RICE UNIVERSITY

VIBRATION OF HINGED CIRCULAR ARCHES

by

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ABSTRACT

The governing differential equations for the vibration of uniform circular arches have been derived considering in addition to the usual actions the effect of rotatory inertia and shear deformation. A trial eigenvalue method is used for determining the natural frequencies and mode shapes. The Runge-Kutta fifth order integration technique is used in this method to perform the integration of the differential equations.

A detailed study has been made of the lowest eight vibration frequencies for a hinged arch with an angle of opening equal to ninety degrees. The effect of slenderness ratio and the influences of the rotatory inertia and shear deformation on frequencies are presented in detail in tables and curves. It is shown that the frequencies found neglecting shear and rotatory inertia can be considered in most regions to be either primarily extensional or flexural but that a coupling exists in certain transition regions. The effects of shear deformation and rotatory inertia, which are most important in and near the transition regions, are to smooth the frequency curves.
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CHAPTER I.

INTRODUCTION

1.1. Object and Scope

The object of this study is to obtain information on the free vibration characteristics of circular arches. Despite the fact that the theory has been available for many years there is a paucity of information on the natural frequencies and modes of vibration of arches, due primarily to the labor required to extract numerical information from the theory.

This thesis contains a derivation of the governing differential equations, the development of a numerical procedure for the solution of these equations, and a detailed study of the lowest eight vibration frequencies for a circular hinged arch with an included angle of 90 degrees.

The governing differential equations consider in addition to the usual actions the effects of rotatory inertia and shear deformation. When rotatory inertia and shear deformation are neglected the equations are consistent with Flugge's theory of cylindrical shells. The equations are derived by both the customary displacement method and also by an energy method. The equations are presented in such a way that it is easy to either consider or neglect rotatory inertia and shearing deformations independently.

A trial eigenvalue method is used for solving the differential equations and determining the frequencies. The Runge-Kutta fifth order integration technique is used in the method to perform the required integrations. The method has been programmed for an IBM 7040
computer. All solutions were obtained on this computer.

A detailed study has been made of the lowest eight vibration frequencies for a hinged arch with an angle of opening of ninety degrees. The effect of slenderness ratio and the influences of rotatory inertia and shear deformation on the frequencies are presented in detail in tables and curves. It is shown that the frequencies found neglecting shear and rotatory inertia can be considered in most regions to be either primarily extensional or primarily flexural, but that a coupling exists in certain transition regions. The effects of rotatory inertia and shear deformation seem to smooth the frequency curves. These effects are most important in and near the transition regions. Their effect is negligible when the slenderness ratio is very large. As has been reported previously, the effects of shear deformation far exceed the effects of rotatory inertia.

1.2. Acknowledgements

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1.3. Nomenclature

Latin Symbols.

- A: cross section area of arch
- a: radius of centroidal axis of arch
- $A_1$: potential energy due to bending and stretching, see Eq. (39.a)
- $A_2$: potential energy due to shear, see Eq. (39.b)
- $A_3$: translatory kinetic energy, see Eq. (39.c)
- $A_4$: rotatory kinetic energy, see Eq. (39.d)
- D: determinant, see Eqs. (66) or (72)
- E: Young's modulus
- F: non-dimensional frequency parameter equal to $mp^2a^2/EA$
- G: shearing modulus
- g: integrand of the Lagrangian, see Eq. (40)
- I: moment of inertia of cross section of arch
- k: non-dimensional dimension parameter, equal to $\frac{r^2}{a^2}$
- L: length of arch
- $L_1$: Lagrangian of the dynamic system, see Eq. (40)
- M: bending moment, see Fig. 2
- m: mass per unit length
- N: normal force, see Fig. 2
- p: angular frequency
- $p_\phi$: translatory inertia force per unit length in circumferential direction, see Eq. (18)
- $p_r$: translatory inertia force per unit length in radial direction, see Eq. (19)
radius of gyration of cross section of arch

rotatory inertia index, equals to unity if rotatory inertia is included, equals to zero if not

shear index; equals to unity if shear deformation is included; equals to zero if not

rotatory inertia force, see Eq. (17)

time variable

shear variable, equal to \(a\beta\)
circumferential displacement

radial displacement

radial coordinate

Greek Symbols

angle of opening of arch, see Fig. 1

shear deformation, see Fig. 1 and Eq. (7.a)

non-dimensional shear parameter, equal to \(\frac{\lambda G}{E}\)

mass per unit volume

extensional strain

constant of proportionality, equal to \(\frac{Q}{G A \beta}\)

normal stress on cross-section

shear stress on cross-section

angular coordinate, see Fig. 1

rotation of the tangent to the centroidal axis, see Fig. 3

rotation of cross-section, see Fig. 3
II. DERIVATION OF FUNDAMENTAL EQUATIONS

2.1. Equations of Equilibrium

Consider the in-plane, small vibrations of the prismatic circular arch shown in Fig. 1. The radius of the centroid of the cross section is denoted as $a$ and the angle of opening as $\alpha$. The angle from the left radius to any point is denoted as $\varphi$. Also shown on Fig. 1 are the positive directions of the radial and tangential displacements of the centroidal axis, $w$ and $v$. In the following the arch will be considered to lie in a vertical plane.

The arch normal cross-section is assumed to be symmetrical about a vertical center-line, as shown in Fig. 1. The vertical distance to any point from the centroid is equal to $z$, as shown. Since we are considering only vibration in the plane of the arch, there exists at each point on the cross-section only a normal stress $\sigma$ and a vertical shearing stress $\tau$. For considerations of equilibrium of the entire section, these stresses can be replaced by three stress resultants, a normal force $N$ acting through the centroid, a bending moment $M$ acting about the horizontal centroidal axis, and a shear $Q$ acting vertically through the centroid. They are defined by the following expressions

\[
N = \int_A \sigma \, dA \quad \text{(1.a)}
\]
\[
M = \int_A \sigma \, zdA \quad \text{(1.b)}
\]
\[
Q = \int_A \tau \, dA \quad \text{(1.c)}
\]
The positive directions are shown in Fig. 2.

Consider now the equilibrium of a small element of the ring. Acting on the element are the transverse shears, $Q$, the circumferential normal forces, $N$, the bending moments, $M$, the rotary inertia forces, $T$, and the translatory inertia forces $p_\phi$ and $p_r$, as shown in Fig. 2.

The equation of equilibrium of forces acting in the tangential direction gives the following equation.

$$N + \frac{\partial N}{\partial \phi} d\phi - N + Q \frac{d\phi}{2} + (Q + \frac{\partial Q}{\partial \phi} d\phi) \frac{d\phi}{2} + a p_\phi d\phi = 0.$$ 

Dividing by $d\phi$ and neglecting higher order terms, we obtain Eq. (2).

$$\frac{\partial N}{\partial \phi} + Q + a p_\phi = 0. \quad (2)$$

The equation of equilibrium of forces acting in the radial directions, obtained in a similar manner, is given by Eq. (3).

$$\frac{\partial Q}{\partial \phi} - N + a p_r = 0. \quad (3)$$

Finally, the equation of equilibrium of moments is given by Eq. (4).

$$\frac{1}{a} \frac{\partial M}{\partial \phi} - Q - T = 0. \quad (4)$$

2.2. Stress Resultants

The extensional strain and the rotation of the tangent to the
centroidal axis are given by Eqs. (5) and (6), which are well-known,

\[ \varepsilon = \frac{1}{a} (v' + w) \quad (5) \]

\[ \psi_1 = \frac{1}{a} (w' - v) \quad (6) \]

in which each prime denotes one derivative with respect to \( \varphi \).

When shearing deformations are considered normal plane cross sections do not remain plane and normal to the centroidal axis during deformation. However, an average rotation of the cross section can be found in the usual manner. The average rotation of the cross section will differ from the rotation of the tangent to the centroidal axis by an angle \( \beta \) (which is proportional to the shearing force) as shown in Fig. 3. Then, the rotation of the cross section is given by Eq. (7.a).

\[ \psi_2 = \frac{1}{a} (w' - v - a\beta) = \frac{1}{a} (w' - v - u) \quad (7.a) \]

where

\[ u = a\beta \quad (7.b) \]

The rate of rotation of the cross section is given by Eq. (8)

\[ \frac{d\psi_2}{ds} = K = \frac{1}{2} \left( w'' - v' - u' \right) . \quad (8) \]

The strain at a point at a distance \( z \) from the centroid is given by the following equation:

\[ \varepsilon(z) = \frac{1}{1 + \frac{z}{a}} \left[ \varepsilon - zK \right] \]

or
\[
\varepsilon(z) = \frac{1}{1 + \frac{z}{a}} \left[ \frac{1}{a} (v' + w) - \frac{z}{a^2} (w'' - v' - u') \right]. 
\] (9)

The stress at a distance \( z \) from the centroid is, therefore as follows:

\[
\sigma(z) = E \varepsilon(z)
\]

\[
\sigma(z) = \frac{E}{1 + \frac{z}{a}} \left[ \frac{1}{a} (v' + w) - \frac{z}{a^2} (w'' - v' - u') \right] 
\] (10)

If \( \left| \frac{z}{a} \right| < 1 \), by the binomial expansion

\[
\frac{1}{1 + \frac{z}{a}} = 1 - \frac{z}{a} + \frac{z^2}{a^2} - \frac{z^3}{a^3} + \ldots 
\] (11)

Using this expansion one obtains the following useful approximate integrals.

\[
\int_A \frac{dA}{1 + \frac{z}{a}} = \int_A dA - \int_A \frac{z}{a} dA + \ldots \approx A + \frac{z}{a} 
\] (12.a)

\[
\int_A \frac{z}{1 + \frac{z}{a}} dA = \int_A z dA - \int_A \frac{z^2}{a} dA + \ldots \approx - \frac{1}{a} 
\] (12.b)

\[
\int_A \frac{z^2}{1 + \frac{z}{a}} dA = \int_A z^2 dA - \int_A \frac{z^3}{a} dA + \ldots \approx 1 
\] (12.c)

These approximations are more accurate for symmetrical sections than for unsymmetrical sections.

The normal force, \( N \), may be expressed as a function of the displacements by substituting in Eq. (1.a) the expression for \( \sigma(z) \).
obtained in Eq. (10), thus,
\[ N = E \int \left[ \frac{1}{a(1 + \frac{a}{b})} (v' + w) - \frac{Z}{a^2 (1 + \frac{a}{b})} (w'' - v' - u') \right] dA \]

and then, integrating over the cross section. Thus
\[ N = \frac{EA}{a} (v' + w) + \frac{EI}{a^3} (v' + w) + \frac{EI}{a^3} (w'' - v' - u') \]
or
\[ N = \frac{EA}{a} (v' + w) + \frac{EI}{a^3} (w'' + w - u') \]
or
\[ N = \frac{EA}{a} \left[ (v' + w) + k (w'' + w - u') \right] \tag{13} \]
in which
\[ k = \frac{I}{Aa^2} = \frac{r^2}{a^2} \tag{14} \]
and \( r \) is the radius of gyration, \( r^2 = I/A \). The corresponding expression for bending moments is obtained in a similar way, as detailed in the following steps.
\[ M = \int z \sigma \, dA \]
\[ M = E \int \left[ \frac{Z}{a(1 + \frac{a}{b})} (v' + w) - \frac{Z^2}{a^2 (1 + \frac{a}{b})} (w'' - v' - u') \right] dA \]
\[ M = - \frac{EI}{a^2} (v' + w) - \frac{EI}{a^2} (w'' - v' - u') \]
\[ M = - \frac{EI}{a^2} (w'' + w - u') \]
\[ M = EA k (w'' + w - u') \tag{15} \]
The shearing force \( Q \) is proportional to the cross-section area \( A \), the shearing modulus, \( G \), and the angle of distortion \( \theta \). Thus

\[ Q = \lambda G A \theta = \lambda G A u/a \quad (16) \]

in which \( \lambda \) is a constant of proportionality. For example, the value of \( \lambda \) for a rectangular section is 5/6. In an I beam \( \lambda \) is approximately equal to \( A_w/A \), where \( A_w \) is the area of the web. For rolled sections \( A_w/A \) ranges from about 0.2 to about 0.5.

2.3. D'Alembert Reversed Inertia Forces

The arch is assumed to be in a state of steady vibration with motion given by the equations

\[
\begin{align*}
\dot{w} &= w \sin pt \\
\dot{v} &= v \sin pt \\
\dot{\psi}_2 &= \dot{\psi}_2 \sin pt \\
N &= N \sin pt, \text{ etc}
\end{align*}
\]

The translatory inertia forces at the instant of maximum deflection are given by Eqs. (17) and (18).

\[
\begin{align*}
\dot{p}_v &= m \dot{v} \\
\dot{p}_w &= m \dot{w}, \quad (17) \\
\dot{p}_r &= m \dot{w}, \quad (18)
\end{align*}
\]

in which \( m \) is mass per unit length.

