) = \left(\left(u_2 + u_1\right)/2, \left(u_2 - u_1\right)/2, 0\right), \mathcal{M}(12) = \left(u_2/2, u_2/2, *\right), \mathcal{M}(13) = \left(u_1/2, *, u_1/2\right).$$

Note that agent 1’s payoff at game $N$, and $\{1,2\}$ are never the same. Recall from Section 1.2.3 that those payoffs should be the same at an absence-proof allocation scheme. Hence, $\mathcal{M}^{\mathcal{K}}$ and $\mathcal{M}$ never coincide on $\mathcal{G}$. Indeed, by Proposition 1.4, $\mathcal{M}^{\mathcal{K}}$ can be described by 4 numbers; $\mathcal{M}_1^{\mathcal{K}}(N) = \mathcal{M}_1^{\mathcal{K}}(12) = X_1$, $\mathcal{M}_2^{\mathcal{K}}(N) = \mathcal{M}_2^{\mathcal{K}}(12) = X_2$, $\mathcal{M}_1^{\mathcal{K}}(13) = x_1$, and $\mathcal{M}_3^{\mathcal{K}}(13) = x_3$. The $\mathcal{K}$-monocus is as follows:

$$X_1 = \frac{2u_2 + u_1}{4}, X_2 = \frac{2u_2 - u_1}{4}, x_1 = \frac{u_1}{2} \quad \text{if} \quad u_2 \geq 5u_1/2.$$  

$$X_1 = \frac{u_2 + 2u_1}{3}, X_2 = \frac{2(u_2 - u_1)}{3}, x_1 = \frac{(4u_1 - u_2)}{3}, x_3 = \frac{(u_2 - u_1)}{3} \quad \text{otherwise.}$$
Excluding the \((K, S, T)\)s with trivially zero \(\Sigma\)-monotonicities, it is an easy exercise to check that
the minimum value for the \(\Sigma\)-monotonicities is attained at the triples \((1,0,13)\) and \((3,0,13)\) in the
first case; \((3,0,13), (13,0,123), \) and \((2,3,123)\) in the second case.

**Example 1.4:** *(Aumann, and Maschler 1985).* A bankruptcy problem is defined by an estate of
size \(E\) to be divided among the claimants, and \(d_i\) denotes the claim of individual \(i\). The associated
pessimistic bankruptcy game is \(\nu(S) = \max\{E - \sum_{i \in N \setminus S} d_i, 0\}\), for all \(S \subseteq N\). Let \(\tilde{G}\) be the set of
all such games where \(d_i \geq 2E/|N|\).

We first define the set of coalitions with minimal excess of the nucleolus:

\[
\mathcal{D}(\mu(N, \nu)) = \arg\min_{S \subseteq N \setminus \{\emptyset\}} e(\mu(N); S)
\]

For any \((N, \nu) \in \tilde{G}\), we know that \(\mathcal{D}(\mu(N, \nu)) = \{\{i\}: i \in N\}\), and the nucleolus is \(\mu_i(N, \nu) = v(N)/|N|\) for all \(i \in N\) (Arin and Iñarra 1998).

**Proposition 1.9:** For any \((N, \nu) \in \tilde{G}\), \(M_i(S) = M_i^{\mathcal{K}}(S) = \nu(S)/|S|\) for all \(S \subseteq N\), and for all \(i \in N\).

**Proof:** Proposition trivially holds for the case \(|N| = 1\) or \(|N| = 2\). Take any \((N, \nu) \in \tilde{G}\) with
\(|N| \geq 3\). We first show that \((S, \nu) \in \tilde{G}\) for all \(S \subseteq N\). By construction of \(\nu(\cdot)\), for any \(S \subseteq N\), the
subgame \((S, \nu)\) is a pessimistic bankruptcy game associated with \(E' = \nu(S)\) and the vector
\(\{d_i\}_{i \in S}\). Consider the subgame \(N \setminus j\) for some \(j \in N\). We know that \(\nu(N \setminus j) = E' = E - d_j\).

Note that \(E' \leq E - \frac{2E}{|N|}\), then \(\frac{2E'}{|N| - 1} \leq \frac{2(|N| - 2)E}{(|N| - 1)|N|} < \frac{2E}{|N|} \leq d_i\) for all \(i \in (N \setminus j)\). Hence, \((N \setminus j, \nu) \in \tilde{G}\). Now, take any \(S \subseteq N\). Remove one agent at a time from \(N\) until we finally reach \(S\). At each
step, the subgame we have is in \(\tilde{G}\). Therefore, \((S, \nu) \in \tilde{G}\), and hence, \(M_i(S) = \nu(S)/|S|\). Also, it
is apparent from the above inequality that for any \(S \subseteq T\) we have \(\nu(T)/|T| \geq \nu(S)/|S|\), and also
if \(\nu(T) > 0\), \(M_i(T) > M_i(S)\) for all \(i \in S\). Hence, \(M\) is PM.
Claim: Take any \((K,S,T) \in \mathcal{K}\). We have \(\Sigma \text{mon}(\mathcal{M},(K,S,T)) \geq \Sigma \text{mon}(\mathcal{M},(\emptyset,i,T)) = e(\mathcal{X}(T);i) = v(T)/|T|\).

Proof of Claim: Let \((K,S,T) \in \mathcal{K}\). We know that \(d_i \geq 2E/|N|\) for all \(i \in T\). Then, \(v(j) = 0\) as \(E - \sum_{i \in N \setminus j} d_i \leq 0\) for all \(j \in N\). As \((T,v) \in \tilde{G}\), and \(\mathcal{D}_1(\mu(T,v)) = \{\{i\}: i \in T\}\) we have \(\Sigma \text{mon}(\mathcal{M},(\emptyset,i,T)) = \Sigma \text{mon}(\mathcal{M},(\emptyset,i,T)) = e(\mathcal{M}(T);i) = v(T)/|T| \leq e(\mathcal{M}(T);S)\) for all \(i \in T, S \subset T\). Also, as \(\mathcal{M}\) is PM we have \(e(\mathcal{M}(T);K) - e(\mathcal{M}(T \setminus S);K) \geq 0\).

Let \(\{T_0, T_1, \ldots, T_m\}\) be a partition of \(2^N\) s.t. for any \(l, T, T' \in T_l\) we have \(v(T')/|T'| = v(T)/|T|\), and for any \(T \in T_l, T' \in T_{l+1}\) we have \(v(T')/|T'| > v(T)/|T|\). Note that \(T_0 = \{T \subseteq N: v(T) = 0\}\) is non-empty as \(\{i\} \in T_0\) for all \(i \in N\), and \(T_m = \{N\}\). Also, for every \((K,S,T) \in \mathcal{K}\) with \(T \in T_0\) we have \(\Sigma \text{mon}(\mathcal{M},(K,S,T)) = 0\).

By Claim, the minimum value for \(\Sigma \text{mon}(\mathcal{M},(K,S,T))\) with \(T \notin T_0\) is attained at \((\emptyset,i,T) \in \mathcal{K}\) with \(T \in T_1\). Suppose for a contradiction \(\mathcal{M}(T) \neq \mathcal{M}^\lambda(T)\) for some \(T \in T_1\). Then, \(e(\mathcal{M}^\lambda(T);i) < v(T)/|T|\) and \(\kappa(\mathcal{M}) > \lambda(\kappa(\mathcal{M}^\lambda))\). Again by the Claim, the minimum value for \(\Sigma \text{mon}(\mathcal{M},(K,S,T))\) with \(T \notin (T_0 \cup T_1)\) is attained at \((\emptyset,i,T) \in \mathcal{K}\) with \(T \in T_2\). Suppose for a contradiction \(\mathcal{M}(T) \neq \mathcal{M}^\lambda(T)\) for some \(T \in T_2\). Then, \(e(\mathcal{M}^\lambda(T);i) < v(T)/|T|\) and \(\kappa(\mathcal{M}) > \lambda(\kappa(\mathcal{M}^\lambda))\). The argument applies recursively, hence we are done. \(\square\)

### 1.3 Exchange Economies with Private Endowments

Basic notions: In order to get rid of notational complexity, we prefer to define the basic concepts and most of the models in words. An exchange economy is formed of a set of individuals \(N\), individualized endowments, a feasible consumption space, and preferences of the individuals on this consumption space. An allocation is a redistribution of the total endowments, possibly with some restrictions on individual consumptions, and a vector of balanced monetary transfers (if
available in the model). An allocation rule assigns an allocation or a set of allocations to all problems in a specified domain.

**Pareto optimality:** An allocation is Pareto optimal (PO) if there is no other feasible allocation that makes an agent strictly better-off without hurting some other agent.

**Core stability:** An allocation is in the core if for any group of agents there is no way to redistribute their total endowment among the group in a way Pareto superior to their allocation.

**Böhm-Bawerk’s horse market:** The traded goods are indivisible identical objects (horses here). Society is formed of potential sellers and buyers. Sellers own a horse each while the buyers own none. Each agent wants to consume at most one horse. Preferences are represented by a number corresponding to reservation price for the sellers, and willingness to pay for the buyers. Monetary transfers are allowed and preferences are quasilinear in money. An allocation is a redistribution of the horses such that each agent has at most one horse, and a vector of transfers that adds up to 0.

**House assignment problem:** *(Shapley and Shubik 1971)* The traded goods are indivisible identical objects (houses here). Each agent owns exactly one house. For each agent, preferences are represented by \( n \) numbers corresponding to their willingness to pay for each house. Monetary transfers are allowed and preferences are quasilinear in money. An allocation is a redistribution of the houses such that each agent gets exactly one house, and a vector of transfers that adds up to 0.

**Housing markets:** *(Shapley and Scarf 1974)* The traded goods are indivisible identical objects (houses here). Each agent owns exactly one house, and his ordinal preference is a linear order over the set of all houses. Monetary transfers are not allowed, and an allocation is a redistribution of the houses among the agents.

**Classical exchange economies:** An economy is a triple \( \varepsilon = (N, e, R) \). \( N \in \mathcal{N} \) denotes the set of individuals, where \( \mathcal{N} \) is the set of all finite subsets of \( \mathbb{Z} \). \( e = \{ e_i \}_{i \in N} \) is the profile of private endowments, where \( e_i \in \mathbb{R}_+^f \) for all \( i \), and \( e^S \) denote the endowment profile restricted to \( S \subseteq N \).
Each individual has a complete and transitive preference relation \( R_i \) on \( \mathbb{R}_+^d \), and \( P_i \) denotes the strict counterpart of \( R_i \). Let \( \mathcal{R} \) denote the set of admissible preferences for each individual. Given a society \( N \in \mathcal{N} \), a preference profile is a vector \( R \in \mathcal{R}^N \), and \( R^S \) is the profile restricted to \( S \subseteq N \). We denote the restriction of the economy \( \varepsilon = (N, e, R) \) to \( S \subseteq N \) by \( \varepsilon^S \) i.e., \( \varepsilon^S = (S, e^S, R^S) \).

Given an economy \( \varepsilon = (N, e, R) \), an allocation \( x \in \mathbb{R}_+^N \) is a vector s.t. \( x_i \in \mathbb{R}_+^d \) for all \( i \in N \), and \( \sum_{i \in N} x_i = \sum_{i \in N} e_i \). An allocation rule \( \varphi(\cdot) \) assigns an allocation to each economy \( \varepsilon \).

### 1.3.1 On the core, the competitive equilibrium, and AP

The competitive equilibrium is without dispute the most fundamental solution concept in exchange economies. In most cases it exists, and ensures Pareto optimality. Moreover, for all the problems we will discuss here, it is well-known that the competitive allocations are always core stable. However, it is not immune to manipulations in all respects. In classical exchange economies, an agent can manipulate the competitive equilibrium by withholding or destroying his endowment as discussed in Postlewaite (1979). There, Postlewaite also discusses a group manipulation strategy. A group of agents may perform a trade prior to coming to the market. With their “new” endowments, they can be better off at the allocation the rule assigns. Agents can also manipulate the competitive equilibrium by misrepresenting their preferences (Hurwicz 1972).

In a Böhm-Bawerk market, competitive allocations are determined by a set of prices which equalizes the number of sellers and buyers that are willing to trade. Here, the set of competitive allocations and the core stable allocations coincide. Manipulation by withholding and destroying has no bite as each seller owns a single indivisible unit. An active coalition that performs pre-trade consists of a subset of both sellers and buyers. This trade will make either the sellers or the buyers unhappy, or everyone remains equally happy as there is a uniform market price. Also, strategy-proof mechanisms that are immune to manipulation by misrepresenting preferences exist (see e.g. Moulin 1995).

---

\( ^6 \) We prefer defining rules as functions instead of correspondences to keep things simple.
In this setting, Shapley and Shubik (1971) discuss some weakness of the core. They say, “The core is based on what a coalition can do, not what it can prevent”. Here is a summary of their argument: There are four suppliers and four buyers. Reservation prices for suppliers in an increasing order, and willingness to pay for buyers in a decreasing order are as follows: 2, 3, 4, 7 for suppliers; 12, 9, 8, 3 for buyers. Here, the competitive price ranges from 4 to 7. If the seller with the reservation price of 4 was not involved, the competitive price would range from 7 to 8. Instead of using his bargaining power in the market directly, by not appearing in the scene, he would help increase the bargaining power of the remaining sellers. Below, we take their example to an extreme case.

**Example 1.5: (Horse market)** There are three potential suppliers with a reservation price of 0, and two potential buyers with a willingness to pay of 1.

The unique competitive price is 0. The buyers get a horse each and pay nothing. Hence, in the unique core allocation, buyers equally share the entire surplus of 2 units. If two of the suppliers stay outside of the market with their horses, the remaining supplier sells his horse at a price of 1 at the unique competitive allocation. As AP implies core stability by the very general definition, there is no absence-proof allocation rule here.

Indeed, the problem in this example induces exactly the same TU cooperative game given in Example 1.1; suppliers correspond to workers, and buyers correspond to firms. We argued in Section 1.2.1 that the formulation of manipulation in TU games by the inequality (1.1) cannot be directly applied here. However, as the valuation for the good is exactly the same for each buyer, after the allocation process, the manipulating coalition cannot create extra surplus by passing a horse from one buyer to another. Hence, the argument that rules out AP in both examples is equivalent. The following example reflects exactly the same idea in house assignment problems.

**Example 1.6: (House assignment)** There are five agents, each endowed with a house. Society is partitioned in two distinct sets, say with three agents in set $W$ and two agents in set $F$. An agent
from $W$ is willing to pay 1 unit for a house owned by an agent from $F$, and 0 for a house owned by an agent from $W$. And vice versa for an agent in $F$.

Obviously, the problem above induces the same TU game with the one in Example 1.1, and all the “reduced” problems induce the same subgames. Again for this specific problem, formulation of AP is the same as in inequality (1.1).

**Example 1.7: (Auction)** There is a single seller who owns a single indivisible object. Assume for simplicity his valuation for the good is 0. There are $n \geq 3$ buyers in the market, and their willingness to pay is as follows: $b_1 > b_2 > \cdots > b_n$.

This is a special case of Böhm-Bawerk’s horse market. In a core allocation buyer 1 gets the good paying the seller at least $b_2$, and other buyers pay nothing. Here, there are a number of possibilities for manipulation. We consider just the following trivial case: All the buyers except $n$ stays out. Agent $n$ gets the good by paying at most $b_n$ units. After the trade, he passes the good to buyer 1, and buyer 1 pays the others, say $b_3/(n-1)$ units. This move makes all the buyers strictly better-off.

There is a similar source of collusion in the auction theory. A manipulating coalition is called a “ring”, and members of a ring never bids against each other. If one of the members get the object, they would perform an (or a series of) unofficial auction(s) afterwards, and make all the members better-off. In the example above, if set of all the buyers forms a ring, none of them would bid against agent $n$, resulting in a similar outcome as we suggested.

**Proposition 1.10:** There is no absence-proof allocation rule in Böhm-Bawerk’s horse market, in a single seller auction with a single object, and in house assignment problems.

Postlewatie (1979) showed that in a classical exchange economy, no allocation rule satisfies withholding-proofness along with Pareto optimality and individual rationality. Note that any core allocation is Pareto optimal and individually rational. In both problems above, AP resembles
withholding-proofness while the withholding entity is a group rather than an individual. Moreover, in Example 1.5, and 1.6 as the manipulating coalition does not utilize the goods outside, specific to these examples, the manipulation argument resembles destruction of endowments. As AP implies core stability, the impossibility result above is not surprising.

Group manipulations, as well as the total secession in the core give rise to some transaction cost related to the means of agreement. This cost grows with the size of the manipulating coalition. In both examples the manipulating coalition consists of only three agents. This number is big compared to the size of the society. However, even if the society is formed of 199 agents, with a 100 of one type and 99 of the other, a coalition of three agents would still manipulate with the same argument.

1.3.2 Housing markets

Here, there is a unique core allocation if individual preferences are strict. This allocation can be implemented by the famous top trading cycle (TTC) algorithm introduced by Shapley and Scarf (1974). The TTC algorithm is as follows: Let every agent in $N$ be represented by a node in a directed graph. From each agent $i$, draw a directed link to the agent $j$ who owns the top house in $i$’s linear order. This will create at least one cycle. Within each cycle perform the trade so that each agent gets his top house. All agents who get their top choice in the first round leaves the scene with their new house (unless $i$’s top choice is his own house, in that case he leaves with his own house). Delete the houses that left the scene from the preference of the agents who were not in a cycle in the first round. Now, apply the same procedure among them. This algorithm stops in a finite number of rounds, and returns the unique core allocation.

The direct revelation mechanism through TTC algorithm (henceforth core mechanism) is also known to be group strategy-proof; that no coalition can gain by jointly misrepresenting their preferences. However, a group of agents can manipulate the core mechanism by performing a
trade prior to the implementation of the mechanism (see Moulin 1995). This move never makes every agent in the coalition strictly better-off.

**Example 1.8:** Consider the following economy where $h_i$ represents the house that agent $i$ owns, and $P_i$ represent agent $i$’s preferences.

<table>
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<tr>
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<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
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<tbody>
<tr>
<td>$h_2$</td>
<td>$h_1$</td>
<td>$h_2$</td>
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<tr>
<td>$h_3$</td>
<td>$h_1$</td>
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<td>$h_1$</td>
<td>$h_3$</td>
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</table>

In the core mechanism, agents 1 and 2 trade houses getting their top choice and agent 3 is left out with his own house. If the coalition $\{2,3\}$ agrees on agent 2 to stay out, agent 1 gets $h_3$ and agent 3 gets $h_1$ in the core mechanism. Afterwards, agent 3 gives $h_3$ to agent 2 in return for $h_2$. This move results in a (weak) Pareto improvement for the coalition $\{2,3\}$. Note that the final outcome after the manipulation can also be achieved by a pre-trade between agents 2 and 3 prior to the implementation of the mechanism.

**Absence-proofness (weak vs. strong)** In the absence of monetary transfers, a weak Pareto improvement (that not all the agents from the manipulating coalition strictly benefits) is not always considered as a true motivation for a group manipulation. One particular reason is that agents have preferences over the houses but not on the allocation. Here, we say a rule is *strongly absence-proof* if it is immune to manipulations by weakly Pareto improving moves. If manipulation requires a strict Pareto improvement, the corresponding stability concept is *weak absence-proofness*.

**Proposition 1.11:** There is no strongly absence-proof allocation rule in housing markets, while core mechanism yields the unique weakly absence-proof allocation rule.

**Proof:** As AP induces core stability, and there is a unique core allocation at each problem, Example 3.4 proves the first statement. Let $N_1$ denote the set of agents that gets his top house at the first round of the TTC mechanism, $N_2$ denote those who get the “restricted” top choice in the
second round and so on. Suppose a coalition $S$ is able to make a strict Pareto improvement by leaving a subgroup $T$ out. Then, $S \cap N_1 = \emptyset$. As $T \cap N_1 = \emptyset$, agents who get their top choice at problem $N \setminus T$ in the core mechanism is again $N_1$. Then, no agent from $N_2$ can get a better house in the absence of $T$. Given, $S \cap N_2 = \emptyset$, by a similar argument we have $S \cap N_3 = \emptyset$, and so on. Uniqueness follows from the fact that there is a unique core allocation. \hfill \Box

### 1.3.3 Classical exchange economies

As we discussed earlier, the competitive equilibrium is proven to be vulnerable to individual and group manipulations in many ways in this specific model. Here, the set of competitive equilibrium allocations (when it exists) is a strict subset of the set of core allocations. However, when the economy is large enough (when the effect of a single agent on the competitive price is negligible) the set of competitive allocations converges to the set of core allocations. As absence-proofness implies core, our intuition tells that a selection from competitive allocations (henceforth Walrasian allocation) is our only hope for an absence-proof allocation rule.

