

A Birman exact sequence for $\text{Aut}(F_n)$

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Abstract

The Birman exact sequence describes the effect on the mapping class group of a surface with boundary of gluing discs to the boundary components. We construct an analogous exact sequence for the automorphism group of a free group. For the mapping class group, the kernel of the Birman exact sequence is a surface braid group. We prove that in the context of the automorphism group of a free group, the natural kernel is finitely generated. However, it is not finitely presentable; indeed, we prove that its second rational homology group has infinite rank by constructing an explicit infinite collection of linearly independent abelian cycles. We also determine the abelianization of our kernel and build a simple infinite presentation for it. The key to many of our proofs are several new generalizations of the Johnson homomorphisms.

1 Introduction

When studying the mapping class group $\text{Mod}(\Sigma)$ of a closed orientable surface Σ , one is led inexorably to the mapping class groups of surfaces with boundary. For instance, often phenomena are “concentrated” on a subsurface of Σ , and thus they “live” in the mapping class group of the subsurface (which is a surface with boundary). The key tool for understanding the mapping class group of a surface with boundary is the *Birman exact sequence*. If S is a surface with p boundary components and \hat{S} is the closed surface that results from gluing discs to the boundary components of S , then there is a natural surjection $\psi: \text{Mod}(S) \rightarrow \text{Mod}(\hat{S})$. Namely, if $f \in \text{Mod}(S)$, then $\psi(f)$ is obtained by extending f over the glued-in discs. Excluding some degenerate low-genus cases, a basic result of Birman [3] shows that $\text{Ker}(\psi)$ is isomorphic to the p -strand braid group $B_p(\hat{S})$ on \hat{S} (or possibly a slight modification of $B_p(\hat{S})$ depending on your conventions for mapping class groups of surfaces with boundary). This is summarized in the exact sequence

$$1 \longrightarrow B_p(\hat{S}) \longrightarrow \text{Mod}(S) \longrightarrow \text{Mod}(\hat{S}) \longrightarrow 1. \quad (1)$$

The purpose of this paper is to develop a similar exact sequence for the automorphism group of a free group.

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Automorphisms of free groups with boundary, motivation. We begin by giving a definition of the “automorphism group of a free group with boundary”. A motivation for this definition is as follows. A prototypical example of where mapping class groups of surfaces with boundary arise is as the stabilizer of a collection of homotopy classes of nonnullhomotopic disjoint simple closed curves $\gamma_1, \dots, \gamma_k$ on a genus g surface Σ_g (such a collection forms a simplex in the *curve complex*). The stabilizer of the γ_i is then essentially the mapping class group of the result of cutting Σ_g along the γ_i . We wish to give an algebraic description of such collections.

Recall that there is a bijection between conjugacy classes in $\pi_1(\Sigma_g)$ and homotopy classes of oriented closed curves on Σ_g . We want to understand which collections of conjugacy classes in $\pi_1(\Sigma_g)$ correspond to collections $\gamma_1, \dots, \gamma_k$ of homotopy classes of disjoint nonnullhomotopic simple closed curves on Σ_g (oriented in some way). The description is simplest if we restrict to collections of γ_i such that $\gamma_1 \cup \dots \cup \gamma_k$ does not separate Σ_g . Elements $a_1, b_1, \dots, a_g, b_g \in \pi_1(\Sigma_g)$ form a *standard basis* for $\pi_1(\Sigma_g)$ if they generate $\pi_1(\Sigma_g)$ and satisfy the surface relation $[a_1, b_1] \cdots [a_g, b_g] = 1$. We then have the following folklore fact. As notation, if G is a group and $g \in G$, then $\llbracket g \rrbracket$ will denote the conjugacy class of g .

Lemma 1.1. *A collection c_1, \dots, c_k of conjugacy classes in $\pi_1(\Sigma_g)$ corresponds to a collection $\gamma_1, \dots, \gamma_k$ of homotopy classes of disjoint oriented nonnullhomotopic simple closed curves on Σ_g such that $\Sigma_g \setminus (\gamma_1 \cup \dots \cup \gamma_k)$ is connected if and only if there exists a standard basis $a_1, b_1, \dots, a_g, b_g$ for $\pi_1(\Sigma_g)$ with $c_i = \llbracket a_i \rrbracket$ for $1 \leq i \leq k$.*

We do not know a reference for Lemma 1.1, so we include a proof in Appendix A.

Lemma 1.1 suggests defining the automorphism group of a free group with boundary to be the subgroup of the automorphism group of a free group fixing the conjugacy classes of some fixed partial basis for the free group.

Automorphisms of free groups with boundary, definition. Let $F_{n,k,l}$ be the free group on letters $\{x_1, \dots, x_n, y_1, \dots, y_k, z_1, \dots, z_l\}$ and set

$$X = \{x_1, \dots, x_n\} \quad \text{and} \quad Y = \{y_1, \dots, y_k\} \quad \text{and} \quad Z = \{z_1, \dots, z_l\}.$$

We define

$$\mathcal{A}_{n,k,l} = \{f \in \text{Aut}(F_{n,k,l}) \mid \llbracket f(v) \rrbracket = \llbracket v \rrbracket \text{ for } v \in Y \cup Z\}.$$

The reason for separating the roles of the y_i and the z_j will become apparent shortly. The group $\mathcal{A}_{n,k,l}$ should be viewed as the automorphism group of a free group on n letters with $k + l$ boundary components.

Remark. There have been many proposals for analogues of the curve complex for $\text{Aut}(F_n)$ (see, e.g., [2, 9, 12, 13, 14, 19]). The groups $\mathcal{A}_{n,k,l}$ form the simplex stabilizers for the action of $\text{Aut}(F_n)$ on the *complex of partial bases*, which is one of these analogues. In [9], the results in the current paper are used to prove some topological results about this complex. The simplex stabilizers of several other curve complex analogues are closely related to the groups $\mathcal{A}_{n,k,l}$.

Remark. The group $\mathcal{A}_{0,k,l}$ is known as the *pure symmetric automorphism group* of the free group $F_{0,k,l}$. It was introduced by McCool [23] and has been the subject of much investigation. McCool was also the first to study the general group $\mathcal{A}_{n,k,l}$, and in [21] he proved that it was finitely presentable.

Remark. In [26], Wahl introduced a slightly different definition of an automorphism group of a free group with boundary. These groups were further studied in papers of Hatcher-Wahl [15] and Jensen-Wahl [16]. The paper [16] also discussed the groups $\mathcal{A}_{n,k,l}$ we consider here and constructed presentations for them; see Theorem 5.2 below. Wahl's groups are abelian extensions of $\mathcal{A}_{n,k,l}$, and it is not hard to deduce from our theorems analogous results for her groups. In the context of the mapping class group, there are likewise two natural ways to deal with boundary components. One can require that mapping classes fix the boundary pointwise or merely require them to fix each boundary component setwise. In the former case (to which Wahl's definition is analogous), the Dehn twists about the boundary components are nontrivial, while in the latter case (to which our definition is analogous) they are trivial.

Remark. Another reasonable definition of the automorphism group of a free group with boundary would require the automorphisms to fix the y_i and z_j on the nose (rather than merely up to conjugacy). It turns out that this leads to a much more poorly behaved theory – see Theorem E below.

The Birman exact sequence. Let N be the subgroup of $F_{n,k,l}$ normally generated by Y . There is an evident isomorphism $F_{n,k,l}/N \cong F_{n,0,l}$, and we will henceforth identify these groups. We thus have a short exact sequence

$$1 \longrightarrow N \longrightarrow F_{n,k,l} \longrightarrow F_{n,0,l} \longrightarrow 1.$$

The group $\mathcal{A}_{n,k,l}$ preserves N , so we get a map $\rho: \mathcal{A}_{n,k,l} \rightarrow \text{Aut}(F_{n,0,l})$ whose image is clearly contained in $\mathcal{A}_{n,0,l} < \text{Aut}(F_{n,0,l})$. The map $\rho: \mathcal{A}_{n,k,l} \rightarrow \mathcal{A}_{n,0,l}$ has a right inverse $\psi: \mathcal{A}_{n,0,l} \rightarrow \mathcal{A}_{n,k,l}$ defined as follows. Consider $f \in \mathcal{A}_{n,0,l}$. Define $\psi(f) \in \mathcal{A}_{n,k,l}$ to be the automorphism which has the following behavior on a generator $s \in X \cup Y \cup Z$.

$$\psi(f)(s) = \begin{cases} s & \text{if } s \in Y, \\ f(s) & \text{if } s \in X \cup Z. \end{cases}$$

It is clear that this defines an element of $\mathcal{A}_{n,k,l}$ such that $\rho(\psi(f)) = f$. We conclude that ρ is surjective.

Let $\mathcal{K}_{n,k,l}$ be the kernel of ρ . We thus have a split short exact sequence

$$1 \longrightarrow \mathcal{K}_{n,k,l} \longrightarrow \mathcal{A}_{n,k,l} \xrightarrow{\rho} \mathcal{A}_{n,0,l} \longrightarrow 1. \tag{2}$$

The exact sequence (2) will be our analogue of the Birman exact sequence (1).

Generators for the kernel. To understand (2), we must study the kernel group $\mathcal{K}_{n,k,l}$. We begin by giving generators for it. For distinct $v, w \in X \cup Y \cup Z$ and $\epsilon = \pm 1$, let $M_{v^\epsilon, w}$ and $C_{v, w}$ be the elements of $\text{Aut}(F_{n,k,l})$ that are defined by

$$M_{v^\epsilon, w}(s) = \begin{cases} vw & \text{if } s = v \text{ and } \epsilon = 1 \\ vw^{-1} & \text{if } s = v \text{ and } \epsilon = -1 \\ s & \text{otherwise} \end{cases} \quad \text{and} \quad C_{v, w}(s) = \begin{cases} wvw^{-1} & \text{if } s = v \\ s & \text{otherwise} \end{cases}$$

for $s \in X \cup Y \cup Z$. Our first result is as follows.

Theorem A. *The group $\mathcal{K}_{n,k,l}$ is generated by the finite set*

$$S_K = \{M_{x^\epsilon, y} \mid x \in X, y \in Y, \epsilon = \pm 1\} \cup \{C_{z, y} \mid z \in Z, y \in Y\} \\ \cup \{C_{y, v} \mid y \in Y, v \in X \cup Y \cup Z, y \neq v\}.$$

The kernel is not finitely presentable. Theorem A might suggest to the reader that $\mathcal{K}_{n,k,l}$ is a well-behaved group, but this hope is dashed by the following theorem.

Theorem B. *If $k \geq 1$ and $n + l \geq 2$, then $\mathcal{K}_{n,k,l}$ is not finitely presentable. In fact, $H_2(\mathcal{K}_{n,k,l}; \mathbb{Q})$ has infinite rank.*

Remark. Theorem B should be contrasted to what happens in the mapping class group of a surface, where the kernel of the Birman exact sequence is finitely presented (and, in fact, has a compact Eilenberg-MacLane space, so all of its homology groups have finite rank).

Our proof of Theorem B actually yields an explicit infinite set of linearly independent classes in $H_2(\mathcal{K}_{n,k,l}; \mathbb{Q})$. These classes arise as *abelian cycles*. If G is a group and $x, y \in G$ are commuting elements, then there is a homomorphism $i: \mathbb{Z}^2 \rightarrow G$ taking the generators of \mathbb{Z}^2 to x and y . The 2-torus is an Eilenberg-MacLane space for \mathbb{Z}^2 , so $H_2(\mathbb{Z}^2; \mathbb{Q}) \cong \mathbb{Q}$. Letting $c \in H_2(\mathbb{Z}^2; \mathbb{Q})$ be the standard generator, the element $i_*(c) \in H_2(G; \mathbb{Q})$ is known as the abelian cycle determined by x and y . Returning to $\mathcal{K}_{n,k,l}$, the assumptions of Theorem B imply that we can find $y \in Y$ and $a, b \in X \cup Z$ such that $a \neq b$. It is easily verified that the elements

$$C_{y, a}^m C_{b, y} C_{y, a}^{-m} \quad \text{and} \quad C_{a, y} C_{b, y}$$

of $\mathcal{K}_{n,k,l}$ commute for all $m \geq 1$. Letting $\mu_m \in H_2(\mathcal{K}_{n,k,l}; \mathbb{Q})$ be the associated abelian cycle, we will prove that the μ_m are linearly independent.

Remark. A priori, it is not even clear that the μ_m are nonzero.

Comment about proof of Theorem B. The key step in proving that the μ_m are linearly independent is the construction of elements $\zeta_m \in H^2(\mathcal{K}_{n,k,l}; \mathbb{Q})$ which are “dual” to the μ_m . The ζ_m are “almost” cup products of elements of H^1 . More precisely, there is a natural subgroup $\mathcal{L} < \mathcal{K}_{n,k,l}$ and a homomorphism $I: \mathcal{L} \rightarrow A$, where A is an infinite rank abelian group. The homomorphism I can be viewed as a generalization of the *Johnson homomorphism*, which is a well-known abelian quotient of the Torelli subgroup of the

mapping class group of a surface (see §2 for details). We construct a sequence of surjections $\alpha'_m: A \rightarrow \mathbb{Z}$, and thus a sequence of surjections $\alpha_m: \mathcal{L} \rightarrow \mathbb{Z}$. The cohomology class $[\alpha_m]$ is then an element of $H^1(\mathcal{L}; \mathbb{Q})$, and we can construct elements

$$\eta_m = [\alpha_m] \cup [\alpha_0] \in H^2(\mathcal{L}; \mathbb{Q}).$$

The elements $\zeta_m \in H^2(\mathcal{K}_{n,k,l}; \mathbb{Q})$ are constructed from η_m by a sort of averaging process similar to the classical transfer map, though the transfer map cannot be used directly since \mathcal{L} is an infinite-index subgroup of $\mathcal{K}_{n,k,l}$.

Relations in the kernel. In spite of Theorem B, it turns out that there are only five basic relations in $\mathcal{K}_{n,k,l}$. These appear as relations R1-R5 in Table 1. We have the following theorem.

Theorem C (Informal). *The group $\mathcal{K}_{n,k,l}$ has a presentation $\langle S_K \mid R \rangle$, where S_K is the generating set from Theorem A and R consists of all relations “of the same type” as relations R1-R5 in Table 1.*

Basic relations for $\mathcal{K}_{n,k,l}$

Let S_K be the generating set from Theorem A. The set R_K is the set of all relations between elements of $F(S_K)$ of the following forms:

R1. Three classes of relations saying that generators commute.