A rotatory inertia couple arises due to the rotation \( \dot{\psi}_2 \) of the cross section. The D'Alembert inertia force on each element \( dA \) of thickness \( ds \) is as follows.
Tangential inertia force = (γ · dA · ds)z \frac{d^2ψ_2}{dt^2}

where γ is the mass density. The couple of these forces is,

\[ \overrightarrow{\mathbf{T}} \, ds = - \, ds \cdot γ \left( \frac{d^2ψ_2}{dx^2} \right) \int_A Z^2 \, dA \]

where \( \overrightarrow{T} \) is the couple per unit length, positive in a counter clockwise sense. Hence,

\[ \overrightarrow{T} = -γ \int \frac{d^2ψ_2}{dt^2} \]

But

\[ γA = m. \]

Then

\[ \overrightarrow{T} = T \sin pt = -m \frac{d^2ψ_2}{dt^2} \]

The maximum value of the rotatory inertia couple is, therefore, as follows.

\[ T = m \frac{1}{A} p^2 \psi_1 = m \frac{1}{A} \frac{p^2}{a} (ω' - ν - u) \]

or

\[ T = mp^2 ak(ω' - ν - u) \quad (19) \]

The positive direction of T is shown in Fig. 2.
2.4. Governing Equations and Boundary Conditions, Set 1

Using Eqs. (13), (15), (16), (17), (18), and (19), one obtains the following equations from Eqs. (2), (3), and (4).

\[
\begin{align*}
\ddot{w} + (1 + \frac{1}{k}) \dot{w} + \frac{1}{k} v'' + \frac{F}{k} v - u'' + \frac{k}{k} u &= 0 \quad (20.a) \\
\ddot{w} + (1 + \frac{1}{k} - \frac{F}{k}) \dot{w} + \frac{1}{k} v' - (1 + \frac{F}{k}) u' &= 0 \quad (20.b) \\
\ddot{w} + (1 + F) \dot{w}' - F v - u'' + \left( \frac{F}{k} - F \right) u &= 0 \quad (20.c)
\end{align*}
\]

where

\[F = \frac{2 m a^2}{E A} \quad (20.d)\]

and

\[\Gamma = \lambda \frac{G}{E} \quad (20.e)\]

Differentiate Eq. (20.b) and subtract Eq. (20.a); then multiply by \(\frac{k}{\Gamma}\). We obtain,

\[\frac{F}{\Gamma} \dot{w} + \frac{F}{\Gamma} v + u'' + u = 0 \quad (20.f)\]

now add Eq. (20.a) to Eq. (20.f); we obtain Eq. (20.g).

\[
\ddot{w}'' + (1 + \frac{1}{k} + \frac{F}{\Gamma}) \dot{w}' + \frac{1}{k} v'' + \left( \frac{F}{k} + \frac{F}{\Gamma} \right) v + (1 + \frac{F}{k}) u = 0 \quad (20.g)
\]
Equation (20.a) minus Eq. (20.c) gives Eq. (20.h)

\[
\left(\frac{1}{k} - F\right)w' + \frac{1}{k} v'' + \left(\frac{F}{k} + F\right)v + F u = 0
\] (20.h)

Equation (20.g) minus Eq. (20.h) gives Eq. (20.i)

\[
w'' + \left(1 + F + \frac{F}{F'}\right)w' + \left(\frac{F}{F'} - F\right)v + \left(1 + \frac{F}{k} - F\right)u = 0
\] (20.i)

Equations (20.i), (20.b), and (20.f) are selected as the most convenient set of governing equations. They are rewritten below as Eqs. (21.a), (21.b), and (21.c).

\[
w'' + \left(1 + F + \frac{F}{F'}\right)w' + \left(\frac{F}{F'} - F\right)v + \left(1 + \frac{F}{k} - F\right)u = 0
\] (21.a)

\[
w'' + \left(1 + \frac{1}{k} - \frac{F}{k}\right)w + \frac{1}{k} v' - \left(1 + \frac{F}{k}\right)u' = 0
\] (21.b)

\[w' + v + \frac{F}{F'} u'' + \frac{F}{F'} u = 0
\] (21.c)

The boundary conditions for Eq. (21) are as follows. For hinged ends the basic conditions are as given in Eqs. (22).

\[
w = 0
\] (22.a)

\[v = 0
\] (22.b)

\[M = 0
\] (22.c)

By means of Eq. (15), Eq. (22c) implies the following.

\[w'' + w - u' = 0
\] (22.d)
Since \( w \) vanishes, by Eq. (22.a), Eq. (22.d) becomes,

\[
\begin{align*}
\frac{d^2w}{dx^2} - u' &= 0 \\
(22.e)
\end{align*}
\]

The hinged end boundary conditions are given by Eqs. (22.a), (22.b), and (22.e).

The basic boundary conditions for fixed ends are given in Eqs. (23).

\[
\begin{align*}
w &= 0 \\
v &= 0 \\
\psi_2 &= 0
\end{align*}
\]

(23.a) (23.b) (23.c)

By means of Eq. (7a), Eq. (23) implies

\[
\begin{align*}
w' - v - u &= 0 \\
(23.d)
\end{align*}
\]

By Eq. (23.b), we get

\[
\begin{align*}
w' - u &= 0 \\
(23.e)
\end{align*}
\]

The fixed end boundary conditions are given in Eqs. (23.a), (23.b) and (23.e).

2.5. **Governing Equations and Boundary Conditions, Set II**

In the following we eliminate \( u \) from the equations so that one set of equations can be used to treat all cases, including both shear deformation and rotatory inertia.

Equation (20.h) times \( k \) gives Eq. (24.a)

\[
(l-Fk)w' + v'' + (F+Fk)v + FK u = 0
\]

(24.a)
Differentiate Eq. (20.1) and add to Eq. (20.2); we obtain Eq. (24.2)

$$w^{'''} + (2 + F + \frac{F}{F})w'' + \left(1 + \frac{1}{k} - \frac{F}{F}\right)w + \left(\frac{1}{k} + \frac{F}{F} - F\right)v' - F u' = 0$$  \hspace{1cm} (24.2)

Eliminate $u$ from Eq. (20.1) and (24.1), we have

$$(1 + F k)w' + v'' + (F + F k)v - F k \left\{w^{'''} + \left(1 + \frac{1}{k} - \frac{F}{F}\right)w + \left(\frac{1}{k} + \frac{F}{F} - F\right)v' \right\} = 0$$ \hspace{1cm} (24.3)

Eliminate $u'$ from Eq. (20.2) and Eq. (24.2), and we have Eq. (24.4).

$$w^{'''} + (2 + F + \frac{F}{F})w'' + \left(1 + \frac{1}{k} - \frac{F}{F}\right)w + \left(\frac{1}{k} + \frac{F}{F} - F\right)v' - F k \left\{w^{'''} + \left(\frac{1}{k} + \frac{F}{F} - F\right)v' \right\} = 0$$ \hspace{1cm} (24.4)

We can rearrange Eqs. (24.3) and (24.4) in the following form:

$$w^{'''} = \left(\frac{F k}{k + F} - 2 - F - \frac{F}{F}\right)w'' + \left[\frac{F (1 + k - F)}{k + F} - \frac{1}{k} - \frac{F}{F}\right]w' +$$

$$+ \left(\frac{F}{k + F} - \frac{1}{k} - \frac{F}{F} + F\right)v' \hspace{1cm} (25.1)$$

and

$$v'' = \left(\frac{F k}{1 + \frac{F}{k} - F}\right)w'' + \left[\frac{F k (1 + \frac{F}{F} + F)}{1 + \frac{F}{k} - F} - 1 + F k\right]w' +$$

$$+ \left[\frac{F k (\frac{F}{F} - F)}{1 + \frac{F}{k} - F} - F - F k\right]v \hspace{1cm} (25.2)$$

The boundary conditions for hinged ends are as follows:

$$w = 0 \hspace{1cm} (26.1)$$

$$v = 0 \hspace{1cm} (26.2)$$

$$w'' - u' = 0 \hspace{1cm} (26.3)$$
By means of Eqs. (20.b) and (26.a), we can write Eq. (26.c) as
\[ w'' + \frac{1}{\Gamma} v' = 0 \] \hspace{1cm} (26.d)

Hence, Eqs. (26.a) (26.b) and (26.d) describe the hinged end conditions.

The boundary conditions for fixed ends are given in Eqs. (27).

\[ w = 0 \] \hspace{1cm} (27.a)
\[ v = 0 \] \hspace{1cm} (27.b)
\[ w' - u = 0 \] \hspace{1cm} (27.c)

By means of Eq. (20.j) and (27.b) we can write Eq. (27.c) as
\[ w' + \frac{k \Gamma}{2k\Gamma + \Gamma^2 + Fk} w''' = 0 \] \hspace{1cm} (27.d)

The vibration mode is either symmetrical or anti-symmetrical about midspan. These two conditions are as follows:

(a) Symmetrical vibration,
\[ w' = 0 \] \hspace{1cm} (28.a)
\[ w''' = 0 \] \hspace{1cm} (28.b)
\[ v = 0 \] \hspace{1cm} (28.c)

(b) Anti-symmetrical vibration,
\[ w = 0 \] \hspace{1cm} (29.a)
\[ w'' = 0 \] \hspace{1cm} (29.b)
\[ v' = 0 \] \hspace{1cm} (29.c)
2.6. **Governing Equations and Boundary Conditions, Set III**

Similar formulas neglecting shear deformations can be derived as follows: First, from Eqs. (13), (15), (17), (18), and (19).

\[
N = \frac{EA}{a} [(v' + w) + k(w'' + w)] \quad (30.a)
\]

\[
M = EA k (w'' + w) \quad (30.b)
\]

\[
P_{\phi} = m p^2 v \quad (30.c)
\]

\[
Pr = m p^2 w \quad (30.d)
\]

and

\[
T = m p^2 ak (w' - v) \quad (30.e)
\]

From Eq. (4), we have,

\[
Q = -\frac{EAk}{a} (w''' + w') - m p^2 ak (w' - v) \quad (30.f)
\]

Substituting Eqs. (30.a) through (30.f) into Eqs. (2) and (3), we have

\[
w''' = (-2 - F)w'' + (-1 - \frac{1}{k} + \frac{F}{k}) w + (F - \frac{1}{k}) v' \quad (31.a)
\]

and

\[
v'' = (-1 + Fk)w' - (F+Fk)v' \quad (31.b)
\]

When both shear deformation and rotatory inertia are neglected, we have the following equations:

\[
T = 0 \quad (32.a)
\]

and

\[
Q = -\frac{EAk}{a} (w''' + w') \quad (32.b)
\]
Substituting Eqs. (30.a) through (30.d), (32.a) and (32.b) into Eqs (2) and (3), we obtain Eqs. (33).

\[ w''' = -2w'' + (-1 - \frac{1}{k} + \frac{F}{k})w - \frac{1}{k}v' \]  \hspace{1cm} (33.a)

\[ v'' = -w' - Fv \]  \hspace{1cm} (33.b)

The boundary conditions for hinged ends are as follows:

\[ w = 0 \]  \hspace{1cm} (34.a)
\[ v = 0 \]  \hspace{1cm} (34.b)
\[ M = 0 \]  \hspace{1cm} (34.c)

By means of Eq. (30.b), Eq. (34.c) implies

\[ w'' + w = 0 \]

by Eq. (34.a), we get

\[ w'' = 0 \]  \hspace{1cm} (34.d)

The boundary conditions for fixed ends are as follows:

\[ w = 0 \]  \hspace{1cm} (35.a)
\[ v = 0 \]  \hspace{1cm} (35.b)

and

\[ w' = 0 \]  \hspace{1cm} (35.c)

2.7. Governing Equations and Boundary Conditions, Set IV

To summarize, we can combine all these cases in one set of formulas by means of indices \( R \) and \( S \). \( R \) or \( S \) is set equal to unity
when rotatory inertia or shear deformation is included, and set equal
to zero when rotatory inertia or shear deformation is not included.

The governing equations are as follows:

\[ W^{\prime\prime\prime} = \left( R \cdot S \cdot \frac{F_k}{k + k} - 2 - R \cdot F - S \cdot \frac{F}{r} \right) W^{\prime\prime} + \]
\[ + \left[ R \cdot S \cdot \frac{F (k + l - F)}{k + k} - \frac{k + l - F}{k} \right] W^{\prime} + \]
\[ + \left[ R \cdot S \cdot \frac{F}{k + k} - \frac{F}{k} - S \cdot \frac{F}{r} + R \cdot F \right] V' \]

\[ V^{\prime\prime} = \left( R \cdot S \cdot \frac{F_k}{1 + \frac{F}{k} - F} \right) W^{\prime\prime} \]
\[ + \left[ R \cdot S \cdot \frac{F_k (1 + \frac{F}{k} + F)}{1 + \frac{F}{k} - F} - 1 + R \cdot F_k \right] W' + \]
\[ + \left[ R \cdot S \cdot \frac{F_k (\frac{F}{k} - F)}{1 + \frac{F}{k} - F} - F - R \cdot F_k \right] V \]

The boundary conditions for hinged ends are given in Eqs. (37)

\[ w = 0 \quad (37.a) \]
\[ v = 0 \quad (37.b) \]
\[ w^{\prime\prime} - S \frac{1}{r} v' = 0 \quad (37.c) \]

The boundary conditions for fixed ends are given in Eqs. (38)

\[ w = 0 \quad (38.a) \]
\[ v = 0 \quad (38.b) \]
\[ w' + S \frac{\Gamma k}{2 \Gamma k + h^2 + F_k} w^{\prime\prime\prime} = 0 \quad (38.c) \]
2.8. **Energy Formulations**

The differential equations derived in the previous section can also be obtained by means of the calculus of variations, i.e.; by minimizing the Langragian, $L_1$, of the dynamical system. The potential energy due to bending and stretching is,

$$A_1 = \frac{\alpha}{2} \int_\phi^\beta \sigma^{(i)} \varepsilon^{(i)} \left( I + \frac{Z}{\alpha} \right) dA \, d\phi$$

or

$$A_1 = \frac{E \alpha}{2} \int_\phi^\beta \frac{1}{1 + \frac{Z}{\alpha}} \left[ \frac{1}{\alpha} \left( v' + w \right) - \frac{Z}{\alpha^2} (w'' - v - u') \right]^2 dA \, d\phi$$

or

$$A_1 = \frac{E \alpha}{2a} \int_\phi^\beta \left[ (v' + w)^2 + k (w'' + w - u')^2 \right] d\phi \quad (39.a)$$

The potential energy due to shear is as follows:

$$A_2 = \frac{\Lambda GA}{2 a} \int_\phi^\beta u^2 \, d\phi$$

or

$$A_2 = \frac{E AF}{2a} \int_\phi^\beta u^2 \, d\phi \quad (39.b)$$

The translatory kinetic energies are as follows:

$$A_3 = \frac{m a p^2}{2} \int_\phi^\beta [v^2 + w^2] \, d\phi$$

or

$$A_3 = \frac{E AF}{2a} \int_\phi^\beta [v^2 + w^2] \, d\phi \quad (39.c)$$
The rotatory kinetic energy is as follows:

\[ A_4 = \frac{m r^3 p^2 \dot{\alpha}}{2} \int_0^\alpha \dot{\psi}^2 \, d\varphi \]

or

\[ A_4 = \frac{EAkF}{2} \int_0^\alpha \left( w' - v - u \right)^2 \, d\varphi \] (39.d)

The Langragian of the dynamical system is defined as follows:

\[ L_1 = A_3 + A_4 - A_1 - A_2 \]

or

\[ L_1 = \frac{EAk}{2} \int_0^\alpha \left[ \frac{F}{k} (v^2 + w^2) + F (w' - v - u)^2 - \frac{1}{k} (v' + w)^2 - (w'' + w - u')^2 - \frac{F}{k} u^2 \right] \, d\varphi \] (40)

Denoting the integrand of the above equation by \( g \), we obtain the following Euler equations as a necessary condition to minimize \( L_1 \).

\[ \frac{d}{d\varphi} \left( \frac{\partial g}{\partial \dot{w}} \right) - \frac{\partial}{\partial w} \left( \frac{\partial g}{\partial \dot{w}} \right) + \frac{\partial g}{\partial \dot{w}} = 0 \] (41.a)

\[ \frac{d}{d\varphi} \left( \frac{\partial g}{\partial \dot{v}} \right) - \frac{\partial}{\partial v} \left( \frac{\partial g}{\partial \dot{v}} \right) = 0 \] (41.b)

\[ \frac{d}{d\varphi} \left( \frac{\partial g}{\partial \dot{u}} \right) - \frac{\partial}{\partial u} \left( \frac{\partial g}{\partial \dot{u}} \right) = 0 \] (41.c)
Substituting $G$ into Eqs. (41.a) to (41.c), we have

$$\begin{align*}
W''' + (2 + F)W'' + \left(1 + \frac{1}{k} - \frac{\alpha}{k}\right)W + \left(\frac{1}{k} - F\right)v' - u''' - (1 + F)u' &= 0 \\
(\frac{1}{k} - F)W' + \frac{1}{k}v'' + \left(\frac{F}{k} + F\right)v + Fu &= 0 \\
W''' + (1 + F)W' - Fv - u'' + \left(\frac{\alpha}{k} - F\right)u &= 0
\end{align*}$$

(42.a) (42.b) (42.c)

Differentiating Eq. (42.c) once, we have,

$$\begin{align*}
W''' + (1 + F)W'' - Fv' - u''' + \left(\frac{\alpha}{k} - F\right)u' &= 0
\end{align*}$$

(42.d)

Subtract Eq. (42.d) from Eq. (42.a). We get Eq. (42.e)

$$\begin{align*}
W'' + \left(1 + \frac{1}{k} - \frac{\alpha}{k}\right)W + \frac{1}{k}v' - (1 + \frac{\alpha}{k})u' &= 0
\end{align*}$$

(42.e)

Add Eqs. (42.b) and (41.c). We get (42.f)

$$\begin{align*}
W'' + \left(1 + \frac{1}{k}\right)W' + \frac{1}{k}v'' + \frac{F}{k}v - u'' + \frac{\alpha}{k}u &= 0
\end{align*}$$

(42.f)

Note that Eqs. (42.c), (42.e) and (42.f) are the same as Eqs. (20.c), (20.b) and (20.a). This means that the formulas derived by the two different approaches are consistent.
In the case of no shear deformation, the Lagrangian takes the form of Eq. (43).

\[
L_1 = \frac{EAk}{2a} \int_0^1 \left[ \frac{F}{k} (v^2 + w^2) + F (w' - v)^2 - \frac{1}{k} (v' + w)^2 - (w'' + w)^2 \right] d\phi
\]  

(43)

Euler's equations give

\[
w' + (2+F)w'' + \left(1 + \frac{1}{k} - \frac{F}{k} \right) w + \left(\frac{1}{k} - F \right) v' = 0
\]  

(44.a)

\[(1-Fk)w' + v'' + (F+Fk)v = 0\]  

(44.b)

which are consistent with Eqs. (31.a) and (31.b).