**Definition 1.2:** An allocation rule $\varphi(\cdot)$ is AP on a domain of preferences $\mathcal{R}$ if for any economy $\varepsilon = (N, e, R)$ with $R \in \mathcal{R}^N$, for any $T \subseteq N$, and $K \subseteq N \setminus T$, there is no $y \in \mathbb{R}_+^{(k+t)\ell}$ with $\sum_{i \in (K \cup T)} y_i = \sum_{i \in K} \varphi_i(e_i^{N \setminus T}) + \sum_{i \in T} e_i$ s.t. $y$ Pareto dominates $\{\varphi_i(\varepsilon)\}_{i \in (K \cup T)}$ for agents in $K \cup T$.

**Remark 1.2:** AP implies PO and core stability by definition. Just set $T = N$, $K = \emptyset$ for PO, and for all $T$ set $K = \emptyset$ for core stability.

The following example illustrates that the divisibility of the goods enlarges the manipulation options. In an economy with just three individuals and even where all agents have the same fine Cobb-Douglas preferences, the Walrasian allocation is manipulable.

**Example 1.9:** $\ell = 2, N = 3, \{e_1, e_2, e_3\} = \{(10,10), (35,5), (15,15)\}$. $u_i = x_i y_i$ for all $i$. 
Check that the Walrasian allocation $\phi(\cdot)$ and the induced utilities for the problem $N$ and $S = \{1,2\}$, with the prices $p(N) = (1,2)$ and $p(S) = (1,3)$ as follows:

$$
\begin{array}{cccccc}
S & \phi_1(S) & \phi_2(S) & \phi_3(S) & u_1(S) & u_2(S) & u_3(S) \\
\{1,2,3\} & (15,7,5) & (22,5,11.25) & (22,5,11.25) & 112.5 & 253.125 & 253.125 \\
\{1,2\} & (20,20/3) & (25,25/3) & * & 400/3 & 625/3 & * \\
\end{array}
$$

Table 1.2

Note that $\phi_1(\{1,2\}) + e_3 = (35,21.6)$. Consider the following redistribution of this total to individuals 1, and 3: $z_1 = (15,7.6)$, and $z_3 = (20,14)$. Hence, the Walrasian allocation is manipulable by agents 1 and 3 as we have, $u_1(z_1) > u_1(N)$ and $u_3(z_3) > u_3(N)$.

**Example 1.10**: There are 11 agents each of whom is endowed with 1 kg of beans and 1 kg of rice. All of the agents have linear preferences as follows: $u_i = r_i + 10b_i$, $u_i = 10r_i + b_i$ for $i = 1, \ldots, 11$.

The unique competitive price in the above economy is $(1,p) = (1,10)$, where $p$ is the price of rice. At this price, agent 1 exploits the entire surplus in the market, and the remaining agents are left with their initial utility level of 11. Let $S = \{2, \ldots, 11\}$, and only $K \in S$ comes to the market while agent 1 is always active in the market. Then, the unique competitive price is $p = |K|$. Hence, all the agents in $K$ has a positive profit from the trade. This profit can be redistributed in a way that makes every agent strictly better-off with respect to their initial utility level as the utility is linear.

**Proposition 1.12**: On the domain of linear preferences and the Cobb-Douglas preferences, the Walrasian allocation rule is not AP.

Thomson (2013) defines the same manipulating argument, and the corresponding stability property withdrawal-proofness, where the manipulating party consists of only two agents. Note that withdrawal-proofness is weaker than AP. He proves that the Walrasian allocation is not
withdrawal-proof on the domain of homothetic preference. So, our results coincide with Thomson (2013).

As almost all other stability arguments, the manipulation argument relies on the perfect knowledge of agents about other agents’ characteristics (endowments and preferences) and the allocation method. Example 1.10 illustrates an important characteristic of manipulation by absence. Note that, there, any subset $S'$ of $S$ can manipulate by leaving any strict subset $T \subset S'$ out of the allocation process. This suggests that, in some instances, agent’s rough idea about the “type” of other agents in the market would be a sufficient motivation to take a manipulating action.

Sertel and Yıldız (1999) discuss the welfare effect of an additional agent that brings new trading opportunities on the existing agents. If the allocation rule always assigns core allocations, we expect some agents to benefit from the appearance of a newcomer (unless in the degenerate case where no existing agent is affected at all). They show that existing agents who have “sufficiently similar types” with the entrant are hurt, and the others benefit as their trading opportunities expands. Their discussion relies on the fact that the population-monotonicity is too demanding in this setting. Hence, we cannot expect all the existing agents to benefit regardless of the entrants type. This fact is quite transparent in Example 1.10. Suppose only agents 2 and 3 are in the market initially. As they are exactly of the same type no trade occurs. If agent 1 arrives in the market, both existing agents would benefit. However, if we add 8 more agents (agents 4 to 11), this would hurt agent 2 and 3, while agent 1 benefits from their arrival.

Just like in TU games, PO and PM implies core stability in this context (see Proposition 1.13 below). However, as we discuss in Example 1.11 below, the logical relation between AP and PM breaks down here. Hence, even the very demanding property PM does not guarantee avoiding manipulation by absence.
**Definition 1.3**: An allocation rule $\varphi(\cdot)$ is PM on a domain of preferences $\mathcal{R}$ if for any economy $\varepsilon = (N, e, R)$ with $R \in \mathcal{R}^N$, for any $S \subseteq N$ and $i \in S$, we have $\varphi_i(\varepsilon)R_i\varphi_i(\varepsilon^S)$.

**Proposition 1.13**: If $\varphi(\cdot)$ is PM and PO on a domain of preferences $\mathcal{R}$, $\varphi(\cdot)$ is a core selection on $\mathcal{R}$.

**Proof**: Let $\varphi(\cdot)$ be PM and PO on $\mathcal{R}$. Take any $\varepsilon = (N, e, R)$ with $R \in \mathcal{R}^N$, let $S \subseteq N$, and $x$ be an allocation in $\varepsilon^S$. By PO we have, $\varphi(\varepsilon^S)$ is not Pareto dominated by $x$. As by PM $\varphi_i(\varepsilon)R_i\varphi_i(\varepsilon^S)$ for all $i \in S$, $\{\varphi_i(\varepsilon)\}_{i \in S}$ is not Pareto dominated by $x$ either. □

**Example 1.11**: $\ell = 2$, $N = 3$, $\{e_1, e_2, e_3\} = \{(10,10), (35,5), (15,15)\}$, $u_i = x_iy_i$ for all $i$.

Consider the following allocation scheme and the induced utilities:

<table>
<thead>
<tr>
<th>$S$</th>
<th>$\varphi_1(S)$</th>
<th>$\varphi_2(S)$</th>
<th>$\varphi_3(S)$</th>
<th>$u_1(S)$</th>
<th>$u_2(S)$</th>
<th>$u_3(S)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1,2,3}</td>
<td>$(10\sqrt{2},5\sqrt{2})$</td>
<td>$(60 - 25\sqrt{2},30 - 25\sqrt{2}/2)$</td>
<td>$(15\sqrt{2},15\sqrt{2}/2)$</td>
<td>100</td>
<td>303.7</td>
<td>225</td>
</tr>
<tr>
<td>{1,3}</td>
<td>$(10\sqrt{3},10\sqrt{3}/3)$</td>
<td>$(45 - 10\sqrt{3},15 - 10\sqrt{3}/3)$</td>
<td>*</td>
<td>100</td>
<td>255.4</td>
<td>*</td>
</tr>
<tr>
<td>{1,3}</td>
<td>(10,10)</td>
<td>*</td>
<td>(15,15)</td>
<td>100</td>
<td>*</td>
<td>225</td>
</tr>
<tr>
<td>{2,3}</td>
<td>*</td>
<td>$(50 - 15\sqrt{5}/2,20 - 6\sqrt{5}/2)$</td>
<td>$(15,5/2,6\sqrt{5}/2)$</td>
<td>*</td>
<td>276.3</td>
<td>225</td>
</tr>
<tr>
<td>{1}</td>
<td>(10,10)</td>
<td>*</td>
<td>*</td>
<td>100</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>{2}</td>
<td>*</td>
<td>(35,5)</td>
<td>*</td>
<td>*</td>
<td>175</td>
<td>*</td>
</tr>
<tr>
<td>{3}</td>
<td>*</td>
<td>*</td>
<td>(15,15)</td>
<td>*</td>
<td>*</td>
<td>225</td>
</tr>
</tbody>
</table>

*Table 1.3*

Note that $\varphi$ is PM, PO and $\varphi_1(\{(1,2)\}) + e_3 \geq (32.3,20.7)$. Consider the redistribution of this total to individuals 1 and 3 as $z_1 = (14.3,7.7)$ and $z_3 = (18,13)$. We have $u_1(z_1) > u_1(N)$, $u_3(z_3) > u_3(N)$, and hence $\varphi$ is not AP.

### 1.4 Fair Division Problems

**Basic Notions**: A set of individuals $N \in \mathcal{N}$ have equal claims on a fixed supply of commonly owned goods $\Omega \in \mathcal{C}$, where $\mathcal{C}$ stands for the consumption space, and $\mathcal{N}$ denotes the set of all finite subsets of $\mathbb{Z}^+$. For now, we do not impose a structure on $\mathcal{C}$ so that we keep our main result as general as possible. However, we want the reader to keep in mind that we are dealing with the
allocation of a bundle of perfectly divisible goods in \( \mathbb{R}_+^k \), or a finite set of indivisible objects, or a combination of both. Two important cases we do not include in Theorem 1.1 is the fair division of a heterogeneous, and divisible commodity (generally known as cake cutting, or land division problem), and the case where we do not necessarily allocate \( \Omega \) wholly. Indeed, with some further specifications the result will still hold. However, we want to keep the idea (and the notation) as simple as possible. A very critical assumption for the manipulation argument to be conceptually proper is that consumption is private, and once an agent receives the good, he has the complete right to consume or transfer it to another agent. So, we are dealing with excludable and rival goods. Moreover, we can safely introduce monetary transfers into the model, and say along with the goods, a total amount \( M \in \mathbb{R} \) of money is distributed. In that case \( M \) is embedded as a separate component in \( C \). Each individual \( i \) in the maximal society has a complete and transitive preference relation \( R_i \) on \( C \), and \( P_i \) denotes the strict counterpart of \( R_i \). \( \mathcal{R} \) denotes the set of admissible preferences for each individual. Given a society \( N \in \mathcal{N} \), a preference profile is a vector \( R \in \mathcal{R}^N \), and \( R^S \) is the profile restricted to \( S \subseteq N \). Then, a fair division model is \((\mathcal{N}, \mathcal{C}, \mathcal{R})\), and a specific problem is simply a triple \((N, \Omega, R)\). We also assume that preferences are strictly increasing in money in case transfers are allowed.

For a fixed \((N, \Omega, R)\), an allocation \( x = (x_i)_{i \in N} \) is a vector s.t. the sum and/or union of the total allocated goods is \( \Omega \). Specifically, in the context of indivisible good, no two agents share the same good while some agents may receive no good at all. Also, no agent receives a negative amount of any divisible good. \( X(N, \Omega, R) \) denotes the set of allocations. For each \( x \in X(S, \Omega, R^S) \), and \( K \subseteq S \subseteq N \), \( \sum_K x(S) \) denotes the sum and/or union of goods (and possibly money) that the coalition \( K \) gets in allocation \( x \) at the “reduced” problem \((S, \Omega, R^S)\) where agents in \( S \) are the only claimants. An allocation \( x \in X(N, \Omega, R) \) is Pareto optimal (PO) if \( x \) is not Pareto dominated by another allocation i.e., there is no \( z \in X(N, \Omega, R) \) such that \( z_i R_i x_i \) for all \( i \in N \) and \( z_j P_j x_j \) for some \( j \in N \).
Given a model \((N, C, R)\), an *allocation rule* \(\varphi\) is a mapping that assigns a *subset of allocations* to each problem \((N, \Omega, R)\). An allocation rule is Pareto optimal if \(\varphi\) assigns PO allocations to all \((N, \Omega, R)\).

In case monetary compensations are not available, for any two consumption bundles \(x, y\) in the consumption space, we say that \(x > y\) if \(x\) is weakly greater in all components, and strictly greater in at least one component. Definition is akin to the vector relation in \(\mathbb{R}^d_+\), and if we have a set of indivisible objects as a component in \(x\) and \(y\), weakly and strictly greater corresponds to the set inclusion relations \(\subseteq\) and \(\subset\), respectively. Also, \(x \gg y\) if \(x\) is strictly greater in all components. When compensations are possible, if two bundles have the same transfer, definitions remain the same. If \(x\) has strictly more money and \(x\) is weakly greater in all other components, we say \(x \gg y\).

1.4.1 On the AP and PM

In an allocation problem with common endowments, absence of a coalition \(S\) in the allocation process means that they renounce their claims. Thus, core has no bite here. However, the partial secession of \(S\), meaning only a strict \(T \subset S\) is left out, can still be profitable. In that sense, (to my knowledge) absence-proofness is the first core-like stability property in the context of fair division (except withdrawal-proofness in Thomson (2013)).

**Definition 1.4:** An allocation rule \(\varphi\) is *manipulable* at a problem \((N, \Omega, R)\) by a coalition of agents \(S \subseteq N\) via absence of \(T \subset S\), if there exist \(y \in \varphi(N, \Omega, R), y' \in \varphi(N \setminus T, \Omega, R^{N \setminus T})\), and \(\{z_i\}_{i \in S} \in X(S, \sum_{S \setminus T} y'(N \setminus T), R^S)\) (a reallocation of what \(S \setminus T\) gets in the allocation \(y'\) at problem \((N \setminus T, \Omega, R^{N \setminus T})\) to the agents in \(S\)) s.t. \(z_i R_i y_i\) for all \(i \in S\), and \(z_j P_j y_j\) for some \(j \in S\).

**Definition 1.5:** Given a model \((N, C, R)\), an allocation rule \(\varphi\) is *absence-proof* (AP) if it is not manipulable at \((N, \Omega, R)\), for any \(N \in \mathcal{N}, R \in \mathcal{R}^N\) and \(\Omega \in \mathcal{C}\).
Note that manipulability is defined here in the weak form; existence of one allocation in the grand game and one allocation in the reduced game is enough. Thus, AP has the strongest possible interpretation. In the examples we provide for our negative results, the allocation rules assign single allocations, although they may assign multiple allocations in general (see CEEI in Section 1.4.2). Hence, this strong interpretation does not affect the results here. Moreover, it enhances the robustness of AP rules we discuss here in terms of stability.

**Proposition 1.14:** Given \((\mathcal{N}, \mathcal{C}, \mathcal{R})\), every AP allocation rule is PO.

**Proof:** Suppose \(\varphi\) is not PO at \((N, \Omega, R)\), and say \(z \in X(N, \Omega, R)\) Pareto dominates \(y \in \varphi(N, \Omega, R)\). Let \(S = N\), and fix a \(T \subset S\). As \(\sum_{N \setminus T} y' = \Omega\), for any \(y' \in \varphi(N \setminus T, \Omega, R^{N \setminus T})\), we have \(z \in X(N, \sum_{N \setminus T} y', R^N)\). Then, \(\varphi\) is manipulable at \((N, \Omega, R)\) by \(N\) via absence of \(T\). \(\square\)

When agents share a fixed supply of goods, it is natural to ask no one to benefit from arrival of additional agents. This is one (strong) interpretation of population-monotonicity as a normative solidarity principle. However, in case monetary compensations are available (and utilities are quasilinear in money), an additional agent who receives a much higher utility from some bundle than all the existing agents may significantly increase the monetary value of the pie to be distributed. Then, it is plausible for an existing agent to benefit from the arrival of the newcomer. In that case, the solidarity principle asks either no existing agent lose, or no one gains. This weaker version appears under different names in the literature; population solidarity, weak population-monotonicity, and even sometimes population-monotonicity (Moulin 1992; Thomson 1995; Tadenuma and Thomson 1993).

**Definition 1.6:** Given a model \((\mathcal{N}, \mathcal{C}, \mathcal{R})\), an allocation rule \(\varphi\) is population-monotonic (PM) if for all \(\Omega \in \mathcal{C}; N, N' \in \mathcal{N}\) with \(N \subseteq N'\), \(R \in \mathcal{R}^{N'}\), \(y \in \varphi(N, \Omega, R^N)\), \(y' \in \varphi(N', \Omega, R)\), we have \(y_i R_i y'_i\) for all \(i \in N\). \(\varphi\) is weakly population-monotonic (wPM), if we have either \(y_i R_i y'_i\) for all \(i \in N\), or \(y'_i R_i y_i\) for all \(i \in N\).
Note that Proposition 1.14 does not hold if we replace AP with PM. Just consider the case
\( \mathcal{C} = \mathbb{R}_{+}^{n} \), and \( \varphi_{i}(N, \Omega, R) = \Omega / n \) for all \( i \in N \). \( \varphi \) is obviously PM for \( \mathcal{R} \) being the domain of all
monotone preferences\(^7\), but not PO for many preference profiles.

**Theorem 1.1:** Given a model \((\mathcal{N}, \mathcal{C}, \mathcal{R})\), if a PO allocation rule \( \varphi \) is PM, then it is also AP.

**Proof:** Fix a model \((\mathcal{N}, \mathcal{C}, \mathcal{R})\), and let \( \varphi \) be PO and PM. Suppose for a contradiction that \( \varphi \) is not AP. Then, for some \((N, \Omega, R), T \subset S \subseteq N, y \in \varphi(N, \Omega, R), y' \in \varphi(N \setminus T, \Omega, R^{N \setminus T})\), there is \( \{z_i\}_{i \in S} \in X(S, \sum_{S \setminus T} y' (N \setminus T), R^S) \) s.t. \( z_i R_i y_i \) for all \( i \in S \), and \( z_j P_j y_j \) for some \( j \in S \). By PM, for all \( i \in N \setminus S \), we have \( y'_i R_i y_i \). Now, consider the following allocation at problem \((N, \Omega, R)\):
\[ x_i = y'_i \] if \( i \in N \setminus S \), and \( x_i = z_i \) if \( i \in S \). Note that \( x \in X(N, \Omega, R) \), and it Pareto dominates \( y \). This contradicts that \( \varphi \) is PO. \( \square \)

Interestingly, although PM works in the opposite direction in TU surplus sharing games and in
fair allocation problems, PM implies AP in both problems. In the context of divisible goods
\((\mathcal{C} = \mathbb{R}_{+}^{n})\), Thomson (2013) introduces withdrawal-proofness. While the manipulation idea is the
same as in AP, the manipulating coalition consists of only two agents; corresponding to \( |S| = 2 \),
and \( |T| = 1 \) in Definition 1.4. Note that this property is weaker than AP. He also relates it to PM,
but from an opposite direction. He argues that if a coalition \( S \) manipulates by withdrawal of agent
\( i \), then the agent who stays in should not be worse off by departure of \( i \) in the restricted problem.
Hence, his welfare should be affected in the manner required by PM. This argument is true.
However, the critical argument for Theorem 1.1 depends on how the agents in \( N \setminus S \) are affected
as the resource is fixed.

Theorem 1.1 provides a sufficient condition to block the possibility of manipulation. Finding a
simple necessary and sufficient condition for AP is not an easy task in general. However, in the

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\(^7\) \( R_i \) on \( \mathbb{R}_{+}^{n} \) is said to be monotone if for any \( z, z' \in \mathbb{R}_{+}^{n} \) with \( z < z' \) we have \( z'R_i z \), and if \( z \ll z' \) we have \( z' P_i z \). Moreover, it is strictly monotone if \( z < z' \) implies \( z' P_i z \).
very simple model of allocating a single object where monetary transfers are available, it is possible (see Section 1.4.3.1).

An easier task is to define a sufficient condition for manipulation. Suppose an additional agent arrives in the allocation process, and an existing agent \( j \) receives no less than what he receives before and also more of some divisible good and/or an extra object and/or extra money. Then, all the existing agents except \( j \) would be willing to compensate the newcomer out of their total allocation and ask him to stay out.

**Proposition 1.15:** Given a model \((\mathcal{N}, \mathcal{C}, \mathcal{R})\), let \( \varphi \) be a allocation rule such that for some \( N, N' \in \mathcal{N} \) with \( N \subseteq N' \), \( j \in N \), \( \Omega \in \mathcal{C} \), \( R \in \mathcal{R}^{N'} \) where \( R_i \) is monotone (strictly monotone)\(^8\) for all \( i \in N' \). If for some \( y \in \varphi(N, \Omega, R^{N}) \), \( y' \in \varphi(N', \Omega, R) \) we have \( y_j \ll y'_j \) (\( y_j < y'_j \)), \( \varphi \) is not AP.