- (1) $[M_{x^\epsilon, v}, M_{w^\delta, y}] = 1$ for $x, w \in X$, $\epsilon, \delta = \pm 1$ and $v, y \in Y$ with $x^\epsilon \neq w^\delta$,
- (2) $[M_{x^\epsilon, y}, C_{v, z}] = 1$ for $x \in X$, $\epsilon = \pm 1$, $y \in Y$, $v \in Y \cup Z$ and $z \in X \cup Y \cup Z$ such that $C_{v, z} \in S_K$, $x \neq z$ and $v \neq y$,
- (3) $[C_{u, v}, C_{w, z}] = 1$ for $u, w \in Y \cup Z$ and $v, z \in X \cup Y \cup Z$ such that $C_{u, v}, C_{w, z} \in S_K$, $u \neq w, z$ and $w \neq v$;

R2. $C_{v, x}^{-\epsilon} M_{x^\epsilon, z} C_{v, x}^\epsilon = C_{v, z} M_{x^\epsilon, z}$ for $\epsilon = \pm 1$, $x \in X$, and $v, z \in Y$ such that $v \neq z$;

R3. $C_{v, z}^\epsilon M_{x^\delta, v} C_{v, z}^{-\epsilon} = M_{x^\delta, z}^{-\epsilon} M_{x^\delta, v} M_{x^\delta, z}^\epsilon$ for $\epsilon, \delta = \pm 1$, $x \in X$, and $v, z \in Y$ such that $v \neq z$;

R4. $C_{v, z}^\epsilon C_{w, v} C_{v, z}^{-\epsilon} = C_{w, z}^{-\epsilon} C_{w, v} C_{w, z}^\epsilon$ for $\epsilon = \pm 1$, $w, v \in Y \cup Z$ and $z \in X \cup Y \cup Z$ such that $C_{v, z}, C_{w, v}, C_{w, z} \in S_K$; and

R5. $C_{y, x}^{-\epsilon} M_{x^{-\epsilon}, y} C_{y, x}^\epsilon = M_{x^\epsilon, y}^{-1}$ for $\epsilon = \pm 1$, $x \in X$, and $y \in Y$.

Table 1: Relations for $\mathcal{K}_{n,k,l}$

Of course, this appears to contradict Theorem B. However, the relations in R1-R5 depend on a choice of basis, and in Theorem C we require the infinite collection of relations of

the forms R1-R5 with respect to all choices of basis. See Theorem 5.6 in §5 for a precise statement. Also see Corollary 5.7, where we deduce that $\mathcal{K}_{n,k,l}$ has a finite L -presentation (a strong kind of recursive presentation).

The abelianization of the kernel. Our next main theorem gives the abelianization of $\mathcal{K}_{n,k,l}$. There are three families of abelian quotients (see §3 for more details)

- Let ψ be the restriction of the natural homomorphism $\text{Aut}(F_{n,k,l}) \rightarrow \text{Aut}(\mathbb{Z}^{n+k+l}) \cong \text{GL}_{n+k+l}(\mathbb{Z})$ to $\mathcal{K}_{n,k,l}$. Then it turns out that the image of ψ is a free abelian subgroup of $\text{GL}_{n+k+l}(\mathbb{Z})$ of rank nk .
- If $n = 0$, then $\mathcal{K}_{n,k,l}$ acts trivially on $F_{n,k,l}^{\text{ab}} = \mathbb{Z}^{n+k+l}$ and thus by definition lies in the subgroup

$$\text{IA}_{n,k,l} = \{f \in \text{Aut}(F_{n,k,l}) \mid f \text{ acts trivially on } F_{n,k,l}^{\text{ab}}\}$$

of $\text{Aut}(F_{n,k,l})$. The group $\text{IA}_{n,k,l}$ is often called the *Torelli subgroup* of $\text{Aut}(F_{n,k,l})$. There is a well-known homomorphism $J: \text{IA}_n \rightarrow \text{Hom}(\mathbb{Z}^{n+k+l}, \bigwedge^2 \mathbb{Z}^{n+k+l})$ known as the *Johnson homomorphism*, and $J(\mathcal{K}_{n,k,l})$ has rank $2kl + k(k-1)$.

- If $n > 0$, then $\mathcal{K}_{n,k,l}$ does not act trivially on $F_{n,k,l}^{\text{ab}}$ so the Johnson homomorphism is not available. Nonetheless, we will construct a sequence of modified versions of the Johnson homomorphism in this case the direct sum of whose images is a free abelian group of rank $2kl + kn$.

The above abelian quotients are all independent of each other, and we prove that they give the entire abelianization of $\mathcal{K}_{n,k,l}$.

Theorem D. *In the case $n = 0$, we have $H_1(\mathcal{K}_{0,k,l}; \mathbb{Z}) \cong \mathbb{Z}^{2kl+k(k-1)}$. If $n > 0$, then $H_1(\mathcal{K}_{n,k,l}; \mathbb{Z}) \cong \mathbb{Z}^{2kn+2kl}$.*

An alternate definition. Our last theorem concerns a possible alternate definition of the automorphism group of a free group with boundary. Define

$$\mathcal{A}'_{n,k,l} = \{f \in \text{Aut}(F_{n,k,l}) \mid f(v) = v \text{ for } v \in Y \cup Z\}.$$

The group $\mathcal{A}'_{n,k,l}$ was first studied by McCool [21], who proved that it was finitely presentable. Just as for $\mathcal{A}_{n,k,l}$, there is a split surjection $\rho': \mathcal{A}'_{n,k,l} \rightarrow \mathcal{A}_{n,0,l}$. Define $\mathcal{K}'_{n,k,l} = \text{Ker}(\rho')$, so we have a short exact sequence

$$1 \longrightarrow \mathcal{K}'_{n,k,l} \longrightarrow \mathcal{A}'_{n,k,l} \xrightarrow{\rho'} \mathcal{A}'_{n,0,l} \longrightarrow 1.$$

One might expect that $\mathcal{K}'_{n,k,l}$ is similar to $\mathcal{K}_{n,k,l}$; however, the following theorem shows that they are quite different.

Theorem E. *Fix $n, k \geq 1$ and $l \geq 0$ such that $n + l \geq 2$. Then the group $\mathcal{K}'_{n,k,l}$ is not finitely generated. In fact, $H_1(\mathcal{K}'_{n,k,l}; \mathbb{Q})$ has infinite rank.*

The proof of this falls out of our proof of Theorem B above. Recall from the proof sketch that one of the key steps was the construction of a subgroup $\mathcal{L} < \mathcal{K}_{n,k,l}$ and a ‘‘Johnson homomorphism’’ $I: \mathcal{L} \rightarrow A$, where A is an abelian group of infinite rank. It turns out that $\mathcal{K}'_{n,k,l} \subset \mathcal{L}$ and the image $I(\mathcal{K}'_{n,k,l})$ has infinite rank.

Prior results. Some special cases of our theorems appear in the literature on the pure symmetric automorphism group. Define $K_m = \mathcal{K}_{0,1,m-1}$. In [8], Collins and Gilbert proved that K_3 is not finitely presentable. In [25], Pettet proved that K_m is finitely generated for all m and that K_m is not finitely presentable for $m \geq 3$. She also calculated the abelianization of K_m . Our proof of the finite generation of $\mathcal{K}_{n,k,l}$ is a generalization of the proof in [25]; however, our proofs of our other results are quite different from those in [8] and [25].

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Outline and conventions. In §2, we give a general construction of a “Johnson crossed homomorphism”. This construction will be used in both §3 and §4. In §3, we calculate the abelianization of $\mathcal{K}_{n,k,l}$, proving Theorem D. In §4, we prove Theorem B, which asserts that $\mathcal{K}_{n,k,l}$ is not finitely presentable if $k \geq 1$ and $n+l \geq 2$. The proof of Theorem E is also contained in §4. Finally, in §5, we determine generators and relations for $\mathcal{K}_{n,k,l}$, proving Theorems A and C. We put §5 at the end because of its heavy use of combinatorial group theory, however, it does not depend on the other sections and can also be read first.

Automorphisms act on the left and compose right-to-left like functions. If G is a group and $g, h \in H$, then we define $g^h = hgh^{-1}$ and $[g, h] = ghg^{-1}h^{-1}$. We note again our conventions that $C_{x,y}(x) = x^y$ and $M_{x,y}(x) = yx$.

2 Johnson crossed homomorphisms

To study $H_1(\mathcal{K}_{n,k,l}; \mathbb{Z})$ and $H_2(\mathcal{K}_{n,k,l}; \mathbb{Q})$, we will need several variants on the well-known Johnson homomorphisms. These homomorphisms were originally introduced in the context of the Torelli subgroup of the mapping class group of a surface by Johnson [17, 18] and have since been generalized to a variety of contexts (see, e.g., [4, 6, 7, 10, 20]). In this section, we give a general framework for studying them.

Recall first that if a group Γ acts on an abelian group M , then a *crossed homomorphism* from Γ to M with respect to this action is a function $\phi: \Gamma \rightarrow M$ satisfying the cocycle identity

$$\phi(g_1g_2) = \phi(g_1) + g_1(\phi(g_2)) \quad (g_1, g_2 \in \Gamma).$$

Crossed homomorphisms are also sometimes referred to as *twisted 1-cocycles* or as *derivations*. The set of all crossed homomorphisms from Γ to M forms an abelian group which we will denote $\text{Der}(\Gamma, M)$. If Γ acts trivially on M , then $\text{Der}(\Gamma, M) = \text{Hom}(\Gamma, M)$.

Now let

$$1 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 1$$

be a short exact sequence of groups and let G be a group which acts on B . Assume the following three conditions hold.

1. The group A is abelian.

2. For $a \in A$ and $g \in G$, we have $g(a) \in A$.
3. The induced action of G on $C = B/A$ is trivial.

To simplify our notation, we will often use additive notation when discussing A , though we will try never to mix additive and multiplicative notation.

For $g \in G$, define a function $\mathcal{J}_g: B \rightarrow A$ as follows. Condition 3 implies that $g(b) \cdot b^{-1} \in A$ for $b \in B$. We can thus define $\mathcal{J}_g(b) = g(b) \cdot b^{-1}$. The group B acts on A by conjugation. We claim that \mathcal{J}_g is a crossed homomorphism. Indeed, for $b_1, b_2 \in B$ we have

$$\begin{aligned} \mathcal{J}_g(b_1 \cdot b_2) &= g(b_1 \cdot b_2) \cdot (b_1 \cdot b_2)^{-1} \\ &= (g(b_1) \cdot b_1^{-1}) \cdot b_1 \cdot (g(b_2) \cdot b_2^{-1}) \cdot b_1^{-1} \\ &= \mathcal{J}_g(b_1) + b_1(\mathcal{J}_g(b_2)). \end{aligned}$$

We can thus define a function

$$\mathcal{J}: G \longrightarrow \text{Der}(B, A)$$

by $\mathcal{J}(g) = \mathcal{J}_g$. The group G acts on $\text{Der}(B, A)$ via the action of G on A . We claim that \mathcal{J} is a crossed homomorphism. Indeed, for $g_1, g_2 \in G$ and $b \in B$ we have

$$\begin{aligned} \mathcal{J}(g_1 \cdot g_2)(b) &= g_1(g_2(b)) \cdot b^{-1} \\ &= g_1(g_2(b) \cdot b^{-1}) \cdot g_1(b) \cdot b^{-1} \\ &= g_1(\mathcal{J}(g_2)(b)) + \mathcal{J}(g_1). \end{aligned}$$

We will call the function \mathcal{J} a *Johnson crossed homomorphism*.

3 The abelianization of $\mathcal{K}_{n,k,l}$

In this section, we calculate the abelianization of $\mathcal{K}_{n,k,l}$. The actual calculation is contained in §3.3. This is preceded by §3.1–3.2, which are devoted to constructing the necessary abelian quotients of $\mathcal{K}_{n,k,l}$. The homomorphisms constructed in §3.1 come from the action of $\mathcal{K}_{n,k,l}$ on $F_{n,k,l}^{\text{ab}}$, and the homomorphisms in §3.2 come from certain Johnson crossed homomorphisms.

To simplify our notation, we define

$$V = F_{n,k,l} = \langle x_1, \dots, x_n, y_1, \dots, y_k, z_1, \dots, z_l \rangle \quad \text{and} \quad \bar{V} = V^{\text{ab}}$$

and

$$V_C = \langle y_1, \dots, y_k, z_1, \dots, z_l \rangle \quad \text{and} \quad \bar{V}_C = V_C^{\text{ab}}$$

and

$$V_Y = \langle y_1, \dots, y_k \rangle \quad \text{and} \quad \bar{V}_Y = V_Y^{\text{ab}}.$$

Also, for $v \in V$ we will denote by $\bar{v} \in \bar{V}$ the image in the abelianization. The notation \bar{V}_C is intended to indicate that this is the subgroup of V generated by basis elements whose conjugacy classes are fixed by $\mathcal{A}_{n,k,l}$.

3.1 The action on homology

The first source of abelian quotients of $\mathcal{K}_{n,k,l}$ is the action of $\mathcal{A}_{n,k,l}$ on \bar{V} . This action restricts to the identity action on \bar{V}_C . If M and N are free \mathbb{Z} -modules and N is a direct summand of M , then denote by $\text{Aut}(M, N)$ the set of automorphisms of M that restrict to the identity on N . With respect to an appropriate choice of basis, elements of $\text{Aut}(M, N)$ are represented by matrices with an identity block in the upper left hand corner and a block of zeros in the lower left hand corner. Let $\pi: \bar{V} \rightarrow \bar{V}/\bar{V}_C$ be the projection. We have a split short exact sequence

$$1 \longrightarrow \text{Hom}(\bar{V}/\bar{V}_C, \bar{V}_Y) \longrightarrow \text{Aut}(\bar{V}, \bar{V}_C) \longrightarrow \text{Aut}(\bar{V}/\bar{V}_Y, \bar{V}_C/\bar{V}_Y) \longrightarrow 1,$$

where $h \in \text{Hom}(\bar{V}/\bar{V}_C, \bar{V}_Y)$ corresponds to the automorphism of \bar{V} that takes $\bar{v} \in \bar{V}$ to $\bar{v} + h(\pi(\bar{v}))$. This fits into a commutative diagram of the form

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{K}_{n,k,l} & \longrightarrow & \mathcal{A}_{n,k,l} & \longrightarrow & \mathcal{A}_{n,0,l} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \text{Hom}(\bar{V}/\bar{V}_C, \bar{V}_Y) & \longrightarrow & \text{Aut}(\bar{V}, \bar{V}_C) & \longrightarrow & \text{Aut}(\bar{V}/\bar{V}_Y, \bar{V}_C/\bar{V}_Y) & \longrightarrow & 1. \end{array}$$

In summary, we have a homomorphism $A: \mathcal{K}_{n,k,l} \rightarrow \text{Hom}(\bar{V}/\bar{V}_C, \bar{V}_Y)$.

Remark. It is easy to write down a formula for A . Namely, if $f \in \mathcal{K}_{n,k,l}$ and $w \in \bar{V}/\bar{V}_C$, then $A(f)(w) = f_*(\bar{v}) - \bar{v} \in \bar{V}_Y$, where $\bar{v} \in \bar{V}$ is any lift of w and $f_*: \bar{V} \rightarrow \bar{V}$ is the induced action.