If both shear deformation and rotatory inertia are neglected, the Langrangian becomes

\[
L_1 = \frac{EAk}{2a} \int_0^1 \left[ \frac{F}{k} (v^2 + w^2) - \frac{1}{k} (v' + w)^2 - (w'' + w)^2 \right] d\phi
\]

(45)

In this case Euler's equations give the following equations:

\[
w'''' + 2w'' + (1 + \frac{1}{k} - \frac{F}{k}) w + \frac{1}{k} v' = 0
\]  

(46.a)

\[w' + v'' + Fv = 0\]  

(46.b)

which are consistent with Eqs. (33.a) and (33.b).
III REVIEW OF THE LITERATURE

Three methods have been used to study the natural frequencies of in-plane vibrations of circular arches: (1) the analytic solution of the differential equations of motion. (2) the Rayleigh-Ritz method and (3) the numerical solution of the differential equations of motion.

The first method was used by Waltking(1), Morley(2), Archer(3), Federhofer(4), and Buckens(5). The second method was used by Den Hartog(6), and Nelson(7). A numerical method was used by Eppink and Veletsos(8). The contributions are summarized below.

Waltking studied the extensional vibrations of hinged-ended uniform circular arches. In this paper, the effects of shear deformation, rotatory inertia and the variations of normal strain along the cross section on the natural frequencies were neglected. Therefore, in the derivation of equations, the author assumed

\[ \beta = u = 0 \]  
\[ T = 0 \]

and

\[ \varepsilon(z) = \varepsilon z K \]

Using Equations (5) and (8), we obtain Eq. (47.d) for the normal stress.

\[ \sigma^z = \frac{1}{a} (v' + w) - \frac{2}{a^2} (w'' - v') \]  

The normal force and the shear are given by the following expressions:

\[ N = \frac{EA}{a} (v' + w) \]
\[ M = \text{E}ak (\omega'' - \omega') \quad (47.f) \]

and the governing differential equation becomes

\[ v^{(\omega)} + (2 + F) v^{(\omega)} + \left( 1 - \frac{F}{k} - F \right) v'' + \left( \frac{F}{k} - \frac{F^2}{k} \right) v = 0 \quad (47.g) \]

The solution of Eq. (47.g) can be taken in the following form:

\[ v = \sum_{i=1}^{3} \left( A_i \cos \eta_i \varphi + B_i \sin \eta_i \varphi \right) \quad (48.a) \]

Then the \( \eta_i \) are the solutions of the following equation,

\[ \eta_i^6 - (2 + F) \eta_i^4 + \left( 1 - \frac{F}{k} - F \right) \eta_i^2 - \left( \frac{F}{k} - \frac{F^2}{k} \right) = 0 \quad (48.b) \]

The author gave the following approximation,

\[ \eta_i^2 \approx \frac{3}{2} + \sqrt{\frac{F}{k}} \quad (49.a) \]

\[ \eta_i^2 \approx \frac{3}{2} - \sqrt{\frac{F}{k}} \quad (49.b) \]

\[ \eta_3^2 \approx F - 1 \quad (49.c) \]

Using the boundary conditions at one end and at the midspan of the arch, the author obtained frequency equations for the case of symmetrical and anti-symmetrical vibrations of the hinged-ended arches. The frequency equation for symmetrical vibration is,

\[ \varepsilon_1 \tan \frac{\eta_1}{2} + \varepsilon_2 \tan \frac{\eta_2}{2} + \varepsilon_3 \tan \frac{\eta_3}{2} = 0 \quad (50.a) \]
and for anti-symmetrical vibration

\[ e_1 \cot \frac{\eta x}{2} + e_2 \cot \frac{\eta x}{2} + e_3 \cot \frac{\eta x}{2} = 0 \] (50.b)

where

\[ e_1 = \eta_2 \eta_3 (\eta_2^3 - \eta_3^2) \left[ 1 + k \left( \eta_1^4 - \frac{F}{k} \right) \right] \] (50.c)

\[ e_2 = \eta_3 \eta_1 (\eta_3^2 - \eta_1^3) \left[ 1 + k \left( \eta_2^4 - \frac{F}{k} \right) \right] \] (50.d)

\[ e_3 = \eta_1 \eta_2 (\eta_1^2 - \eta_2) \left[ 1 + k \left( \eta_3 - \frac{F}{k} \right) \right] \] (50.e)

The mode shapes and the curves of frequencies were given for varying parameters \( \frac{L_0}{k} \) and \( \frac{f_0}{L_0} \), where \( L_0, f_0 \) were the span length and the height of the arches.

Morley considered the inextensible vibrations of cut circular rings. In his paper the effects of shear deformation, rotatory inertia and extension of the centroidal axis were neglected, i.e.,

\[ \beta = 0 \] (51.a)

\[ T = 0 \] (51.b)

and

\[ w + v' = 0 \] (51.c)

Also, the curved beam effect was neglected so that the expressions for stress, moments, etc., are exactly like those of Waltking. With these assumptions, the governing differential equation takes the following
For symmetrical vibrations, the solution is given by Eq. (53).

\[ M = \sum_{i=1}^{3} b_i \cos \theta_i \phi \]  \hspace{1cm} (53)

The characteristic equation is

\[ \eta_i^6 - 2 \eta_i^4 + \left( 1 - \frac{F}{k} \right) \eta_i^2 - \frac{F}{k} = 0 \]  \hspace{1cm} (54)

Boundary conditions of \( M(\pi) = 0, Q(\pi) = 0, \) and \( N(\pi) = 0 \) gave the following determinantal equation:

\[
\begin{vmatrix}
1 & 1 & 1 \\
\eta_1 (\eta_1^2-1)^2 \tan \eta_1 \pi & \eta_2 (\eta_2^2-1)^2 \tan \eta_2 \pi & \eta_3 (\eta_3^2-1)^2 \tan \eta_3 \pi \\
(\eta_1^2-1)^2 & (\eta_2^2-1)^2 & (\eta_3^2-1)^2
\end{vmatrix} = 0 \]  \hspace{1cm} (55)

Equation (54) gave

\[ \eta_1^2 + \eta_2^2 + \eta_3^2 = 1 \]  \hspace{1cm} (56.a)

\[ \eta_1^2 \eta_2^2 + \eta_2^2 \eta_3^2 + \eta_3^2 \eta_1^2 = 1 - \frac{F}{k} \]  \hspace{1cm} (56.b)

\[ \eta_1^2 \eta_2^2 \eta_3^2 = F \]  \hspace{1cm} (56.c)
So that

\[ \eta_2^2 = \frac{2 - \eta_i^2}{2} \left\{ |1 - \left| 1 - \frac{4 \left(1 - \eta_i^2\right)^2}{(1 + \eta_i)(2 - \eta_i^2)} \right|^{\frac{1}{2}} \right\} \]

or

\[ \eta_2^2 \approx - \eta_i^2 + 3 \]  \hspace{1cm} (57.a)

and

\[ \eta_3^2 = \frac{2 - \eta}{2} \left\{ |1 - \left| 1 - \frac{4 \left(1 - \eta_i^2\right)^2}{(1 + \eta_i)(2 - \eta_i^2)} \right|^{\frac{1}{2}} \right\} \]

or

\[ \eta_3^2 \approx - 1 \]  \hspace{1cm} (57.b)

The determinantal equation reduces to the following form.

\[ \eta_1 \tan \eta_1 \pi \approx \eta_2 \tan \eta_2 \pi \]  \hspace{1cm} (57.c)

Using Eqs. (57.a), (57.b), and (57.c), the author found

\[ \eta_1 \approx \left( n - \frac{1}{4} \right) + \frac{3}{4\pi (n - \frac{1}{4})^2} \]  \hspace{1cm} (57.d)

where \( n \) is an integer. Hence \( n_2, n_3 \) and \( F \) could be obtained.

A similar solution was made for the anti-symmetrical vibrations.

The frequencies of the first ten modes for both cases were given in the paper.

Philipson\(^{(9)}\) extended the classical analysis of Love and Waltking
to the extensional vibration, including rotatory inertia. He found that
the rotatory terms neglected in Waltking's paper are of the same order
of magnitude as the extensional terms and that Love's equation is valid
if the following criterion holds,

\[ F \ll 1. \]

Archer studied the inextensible vibrations of a fixed-ended
arch. He neglected shear deformations and rotatory inertia, but in-
cluded a damping effect. He assumed the centerline to be inextensional.

The equations of equilibrium became

\[
\frac{\partial N}{\partial \varphi} + Q = ma \frac{\partial^2 v}{\partial t^2} + K a \frac{\partial v}{\partial t} \quad (58. a)
\]

\[
\frac{\partial Q}{\partial \varphi} - N = ma \frac{\partial^2 w}{\partial t^2} + K a \frac{\partial w}{\partial t} \quad (58. b)
\]

\[
\frac{1}{a} \frac{\partial M}{\partial \varphi} = Q \quad (58. c)
\]

where \( K \) is the viscous damping coefficient.

The moment-curvature relation was as follows:

\[
M = - \frac{EI}{a^2} (w'' + w) \quad (58. d)
\]

Finally, the governing differential equation was

\[
\frac{EI}{a^4} \left( \frac{\partial^6 v}{\partial \varphi^6} + 2 \frac{\partial^4 v}{\partial \varphi^4} + \frac{\partial^2 v}{\partial \varphi^2} \right) = ma \frac{\partial^2 v}{\partial t^2} \left( v - \frac{\partial^2 v}{\partial \varphi^2} \right) + K \frac{\partial}{\partial t} \left( v - \frac{\partial^2 v}{\partial \varphi^2} \right) \quad (59. a)
\]

Set

\[
v(\varphi, t) = \sum_n V_n(\varphi) T_n(t)
\]
Equation (59) can then be expressed as follows:

$$\frac{EI}{ma^4} \left( \frac{\partial^6 v}{\partial q^6} + 2 \frac{\partial^4 v}{\partial q^4} + \frac{\partial^2 v}{\partial q^2} \right) = \rho_n \left( \frac{\partial^2 \bar{v}_n}{\partial t^2} - \bar{V}_n \right)$$  \hspace{1cm} (59.b)

$$\frac{\partial^2 \bar{v}_n}{\partial t^2} + \frac{k}{m} \frac{\partial \bar{v}_n}{\partial t} + \rho_n^2 \bar{V}_n = 0$$  \hspace{1cm} (59.c)

Equation (60.a) was solved for fixed end boundary conditions at each end. A six by six determinant was set equal to zero for calculating frequencies. The four lowest frequencies for \( a = 180, 234, 270, 324, \) and 360 degrees are tabulated. The author also solved the problem of a semi-circular fixed-ended arch initially at rest in a medium when one end is suddenly displaced tangentially by a small amount.

Buckens studied the flexural vibrations of a complete ring by taking the effects of extensibility, shear deformation and rotatory inertia into account and computed the deviations from the classical frequency formula. He shows that the shear effect plays a predominant role in the deviations from the classical formula, and is more important than the curvature effect and rotatory inertia. Extensibility influence was found to be small.

Federhofer derived the differential equations including shear deformation and rotatory inertia by minimizing the Lagrangian of the corresponding dynamical system. His Lagrangian was the same as Eq. (40).

Federhofer studied the solution of the complete ring. The solution of the differential equation is given by Eq. (48.a). Substituting this solution into the differential equation, he obtained
the frequency equation of a ring as follows:

\[ p^2 = \frac{p^2_0}{1 + e_0 + e_1 + e_2} \]  \hspace{1cm} (60.a)

where \( p_0 \) is the frequency obtained by R. Hoppe for a ring neglecting extension of the centerline of the ring, rotatory inertia and shear deformation

\[ p^2_0 = \frac{\eta^2 (\eta^2 - 1)^2}{\eta^2 + 1} \frac{E I}{m a^4} \]  \hspace{1cm} (60.b)

and \( e_0, e_1, e_2 \) are the influences of the extension of the centerline, rotatory inertia and shear deformation.

The author gave frequencies for the first ten modes for rings in which \( k \) equaled \( \frac{1}{1000} \) and \( \frac{1}{300} \) and \( \Gamma = \frac{1}{3} \). More discussions and results are given in Federhofer's book.\(^{(10)}\)

Den Hartog derived approximate formulas for the lowest frequency of vibration in the first extensional and first inextensional modes for hinged-ended and fixed-ended circular arches. The effects of shear deformation and rotatory inertia were neglected. The potential energy for the inextensional case was

\[ P.E. = \frac{E I}{2a^3} \int_{\phi_0}^{\phi} \left( w + \frac{\partial^2 w}{\partial \phi^2} \right)^2 d\phi \]

and for the extensional case was,

\[ P.E. = \frac{E A}{2a} \int_{\phi_0}^{\phi} \left( w + \frac{\partial v}{\partial \phi} \right)^2 d\phi + \frac{E I}{2a^3} \int_{\phi_0}^{\phi} \left( w + \frac{\partial^2 w}{\partial \phi^2} \right)^2 d\phi \]
The kinetic energy for both cases was as follows:

$$k.E. = \frac{ma}{2} \int_{0}^{\alpha} \left( \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 v}{\partial t^2} \right) d\phi$$

For each case the author, using a Rayleigh-Ritz approach, represented $w$ and $v$ in a finite series which satisfied boundary conditions and constraint conditions. Curves were given for frequency versus the parameters $\frac{a}{r}$ and angle of opening $a$.