**Proof:** Let \( T = N' \setminus N \) and \( S = N' \setminus \{j\} \). Note that \( T \subset S \) and \( S \setminus T = N \setminus j \). Then, we have \( \sum_S y'(N') = \Omega - y'_j \ll (\prec) \Omega - y_j = \sum_{S \setminus T} y(N) \). By monotonicity (strict monotonicity) of preferences, \( \varphi \) is manipulable at the problem \( N' \) by coalition \( S \) via absence of \( T \). \( \square \)

### 1.4.2 Perfectly divisible goods with no monetary transfers

Here, we have \( \mathcal{C} = \mathbb{R}^e_+ \), and for a specific problem \((N, \Omega, R)\), an allocation \( x \in \mathbb{R}^n_+ \) is a vector s.t. \( x_i \in \mathbb{R}^e_+ \) for all \( i \in N \) and \( \sum_{i \in N} x_i = \Omega \). Three important solutions for the underlying problem are competitive equilibrium with equal incomes (CEEI), the \( \Omega \)-egalitarian equivalent (\( \Omega \)-EE) allocation proposed by Pazner and Schmeidler (1978), and the sequential priority (SP) solution. Given \((N, \Omega, R)\), CEEI is the set of competitive equilibrium allocations of the economy where each individual is initially endowed with \( \Omega/n \). The \( \Omega \)-EE allocation is such that each individual is indifferent between his allocation and \( \lambda \Omega \) for some number \( \lambda \). \( \Omega \)-EE picks the highest number \( \lambda^* \) such that a corresponding egalitarian equivalent feasible allocation exists, and assigns one of

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\(^8\) Standard adaptations of these properties in consumption space \( \mathbb{R}^e_+ \) to \( \mathcal{C} \). See footnote 7.
those allocations among which all the individuals are indifferent. On the domain of strictly monotonic and continuous preferences $\Omega$-$EE$ allocation is well-defined and PO. A nice feature of the $\Omega$-$EE$ allocation is that on that domain, it is PM (see e.g. Moulin 1995), while $CEEI$ is not (Chichilnisky and Thomson 1987).

Unlike the other two solutions, the $SP$ solution is not anonymous. Given a society $N$, fix a strict priority ordering of the agents in $N$. Then, at each problem $(S, \Omega, R)$ with $S \subseteq N$, just assign $\Omega$ to the agent in $S$ who precedes others in the order. Given a maximal society $N$, and an order on $N$, the solution is well-defined for each problem at a subsociety $S \subseteq N$, and only at those $S$. Hence, it resembles the allocation schemes in Section 1.2.1. Note that on any domain of preferences this solution is trivially PM. However, it may not be efficient on the domain of monotone preferences. Just consider the case where the first agent in the order is indifferent between any two bundles in $\mathbb{R}_+^\ell$. Moreover, if the second agent in that order has a strictly monotonic preference, these two agents can manipulate $SP$ by the absence of the first agent. Note that on the domain of strictly monotone preferences, the $SP$ solution is also PO.

**Corollary to Theorem 1.1:**

(i) The $\Omega$-$EE$ allocation rule is AP on the domain of continuous and strictly monotonic preferences.

(ii) The $SP$ solution is AP on the domain of strictly monotonic preferences.

As in exchange economies, competitive idea is vulnerable to manipulation by absence. The following example is used to show that the $CEEI$ is not PM on the domain of strictly monotonic preferences by Chichilnisky and Thomson (1987).

**Example 1.12:** Let $\ell = 2, \Omega = (24,24), |N| = 4$

$$u_i = \min\{2x + 8, y\}, u_i = \min\{18x + 100, 25y + 132\} \text{ for } i = 2,3,4.$$
The CEEI gives (2,12) to agent 1 at game N, and (1,10) at game \{1,2,3\}. Note that agent 1 gets less in each good when agent 4 leaves the game.

**Corollary to Proposition 1.15:** CEEI is not AP on the domain of continuous and strictly monotonic preferences.

If everyone has an equal right on the common endowment, it is fair to give them an equal share, but that is not efficient in general. If we assume that individuals seek for welfare improving trade opportunities, they would end up in a competitive equilibrium of the economy where the initial endowment of each individual is \( \Omega/n \). Indeed, the CEEI is the summary of this process. Although CEEI is not AP in general, it is less vulnerable to manipulation compared to the competitive allocation in exchange economies. We adapt the story in Example 1.10 to a fair division problem.

**Example 1.13:** An individual is planning to give away a total of 11 kgs of beans and 11 kgs of rice as a food aid. There are 11 potential poor individuals in his neighborhood. The donor announces to these people that he will divide the total among those who appear at his door at a certain time, truly assuming that they will trade afterwards. Beans and rice are necessity for these people, and hence substitutes. Agent 1 prefers beans and the other 10 agents prefer rice. Assume wlog that \( u_1 = r_1 + 10b_1, u_i = 10r_i + b_i \) for \( i = 1, ..., 11 \).

Recall from Section 1.3 that if all agents appear at the door, in the final CEEI outcome, agent 1 gets 11 kgs of beans with a utility level of 110, and all the other receive 1.1 kg of rice each with a utility level of 11. If only \( K \subset S = \{2, ..., 11\} \) appears, each agent in \( S \) receives \( \Omega/|S| \), and hence no group of agents is able to compensate the loss of agent 1 due to his absence. If agent 1 and \( K \subset S \) appear, it is easy to check that agent 1 receives 11 kgs of beans and the remaining agents get \( 11/|S| \) kgs of rice. Apparently, no group has a motivation for manipulation in this case too. Here, the CEEI outcome is easy to implement for the donor, efficient and immune to manipulation by absence of some agents.
A plausible rule, well-defined on all finite societies, that is AP but not PM is yet to be explored. However, if we have a maximal society and we adopt allocation schemes as a solution concept, AP solutions violating PM exist. The definition of allocation schemes here is parallel to the one in Section 1.2.1 i.e., given a problem \((N, \Omega, R)\), an allocation scheme assigns an allocation \(X(S) \in \mathbb{R}_+^\ell\) for any problem \((S, \Omega, R^S)\) with \(S \subseteq N\). The allocation scheme \(X\) in Table 1.4 is AP, but not PM.

Example 1.14: Consider the story in Example 1.13 where the parameters and the preferences are as follows: \(\ell = 2, \Omega = (4,4), |N| = 3\).

\[
\begin{align*}
    u_1 &= \min\{28x + y, 10y + x\}, u_2 = \min\{10x + y, 28y + x\}, u_3 = xy
\end{align*}
\]

<table>
<thead>
<tr>
<th>(S)</th>
<th>(X_1(S))</th>
<th>(X_2(S))</th>
<th>(X_3(S))</th>
</tr>
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<td>(1.5, 0.5)</td>
<td>(2, 2)</td>
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<td>(3, 1)</td>
</tr>
<tr>
<td>{2, 3}</td>
<td>*</td>
<td>(3, 1)</td>
<td>(1, 3)</td>
</tr>
</tbody>
</table>

Table 1.4

It is easy to check that \(X(S)\) is PO at each \(S\). As \(u_3(2,2) > u_3(3,1)\), \(X\) is not PM. To see that \(X\) is not manipulable by \(S = \{1,2\}\) via absence of 1, note that agent 2 gets \(X_2(\{2,3\}) = (3,1)\). Then, agent 1 can get a maximum of 13 unit of utility in any redistribution of \((3,1)\) among agents 1 and 2, while agent 1 already gets 15.5 unit of utility at problem \(N\). The argument is similar for the absence of agent 2. Agent 3 will form a coalition with neither agent 1 nor agent 2 as in any case \((1,3)\) or \((3,1)\) will be redistributed among the manipulating coalition, and agent 3 gets at most 3 units of utility while he gets 4 units at the problem \(N\). Hence, \(X\) is AP.

1.4.3 Models with monetary transfers

In models with quasilinear utilities and monetary transfers, for each problem \((N, \Omega, R)\) there is an associated stand-alone TU cooperative game defined by the characteristic function \(v(S) = \max\{\sum_{i \in S} u_i(x_i): x \in X(S, \Omega, R^S)\}\). This game is not directly useful in AP analysis as we
discussed in Section 1.2.1. However, it is useful to find PM solutions. Moulin (1990b) proved that PM is not compatible with PO in general, in the allocation of a perfectly divisible bundle of goods from $\mathbb{R}^\ell_+$ (for $\ell > 2$). A similar impossibility was established in Beviá (1996a) in the context of indivisible goods where an agent is allowed to receive more than one good. However, introducing a substitutability\(^9\) axiom on the individual preferences and the “joint preference”, Moulin (1992) in the context of divisible goods, and Beviá (1996a) in the context of indivisible goods showed that the associated TU game is concave. Hence, the Shapley solution is PM by Sprumont (1990). Here, the Shapley solution is simply the set of allocations that yield the final utilities equal to the Shapley value of the stand-alone game. Moulin’s result is indeed a bit more general, and also implies that in case of allocating finite objects where each agent receives at most one object (a special case is allocating a single object with monetary transfers), Shapley solution is also PM. In that case if $|N| < |\Omega|$, Theorem 1.1 does not immediately imply that the Shapley solution is AP. However, with some little adjustments it does capture this case also.

**Open question 2:** Is there an AP rule in the general contexts studied in Moulin (1990b) and Beviá (1996a)?

For the special case of distributing two objects (where an agent can get both) we show in Chapter 3 that on the domain of monotone preferences, PM (and hence AP) solutions always exist. In particular, a hybrid Shapley solution is PM.

The case of single object is equivalent to the well-known airport problems (cost sharing) and this problem admits several interesting PM solutions including the Shapley value, nucleolus and the Dutta-Ray solution of the associated game (see Thomson (2007) for a survey).

Stating the corollaries to Theorem 1.1, we turn back to our primary question: to understand the difference between AP and PM. The difference between the two properties is not obvious in

\(^9\) In perfectly divisible case, two goods $j$, and $k$ are substitutes in the utility function $u$ if marginal utility from consuming good $k$ decreases by an increase in the consumption of good $j$. The idea is similar for the indivisible case.
general. However, in the simple model of allocating a single indivisible object, we give a compact characterization of AP rules. This characterization makes it easy to read if a rule is AP or not, moreover it makes the difference between the two properties clearly visible.

1.4.3.1 A single indivisible object

A single object is to be distributed to a set of individuals $N$. A well-known example is an inheritance problem where an estate is to be assigned to one of the heirs, and monetary transfers are available to compensate the ones who do not get the object. For simplicity we take $M = 0$ (our results still hold for $M 
eq 0$ with minor adjustments). The value of the object for each individual $i$ is $a_i \geq 0$, and a problem is a tuple $(N, a)$ with $a = \{a_i\}_{i \in N}$. A solution to the problem $(N, a)$ is a tuple $\{ (\delta_i, t_i) \}_{i \in N}$, where $\delta_i = 0$ for those who do not get the object at the solution, and if $j$ gets the object $\delta_j = 1$. Also, $t_i$ represents the monetary transfer with $\sum_{i \in N} t_i = 0$. Agents preferences are quasilinear in money, i.e., $u_i(\delta_i, t_i) = \delta_i a_i + t_i$ for all $i \in N$. Given a problem $(N, a)$ and a subset $S \subseteq N$, $(S, a^S)$ denotes the restricted problem where $a^S_i = a_i$ for all $i \in S$, and $\bar{a}^S = \max_{i \in S} a_i$.

As we already discussed, individual rationality (IR: that each agent ends up with a non-negative utility) is conceptually a requirement for AP, and PO is a consequence of AP. At a solution satisfying both properties, an agent $j$ with $a_j = \bar{a}^N$ gets the object and compensate the others with monetary transfers such that $t_i \geq 0$ for all $i \neq j$ and $u_j = \bar{a}^N - \sum_{i \in N \setminus j} t_i \geq 0$. Therefore, such a solution induces a unique set of numbers $\{u_i\}_{i \in N}$ with $u_i \geq 0$ and $\sum_{i \in N} u_i = \bar{a}^N$. Conversely, any such set of numbers represents a set of PO and IR solutions among which all individuals are indifferent, and a single solution if there is only one $i$ with $a_i = \bar{a}^N$.

An allocation rule $\varphi$ assigns a solution to each problem $(N, a)$. Given a problem $(N, a)$ and an allocation rule, $\{ (\delta_i(S), t_i(S)) \}_{S \subseteq N, i \in S}$ denotes the solutions that the allocation rule assigns to problem $(S, a^S)$ for all $S \subseteq N$, $\{u_i(S)\}_{S \subseteq N, i \in S}$ denotes the induced final utilities.
The main difficulty in analyzing absence-proofness is the complexity of reallocation opportunities after the allocation process. In this model however, the most beneficial reallocation is simple. If the agents from the manipulating coalition who stays in the problem do not receive the object, all we need to know is the total transfer they receive. If one of them, say agent $j$, receives the object in the sub-problem, the best he can do is to pass the object to an agent, say agent $k$, with the highest valuation among the agents that stay out, and that is in case $a_k > a_j$.

**Proposition 1.16:** Let $\varphi$ be an allocation rule that induces $\{(\delta_i(S), t_i(S))\}_{S \subseteq N, i \in S}$, and $\{u_i(S)\}_{S \subseteq N, i \in S}$ at problem $(N, a)$. Then, the following are equivalent;

(i) $\varphi$ is AP at $(N, a)$.

(ii) $\varphi$ is PO at $(N, a)$, and also, if $|N| \geq 3$; $\forall S \subseteq N$ we have;

$$u_i(N) \leq u_i(S) \text{ if } \delta_i(S) = 0, \text{ and } u_j(N) - u_j(S) \leq \bar{a}^N - \bar{a}^S \text{ if } \delta_j(S) = 1.$$

**Proof:**

(i) $\Rightarrow$ (ii): By Proposition 1.14 $\varphi$ is PO. Suppose for a contradiction for some $j \in S$ with $\delta_j(S) = 0$ we have $u_j(N) > u_j(S)$, or $\delta_j(S) = 1$ and we have $u_j(N) - u_j(S) > \bar{a}^N - \bar{a}^S$. There are four cases depending on whether $\delta_j(N) = 0$, or $\delta_j(N) = 1$. In each case it is easy to check that $N \setminus j$ can manipulate $\varphi$ by absence of $N \setminus S$.

(ii) $\Rightarrow$ (i): Take any $K \subset S \subset N$, let $j \in S$ with $\delta_j(S) = 1$ and (ii) hold. We need to show that $K \cup (N \setminus S)$ does not Pareto improve upon its allocation by leaving $N \setminus S$ out of the problem. Consider first the case that $j \notin K$. Then, $K \cup N \setminus S$ has only the transfers but not the object if $N \setminus S$ leaves, and the total money to be redistributed is $\bar{a}^S - \sum_{i \in S \setminus K} u_i(S)$. By (ii), $\bar{a}^S - \sum_{i \in (S \setminus K) \setminus j} u_i(S) - u_j(S) \leq \bar{a}^N - \sum_{i \in (S \setminus K) \setminus j} u_i(N) - u_j(N) = \sum_{i \in (K \cup (N \setminus S))} u_i(N)$. Now, consider the case $j \in K$. The total utility of $K$ in problem $(S, \bar{a}^S)$ is $\bar{a}^S - \sum_{i \in S \setminus K} u_i(S)$. By a redistribution of the transfers and the object $K \cup (N \setminus S)$ can increase this total utility by a
maximum of \( \bar{a}^N - \bar{a}^S \), and this maximum is reached if there is \( k \in N \setminus S \) with \( a_k = \bar{a}^N \). By (ii) we have \( \bar{a}^N - \bar{a}^S + \bar{a}^S - \sum_{t \in S \setminus K} u_t(S) \leq \bar{a}^N - \sum_{t \in S \setminus K} u_t(N) = \sum_{t \in (K \cup N \setminus S)} u_t(N) \). \( \square \)

Proposition 1.16 makes the difference between AP and PM very clear. PM requires that when a group of agents leave, the utility of none of the agents should decrease. AP, however, allows in such a case, the utility of only one individual (the one who gets the object in the sub-problem) to decrease, and the upper-bound in the change of the utility of this agent is \( \bar{a}^N - \bar{a}^S \).

The interesting exercise here is to find out solutions that are AP but not PM. The following solution \( X^{SO}(\cdot) \) (serial oligarchy) is an allocation scheme rather than an allocation rule. It is well-defined on a fixed maximal society and all the sub-problems.

**Definition 1.7**: Fix a maximal society \( N \), and define a linear order on \( N \). Given a problem \((N, a)\); for any problem \((S, a^S)\), assign the object to individual \( j \) s.t. \( j \) is the first in the order with \( a_j = \bar{a}^S \). In the reduced profile \( a^{S \setminus j} \), pick the individual \( j' \) s.t. \( j' \) is the first in the order with \( a_{j'} = \bar{a}^{S \setminus j} \). Then, distribute \( \bar{a}^S \) equally among this two agents, i.e. \( X_j^{SO} = (1, -\bar{a}^S/2) \), \( X_{j'}^{SO} = (0, \bar{a}^S/2) \), and \( X_i^{SO} = (0, 0) \) for all \( i \in N \setminus \{j, j'\} \).

Note that the fixed order serves in breaking ties. It is relevant only if there are at least two agents with the highest valuation in the profile, or there is a single agent with the highest valuation, and there are at least two individuals with the second highest valuation.

**Proposition 1.17**: Given a maximal society \( N \), and a linear order on \( N \), the associated serial oligarchy rule \( X^{SO}(\cdot) \) is AP, but not PM.

**Proof**: Let \( |N| = 3 \), and \((a_1, a_2, a_3) = (2, 6, 10)\). Regardless of the fixed order, final utilities for agent 2 induced by \( X^{SO}(\cdot) \) are \( u_2(N) = 5 \), and \( u_2(\{1, 2\}) = 3 \). Hence, \( X^{SO}(\cdot) \) is not PM. Now, fix a maximal society \( N \), a linear order on \( N \), a problem \((S, a^S)\), and suppose a group of individuals leave. In the reduced problem, say \( S' \), the utility of only the agent \( j \) who gets the object in \( S' \) may decrease by definition of \( X^{SO}(\cdot) \) regardless of the order. That happens only in
case $j$ does not get the object, but receives a transfer in problem $S$. In that case, the decrease in his utility is $(\bar{a}^S - \bar{a}^{S'})/2$. Hence, by Proposition 1.16, $X^{S0}(\cdot)$ is AP. □

A simple but very compelling fairness property is equal treatment of equals (ETE). A solution that satisfies ETE does not discriminate the agents with the same valuation, i.e., for any problem $(N, a)$ the final utilities induced by the rule satisfy $u_i(N, a) = u_j(N, a)$ for any $i, j \in N$ with $a_i = a_j$. PO dictates the assignment of the object to and agent with the highest valuation. In case there are several such agents, this assignment is critical to analyze AP. If in addition to IR and PO we impose ETE, all we need to know is the final utilities of the agents to check for AP. Hence, we can now use utility distributions as a solution object, which by definition satisfies PO and IR.

All the following results hold for single-valued allocation rules that satisfy ETE.

**Definition 1.8**: A utility distribution $U(\cdot)$ is a mapping from the set of all problems to $\mathbb{R}^n_+$ s.t. for each $(N, a)$, $\sum_{i \in N} U_i(N, a) = \bar{a}^N$.

**Proposition 1.18**: Let $U(\cdot)$ be a utility distribution that satisfies ETE. Fix a problem $(N, a)$, and order the individuals s.t. $a_1 \leq a_2 \leq \cdots \leq a_n$. Suppose we add agent $j$ to the problem. In case $a_n = a_{n-1}$ or $a_j < a_n$, AP dictates utility of the pre-existing agents not to increase. In case $a_{n-1} < a_n < a_j$, AP dictates utility of only agent $n$ can increase, and the maximum increase in his utility is $a_j - a_n$. Moreover, these are sufficient conditions for $U(\cdot)$ (any allocation rule $\varphi$ that yields the same final utilities with $U(\cdot)$) to satisfy AP.

**Proof**: Necessity of the conditions directly follows from Proposition 1.16 and ETE. To see the sufficiency, take a problem $(N, a)$, a subproblem $(S, a^S)$, and suppose the conditions above hold. In case $\bar{a}^N = \bar{a}^S$, the condition ii. in Proposition 1.16 trivially holds. In case $\bar{a}^N > \bar{a}^S$, add agents in $N \setminus S$ recursively to the problem $(S, a^S)$, starting with some $j \in N$ with $a_j = \bar{a}^N$. It is easy to see that condition ii. holds here too. □
One of the main themes in the fair division literature is the compatibility of the monotonicity properties with different fairness criteria. Alkan (1994) showed that *envy-freeness* (EF) is not compatible with PM. Tadenuma and Thomson (1993) replaced PM with wPM and proposed a rule that satisfy both wPM, and EF. It corresponds to equal division of $\bar{a}^N$ at each problem here, i.e. $u_i(N, a) = \bar{a}^N/n$, for all $i \in N$. To see that this rule is not AP, consider the problem $(N, a)$ with $n = 3$, $(a_1, a_2, a_3) = (2,2,6)$. Note that each agent gets 2 at problem $(N, a)$, while at problem $(S, a^S)$ with $S = \{1,2\}$ each agent gets 1.

**Remark 1.3:** wPM does not imply AP.

**Definition 1.9:** A solution $\{\{(\delta_i, t_i)\}_{i \in N}\}$ to the problem $(N, a)$ is *envy-free* (EF) if for all $i, j \in N$, $u_i(\delta_i, t_i) \geq u_i(\delta_j, t_j)$. An allocation rule $\varphi$ is EF if it assigns an EF solution to all problems.