3.2 The Johnson homomorphisms on $\mathcal{K}_{n,k,l}$

The second source of abelian quotients of $\mathcal{K}_{n,k,l}$ are homomorphisms constructed from Johnson crossed homomorphisms in the sense of §2. Let $V'_C < V$ be the subgroup generated by V_C and $[V, V]$. We then have a short exact sequence

$$1 \longrightarrow [V, V] \longrightarrow V'_C \longrightarrow \bar{V}_C \longrightarrow 1. \quad (3)$$

It is well-known that $[V, V]/[V, [V, V]] \cong \bigwedge^2 \bar{V}$ (see, e.g., [17]). Taking the quotient of the groups in the short exact sequence (3) by $[V, [V, V]]$, we thus get a short exact sequence

$$1 \longrightarrow \bigwedge^2 \bar{V} \longrightarrow V'_C/[V, [V, V]] \longrightarrow \bar{V}_C \longrightarrow 1. \quad (4)$$

It is clear that the group $\mathcal{K}_{n,k,l}$ acts on V'_C and thus on $V'_C/[V, [V, V]]$. Also, the action of $\mathcal{K}_{n,k,l}$ on \bar{V}_C is trivial. By the method of §2, we thus get a Johnson crossed homomorphism $J: \mathcal{K}_{n,k,l} \rightarrow \text{Der}(\bar{V}_C, \bigwedge^2 \bar{V})$. The action of \bar{V}_C on $\bigwedge^2 \bar{V}$ is trivial, so $\text{Der}(\bar{V}_C, \bigwedge^2 \bar{V}) = \text{Hom}(\bar{V}_C, \bigwedge^2 \bar{V})$. Summing up, we have constructed a Johnson crossed homomorphism

$$J: \mathcal{K}_{n,k,l} \longrightarrow \text{Hom}(\bar{V}_C, \bigwedge^2 \bar{V}).$$

If $n = 0$, then $\bar{V} = \bar{V}_C$ and thus $\mathcal{K}_{n,k,l}$ acts trivially on $\bigwedge^2 \bar{V}$. This implies that in this case J is actually a homomorphism. If $n \geq 1$, however, then it is only a crossed

homomorphism. This can be fixed as follows. Consider $c \in Y \cup Z$. Let $J_c: \mathcal{K}_{n,k,l} \rightarrow \bigwedge^2 \bar{V}$ be the composition of J with the restriction map from $\text{Hom}(\bar{V}_C, \bigwedge^2 \bar{V})$ to

$$\text{Hom}(\langle \bar{c} \rangle, \bigwedge^2 \bar{V}) \cong \text{Hom}(\mathbb{Z}, \bigwedge^2 \bar{V}) \cong \bigwedge^2 \bar{V}.$$

For $f \in \mathcal{K}_{n,k,l}$, there exists some $w \in V$ such that $f(c) = wcw^{-1}$. By definition, $J_c(f)$ is the image of $f(c) \cdot c^{-1} = [w, c]$ in $\bigwedge^2 \bar{V}$. Letting \bar{c} and \bar{w} be the images of c and w in \bar{V} , respectively, we get that $J_c(f) = \bar{w} \wedge \bar{c}$. In other words, the image of J_c lies in $\bar{V} \wedge \bar{c} \subset \bigwedge^2 \bar{V}$, which is isomorphic to $\bar{V}/\langle \bar{c} \rangle$ via the isomorphism that takes $x \in \bar{V}/\langle \bar{c} \rangle$ to $\tilde{x} \wedge \bar{c}$, where $\tilde{x} \in \bar{V}$ is any lift of x .

So we have described a crossed homomorphism

$$J_c: \mathcal{K}_{n,k,l} \longrightarrow \bar{V}/\langle \bar{c} \rangle.$$

If we further suppose that $c \in Z$, then the fact that f is in $\mathcal{K}_{n,k,l}$ implies that \bar{w} is in $\langle \bar{V}_Y, \bar{c} \rangle$ (otherwise wcw^{-1} will not become c when the elements of Y are deleted from it). So in the case that $c \in Z$, we have

$$J_c: \mathcal{K}_{n,k,l} \longrightarrow \langle \bar{V}_Y, \bar{c} \rangle / \langle \bar{c} \rangle.$$

Since $\langle \bar{V}_Y, \bar{c} \rangle \subset \bar{V}_C$, the group $\mathcal{K}_{n,k,l}$ acts trivially on $\langle \bar{V}_Y, \bar{c} \rangle / \langle \bar{c} \rangle$. This implies that J_c is a homomorphism when $c \in Z$. Tracing through the above identifications, we may calculate $J_c(f)$ for $f \in \mathcal{K}_{n,k,l}$ as follows. Write $f(c) = wcw^{-1}$ for some $w \in V$. Then $\bar{w} \in \langle \bar{V}_Y, \bar{c} \rangle$ and $f(c)$ is the image of \bar{w} in $\langle \bar{V}_Y, \bar{c} \rangle / \langle \bar{c} \rangle$.

On the other hand, if $c \in Y$, then we are forced to take a further quotient to get an honest homomorphism. By definition $\mathcal{K}_{n,k,l}$ acts trivially on \bar{V}/\bar{V}_Y . Letting

$$J'_c: \mathcal{K}_{n,k,l} \rightarrow \bar{V}/\langle \bar{V}_Y, \bar{c} \rangle$$

be the composition of J_c (with its target redefined via the above isomorphism to be $\bar{V}/\langle \bar{c} \rangle$) with the projection $\bar{V}/\langle \bar{c} \rangle \rightarrow \bar{V}/\langle \bar{V}_Y, \bar{c} \rangle$, it follows that J'_c is a homomorphism. Tracing through the above identifications, we may calculate $J'_c(f)$ for $f \in \mathcal{K}_{n,k,l}$ as follows. Write $f(c) = wcw^{-1}$ for some $w \in V$. Then $J'_c(f)$ is the image of w in $\bar{V}/\langle Y, \bar{c} \rangle$.

3.3 Calculating the abelianization

We now have all the pieces necessary to prove the following two theorems. Together they imply Theorem D.

Theorem 3.1 (Abelianization of $\mathcal{K}_{n,k,l}$, $n = 0$). *Consider $k \geq 1$ and $l \geq 0$. The image of the homomorphism*

$$J: \mathcal{K}_{0,k,l} \longrightarrow \text{Hom}(\bar{V}, \bigwedge^2 \bar{V})$$

is the abelianization of $\mathcal{K}_{0,k,l}$. This image is isomorphic to \mathbb{Z}^r with $r = 2kl + k(k-1)$.

Proof. Define

$$C = \{C_{y_i, y_j} \mid 1 \leq i, j \leq k, i \neq j\} \quad \text{and} \quad C' = \{C_{z_i, y_j} \mid 1 \leq i \leq l, 1 \leq j \leq k\}$$

$$\text{and} \quad C'' = \{C_{y_i, z_j} \mid 1 \leq i \leq k, 1 \leq j \leq l\}.$$

By Theorem A (proved in §5 below), the group $\mathcal{K}_{0, k, l}$ is generated by $C \cup C' \cup C''$. Since $C \cup C' \cup C''$ has $k(k-1) + 2kl$ elements, it is enough to prove that the images under J of the elements of $C \cup C' \cup C''$ are linearly independent. This follows from the following three calculations. Checking these calculations is easy and left to the reader.

- Consider $C_{y_i, y_j} \in C$. Then $J(C_{y_i, y_j}) \in \text{Hom}(\bar{V}, \wedge^2 \bar{V})$ is the map with the following behavior for $s \in \{y_1, \dots, y_k, z_1, \dots, z_l\}$.

$$J(C_{y_i, y_j})(\bar{s}) = \begin{cases} \bar{y}_i \wedge \bar{y}_j & \text{if } s = \bar{y}_i, \\ 0 & \text{otherwise.} \end{cases}$$

- Consider $C_{z_i, y_j} \in C'$. Then $J(C_{z_i, y_j}) \in \text{Hom}(\bar{V}, \wedge^2 \bar{V})$ is the map with the following behavior for $s \in \{y_1, \dots, y_k, z_1, \dots, z_l\}$.

$$J(C_{z_i, y_j})(\bar{s}) = \begin{cases} \bar{z}_i \wedge \bar{y}_j & \text{if } s = \bar{z}_i, \\ 0 & \text{otherwise.} \end{cases}$$

- Consider $C_{y_i, z_j} \in C$. Then $J(C_{y_i, z_j}) \in \text{Hom}(\bar{V}, \wedge^2 \bar{V})$ is the map with the following behavior for $s \in \{y_1, \dots, y_k, z_1, \dots, z_l\}$.

$$J(C_{y_i, z_j})(\bar{s}) = \begin{cases} \bar{y}_i \wedge \bar{z}_j & \text{if } s = \bar{y}_i, \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

Theorem 3.2 (Abelianization of $\mathcal{K}_{n, k, l}$, $n \geq 1$). Consider $n \geq 1$, $k \geq 1$, and $l \geq 0$. The image of the homomorphism

$$A \oplus \left(\bigoplus_{c \in Y} J'_c \right) \oplus \left(\bigoplus_{c \in Z} J_c \right) : \mathcal{K}_{n, k, l} \longrightarrow \text{Hom}(\bar{V}/\bar{V}_C, \bar{V}_Y) \oplus \left(\bigoplus_{c \in Y} \bar{V}/\langle \bar{V}_Y, \bar{c} \rangle \right) \oplus \left(\bigoplus_{c \in Z} \langle \bar{V}_Y, \bar{c} \rangle / \langle \bar{c} \rangle \right)$$

is the abelianization of $\mathcal{K}_{n, k, l}$. This image is isomorphic to \mathbb{Z}^r for $r = 2kn + kl$.

Proof. Define

$$M = \{M_{x_i, y_j} \mid 1 \leq i \leq n, 1 \leq j \leq k\} \quad \text{and} \quad M' = \{M_{x_i^{-1}, y_j} \mid 1 \leq i \leq n, 1 \leq j \leq k\}$$

and

$$C = \{C_{y_i, y_j} \mid 1 \leq i, j \leq k, i \neq j\} \quad \text{and} \quad C' = \{C_{z_i, y_j} \mid 1 \leq i \leq l, 1 \leq j \leq k\}$$

$$\text{and} \quad C'' = \{C_{y_i, z_j} \mid 1 \leq i \leq k, 1 \leq j \leq l\} \quad \text{and} \quad C''' = \{C_{y_i, x_j} \mid 1 \leq i \leq k, 1 \leq j \leq n\}.$$

By Theorem A, the group $\mathcal{K}_{n,k,l}$ is generated by $M \cup M' \cup C \cup C' \cup C'' \cup C'''$.

We have relations

$$M_{x_i, y_j} M_{x_i^{-1}, y_j} = [M_{x_i, y_j}, C_{y_j, x_i}] \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq k$$

and

$$C_{y_i, y_j} = [C_{y_i, x_1}^{-1}, M_{x_1, y_j}] \quad \text{for } 1 \leq i, j \leq k \text{ such that } i \neq j.$$

When $\mathcal{K}_{n,k,l}$ is abelianized, therefore, the generators in M' and C become redundant. Since M has kn elements, C''' has kn elements, and C' and C'' have kl elements, it follows that the rank of the abelianization of $\mathcal{K}_{n,k,l}$ is at most $2kn + 2kl$. To prove the theorem, it is thus enough to show that the images under our map of the elements of $M \cup C' \cup C'' \cup C'''$ are linearly independent. This follows from the following four calculations. For $c \in Y \cup Z$, let $q_c: \langle \bar{V}_Y, \bar{c} \rangle \rightarrow \langle \bar{V}_Y, \bar{c} \rangle / \langle \bar{c} \rangle$ and $q'_c: \bar{V} \rightarrow \bar{V} / \langle \bar{V}_Y, \bar{c} \rangle$ be the natural maps.

- Clearly $J'_c(M) = 0$ for $c \in Y$ and $J_c(M) = 0$ for $c \in Z$. Also, the elements of M project under A to a basis for the abelian group $\text{Hom}(\bar{V}/\bar{V}_C, \bar{V}_Y)$.
- Consider $C_{z_i, y_j} \in C'$. Then $A(C_{z_i, y_j}) = 0$, $J'_c(C_{z_i, y_j}) = 0$ for $c \in Y$ and $J_c(C_{z_i, y_j}) = 0$ for $c \in Z \setminus \{z_i\}$. Also, $J_{z_i}(C_{z_i, y_j}) = q_{z_i}(y_j) \neq 0$.
- Consider $C_{y_i, z_j} \in C''$. Then $A(C_{y_i, z_j}) = 0$, $J_c(C_{y_i, z_j}) = 0$ for $c \in Z$ and $J'_c(C_{y_i, z_j}) = 0$ for $c \in Y \setminus \{y_i\}$. Also, $J'_{y_i}(C_{y_i, z_j}) = q'_{y_i}(z_j) \neq 0$.
- Consider $C_{y_i, x_j} \in C'''$. Then $A(C_{y_i, x_j}) = 0$, $J_c(C_{y_i, x_j}) = 0$ for $c \in Z$ and $J'_c(C_{y_i, x_j}) = 0$ for $c \in Y \setminus \{y_i\}$. Also, $J'_{y_i}(C_{y_i, x_j}) = q'_{y_i}(x_j) \neq 0$. \square

4 The groups $\mathcal{K}_{n,k,l}$ are not finitely presentable

The goal of this section is to prove Theorem B. Along the way (in §4.2), we will also prove Theorem E. Let us recall the setup of Theorem B. We assume $n, k, l \in \mathbb{Z}$ are chosen such that $k \geq 1$ and $n + l \geq 2$. This implies that there exists some $y \in Y$ and $a, b \in X \cup Z$ such that $a \neq b$. Consider $m \geq 1$. The elements

$$C_{y,a}^m C_{b,y} C_{y,a}^{-m} \quad \text{and} \quad C_{a,y} C_{b,y}$$

of $\mathcal{K}_{n,k,l}$ commute and thus determine an abelian cycle $\mu_m \in H_2(\mathcal{K}_{n,k,l}; \mathbb{Q})$. Theorem B asserts that $H_2(\mathcal{K}_{n,k,l}; \mathbb{Q})$ has infinite rank, which will follow from the fact that the μ_m are linearly independent. The skeleton of the proof of this is in §4.1, which concludes with an outline of the remainder of this section. Because we will not change our hypotheses on n , k and l in this section, we will abbreviate $\mathcal{K}_{n,k,l}$ as \mathcal{K} .

4.1 Skeleton of proof

Recall that there is a bilinear pairing (the *cap product* or *evaluation pairing*)

$$\omega: H^2(\mathcal{K}; \mathbb{Q}) \times H_2(\mathcal{K}; \mathbb{Q}) \longrightarrow \mathbb{Q}.$$

In §4.2 – 4.5, we will prove the following lemma.

Lemma 4.1. *For $r \geq 1$, there exist elements $[\zeta_r] \in H^2(\mathcal{K}; \mathbb{Q})$ such that*

$$\omega([\zeta_r], \mu_m) = \begin{cases} 2 & \text{if } m = r, \\ 0 & \text{if } m \neq r, \end{cases}$$

for all $m \geq 1$.

Remark. We denote our cohomology class $[\zeta_r]$ because it will be defined by an explicit cocycle ζ_r on \mathcal{K} .

Lemma 4.1 immediately implies Theorem B. Indeed, if

$$\sum_{i=1}^{\infty} c_i \mu_i = 0 \quad (c_m \in \mathbb{Q}, \text{ only finitely many } c_m \text{ nonzero}),$$

then we obtain that

$$0 = \omega\left(\sum_{i=1}^{\infty} [\zeta_r], c_i \mu_i\right) = \sum_{i=1}^{\infty} c_i \omega([\zeta_r], \mu_i) = 2c_r$$

for all $r \geq 1$, as desired.