Nelson considered the extensional vibrations of hinged-ended circular arches. The effects of shear deformation, rotatory inertia and variations of normal stress along the cross section were neglected. The potential energy was assumed as follows:

$$P.E. = \frac{EI}{2A} \int_{0}^{\alpha} (w'' - v')^2 d\phi + \frac{EA}{2A} \int_{0}^{\alpha} (w'' + v')^2 d\phi$$

This expression is based upon neglect of the curved beam effect just as was Waltking's work. The Rayleigh-Ritz method was used in conjunction with Lagrangian multipliers to obtain natural frequency equations in the form of infinite series. The curves of frequency versus angle of opening $\alpha$ with fixed $\frac{a\gamma}{r\pi}$ were given and the accuracy of the inextensional approximation and frequency cross over between symmetric and anti-symmetric mode shapes were discussed. Very limited data was given.

Eppink and Veletsos developed a discrete model analysis for determining the natural frequencies and buckling loads of circular arches. The effects of shear deformation, rotatory inertia and the curved beam effect were neglected. The authors approximated the continuous arch by a series of bars and joints which had a finite number of degrees of freedom. By comparing the solutions with Waltking and
Den Hartog it was shown that excellent results could be obtained with relatively few bars. The primary purpose of this work was to develop an arch analysis which could be used to determine the elastic and inelastic response of arch ribs to transient dynamic forces.
IV. NUMERICAL METHOD

The governing differential equations (36.a) and (36.b) can be split into six first order differential equations as follows:

\[ w' = WA \] (61.a)
\[ WA' = WB \] (61.b)
\[ WB' = WC \] (61.c)

\[
WC' = \left( R \cdot S \cdot \frac{F_k}{\Gamma + k} - 2 - R \cdot F - S \cdot \frac{F}{\Gamma} \right) WB + \\
+ \left[ R \cdot S \cdot \frac{F \left( 1 + k - F \right)}{\Gamma + k} - \frac{1 + k - F}{k} \right] w + \\
+ \left( R \cdot S \cdot \frac{F}{\Gamma + k} - \frac{1}{k} - S \cdot \frac{F}{\Gamma} + R \cdot F \right) VA \] (61.d)

\[ v' = VA \] (62.a)

\[
VA' = \left( R \cdot S \cdot \frac{F_k}{1 + \frac{k}{F} - F} \right) WC + \\
+ \left[ R \cdot S \cdot \frac{F_k \left( 1 - F - F \right)}{1 + \frac{k}{F} - F} + R \cdot F_k \right] WA + \\
+ \left[ R \cdot S \cdot \frac{F_k \left( F - F \right)}{1 + \frac{k}{F} - F} - R \cdot F_k - F \right] v \] (62.b)

Among the six variables at the initial end, we know three of them. For example, for hinged-ended arches we have,

\[ w = 0 \] (63.a)
\[ v = 0 \] (63.b)
\[ WB - S \cdot \frac{1}{\Gamma} \cdot VA = 0 \] (63.c)

and for fixed ended arches, we have,

\[ w = 0 \] (64.a)
\[ v = 0 \] (64.b)
\[ WA + S \cdot \frac{F_k}{2\Gamma k + 1^2 + F_k} WC = 0 \] (64.c)
The remaining three variables are not known. We assume three sets of independent initial conditions as follows:

<table>
<thead>
<tr>
<th>Set No.</th>
<th>w</th>
<th>WA</th>
<th>WB</th>
<th>WC</th>
<th>v</th>
<th>VA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>sl</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

*Initial Conditions For Hinged-Ended Arches*

Thus, the problem is decomposed into a linear combination of three linearly independent homogeneous initial value problems with initial conditions given. We assume a frequency $F$ and integrate numerically Eqs. (61) and (62) three times using the three sets of initial conditions as described in the tables above. The numerical method we use here is the Runge-Kutta's method which is described in the following table.
\[ y = f(x_1) \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
<th>( k_\mu = hf(x_1, y) )</th>
<th>Correction</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0 )</td>
<td>( y_0 )</td>
<td>( k_1 )</td>
<td></td>
</tr>
</tbody>
</table>
| \( x_0 + \frac{h}{2} \) | \( y_0 + \frac{k_1}{2} \) | \( k_2 \) | \( k = \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \)
| \( x_0 + \frac{h}{2} \) | \( y_0 + \frac{k_2}{2} \) | \( k_3 \) | |
| \( x_0 + h \) | \( y_0 + k_3 \) | \( k_4 \) | |
| \( x_1 = x_0 + h \) | \( y_1 = y_0 + k \) | | |

**Runge-Kutta Scheme For First Order Differential Equations**

Assume the first set of initial conditions are \( W_1(0), \) \( WA_1(0), \) \( WB_1(0), \) \( WC_1(0), \) \( VL(0), \) \( VA_1(0), \) using the Runge-Kutta numerical integration method we can evaluate \( W_1(\phi), \) \( WA_1(\phi), \) \( WB_1(\phi), \) \( WC_1(\phi), \) \( VL(\phi), \) \( VA_1(\phi) \) at any coordinate \( \phi \).

\[
\begin{align*}
W_1(0) & \quad W_1(\phi) \\
WA_1(0) & \quad WA_1(\phi) \\
WB_1(0) & \quad WB_1(\phi) \\
WC_1(0) & \quad WC_1(\phi) \\
VL(0) & \quad VL(\phi) \\
VA_1(0) & \quad VA_1(\phi)
\end{align*}
\]

Similarly for the second and third sets of initial conditions.

\[
\begin{align*}
W_2(0) & \quad W_2(\phi) \\
WA_2(0) & \quad WA_2(\phi) \\
WB_2(0) & \quad WB_2(\phi) \\
WC_2(0) & \quad WC_2(\phi) \\
V_2(0) & \quad V_2(\phi) \\
VA_2(0) & \quad VA_2(\phi)
\end{align*}
\]
If the frequency is a characteristic value, the conditions at the midspan \((\omega = \frac{\alpha}{2})\) satisfy the following equations.

**Anti-symmetrical vibrations:** The boundary conditions at midspan are as follows:

\[
\begin{align*}
w\left(\frac{\alpha}{2}\right) &= W1\left(\frac{\alpha}{2}\right) + C2 \cdot W2\left(\frac{\alpha}{2}\right) + C3 \cdot W3\left(\frac{\alpha}{2}\right) = 0 \\
W_b\left(\frac{\alpha}{2}\right) &= W_b1\left(\frac{\alpha}{2}\right) + C2 \cdot W_b2\left(\frac{\alpha}{2}\right) + C3 \cdot W_b3\left(\frac{\alpha}{2}\right) = 0 \\
V_a\left(\frac{\alpha}{2}\right) &= V_a1\left(\frac{\alpha}{2}\right) + C2 \cdot V_a2\left(\frac{\alpha}{2}\right) + C3 \cdot V_a3\left(\frac{\alpha}{2}\right) = 0
\end{align*}
\]

In the above equations \(C2\) and \(C3\) are proportionality constants.

For a solution of this set of three linear homogeneous equations to exist, it is necessary that the determinant of the coefficients vanish.

\[
D = \begin{vmatrix}
w1\left(\frac{\alpha}{2}\right) & w2\left(\frac{\alpha}{2}\right) & w3\left(\frac{\alpha}{2}\right) \\
w_b1\left(\frac{\alpha}{2}\right) & w_b2\left(\frac{\alpha}{2}\right) & w_b3\left(\frac{\alpha}{2}\right) \\
v_a1\left(\frac{\alpha}{2}\right) & v_a2\left(\frac{\alpha}{2}\right) & v_a3\left(\frac{\alpha}{2}\right)
\end{vmatrix} = 0 \quad (66)
\]

If the determinant \(D\) above vanishes and if,

\[
\begin{vmatrix}
w2\left(\frac{\alpha}{2}\right) & w3\left(\frac{\alpha}{2}\right) \\
w_b2\left(\frac{\alpha}{2}\right) & w_b3\left(\frac{\alpha}{2}\right)
\end{vmatrix} \neq 0 \quad (67)
\]
then the proportionality constants can be determined by the following equation.

\[
C_2 = \frac{-W_1\left(\frac{\alpha}{2}\right), W_3\left(\frac{\alpha}{2}\right)}{-W_B1\left(\frac{\alpha}{2}\right), WB3\left(\frac{\alpha}{2}\right)}
\]

\[
C_3 = \frac{W_2\left(\frac{\alpha}{2}\right), -W_1\left(\frac{\alpha}{2}\right)}{W_B2\left(\frac{\alpha}{2}\right), -W_B1\left(\frac{\alpha}{2}\right)}
\]

If the determinant \( D \) vanishes but

\[
\begin{vmatrix}
W_2\left(\frac{\alpha}{2}\right), & W_3\left(\frac{\alpha}{2}\right) \\
W_B2\left(\frac{\alpha}{2}\right), & WB3\left(\frac{\alpha}{2}\right)
\end{vmatrix} = 0
\]

(69.a)
then the proportionality constants can be determined by the following equations.

\[
C_2 = \frac{\begin{vmatrix} -W_1(\frac{\alpha}{2}) & W_3(\frac{\alpha}{2}) \\ -V_{A1}(\frac{\alpha}{2}) & V_{A3}(\frac{\alpha}{2}) \\ W_2(\frac{\alpha}{2}) & W_3(\frac{\alpha}{2}) \\ V_{A2}(\frac{\alpha}{2}) & V_{A3}(\frac{\alpha}{2}) \end{vmatrix}}{\begin{vmatrix} -W_1(\frac{\alpha}{2}) & W_3(\frac{\alpha}{2}) \\ -V_{A1}(\frac{\alpha}{2}) & V_{A3}(\frac{\alpha}{2}) \\ W_2(\frac{\alpha}{2}) & W_3(\frac{\alpha}{2}) \\ V_{A2}(\frac{\alpha}{2}) & V_{A3}(\frac{\alpha}{2}) \end{vmatrix}} \]  

\[
C_3 = \frac{\begin{vmatrix} W_2(\frac{\alpha}{2}) & -W_1(\frac{\alpha}{2}) \\ V_{A2}(\frac{\alpha}{2}) & -V_{A1}(\frac{\alpha}{2}) \\ W_2(\frac{\alpha}{2}) & W_3(\frac{\alpha}{2}) \\ V_{A2}(\frac{\alpha}{2}) & V_{A3}(\frac{\alpha}{2}) \end{vmatrix}}{\begin{vmatrix} W_2(\frac{\alpha}{2}) & -W_1(\frac{\alpha}{2}) \\ V_{A2}(\frac{\alpha}{2}) & -V_{A1}(\frac{\alpha}{2}) \\ W_2(\frac{\alpha}{2}) & W_3(\frac{\alpha}{2}) \\ V_{A2}(\frac{\alpha}{2}) & V_{A3}(\frac{\alpha}{2}) \end{vmatrix}} \]  

**Symmetrical vibrations:** The boundary conditions at midspan are as follows:

\[
W_{A}(\frac{\alpha}{2}) = W_{A1}(\frac{\alpha}{2}) + C_2 \cdot W_{A2}(\frac{\alpha}{2}) + C_3 \cdot W_{A3}(\frac{\alpha}{2}) = 0 \]  

\[
W_{C}(\frac{\alpha}{2}) = W_{C1}(\frac{\alpha}{2}) + C_2 \cdot W_{C2}(\frac{\alpha}{2}) + C_3 \cdot W_{C3}(\frac{\alpha}{2}) = 0 \]  

\[
v(\frac{\alpha}{2}) = v_1(\frac{\alpha}{2}) + C_2 \cdot v_2(\frac{\alpha}{2}) + C_3 \cdot v_3(\frac{\alpha}{2}) = 0 \]
As before, if the correct frequency has been assumed the following determinant will vanish.

\[
D = \begin{vmatrix}
W_{A1}(\frac{\alpha}{2}), & W_{A2}(\frac{\alpha}{2}), & W_{A3}(\frac{\alpha}{2}) \\
W_{C1}(\frac{\alpha}{2}), & W_{C2}(\frac{\alpha}{2}), & W_{C3}(\frac{\alpha}{2}) \\
V_{1}(\frac{\alpha}{2}), & V_{2}(\frac{\alpha}{2}), & V_{3}(\frac{\alpha}{2})
\end{vmatrix} = 0 \quad (72)
\]

If \( D = 0 \) and if

\[
\begin{vmatrix}
W_{A2}(\frac{\alpha}{2}), & W_{A3}(\frac{\alpha}{2}) \\
V_{2}(\frac{\alpha}{2}), & V_{3}(\frac{\alpha}{2})
\end{vmatrix} \neq 0 \quad (73)
\]

then the proportionality constants can be found by the following equation.

\[
C_{2} = \frac{\begin{vmatrix}
-W_{A1}(\frac{\alpha}{2}), & -W_{A3}(\frac{\alpha}{2}) \\
-V_{1}(\frac{\alpha}{2}), & V_{3}(\frac{\alpha}{2}) \\
W_{A2}(\frac{\alpha}{2}), & W_{A3}(\frac{\alpha}{2}) \\
V_{2}(\frac{\alpha}{2}), & V_{3}(\frac{\alpha}{2})
\end{vmatrix}}{\begin{vmatrix}
W_{A2}(\frac{\alpha}{2}), & W_{A3}(\frac{\alpha}{2}) \\
V_{2}(\frac{\alpha}{2}), & V_{3}(\frac{\alpha}{2})
\end{vmatrix}} \quad (74.a)
\]

\[
C_{3} = \frac{\begin{vmatrix}
W_{A2}(\frac{\alpha}{2}), & -W_{A1}(\frac{\alpha}{2}) \\
V_{2}(\frac{\alpha}{2}), & -V_{1}(\frac{\alpha}{2}) \\
W_{A2}(\frac{\alpha}{2}), & W_{A3}(\frac{\alpha}{2}) \\
V_{2}(\frac{\alpha}{2}), & V_{3}(\frac{\alpha}{2})
\end{vmatrix}}{\begin{vmatrix}
W_{A2}(\frac{\alpha}{2}), & W_{A3}(\frac{\alpha}{2}) \\
V_{2}(\frac{\alpha}{2}), & V_{3}(\frac{\alpha}{2})
\end{vmatrix}} \quad (74.b)
\]
However, if

\[
\begin{vmatrix}
\text{WA}_2(\frac{\alpha}{2}) & \text{WA}_3(\frac{\alpha}{2}) \\
\text{V}_2(\frac{\alpha}{2}) & \text{V}_3(\frac{\alpha}{2})
\end{vmatrix} = 0
\]

\[(75.a)\]

but

\[
\begin{vmatrix}
\text{WA}_2(\frac{\alpha}{2}) & \text{WA}_3(\frac{\alpha}{2}) \\
\text{WC}_2(\frac{\alpha}{2}) & \text{WC}_3(\frac{\alpha}{2})
\end{vmatrix} \neq 0
\]

\[(75.b)\]

then the proportionality constants can be determined from the following equations.