Envy-freeness is a pretty strong condition, especially in this model. It is well-known that EF implies PO (see for example Tadenuma and Thomson (1993)). To derive our next result, it suffices to know two simple properties that EF implies. One of them is ETE, and the second is that any EF allocation assigns an equal share of transfers to those who do not get the object. Both properties follow immediately from Definition 1.9.

**Proposition 1.19:** There is no allocation rule that satisfy both EF and AP.

**Proof:** Let $\varphi$ be EF, and consider the problem $(N, a)$ with $n = 4$ and $(a_1, a_2, a_3, a_4) = (2,2,16,16)$. The unique EF allocation induces the utilities $u_i(N) = 4$ for all $i \in N$, and the unique EF allocation induces the utilities $u_i(S) = 1$ at problem $(S, a^S)$ for $S = \{1,2\}$, and for all $i \in S$. As one of the agents does not get the object at problem $S$, by Prop. 1.16 $\varphi$ is not AP. □

Moulin (1990b) showed that the key property that causes the incompatibility of PM and EF is the free access upper bound (FAU). PM implies FAU, which simply says that the final utility of an agent should be less than his valuation, i.e. $U_i(N, a) < a_i$. A stronger property that PM implies is that $U$ is in the stand-alone core (SAC): no coalition $S$ in total can get more than what they get
in problem \((S, a^S)\), i.e. \(\sum_{i \in S} U_i(N, a) \leq \bar{a}^S\). Now, order individuals s.t. \(a_1 \leq a_2 \leq \cdots \leq a_n\). Then, in particular, SAC implies \(\sum_{i=1}^{k} u_i(N, a) \leq a_k\). We now introduce a similar necessary condition for AP.

**Proposition 1.20:** Fix a utility distribution \(U(\cdot)\) that satisfies ETE, and a problem \((N, a)\). Order the individuals s.t. \(a_1 \leq a_2 \leq \cdots \leq a_n\). If \(\varphi\) is AP, then \(U(\cdot)\) satisfies the following:

\[
\sum_{i=1}^{k} U_i(N, a) \leq a_{k+1} \quad \text{for all } k \leq n - 1
\]

**(1.9)**

**Proof:** Let everything be given as in the statement of the proposition. As \(U_i(N) \geq 0\) for all \(i \in N\), we have the desired inequality for \(k = n - 1\). Let \(k < n - 1\), and suppose for a contradiction that \(\sum_{i=1}^{k} U_i(N, a) > a_{k+1}\). Let \(S = \{a_1, \ldots, a_{k+1}\}\), consider the problem \((S, a^S)\). Note that \(\sum_{i \in S} U_i(S, a^S) = a_{k+1}\). By Proposition 1.18, we have \(U_i(N, a) \leq U_i(S, a^S)\) for all \(i \in \{1, \ldots, k\}\). Then, \(U_{k+1}(S, a^S) < 0\). □

Now, using the condition (1.9), we will construct a utility distribution that is AP, but not PM (not even wPM). To avoid notational complexity, we will just introduce and explain it on an example first of all to make it easy to read, and also to explain our choice of distribution when there are several agents with the same valuation. We see below that this choice is critical.

**Example 1.15:** Let \(n = 8\), \((a_1, \ldots, a_8) = (2,3,5,6,15,15,15,18)\).

Our utility distribution \(\bar{U}\) is as follows: Start by assigning the agent with lowest valuation, his value, i.e. \(\bar{U}_1 = 2\). Continue with agent 2. If his valuation plus \(\bar{U}_1\) does not exceed \(a_3\) (so that (1.9) is not violated) assign his valuation to agent 2, \(\bar{U}_2 = 3\). Now, note that \(\bar{U}_1 + \bar{U}_2 + a_3\) exceeds \(a_4\), so we give agent 3 the maximum share that does not violate (1.9), \(\bar{U}_3 = 1\). It is safe to assign his valuation to agent 4 as \(\sum_{i=1}^{3} \bar{U}_i + a_4 \leq a_5\), hence \(\bar{U}_4 = 6\). Assigning 5, 6 and 7 equal shares without violating AP is critical. Note that \(\sum_{i=1}^{4} \bar{U}_i = 12\) and applying the argument we used up to now yields 1.5 for each of these agents. If each gets this share, (1.9) is not violated.
and this is the maximum each can get (check (1.9) for $k = 6$). But suppose we do not have agent 8 initially. Then, agents 5, 6 and 7 gets a share of $(15 - 12)/3 = 1$ in the problem \( (N \setminus \{8\}, a^{N \setminus \{8\}}) \). When agent 8 appears and if we give them 1.5 each, share of 3 of these agents increase and this violates AP as only one of them gets the object in the problem without agent 8. Therefore, what we do here is to give first 7 agents a total of 15 instead of 18. Hence, \( \bar{U}_5 = \bar{U}_6 = \bar{U}_7 = 1 \). We can generalize this idea as follows: Suppose we have \( a_1 \leq \cdots < a_j = \cdots = a_{j+m-1} < \cdots \), and our procedure has already assigned \( \{\bar{U}_i\}_{i \leq j-1} \). Then, \( \bar{U}_j = \cdots \bar{U}_{j+m-1} = (a_j - \sum_{i=1}^{j-1} \bar{U}_i)/m \). We give the remains to agent 8, \( \bar{U}_8 = 3 \). Note that had the vector of valuations be \( (2,2,3...) \), we would assign the first two agents 1 each and continue.

**Proposition 1.21:** \( \bar{U} \) is AP, and is neither wPM nor in the SAC.

**Proof:** To see that \( \bar{U} \) is AP, take a problem \( (N, a) \), and order the individuals s.t. \( a_1 \leq a_2 \leq \cdots \leq a_n \). Consider first adding an agent \( j \) to the problem with \( a_j \leq \bar{a}^N \) and suppose \( a_i < a_j < a_{i+1} \). This will not change the share of any agent except that of agents \( i \) and \( i + 1 \), while the share of these agents would possibly decrease but not increase. If \( a_i = a_j \), this will not affect the share of agents whose valuation is not equal to \( a_i \) , and the share of all agents whose valuation is equal to \( a_i \) decreases. Now, suppose \( a_j > \bar{a}^N \). This would possibly affect the share of agents with valuation \( \bar{a}^N \) only and not the others. If there is more than one agent with valuation equal to \( \bar{a}^N \) their share do not change either. Let \( a_{n-1} < a_n \), and note that \( n \) agents already get a total of \( \bar{a}^N \) in problem \( (N, a) \). After \( j \) arrives, applying (1.9) for \( k = n \), the share of agent \( n \) can increase by at most \( a_j - \bar{a}^N \). Then, Proposition 1.18 implies that \( \bar{U} \) is AP.

\( \bar{U} \) is clearly not in the SAC. In order to see that \( \bar{U} \) is not wPM, consider the following problem \( (N, a) \) with \( n = 5 \), \( (a_1, \ldots, a_5) = (2,3,4,5,10) \). We have \( \bar{U}(N,a) = (2,2,1,5,0) \). Now, let \( S = \{1,2,4\} \). We have \( \bar{U}_1(S) = 2, \bar{U}_2(S) = 3, \) and \( \bar{U}_4(S) = 0 \) at the problem \( (S, a^S) \). □

**Remark 1.4:** By Remark 1.3 and Proposition 1.21, AP and wPM are independent properties.
1.5 Appendix

Proof of Proposition 1.4: Necessity. Let $\mathcal{X}$ be an APAS. As it is a core selection (hence efficient at each $S$), and $(a_1, b_2)$ is an optimal pair at game $N$, by Lemma 1.1, we have $X_1(N) + X_2(N) = u_2$, $X_3(N) = 0$, and also $X_1(N) + X_3(N) = X_1(N) \geq u_1$. Also, by Lemma 1.2, $X_1(N) = X_1(1,2)$ and $X_2(N) = X_2(1,2)$. Suppose $X_3(1,3) > X_2(N)$. Then, $b_2$ and $b_3$ would be better off by absence of $b_2$ at game $N$.

Sufficiency. Here, the only critical agent is $b_2$. \{a_1, b_2\} would not be better off if $b_2$ leaves as they would get at most $u_1$, while they get together $u_2$ at game $N$. \{b_2, b_3\} would not be better off if $b_2$ leaves as $X_2(N) \geq X_3(1,3)$. □

Proof of Proposition 1.6: Necessity. Let $\mathcal{X}$ be an APAS. To see $u_3 \geq u_1 + u_2$ holds in Case 1, note that by Proposition 1.4 applied to game $\{2,3,4\}$ we have, $X_2(2,3) \geq u_2$; and applied to game $\{1,2,3\}$ we have, $X_3(2,3) \geq u_1$. Then by efficiency of $\mathcal{X}$, $X_2(2,3) + X_3(2,3) = u_3 \geq u_1 + u_2$.

The argument for Case 2.1 (i) is similar. To see $u_3 + u_4 \geq 3u_2$ holds in Case 1, note that by Proposition 1.4 applied to game $\{1,2,4\}$ we have, $X_4(1,2,4) \geq u_2$, and hence $u_4 - u_2 \geq X_1(1,2) \geq X_2(2,4)$. By the same argument for game $\{2,3,4\}$ we have, $u_3 - u_2 \geq X_4(2,4)$.

Then, by efficiency, $u_4 - u_2 + u_3 - u_2 \geq u_2 = X_2(2,4) + X_4(2,4)$. The argument for Case 2.1 (ii) is similar. To see $u_3 \geq u_1 + u_2$ holds in Case 2.2, note that applying Proposition 1.4 to game $\{1,2,3\}$ we have $X_3(2,3) \geq u_1$; and applying it to $\{2,3,4\}$ we have, $X_2(2,3,4) \geq u_2$. Therefore, $u_3 - u_2 \geq X_4(2,3,4)$ and $X_3(2,3,4) = 0$. If $u_1 > u_3 - u_2$, then $b_3$ and $b_4$ would be better off by absence of $b_4$ at game $\{2,3,4\}$. Argument is similar for Case 2.2 (ii). □

Sufficiency. We will show that the allocation scheme defined in Table 1.1 is an APAS, and then define allocation schemes for Case 2.1, and Case 2.2 in Table 1.5, and Table 1.6, respectively. Proof for those is similar to Case 1, so we omit it. Note that for all subgames that do not appear at the tables $v(S) = 0$. The optimal assignments (not necessarily unique) at game $N$ are
\((a_1, b_4), (a_2, b_3)\) in Case 1 and 2.2, and \((a_1, b_3), (a_2, b_4)\) in Case 2.1. Check that \(X(S)\) is efficient at every subgame \(S \subseteq N\) in all cases. For all 2-person subgames efficiency and that \(v(i) = 0\) for all \(i\) implies \(X\) is not manipulable. For 3-person subgames we use Proposition 1.4 to show absence-proofness. We will use the Claim 1 below to prove that \(X\) is not manipulable at game \(N\). Now, fix an allocation scheme \(X\) on \((N, v), T \subseteq N\) and define the set of agents in \(T\) whose payoff decrease at game \(N\) w.r.t game \(T\) as \(K(X; T, N) = \{i \in T: X_i(T) > X_i(N)\}\). We say \(X\) is monotone from \(T\) to \(N\) if the set \(K(X; T, N)\) is empty.

**Claim 1:** Let \(X(N)\) be a core allocation. \(X\) is manipulable by absence of \(N \setminus T\) at game \(N\) if and only if \(X\) is not monotone from \(T\) to \(N\) and \(K(X; T, N) \cup N \setminus T\) manipulates \(X\) at game \(N\) by absence of \(N \setminus T\).

**Proof of Claim 1:** Let \(X(N)\) be a core allocation, and \(K(X; T, N)\) be empty. Then, for any \(K' \subseteq T\) we have, \(\sum_{i \in K} X_i(T) \leq \sum_{i \in K} X_i(N)\), also as \(X(N) \in C(N, v)\), we have \(v(N \setminus T) \leq \sum_{i \in N \setminus T} X_i(N)\). Hence, (1.1) holds for \(N \setminus T \subseteq (K' \cup N \setminus T) \subseteq N\). Now, let \(K(X; T, N)\) be nonempty, and \(K' \subseteq T\) s.t. \(K' \cup N \setminus T\) manipulates \(X\) at game \(N\) by absence of \(N \setminus T\). Then, we have \(\sum_{i \in K} (X_i(T) - X_i(N)) > \sum_{i \in N \setminus T} X_i(N) - v(N \setminus T)\). Note that the expression on the left hand side of the inequality is maximized by \(K' = K(X; T, N)\).

**Case 1:** To see that \(X\) in Table 1.1 is an APAS at game \(\{1, 2, 3\}\), check that \(X_3(1, 2, 3) = X_3(2, 3) = u_3 - u_2 \geq u_1\) by condition (i), \(X_1(1, 2, 3) = 0\), and \(X_2(2, 3) = X_2(1, 2, 3) = u_2 \geq u_1/2 = X_1(1, 3)\). To see that \(X\) is an APAS at game \(\{1, 3, 4\}\), check that \(X_1(1, 3, 4) = X_1(1, 4) = u_4 - u_2 \geq u_1\) by condition (i), \(X_3(1, 3, 4) = 0\), and \(X_4(1, 4) = X_4(1, 3, 4) = u_2 \geq u_1/2 = X_3(1, 3)\). To see that \(X\) is an APAS at game \(\{2, 3, 4\}\), check that \(X_2(2, 3, 4) = X_2(2, 3) \geq u_2, X_4(2, 3, 4) = 0\), and also \(X_3(2, 3) = X_3(2, 3, 4) = u_3 - u_2 \geq \min\{u_3 - u_2, u_2/2\} = X_4(1, 4)\). To see that \(X\) is an APAS at game \(\{1, 2, 4\}\) check that \(X_4(1, 2, 4) = X_4(1, 4) \geq u_2, X_2(1, 2, 4) = 0\).
Also, in case $u_3 - u_2 \geq u_2/2$ we have, $X_1(1,4) = X_1(1,2,4) = u_4 - u_2 \geq u_2/2 = X_2(2,4)$, and otherwise $X_1(1,4) = X_1(1,2,4) = u_4 - u_2 \geq 2u_2 - u_3 = X_2(2,4)$ by condition (ii).

To see $\mathcal{X}$ is not manipulable at game $N$, first note that by Lemma 1, $\mathcal{X}(N)$ is in the core of game $N$ as $\mathcal{X}_1(N) + \mathcal{X}_3(N) \geq u_3 - u_2 \geq u_1$ by condition (i), $\mathcal{X}_2(N) + \mathcal{X}_4(N) \geq u_2$, and the optimal pairs get the exact surplus they create. Check that $\mathcal{X}$ is monotone from any $T$ to $N$ as by (i) we have $u_4 - u_2 \geq u_3 - u_2 \geq u_1/2$. Then, by Claim 1 we are done. □

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<th>$S$</th>
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Table 1.5

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Table 1.6
2 Strongly Stable and Responsive Cost Sharing Solutions for Minimum Cost Spanning Tree Problems

2.1 Introduction

We consider a minimum cost spanning tree (mcst) problem where agents need to be connected to a source, and there is a fixed cost of the links connecting any two agents and any agent to the source. Agents do not care through which links they are connected. Then, to connect all agents to the source, efficiency requires that the links used in the connection form a spanning tree. Such a tree with the minimal cost (a mcst) can be constructed and its cost can be calculated by Prim’s algorithm (see Section 2.2 for details).

Many authors proposed several interesting solutions to distribute the efficient cost. Bird solution (B) (Bird 1976) and Dutta Kar solution (DK) (Dutta and Kar 2004) are among those. These solutions have been criticized as they lack many desired fairness criteria.\(^\text{10}\) However, both solutions satisfy the stand alone core stability, i.e., no group of agents has an incentive to secede from cooperation and construct their own connection to the source. Moreover, they are very easy to calculate when there is a unique mcst.

Özsoy (2006) introduced a manipulation idea by merging. A group of agents \(S\) leave the scene and covertly connect the source through another agent \(i\), hiding their existence from \(N \setminus S\) except \(i\). If the minimal connection cost of \(S\) to agent \(i\) plus \(i\)’s cost share at solution \(\varphi\) at the reduced problem \(N \setminus S\) is less than the total cost share of \(S \cup i\) at the solution \(\varphi\) at the original problem, \(\varphi\) is said to be manipulable by covertly merging. She defined the related stability property covert-merge-proofness and showed that no solution is stable in that sense. Here, we define the similar

\(^{10}\) See Bergantiños and Vidal-Puga (2007) for an extensive axiomatic analysis on these solutions.
manipulation idea that we discuss in Chapter 1, and the corresponding stability property absence-proofness. Consider the following problem where the set of agents is $N = \{1,2,3\}$ and $\omega$ is the source:

![Graph](image)

Figure 2.1

Note that the unique $mest$ is $\{(\omega,3)(3,2)(2,1)\}$. The Bird solution and the Dutta Kar solution yield the cost allocation vectors $B(N) = (2,3,9)$, and $DK(N) = (9,2,3)^{11}$. Suppose now agent 3 leaves or had never appeared in the first place. Then, the solutions yield $B(\{1,2\}) = (10,2,\ast)$, and $DK(\{1,2\}) = (2,10,\ast)$ at the reduced problem where only agent 1 and 2 cooperates to connect to the source. Note that the individual connection cost to the source for agent 3 is 9. Consider first the Bird solution. If agents 2 and 3 agree to keep agent 3 away from the scene, they can connect to the source at a total cost of $2 + 9 = 11$, while their total cost share is 12 had 3 not been left aside. A similar argument holds for agents 1 and 3 for the Dutta Kar solution. Hence, both solutions are manipulable by the group of agents $\{2,3\}$ and $\{1,3\}$, respectively, via absence of agent 3.

In this cost sharing setting, manipulation and absence-proofness is the exact dual of the formulation in (1.1) and our first critique is against the solutions that fails separability (SEP). Suppose two groups of agents $S$ and $T$ decide to connect to the source jointly. In case there is no

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^{11}$ See Section 2.3 for definition of all the properties and the basic solutions that we mention here in the introduction.
cost saving, i.e. own connection cost of $S$ plus own connection cost of $T$ is equal to the efficient connection cost of $S \cup T$, cost shares of agents should remain the same. Any solution that violates SEP also violates AP (see Proposition 2.2).

As in Chapter 1, our main contribution by a very simple argument is that all population monotonic solutions satisfy AP. Population monotonicity (PM) requires that no one should pay more when additional agents arrive. Note that if PM holds, cost share of $S \setminus T$ do not decrease when $T$ leaves. Again by the same argument, $T$’s own connection cost is no less than $\sum T \varphi_i(N)$. Therefore, PM implies AP.

Several population monotonic, and hence absence-proof, solutions are defined in this context (Feltkamp et al. 1994; Norde et al. 2001; Brânzei et al. 2004; Tijs et al. 2006; Bergantños and Vidal-Puga 2007; Bergantños and Lorenzo-Freire 2008; Bogomolania and Moulin 2010; Bergantños and Vidal-Puga 2012). Among them, the celebrated (as many calls it) folk solution singles out, satisfying compelling fairness properties such as continuity, cost monotonicity and ranking (see e.g. Bogomolnaia and Moulin (2010)). It is also known as equal remaining obligation solution (Feltkamp et al. 1994) or $P$-value (Brânzei et al. 2004).

Bogomoláia and Moulin (2010) criticized the folk solution as in some instances where the cost data strongly suggest that an agent should receive a strictly less cost share from another, it treats the agents equally. Now, write $c_{ij}$ for the cost of connecting agents $i, j$, and $c_{owi}$ for the cost of connecting $i$ to the source $\omega$. Consider the following problem with $n$ agents:

$$c_{owi} = c_1 \text{ for all } i; \quad c_{ij} = c_2 \text{ for all } i, j \geq 2; \quad c_{1i} = 0 \text{ for all } i \geq 2 \quad (2.1)$$

Bogomoláia and Moulin (2010) discussed the case $c_1 = 10$, $c_2 = 1$. Note that in the absence of agent 1, the efficient cost of connecting the other agents to the source is 18, and agent 1 comes with 8 units of overall cost saving where the mcst is a star and agent 1 is at the central position. Folk solution is reductionist in the sense that it does not take the cost of the links that do not appear in a mcst into account, and yields 1 unit of cost share to each agent. Bogomoláia and
Moulin (2010) asks agent 1 to receive a strictly less cost share than to that of others in this case, and formulate the related property strict ranking 1. They defined another strict ranking property and a strict cost monotonicity property, and proposed a family of solutions that satisfy all those, together with the basic fairness properties; continuity, cost monotonicity and ranking, except PM. There, whether all these properties are compatible with PM or not is left as an open question.