The outline of the proof of Lemma 4.1 is as follows. In §4.2, we will construct a certain subgroup $\mathcal{L} < \mathcal{K}$ and a family of homomorphisms from \mathcal{L} to an infinite rank free abelian group A . These homomorphisms are constructed from a Johnson crossed homomorphism. This section also contains a proof of Theorem E. In §4.4, we will compose one of these homomorphisms with a family of maps from A to \mathbb{Z} to obtain a sequence of surjective homomorphisms

$$\alpha_i: \mathcal{L} \longrightarrow \mathbb{Z} \quad (i \geq 0).$$

The maps α_i determine elements $[\alpha_i] \in H^1(\mathcal{L}; \mathbb{Q})$, so we can use the cup product to obtain cohomology classes $[\alpha_r] \cup [\alpha_0] \in H^2(\mathcal{L}; \mathbb{Q})$ for $r \geq 1$. We will use a sort of averaging process to construct an element $[\zeta_r] \in H^2(\mathcal{K}; \mathbb{Q})$ out of $[\alpha_r] \cup [\alpha_0] \in H^2(\mathcal{L}; \mathbb{Q})$. The calculation in Lemma 4.1 is then contained in §4.5. Our calculations in §4.4–4.5 will (alas) have to be performed at the chain level, so between §4.2 and §4.4 we have §4.3, which reviews some chain-level information about group cohomology.

4.2 The group \mathcal{L} and its Johnson homomorphism

The goal of this section is to define a subgroup \mathcal{L} of \mathcal{K} with infinitely many homomorphisms $\mathcal{L} \rightarrow \mathbb{Z}$; we use these to construct our desired cocycles on \mathcal{K} . In fact, we define a crossed homomorphism from \mathcal{K} to an infinite rank abelian group, and by restriction get homomorphisms on a subgroup. In order to make these definitions, we must first investigate some subgroups of V that arise naturally given the special role of Y .

Recall that we have fixed some $y \in Y$. Also, recall that

$$V = F_{n,k,l} \quad \text{and} \quad \bar{V} = V^{\text{ab}} \quad \text{and} \quad V_Y = \langle Y \rangle \subset V \quad \text{and} \quad \bar{V}_Y = V_Y^{\text{ab}}.$$

Define N_Y to be the normal closure of Y in V . We have a short exact sequence

$$1 \longrightarrow N_Y \longrightarrow V \longrightarrow V/N_Y \longrightarrow 1. \quad (5)$$

The group \mathcal{K} acts on V . This action preserves N_Y , and the induced action on V/N_Y is trivial by the definition of \mathcal{K} . To construct a Johnson crossed homomorphism out of this situation, we will need to quotient the groups in (5) by a subgroup $M \triangleleft V$ which is preserved by \mathcal{K} such that $M < N_Y$ and such that N_Y/M is abelian. Of course, we could use $[N_Y, N_Y]$, but it turns out that a larger M gives a Johnson crossed homomorphism with better properties.

Let $Y' = Y \setminus \{y\}$ and $N_{Y'} < V$ be the normal closure of Y' . Observe that each of the subgroups

$$N_{Y'}, \quad \text{and} \quad [N_Y, N_Y] \quad \text{and} \quad [N_Y, [V, V]] \quad (6)$$

is normal and preserved by \mathcal{K} . Let $M < V$ be the subgroup generated by the subgroups in (6). It is clear that M is a normal subgroup. We have a short exact sequence

$$1 \longrightarrow N_Y/M \longrightarrow V/M \longrightarrow V/N_Y \longrightarrow 1.$$

Associated to this is a Johnson crossed homomorphism

$$I: \mathcal{K} \longrightarrow \text{Der}(V/M, N_Y/M).$$

Our next order of business is to find a subgroup $\mathcal{L} < \mathcal{K}$ such that \mathcal{L} acts trivially on N_Y/M . The restriction of I to \mathcal{L} will then be an actual homomorphism. We need the following lemma about N_Y/M .

Lemma 4.2. *If $\tilde{v} \in V$ and $\tilde{v}' \in V$ have the same image in $\overline{V}/\overline{V}_Y$, then the images of $y^{\tilde{v}}$ and $y^{\tilde{v}'}$ in N_Y/M are equal.*

Proof. By assumption, we can find $w_1 \in [V, V]$ and $w_2 \in N_Y$ such that $\tilde{v} = \tilde{v}'w_1w_2$. It then follows that

$$y^{\tilde{v}} = y^{\tilde{v}'w_1w_2} = ((y^{w_2})^{w_1})^{\tilde{v}'}. \quad (7)$$

Modulo M , we have $y^{w_2} = y$ and $y^{w_1} = y$. It follows that modulo M , the expression in (7) equals $y^{\tilde{v}'}$, as desired. \square

For $v \in \overline{V}/\overline{V}_Y$, pick some $\tilde{v} \in V$ that projects to v and define y^v to equal the image of $y^{\tilde{v}}$ in N_Y/M . Lemma 4.2 implies that y^v is independent of the choice of \tilde{v} . We then have the following.

Lemma 4.3. *The infinite set $\{y^v \mid v \in \overline{V}/\overline{V}_Y\}$ is a basis for the abelian group N_Y/M .*

Proof. Let A be the free abelian group with basis the formal symbols $\{y^v \mid v \in \overline{V}/\overline{V}_Y\}$. Using standard ‘‘covering spaces of graphs’’ arguments, it is easy to see that N_Y is a free group with free basis $S = \{y^{\tilde{v}} \mid \tilde{v} \in \langle X, Z \rangle\}$. Let $\rho: N_Y \rightarrow A$ be the projection taking $y^{\tilde{v}} \in S$ to $y^v \in A$, where $v \in \overline{V}/\overline{V}_Y$ is the image of S . It is clear that $M < \text{Ker}(\rho)$, and Lemma 4.2 implies that $\text{Ker}(\rho) < M$. The lemma follows. \square

Since \mathcal{K} preserves M , it acts on N_Y/M . Recall that in §3.2, we constructed a homomorphism $J'_y: \mathcal{K} \rightarrow \overline{V}/\overline{V}_Y$. Tracing through the definitions, we see that

$$f(y^v) = y^{v+J'_y(f)} \quad (v \in \overline{V}/\overline{V}_Y, f \in \mathcal{K})$$

We define $\mathcal{L} = \text{Ker}(J'_y)$. This lets us obtain the following result.

Corollary 4.4. *The group $\mathcal{L} < \mathcal{K}$ acts trivially on N_Y/M , and hence the restriction of the Johnson crossed homomorphism $I: \mathcal{K} \rightarrow \text{Der}(V/M, N_Y/M)$ to \mathcal{L} is a homomorphism.*

The target $\text{Der}(V/M, N_Y/M)$ of I is rather complicated, so our next order of business is to break it into understandable pieces. The key will be the following lemma, whose proof is trivial and thus omitted.

Lemma 4.5. *Let G be an infinite cyclic group with generator t . Also, let A be an abelian group upon which G acts. The map $i: \text{Der}(G, A) \rightarrow A$ defined by $i(\phi) = \phi(t)$ is then an isomorphism.*

For $s \in X \cup Z$, its image $s' \in V/M$ generates an infinite cyclic subgroup. We can thus compose $I: \mathcal{L} \rightarrow \text{Der}(V/M, N_Y/M)$ with the restriction map

$$\text{Der}(V/M, N_Y/M) \rightarrow \text{Der}(\langle s' \rangle, N_Y/M) \cong N_Y/M$$

to obtain a homomorphism $I_s: \mathcal{L} \rightarrow N_Y/M$. The value of I_s is easy to calculate. Namely, for $f \in \mathcal{L}$ we have $f(s) \cdot s^{-1} \in N_Y$, and $I_s(f)$ is the image of $f(s) \cdot s^{-1}$ in N_Y/M .

At this point, we have developed enough machinery to prove Theorem E, whose we recall statement is as follows. Recall that

$$\mathcal{A}'_{n,k,l} = \{f \in \text{Aut}(F_{n,k,l}) \mid f(v) = v \text{ for } v \in Y \cup Z\}.$$

There is an obvious surjective map $\rho: \mathcal{A}'_{n,k,l} \rightarrow \mathcal{A}'_{n,0,l}$, and $\mathcal{K}'_{n,k,l} = \text{Ker}(\rho)$. Theorem E asserts that $H_1(\mathcal{K}'_{n,k,l}; \mathbb{Q})$ has infinite rank if $n, k \geq 1$ and $n + l \geq 2$.

Proof of Theorem E. By assumption, we can find $s \in X$ and $t \in X \cup Z$ such that $s \neq t$. Observe that $\mathcal{K}'_{n,k,l} \subset \mathcal{L}$. It follows that I_s restricts to a homomorphism $I_s: \mathcal{K}'_{n,k,l} \rightarrow N_Y/M$. It is enough to show that $I_s(\mathcal{K}'_{n,k,l})$ contains an infinite set of independent elements. Now, for $m \in \mathbb{Z}$ define $h_m = C_{y,t}^m M_{s,y} C_{y,t}^{-m}$. It is easily verified that $h_m \in \mathcal{K}'_{n,k,l}$. Also, $I_s(h_m)$ is the image of

$$h_m(s)s^{-1} = (t^m y t^{-m} s) s^{-1} = t^m y t^{-m}$$

in N_Y/M . Letting T be the image of t in $\overline{V}/\overline{V}_Y$, we see that $I_s(h_m) = y^{mT}$. The set $\{y^{mT} \mid m \in \mathbb{Z}\}$ is an infinite set of independent elements of N_Y/M , and we are done. \square

4.3 Group cohomology at the chain level

This section contains some standard results about group cohomology, all of which are contained in [5] (although with different notation).

The standard cochain complex. Let G be a group. Denote by $C^n(G; \mathbb{Q})$ the \mathbb{Q} -vector space consisting of all functions $\phi: G^{n+1} \rightarrow \mathbb{Q}$ which are G -equivariant in the sense that $\phi(g \cdot g_0, \dots, g \cdot g_n) = \phi(g_0, \dots, g_n)$ for all $g, g_0, \dots, g_n \in G$. Elements of $C^n(G; \mathbb{Q})$ are known as n -cochains on G . They fit into a cochain complex

$$\dots \longleftarrow C^{n+1}(G; \mathbb{Q}) \longleftarrow C^n(G; \mathbb{Q}) \longleftarrow C^{n-1}(G; \mathbb{Q}) \longleftarrow \dots \quad (8)$$

whose differential $\delta: C^n(G; \mathbb{Q}) \rightarrow C^{n+1}(G; \mathbb{Q})$ is given by

$$\delta(\phi)(g_0, \dots, g_{n+1}) = \sum_{i=0}^{n+1} (-1)^i \phi(g_0, \dots, \hat{g}_i, \dots, g_{n+1}).$$

The cohomology groups of (8) are $H^*(G; \mathbb{Q})$. If $\phi \in C^n(G; \mathbb{Q})$ is a cocycle, then we will denote by $[\phi]$ the associated element of $H^n(G; \mathbb{Q})$.

Remark. The standard cochain complex is a complex of homomorphisms from the *standard resolution* of \mathbb{Z} over $\mathbb{Z}G$ to \mathbb{Q} . The standard resolution is a chain complex whose n^{th} graded piece has a basis consisting of $(n+1)$ -tuples of elements of G . We note that there is a common alternate notation for the simplices in the standard resolution known as “bar notation.” The use of bar notation leads to a different description of the standard cochain complex and a different formula for the coboundary. We are *not* using bar notation; our reason for avoiding it is that our averaging construction is clearer without it.

First cohomology. The \mathbb{Q} -vector space $H^1(G; \mathbb{Q})$ is isomorphic to the space of homomorphisms $G \rightarrow \mathbb{Q}$. Given a homomorphism $f: G \rightarrow \mathbb{Q}$, the associated element of $C^1(G; \mathbb{Q})$ is given by

$$\phi(g_0, g_1) = f(g_1) - f(g_0) \quad (g_0, g_1 \in G).$$

We will denote the element of $H^1(G; \mathbb{Q})$ corresponding to f by $[f]$.

Cup products. There is a cup product map $\cup: H^p(G; \mathbb{Q}) \otimes H^q(G; \mathbb{Q}) \rightarrow H^{p+q}(G; \mathbb{Q})$. If $\phi \in C^p(G; \mathbb{Q})$ and $\phi' \in C^q(G; \mathbb{Q})$ are cocycles, then $[\phi] \cup [\phi'] = [\phi'']$, where $\phi'' \in C^{p+q}(G; \mathbb{Q})$ is the cochain given by

$$\phi''(g_0, \dots, g_{p+q}) = \phi(g_0, \dots, g_p) \phi'(g_p, \dots, g_{p+q}).$$

Evaluating on abelian cycles. For all $n \geq 0$, there is a bilinear pairing

$$\omega: H^n(G; \mathbb{Q}) \times H_n(G; \mathbb{Q}) \rightarrow \mathbb{Q}.$$

We will need a formula for this in the following situation. Let $f_1, f_2: G \rightarrow \mathbb{Z}$ be homomorphisms. Also, let $g_1, g_2 \in G$ be commuting elements. There is then a homomorphism $\mathbb{Z}^2 \rightarrow G$ taking the generators of \mathbb{Z}^2 to the g_i . Let $c \in H_2(G; \mathbb{Q})$ be the image of the standard generator of $H_2(\mathbb{Z}^2; \mathbb{Q}) \cong \mathbb{Q}$ under the induced map $H_2(\mathbb{Z}^2; \mathbb{Q}) \rightarrow H_2(G; \mathbb{Q})$. We will call c the *abelian cycle* determined by g_1 and g_2 . Using the cup product structure on $H^*(\mathbb{Z}^2; \mathbb{Z})$, we then have

$$\omega([f_1] \cup [f_2], c) = f_1(g_1) f_2(g_2) - f_1(g_2) f_2(g_1).$$

Relating different groups. Assume now that H is a subgroup of G . We then have a diagram

$$\begin{array}{ccccccc}
\dots & \longleftarrow & C^{n+1}(G; \mathbb{Q}) & \longleftarrow & C^n(G; \mathbb{Q}) & \longleftarrow & C^{n-1}(G; \mathbb{Q}) & \longleftarrow & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longleftarrow & C^{n+1}(H; \mathbb{Q}) & \longleftarrow & C^n(H; \mathbb{Q}) & \longleftarrow & C^{n-1}(H; \mathbb{Q}) & \longleftarrow & \dots
\end{array} \tag{9}$$

of cochain complexes inducing the natural map $H^*(G; \mathbb{Q}) \rightarrow H^*(H; \mathbb{Q})$. Here the map $C^n(G; \mathbb{Q}) \rightarrow C^n(H; \mathbb{Q})$ is simply restriction.