\[
C_2 = \frac{
\begin{vmatrix}
-\text{WA}_1(\frac{\alpha}{2}) & \text{WA}_3(\frac{\alpha}{2}) \\
-\text{WC}_1(\frac{\alpha}{2}) & \text{WC}_3(\frac{\alpha}{2})
\end{vmatrix}
}{
\begin{vmatrix}
\text{WA}_2(\frac{\alpha}{2}) & \text{WA}_3(\frac{\alpha}{2}) \\
\text{WC}_2(\frac{\alpha}{2}) & \text{WC}_3(\frac{\alpha}{2})
\end{vmatrix}
}\]

\[(76.a)\]

\[
C_3 = \frac{
\begin{vmatrix}
\text{WA}_2(\frac{\alpha}{2}) & -\text{WA}_1(\frac{\alpha}{2}) \\
\text{WC}_2(\frac{\alpha}{2}) & -\text{WC}_1(\frac{\alpha}{2})
\end{vmatrix}
}{
\begin{vmatrix}
\text{WA}_2(\frac{\alpha}{2}) & \text{WA}_3(\frac{\alpha}{2}) \\
\text{WC}_2(\frac{\alpha}{2}) & \text{WC}_3(\frac{\alpha}{2})
\end{vmatrix}
}\]

\[(76.b)\]

The eigenmode is determined from the following equations:

\[
w(\varphi) = W_1(\varphi) + C_2 \cdot W_2(\varphi) + C_3 \cdot W_3(\varphi)
\]

\[(77.a)\]

\[
v(\varphi) = V_1(\varphi) + C_2 \cdot V_2(\varphi) + C_3 \cdot V_3(\varphi)
\]

\[(77.b)\]
If the assumed frequency is not a characteristic value, D does not vanish. We note its sign, then assume another frequency by adding a small amount to the previous one and recalculate D. If D does not change sign, we keep trying again until D changes sign. If D changes sign, we know a characteristic value lies between the last two trial frequencies. Now using the linear interpolation method shown below, we can quickly converge to the characteristic value.

The criteria used for convergence to a solution is as follows: either the absolute value of the determinant is less than 0.0001

$$|D| < 0.0001$$

or two frequencies $F_1$ and $F_2$ for which the determinants are of opposite sign

$$D_1 D_2 < 0$$

are sufficiently close that

$$\left| \frac{F_2 - F_1}{F_1} \right| < 0.0001$$

If $F_1$ and $F_2$ do not satisfy either of these criteria a linear interpolation is made to determine a new trial frequency $F_3$,

$$F_3 = \frac{F_2 |D_1| + F_1 |D_2|}{|D_1| + |D_2|}$$

and the corresponding determinant $D_3$ is found. Then, the criteria above are applied to $D_3$ and to whichever of $D_1$ or $D_2$ is of the opposite sign.
If the criteria are not satisfied, a new value $F_4$ is found by linear interpolation, and so forth. Typical Variations of $D$ with $F$ and the variations at one of the transition regions is shown in Fig. 4.

The convergence of the numerical solution as the number of division points in the entire arch is increased as shown in Table 1. The number of divisions shown in the left column are for the entire arch. However, only half this number of divisions are used in the procedure as the solution is made from support to centerline. It can be seen in Table 1 that the frequencies converge uniformly as the number of divisions are increased. For this case the first mode frequencies converge to four figure accuracy with 24 divisions, the third mode frequencies converge to four figure accuracy in 40 to 60 divisions and the fourth frequency is very nearly converged with 60 divisions.

The figures tabulated and plotted in this report were computed using 40 divisions in the complete arch, or 20 divisions in the half. It can be seen that this number of divisions gives very high accuracy.

The flow chart and Fortran program for the numerical method used herein are given in Appendix 1.
V. PRESENTATION OF RESULTS

5.1. Solutions of Classical Theory

Solutions of the classical theory were obtained for hinged ended uniform circular arches with an angle of opening equal to ninety degrees and slenderness ratios varying from 12 to 380. These solutions are shown in Table 2 and Figs. 5 and 6. The figures show that the curves are a combination of plateaus and diagonal lines. The plateaus are related to flexural vibrations and the diagonal lines are related to extensional vibrations. The first four modes of antisymmetrical flexural vibrations with slenderness ratio equal to 377 are shown in Fig. 7. Note that all four modes are located on plateaus. The flexural modes are very similar to those in a straight beam. Figure 8 shows the flexural vibrations in the symmetrical case. The fact that the flexural modes on the plateaus with same height are the same is shown in Fig. 9. The extensional modes on the diagonal lines are studied in Figs. 10, 11, 12 and 13. The extensional modes turn out to be the same on the same diagonal lines except for the first extensional vibrations of the symmetrical case, which are the breathing modes, as shown in Fig. 12. The change of mode shapes in transition regions is shown in Fig. 14.

5.2. Approximate Formulas For Cases Neglecting Shear and Rotatory Inertia

(a) Flexural modes

Assume the arch is inextensible. The Lagrangian of the dynamical system is given in Eq. (78).

$$ L_1 = \int_0^\Delta \left[ \frac{F}{k} (\dot{v}^2 + w^2) - (\dot{w}'' + w')^2 \right] dq $$

(78)
We use the Rayleigh-Ritz method to find the approximate solutions.

(i) Anti-symmetric case:

Assume \( w = A \sin \frac{n \pi \rho}{\alpha} \)  

(79.a)

and

\[ v = A \frac{\kappa}{\varepsilon} \left( 1 - \cos \frac{n \pi \rho}{\alpha} \right) \]  

(79.b)

where

\[ n = 2, 4, 6, \ldots \]

These deflection functions satisfy the conditions of inextensibility and they satisfy the geometric boundary conditions. Set \( \frac{\partial L}{\partial A} = 0 \).

This gives

\[ F = \frac{m p^2 a^2}{E A} = \kappa \left( \frac{n}{\alpha} \right)^4 \frac{n^2 \left( n^2 - \left( \frac{\alpha}{\pi} \right)^2 \right)}{3 \left( \frac{n}{\pi} \right)^2 + n^2} \]  

(80.a)

or

\[ p \frac{\sqrt{m L^4}}{E I} = \frac{n \pi^2 \left( n^2 - \left( \frac{\alpha}{\pi} \right)^2 \right)}{\sqrt{n^2 + 3 \left( \frac{\alpha}{\pi} \right)^2}} \]  

(80.b)

For \( \alpha = \frac{\pi}{2} \)

\[ p \frac{\sqrt{m L^4}}{E I} = \frac{n \pi^2 \left( n^2 - \frac{1}{4} \right)}{\sqrt{n^2 + \frac{3}{4}}} \]  

(81)
The first five frequencies computed from Eq. (81) for \( \alpha = \frac{\pi}{2} \) are as follows:

<table>
<thead>
<tr>
<th>n</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p \sqrt{\frac{mL^4}{EI}} )</td>
<td>33.96</td>
<td>151.9</td>
<td>349.2</td>
<td>625.5</td>
<td>980.8</td>
</tr>
</tbody>
</table>

(ii) Symmetric case:

Assume: \( w = A \sin \frac{n \pi \omega}{\alpha} \) \hspace{1cm} (82.a)
and

\[ v = A \frac{\alpha}{n \pi} \cos \frac{n \pi \phi}{\alpha} \quad \text{for} \quad \frac{\alpha}{2n} < \phi < \frac{\alpha}{2n} \] \hspace{1cm} (82.b)

and

\[ v = 0 \quad \text{for} \quad 0 \leq \phi \leq \frac{\alpha}{2n} \quad \text{and} \quad \frac{\alpha}{2n} < \phi < \alpha \]

where

\[ n = 1, 3, 5 \ldots \]

Set \( \frac{\partial L}{\partial A} = 0 \). This gives

\[ F = k \left( \frac{n}{\alpha} \right)^4 \frac{n^2 \left( n^2 - \left( \frac{\alpha}{n} \right)^2 \right)^2}{3 \left( \frac{\alpha}{n} \right)^2 + n^2} \] \hspace{1cm} (83.a)

Hence,

\[ p \sqrt{\frac{mL^4}{EI}} = \frac{n \pi^2 \left| n^2 - \left( \frac{\alpha}{n} \right)^2 \right|}{\sqrt{n^2 + \frac{n-1}{n} \left( \frac{\alpha}{n} \right)^2}} \] \hspace{1cm} (83.b)

For \( \alpha = \frac{\pi}{2} \)
The first five frequencies computed from Eq. (84) for $\alpha = \pi/2$

are as follows:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \sqrt{\frac{mL^4}{EI}}$</td>
<td>7.40</td>
<td>85.57</td>
<td>243.3</td>
<td>480.1</td>
<td>795.9</td>
</tr>
</tbody>
</table>

(b) Extensional modes

We use the Rayleigh-Ritz method to find the approximate solutions. Equation (85) gives the appropriate Lagrangian:

$$L_i = \int \left[ \frac{v'^2 + w'^2}{2} - (v' + w')^2 \right] d\varphi$$  \hspace{1cm} (85)

(i) Anti-symmetric case:

Assume $v = A \sin \frac{n\pi \theta}{\alpha}$  \hspace{1cm} (86.a)

and

$$w = A \frac{\alpha}{n\pi} \cos \frac{n\pi \varphi}{\alpha} \quad \text{for} \quad \frac{\alpha}{2\pi} \leq \varphi \leq \frac{\alpha}{2n} \quad (86.b)$$

and

$$w = 0 \quad \text{for} \quad 0 \leq \varphi \leq \frac{\alpha}{2n} \quad \text{and} \quad \frac{\alpha}{2n} \leq \varphi \leq \frac{\alpha}{2n}$$

where

$n = 1, 3, 5, \ldots$
These functions satisfy the conditions of zero bending moment. Substitute Eqs. (86.a) into Eq. (85) and set \( \frac{\partial L_1}{\partial A} = 0 \), we have

\[
F = \left( \frac{n\pi}{\alpha} \right)^2 \frac{1}{1 + \left( \frac{\alpha}{n\pi} \right)^2 \frac{n-1}{n}}
\]

(87.a)

If we denote

\[
\phi_n = C \frac{L}{r}
\]

(87.b)

then

\[
C = \frac{\pi}{2} \sqrt{\left( \frac{n\pi}{\alpha} \right)^2 + \left( 2 + \left( \frac{\alpha}{n\pi} \right)^2 \frac{n-1}{n} \right)}
\]

(87.c)

For \( \alpha = \frac{\pi}{2} \)

\[
C = \frac{\pi}{2} \sqrt{4n^2 + \left( 2 + \left( \frac{1}{2n} \right)^2 \frac{n-1}{n} \right)}
\]

(88)

The first five frequencies obtained from Eq. (88) for \( \alpha = \frac{\pi}{2} \) are listed as follows:

<table>
<thead>
<tr>
<th>N</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>3.14</td>
<td>9.51</td>
<td>15.77</td>
<td>22.04</td>
<td>28.31</td>
</tr>
</tbody>
</table>

(ii) Symmetrical case:

Assume \( v = A \sin \frac{n\pi y}{\alpha} \)
and

\[ w = \frac{A_w}{n} \left( \cos \frac{n \pi \xi}{a} - 1 \right) \tag{89.b} \]

where

\[ n = 2, 4, 6, \ldots \]

Substitute Eqs. (89.a) and (89.b) into Eq. (85) and set \( \frac{\beta L_1}{\theta_A} = 0 \), we have

\[ F = \frac{n^4 (\frac{\pi}{a})^2 + 3 (\frac{\pi}{a})^2 + 2 n^2}{n^2 + 3 (\frac{\pi}{a})^2} \tag{90.a} \]

Then, the value of \( C \) by Eq. (89.b) is

\[ C = \frac{\pi}{2} \sqrt{\frac{n^4 (\frac{\pi}{a})^2 + 3 (\frac{\pi}{a})^2 + 2 n^2}{n^2 + 3 (\frac{\pi}{a})^2}} \tag{90.b} \]

For \( \xi = \frac{\pi}{2} \)

\[ C = \frac{\pi}{2} \sqrt{\frac{16 n^4 + 8 n^2 + 3}{4 n^2 + 3}} \tag{91} \]

The first five frequencies obtained from Eq. (91) are as follows:

<table>
<thead>
<tr>
<th>N</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>6.15</td>
<td>12.48</td>
<td>18.79</td>
<td>25.08</td>
<td>31.38</td>
</tr>
</tbody>
</table>

(iii) Breathing Modes:

In the symmetric vibration, the deformation, corresponding to the lowest natural mode of the straight bar, which does not have
modes except at the ends, becomes extensional as soon as the bar becomes curved. An approximate formula was found by Den Hartog as follows:

For $\alpha = \frac{\pi}{2}$

$$p \sqrt{\frac{mL^4}{EI}} = \frac{\pi}{2} \sqrt{0.82 \left( \frac{L}{r} \right)^2 + 2.25 \pi^4}$$

<table>
<thead>
<tr>
<th>$\frac{L}{r}$</th>
<th>60</th>
<th>120</th>
<th>180</th>
<th>240</th>
<th>300</th>
<th>360</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \sqrt{\frac{mL^4}{EI}}$</td>
<td>85.67</td>
<td>170.9</td>
<td>256.1</td>
<td>341.6</td>
<td>426.8</td>
<td>512.1</td>
</tr>
</tbody>
</table>

The curve is shown in Fig. 17 which corresponds to the first extensional mode in the diagram. This does not agree closely with the exact solution for the higher modes. As noted earlier the shape of the breathing mode changes significantly as the frequency increases.

(c) Conclusion

Figures 15 and 16 show the curves obtained by the approximate formulas and by the exact solutions. It seems the exact solutions are composed of plateaus and diagonal lines formed by the approximate formulas. They turn from diagonal lines to plateaus or vice versa around the intersecting points of the curves obtained by the approximate solutions. So we can predict reasonably well any high frequency by means of these approximate formulas.
5.3. **Effects of rotatory inertia and shear**

For a structure, rotatory inertia gives additional inertia force and shear deformation increases flexibility. Both of these effects lower the value of the natural frequencies.

Four families of curves are shown in Figs. 17 to 20. Each family consist of the first five frequencies for symmetrical or anti-symmetrical vibrations. The families of curves obtained by including both rotatory inertia and shear deformation, with \( \Gamma = 0.3 \) and 0.1, and with rotatory inertia but neglecting shear deformations are drawn together with the family of curves obtained by classical theory. The curves demonstrate that the effects of rotatory inertia and shear deformation tend to smooth the curves and are more prominent around the convex corners. The effects become more important where the slenderness ratio is small.