Norde (2013) is the first to respond this question and proposed the cost adjusted folk solution (CAF) meeting all the properties mentioned above. CAF yields cost allocations very close to the folk solution. In particular, each agent receives a cost share between 97 and 103 percent of his cost share at the folk solution. Moreover, this interval shrinks rapidly as the number of agents $n$ increase. Consider the problem in (2.1) where $c_1 = c_2 = 100$. Note that agent 1 brings 800 units of cost saving (decreasing the efficient cost from 900 to 100 units). It is natural to ask from a fair allocation to significantly discriminate the cost shares of agent 1 from that of others. One may even argue that agent 1 should receive a negative cost share (a subsidy) (see Trudeau (2012)).

However, we follow Bogomolnaia and Moulin (2010), and restrict ourselves to nonnegative cost shares. The CAF solution yields approximately a cost share of 9.96 for agent 1 and 10.004 for others while the folk solution yields a 10 unit share to each agent.

Here, we do not impose any normative principle about how much the solution should favor agent 1 in the above example. Instead, we define a family of solutions which on the one end yields (almost) 10 units to each agent at the above problem and on the other end (almost) 0 to agent 1, depending on a continuous parameter. Moreover, this family carries all the fairness properties Bogomolania and Moulin (2010) asked for, except PM. However, it meets the weaker but more compelling stability property, absence-proofness.

In Section 2.2 and 2.3, we give the setting; define basic properties and solutions, respectively. In section 2.4 we define absence-proofness, give an alternative interpretation of AP, and show that PM implies AP, and AP implies SEP. In Section 2.5, we define partial solutions on the elementary cost vectors where cost of the links are either 0 or 1, and the extension of these
solutions on general cost vectors. We show that for any partial solution satisfying independence of irrelevant components, the extended solution is absence proof, and then propose a family of solutions as an alternative to the folk solution.

2.2 The Setting

Let \( N = \{1, \ldots, n\} \) be the set of agents and \( \omega \) donate the source to which agents need to get connected. There is a nonnegative cost to connect each agent \( i \in N \) to the source and to the other agents which is denoted by \( c_{ik} \) for \( k \in N \cup \{\omega\} \setminus i \). Let \( (N \cup \{\omega\})(2) \) denote the set of all non-ordered pairs \( i, j \) in set \( N \cup \{\omega\} \). Therefore, a mstc problem is a triple \( (\omega, N, c) \), where \( c = (c_{ij})_{i,j \in (N \cup \{\omega\})(2)} \in C \) and \( C \) denotes the set of all cost vectors for \( N \cup \{\omega\} \). We omit \( \omega \) as it is fixed and simply speak of the problem \( (N, c) \). The reduced problem for a subset \( S \subseteq N \) of agents is denoted by \( (S, c_S) \), where \( c_S = (c_{ij})_{i,j \in (S \cup \{\omega\})(2)} \). By abuse of notation we will use \( (S, c) \) when we speak of a subproblem.

Given a problem \( (N, c) \), an edge \( e = e_{ij} \) represents a connection between \( i, j \in N \cup \{\omega\} \) with the cost \( c_e = c_{ij} \). A spanning tree \( g \) is a non-directed graph with \( n \) edges that connects all the elements in \( N \cup \{\omega\} \) and the cost of \( g \) is \( c(g) = \sum_{e \in g} c_e \).

The minimal cost of connecting \( n \) agents to the source is \( \nu(N, c) = \min_{g \in \Gamma(N, c)} c(g) \), where \( \Gamma(N, c) \) denotes the set of all spanning trees, and can be computed by Prim’s algorithm (Prim 1957): At the first step, among the edges that connect agents to the source, pick an edge \( e_{\omega i} \) with the cheapest cost, and say \( i \) is connected to the source. At the second step, pick an edge \( e_{kh} \) with the minimum cost, where \( k \in \{i, \omega\} \) and \( h \in N \setminus i \), and say \( h \) is connected to the source. At each step, continue to connect a new agent to the agents that are connected in the previous steps or to the source directly using the same method. This procedure ends in \( n \) steps and returns a mstc. Note that this procedure might not return a unique tree.
A cost allocation at problem \((N, c)\) is a vector \(y \in \mathbb{R}^n_+\) such that \(\sum_{i \in N} y_i = v(N, c)\) and a solution \(\varphi\) specifies a cost allocation for each problem. Note that we restrict ourselves to nonnegative cost shares while negative cost shares may be reasonable for some instances (see Trudeau (2012), Kar (2002)).

2.3 Basic properties and solutions

In this context, there are two interpretations of the most fundamental incentive compatibility property, stand alone core stability. Strict stand alone cost of a coalition \(S \subseteq N\) \((\nu(S, c_S))\) is the minimal cost of connecting all agents in \(S\) to the source using only the links in \(S \cup \{\omega\}\). Stand alone cost of \(S\) \((\bar{\nu}(S, c))\) is the minimal cost of connecting agents in \(S\) to the source using links in \(N \cup \{\omega\}\). The nature of the problem determines the right choice between two cost sharing game \(\nu\) and \(\bar{\nu}\). If we are interested in only core stability, this choice is irrelevant as under the assumption of nonnegative cost shares, both games yield the same set of core allocations (see e.g. Sharkey (1995)). However, this choice is critical here and we will speak of the strict stand alone cost throughout the paper and write \(\nu(S, c)\) by abuse of notation.

Definition 2.1: Given \((N, c)\), an allocation \(y\) is core stable if \(\sum_{i \in S} y_i \leq \nu(S, c)\) for all \(S \subseteq N\), and a solution \(\varphi\) is a core selection (CS) if it always assigns a core stable allocation.

Bird (1976) introduced the first core selection in this context. Assume first that there is a unique \(mcst\) \(g\). To each agent \(i\), the Bird solution \((B)\) assigns the cost of the edge adjacent to \(i\) on the unique path form \(i\) to \(\omega\) in \(g\). If there are multiple \(mcst\)'s, cost share of \(i\) can be calculated by taking the average of the cost shares calculated for each \(mcst\).

Cost Monotonicity (CM). For all \((N, c), (N, c')\) and \(i \in N, k \in N \cup \{\omega\}\):

\[\{c_{ik} < c'_{ik} \text{ and } c_e = c'_e \text{ for all } e \neq e_{ik}\} \Rightarrow \varphi_i(N, c) \leq \varphi_i(N, c')\]
Despite the ease of its calculation, Dutta and Kar (2004) criticized the Bird solution as it is not cost monotonic (CM). Besides its fairness aspect, CM is also considered as an incentive compatibility property. In case it does not hold, agent $i$ would find it profitable to announce the cost of his link more than its actual value if it is private information. If the information is public, violation of CM would kill incentives to decrease the connection costs.

The Dutta Kar solution (DK) is calculated through Prim’s algorithm. Assume first that there is a unique mcst $g$. Let $\bar{c}_m$ be the cost of the most expensive edge constructed in the first $m$ steps of the algorithm. Suppose agent $i$ is connected to the source at step $m$, and the edge $e_{kh}$ is constructed at step $m + 1$. Then, DK assigns the minimum of $c_{kh}$ and $\bar{c}_m$ to agent $i$. If there are multiple mcst’s, cost share of $i$ can be calculated by taking the average of the cost shares calculated for each mcst.

Both solutions are easy to calculate with a unique mcst, even for problems with multiple mcst’s they satisfy the following axiom.\textsuperscript{12}

**Polynomial Complexity (PC).** For all $(N, c)$, $\varphi(N, c)$ is computed by an algorithm polynomial in $n = |N|$.

Naturally, we expect an agent $i$ to pay less than $j$ if $i$ is more efficient in connecting to any other agent or to the source compared to $j$.

**Ranking (RKG).** For all $(N, c); i, j \in N; \left\{ c_{ik} \leq c_{jk} \text{ for all } k \in N \cup \{\omega\} \setminus \{i, j\} \right\} \Rightarrow \varphi_i \leq \varphi_j$

Note that RKG implies equal treatment of equals (ETE) that the cost shares of two agents $i, j$ should be the same whenever $c_{ik} = c_{jk}$ for all $k \in N \cup \{\omega\} \setminus \{i, j\}$. Both $B$, and DK obviously satisfy ETE. However, $B$ fails RKG while DK meets RKG\textsuperscript{13}. Hence, we can interpret DK as a refinement\textsuperscript{14} of $B$ not only in terms of CM but also in terms of RKG.

\textsuperscript{12} See e.g. Bogomolnai and Moulin (2010).

\textsuperscript{13} Consider the problem $(c_{w1}, c_{w2}, c_{12}) = (2,3,1)$. $B(N, c) = (2,1)$ violates ranking. For DK the idea is roughly as follows: Pick $i,j$ s.t. $c_{ik} \leq c_{jk}$ for all $k \in N \cup \{\omega\} \setminus \{i, j\}$. Let $m_i, m_j$ denote the steps of Prim’s
An important critique to these solutions is that in many instances, for a miniscule change in the cost of one link only, there is a substantial change in the cost share of an agent. Also, an agent would be worse off with the arrival of an additional agent. Hence, both solutions fail continuity and population monotonicity. Moreover, they fail separability (weaker than PM): If two sets of agents merge to connect the source jointly and there is no cost saving from the merger, cost shares remain the same for all agents (see e.g. Bergantiños and Vidal-Puga (2007)).

**Continuity (CO).** For all \((N, c), \varphi(N, c)\) is a continuous function of \(c\).

**Population Monotonicity (PM).** For all \((N, c), S \subseteq N\) and \(i \in S\), \(\varphi_i(N, c) \leq \varphi_i(S, c)\).

**Separability (SEP).** For all \((N, c), S \subseteq N : \{v(N, c) = v(S, c) + v(N \setminus S, c)\} \Rightarrow \varphi_i(N, c) = \varphi_i(S, c)\) for all \(i \in S\).

The folk solution satisfies all the properties discussed above (see e.g. Bogomolnaia and Moulin (2010)). Among several different descriptions, Bergantiños and Vidal-Puga (2007) uses the *irreducible cost matrix* that is the smallest cost matrix \(c^*\) below \(c\) such that \(v(N, c) = v(N, c^*)\). Then, the *folk solution* is \(B(N, c^*)\).

Bogomolnaia and Moulin (2010) criticized the folk solution for it ignores a substantial amount of data in the cost vector and call it reductionist.

**Reductionism (RED).** For all \((N, c), \varphi(N, c) = \varphi(N, c^*)\).

They argue that the folk solution (and any reductionist solution) fails to rank the cost shares strictly when such ranking is compelling in certain cases. They define strict versions of RKG on

Note that when there is a unique mcst, both solution yields allocation vectors consisting of same \(n\) numbers while they differ in the allocation of these shares to the agents (see the example in Figure 1).
the domain of cost vectors $\mathcal{D} = \{ c \in \mathcal{C} : c_{kl} < c_{om} \text{ for all } k, l, m \text{ (not necessarily distinct)} \}$, where connecting any two agent is cheaper than connecting an agent to the source.\footnote{Note that on this domain $\psi(N, c)$ is equal to the minimum cost of connecting $n$ agents to each other plus $\min_{i \in N} c_{oi}$. See Bogomolnaia and Moulin (2010) for a detailed justification of these properties on this domain.}

**Strict Ranking$_1$ ($SRK_1$).** For any $(N, c)$ s.t. $c \in \mathcal{D}$, and for all $i, j \in N$:

$$\{ c_{ik} < c_{jk} \text{ for all } k \in N \setminus \{ i, j \} \text{ and } c_{oi} \leq c_{oj} \} \implies \varphi_i < \varphi_j$$

**Strict Ranking$_2$ ($SRK_2$).** For any $(N, c)$ s.t. $c \in \mathcal{D}$, and for all $i, j \in N$:

$$\{ c_{ik} \leq c_{jk} \text{ for all } k \in N \setminus \{ i, j \} \text{ and } c_{oi} < c_{oj} \} \implies \varphi_i < \varphi_j$$

**Strict Cost Monotonicity ($SCM$).** For any $(N, c)$ s.t. $c \in \mathcal{D}$, and for all $i \in N$, $\varphi_i(N, c)$ is strictly increasing in each coordinate $c_{ik}, k \in N \cup \{ \omega \} \setminus \{ i \}$.

### 2.4 Absence-proofness: A strong stability property

Stand alone core stability ensures that no coalition finds it profitable to secede from the cooperation and connect the source on their own. Here, we define an alternative but a related way for a coalition to improve upon their allocation $\varphi(N, c)$, and the associated stability concept. Instead of fully seceding, a coalition $S = K \cup T$ can partially secede and be better off. In particular, $T$ does not appear at the scene and connects to the source using only the links in $T \cup \{ \omega \}$. $K$ cooperates with agents in $N \setminus S$ to connect to the source using the links in $(N \setminus T) \cup \{ \omega \}$. If the total cost share of $S$ at $\varphi(N, c)$ is strictly more than $T$’s own connection cost plus the total cost share of $K$ at $\varphi(N \setminus T, c)$, $S$ would profit from a partial secession.

**Definition 2.2:** A solution $\varphi$ is absence-proof (AP)$^{16}$, if for all $(N, c), T \subseteq N, K \subseteq N \setminus T$:

$$\sum_{i \in K \cup T} \varphi_i(N, c) \leq \sum_{i \in K} \varphi_i(N \setminus T, c) + v(T, c) \tag{2.2}$$

**Remark 2.1:** Note that any AP solution is a core selection; just set $K = \emptyset$.

\footnote{We use AP as an abbreviation to both absence-proof and absence-proofness, whichever it fits.}
Generally, stability properties are interpreted as arguments preventing the cooperation from braking up. We now consider the situation from an opposite angle. Suppose two sets of agents $S$ and $S'$ are connecting to the source separately and the links are reconstructed periodically. For example, agents in $S$ live in suburbs a little south of the northwest and $S'$ live in suburbs a little east of the northwest of a big city. The groups $S$ and $S'$ separately carpool to commute to downtown directly. $S$ and $S'$ discovered an alternative option; meet at a point exactly on the northwest and then commute using a single shuttle bus. Suppose this merger yields an overall cost saving, i.e., $\delta(S, S') = v(S, c) + v(S', c) - v(N, c) \geq 0$. Then, AP requires that no coalition from one of these groups has a cost saving more than the total cost saving from the merger. Moreover, that is all AP asks for, exactly as in the surplus sharing TU games (Proposition 1.2).

**Proposition 2.1:** A solution $\phi$ is absence-proof if and only if for any $(N, c)$; for all $S, S' \subseteq N$ such that $N = S \cup S'$, $S \cap S' = \emptyset$ and for all $K \subseteq S$ we have,

$$\sum_{i \in K} (\phi_i(S, c) - \phi_i(N, c)) \leq \delta(S, S') \tag{2.3}$$

Suppose an efficient $\phi$ is not a core selection, i.e., $\sum_{i \in S} \phi_i(N, c) > v(S, c)$ for some $S$. Then, agents in $S$ unanimously raise a credible objection to the merger between $S$ and $N \setminus S$ as some reallocation of $\phi_i(S, c) = v(S, c)$ to agents in $S$ would make all strictly better off compared to their allocation after the merger. Now, let $\phi$ be a core selection but fails (2.3) for some $S, K \subseteq S$ and $S' = N \setminus S$. Then, agents in $N \setminus K$ unanimously objects to the merger between $S$ and $N \setminus S$, and their objection is credible as now a reallocation of their total cost share before the merger among $N \setminus K$ would make all strictly better off. If (2.3) holds, there is no credible objection as neither $N \setminus S$ nor $N \setminus K$ unanimously objects.

Following directly from the idea in (2.3), our first critique is against the solutions that fail SEP, particularly against the Bird and the Dutta-Kar solution.

**Proposition 2.2:** Any solution that fails SEP also fails AP.
Proof: Fix a core selection \( \varphi ; (N, c) \), \( S \subseteq N \), \( i \in S \) s.t. \( v(N, c) = v(S, c) + v(N \setminus S, c) \) and \( \varphi_i(N, c) \neq \varphi_i(S, c) \). As \( \varphi \) is CS, \( \sum_{i \in T} \varphi_i(N, c) = v(T, c) \) for \( T \in \{ S, N \setminus S \} \). Then, there is \( j \in S \) s.t. \( \varphi_j(S, c) - \varphi_j(N, c) > 0 = \delta(S, N \setminus S) \). \( \square \)

**Corollary 2.1:** The Bird and the Dutta-Kar solutions fail AP.

Note that for any core selection, when agents \( T \) leave the scene and connect on their own, their total will not decrease. So, for \( S = K \cup T \) to improve upon their allocation proposed by a core selection at problem \((N, c)\), the cost share of \( K \) should decrease in the restricted problem. This fact is summarized in the following proposition.

**Proposition 2.3:** Any solution that meets PM also meets AP.

**Proof:** Let \( \varphi \) satisfy PM. For any \( T \subseteq N \), we have \( \sum_{i \in T} \varphi_i(N, c) \leq \sum_{i \in T} \varphi_i(T, c) = v(T, c) \). PM also implies \( \sum_{i \in K} \varphi_i(N, c) \leq \sum_{i \in K} \varphi_i(N \setminus T, c) \). Summing up those two, (2.2) holds. \( \square \)

**Corollary 2.2:** The folk solution is AP.

Population monotonicity has been interpreted as a normative solidarity concept. As an additional agent always brings nonnegative cost savings, no one should be worse off. By Proposition 2.3, we can interpret population monotonicity as a strong stability property as well. However, violation of PM does not necessarily mean an opportunity for manipulation. Suppose two groups \( S, S' \) decides to connect to the source jointly. Everyone except \( i \in S \) has two units of cost saving while \( i \) loses 1 unit and \( i \)'s cost share after the merger is less than \( c_{oi} \). Then, \( i \) cannot convince anyone to object to the merger and he cannot raise a credible objection himself.

In this context, several authors defined families of population monotonic solutions that contain the folk solution. Obligation rules (Tijs et al. 2006) and optimistic weighted Shapley solutions (Bergantiños and Lorenzo-Freire 2008), which is a subset of the obligation rules, are among those. Recently, Bergantiños and Vidal-Puga (2012) introduced a family that contains the obligation rules. This family consists of all population monotonic solutions that satisfy strong
cost monotonicity\textsuperscript{17} (also known as solidarity). All these solutions are reductionist as solidarity implies RED\textsuperscript{18}.

Bogomolnaia and Moulin (2010) pioneered the search for non-reductionist solutions seeking the properties SRK\textsubscript{1}, SRK\textsubscript{2}, and SCM. Taking CS, CO, RKG and CM as basic fairness standard, they propose families of solutions that satisfy some combinations of SRK\textsubscript{1, 2}, SCM and PM. However, none of them satisfy all these properties simultaneously, and they left this issue as an open question. Norde (2013) defined the cost adjusted folk solution which discriminates the cost shares of agents very slightly while a significant discrimination is compelling in some cases. In the next section, we define solutions more responsive to the asymmetries in the cost data compared to CAF at the cost of the solidarity aspect of PM.

### 2.5 Achieving SRK\textsubscript{1}, SRK\textsubscript{2}, SCM and AP

Here, we follow Bogomolnaia and Moulin (2010) and first define elementary cost vectors $\hat{c}$. Every cost vector $c$ can be written by integrating out these elementary cost vectors in a certain way (see (2.4)). Then, we define partial solutions that assign allocations to each problem with elementary cost vectors, given $(N, c)$ (see (2.5)). Finally, we introduce solutions that can be written as an extension of these partial solutions (see (2.6)). For a more detailed argument on (2.4), (2.5), and (2.6) defined below; and how core stability, polynomial complexity and cost monotonicity extend from partial solutions to their extensions, we refer to Bogomolnaia and Moulin (2010).

A cost vector $\hat{c} \in C^b$ is elementary if $\hat{c}_{ij} \in \{0,1\}$ for all $i, j \in N \cup \{\omega\}$, where $C^b$ represents the set of all elementary cost vectors. For $\hat{c} \in C^b$, $G(\hat{c})$ represents the graph of free edges among the elements in $N \cup \{\omega\}$: $G(\hat{c}) = \{e_{ij} : \hat{c}_{ij} = 0\}$. We say $i$ and $j$ are connected if there is a path

\textsuperscript{17} For all $(N, c), (N, c')$: $c \leq c'$ and $c_e < c'_e$ for some $e$ \implies $\varphi_i(N, c) \leq \varphi_i(N, c')$ for all $i \in N$.

\textsuperscript{18} See e.g. Bogomolnaia and Moulin (2010).
between \( i \) and \( j \) consisting of only free edges. \( \mathcal{A}(\hat{c}) \) denotes the set of connected components in \( G(\hat{c}) \) and a particular element is \( A \). Also, \( A_i(\hat{c}) \) is the connected component \( i \) belongs to.