There is a cochain complex naturally lying between the cochain complexes in (9). Denote by $C^n(G, H; \mathbb{Q})$ the \mathbb{Q} -vector space consisting of all functions $\phi: G^{n+1} \rightarrow \mathbb{Q}$ that are H -equivariant in the sense that $\phi(g \cdot g_0, \dots, g \cdot g_n) = \phi(g_0, \dots, g_n)$ for all $g \in H$ and $g_0, \dots, g_n \in G$. Observe that $C^n(G; \mathbb{Q}) \subset C^n(G, H; \mathbb{Q})$. The spaces $C^n(G, H; \mathbb{Q})$ fit into a cochain complex

$$\dots \longleftarrow C^{n+1}(G, H; \mathbb{Q}) \longleftarrow C^n(G, H; \mathbb{Q}) \longleftarrow C^{n-1}(G, H; \mathbb{Q}) \longleftarrow \dots$$

whose cohomology groups are easily seen to equal $H^*(H; \mathbb{Q})$. Further, we have maps $C^n(G, H; \mathbb{Q}) \rightarrow C^n(H; \mathbb{Q})$ by restriction. So we have a commutative diagram of the form

$$\begin{array}{ccccccc}
\dots & \longleftarrow & C^{n+1}(G; \mathbb{Q}) & \longleftarrow & C^n(G; \mathbb{Q}) & \longleftarrow & C^{n-1}(G; \mathbb{Q}) & \longleftarrow & \dots \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longleftarrow & C^{n+1}(G, H; \mathbb{Q}) & \longleftarrow & C^n(G, H; \mathbb{Q}) & \longleftarrow & C^{n-1}(G, H; \mathbb{Q}) & \longleftarrow & \dots \tag{10} \\
& & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longleftarrow & C^{n+1}(H; \mathbb{Q}) & \longleftarrow & C^n(H; \mathbb{Q}) & \longleftarrow & C^{n-1}(H; \mathbb{Q}) & \longleftarrow & \dots
\end{array}$$

The maps $C^n(G, H; \mathbb{Q}) \rightarrow C^n(H; \mathbb{Q})$ induce a chain homotopy equivalence. We will need an explicit formula for an inverse chain homotopy equivalence. Fix a set of right coset representatives for H in G . Assume that the representative of the trivial coset is 1. For $g \in G$, denote by \hat{g} the coset representative of the coset $H \cdot g$. Our inverse chain homotopy equivalence is then given by the maps $j: C^n(H; \mathbb{Q}) \rightarrow C^n(G, H; \mathbb{Q})$ defined by

$$j(\phi)(g_0, \dots, g_n) = \phi(g_0(\hat{g}_0)^{-1}, \dots, g_n(\hat{g}_n)^{-1}) \quad (g_0, \dots, g_n \in G).$$

This formula makes sense since $g(\hat{g})^{-1} \in H$ for all $g \in G$. It is easy to see that $j(\phi)$ is H -equivariant and compatible with the coboundary maps.

Remark. If H is normal in G , then the diagonal action of G on $C^n(G, H; \mathbb{Q})$ descends to an action of G/H . By definition, the cochains $C^n(G; \mathbb{Q})$ are simply the invariants $C^n(G, H; \mathbb{Q})^{G/H}$, that is, the subset of cochains that are fixed by G/H . If G/H is finite, then an invariant element can be found simply by taking the sum (or average) of the G/H -orbit of any given element of $C^n(G, H; \mathbb{Q})$. The procedure of taking a cocycle in $C^n(H; \mathbb{Q})$, representing its cohomology class with a cocycle in $C^n(G, H; \mathbb{Q})$, and summing its orbit to get a cocycle in $C^n(G, \mathbb{Q})$ defines a map $H^n(H; \mathbb{Q}) \rightarrow H^n(G; \mathbb{Q})$ called the *transfer map*.

Our construction below builds a cocycle formally using the same procedure that defines the transfer map; however, our subgroup is not of finite index so there is no transfer map to define. As we will see below, our construction works because we use cocycles that satisfy a kind of local finiteness condition.

4.4 The cocycles ζ_r

Recall that we have fixed some $y \in Y$ and some $a, b \in X \cup Z$ such that $a \neq b$. Let A and B be the images of a and b in $\overline{V}/\overline{V}_Y$, respectively. Lemma 4.3 says that $\{y^v \mid v \in \overline{V}/\overline{V}_Y\}$ is a basis for the free abelian group N_Y/M . For $r \in \mathbb{Z}$, let $\alpha'_r: N_Y/M \rightarrow \mathbb{Z}$ be the homomorphism such that

$$\alpha'_r(y^v) = \begin{cases} 1 & \text{if } v = rA \\ 0 & \text{otherwise} \end{cases} \quad (v \in \overline{V}/\overline{V}_Y).$$

and let $\alpha_r: \mathcal{L} \rightarrow \mathbb{Z}$ be the composition

$$\mathcal{L} \xrightarrow{I_b} N_Y/M \xrightarrow{\alpha'_r} \mathbb{Z}.$$

We thus have elements $[\alpha_r] \in H^1(\mathcal{L}; \mathbb{Q})$. By the formulas in §4.3, the cohomology class $[\alpha_r] \cup [\alpha_0] \in H^2(\mathcal{L}; \mathbb{Q})$ can be represented by the cocycle $\eta_r \in C^2(\mathcal{L}; \mathbb{Q})$ defined by the formula

$$\eta_r(g_0, g_1, g_2) = (\alpha_r(g_1) - \alpha_r(g_0))(\alpha_0(g_2) - \alpha_0(g_1)) \quad (g_0, g_1, g_2 \in \mathcal{L}).$$

Our goal is to modify η_r so that it can be extended to a cocycle on \mathcal{K} . Using the recipe in §4.3, we first extend η_r to a cocycle $\kappa_r \in C^2(\mathcal{K}, \mathcal{L}; \mathbb{Q})$ using a chain homotopy equivalence defined in terms of a set of coset representatives. We will need to use a special set of coset representatives which we now describe. It is not hard to see that $J'_y: \mathcal{K} \rightarrow \overline{V}/\overline{V}_Y$ is surjective. Then by the definition of \mathcal{L} as $\text{Ker}(J'_y)$, we have a short exact sequence:

$$1 \longrightarrow \mathcal{L} \longrightarrow \mathcal{K} \xrightarrow{J'_y} \overline{V}/\overline{V}_Y \longrightarrow 1.$$

We choose a normalized section σ to J'_y , so that $\sigma: \overline{V}/\overline{V}_Y \rightarrow \mathcal{K}$ is a set map that is a right-inverse to J'_y and such that $\sigma(1) = 1$. Now, the set

$$S = \{C_{y,s} \mid s \in X \cup Z\} \subset \mathcal{K}$$

projects under J'_y to a basis for the free abelian group $\overline{V}/\overline{V}_Y$. We can therefore choose $\sigma(x) \in \mathcal{K}$ such that $\sigma(x)$ is contained in the subgroup generated by S . Next, for $g \in \mathcal{K}$ let $\hat{g} = \sigma(J'_y(g)) \in \mathcal{K}$. So by construction, for any g in \mathcal{K} , the element \hat{g} is a representative of the coset $\mathcal{L} \cdot g$. Our formula for κ_r is then

$$\begin{aligned} \kappa_r(g_0, g_1, g_2) &= \eta_r(g_0 \hat{g}_0^{-1}, g_1 \hat{g}_1^{-1}, g_2 \hat{g}_2^{-1}) \\ &= (\alpha_r(g_1 \hat{g}_1^{-1}) - \alpha_r(g_0 \hat{g}_0^{-1}))(\alpha_0(g_2 \hat{g}_2^{-1}) - \alpha_0(g_1 \hat{g}_1^{-1})). \end{aligned}$$

Now, if $\kappa_r \in C^2(\mathcal{K}, \mathcal{L}; \mathbb{Q})$ was an element of $C^2(\mathcal{K}; \mathbb{Q}) \subset C^2(\mathcal{K}, \mathcal{L}; \mathbb{Q})$, then it would define a cohomology class in $H^2(\mathcal{K}; \mathbb{Q})$. Alas, this is not true. However, we will be able to average out the action of $\overline{V}/\overline{V}_Y$ on κ_r (as hinted in Remark 9) to get a cocycle that is honestly an element of $C^2(\mathcal{K}; \mathbb{Q})$.

We now consider that action. The diagonal left action of \mathcal{K} on $C^2(\mathcal{K}, \mathcal{L}; \mathbb{Q})$ descends to an action of $\overline{V}/\overline{V}_Y$, because the cochains in this group are already \mathcal{L} -invariant. Specifically, for $x \in \overline{V}/\overline{V}_Y$, the action is given by

$$(x \cdot \kappa_r)(g_0, g_1, g_2) = \kappa_r(\sigma(x)^{-1}g_0, \sigma(x)^{-1}g_1, \sigma(x)^{-1}g_2) \quad (g_0, g_1, g_2 \in \mathcal{K}).$$

We can unravel our expression for $x \cdot \kappa_r$ by considering an appropriate action of $\overline{V}/\overline{V}_Y$ on $\text{Hom}(\mathcal{L}, \mathbb{Z})$. For $\beta \in \text{Hom}(\mathcal{L}, \mathbb{Z})$ and $x \in \overline{V}/\overline{V}_Y$, define

$$(x \cdot \beta)(g) = \beta(\sigma(x)^{-1}g\sigma(x)).$$

This is clearly a well-defined left action: \mathcal{K} acts on $\text{Hom}(\mathcal{L}, \mathbb{Z})$ by conjugating inputs, and since the action of conjugation by \mathcal{L} on $\text{Hom}(\mathcal{L}, \mathbb{Z})$ is trivial, the action descends. Now we can better explain $x \cdot \kappa_r$ with the following lemma.

Lemma 4.6. *For all $g_0, g_1, g_2 \in \mathcal{K}$, we have*

$$(x \cdot \kappa_r)(g_0, g_1, g_2) = ((x \cdot \alpha_r)(g_1 \hat{g}_1^{-1}) - (x \cdot \alpha_r)(g_0 \hat{g}_0^{-1}))((x \cdot \alpha_0)(g_2 \hat{g}_2^{-1}) - (x \cdot \alpha_0)(g_1 \hat{g}_1^{-1})).$$

Proof. From the definitions, it is enough to show that for any $r \in \mathbb{Z}$ and $g \in \mathcal{K}$, we have $\alpha_r(h \hat{h}^{-1}) = (x \cdot \alpha_r)(g \hat{g}^{-1})$, where $h = \sigma(x)^{-1}g$. As noted before, S projects to a basis for the free abelian group $\overline{V}/\overline{V}_Y$. By the definition of σ , we know \hat{h}^{-1} and $\hat{g}^{-1}\sigma(x)$ are in $\langle S \rangle$ and map to the same element of $\overline{V}/\overline{V}_Y$ under J'_y . Since J'_y maps S to a basis for its image (a free abelian group), the kernel of $J'_y|_{\langle S \rangle}$ is $[\langle S \rangle, \langle S \rangle]$. Thus there exists some $w \in [\langle S \rangle, \langle S \rangle]$ such that $\hat{h}^{-1} = \hat{g}^{-1}\sigma(x)w$. Of course $\alpha_k(w) = 0$. Then

$$\alpha_r(h \cdot \hat{h}^{-1}) = \alpha_r(\sigma(x)^{-1}g \hat{g}^{-1}\sigma(x)w) = \alpha_r(\sigma(x)^{-1}g \hat{g}^{-1}\sigma(x)) = (x \cdot \alpha_r)(g \hat{g}^{-1}),$$

as desired. □

This brings us to the following important lemma.

Lemma 4.7. *Fix $r \in \mathbb{Z}$ and $g \in \mathcal{L}$. There then only exist finitely many $x \in \overline{V}/\overline{V}_Y$ such that $(x \cdot \alpha_r)(g) \neq 0$.*

Proof. Write

$$I_b(g) = y^{v_1} + \cdots + y^{v_m} \quad (v_i \in \overline{V}/\overline{V}_Y).$$

For $x \in \overline{V}/\overline{V}_Y$, we then have

$$I_b(\sigma(x)^{-1}g\sigma(x)) = y^{v_1+x} + \cdots + y^{v_m+x}.$$

It follows that $(x \cdot \alpha_r)(g) = 0$ unless $x = rA - v_i$ for some $1 \leq i \leq m$. □

We now define $\zeta_r: (\mathcal{K})^3 \rightarrow \mathbb{Q}$ by the formula

$$\zeta_r(g_0, g_1, g_2) = \sum_{x \in \overline{V}/\overline{V}_Y} (x \cdot \kappa_r)(g_0, g_1, g_2) \quad (g_0, g_1, g_2 \in \mathcal{K}).$$

A priori this infinite sum does not make sense; however, Lemmas 4.6 and 4.7 imply that only finitely many terms of it are nonzero for any particular choice of g_i . Since $x \cdot \kappa_r$ is a cocycle in $C^2(\mathcal{K}, \mathcal{L}; \mathbb{Q})$ for all $x \in \overline{V}/\overline{V}_Y$, we have that ζ_r is a cocycle in $C^2(\mathcal{K}, \mathcal{L}; \mathbb{Q})$. Moreover, by construction we have $\zeta_r(\sigma(x)^{-1}g_0, \sigma(x)^{-1}g_1, \sigma(x)^{-1}g_2) = \zeta_r(g_0, g_1, g_2)$ for all $x \in \overline{V}/\overline{V}_Y$ and $g_0, g_1, g_2 \in \mathcal{K}$. This implies the following lemma.

Lemma 4.8. *For all $r \in \mathbb{Z}$, we have that ζ_r is a cocycle in $C^2(\mathcal{K}; \mathbb{Q})$. Hence ζ_r defines a cohomology class $[\zeta_r] \in H^2(\mathcal{K}; \mathbb{Q})$.*

4.5 Evaluating $[\zeta_r]$ on η_m : the proof of Lemma 4.1

We now conclude this section by proving Lemma 4.1. Let us recall the setup. For $m \in \mathbb{Z}$, define $f_m = C_{y,a}^m C_{b,y} C_{y,a}^{-m}$ and $g = C_{a,y} C_{b,y}$. It is easily verified that f_m and g commute and hence determine an abelian cycle $\eta_m \in H_2(\mathcal{K}; \mathbb{Q})$. Next, recall that there is an evaluation pairing

$$\omega: H^2(\mathcal{K}; \mathbb{Q}) \times H_2(\mathcal{K}; \mathbb{Q}) \longrightarrow \mathbb{Q}.$$

We must prove that for $r, m \geq 1$, we have

$$\omega([\zeta_r], \eta_m) = \begin{cases} 2 & \text{if } r = m \\ 0 & \text{otherwise} \end{cases}$$

Observe first that $f_m, g \in \mathcal{L}$. Using the formula from §4.3, we see that

$$\omega([\zeta_r], \eta_m) = \sum_{x \in \overline{V}/\overline{V}_Y} ((x \cdot \alpha_r)(f_m) \cdot (x \cdot \alpha_0)(g) - (x \cdot \alpha_r)(g) \cdot (x \cdot \alpha_0)(f_m)).$$

We must evaluate the various terms in this expression. Observe that $I_b(f_m)$ is the image of

$$f_m(b)b^{-1} = (a^m y a^{-m} b a^m y^{-1} a^{-m}) b^{-1} = y^{a^m} (y^{b a^m})^{-1}$$

in $\overline{V}/\overline{V}_Y$. As usual, A and B denote the images of a and b in $\overline{V}/\overline{V}_Y$, so we get that $I_b(f_m) = y^{m A} - y^{m A + B}$. Similarly, $I_b(g)$ is the image of

$$g(b)b^{-1} = (y b y^{-1}) b^{-1} = y (y^b)^{-1}$$

in $\overline{V}/\overline{V}_Y$, so $I_b(g) = y - y^B$. From these two calculations, we obtain that

$$(x \cdot \alpha_s)(f_m) = \begin{cases} 1 & \text{if } x = (s - m)A \\ -1 & \text{if } x = (s - m)A - B \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (x \cdot \alpha_s)(g) = \begin{cases} 1 & \text{if } x = sA \\ -1 & \text{if } x = sA - B \\ 0 & \text{otherwise} \end{cases}$$

for all $s \in \mathbb{Z}$ and $x \in \overline{V}/\overline{V}_Y$. This implies that

$$(x \cdot \alpha_r)(f_m) \cdot (x \cdot \alpha_0)(g) = \begin{cases} 1 & \text{if } r = m \text{ and either } x = 0 \text{ or } x = -B \\ 0 & \text{otherwise} \end{cases}$$

$$(x \cdot \alpha_r)(g) \cdot (x \cdot \alpha_0)(f_m) = \begin{cases} 1 & \text{if } r = -m \text{ and either } x = rA \text{ or } x = rA - B \\ 0 & \text{otherwise} \end{cases}$$

This allows us to conclude that for $r, m \geq 1$, we have

$$\omega(\zeta_r, \eta_m) = \sum_{x \in \overline{V}/\overline{V}_Y} ((x \cdot \alpha_r)(f_m) \cdot (x \cdot \alpha_0)(g) - (x \cdot \alpha_r)(g) \cdot (x \cdot \alpha_0)(f_m)) = \begin{cases} 2 & \text{if } r = m \\ 0 & \text{otherwise,} \end{cases}$$

as desired.