The numerical values obtained are listed in Tables 3, 4, and 5.

5.4. **Summary**

This thesis makes the following contributions.

1. Solutions have been obtained for the classical theory for a hinged arch with angle of opening of 90 degrees for a wide range of slenderness ratios. These solutions are given in Figs. 5 and 6. The frequencies are affected by the combination of flexural vibration and extensional vibration. In certain regions, one of them is more dominant. We can use a set of approximate formulas to predict frequencies and modes.

2. The effect of rotatory inertia and shear deformation tend to smooth the curves. Their effects are predominant around the convex corners.
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TABLE 1

Convergence of the Numerical Method

\[ \alpha = 87.2^\circ \quad \frac{L}{r} = 110.3 \]

All calculations for theory neglecting rotatory inertia and shear. Arch is simply supported.

<table>
<thead>
<tr>
<th>Number of Divisions in Complete Arch</th>
<th>Values of Frequency Coefficient, ( \sqrt{\frac{4L^4}{EI}} )</th>
<th>Anti-Symmetrical Modes</th>
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<td>60</td>
<td>64.58</td>
<td>131.5</td>
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Eppink Values Obtained By Extrapolation

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<td>10 and 12</td>
<td>64.48</td>
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<td>16 and 20</td>
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Values Obtained From Waltking's Paper

|        |        |        |        |        |
|--------|--------|--------|--------|
| \( \infty \) | 64.5   | 132    |        |        |
|         |        |        | 28.1   | 123    |
TABLE 2
Summary of Solutions of Classical Theory

Arch is simply supported with $\alpha = 90^\circ$

<table>
<thead>
<tr>
<th>Ratio $L/r$</th>
<th>Ratio $a/r$</th>
<th>Frequency Coefficient, $p\sqrt{mL^4/EI}$</th>
</tr>
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<td>Frequency Coefficient, ( p \sqrt{\frac{mL^2}{EI}} )</td>
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<td>785.4</td>
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</table>
**TABLE 3**

**SUMMARY OF SOLUTIONS FOR THEORY INCLUDING ROTATORY INERTIA BUT NEGLECTING SHEARING DEFORMATIONS**

Arch is simply supported with $\alpha = 90^\circ$

<table>
<thead>
<tr>
<th>Ratio $\frac{L}{r}$</th>
<th>Ratio $\frac{a}{r}$</th>
<th>Frequency Coefficient $p \sqrt{\frac{ML^4}{EI}}$</th>
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<td>77.49</td>
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<td>79.45</td>
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<tr>
<td>377.0</td>
<td>240</td>
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TABLE 4
SUMMARY OF SOLUTIONS FOR THEORY INCLUDING ROTATORY INERTIA AND SHEAR DEFORMATION

\( \Gamma = 0.30 \)

Arch is simply supported with \( \alpha = 90^\circ \)

<table>
<thead>
<tr>
<th>Ratio ( \frac{L}{\gamma} )</th>
<th>Ratio ( \frac{\alpha}{\gamma} )</th>
<th>Frequency Coefficient, ( p \sqrt{\frac{m}{L^4}} / EI )</th>
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<td>17.29 46.22 73.12 88.89 90.79</td>
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<td>15</td>
<td>33.00 67.24 144.5 155.2 235.3</td>
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<td>47.12</td>
<td>30</td>
<td>60.45 85.97 201.3 303.2 353.5</td>
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<td>70.69</td>
<td>45</td>
<td>72.81 111.0 221.7 404.6 457.9</td>
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<td>94.25</td>
<td>60</td>
<td>76.33 142.4 231.7 433.0 607.6</td>
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<td>141.4</td>
<td>90</td>
<td>78.45 201.6 248.4 457.8 737.2</td>
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<td>79.49 233.5 379.5 479.6</td>
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<td>79.66 235.4 445.9 509.1</td>
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<td>79.75 236.3 464.4 584.9</td>
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TABLE 5
SUMMARY OF SOLUTIONS FOR THEORY INCLUDING ROTATORY INERTIA AND SHEAR DEFORMATION

\( \Gamma = 0.10 \)

Arch is simply supported with \( \alpha = 90^\circ \)

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<th>Ratio ( \frac{R}{\tau} )</th>
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<td></td>
<td>First    Second  Third  Fourth  Fifth</td>
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<td>15</td>
<td>32.30    53.75   103.05  148.52  156.0</td>
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<td>47.12</td>
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<td>57.91    80.97   165.3   265.3   305.0</td>
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<tr>
<td>70.69</td>
<td>45</td>
<td>69.74    109.7   197.5   337.5   455.0</td>
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<tr>
<td>94.25</td>
<td>60</td>
<td>74.16    141.6   215.6   381.9   568.7</td>
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<td>141.4</td>
<td>90</td>
<td>77.36    199.1   242.1   428.3   667.0</td>
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<td>188.5</td>
<td>120</td>
<td>78.49    223.0   293.5   450.5   715.7</td>
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<tr>
<td>219.9</td>
<td>140</td>
<td>78.87    228.0   336.2   460.2   735.2</td>
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<td>251.3</td>
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<td>79.13    230.6   378.4   469.3   735.2</td>
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<tr>
<td>341.2</td>
<td>200</td>
<td>79.42    233.5   441.7   505.7   735.2</td>
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<tr>
<td>377.0</td>
<td>240</td>
<td>79.59    234.9   459.7   583.9   735.2</td>
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</table>
Fig. 1: COORDINATE AND DISPLACEMENT NOTATIONS

Fig. 2: STRESS RESULTANTS AND LOADS

Fig. 3: SHEAR DEFORMATION
TYPICAL REGION \( \left( \frac{L}{r} = 298.5 \right) \)

TRANSITION REGION \( \left( \frac{L}{r} = 157.1 \right) \)

Fig. 4: TYPICAL PLOTS OF DETERMINANT VS. FREQUENCY
Fig. 5: NATURAL FREQUENCIES OF SYMMETRIC VIBRATIONS OF HINGED-ENDED UNIFORM CIRCULAR ARCHES (CLASSICAL THEORY)
Fig. 6: NATURAL FREQUENCIES OF ANTI-SYMMETRIC VIBRATIONS OF HINGED-ENDED UNIFORM CIRCULAR ARCHES. (CLASSICAL THEORY)
Fig. 7: TYPICAL ANTI-SYMMETRICAL FLEXURAL MODES
Fig. 8: TYPICAL SYMMETRICAL FLEXURAL MODES
Fig. 9: MODES ON THE PLATEAUS WITH THE SAME HEIGHT
(ANTI-SYMMETRIC)
Fig. 10: THE FIRST EXTENSIONAL MODE OF ANTI-SYMMETRICAL VIBRATION

(a) Fourth Mode at $\frac{L}{r} = 149.2$, $p \sqrt{\frac{mL^4}{EI}} = 517.5$

(b) Third Mode at $\frac{L}{r} = 70.69$, $p \sqrt{\frac{mL^4}{EI}} = 245.2$

(c) Second Mode at $\frac{L}{r} = 23.56$, $p \sqrt{\frac{mL^4}{EI}} = 81.49$
(a) Fifth Mode at $\frac{L}{r} = 54.98$, $p \sqrt{\frac{mL^4}{EI}} = 524.6$

(b) Fourth Mode at $\frac{L}{r} = 31.42$, $p \sqrt{\frac{mL^4}{EI}} = 299.2$

(c) Third Mode at $\frac{L}{r} = 11.78$, $p \sqrt{\frac{mL^4}{EI}} = 110.7$

Fig. 11: THE SECOND EXTENSIONAL MODE OF ANTI-SYMMETRICAL VIBRATION
Fig. 12: THE FIRST EXTENSIONAL MODE OF SYMMETRICAL VIBRATION (BREATHING MODE)
(a) Fifth Mode at $\frac{L}{r} = 102.1$, $p \sqrt{\frac{mL^4}{EI}} = 659.4$

(b) Fourth Mode at $\frac{L}{r} = 54.98$, $p \sqrt{\frac{mL^4}{EI}} = 355.2$

(c) Third Mode at $\frac{L}{r} = 23.56$, $p \sqrt{\frac{mL^4}{EI}} = 152.5$

Fig. 13: THE SECOND EXTENSIONAL MODE OF SYMMETRICAL VIBRATION
Fig. 14: CHANGE OF MODE SHAPES IN TRANSITION REGION
(THIRD MODES)
Fig. 15: COMPARISON OF APPROXIMATE AND EXACT SOLUTIONS FOR ANTI-SYMMETRIC VIBRATIONS - CLASSICAL THEORY
Fig. 16: COMPARISON OF APPROXIMATE AND EXACT SOLUTIONS FOR SYMMETRIC VIBRATIONS - CLASSICAL THEORY
Fig. 17: EFFECT OF SHEAR DEFORMATION AND ROTATORY INERTIA FOR SYMMETRICAL VIBRATIONS (FIRST, SECOND AND THIRD MODES)
Fig. 18: EFFECT OF SHEAR DEFORMATION AND ROTATORY INERTIA FOR SYMMETRICAL VIBRATIONS (FOURTH AND FIFTH MODES)
Fig. 19: EFFECT OF SHEAR DEFORMATION AND ROTATORY INERTIA FOR ANTI-SYMMETRICAL VIBRATIONS (FIRST, SECOND, AND THIRD MODES)
Fig. 20: EFFECT OF SHEAR DEFORMATION AND ROTATORY INERTIA FOR ANTI-SYMMETRICAL VIBRATIONS (FOURTH AND FIFTH MODES)
APPENDIX I

FLOW CHART AND FORTRAN PROGRAM
1. Flow Chart

Read a Card

J = 1, NP

K = 1

X1 = BG

X2 = X1 + TV

Find conditions at midspan

SUBROUTINE SOLVE

SUBROUTINE DETER

Compute determinant (D1, D2)

\[ T \]

\[ D1 < 10^{-4} \]

\[ D2 < 10^{-4} \]

D1 \cdot D2 < 0

PP = \frac{X2 \cdot |D1| + X1 \cdot |D2|}{|D1| + |D2|}

Compute Determinant (D)

\[ |D| < 10^{-4} \]

\[ |D| > 10^{-4} \]

\[ D1 \cdot D > 0 \]

\frac{|PP - X|}{|PP|} < 10^{-5}

X2 = pp

D2 = D

Output

K ≥ ND

K = K + 1

X1 = pp + TV

22

END
2. Input Data

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<tr>
<th>Column</th>
<th>Data</th>
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<tr>
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<tr>
<td>9-12</td>
<td>IHOF</td>
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<td>ISOA</td>
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<td>BG</td>
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<td>TV</td>
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<td>SH</td>
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<td>61-71</td>
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The various data terms will be defined in the computer program of next section.

3. Fortran Program List.
COMPUTER PROGRAM OF VIBRATIONS OF ARCHES

*****************************************************************

N--NUMBER OF DIVIDING POINTS OF HALF ARCH
NP--NUMBER OF PROBLEMS
ND--NUMBER OF FREQUENCIES IN EACH PROBLEM
BG--BEGINNING TRIAL FREQUENCY
TV--INTERVAL OF TRIAL FREQUENCY
A--RADIUS OF ARCH
R--RADIUS OF GYRATION
E--YOUNG S MODULUS
G--SHEAR MODULUS
K--COEFFICIENT OF SHAPE
AOR = A/R
SH = K*G/E
PP--E*AREA/MA2

M--MASS/UNIT LENGTH

IHOF = 1  HINGED EDGES
IHOF = 2  FIXED EDGES
ISOA = 1  SYMMETRICAL ARCHES
ISOA = 2  ANTI-SYMMETRICAL ARCHES
SGR = 1.  WITH ROTATORY INERTIA
SGR = 0.  WITHOUT ROTATORY INERTIA
SGS = 1.  WITH SHEAR
SGS = 0.  WITHOUT SHEAR