Given any problem \((N, c)\), and \( t \in \mathbb{R}_+ \) we define \( c^t \in \mathcal{C}^b \) such that \( c^t_{ij} = 0 \) if \( c_{ij} < t \) and \( c^t_{ij} = 1 \) if \( c_{ij} \geq t \). Also, let \( \bar{c} \) represent the cost of the most expensive edge in \( N \cup \{\omega\} \), while \( \bar{c}_S \) represent the cost of the most expensive edge in \( S \cup \{\omega\} \), i.e. \( \bar{c}_S = \max_{e \in (S \cup \{\omega\})} c_e \). Then, the cost vector \( c_S \) can be written as follows:

\[
c_S = \int_0^{\bar{c}_S} c_S^t dt
\]

(2.4)

Let \( \psi^b(N, c) \) be a partial solution that assigns a cost allocation to problem \((N, \hat{c})\) for all \( \hat{c} \in \mathcal{C}^b \), and for all \((N, c)\):

\[
\psi^b(N, \hat{c}, c) \in \mathbb{R}_+^n \quad \text{and} \quad \sum_{i \in N} \psi^b_i(N, \hat{c}, c) = v(N, \hat{c})
\]

(2.5)

For any partial solution \( \psi^b \) as defined in (2.5), we can write the extension of this solution to the problem \((N, c)\) as follows:

\[
\psi(N, c) = \int_0^{\bar{c}} \psi^b(N, c^t, c) dt
\]

(2.6)

\textbf{Remark 2.2:} Note that for any \( r \geq \bar{c} \), as \( \sum_{i \in N} \psi^b_i(N, \hat{c}, c) = v(N, \hat{c}) \) for all \( \hat{c} \in \mathcal{C}^b \), we have

\[
\psi(N, c) = \int_0^{\bar{c}} \psi^b(N, c^t, c) dt = \int_0^{\bar{c}} \psi^b(N, c^t, c) dt
\]

\( \psi(N, c) \) defined by (2.5) and (2.6) is a legitimate solution to problem \((N, c)\): \( \psi(N, c) \in \mathbb{R}_+^n \) and \( \int_0^{\bar{c}} v(N, c^t) dt = v(N, c) \). It is continuous if \( \psi^b(N, \hat{c}, c) \) is continuous in \( c \) for all \( \hat{c} \in \mathcal{C}^b \), and is of polynomial complexity if \( \psi^b(N, \hat{c}, c) \) is for all \( \hat{c} \). Moreover, \( \psi \) is a core selection if (but not only if) for all \( \hat{c} \in \mathcal{C}^b \), \( c \in \mathcal{C} \), \( A \in \mathcal{A}(\hat{c}) \), \( i \in A \):
\[
\sum_{i \in A} \psi^b_i(N, \hat{\epsilon}, c) = 1 \quad \text{if } \omega \notin A; \quad \psi^b_i(N, \hat{\epsilon}, c) = 0 \quad \text{if } \omega \in A
\] (2.7)

Cost monotonicity also extend from \( \psi^b \) to \( \psi \) in the following way:

*Cost Monotonicity.* \( \psi \) satisfies CM if for all \( \hat{\epsilon}, \hat{\epsilon}' \in C^b, c, c' \in C, i \in N, k \in N \cup \{ \omega \} \setminus i \):

\[
\{ c_{ik} < c'_{ik} \text{ and } c_e = c'_e \text{ for all } e \neq e_{ik} \} \Rightarrow \psi^b_i(N, \hat{\epsilon}, c) \leq \psi^b_i(N, \hat{\epsilon}', c') \quad \text{and}
\]
\[
\{ \hat{c}_{ik} < \hat{c}'_{ik} \text{ and } \hat{c}_e = \hat{c}'_e \text{ for all } e \neq e_{ik} \} \Rightarrow \psi^b_i(N, \hat{\epsilon}, c) \leq \psi^b_i(N, \hat{\epsilon}', c)
\]

Now, fix two continuous, strictly positive and weakly increasing functions \( f, g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) and consider the following partial solutions \( \bar{\psi}^b \) s.t. for any \( (N, c), \hat{\epsilon} \in C^b, A \in \mathcal{A}(\hat{\epsilon}) \) and \( i \in A \):

\[
\bar{\psi}^b_i(N, \hat{\epsilon}, c) = \frac{f(c_{\omega i}) \cdot \prod_{j \in N \setminus i} g(c_{ij})}{\sum_{k \in A} f(c_{\omega k}) \cdot \prod_{j \in N \setminus k} g(c_{kj})} \quad \text{if } \omega \notin A; \quad \psi^b_i(N, \hat{\epsilon}, c) = 0 \quad \text{if } \omega \in A
\] (2.8)

**Proposition 2.4:** (*Moulin and Bogomolnaia (2010)*) If \( f \) and \( g \) are strictly increasing, all solutions \( \bar{\psi} \) defined (2.6) and (2.8) are core selections meeting CO, CM, RKG, SRK\(_1\), SRK\(_2\), and SCM but fail PM.

Bogomolnaia and Moulin showed if \( f \) is strictly increasing and \( g \) is constant, \( \bar{\psi} \) satisfies all but SRK\(_1\) and SCM. Consider the following 3-person problem: \( (c_{\omega 1}, c_{\omega 2}, c_{\omega 3}, c_{12}, c_{13}, c_{23}) = (4,4,2,0,10,20) \). Note that any solution \( \varphi \) meeting SEP and RKG (and hence ETE) yields \( \varphi(N, c) = (2,2,2) \).

**Remark 2.3:** If \( g \) is strictly increasing, all solutions \( \bar{\psi} \) defined (2.6) and (2.8) fail SEP (and hence AP).

Now we define a property on the partial solutions \( \psi^b \) that ensures the extension of the partial solution is absence-proof. Let for any \( \hat{\epsilon} \), for any \( A \in \mathcal{A}(\hat{\epsilon}), \hat{\epsilon}_A = (\hat{\epsilon}_{ij})_{ij \in A(\omega_{\{\omega\}(2)}}

**Definition 2.3:** A partial solution satisfies *independence of irrelevant components (IIC)* if for all \( (N, c), (N', c'); \hat{\epsilon}, \hat{\epsilon}' \in C^b; A \in \mathcal{A}(\hat{\epsilon}), A' \in \mathcal{A}(\hat{\epsilon}') \) such that \( A = A', c_A = c'_A \), and \( \hat{\epsilon}_A = \hat{\epsilon}'_A \).
\[
\psi_i^b(N, \hat{c}, c) = \psi_i^b(N', \hat{c}', c') \text{ for all } i \in A \setminus \{\omega\}. \quad (2.9)
\]

**Lemma 2.1:** Let \( \psi \) be the extension of a partial solution \( \psi^b \) as defined in (2.6). If \( \psi^b \) satisfies IIC, \( \psi \) is a core selection.

**Proof:** Let \( \psi^b \) satisfy IIC. Fix \((N, c), \hat{c} \in C^b\), and \( A \in A(\hat{c}) \). For \( S = A \setminus \{\omega\} \), we have
\[
\sum_{i \in A} \psi_i^b(N, \hat{c}, c) = \sum_{i \in A} \psi_i^b(S, \hat{c}_S, c) = \nu(S, \hat{c}_S).
\]
Note that \( \nu(S, \hat{c}_S) = 1 \) if \( \omega \notin A \), and \( \nu(S, \hat{c}_S) = 0 \) if \( \omega \in A \). Therefore, (2.7) holds. \( \square \)

The idea in IIC is as follows: By Lemma 2.1, for any \( \hat{c} \in C^b \) and the connected component \( A(\hat{c}) \), \( \psi^b \) distributes a total of 1 if \( \omega \notin A \) and 0 if \( \omega \in A \) to the agents in \( A \). Given a problem \((N, c)\), any partial solution \( \psi^b \) that distributes this cost among agents in \( A \) only as a function of the cost of links in \( A \cup \{\omega\} \) \((c_A, \hat{c}_A)\) satisfy IIC.

**Proposition 2.5:** Let \( \psi \) be the extension of a partial solution \( \psi^b \). If \( \psi^b \) satisfies IIC, \( \psi \) is AP.

**Proof:** Let everything be as in the statement of the proposition. Fix \((N, c), T \subset N \) and let \( K_+ = \{i \in N \setminus T: \psi_i(N, c) > \psi_i(N \setminus T, c)\} \). Note that if for some \( K \in N \setminus T \), \( K \cup T \) is able to manipulate \( \psi \) by the absence of \( T \), then \( K_+ \cup T \) can manipulate, too. Reordering (2.2), and by Remark 2.2, as \( \hat{c} \geq \bar{c}_T, \bar{c}_{N \setminus T} \), it suffices to show:
\[
\sum_{i \in T} \int_0^\hat{c} \left[ \psi_i^b(T, c_T^t, c) - \psi_i^b(N, c_T^t, c) \right] dt \geq \sum_{i \in K_+} \int_0^\hat{c} \left[ \psi_i^b(N, c_T^t, c) - \psi_i^b(N \setminus T, c_{N \setminus T}^t, c) \right] dt
\]
Define \( K_+(t) = \{i \in N \setminus T: \psi_i^b(N, c_t^t, c) > \psi_i^b(N \setminus T, c_{N \setminus T}^t, c)\} \). Note that;
\[
\sum_{i \in K_+(t)} \int_0^\hat{c} \left[ \psi_i^b(N, c_t^t, c) - \psi_i^b(N \setminus T, c_{N \setminus T}^t, c) \right] dt \geq \sum_{i \in K_+} \int_0^\hat{c} \left[ \psi_i^b(N, c_t^t, c) - \psi_i^b(N \setminus T, c_{N \setminus T}^t, c) \right] dt
\]
Now, combining the two inequalities above, it suffices to show that for any \( t \leq \bar{c} \), (2.10) holds.
\[
\sum_{i \in T} \psi_i^b(T, c_T^t, c) + \sum_{i \in K_+(t)} \psi_i^b(N \setminus T, c_{N \setminus T}^t, c) \geq \sum_{i \in T \cup K_+(t)} \psi_i^b(N, c^t, c) \tag{2.10}
\]

Let \(\mathcal{A}(t) = \mathcal{A}(c^t)\), and \(\mathcal{A}^T(t) = \{A \in \mathcal{A}(t) : A \cap T \neq \emptyset\}\). Note that by construction of the graph \(G(\cdot)\), for any \(t\) and \(A \in \mathcal{A}(t) \setminus \mathcal{A}^T(t)\) we have \(A \in \mathcal{A}(c_{N \setminus T}^t)\). Then, by IIC of \(\psi^b\), for any \(t\), \(A \in \mathcal{A}(t) \setminus \mathcal{A}^T(t)\) and for all \(i \in A \setminus \{\omega\}\) we have \(\psi_i^b(N, c^t, c) = \psi_i^b(N \setminus T, c_{N \setminus T}^t, c)\), and hence, \(A \cap K_+(t) = \emptyset\). Then, we can rewrite the inequality (2.10) as follows:

\[
\sum_{i \in U_{\mathcal{A} \cap \mathcal{A}^T(t) \setminus (T \cap A)}} \psi_i^b(T, c_T^t, c) + \sum_{i \in U_{\mathcal{A} \cap \mathcal{A}^T(t) \setminus (K_+(t) \cap A)}} \psi_i^b(N \setminus T, c_{N \setminus T}^t, c) \geq \sum_{i \in U_{\mathcal{A} \cap \mathcal{A}^T(t) \setminus (K_+(t) \cup T) \cap A}} \psi_i^b(N, c^t, c)
\]

Note that if for each \(A \in \mathcal{A}^T(t)\), the inequality above holds, then taking the union of \(A\) in \(\mathcal{A}^T(t)\) we have the desired result. Now, fix any \(A \in \mathcal{A}^T(t)\).

Case 1: \(\omega \notin A\). Agents in \(T \cap A\) constitute at least 1 connected component in \(G(c_T^t)\). Then, by Lemma 2.1, we have:

\[
\sum_{i \in (T \cap A)} \psi_i^b(T, c_T^t, c) \geq 1 \geq \sum_{i \in [(K_+(t) \cup T) \cap A]} \psi_i^b(N, c^t, c)
\]

Case 2: \(\omega \in A\). Then, again by Lemma 2.1, we have:

\[
\sum_{i \in [(K_+(t) \cup T) \cap A]} \psi_i^b(N, c^t, c) = 0
\]

\[\square\]

We are inspired by Bogomolnaia and Moulin (2010) for the solutions defined below by (2.11). Let \(f : \mathbb{R}_+ \rightarrow \mathbb{R}_{++}\) be a continuous, strictly positive\(^{19}\) and weakly increasing function. Fix a number \(\lambda, 1 \leq \lambda \leq \infty\). For all \(\hat{c} \in C^b\), and \(i \in N\) define \(\delta_i(\hat{c}) = |\{e_{ij} \in A_i(\hat{c}) : \hat{c}_{ij} = 1\}|\) that is

\[^{19}\text{\(f(0) = 0\) and \(f\) is strictly positive otherwise}\]
the number of non-null links \( i \) has in \( A_i \), and for all \( A \in \mathcal{A}(\tilde{c}) \), \( A^\delta(\tilde{c}) = \arg \max_{k \in A} \delta_k(\tilde{c}) \) that is the set of agents in \( A \) that has highest number of non-null links. Now, define \( \psi_i^{b, \lambda} \) successively for \( 1 \leq \lambda < \infty \) and \( \lambda = \infty \) as follows:

\[
\psi_i^{b, \lambda}(N, \tilde{c}, c) = \frac{f(c_{\omega i}) \cdot \lambda^{\delta_i(\tilde{c})}}{\sum_{k \in A_i(\tilde{c})} f(c_{\omega k}) \cdot \lambda^{\delta_k(\tilde{c})}} \quad \text{if} \quad \omega \notin A_i(\tilde{c}); \quad \psi_i^{b, \lambda}(\cdot) = 0 \quad \text{otherwise}
\]

(2.11)

\[
\psi_i^{b, \infty}(N, \tilde{c}, c) = \frac{f(c_{\omega i})}{\sum_{k \in A_i^\delta(\tilde{c})} f(c_{\omega k})} \quad \text{if} \quad i \in A_i^\delta(\tilde{c}) \quad \text{and} \quad \omega \notin A_i(\tilde{c}); \quad \psi_i^{b, \infty}(\cdot) = 0 \quad \text{otherwise}
\]

**Remark 2.4:** If \( f \) is constant, \( \psi^L \) is the folk solution.²⁰

**Proposition 2.6:**

(1) All solutions \( \psi^L \) defined by (2.6), (2.11) are core selections meeting CO, RKG, CM, PC and AP.

(2) If \( \lambda > 1 \), they satisfy SRK₁ but fail PM.

(3) If \( f \) is strictly increasing, they all satisfy SRK₂, and they satisfy SCM for \( 1 < \lambda < \infty \).

**Proof:** (1) Note that (2.7) and IIC holds for \( \psi_i^{b, \lambda} \) for all \( \lambda \) by construction. Also, \( \psi_i^{b, \lambda} \) is continuous in \( c \) for all \( \tilde{c} \) and it clearly meets PC. Hence, all \( \psi^L \) are core selections meeting CO, PC and AP.

For RKG, fix \( c; i, j \in N \) such that \( c_{i k} \leq c_{j k} \) for all \( k \in N \setminus \{i, j\} \). If \( |A_j(c^t)| \neq 1, i \in A_j(c^t), \) and \( \delta_i(c^t) \leq \delta_j(c^t) \) for all such \( t \). Also, if \( |A_j(c^t)| = 1, \psi_i^{b, \lambda}(c^t) \leq \psi_j^{b, \lambda}(c^t) = 1 \) Then, \( j \) receives no less than \( i \) at each \( t \) in \( \psi_i^{b, \lambda}(c^t) \) as \( f \) is weakly increasing and \( \lambda \geq 1 \).

For CM, we give the proof for only \( \lambda < \infty \). For \( \lambda = \infty \), the argument is the same, or simpler.

In case we fix \( \tilde{c} \), we only need to check the case \( c_{\omega i} \) is increased to \( c'_{\omega i} \) as \( \delta_k(\tilde{c}) \) is fixed for all \( k \in N \). Since \( f \) is weakly increasing, we are done. Now, suppose the cost of only one link \( \tilde{c}_e \) is increased from 0 to \( \tilde{c}_e' = 1 \) while the rest of \( \tilde{c}, c \) is fixed. Then, if for one vertex \( i \) in \( e, \tilde{c}_{\omega i} = 0, \)

²⁰The folk solution is the extension of the following partial solution: \( \psi_i^p(N, \tilde{c}, c) = \frac{1}{|A|} \quad \text{if} \quad \omega \notin A \) and \( \psi_i^p(N, \tilde{c}, c) = 0 \quad \text{if} \quad \omega \in A \). See Bogomolnaia and Moulin (2010).
\( \psi_t^{b,\lambda} \) cannot decrease as \( \psi_t^{b,\lambda}(N, \hat{\epsilon}, c) = 0 \). Suppose now \( \hat{\epsilon}_{oi} = \hat{\epsilon}_{oj} = 1, \hat{\epsilon}_{ij} \) is increased from 0 to \( \hat{\epsilon}_{ij}' = 1 \). Consider first the case \( j \in A_i(\hat{\epsilon}') \). Let \( \gamma_t(\hat{\epsilon}) = f(c_{oi}) \cdot \lambda^{\delta_i(\hat{\epsilon})} \). Note that \( \gamma_h(\hat{\epsilon}') = \lambda \gamma_h(\hat{\epsilon}) \) for \( h \in \{i, j\} \) and \( \gamma_k(\hat{\epsilon}') = \gamma_k(\hat{\epsilon}) \) otherwise. Check that for any nonnegative \( n \) numbers \( x_1, x_2, \ldots, x_n \) and \( p \geq 1, \)

\[
[x_1/\sum_{s=1}^{n} x_s] \leq [px_1/(px_1 + px_2 + \sum_{s=3}^{n} x_s)] \tag{2.12}
\]

Now, consider the case \( j \notin A_i(\hat{\epsilon}') = M_1 \). Then, \( A_j(\hat{\epsilon}') = M' \) and \( M \) partitions \( A_i(\hat{\epsilon}) \) with \( |M| = m \) and \( |M'| = m' \). Note that \( \gamma_i(\hat{\epsilon}') = \gamma_i(\hat{\epsilon})/\lambda^{(m'-1)} \) and \( \gamma_k(\hat{\epsilon}') = \gamma_k(\hat{\epsilon})/\lambda^{m'} \) for all \( i \neq k \in M \). Similarly, \( \gamma_j(\hat{\epsilon}') = \gamma_j(\hat{\epsilon})/\lambda^{m-1} \) and \( \gamma_k(\hat{\epsilon}') = \gamma_k(\hat{\epsilon})/\lambda^{m} \) for all \( j \neq k \in M' \). Then,

\[
\psi_t^{b,\lambda}(N, \hat{\epsilon}, c) = \lambda \gamma_t(\hat{\epsilon})[(\Sigma_{i \neq k \in M} \gamma_k(\hat{\epsilon})) + (\Sigma_{k \in M} \gamma_k(\hat{\epsilon}))] \geq \gamma_t(\hat{\epsilon})/[(\Sigma_{k \in M} \gamma_k(\hat{\epsilon}))] \geq \psi_t^{b,\lambda}(N, \hat{\epsilon}, c) \tag{2.13}
\]

(2) For SRK1, let \( c \in D; i, j \in N \) such that \( c_{ik} < c_{jk} \) for all \( k \in N \setminus \{i, j\} \) and \( c_{oi} \leq c_{oj} \). We already showed \( \psi_t^{b,\lambda}(c,t) \leq \psi_j^{b,\lambda}(c,t) \) at each \( t \). Consider \( k \in \arg \max_{i \in N \setminus \{j\}} c_{il} \) and recall that for all \( t \in (c_{ik}, c_{jk}], \omega \notin A_i(c_t) \). Then, if \( t \in (c_{ik}, c_{jk}] \), in case \( j \in A_i(c_t) \), we have \( \delta_i(c_t) < \delta_j(c_t) \) implying \( \psi_t^{b,\lambda}(c_t) < \psi_j^{b,\lambda}(c_t) \) as \( f \) is weakly increasing. In case \( j \notin A_i(c_t) \), we have \( \psi_t^{b,\lambda}(c_t) < \psi_j^{b,\lambda}(c_t) = 1 \) for \( \lambda < \infty \). For \( \lambda = \infty \), if \( i \in A_i^{\delta}(c_t) \), all \( l \in N \setminus \{i, j\} \) are also in \( A_i^{\delta}(c_t) \) as \( \delta_i(c_t) = 0 \). Then, again we have \( \psi_t^{b,\lambda}(c_t) < \psi_j^{b,\lambda}(c_t) = 1 \). Therefore, we have the desired result by definition (2.6).

To see PM fails, let \( (N, h; c), h \notin N \) be such that \( c_{1h} = c_{wo} = 1 \) for all \( i \in N \cup h \). \( c_{il} = c_{ij} = c_{ih} = 0 \) for all \( i, j = 2, \ldots, n \). In the absence of \( h \) each agent 1 pays \( 1/n \). When \( h \) is present, agent 1 pays \( \frac{\lambda}{2\lambda + n - 1} \). Therefore, PM requires \( \frac{\lambda}{2\lambda + n - 1} \leq \frac{1}{n} \Leftrightarrow \lambda \leq \frac{n - 1}{n - 2} \).