5 The combinatorial group theory of $\mathcal{K}_{n,k,l}$

The goal in this section is to prove Theorem A (which gives generators for $\mathcal{K}_{n,k,l}$) and Theorem 5.6 in §5.2 below (which gives relations for $\mathcal{K}_{n,k,l}$, making precise Theorem C). We begin in §5.1 by discussing a presentation of $\mathcal{A}_{n,k,l}$ due to Jensen–Wahl. Next, in §5.2 we prove Theorems A and 5.6. These proofs will depend on a calculation which is contained in §5.3.

Throughout this section, we will make use of the notation

$$X = \{x_1, \dots, x_n\} \quad \text{and} \quad Y = \{y_1, \dots, y_k\} \quad \text{and} \quad Z = \{z_1, \dots, z_l\}.$$

5.1 A presentation for $\mathcal{A}_{n,k,l}$

For $1 \leq i, j \leq n$ such that $i \neq j$ and $1 \leq i' \leq n$, let $P_{i,j}$ and $I_{i'}$ be the elements of $\mathcal{A}_{n,k,l}$ that have the following behavior for $v \in X \cup Y \cup Z$.

$$P_{i,j}(v) = \begin{cases} x_j & \text{if } v = x_i \\ x_i & \text{if } v = x_j \\ v & \text{otherwise} \end{cases} \quad \text{and} \quad I_{i'}(v) = \begin{cases} x_{i'}^{-1} & \text{if } v = x_{i'} \\ v & \text{otherwise} \end{cases}$$

The automorphisms of the form $P_{i,j}$ are *swaps* and the automorphisms of the form I_i are *inversions*. Elements of the form $M_{w^\epsilon, v}$ for $w, v \in X \cup Y \cup Z$ are *Nielsen moves* and elements of the form $C_{w, v}$ for $v \in X \cup Y \cup Z$ and $w \in X \cup Z$ are *conjugation moves*. Throughout the rest of the paper, we will tacitly identify $P_{i,j}$ and $P_{j,i}$. We will make an extension of our notation to simplify the statement of certain relations (N2 and Q3) given below: for $v, w \in X \cup Y \cup Z$ and $\epsilon = \pm 1$, we define the symbols

$$M_{v^\epsilon, w^{-1}} = M_{v^\epsilon, w}^{-1} \quad \text{and} \quad C_{v, w^{-1}} = C_{v, w}^{-1}.$$

The following theorem is a restatement of Nielsen's classical presentation [24] for $\text{Aut } F_n$ in our notation and using our composition convention (right to left). A good reference for

Nielsen's presentation is its verification by McCool [22]. We use McCool's terminology N1-5 for the five classes of relations in Nielsen's presentation. We restrict our attention to $\text{Aut}(F(X)) = \mathcal{A}_{n,0,0}$.

Theorem 5.1 (Nielsen [24]). *The group $\text{Aut}(F(X))$ has the presentation $\langle S_N \mid R_N \rangle$, where*

$$S_N = \{P_{a,b} \mid 1 \leq a, b \leq n, a \neq b\} \cup \{I_a \mid 1 \leq a \leq n\} \\ \cup \{M_{x_a^\epsilon, x_b} \mid 1 \leq a, b \leq n, a \neq b, \epsilon = \pm 1\},$$

and R_N is given in Table 2.

Nielsen's relations for $\text{Aut}(F(X))$

The relations R_N consist of the following, where $\epsilon, \delta \in \{1, -1\}$ and indices a, b, c, d are assumed to be distinct elements of $\{1, \dots, n\}$ unless stated otherwise:

N1. relations for the subgroup generated by inversions and swaps, a signed permutation group:

- $I_a^2 = 1$ and $[I_a, I_b] = 1$,
- $P_{a,b}^2 = 1$, $[P_{a,b}, P_{c,d}] = 1$, and $P_{a,b}P_{b,c}P_{a,b}^{-1} = P_{a,c}$,
- $P_{a,b}I_aP_{a,b}^{-1} = I_b$ and $[P_{a,b}, I_c] = 1$;

N2. relations for conjugating Nielsen moves by inversions and swaps, coming from the natural action of inversions and swaps on $X^{\pm 1}$:

- $P_{a,b}M_{x^\epsilon, y}P_{a,b}^{-1} = M_{P_{a,b}(x^\epsilon), P_{a,b}(y)}$ for $x, y \in X$,
- $I_aM_{x^\epsilon, y}I_a^{-1} = M_{I_a(x^\epsilon), I_a(y)}$ for $x, y \in X$;

N3. $M_{x_a^{-1}, x_b}^{-1} M_{x_b^{-1}, x_a} M_{x_a, x_b} = I_a P_{a,b}$ and $M_{x_a^{-1}, x_b} M_{x_b, x_a} M_{x_a, x_b}^{-1} = I_b P_{a,b}$;

N4. $[M_{x_a^\epsilon, x_b}, M_{x_c^\delta, x_d}] = 1$ with a, b, c, d not necessarily all distinct, such that $a \neq b$, $c \neq d$, $x_a^\epsilon \notin \{x_c^\delta, x_d, x_d^{-1}\}$ and $x_c^\delta \notin \{x_b, x_b^{-1}\}$; and

N5. $M_{x_b^\epsilon, x_a} M_{x_c^\delta, x_b}^\epsilon = M_{x_c^\delta, x_b}^\epsilon M_{x_b^\epsilon, x_a} M_{x_c^\delta, x_a}$.

Table 2

We now turn to $\mathcal{A}_{n,k,l}$. The following presentation is due to Jensen-Wahl [16], who made use of an algorithm for generating such a presentation due to McCool [21].

Theorem 5.2 (Jensen-Wahl, [16]). *The group $\mathcal{A}_{n,k,l}$ has a finite presentation $\langle S \mid R \rangle$ where the generating set S consists of the following elements:*

- $P_{i,j}$ for $1 \leq i, j \leq n$ such that $i \neq j$,
- I_i for $1 \leq i \leq n$,
- $M_{x^\epsilon, v}$ for $\epsilon = \pm 1$, $x \in X$, and $v \in X \cup Y \cup Z$ such that $v \neq x$,
- $C_{v, w}$ for $v \in Y \cup Z$ and $w \in X \cup Y \cup Z$ such that $v \neq w$,

and the relations R are given in Table 3.

Remark. In Jensen–Wahl’s original version of the presentation, the relations Q1 were not given explicitly; rather they specified that Q1 should consist of some set of relations for $\text{Aut}(F(X))$ with respect to the generating set S_N . Of course, Nielsen’s relations suffice. We have made minor changes to the presentation that are necessary for giving it in our notation and with our conventions; this includes renumbering the subclasses of relations Q3 and Q4.

Remark. In each of the classes of relation Q3.1–4, there are some relations that state that Nielsen moves and conjugation moves not in S_N commute with swaps and inversions that share no indices in common. These relations do not appear in Jensen–Wahl’s original presentation; we believe their omission to be an error in [16], although a largely inconsequential one. A careful reading of Jensen–Wahl’s proof of this theorem indicates the need for these extra relations.

5.2 Generators and relations for $\mathcal{K}_{n,k,l}$

Our main tool for studying the combinatorial group theory of $\mathcal{K}_{n,k,l}$ is a certain special presentation for $\mathcal{A}_{n,k,l}$ which is a slight modification of Jensen–Wahl’s presentation from Theorem 5.2. Our presentation is inspired by the split exact sequence

$$1 \longrightarrow \mathcal{K}_{n,k,l} \longrightarrow \mathcal{A}_{n,k,l} \longrightarrow \mathcal{A}_{n,0,l} \longrightarrow 1$$

from §1. Since this sequence is split, $\mathcal{A}_{n,k,l}$ is the semidirect product of $\mathcal{A}_{n,0,l}$ and $\mathcal{K}_{n,k,l}$. Our presentation resembles the standard presentation for a semidirect product.

We begin by discussing our generating set, which consists of $S_Q \cup S_K$ where S_Q and S_K are defined by

$$\begin{aligned} S_Q = & \{P_{i,j} \mid 1 \leq i, j \leq n, i \neq j\} \cup \{I_i \mid 1 \leq i \leq n\} \cup \{C_{z,v} \mid v \in X \cup Z, z \in Z\} \\ & \cup \{M_{x^\epsilon, v} \mid x \in X, v \in X \cup Z, x \neq v, \epsilon = \pm 1\} \end{aligned}$$

and

$$\begin{aligned} S_K = & \{M_{x^\epsilon, y} \mid x \in X, y \in Y, \epsilon = \pm 1\} \cup \{C_{z,y} \mid z \in Z, y \in Y\} \\ & \cup \{C_{y,v} \mid y \in Y, v \in X \cup Y \cup Z, y \neq v\}. \end{aligned}$$

Jensen and Wahl's relations for $\mathcal{A}_{n,k,l}$

The relations R consist of the following:

- Q1. all the relations R_N for $\text{Aut}(F(X))$ from Nielsen's presentation, which are relations between among elements of S_N , a subset of S ;
- Q2. relations that certain generators commute:
- (1) $[M_{x^\epsilon, v}, M_{w^\delta, z}] = 1$ for $\epsilon, \delta = \pm 1$, $x, v, w \in X$ and $z \in Y \cup Z$, with $x^\epsilon \neq w^\delta$ and $x \neq v$;
 - (2) $[M_{x^\epsilon, v}, C_{w, z}] = 1$ for $\epsilon = \pm 1$, $x, v \in X$ and $w, z \in Y \cup Z$, with $w \notin \{v, z\}$ and $x \neq v$;
 - (3) $[C_{u, v}, C_{w, z}] = 1$ for $u, v, w, z \in Y \cup Z$ with $u \notin \{v, w, z\}$ and $w \notin \{v, z\}$;
- Q3. relations conjugating other generators by swaps and inversions:
- (1) $P_{i, j} M_{x^\epsilon, z} P_{i, j}^{-1} = M_{P_{i, j}(x^\epsilon), z}$ for $\epsilon = \pm 1$, $x \in X$, $z \in Y \cup Z$, and $1 \leq i, j \leq n$ such that $i \neq j$;
 - (2) $P_{i, j} C_{z, x} P_{i, j}^{-1} = C_{z, P_{i, j}(x)}$ for $x \in X$, $z \in Y \cup Z$, and $1 \leq i, j \leq n$ such that $i \neq j$;
 - (3) $I_i M_{x^\epsilon, z} I_i^{-1} = M_{I_i(x^\epsilon), z}$ for $\epsilon = \pm 1$, $x \in X$, $z \in Y \cup Z$, and $1 \leq i \leq n$;
 - (4) $I_i C_{z, x} I_i^{-1} = C_{z, I_i(x)}$ for $x \in X$, $z \in Y \cup Z$, and $1 \leq i \leq n$;
- Q4. (1) $M_{x^\epsilon, w}^{-\delta} M_{w^\delta, v} M_{x^\epsilon, w}^\delta = M_{x^\epsilon, v} M_{w^\delta, v}$ for $\epsilon, \delta = \pm 1$, $x, w \in X$, and $v \in X \cup Y \cup Z$ such that $x \neq w$, $x \neq v$, and $w \neq v$;
- (1') $C_{z, x}^{-\epsilon} M_{x^\epsilon, v} C_{z, x}^\epsilon = C_{z, v} M_{x^\epsilon, v}$ for $\epsilon = \pm 1$, $x \in X$, $z \in Y \cup Z$, and $v \in X \cup Y \cup Z$ such that $x \neq v$ and $z \neq v$;
- (2) $C_{z, v}^\epsilon M_{x^\delta, z} C_{z, v}^{-\epsilon} = M_{x^\delta, v}^{-\epsilon} M_{x^\delta, z} M_{x^\delta, v}^\epsilon$ for $\epsilon, \delta = \pm 1$, $x \in X$, $z \in Y \cup Z$, and $v \in X \cup Y \cup Z$ such that $x \neq v$ and $z \neq v$;
- (2') $C_{z, v}^\epsilon C_{w, z} C_{z, v}^{-\epsilon} = C_{w, v}^{-\epsilon} C_{w, z} C_{w, v}^\epsilon$ for $\epsilon = \pm 1$, $z, w \in Y \cup Z$, and $v \in X \cup Y \cup Z$ such that $z \neq w$, $z \neq v$, and $w \neq v$; and
- Q5. $C_{v, x}^{-\epsilon} M_{x^{-\epsilon}, v} C_{v, x}^\epsilon = M_{x^\epsilon, v}^{-1}$ for $\epsilon = \pm 1$, $x \in X$, and $v \in Y \cup Z$.
-

Table 3

$t \in S_K$	$s \in S_Q^{\pm 1}$	sts^{-1}	Relations Used
$M_{x_a^\epsilon, y}$	$M_{x_a^\delta, x_b}^\delta$	$C_{y, x_b}^{-\delta} M_{x_a^\epsilon, y} C_{y, x_b}^\delta$	Q4.2
	$M_{x_a^\delta, z_i}^\delta$	$C_{y, z_i}^{-\delta} M_{x_a^\epsilon, y} C_{y, z_i}^\delta$	Q4.2
	$M_{x_b^\delta, x_a}^\epsilon$	$M_{x_a^\epsilon, y} C_{y, x_a}^{-\epsilon} M_{x_b^\delta, y}^{-1} C_{y, x_a}^\epsilon$	Q5, Q2.1-2, Q4.1
	$M_{x_b^\delta, x_a}^{-\epsilon}$	$M_{x_a^\epsilon, y} M_{x_b^\delta, y}$	Q4.1
	C_{z_i, x_a}^ϵ	$M_{x_a^\epsilon, y} C_{y, x_a}^{-\epsilon} C_{z_i, y}^{-1} C_{y, x_a}^\epsilon$	Q5, Q2.2-3, Q4.1'
	$C_{z_i, x_a}^{-\epsilon}$	$M_{x_a^\epsilon, y} C_{z_i, y}$	Q4.1'
	$P_{a, b}$	$M_{x_b^\epsilon, y}$	Q3.1
	I_a	$M_{x_a^{-\epsilon}, y}$	Q3.3
C_{y, x_a}	$M_{x_a^\delta, x_b}^\delta$	$(C_{y, x_a}^\epsilon C_{y, x_b}^\delta)^\epsilon$	Q4.1'
	$M_{x_a^\delta, z_i}^\delta$	$(C_{y, x_a}^\epsilon C_{y, z_i}^\delta)^\epsilon$	Q4.1'
	$P_{a, b}$	C_{y, x_b}	Q3.2
	I_a	C_{y, x_a}^{-1}	Q3.4
$C_{z_i, y}$	C_{z_i, x_a}^ϵ	$C_{y, x_a}^{-\epsilon} C_{z_i, y} C_{y, x_a}^\epsilon$	Q4.2'
	$M_{x_a^\delta, z_i}^\delta$	$C_{z_i, y} [M_{x_a^\delta, y}, C_{y, z_i}^{-\delta}]$	Q4.2 (twice)
	C_{z_i, z_j}^ϵ	$C_{y, z_j}^{-\epsilon} C_{z_i, y} C_{y, z_j}^\epsilon$	Q4.2'
	C_{z_j, z_i}^ϵ	$C_{z_i, y} [C_{z_j, y}, C_{y, z_i}^{-\epsilon}]$	Q4.2' (twice)
C_{y, z_i}	C_{z_i, x_a}^ϵ	$C_{y, x_a}^{-\epsilon} C_{y, z_i} C_{y, x_a}^\epsilon$	Q4.2'
	C_{z_i, z_j}^ϵ	$C_{y, z_j}^{-\epsilon} C_{y, z_i} C_{y, z_j}^\epsilon$	Q4.2'

Table 4: For $s \in S_Q^{\pm 1}$ and $t \in S_K$, this table gives sts^{-1} in terms of the generating set S_K . If there is no entry for a particular $s \in S_Q^{\pm 1}$ and $t \in S_K$, then $sts^{-1} = s^{-1}ts = t$ (and these are relations in Q2). In this table, $y \in Y$, $\epsilon, \delta = \pm 1$, $1 \leq a, b \leq n$ and $1 \leq i, j \leq l$.