ANGLE IN DEGREE

*****************************************************************
COMMON PP, ZZ, SH, ALPHA, SGR, SGS
DIMENSION W(50), V(50), W1(50), V1(50), W2(50), V2(50), W3(50), V3(50),
COORD(50)
DOUBLE PRECISION G1, G2, G3, H1, H2, H3, O1, O2, O3, P1, P2, P3, Q1, Q2, Q3, R1,
R2, R3, CC2, CC3
READ (5, 1) NP
1 FORMAT (15)
   DO 22 J = 1, NP
   READ (5, 2) N, ND, IHOF, ISOA, BG, TV, SGR, SGS, AOR, SH, ANGLE
2 FORMAT (414, F7.4, 3F5.2, 3F11.6)
   AN = N
   NN = N - 1
   ALPHA = ANGLE * 0.017453293
   BOR = AOR * ALPHA
   SOR = 2. * SIN (0.5 * ALPHA) * AOR
   ALR = ALPHA * BOR
   ZZ = 1. / (AOR * AOR)
   IF (IHOF .EQ. 1) GO TO 1001
   WRITE (6, 1002)
1002 FORMAT (1H1, 11HFIXED EDGES)
   GO TO 1004
1001 WRITE (6, 1003)
1003 FORMAT (1H1, 12HHINGED EDGES)
1004 IF (ISOA .EQ. 1) GO TO 1005
   WRITE (6, 1008)
1008 FORMAT (1H1, 20HANTISYMMETRICAL MODE//1H0, 13HFUGGGE THEORY)
   GO TO 1006
1005 WRITE (6, 1007)
1007 FORMAT (1H1, 16HSYMMETRICAL MODE//1H0, 13HFUGGGE THEORY)
1006 IF (SGS .EQ. 1) GO TO 1009
   WRITE (6, 1010)
1009 FORMAT (1H1, 25HWITHOUT SHEAR DEFORMATION)
   GO TO 1012
1010 WRITE (6, 1011)
1011 IF (ISOA .EQ. 1) GO TO 1013
   WRITE (6, 1014)
1012 FORMAT (1H1, 24HWITH ROTATORY INERTIA)
   GO TO 1016
1013 WRITE (6, 1015)
1014 FORMAT (1H1, 25HWITHOUT ROTATORY INERTIA)
1015 WRITE (6, 1017) ALFA, ANGLE, SH, BOR, SOR, ALR, AOR, NN
1017 FORMAT (1H0, 30HRUNGE KUTTA INTEGRATION METHOD//1H1, 8HALPHA =
   1F11.6, 13HRADIANS = , F11.6, 10HDegrees, 10X, 12HLAMDA*G/E = ,
   2F11.6, 7HDEP/R = , F11.6, 10X, 7HSP/R = , F11.6, 10X, 12HALPHA*L/R = ,
   4F11.6, 10X, 6HAP/R = , F11.6/23HNUMBER OF DIVISIONS = , I4//)
   K = 1
   X1 = BG
3 PP = X1
   CALL DETER(IHOF, DETA, DETS, A1, A2, A3, B1, B2, B3, C1, C2, C3, D1, D2, D3,
   E1, E2, E3, F1, F2, F3, G1, G2, G3, H1, H2, H3, O1, O2, O3, P1, P2, P3, Q1, Q2, Q3,
   R1, R2, R3, W1, W2, W3, V1, V2, V3, N)
   IF (ISOA .EQ. 1) GO TO 4
   Y1 = DETA
   GO TO 5
4 Y1 = DETS
5 DET = Y1
   WRITE (6, 1031) PP, DET
1031 FORMAT (1H1, 2E30.6)
6 X2 = X1 TV
   PP = X2
   CALL DETER(IHOE,DETA,DETS,A1,A2,A3,B1,B2,B3,C1,C2,C3,D1,D2,D3,E1,E2,E3,F1,F2,F3,G1,G2,G3,H1,H2,H3,P1,P2,P3,Q1,Q2,Q3,R1,R2,R3,W1,W2,W3,V1,V2,V3,N)
   IF(ISOA EQ. 1) GO TO 7
   Y2 = DETA
   GO TO 8
7 Y2 = DETS
8 DET = Y2
   WRITE (6,1032) PP, DET
   1032 FORMAT (IH,2E30.6)
   IF(ABS(DET) L.T. 0.0001) GO TO 15
   IF(Y1*Y2 L.T. 0.) GO TO 9
   X1 = X2
   Y1 = Y2
   GO TO 6
9 PP = (X2*ABS(Y1)+X1*ABS(Y2))/(ABS(Y1)+ABS(Y2))
   CALL DETER(IHOE,DETA,DETS,A1,A2,A3,B1,B2,B3,C1,C2,C3,D1,D2,D3,E1,E2,E3,F1,F2,F3,G1,G2,G3,H1,H2,H3,P1,P2,P3,Q1,Q2,Q3,R1,R2,R3,W1,W2,W3,V1,V2,V3,N)
   IF(ISOA EQ. 1) GO TO 10
   Y3 = DETA
   GO TO 11
10 Y3 = DETS
11 DET = Y3
   WRITE (6,1033) PP, DET
   1033 FORMAT (IH,2E30.6)
   IF(ABS(DET) L.T. 0.0001) GO TO 15
   IF(Y1*Y3 L.T. 0.) GO TO 12
   IF(ABS((PP-X1)/PP) L.T. 0.0001) GO TO 15
   X2 = PP
   Y2 = Y3
   GO TO 9
12 IF(ABS((X2-PP)/PP) L.T. 0.0001) GO TO 15
   X1 = PP
   Y1 = Y3
   GO TO 9
15 IF(ISOA EQ. 1) GO TO 16
   CHC = G2*R3-G3*R2
   IF(ABS(CHC) L.T. 0.0001) GO TO 17
   CC2 = (G3*R1-G1*R3)/CHC
   CC3 = (G1*R2-G2*R1)/CHC
   RES = Q1+CC2*Q2+CC3*Q3
   GO TO 18
17 CC2 = (G3*Q1-G1*Q3)/(G2*Q3-G3*Q2)
   CC3 = (G1*Q2-G2*Q1)/(G2*Q3-G3*Q2)
   RES = R1+CC2*R2+CC3*R3
   GO TO 18
16 CHC = Q2*H3-Q3*H2
   IF(ABS(CHC) L.T. 0.0001) GO TO 19
   CC2 = (Q3*H1-Q1*H3)/CHC
   CC3 = (Q1*H2-Q2*H1)/CHC
   RES = P1+CC2*P2+CC3*P3
   GO TO 18
19 CC2 = (Q3*P1-Q1*P3)/(Q2*P3-Q3*P2)
   CC3 = (Q1*P2-Q2*P1)/(Q2*P3-Q3*P2)
   RES = H1+CC2*H2+CC3*H3
18 CCC2 = CC2
CCC3 = CC3
DO 20 I = 1,N
AI = I
COORD(I) = (AI-1.)/(AN-1.)
W(I) = W(I)+CCC2*W2(I)+CCC3*W3(I)
20 V(I) = V(I)+CCC2*V2(I)+CCC3*V3(I)

GG1 = G1
GG2 = G2
GG3 = G3
HH1 = H1
HH2 = H2
HH3 = H3
OO1 = O1
OO2 = O2
OO3 = O3
PP1 = P1
PP2 = P2
PP3 = P3
QQ1 = Q1
QQ2 = Q2
QQ3 = Q3
RR1 = R1
RR2 = R2
RR3 = R3
AA = A1+CCC2*A2+CCC3*A3
BB = B1+CCC2*B2+CCC3*B3
CC = C1+CCC2*C2+CCC3*C3
DD = D1+CCC2*D2+CCC3*D3
EE = E1+CCC2*E2+CCC3*E3
FF = F1+CCC2*F2+CCC3*F3
GG = G1+CCC2*G2+CCC3*G3
HH = H1+CCC2*H2+CCC3*H3
OO = O1+CCC2*O2+CCC3*O3
PPT = P1+CCC2*P2+CCC3*P3
QQ = Q1+CCC2*Q2+CCC3*Q3
RR = R1+CCC2*R2+CCC3*R3
PPA = PP*OR*OR
PPL = PPA*ALPHA**4
WRITE (6,1020) K,PPL,PPA,PP
1020 FORMAT(/1HO,I3,17H TH FREQ. SQ. = ,E15.6,13H EI/MA4 = ,
1E15.6,13H EI/MA4 = ,E15.6,13H E*AREA/MA2)

FPL = SQRT(PPL)
FPA = SQRT(PPA)
FP = SQRT(PP)
WRITE (6,1024) K,FPL,FPA,FP
1024 FORMAT(/1H I3,17H TH FREQUENCY = ,E15.6,10X,3H = ,E15.6,10X,
13H = ,E15.6//35HOCOMPONENT SOLUTION INPUT AT END//)
WRITE (6,1021) A1,A2,A3,AA,B1,B2,B3,BB,C1,C2,C3,CC,D1,D2,D3,DD,
1E1,E2,E3,EE,F1,F2,F3,FF
1021 FORMAT(1H I3,17H TH W1 = ,E15.8,10X,7H W2 = ,E15.8,10X,7H W3 = ,
1E15.8,10X,7H W4 = ,E15.8,10X,7H W5 = ,E15.8,10X,7H W6 = ,E15.8,
210X,7H W7 = ,E15.8,10X,7H W8 = ,E15.8,10X,7H W9 = ,E15.8,
37HD2W2 = ,E15.8,10X,7HD2W3 = ,E15.8,10X,7HD2WT = ,E15.4/1H ,
47HD3W1 = ,E15.8,10X,7HD3W2 = ,E15.8,10X,7HD3W3 = ,E15.8,10X,
57HD3WT = ,E15.4/1H ,7H V1 = ,E15.8,10X,7H V2 = ,E15.8,10X,
67H V3 = ,E15.8,10X,7H VT = ,E15.4/1H ,7H DV1 = ,E15.8,10X,
77H DV2 = ,E15.8,10X,7H DV3 = ,E15.8,10X,7H DVT = ,E15.4//)
WRITE (6,1022)
1022 FORMAT(1H I3,23X,17HOUTPUT AT MIDSPAN//)
WRITE (6,1021) GG1,GG2,GG3,GG,HH1,HH2,HH3,HH,001,002,003,00,PP1,
1PP2,PP3,PTT,QQ1,QQ2,QQ3,QQ,RR1,RR2,RR3,RR
WRITE (6,1023) DET,RES,CCC2,CCC3,((COORD(I),W(I),V(I)),I = 1,N)
1023 FORMAT(1H0,6HDET = ,E15.6,10X,10HRESIDUE = ,E15.6,10X,6HPF2 = ,
1E15.6,10X,6HPF3 = ,E15.6,1E15.6///1H0,40X,1OHMOOE SHAPE///1H0,14X,
216HANGLE/HALF ALPHA,22X,1HW,30X,1HV///1H ,15X,E15.4,15X,E15.4,
315X,E15.4))
IF(K .GE. ND) GO TO 22
K = K+1
X1 = PP+TV
GO TO 3
22 CONTINUE
STOP
END
SUBROUTINE DETER(IHQF,DETA,DETS,A1,A2,A3,B1,B2,B3,C1,C2,C3,D1,D2,D3,E1,E2,E3,F1,F2,F3,G1,G2,G3,H1,H2,H3,O1,O2,O3,P1,P2,P3,Q1,Q2,Q3,R1,R2,R3,W1,W2,W3,V1,V2,V3,N)

COMMON PP,ZZ,SH,ALPHA,SGR,SGS
DIMENSION W1(N),W2(N),W3(N),V1(N),V2(N),V3(N)
DOUBLE PRECISION G1,G2,G3,H1,H2,H3,O1,O2,O3,P1,P2,P3,Q1,Q2,Q3,R1,R2,R3

IF(IHQF .EQ. 1)
   A1 = 0.
   B1 = 0.
   C1 = 1.
   D1 = 0.
   E1 = 0.
   F1 = 0.
   A2 = 0.
   B2 = 0.
   C2 = 0.
   D2 = 0.
   E2 = 0.
   F2 = 1.
   A3 = 0.
   B3 = -SGS*ZZ*SH/(2.*ZZ*SH*SH+ZZ*PP)
   C3 = 0.
   D3 = 1.
   E3 = 0.
   F3 = 0.
   GO TO 2
    1 A1 = 0.
   B1 = 1.
   C1 = 0.
   D1 = 0.
   E1 = 0.
   F1 = 0.
   A2 = 0.
   B2 = 0.
   C2 = 0.
   D2 = 1.
   E2 = 0.
   F2 = 0.
   A3 = 0.
   B3 = SGS/SH
   C3 = 0.
   D3 = 0.
   E3 = 0.
   F3 = 1.
   2 CALL SOLVE(A1,B1,C1,D1,E1,F1,G1,H1,O1,P1,Q1,R1,W1,V1,N)
   CALL SOLVE(A2,B2,C2,D2,E2,F2,G2,H2,O2,P2,Q2,R2,W2,V2,N)
   DETA = G1*O2*R3+G3*O1*R2+G2*O3*R1-G1*O3*R2-G2*O1*R3
   DETS = H1*P2*Q3+H3*P1*Q2+H2*P3*Q1-H3*P2*Q1-H1*P3*Q2-H2*P1*Q3
   RETURN
END
COMMON PP,ZZ,SH,ALPHA,SGR,SGS
DIMENSION W(N),V(N)
DOUBLE PRECISION G,H,O,P,Q,R
FTN(S,T,U) = (SGR*SGS*PP*(ZZ+1.-PP)/(ZZ+SH)-(ZZ+1.-PP)/ZZ)*S
1+SGR*SGS*PP*Z/(ZZ+SH)-2.-SGR*PP-SGS*PP/SH)*T
2+(SGR*SGS*PP/ZZ+SH-SGR*PP)**U
GTN(X,Y,Z) = SGR*SGS*PP*Z/(1.+SH/ZZ-P)1+SGR*PP*Z1/ZZ+P)
AN = N-1
HH = 0.5*ALPHA/AN
W(1) = A
WA = B
WB = C
WC = D
V(1) = E
VA = F
NI = N-1
DO 100 I = 1,NI
A1 = HH*WA
B1 = HH*WB
C1 = HH*WC
D1 = HH*FTN(W(I),WB,VA)
E1 = HH*VA
F1 = HH*GTN(WA,WC,V(I))
W1 = w(I)+0.5*A1
WA1 = WA+0.5*B1
WB1 = WB+0.5*C1
WC1 = WC+0.5*D1
V1 = V(I)+0.5*E1
VA1 = VA+0.5*F1
A2 = HH*WA1
B2 = HH*WB1
C2 = HH*WC1
D2 = HH*FTN(W1,WB1,VA1)
E2 = HH*VA1
F2 = HH*GTN(WA1,WC1,V1)
W2 = w(I)+0.5*A2
WA2 = WA+0.5*B2
WB2 = WB+0.5*C2
WC2 = WC+0.5*D2
V2 = V(I)+0.5*E2
VA2 = VA+0.5*F2
A3 = HH*WA2
B3 = HH*WB2
C3 = HH*WC2
D3 = HH*FTN(W2,WB2,VA2)
E3 = HH*VA2
F3 = HH*GTN(WA2,WC2,V2)
W3 = w(I)+A3
WA3 = WA+B3
WB3 = WB+C3
WC3 = WC+D3
V3 = V(I)+E3
VA3 = VA+F3
A4 = HH*WA3
B4 = HH*WB3
C4 = HH*WC3
D4 = HH*FTN(W3, WB3, VA3)
E4 = HH*VA3
F4 = HH*GTN(WA3, WC3, V3)
WA = WA + (B1+2.*B2+2.*B3+B4)/6.*
WB = WB + (C1+2.*C2+2.*C3+C4)/6.*
WC = WC + (D1+2.*D2+2.*D3+D4)/6.*
V(I+1) = V(I) + (F1+2.*F2+2.*F3+F4)/6.*
100 VA = VA + (F1+2.*F2+2.*F3+F4)/6.*
G = W(N)
H = WA
O = WB
P = WC
Q = V(N)
R = VA
RETURN
ENC
**COMPUTER PROGRAM OF VIBRATIONS OF ARCHES**

**N**--NUMBER OF DIVIDING POINTS OF HALF ARCH

**NP**--NUMBER OF PROBLEMS

**ND**--NUMBER OF FREQUENCIES IN EACH PROBLEM

**BG**--BEGINNING TRIAL FREQUENCY

**TV**--INTERVAL OF TRIAL FREQUENCY

**A**--RADIUS OF ARCH  \( R \)--RADIUS OF GYRATION

**E**--YOUNG'S MODULUS \( G \)--SHEAR MODULUS \( K \)--COEFFICIENT OF SHAPE

\( AOR = A/R \)

\( SH = K*G/E \)