(3) For SRK2, let \( c \in D; i, j \in N \) such that \( c_{ik} \leq c_{jk} \) for all \( k \in N \setminus \{i, j\} \) and \( c_{oi} < c_{oj} \). Recall that \( \bar{r} = \max_{k, l \in N} c_{kl} < r = \min_{k \in N} c_{wo} \) as \( c \in D \).
Then, for any \( t \in (\tilde{r}, r] \) we have \( \delta_k(c^t) = 0 \) for all \( k \in N \), \( \psi_i^{b\lambda}(c^t) = \frac{f(c_{oi})}{\sum_{k\in N} f(c_{ok})} \), and hence, \( \psi_i^{b\lambda}(c^t) < \psi_j^{b\lambda}(c^t) \) if \( f \) is strictly increasing. Thus, we have the desired result by definition (2.6).

For SCM, let \( f \) be strictly increasing, \( c \in D \). Suppose first \( c_{oi} \) increase to \( \tilde{c}_{oi} \) and the rest of \( c \) is fixed. Let \( \tilde{r}, r \) be as defined just above. Note that \( c^t = \tilde{c}^t \) for all \( t < r \). Then, as \( f \) is strictly increasing, \( \psi_i^{b\lambda}(c^t) \leq \psi_i^{b\lambda}(\tilde{c}^t) \) for all \( t \leq \tilde{r} \) and for all \( \lambda \). Also, for any \( t \in (\tilde{r}, r] \) we have \( \delta_k(c^t) = 0 \) for all \( k \in N \), and hence, \( \psi_i^{b\lambda}(c^t) = \frac{f(c_{oi})}{\sum_{k\in N} f(c_{ok})} < \frac{f(\tilde{c}_{oi})}{\sum_{k\in N} f(\tilde{c}_{ok})} = \psi_i^{b\lambda}(\tilde{c}^t) \). Then, we have \( \psi_i^{\lambda}(c^t) < \psi_i^{\lambda}(\tilde{c}^t) \) by (2.6). Now, let only \( c_{ij} \) increase to \( \tilde{c}_{ij} < r \) so that \( \tilde{c} \in D \), and the rest of \( c \) is fixed. Note that \( c^t = \tilde{c}^t \) for all \( t \leq \tilde{r} \) and \( t > \tilde{c}_{ij} \). For all those \( t \) as \( \delta_k(c^t) \) and \( c_{ok} \) remains the same for all \( k \in N \), \( \psi_i^{b\lambda}(c^t) = \psi_i^{b\lambda}(\tilde{c}^t) \). For \( t \in (\tilde{r}, r] \) we have two cases: \( j \in A_i(\tilde{c}^t) \) and \( j \notin A_i(\tilde{c}^t) \). Then, the proof mimics that of CM and inequalities (2.12) and (2.13) hold strictly for \( 1 < \lambda < \infty \) as \( f(c_{ok}) > 0 \) for all \( k \in N \). \( \square \)
3 Population Monotonicity in Fair Division of Multiple Indivisible Goods

3.1 Introduction

We consider the fair division problem where individuals have equal claims on a set of indivisible items, and each agent can get more than one good. We also allow for balanced monetary transfers among agents.

Many authors have studied population monotonicity for the fair division problem (Moulin (1990b, 1992), Beviá (1996a,b), Tadenuma and Thomson (1993)). It is mostly interpreted as a solidarity principle. Among the two well-known versions, upon the arrival of an additional agent, the stronger notion (PM) asks no one to be better-off, while the weaker one (wPM) requires everyone to be affected in the same direction; either everyone loose or everyone gains.

The essence of the solidarity idea, indeed, lies in the weaker version. In case there is no production, no monetary transfers among the agents, and agents have monotone preferences, when an extra claimant appears in the allocation process of a fixed supply of consumption goods, unambiguously, he creates a burden on the existing agents. There, both PM and wPM asks no existing agents to be better-off. However, if utility is transferable, an extra claimant who receives more utility from some bundle that is not very desirable for the existing agents would possibly be beneficial to the society. Therefore, in our setting wPM is more suitable as a solidarity principle.

In a very general setting where there is no restriction on the individual preferences, Beviá (1996a) showed that PM is incompatible with efficiency for problems with more than 4 goods. Moreover, PM is incompatible with one of the most important fairness criterion; envy-freeness (Alkan 1994, Moulin 1990b). For all these reasons, research on PM in this context is limited.
As we already discussed in Chapter 1, wPM does not imply absence-proofness while PM does. Here, we put forward the stability aspect of population monotonicty besides the solidarity aspect, and try to make an extensive analysis of population monotonicty in this setting.

Aside from the negative result, Beviá (1996a) also showed that when the domain of preference profiles satisfies “substitutability”, the induced transferable utility (TU) game is concave and hence the Shapley solution (Shapley 1962) is PM (Sprumont 1990). However, substitutability is not defined on the Cartesian product of individual preference domains. Assuming free disposability (monotone preferences), we try to stretch Beviá’s both the positive and the negative results for different number of goods under different Cartesian product domains where individual preferences are submodular, subadditive, and superadditive.

In Section 3.2, we give the general setting. In Section 3.3, we define some well-known individual preference domains and basic properties: symmetry, continuity, equal split guarantee; and population monotonicty. In Section 3.4, we define two well-known symmetric and continuous solutions, the Shapley solution and the constrained egalitarian (Dutta and Ray 1989), which are also PM in concave games. In Section 3.5, we analyze problems with three or more goods. Here, PM and efficiency are incompatible on the superadditive domain. Also, neither submodular nor subadditive preferences induce concave TU games. Moreover, the Shapley solution is not PM on any of these domains.

In Section 3.6, we first show that when each agent has subadditive preferences the Shapley solution and the egalitarian solution are PM. On the full domain of monotone preferences we can write the efficient surplus as the summation of surplus derived from two problems where each problem induces concave TU games: a 2-goods problem with subadditive preferences and a single good problem. We define hybrid solutions as the summation of two solutions to those problems. However, dynamics of a change in population is not trivial in our construction. A hybrid solution is PM if both solutions are PM and solution to the single good problem satisfies
additive scale monotonicity\textsuperscript{21}. The hybrid Shapley solution is PM on the full domain while the hybrid egalitarian solution is not. Finally, we show that the equal split guarantee is not compatible with PM for problems with two or more goods.

### 3.2 The setting

A finite set of commonly owned indivisible goods denoted by $\Omega$ is distributed to a set of individuals denoted by $N \in \mathcal{N}$ where $\mathcal{N}$ is the set of all finite potential societies and $|N| = n$. We consider a general model where agents can get multiple goods. Monetary transfers are available, and agents’ preferences are quasilinear in money. Given $(N, \Omega)$, each agent $i \in N$ receives $u_i(A)$ units of utility from the bundle $A \subseteq \Omega$, while also receiving $m \in \mathbb{R}$ units of money yields $u_i(A, m) = u_i(A) + m$. We also assume free disposability (monotone preferences), i.e., $u_i(A) \leq u_i(B)$ for all $A \subseteq B$. By convention $u_i(\emptyset) = 0$. A list of preferences $\{u_i\}_{i \in N}$ is denoted by $u$, and $u^S$ denotes the restricted list to $S \subseteq N$. A fair division problem is a triple $(N, \Omega, u)$. $\epsilon$ denotes the set of all problems (with monotone preferences), and $\epsilon_m$ denotes all problems with $m$ goods.

Given a problem with monetary transfers, an allocation consists of two components: Assignment of the objects to the agents and balanced monetary transfers among the agents. An assignment is a mapping $\sigma: N \rightarrow 2^\Omega$ such that $\sigma_i \cap \sigma_j = \emptyset$ for all $i, j \in N$, and $\bigcup_{i \in N} \sigma_i = \Omega$, while some agents may receive no good with $\sigma_i = \emptyset$. A vector of balanced monetary transfers is $m \in \mathbb{R}^n$ s.t. $\sum_{i \in N} m_i = 0$.

An assignment $\sigma$ is efficient if $\sum_{i \in N} u_i(\sigma_i) \geq \sum_{i \in N} u_i(\sigma_i')$ for all assignments $\sigma': N \rightarrow 2^\Omega$. By quasilinearity, an allocation $(\sigma, m)$ is efficient if and only if $\sigma$ is efficient. Here, we are only interested in individually rational (IR) allocations, i.e., $u_i(\sigma_i) + m_i \geq 0$ for all $i \in N$. A solution $\phi$ is a mapping such that in a given domain of problems, it assigns a set of allocations to each

\textsuperscript{21} See Section 3.6 for a definition.
problem. \(\varphi\) is efficient (or IR) if it always yields efficient (or IR) allocations, and is *single valued* if for any \((N, \Omega, u)\), for any \((\sigma, m), (\sigma', m') \in \varphi(N, \Omega, u)\) we have \(u_i(\sigma_i) + m_i = u_i(\sigma'_i) + m'_i\) for all \(i \in N\). Then, \(\varphi^i(N, \Omega, u)\) denotes the final utility of agent \(i\) at solution \(\varphi\). Throughout this paper, for simplicity, we will write down some properties and results for single valued solutions only. However, this simplification does not alter any result stated here.

Each problem \((N, \Omega, u)\) induces a *TU cooperative game* \(\nu: 2^N \to \mathbb{R}_+\) with \(\nu(S) = \sum_{i \in S} u_i(\sigma_i)\) where \(\sigma\) is an efficient assignment at the problem \((S, \Omega, u^S)\). We also write \(\nu(S, A) = \sum_{i \in S} u_i(\sigma_i)\) if \(\sigma\) is an efficient assignment at the problem \((S, A, u^S)\) with \(A \subseteq \Omega\). By convention \(\nu(\emptyset) = 0\). A game \(\nu\) is *concave* if \(\nu(S \cup \{i, j\}) - \nu(S \cup \{j\}) \leq \nu(S \cup \{i\}) - \nu(S)\) for all \(S \subseteq N: i, j \in N \setminus S\).

### 3.3 Preferences and basic properties

Throughout this paper we will consider the following basic types of preferences:

**Definition 3.1:** A utility function \(u_i: 2^\Omega \to \mathbb{R}_+\)

(i) is *submodular* if for all \(A, B \subseteq \Omega\), \(u_i(A \cup B) + u_i(A \cap B) \leq u_i(A) + u_i(B)\).

(ii) has *decreasing marginal returns* if for all \(A \subseteq B \subseteq \Omega\), \(\alpha \in A\),

\[ u_i(B) - u_i(B \setminus \{\alpha\}) \leq u_i(A) - u_i(A \setminus \{\alpha\}). \]

(iii) is *subadditive* if for all \(A, B \subseteq \Omega\), \(u_i(A \cup B) \leq u_i(A) + u_i(B)\).

(iv) is *additively separable* if for all \(A \subseteq \Omega\), \(u_i(A) = \sum_{\alpha \in A} u_i(\alpha)\).

(v) is *superadditive* if for all \(A, B \subseteq \Omega\), \(u_i(A) + u_i(B) \leq u_i(A \cup B)\).

Properties (ii) and (iii) are equivalent, and reflect the concavity of \(u_i\) (see e.g. Gul and Stacchetti (1999)). Subadditive preference domain contains the submodular preferences. Both subadditive and superadditive preference domain contains the additively separable preferences. In case of a single object, all properties trivially hold. Also, for \(|\Omega| = 2\), subadditivity and submodularity coincides.
Definition 3.2: A solution \( \varphi \) is

(i) **population monotonic** (PM) on \( \varepsilon \subseteq \varepsilon \) if for all \( (N', \Omega, u), (N, \Omega, u^N) \in \varepsilon' \) with \( N \subseteq N' \), and for all \( i \in N \), we have \( u_i(\sigma_i) + m_i \geq u_i(\sigma'_i) + m'_i \) for all \( (\sigma', m') \in \varphi(N', \Omega, u) \) and \( (\sigma, m) \in \varphi(N, \Omega, u^N) \).

(ii) **weakly population monotonic** (wPM) on \( \varepsilon \subseteq \varepsilon \) if for all \( (N', \Omega, u), (N, \Omega, u^N) \in \varepsilon' \) with \( N \subseteq N' \), and for all \( (\sigma', m') \in \varphi(N', \Omega, u) \) and \( (\sigma, m) \in \varphi(N, \Omega, u^N) \) we have either

\[ u_i(\sigma_i) + m_i \leq u_i(\sigma'_i) + m'_i \text{ for all } i \in N \text{ or } u_i(\sigma_i) + m_i \geq u_i(\sigma'_i) + m'_i \text{ for all } i \in N. \]

We already discussed in Chapter 1 that PM implies AP in this setting while wPM does not. In this chapter we put forward the stability aspect of population monotonicity besides the solidarity aspect, and therefore use PM as our main principle. Where we have positive results, along with the PM we also look for solutions with basic fairness criteria. The most basic one is that a solution does not discriminate agents by their names and cares for only their preferences. We also do not want any significant jumps in the final utilities when there is a miniscule change in the utility profile.

**Symmetry.** A single valued solution \( \varphi \) satisfies symmetry if for any two problems \( (N, \Omega, u) \) and \( (N, \Omega, u') \) s.t. \( u_i = u'_j, u_j = u'_i \) for some \( i, j \in N \), and \( u_k = u'_k \) for all \( k \in N \setminus \{i, j\} \) we have;

\[ \varphi^i(N, \Omega, u) = \varphi^j(N, \Omega, u'), \varphi^j(N, \Omega, u) = \varphi^i(N, \Omega, u'), \text{ and } \varphi^k(N, \Omega, u) = \varphi^k(N, \Omega, u') \text{ for all } k \in N \setminus \{i, j\}. \]

**Continuity.** A single valued solution \( \varphi \) is continuous if for any fixed \( (N, \Omega) \), \( \varphi^i \) is a continuous function of \( u \).

Another important fairness idea is that no agent is worse off compared to receiving \( (1/n)^{th} \) of the goods. For the indivisible goods, \( (1/n)^{th} \) is not well defined. Beviá (1996c) discusses this issue and uses identical preference lower bound (IPLB) to apply the idea. In our case, for each agent this lower bound corresponds to \( (1/n)^{th} \) of the efficient surplus generated at the
hypothetical problem where everybody else has the same preferences. Here, we ask for a less demanding lower bound.

**Equal Split Guarantee (ESG).** A single valued solution $\varphi$ satisfies ESG if for all $i \in N$, $\varphi^i(N, \Omega, u) \geq u_i(\Omega)/n$.

In the worst case scenario, an agent receives all the goods and then compensates others so that everybody gets the same final utility. In case $|\Omega| = 1$, where PM and ESG are compatible, ESG and IPLB coincides. When $|\Omega| \geq 2$, PM is not even compatible with the weaker property, ESG.\(^{22}\)

### 3.4 Concave TU games and solutions

In allocation problems where utility is transferable, an important source in the design of solutions is the set of algorithms that calculate the efficient surplus (or cost). However, excluding the special case where utilities are additively separable\(^{23}\), finding the efficient surplus is not an easy task. Indeed, it is an NP hard problem for submodular preference domain (Feige 2009). If symmetry and continuity is desired as a minimal fairness principle, a remedy to define efficient solutions is to borrow them from TU game literature such as the Shapley solution and the egalitarian solution (Dutta and Ray 1989).

Concave games are of special importance for two reasons. First, both solutions are population monotonic on concave games. Moreover, the egalitarian solution yields the unique vector that Lorenz dominates every other vector within the dual core of the game.

**The Shapley solution (Sh).** Given a problem $(N, \Omega, u), (\sigma, m) \in Sh(N, \Omega, u)$ if and only if for all $i \in N$, $u_i(\sigma_i) + m_i = Sh^i(v)$ where $v$ is the induced TU game.\(^{24}\)

---

\(^{22}\) See Section 3.6.1.

\(^{23}\) Just assign each good to some agent who gets the maximum utility from that good.

\(^{24}\) Given any TU game $(N, v), Sh^i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!} \left( v(S \cup \{i\}) - v(S) \right)$. 

78
Before we define the egalitarian solution, we need to define the dual core, which is closely related to population monotonicity.

**Stand-alone Core (SAC).** An allocation \((\sigma, m)\) is in the SAC if \(\sum_{i \in S} u_i(\sigma_i) + m_i \leq v(S)\) for all \(S \subseteq N\). A single valued solution \(\phi\) is in the SAC on \(\varepsilon' \subseteq \varepsilon\) if for all \((N, \Omega, u) \in \varepsilon'\), and for all \(S \subseteq N\), \(\sum_{i \in S} \phi^i(N, \Omega, u) \leq v(S)\).

Note that population monotonic solutions are always in the SAC as otherwise for some \(S\), at least one agent from \(S\) is strictly worse-off when \(N \setminus S\) leaves.

Dutta and Ray (1989) defined the Lorenz core and related egalitarian solution in the TU game context. They also defined an algorithm to calculate the solution for convex TU games. The dual algorithm for concave games, defined by Klijn et. al (2001), is as follows: Fix a concave TU game \((N, v)\) and for any \(S \subseteq N\), \(e(S, v) = v(S)/|S|\), so that \(e(S, v)\) is the average worth of \(S\) under \(v\). Define \(v_1 = v\).

**Step 1:** Define by \(S_1\) the unique coalition such that (i) \(e(S_1, v_1) \leq e(S, v_1)\) for all \(S \subseteq N\); (ii) \(|S_1| > |S|\) for all \(S \neq S_1\) such that \(e(S_1, v_1) = e(S, v_1)\); so that \(S_1\) is the largest coalition with the lowest average worth (under concavity \(S_1\) exists). Define

\[
Eg^i(v) = e(S_1, v_1) \quad \text{for all } i \in S_1. \quad (3.1)
\]

**Step k:** Suppose that \(S_1 \ldots S_{k-1}\) have been defined recursively and \(\bigcup_{p=1}^{k-1} S_p \neq N\). Define a new game with the set of agents \(N^k = N \setminus \bigcup_{p=1}^{k-1} S_p\). For all \(S \subseteq N^k\), define \(v_k(S) = v_{k-1}(S \cup S_{k-1}) - v_{k-1}(S_{k-1})\). This new game \((N^k, v_k)\) is concave. Let \(S_k\) be the largest coalition with the lowest average worth and define

\[
Eg^i(v) = e(S_k, v_k) \quad \text{for all } i \in S_k. \quad (3.2)
\]

Now we define a sufficient condition for PM on the domain of preference profiles.
Substitutability. Given a problem \((N, \Omega, u)\), and \(S \subseteq N\), \(v(S, \cdot)\) satisfies substitutability if for all \(A \subseteq B \subseteq \Omega\) and \(\alpha \notin B\), \(v(S, B \cup \{\alpha\}) - v(S, B) \leq v(S, A \cup \{\alpha\}) - v(S, A)\).

Proposition 3.1: (Beviá 1996a). If \(\varepsilon^{subs} \subseteq \varepsilon\) is such that for each \((N, \Omega, u)\), \(S \subseteq N\), \(v(S, \cdot)\) satisfies substitutability, the induced game \(v(\cdot)\) is concave for all \((N, \Omega, u) \in \varepsilon^{subs}\).

Corollary 3.1: The Shapley solution and the egalitarian solution are PM on \(\varepsilon^{subs}\).

Both solutions are symmetric, and the Shapley value is continuous in \(v\) by definition. Egalitarian solution is also continuous in \(v\) (see e.g. Hougaard et al (2005)). As \(v\) is continuous in \(u\), both solutions satisfy continuity.

3.5 Problems with \(|\Omega| \geq 3\)

Here, we want to emphasize two main points. First one is that the incompatibility between the population monotonicity and efficiency is more serious than it was proven in Beviá (1996a). Secondly, we feel that the positive result (Proposition 3.1) needs some clarification in terms of the Cartesian product domain of profiles for which the result holds.

The incompatibility result in Beviá (1996a) is based on an example with 4 goods where individuals are allowed to have non-monotonic preferences (no free disposal). However, if we allow non-monotonic preferences, incompatibility prevails even in a 2-person, 2-good problem. Consider the following problem;

Example 3.1: \(u_1(\alpha) = u_2(\beta) = 3\), \(u_1(\beta) = u_2(\alpha) = 0\), \(u_1(\alpha \beta) = u_2(\alpha \beta) = 2\)

Note that \(\varphi^1(N) + \varphi^2(N) = 6\) for any efficient solution. Hence, for at least one \(i\) we have \(\varphi^i(N) \geq 3\). However, each agent gets 2 units of utility at the efficient solution when they are the only claimants. Thus, achieving PM is not possible here.
We strengthen this negative result in two dimensions. Incompatibility between PM and efficiency prevails for 3-goods problem even in economies with monotone preferences. Let \( \varepsilon_{sup} \) represent the set of problems where each agent has superadditive preferences.

**Proposition 3.2:** No efficient solution is population monotonic on \( \varepsilon' \supseteq \varepsilon_{m}^{sup} \) for \( m \geq 3 \).