Of course, $S_Q \cup S_K$ is the generating set from Theorem 5.2. Clearly $S_K \subset \mathcal{K}_{n, k, l}$, and by Theorem 5.2 the set S_Q generates $\mathcal{A}_{n, 0, l} \subset \mathcal{A}_{n, k, l}$. Table 4 gives an expression in the generating set S_K for sts^{-1} and $s^{-1}ts$, for each choice of $s \in S_Q$ and $t \in S_K$. The correctness of these expressions as equations in $\mathcal{A}_{n, k, l}$ can be verified by direct computation, or by using the relations in Theorem 5.2. We have provided hints for the reader who wishes to derive these relations from the earlier ones. With Table 4, we can prove Theorem A, which we recall asserts that S_K generates $\mathcal{K}_{n, k, l}$.

Proof of Theorem A. In the presentation for $\mathcal{A}_{n, k, l}$ from Theorem 5.2, setting $t = 1$ for all $t \in S_K$ results in the presentation from the same theorem for $\mathcal{A}_{n, 0, l}$. This implies that S_K normally generates $\mathcal{K}_{n, k, l}$. The calculations in Table 4 then show that the subgroup of $\mathcal{A}_{n, k, l}$ generated by S_K is normal, so S_K generates $\mathcal{K}_{n, k, l}$. \square

We now turn to the relations for our new presentation of $\mathcal{A}_{n, k, l}$, which consist of $R_Q \cup$

$R_K \cup R_{\text{conj}}$, where R_Q , R_K , and R_{conj} are as defined below. First, $R_Q \subset F(S_Q)$ is the subset of the relations R from Theorem 5.2 that are in $F(S_Q)$. By inspecting Theorem 5.2, it is apparent that $\langle S_Q \mid R_Q \rangle \cong \mathcal{A}_{n,0,l}$, since these are exactly the relations that appear when $k = 0$. Next, let

$$f: S_Q^{\pm 1} \times S_K \rightarrow F(S_K) \quad (11)$$

be the function that takes $(s^\epsilon, t) \in S_Q^{\pm 1} \times S_K$ to the expression for $s^\epsilon t s^{-\epsilon}$ given by Table 4. Define

$$R_{\text{conj}} = \{s^\epsilon t s^{-\epsilon} (f(s^\epsilon, t))^{-1} \mid s \in S_Q, t \in S_K, \epsilon = \pm 1\}.$$

Finally, we define R_K to be the subset of the relations R from Theorem 5.2 that lie entirely within $F(S_K)$. The reader can check that this coincides with the set of relations R1-R5 from §1.

We can now state our presentation.

Proposition 5.3. *We have $\mathcal{A}_{n,k,l} \cong \langle S_K \cup S_Q \mid R_K \cup R_Q \cup R_{\text{conj}} \rangle$.*

Before proving this, we record a few additional relations. The relations C2 are used to prove Proposition 5.3; the relations C1 will be used later.

Lemma 5.4. *The following relations in S_K follow from R_K and R_{conj} , and therefore hold in $\langle S_K \cup S_Q \mid R_K \cup R_Q \cup R_{\text{conj}} \rangle$:*

- C1. (1) $[C_{y,v}^\epsilon M_{x_a^\delta, y} C_{y,v}^{-\epsilon}, M_{x_b^\zeta, y}] = 1$ for $\epsilon, \delta, \zeta = \pm 1$, $x_a, x_b \in X$, $y \in Y$ and $v \in X \cup Z$ with $x_a^\delta \neq x_b^\zeta$, $v \notin \{x_a, x_b, y\}$.
- (2) $[C_{y,v}^\epsilon M_{x^\delta, y} C_{y,v}^{-\epsilon}, C_{z,y}] = 1$ for $\epsilon, \delta = \pm 1$, $x \in X$, $y \in Y$, $z \in Y \cup Z$ and $v \in X \cup Z$, with $z \neq y$ and $v \notin \{z, x, y\}$.
- (3) $[C_{y,v}^\epsilon C_{z,y} C_{y,v}^{-\epsilon}, C_{w,y}] = 1$ for $\epsilon = \pm 1$, $y \in Y$, $z, w \in Y \cup Z$ and $v \in X \cup Z$ with $w \neq u$, $y \notin \{z, w\}$ and $v \notin \{z, w, y\}$.
- C2. (1) $[C_{y,z}^\epsilon M_{x^\delta, y} C_{y,z}^{-\epsilon}, C_{z,y} M_{x^\delta, y}] = 1$ for $\epsilon, \delta = \pm 1$, $y \in Y$, $z \in Z$ and $x \in X$.
- (2) $[C_{y,z}^\epsilon C_{w,y} C_{y,z}^{-\epsilon}, C_{z,y} C_{w,y}] = 1$ for $\epsilon = \pm 1$, $y \in Y$, and $w, z \in Z$ with $w \neq z$.

Proof. To derive C1.1, start with the relation $[M_{x_a^\delta, y}, M_{x_b^\zeta, y}] = 1$ from R1.1. We conjugate this relation by $M_{x_a^\delta, v}^{-\epsilon}$ and move the conjugating elements into the commutator. The relation C1.1 follows by modifying the conjugates of commutator factors using relations from R_{conj} . The derivations for C1.2 and C1.3 are completely parallel.

To derive C2.1, start with the relation $[M_{x^\delta, y}, C_{v,y}] = 1$ from R1.2. We conjugate by $M_{x^\delta, v}^{-\epsilon}$, move the conjugating elements into the commutator, and apply relations from R_{conj} to get

$$[C_{y,v}^\epsilon M_{x^\delta, y} C_{y,v}^{-\epsilon}, C_{v,y} M_{x^\delta, y} C_{y,v}^\epsilon M_{x^\delta, y}^{-1} C_{y,v}^{-\epsilon}] = 1.$$

This is a product of the form $[g, hg^{-1}] = 1$, so of course the corresponding expression $[g, h] = 1$ follows, which is relation C2.1. The derivation of C2.2 is parallel. \square

Proof of Proposition 5.3. Since all the relations in $R_K \cup R_Q \cup R_{\text{conj}}$ are relations in $\mathcal{A}_{n,k,l}$, it is enough to show that all the relations in Theorem 5.2 are consequences of the relations $R_K \cup R_Q \cup R_{\text{conj}}$. In fact, almost all the relations in Theorem 5.2 already lie in $R_K \cup R_Q \cup R_{\text{conj}}$. As the reader can easily check, there are three exceptions.

- Relation Q4.1' when $v \in Y$ and $z \in X \cup Z$. Recall that this asserts that

$$C_{v,x}^{-\epsilon} M_{x^\epsilon,z} C_{v,x}^\epsilon = C_{v,z} M_{x^\epsilon,z}.$$

Rearranging the terms, we get

$$M_{x^\epsilon,z} C_{v,x}^\epsilon M_{x^\epsilon,z}^{-1} = C_{v,x}^\epsilon C_{v,z},$$

which lies in R_{conj} (it is written in Table 4 as $M_{x^\epsilon,z} C_{v,x} M_{x^\epsilon,z}^{-1} = (C_{v,x}^\epsilon C_{v,z})^\epsilon$).

- Relation Q4.2 when $v \in Z$ and $z \in Y$. This asserts that

$$C_{v,z}^\epsilon M_{x^\delta,v} C_{v,z}^{-\epsilon} = M_{x^\delta,z}^{-\epsilon} M_{x^\delta,v} M_{x^\delta,z}^\epsilon.$$

Multiplying on the left by $M_{x^\delta,v}^{-1}$ and applying two relations from R_{conj} , we see that we must prove that

$$([C_{z,v}, M_{x^\delta,z}^{-1}] C_{v,z})^\epsilon C_{v,z}^{-\epsilon} = C_{z,v} M_{x^\delta,z}^{-\epsilon} C_{z,v}^{-1} M_{x^\delta,z}^\epsilon.$$

For $\epsilon = 1$, this is trivially true. For $\epsilon = -1$, it states that

$$C_{v,z}^{-1} M_{x^\delta,z}^{-1} C_{z,v} M_{x^\delta,z} C_{z,v}^{-1} C_{v,z} = C_{z,v} M_{x^\delta,z} C_{z,v}^{-1} M_{x^\delta,z}^{-1}.$$

By relation R1.2, we commute $M_{x^\delta,z}^{-1}$ past $C_{v,z}^{-1}$ to get

$$(M_{x^\delta,z}^{-1} C_{v,z}^{-1})(C_{z,v} M_{x^\delta,z} C_{z,v}^{-1}) C_{v,z} = C_{z,v} M_{x^\delta,z} C_{z,v}^{-1} M_{x^\delta,z}^{-1}.$$

By relation C2.1 from Lemma 5.4, we can commute $M_{x^\delta,z}^{-1} C_{v,z}^{-1}$ past $C_{z,v} M_{x^\delta,z} C_{z,v}^{-1}$. Then the resulting relation is obviously true.

- Relation Q4.2' when v and w are in Z and z is in Y . This asserts that

$$C_{v,z}^\epsilon C_{w,v} C_{v,z}^{-\epsilon} = C_{w,z}^{-\epsilon} C_{w,v} C_{w,z}^\epsilon.$$

Multiplying on the left by $C_{w,v}^{-1}$ and applying two relations from R_{conj} , we see that we must prove that

$$([C_{z,v}, C_{w,z}^{-1}] C_{v,z})^\epsilon C_{v,z}^{-\epsilon} = (C_{z,v} C_{w,z} C_{z,v}^{-1})^{-\epsilon} C_{w,z}^\epsilon.$$

For $\epsilon = 1$, this is clearly true. For $\epsilon = -1$, this becomes

$$C_{v,z}^{-1} C_{w,z}^{-1} C_{z,v} C_{w,z} C_{z,v}^{-1} C_{v,z} = C_{z,v} C_{w,z} C_{z,v}^{-1} C_{w,z}^{-1}.$$

Relation R1.3 allows us to commute $C_{v,z}^{-1}$ past $C_{w,z}^{-1}$, and the relation becomes

$$(C_{w,z}^{-1} C_{v,z}^{-1})(C_{z,v} C_{w,z} C_{z,v}^{-1}) C_{v,z} = C_{z,v} C_{w,z} C_{z,v}^{-1} C_{w,z}^{-1}.$$

Then by relation C2.2 from Lemma 5.4, we can commute $C_{w,z}^{-1} C_{v,z}^{-1}$ past $C_{z,v} C_{w,z} C_{z,v}^{-1}$ and the resulting expression is trivially true. \square

From Proposition 5.3, one might be led to think that $\mathcal{K}_{n,k,l}$ is isomorphic to $\langle S_K \mid R_K \rangle$. If it were, then the standard presentation for a semidirect product would yield Proposition 5.3. However, this is false (recall that we earlier proved Theorem B, which asserts that $\mathcal{K}_{n,k,l}$ is not finitely presented). The problem is that $\mathcal{A}_{n,0,l}$ does not act on $\langle S_K \mid R_K \rangle$; the obvious candidate for an action from Proposition 5.3 does not preserve R_K .

To fix this, we will have to add some additional relations. Recall that above (see (11)) we defined a function $f: S_Q^{\pm 1} \times S_K \rightarrow F(S_K)$. For fixed $s \in S_Q^{\pm 1}$, we have $f(s, \cdot): S_K \rightarrow F(S_K)$. Since $F(S_K)$ is a free group, this extends to a homomorphism $f(s, \cdot): F(S_K) \rightarrow F(S_K)$. These maps assemble to a set map $f: S_Q^{\pm 1} \rightarrow \text{End}(F(S_K))$. A tedious but elementary calculation shows that for all $t \in S_Q$ and $s \in S_K$, we have

$$f(t^{-1}, f(t, s)) = s \quad \text{and} \quad f(t, f(t^{-1}, s)) = s.$$

This lets us deduce two things. First, $f(S_Q^{\pm 1}) \subset \text{Aut}(F(S_K))$. Second, $f(t^{-1}) = f(t)^{-1}$. This second fact lets us extend f to a homomorphism $f: F(S_Q) \rightarrow \text{Aut}(F(S_K))$. In other words, f induces an action of $F(S_Q)$ on $F(S_K)$. We will still write f as a two-variable function $f: F(S_Q) \times F(S_K) \rightarrow F(S_K)$. Finally, we construct our extended set of relations.

$$\hat{R}_K = \{f(w, r) \mid w \in F_Q, r \in R_K\} \subset F(S_K).$$

One should think of $f(w, r)$ here as the ‘‘conjugate’’ of the relation r by w . By construction, $F(S_Q)$ acts on $\langle S_K \mid \hat{R}_K \rangle$. We will prove the following proposition in §5.3.

Proposition 5.5. *The action of $F(S_Q)$ on $\langle S_K \mid \hat{R}_K \rangle$ descends to an action of $\mathcal{A}_{n,0,l}$ on $\langle S_K \mid \hat{R}_K \rangle$.*

We can now give a presentation for $\mathcal{K}_{n,k,l}$.