**PP**--E*AREA/MA2

**M**--MASS/UNIT LENGTH

\( IHOF = 1 \)--HINGED EDGES \( IHOF = 2 \)--FIXED EDGES

\( ISOA = 1 \)--SYMMETRICAL ARCHES \( ISOA = 2 \)--ANTI-SYMMETRICAL ARCHES

\( SGR = 1 \)--WITH ROTATORY INERTIA \( SGR = 0 \)--WITHOUT ROTATORY INERTIA

\( SGS = 1 \)--WITH SHEAR \( SGS = 0 \)--WITHOUT SHEAR

**ANGLE IN DEGREE**

******************************************************************************
C

MAIN PROGRAM
COMMON PP,ZI,SH,ALPHA,SGR,SGS
DIMENSION W(50),V(50),W1(50),V1(50),W2(50),V2(50),W3(50),V3(50),
100R(50)
DOUBLE PRECISION G1,G2,G3,H1,H2,H3,O1,O2,O3,P1,P2,P3,Q1,Q2,Q3,R1,
R2,R3,CC2,CC3
READ (5,1) NP
1 FORMAT (15)
DO 22 J = 1,NP
READ (5,2) N,J,NO,IHOF,ISOA,BG,TV,SGR,SGS,AOR,SH,ANGLE
2 FORMAT (414,F7.4,3F5.2,3F11.6)
AN = N
NN = N-1
ALPHA = ANGLE*0.017453293
BOR = AOR*ALPHA
SOR = 2.5*SIN(0.5*ALPHA)*AOR
ALR = ALPHA*BOR
ZZ = 1./(AOR*AOR)
IF(IHOF.EQ.1) GO TO 1001
WRITE (6,1002)
1002 FORMAT (1H1, 11HFIXED EDGES)
GO TO 1004
1001 WRITE (6,1003)
1003 FORMAT (1H1, 12HHINGED EDGES)
1004 IF(ISOA.EQ.1) GO TO 1005
WRITE (6,1008)
1008 FORMAT (1H20HANTISYMMETRICAL MODE//1H0,13HFLUGGE THEORY)
GO TO 1006
1005 WRITE (6,1007)
1007 FORMAT (1H1, 16HSYMMETRICAL MODE//1H0,13HFLUGGE THEORY)
1006 IF(SGS.EQ.1) GO TO 1009
WRITE (6,1010)
1010 FORMAT (1H1, 25HWITHOUT SHEAR DEFORMATION)
GO TO 1012
1009 WRITE (6,1011)
1011 FORMAT (1H1, 22HWITH SHEAR DEFORMATION)
1012 IF(SGR.EQ.1) GO TO 1013
WRITE (6,1014)
1014 FORMAT (1H1, 24HWITHOUT ROTATORY INERTIA)
GO TO 1016
1013 WRITE (6,1015)
1015 FORMAT (1H1, 21HWITH ROTATORY INERTIA)
1016 WRITE (6,1017) ALPHA,ANGLE,SH,BOR,SOR,ALR,AOR,NN
1017 FORMAT (1H0, 30HRUNGE KUTTA INTEGRATION METHOD//1H , 8HALPHA =
1F11.6, 13H RADIANS = ,F11.6, 10H DEGREES, 10X,12HLAMDA*G/E = ,
2F11.6/7HOL/R = ,F11.6, 10X,7HSP/R = ,F11.6, 10X,12HALPHA*L/R = ,
4F11.6, 10X,6HA/R = ,F11.6/23HONUMBER OF DIVISIONS = ,I4//)
K = 1
X1 = BG
3 PP = X1
CALL DETER(IHOF,DETA,DETS,A1,A2,A3,B1,B2,B3,C1,C2,C3,D1,D2,D3,
E1,E2,E3,F1,F2,F3,G1,G2,G3,H1,H2,H3,O1,O2,O3,P1,P2,P3,Q1,Q2,Q3,
R1,R2,R3,W1,W2,W3,V1,V2,V3,N)
IF(ISOA .EQ. 1) GO TO 4
Y1 = DETA
GO TO 5
4 Y1 = DETS
5 DETER (1H ,2E30.6)
1031
IF(ABS(DET) .LT. 0.0001) GO TO 15
6 X2 = X1 + TV
   PP = X2
   CALL DETER(IHOF, DETA, DETS, A1, A2, A3, B1, B2, B3, C1, C2, C3, D1, D2, D3,
   E1, E2, E3, F1, F2, F3, G1, G2, G3, H1, H2, H3, O1, O2, O3, P1, P2, P3, Q1, Q2, Q3,
   R1, R2, R3, W1, W2, W3, V1, V2, V3, N)
   IF(ISOA .EQ. 1) GO TO 7
   Y2 = DETA
   GO TO 8
7 Y2 = DETS
8 DET = Y2
   WRITE (6, 1032) PP, DET
1032 FORMAT (1H,7E30.6)
   IF(ABS(DET) .LT. 0.0001) GO TO 15
   IF(Y1 .EQ. 0.) GO TO 9
   X1 = X2
   Y1 = Y2
   GO TO 6
9 PP = (X2*ABS(Y1)+X1*ABS(Y2))/(ABS(Y1)+ABS(Y2))
   CALL DETER(IHOF, DETA, DETS, A1, A2, A3, B1, B2, B3, C1, C2, C3, D1, D2, D3,
   E1, E2, E3, F1, F2, F3, G1, G2, G3, H1, H2, H3, O1, O2, O3, P1, P2, P3, Q1, Q2, Q3,
   R1, R2, R3, W1, W2, W3, V1, V2, V3, N)
   IF(ISOA .EQ. 1) GO TO 10
   Y3 = DETA
   GO TO 11
10 Y3 = DETS
11 DET = Y3
   WRITE (6, 1033) PP, DET
1033 FORMAT (1H,7E30.6)
   IF(ABS(DET) .LT. 0.0001) GO TO 15
   IF(Y1 .EQ. 0.) GO TO 12
   PP = (X2-PP)/PP
   Y2 = Y3
   GO TO 9
12 IF(ABS((X2-PP)/PP) .LT. 0.0001) GO TO 15
   X1 = PP
   Y1 = Y3
   GO TO 9
15 IF(ISOA .EQ. 1) GO TO 16
   CHC = G2*R3-G3*R2
   IF(ABS(CHC) .LT. 0.0001) GO TO 17
   CC2 = (G3*R1-G1*R3)/CHC
   CC3 = (G1*R2-G2*R1)/CHC
   RES = Q1+CC2*Q2+CC3*Q3
   GO TO 18
17 CC2 = (G3*C1-G1*C3)/(G2*C3-G3*C2)
   CC3 = (G1*C2-G2*C1)/(G2*C3-G3*C2)
   RES = R1+CC2*R2+CC3*R3
   GO TO 18
16 CHC = Q2*H3-Q3*H2
   IF(ABS(CHC) .LT. 0.0001) GO TO 19
   CC2 = (Q3*H1-Q1*H3)/CHC
   CC3 = (Q1*H2-Q2*H1)/CHC
   RES = P1+CC2*P2+CC3*P3
   GO TO 18
19 CC2 = (Q3*P1-Q1*P3)/(Q2*P3-Q3*P2)
   CC3 = (Q1*P2-Q2*P1)/(Q2*P3-Q3*P2)
   RES = H1+CC2*H2+CC3*H3
18  \( CCC2 = CC2 \)
\( CCC3 = CC3 \)
DO 20 I = 1,N
AI = I
\( COOR(I) = (AI-1.)/(AN-1.) \)
\( W(I) = W1(I)+CCC2*W2(I)+CCC3*W3(I) \)
20  \( V(I) = V1(I)+CCC2*V2(I)+CCC3*V3(I) \)
GG1 = G1
GG2 = G2
GG3 = G3
HH1 = H1
HH2 = H2
HH3 = H3
OO1 = O1
OO2 = O2
OO3 = O3
PP1 = P1
PP2 = P2
PP3 = P3
QQ1 = Q1
QQ2 = Q2
QQ3 = Q3
RR1 = R1
RR2 = R2
RR3 = R3
AA = A1+CCC2*A2+CCC3*A3
BB = B1+CCC2*B2+CCC3*B3
CC = C1+CCC2*C2+CCC3*C3
DD = D1+CCC2*D2+CCC3*D3
EE = E1+CCC2*E2+CCC3*E3
FF = F1+CCC2*F2+CCC3*F3
GG = G1+CC2*G2+CC3*G3
HH = H1+CC2*H2+CC3*H3
OO = O1+CC2*O2+CC3*O3
PPT = P1+CC2*P2+CC3*P3
QQ = Q1+CC2*Q2+CC3*Q3
RR = R1+CC2*R2+CC3*R3
PPA = PP*VOR*VOR
PPL = PPA*ALPHA**4
WRITE (6,1020) K,PPL,PPA,PP
1020 FORMAT(1H0,I3,17H TH FREQUENCY = ,E15.6,13H  E1/ML4 = ,E15.6,13H  E/MA4 = ,E15.6,13H  E*AREA/MA2)
FPL = SQRT(PPL)
FPA = SQRT(PPA)
FP = SQRT(PP)
WRITE (6,1024) K,FPL,FPA,FP
1024 FORMAT(1H0,I3,17H TH FREQUENCY = ,E15.6,10X,3H = ,E15.6,10X,13H = ,E15.6/35HCOMPONENT SOLUTION INPUT AT END//)
WRITE (6,1021) A1,A2,A3,AA,B1,B2,B3,BB,CI,C2,C3,CC,DI,D2,D3,DD,
1021 FORMAT(1H0,23X,17HOUTPUT AT MIDSPAN//)
1022 FORMAT(1H0,23X,17HOUTPUT AT MIDSPAN//)
WRITE (6,1021) GG1,GG2,GG3,GG,HH1,HH2,HH3,HH,OO1,OO2,OO3,OO,PP1,
           PP2,PP3,PPT,QQ1,QQ2,QQ3,QQ,RR1,RR2,RR3,RR
WRITE (6,1023) DET,RES,CCC2,CCC3,(COORD(I),W(I),V(I)),I = 1,N
1023 FORMAT(1H0,6HDET,,E15.6,10X,10HRESIDUE = ,E15.6,10X,6HPF2 = ,
           E15.6,10X,6HPF3 = ,E15.6//1H0,40X,10HMODE SHAPE//1H0,14X,
           216HANGLE/HALF ALPHA,22X,1HW,30X,1HV//1H4,15X,E15.4,15X,E15.4,
           315X,E15.4))
   IF(K .GE. ND) GO TO 22
   K = K+1
   X1 = PP+TV
GO TO 3
22 CONTINUE
STOP
END
SUBROUTINE DETER(IHOF,DETA,DETS,A1,A2,A3,B1,B2,B3,C1,C2,C3,D1,D2,D3,E1,E2,E3,F1,F2,F3,G1,G2,G3,H1,H2,H3,O1,O2,O3,P1,P2,P3,Q1,Q2,Q3,R1,R2,R3,W1,W2,W3,V1,V2,V3,N)
COMMON PP,ZZ,SH,ALPHA,SGR,SGS
DIMENSION W1(N),W2(N),W3(N),V1(N),V2(N),V3(N)
DOUBLE PRECISION G1,G2,G3,H1,H2,H3,O1,O2,O3,P1,P2,P3,Q1,Q2,Q3,R1,R2,R3
    IF(IHOF.EQ.1) GO TO 1
    A1 = 0.
    B1 = 0.
    C1 = 1.
    D1 = 0.
    E1 = 0.
    F1 = 0.
    A2 = 0.
    B2 = 0.
    C2 = 0.
    D2 = 0.
    E2 = 0.
    F2 = 1.
    A3 = 0.
    B3 = -SGS*ZZ*SH/(2.*ZZ*SH+SH*SH+ZZ*PP)
    C3 = 0.
    D3 = 1.
    E3 = 0.
    F3 = 0.
    GO TO 2
1   A1 = 0.
    B1 = 1.
    C1 = 0.
    D1 = 0.
    E1 = 0.
    F1 = 0.
    A2 = 0.
    B2 = 0.
    C2 = 0.
    D2 = 1.
    E2 = 0.
    F2 = 0.
    A3 = 0.
    B3 = SGS/SH
    C3 = 0.
    D3 = 0.
    E3 = 0.
    F3 = 1.
2   CALL SOLVE(A1,B1,C1,D1,E1,F1,G1,H1,O1,P1,Q1,R1,W1,V1,N)
   CALL SOLVE(A2,B2,C2,D2,E2,F2,G2,H2,O2,P2,Q2,R2,W2,V2,N)
DETS = H1*P2*Q3*H3*P1*Q2+H2*P3*Q1-H3*P2*Q1-H1*P3*Q2-H2*P1*Q3
RETURN
END
COMMON PP, ZZ, SH, ALPHA, SGR, SGS
DIMENSION W(N), V(N)
DOUBLE PRECISION G, H, O, P, Q, R
FTN(S, T, U) = (SGR*SGS*PP*(ZZ+1.-PP)/(ZZ+SH)-(ZZ+1.-PP)/ZZ)*S
1+SGR*SGS*PP*ZZ/(ZZ+SH)-2.-SGR*PP-SGS*PP/SH)*T
GTN(X, Y, Z) = SGR*SGS*PP*ZZ*Y/(1.+SH/ZZ-PP)
1+SGR*SGS*PP*ZZ*(1.+PP+PP/SH)/(1.+SH/ZZ-PP)-1.*SGR*PP*ZZ)*X
2+SGR*SGS*PP*ZZ*(PP/SH-PP)/(1.+SH/ZZ-PP)-PP-SGR*ZZ*PP)*Z
AN = N-1
HH = 0.5*ALPHA/AN
W(1) = A
WA = B
WB = C
WC = D
V(1) = E
VA = F
NI = N-1
DO 100 I = 1, NI
A1 = HH*WA
B1 = HH*WB
C1 = HH*WC
D1 = HH*FTN(W(I), WB, VA)
E1 = HH*VA
F1 = HH*GTN(WA, WC, V(I))
W1 = W(I) + 0.5*A1
WA1 = WA + 0.5*B1
WB1 = WB + 0.5*C1
WC1 = WC + 0.5*D1
V1 = V(I) + 0.5*E1
VA1 = VA + 0.5*F1
A2 = HH*WA1
B2 = HH*WB1
C2 = HH*WC1
D2 = HH*FTN(W1, WB1, VA1)
E2 = HH*VA1
F2 = HH*GTN(WA1, WC1, V1)
W2 = W(I) + 0.5*A2
WA2 = WA + 0.5*B2
WB2 = WB + 0.5*C2
WC2 = WC + 0.5*D2
V2 = V(I) + 0.5*E2
VA2 = VA + 0.5*F2
A3 = HH*WA2
B3 = HH*WB2
C3 = HH*WC2
D3 = HH*FTN(W2, WB2, VA2)
E3 = HH*VA2
F3 = HH*GTN(WA2, WC2, V2)
W3 = W(I) + A3
WA3 = WA + B3
WB3 = WB + C3
WC3 = WC + D3
V3 = V(I) + E3
VA3 = VA + F3
A4 = HH*WA3
B4 = HH*WB3
C4 = HH*WC3
D4 = HH*FTN(\(w_3, w_B, v_A\))
E4 = HH*VA
F4 = HH*GTN(\(w_A, w_C, v_3\))

\[ W(l+1) = W(l) + \left( A1 + 2.*A2 + 2.*A3 + A4 \right)/6. \]
\[ W_A = W_A + \left( B1 + 2.*B2 + 2.*B3 + B4 \right)/6. \]
\[ W_B = W_B + \left( C1 + 2.*C2 + 2.*C3 + C4 \right)/6. \]
\[ W_C = W_C + \left( D1 + 2.*D2 + 2.*D3 + D4 \right)/6. \]
\[ V(l+1) = V(l) + \left( E1 + 2.*E2 + 2.*E3 + E4 \right)/6. \]
100 \[ V_A = V_A + \left( F1 + 2.*F2 + 2.*F3 + F4 \right)/6. \]
G = \(W(N)\)
H = \(W_A\)
O = \(W_B\)
P = \(W_C\)
Q = \(V(N)\)
R = \(V_A\)
RETURN
ENC