**Proof:** Let \( |\Omega| = 3 \), and consider the following preferences:

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<tr>
<td>( u_1(A) )</td>
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<td>2</td>
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<td>( u_2(A) )</td>
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<td>( u_3(A) )</td>
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<td>( u_4(A) )</td>
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Consider the problem \((S, \Omega, u)\) with \( S = \{1,2,3\} \). At the unique efficient assignment \( \sigma_1 = \alpha\beta \), \( \sigma_2 = \gamma \) and \( v(S) = v(12) = 4 \). Also, \( v(13) = v(23) = 2 \). Let \( \varphi \) be efficient, population monotonic and \( (\sigma, m) \in \varphi(S, \Omega, u) \). By PM, \( (\sigma, m) \) is in the SAC. Then, \( u_1(\sigma_1, m_1) + u_3(\sigma_3, m_3) \leq 2 \), \( u_2(\sigma_2, m_2) + u_3(\sigma_3, m_3) \leq 2 \). Hence, we have \( u_1(\sigma_1, m_1) = u_2(\sigma_2, m_2) = 2 \), and \( u_3(\sigma_3, m_3) = 0 \). Then, \( \varphi^1([1,3]) = 2 \) again by PM. Now, consider the problem \((S', \Omega, u)\) with \( S' = \{1,3,4\} \). By a similar argument we have \( \varphi^2([1,3]) = 2 \). Therefore, we have the desired contradiction. For \( |\Omega| > 3 \), use the same profile and add dummy goods such that it brings 0 extra utility to all bundles for all agents. \( \Box \)

Cartesian product of two special types of submodular preferences constitutes substitutable domains. First one is \( u_i(A) = \max_{\alpha \in A} u_i(\alpha) \) for all agents. Then, our problem is equivalent to the allocation problem where each agent can get at most one good. Moulin (1992) showed for this case that the utility profile is substitutable. Hence, the induced game is concave, and the Shapley and the egalitarian solutions are PM.

Additively separable preferences also constitute a substitutable domain. Given a problem \((N, \Omega, u)\) with \( u_i \) is additively separable for all \( i \in N \), for a fixed \( S \subseteq N \), marginal contribution of
to the efficient surplus \( v(S, r) \) is constant and equal to \( \max_{i \in S} u_i(\alpha) \). However, in this case, instead of the general approach of distributing many goods at once, we can just think of the problem as distributing \( |\Omega| \) goods separately. The solution to the general problem can be defined as the sum of solutions to \( |\Omega| \) separate problems. Obviously, if at each single good allocation problem the solution is PM, then their summation is also PM. Hence, for this domain of preferences there is a variety of efficient and PM solutions.\(^{25}\)

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<tr>
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<td>2</td>
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*Table 3.1*

We know that substitutability is a sufficient condition for the concavity of the induced game, and hence for the existence of a PM solution. It also requires that individual preferences are submodular. However, submodularity of the individual preferences is not a necessary condition for concavity. Although agent 1’s preference is not submodular in the 3-person problem in Table 3.1, the induced game is concave.\(^{26}\)

What if all individuals have submodular preferences? Note that additively separable preferences constitute the border between the submodular (concave) and supermodular (convex) utility functions. At a first glance, if the least concave utility functions in submodular domain induce concave games, one may expect that when concavity becomes more severe, the induced game would be still concave. However, this intuition fails. Consider the 4-person, 3-goods problem in Table 3.2. The induced game is not concave as \( v(N) - v(124) = 2 > v(123) - v(12) = 1 \) while all agents have submodular utility functions. Also, concavity of the game is a

\(^{25}\) See Section 3.6.

\(^{26}\) \( v(1) = 3, v(2) = 2, v(3) = 2, v(12) = 4, v(13) = 3, v(23) = 4, v(123) = 4. \)
necessary but not a sufficient condition for Shapley solution to be population monotonic\(^{27}\). The same example also illustrates that the Shapley solution is not PM on the domain of submodular preferences. Here, we have \(Sh^3(\{1,2,3\}) \cong 4,67\), while \(Sh^3(N) \cong 4,92\).

<table>
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Table 3.2

### 3.6 Problems with \(|\Omega| \leq 2\)

The problem of allocating a single indivisible object is studied by several authors. However, all population monotonic solutions to this problem (except the Shapley solution) are defined for the well-known airport (cost sharing) problem (see Thomson (2007) for a survey). This problem yields concave TU games and admits several population monotonic solutions other than the Shapley solution and the constrained egalitarian solution. It is easy to check that these two solutions also satisfy ESG when \(|\Omega| = 1\). Our intention here is to work on the 2-goods case. However, to construct PM solutions for this case we need to define a property on solutions to the single good problem.

For the single good problem, preference of agent \(i\) is represented by a number \(u_i\), and \((N, u)\) represents a problem. Here, for simplicity we assume the ordered profile \(u_1 \leq u_2 \ldots \leq u_n\). Efficiency dictates assigning the good to an agent with the highest \(u_i\). Note that \(v(N) = u_n\). Then, for any efficient, individually rational, and single valued solution we have \(\Sigma_{i \in N} \phi^i(N,u) = u_n\), and \(\phi^i \geq 0\) for all \(i \in N\).

\(^{27}\) Consider the following game: \(v(1) = 10, v(2) = 8, v(3) = 6, v(12) = 14, v(13) = 13, v(23) = 11, v(123) = 18\). Note that agent 1’s contribution to \(S = \{1,2\}\) is 6 while he contributes 7 to \(N\). It is easy to check that the Shapley value of this game is population monotonic.
Given any ordered problem \((N, u)\), by the convention that \(u_0 = 0\), Shapley solution is calculated as follows:

\[
Sh^i = \sum_{k=1}^{i} \frac{u_k - u_{k-1}}{|N|-(k-1)} \text{ for all } i \in N
\] (3.3)

**Definition 3.3:** A solution \(\varphi\) is *additively scale monotonic (ASM)* if for any \((N, u), (N, u') \in \varepsilon_1\) such that for some \(t \in \mathbb{R}_+\), \(u'_i = u_i + t\) for all \(i \in N\), we have \(\varphi^i(N, u') \geq \varphi^i(N, u)\) for all \(i \in N\).

**Lemma 3.1:** The Shapley solution is ASM while the egalitarian solution is not.

**Proof:** Without loss of generality, take any ordered problem \((N, u) \in \varepsilon_1\), and \(t \in \mathbb{R}_+\) so that \((N, u')\) is defined as in the statement above. Note that the order is preserved at the profile \(u'\). Then, for each \(j > 1\), \(u_j - u_{j-1} = u'_j - u'_{j-1}\), and \(u_i - u_0 + t = u'_i - u'_0\). Therefore, \(Sh^i(N, u') = Sh^i(N, u) + t/n\) for all \(i \in N\), by (3.3).

To see that the egalitarian solution is not ASM, consider the following profiles for \(n = 3\): \(u = (0,0,2)\), and \(u' = (1,1,3)\). Note that \(Eg^3(N, u) = 2 > 1 = Eg^3(N, u')\). \(\square\)

Additive scale monotonicity is a fairly weak property. A trivial example of a PM solution that is also ASM is the equal distribution of the efficient surplus \(u_n\) among the agents with the highest valuation.

### 3.6.1 The case \(|\Omega| = 2\)

In this case, we have a more clear way of partitioning the monotone preferences in a useful way for the purpose of our analysis. Here, the monotone preferences are either superadditive, i.e., \(u_i(\alpha) + u_i(\beta) \leq u_i(\alpha\beta)\), or subadditive, i.e., \(\max\{u_i(\alpha), u_i(\beta)\} \leq u_i(\alpha\beta) \leq u_i(\alpha) + u_i(\beta)\), or both (additively separable), i.e., \(u_i(\alpha) + u_i(\beta) = u_i(\alpha\beta)\). Also, subadditivity and submodularity coincides. Moreover, they are not only necessary but also sufficient for substitutability. Now, let \(\varepsilon_2^{sub}\) be the set of all problems such that \(u_i\) is monotone and subadditive for all \(i \in N\).
**Proposition 3.3:** For any \((N, \Omega, u) \in s^\text{sub}_2\) the induced TU game \(v\) is concave.

**Proof:** Let \((N, \Omega, u) \in s^\text{sub}_2\) and \(S \subseteq N\). It suffices to show that \(v(S, \cdot)\) satisfies substitutability. The only relevant case is \(A = \emptyset\), and \(B = \{\beta\}\). Hence, we need to show \(v(S, \{\alpha\beta\}) = v(S, \{\beta\}) \leq v(S, \{\alpha\})\). Note that \(v(S, \{\alpha\}) = \max_{i \in S} u_i(\alpha)\), and \(v(S, \{\alpha\beta\}) = u_i(\alpha) + u_j(\beta)\) for some distinct \(i, j \in S\) or \(v(S, \{\alpha\beta\}) = u_i(\alpha\beta)\) for some \(i \in S\). Then, by definition, subadditivity implies \(v(S, \{\alpha\beta\}) \leq \max_{i \in S} u_i(\alpha) + \max_{i \in S} u_i(\beta)\). □

**Corollary 3.2:** The Shapley solution and the egalitarian solution are PM on \(s^\text{sub}_2\).

Unlike the general case \(|\Omega| \geq 3\), here, even if some or all agents have superadditive preferences, we have population monotonic solutions. We now introduce some extra notation.

Fix a problem \((N, \Omega, u) \in s_2\) and let \(v(\cdot)\) be the induced TU game. For any \(i \in N\), define \(\bar{u}_i(\alpha\beta) \equiv \min\{u_i(\alpha\beta), u_i(\alpha) + u_i(\beta)\}\), while \(\bar{u}_i(c) = u_i(c)\) for \(c = \alpha, \beta\). Then, we have the perturbed problem \((N, \Omega, \bar{u})\) and the associated TU game \(\bar{v}(\cdot)\). The only difference between the two problems is the utilities of agents who have (strictly) superadditive utilities at the original problem. For such an agent the only difference is \(\bar{u}_i(\alpha\beta) = u_i(\alpha) + u_i(\beta) < u_i(\alpha\beta)\). Then, by construction, we have \((N, \Omega, \bar{u}) \in s^\text{sub}_2\). Now, define \(\tilde{u}_i \equiv \max(u_i(\alpha\beta) - \bar{v}(N), 0)\) , and \(\tilde{v}(N) \equiv \max_{i \in N} \tilde{u}_i\).

**Lemma 3.2:** For any \((N, \Omega, u) \in s_2\), we have \(v(N) = \bar{v}(N) + \tilde{v}(N)\).

**Proof:** Take any \((N, \Omega, u) \in s_2\). Consider first the case \(v(N) = u_j(\alpha) + u_k(\beta)\) for some distinct \(j, k \in N\). Then, \(u_j(\alpha\beta) \leq u_j(\alpha) + u_k(\beta)\) for all \(i \in N\), and also \(\bar{v}(N) = u_j(\alpha) + u_k(\beta)\). Hence, \(\tilde{u}_i = 0\) for all \(i \in N\) and \(\tilde{v}(N) = 0\). Now, consider the case \(v(N) = u_j(\alpha\beta)\) for some \(j \in N\). If \(u_j(\alpha\beta) = \bar{u}_j(\alpha\beta)\), again we have \(\bar{v}(N) = v(N)\) and \(\tilde{v}(N) = 0\). Now, let \(u_j(\alpha\beta) > \bar{u}_j(\alpha\beta)\). Then, we have \(\tilde{v}(N) = u_j(\alpha\beta) - \bar{v}(N)\). □
Consider the 4-person problem in Table 3.3 which clarifies our construction. Note that 
\[ v(N) = u_1(\alpha \beta) = 10, \quad \bar{v}(N) = u_2(\alpha) + u_3(\beta) = 8. \]
An efficient, individually rational, and single valued solution to the problem \((N, \Omega, u)\) is just a nonnegative distribution of \(v(N) = 10\) units of surplus to the 4 agents. We aim to write a solution as the summation of two solutions to two different problems. As the first component of our hybrid solution to the problem \((N, \Omega, u)\), we pick a solution \(\bar{\phi}(N, \Omega, \bar{u})\). Then, \(\bar{v}(N) = 8\) units of surplus is distributed by \(\bar{\phi}\). By Lemma 3.2, the remaining to distribute is \(\bar{v}(N)\). Note that by definition, \(\bar{v}(N) = \bar{u}_j\) for some \(j \in N\) and in that case \(\bar{u}_j \geq \bar{u}_i\) for all \(i \in N\). Also, \(\bar{u}_i \geq 0\) for all \(i \in N\). Then, we can think of distributing \(\bar{v}(N)\) as allocation of a single good where agents’ valuations are \(\bar{u}_i\). Let us call this problem \((N, \bar{u})\). Then, the remaining \(\bar{v}(N) = 2\) units of surplus is distributed by some \(\bar{\phi}(N, \bar{u})\).

**Definition 3.4:** Given any two solutions \(\phi\) on \(\varepsilon_2^{\text{sub}}\), and \(\bar{\phi}\) on \(\varepsilon_1\), \(\phi\) is a hybrid (of \(\phi\) and \(\bar{\phi}\)) solution on \(\varepsilon_2\) if for all \((N, \Omega, u) \in \varepsilon_2\) we have, \((\sigma, m) \in \varphi(N, \Omega, u)\) if \(u_i(\sigma, m_i) = \bar{\phi}^i(N, \Omega, \bar{u}) + \bar{\phi}^i(N, \bar{u})\) for all \(i \in N\).

Note that a hybrid solution is single valued and well-defined by Lemma 3.2.

**Theorem 3.1:** Let \(\phi\) be a hybrid solution such that for all \((N, \Omega, u) \in \varepsilon_2, i \in N, \phi^i = \bar{\phi}^i + \bar{\phi}^i\) where \(\bar{\phi}\) is a solution to \((N, \bar{u})\), and \(\phi\) is a solution to \((N, \bar{u})\).

(i) \(\phi\) is efficient, symmetric and continuous if both \(\bar{\phi}\) and \(\bar{\phi}\) are efficient, symmetric and continuous.

(ii) \(\phi\) is population monotonic if both \(\bar{\phi}\) and \(\bar{\phi}\) are population monotonic, and \(\bar{\phi}\) is ASM.

<table>
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*Table 3.3*
Proof: We skip the trivial argument for efficiency and symmetry. Let \( \tilde{v} \) and \( \hat{v} \) be continuous in \( \tilde{u} \) and \( \hat{u} \), respectively. By construction \( \tilde{u} \) is continuous in \( u \). Also, we know that \( \tilde{v}(N) = \max\{\max_{i \in J} \tilde{u}_i(\alpha) + \hat{u}_j(\beta), \max_i \tilde{u}_i(\alpha \beta)\} \) is a continuous function of \( \tilde{u} \), and hence, it is continuous in \( u \). Then, by definition \( \tilde{u} = \{\max(u_i(\alpha \beta) - \tilde{v}(N), 0)\}_{i \in N} \) is continuous in \( u \). Therefore, both \( \tilde{v} \) and \( \hat{v} \), and hence, \( \phi \) is continuous in \( u \).

Now, let \( \tilde{v} \) and \( \hat{v} \) be population monotonic, and \( \hat{v} \) be ASM. Take any \( (N \cup h, \Omega, u) \in \varepsilon_2 \). Note that by construction \( \tilde{u}_i \) is independent of the profile (the set of agents in the problem), and \( \tilde{u}_i = \tilde{u}_i^N \) for all \( i \in N \). Hence, we have \( \tilde{v}^i(N \cup h, \Omega, \tilde{u}) \leq \tilde{v}^i(N, \tilde{u}^N) \) for all \( i \in N \) as \( \tilde{v} \) is PM.

Let \( \tilde{u}(N) \) represent the profile derived from the problem \( (N, \Omega, u^N) \), and \( \tilde{u}^N \) be the restricted profile of \( \tilde{u} \) to the agents in \( N \). Here, \( \tilde{u}(N) \) is not necessarily equal to \( \tilde{u}^N \). Define \( K^0 \equiv \{i \in N: \tilde{v}(N \cup h) < u_i(\alpha \beta)\} \). Note that for \( i \in N \), \( \tilde{u}_i > 0 \) only if \( i \in K^0 \). By PM (and hence SAC) we have \( \tilde{v}^i(N \cup h, \tilde{u}) = 0 \) for all \( i \in N \setminus K^0 \). Hence, to complete the proof it suffices to show \( \tilde{v}^i(N \cup h, \tilde{u}) \leq \tilde{v}^i(N, \tilde{u}(N)) \) for all \( i \in K^0 \).

Consider first the case \( \tilde{v}(N \cup h) = \tilde{v}(N) \). Note that \( \tilde{u}_i = \tilde{u}_i(N) \) for all \( i \in N \), and we have the desired inequality as \( \tilde{v} \) is PM. Now, let \( \tilde{v}(N \cup h) - \tilde{v}(N) \equiv \Delta \tilde{v} > 0 \). Define \( K^1 \equiv \{i \in N: \tilde{v}(N \cup h) = u_i(\alpha \beta)\}, K^2 \equiv \{i \in N: 0 < \tilde{v}(N \cup h) - u_i(\alpha \beta) < \Delta \tilde{v}\} \), and \( K = K^0 \cup K^1 \). Then, \( \tilde{u}_i(N) = \tilde{u}_i + \Delta \tilde{v} \) for all \( i \in K, 0 < \tilde{u}_i(N) < \Delta \tilde{v} \) for all \( i \in K^2 \), and \( \tilde{u}_i(N) = 0 \) otherwise. Now, consider the problem \( (K, \tilde{u}^K) \). Note that PM implies the dummy property that if we add or remove agents with zero utility, allocation to the other agents does not change. Hence, we have \( \tilde{v}^i(N \cup h, \tilde{u}) = \tilde{v}^i(K \cup h, \tilde{u}^{K \cup h}) \leq \tilde{v}^i(K, \tilde{u}^K) \) for all \( i \in K \), by PM. Let \( \{i^1, i^2, ..., i^m\} \) be the partition of \( K^2 \) such that \( i^1 \) is the set of agents in \( K^2 \) with the highest \( u_i(\alpha \beta) \), \( i^2 \) is the agents with the 2nd highest \( u_i(\alpha \beta) \), etc. Also, define \( t_0 = \tilde{v}(N \cup h) - u_j(\alpha \beta) \) for \( j \in i^1 \), \( t_p = u_j(\alpha \beta) - u_k(\alpha \beta) \) for \( j \in i^p, k \in i^{p+1} \) and \( p < m \), and \( t_m = u_j(\alpha \beta) - \tilde{v}(N) \) for \( j \in i^m \).

Consider the problem \( (K \cup i^1, \tilde{u}^\prime) \) where \( \tilde{u}^\prime_i = \tilde{u}_i + t_0 \) for \( i \in K \) and \( \tilde{u}^\prime_i = 0 \) for \( i \in i^1 \). By ASM and the dummy property we have \( \tilde{v}^i(K \cup i^1, \tilde{u}^\prime) = \tilde{v}^i(K, \tilde{u}^K) \geq \tilde{v}^i(K, \tilde{u}^K) \) for all \( i \in K \).
Then, add $t_1$ to utilities of all agents in problem $(K \cup i^1, \hat{u}^1)$ and then add the agents in $i^2$ to the problem assigning them 0 utilities. By the same argument, no agent in $K$ is worse off compared to $\hat{\phi}^i(K \cup i^1, \hat{u}^1)$ and hence to $\hat{\phi}^i(K, \hat{u}^K)$. Recursively applying the same argument where at the last step we add $t_m$ to utilities of agents in $K \cup K^2$ and add the remaining agents $N \setminus (K \cup K^2)$ with zero utilities, we reach the problem $\hat{\phi}^i(N, \hat{u}(N))$. As at each step, none of the agents in $K$ gets worse off, we have $\hat{\phi}^i(N, \hat{u}(N)) \geq \hat{\phi}^i(K, \hat{u}^K) \geq \hat{\phi}^i(N \cup h, \hat{u})$ for all $i \in K$. \qed

**Corollary 3.3:** The hybrid Shapley solution $(\hat{Sh}(N, \Omega, u) = Sh(N, \Omega, \hat{u}) + Sh(N, \hat{u}))$ is efficient, symmetric, continuous and population monotonic.

The egalitarian solution is population monotonic for each component of the hybrid solution. However, the hybrid egalitarian solution is not PM.

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*Table 3.4*

Consider the problem in Table 3.4. Note that $\bar{u}(N) = 2$, and $\bar{u}(\{123\}) = 0$. As the egalitarian solution is in the SAC, agents 1, 2, and 3 get 0 at both problems $(N, \Omega, \bar{u})$ and $(\{123\}, \Omega, \bar{u}(\{123\}))$. Also, $E_1(N, \hat{u}) = 4$, and $E_1(\{123\}, \hat{u}(\{123\})) = 2$. Thus, agent 1’s final utility decreases when agent 4 leaves.

Finally, we show that ESG is not compatible with PM. Consider the problem in Table 3.5. For any population monotonic solution, we have $\varphi^1 + \varphi^3 \leq \nu(13) = 10$ and $\varphi^2 + \varphi^3 \leq \nu(23) = 10$. Also, $\varphi^1 + \varphi^2 + \varphi^3 = 20$. Therefore, $\varphi^3 = 0$, contradicting ESG.
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*Table 3.5*
References