Theorem 5.6. *We have $\mathcal{K}_{n,k,l} \cong \langle S_K \mid \hat{R}_K \rangle$.*

Proof. Set $\Gamma = \langle S_K \mid \hat{R}_K \rangle$. Using Proposition 5.5, we can form the semidirect product of Γ and $\mathcal{A}_{n,0,l}$. This fits into the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma & \longrightarrow & \Gamma \rtimes \mathcal{A}_{n,0,l} & \longrightarrow & \mathcal{A}_{n,0,l} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow \cong & & \\ 1 & \longrightarrow & \mathcal{K}_{n,k,l} & \longrightarrow & \mathcal{A}_{n,k,l} & \longrightarrow & \mathcal{A}_{n,0,l} & \longrightarrow & 1 \end{array}$$

Now, $\Gamma \rtimes \mathcal{A}_{n,0,l}$ has the presentation $\langle S_K \cup S_Q \mid \hat{R}_K \cup R_Q \cup R_{\text{conj}} \rangle$. However, using the relations R_{conj} we see that the relations in $\hat{R}_K \setminus R_K$ are redundant. Using Proposition 5.3, we deduce that

$$\Gamma \rtimes \mathcal{A}_{n,0,l} \cong \langle S_K \cup S_Q \mid R_K \cup R_Q \cup R_{\text{conj}} \rangle \cong \mathcal{A}_{n,k,l}.$$

The five lemma then implies that $\Gamma \cong \mathcal{K}_{n,k,l}$, as desired. \square

In [1], Bartholdi defined an L -presentation to be an expression

$$\langle S \mid Q \mid \Phi \mid R \rangle$$

where S is a set of generators, Φ is a set of endomorphisms of $F(S)$, and Q and R are sets of relations in $F(S)$. The group presented by the L -presentation above is $F(S)/N$, where $N \triangleleft F(S)$ is the normal closure of the set

$$Q \cup \bigcup_{\phi \in \Phi^*} \phi(R).$$

Here Φ^* denotes the closure of $\Phi \cup \{\text{id}_{F(S)}\}$ under composition. An L -presentation is *finite* if S , Q , Φ and R are finite, *ascending* if $Q = \emptyset$, and *injective* if Φ consists of injective endomorphisms.

Corollary 5.7. *The group $\mathcal{K}_{n,k,l}$ has the following finite L -presentation, which is also ascending and injective.*

$$\mathcal{K}_{n,k,l} \cong \langle S_K \mid \emptyset \mid \{f(t)\}_{t \in S_Q^{\pm 1}} \mid R_K \rangle.$$

Proof. This is an immediate consequence of Theorem 5.6 and the definition of \hat{R}_K . \square

5.3 Proof of Proposition 5.5

In this subsection, we again use Γ to denote $\langle S_K \mid \hat{R}_K \rangle$. We prove here that $\mathcal{A}_{n,0,l}$ acts on Γ . Our proof uses the presentation of $\mathcal{A}_{n,0,l}$ directly; we would prefer a more conceptual proof but we were unable to find one.

Lemma 5.8. *The relations C1.1-3 and C2.1-2 from Lemma 5.4 hold in Γ .*

Proof. Inspecting the proof of Lemma 5.4, we see that all relations C1.1-3 and C2.1-2 follow easily from relations in $\{f(t, r) \mid t \in S_Q^{\pm 1}, r \in R_K\}$, which is a subset of \hat{R}_K . \square

For each generator $s \in S_Q \cup S_K$, we define two subsets, the *support*, denoted $\text{supp}(s) \subset (X \cup Y \cup Z)^{\pm 1}$, and the *multiplier set*, denoted $\text{mult } s \subset X \cup Y \cup Z$. We define these as follows:

- $\text{supp}(I_a) = \{x_a, x_a^{-1}\}$ and $\text{mult}(I_a) = \{x_a\}$,
- $\text{supp}(P_{a,b}) = \{x_a, x_a^{-1}, x_b, x_b^{-1}\}$ and $\text{mult}(P_{a,b}) = \{x_a, x_b\}$,
- $\text{supp}(M_{x^\epsilon, v}) = \{x^\epsilon\}$ and $\text{mult}(M_{x^\epsilon, v}) = \{v\}$, and
- $\text{supp}(C_{w, v}) = \{w, w^{-1}\}$ and $\text{mult}(C_{w, v}) = \{v\}$.

For s with $s^{-1} \in S_K \cup S_Q$, define $\text{supp}(s) = \text{supp}(s^{-1})$ and $\text{mult}(s) = \text{mult}(s^{-1})$. For $g \in F(S_K \cup S_Q)$ with $g = s_1 \cdots s_m$ (as reduced words) for $s_1, \dots, s_m \in (S_K \cup S_Q)^{\pm 1}$, define $\text{supp}(g) = \bigcup_{i=1}^m \text{supp}(s_i)$ and $\text{mult}(g) = \bigcup_{i=1}^m \text{mult}(s_i)$.

The following two lemmas can easily be proven using the definition of f and induction on word length. The proofs are left to the reader.

Lemma 5.9. *Suppose $s \in F(S_K)$ and $t \in F(S_Q)$. If $\text{supp}(s) \cap \text{supp}(t) = \emptyset$, $\text{supp}(s) \cap \text{mult}(t)^{\pm 1} = \emptyset$ and $\text{supp}(t) \cap \text{mult}(s)^{\pm 1} = \emptyset$, then $f(t, s) = s$ as elements of $F(S_K)$.*

Lemma 5.10. *Suppose $s \in F(S_K)$ and $t \in F(S_Q)$. Then $\text{supp}(f(t, s)) \subset \text{supp}(s) \cup \text{supp}(t) \cup \text{mult}(s)^{\pm 1}$ and $\text{mult}(f(t, s)) \subset \text{mult}(s) \cup \text{mult}(t)$.*

Proof of Proposition 5.5. Recall that the goal is to show that f defines an action of $\mathcal{A}_{n,0,l} \cong \langle S_Q \mid R_Q \rangle$ on Γ . From the discussion preceding the statement of Proposition 5.5, we know that f defines a homomorphism $F(S_Q) \rightarrow \text{Aut}(\Gamma)$. To prove the proposition, it is enough to show that each relation in R_Q fixes each generator in S_K as an element of Γ under this action. We do this in several cases, depending on which relations in R_Q we are considering.

Case 1. *Relations from Q3, N1 and N2.*

In the action of $F(S_Q)$ on $F(S_K)$, these relations fix each element of S_K . This can be verified by direct computation, but it is also easy to see for structural reasons. The subgroup \tilde{P} of $F(S_Q)$ generated by swaps and inversions acts on $F(S_Q)$ and $F(S_K)$ directly through the action of a signed permutation group P , and on $\text{Aut}(F(S_K))$ diagonally. This implies that the relations N1 act trivially on S_K , because these relations hold in P . The action homomorphism $f: F(S_Q) \rightarrow \text{Aut}(F(S_K))$ is equivariant with respect to these actions because the definitions in Table 4 are uniform with respect to the subscripts of the generators. The map f sends the conjugation action of \tilde{P} on $F(S_Q)$ to the diagonal action on $\text{Aut}(F(S_K))$. Then all the relations from N2 and Q3 act trivially on $F(S_K)$ because of the equivariance of $f: F(S_Q) \rightarrow \text{Aut}(F(S_K))$.

Case 2. *Relations from Q2 and N4.*

These are exactly the relations in R_Q that state that certain pairs of generators t_1, t_2 from S_Q commute, where each t_i is a Nielsen move or a conjugation move. Instead of enumerating the numerous subcases for this case, we treat many cases simultaneously by analyzing the supports and multiplier sets of the generators.

In each case, we can write our relation as $t_1 t_2 = t_2 t_1$ for some $t_1, t_2 \in S_Q$. By inspecting relations Q2 and N4, we see that this implies

$$\text{supp}(t_1) \cap \text{supp}(t_2) = \text{supp}(t_1) \cap \text{mult}(t_2)^{\pm 1} = \text{supp}(t_2) \cap \text{mult}(t_1)^{\pm 1} = \emptyset. \quad (12)$$

By definition, we have $\text{supp}(t_i) \cup \text{mult}(t_i) \subset (X \cup Z)^{\pm 1}$ for $i = 1, 2$. Suppose $s \in S_K$. If $\text{supp}(s) \cup \text{mult}(s) \subset Y^{\pm 1}$, then both $t_1 t_2$ and $t_2 t_1$ fix s by Lemma 5.9. So we suppose this is not the case; however, we must have $\text{supp}(s) \subset Y^{\pm 1}$ or $\text{mult}(s) \subset Y$.

If $\text{supp}(s) \subset Y^{\pm 1}$, this means $s = C_{y,v}$ for some $y \in Y$ and $v \in X \cup Z$. In particular, this means $(\text{mult}(t_i)^{\pm 1} \cup \text{supp}(t_i)) \cap \text{supp}(s) = \emptyset$ for $i = 1, 2$. If the relation acts nontrivially on s , then at least one t_i , say t_1 , satisfies $f(t_1, s) \neq s$. Then by Lemma 5.9, we have that $\text{supp}(t_1) \subset \{v, v^{-1}\}$. From the definition of f , we know $\text{supp}(f(t_1, s)) = \{y, y^{-1}\}$ and $\text{mult}(f(t_1, s)) \subset \{v\} \cup \text{mult}(t_1)$. From Equation (12), we know $\text{supp}(t_2) \cap \text{mult}(t_1)^{\pm 1} = \emptyset$. Then since $\text{mult}(t_2)^{\pm 1} \cap \text{supp}(f(t_1, s)) = \emptyset$, Lemma 5.9 implies that either $\text{supp}(t_2) \subset \{v, v^{-1}\}$ or else $f(t_2, s) = s$ and $f(t_2 t_1, s) = f(t_1, s)$. Since the latter case implies that

$f(t_1 t_2, s) = f(t_2 t_1, s)$, we assume that $t_i \in \{v, v^{-1}\}$ for $i = 1, 2$. By Equation (12), the only way this is possible is if $s = C_{y, x_a}$, $t_1 = M_{x_a^\epsilon, y}$ and $t_2 = M_{x_a^{-\epsilon}, y}$ for some $x_a^\epsilon \in X^{\pm 1}$. Direct computation shows that $f(t_2 t_1, s) = f(t_1 t_2, s)$ in $F(S_K)$ in this case.

Now suppose that $\text{mult}(s) \subset Y$. First we suppose that $\text{supp}(s) \cap \text{supp}(t_1) \neq \emptyset$; in fact, this implies that $\text{supp}(s) = \text{supp}(t_1)$. Then by Lemma 5.10, $\text{supp}(f(t_1, s)) \subset \text{supp}(s) \cup \text{mult}(s)^{\pm 1}$ and $\text{mult}(f(t_1, s)) \subset \text{mult}(t_1) \cup \text{mult}(s)$. Then $(\text{supp}(t_2) \cup \text{mult}(t_2)^{\pm 1}) \cap \text{supp}(g) = \emptyset$ and $(\text{supp}(g) \cup \text{mult}(g)^{\pm 1}) \cap \text{supp}(t_2) = \emptyset$ for $g = s$ and for $g = f(t_1, s)$ (this uses Equation (12)). Then $f(t_1 t_2, s) = f(t_2 t_1, s)$ by Lemma 5.9. Swapping the roles of t_1 and t_2 , we can now suppose that $\text{supp}(s) \cap \text{supp}(t_i) = \emptyset$ for $i = 1, 2$.

Still supposing that $\text{mult}(s) \subset Y$, we suppose (as above) that $f(t_1, s) \neq s$. Then this implies that $\text{supp}(s) \subset \text{mult}(t_1)^{\pm 1}$, $\text{supp}(f(t_1, s)) \subset \text{supp}(t_1) \cup \text{supp}(s) \cup Y^{\pm 1}$ and $\text{mult}(f(t_1, s)) \subset \text{mult}(t_1) \cup Y$. If $\text{supp}(s) \cap \text{mult}(t_2)^{\pm 1} = \emptyset$, Then $(\text{supp}(t_2) \cup \text{mult}(t_2)^{\pm 1}) \cap \text{supp}(g) = \emptyset$ and $(\text{supp}(g) \cup \text{mult}(g)^{\pm 1}) \cap \text{supp}(t_2) = \emptyset$ for $g = s$ and for $g = f(t_1, s)$ implying that $f(t_1 t_2, s) = f(t_2 t_1, s)$. So we also assume that $\text{supp}(s) \subset \text{mult}(t_2)^{\pm 1}$. Then there are only nine remaining cases; these are given in Table 5.

$s \in S_K$	$t_1 \in S_Q^{\pm 1}$	$t_2 \in S_Q^{\pm 1}$	$f(t_2 t_1, s)^{-1} f(t_1 t_2, s)$
$M_{x_a, y}$	$M_{x_b^\epsilon, x_a}$	$M_{x_c^\delta, x_a}$	$C_{y, x_a}^{-1} [M_{x_b^\epsilon, y}, M_{x_c^\delta, y}] C_{y, x_a}$
	$M_{x_b^\epsilon, x_a}$	C_{z_i, x_a}	$C_{y, x_a}^{-1} [M_{x_b^\epsilon, y}, C_{z_i, y}] C_{y, x_a}$
	C_{z_i, x_a}	C_{z_j, x_a}	$C_{y, x_a}^{-1} [C_{z_i, y}, C_{z_j, y}] C_{y, x_a}$
$M_{x_a^{-1}, y}$	$M_{x_b^\epsilon, x_a}$	$M_{x_c^\delta, x_a}$	$[M_{x_b^\epsilon, y}^{-1}, M_{x_c^\delta, y}^{-1}]$
	$M_{x_b^\epsilon, x_a}$	C_{z_i, x_a}	$[M_{x_b^\epsilon, y}^{-1}, C_{z_i, y}^{-1}]$
	C_{z_i, x_a}	C_{z_j, x_a}	$[C_{z_i, y}^{-1}, C_{z_j, y}^{-1}]$
$C_{z_i, y}$	$M_{x_a^\epsilon, z_i}^{-1}$	$M_{x_b^\delta, z_i}^{-1}$	$[C_{y, z_i}, M_{x_a^\epsilon, y}], [C_{y, z_i}, M_{x_b^\delta, y}]$
	$M_{x_a^\epsilon, z_i}^{-1}$	C_{z_j, z_i}^{-1}	$[C_{y, z_i}, M_{x_a^\epsilon, y}], [C_{y, z_i}, C_{z_j, y}]$
	C_{z_j, z_i}^{-1}	C_{z_m, z_i}^{-1}	$[C_{y, z_i}, C_{z_j, y}], [C_{y, z_i}, C_{z_m, y}]$

Table 5: This table includes the computations relevant to Case 2 of Proposition 5.5. In each row, we compute the difference of the actions of the two sides of the relation $t_1 t_2 = t_2 t_1$ on s , as an element of $F(S_K)$. This includes the only cases where $\text{mult}(s) \in Y$ and inspection of the supports of the generators does not immediately imply that the relation fixes s .

To finish this case, we must show that the expressions for $f(t_2 t_1, s)^{-1} f(t_1 t_2, s)$ in Table 5 are always trivial in Γ . The entries for $s = M_{x_a^{\pm 1}, y}$ are trivial by relations R1.1-3; those for $s = C_{z_i, y}$ are trickier. We consider the terms from one of the entries:

$$[C_{y, z_i}, M_{x_a^\epsilon, y}][C_{y, z_i}, M_{x_b^\delta, y}] = (C_{y, z_i} M_{x_a^\epsilon, y} C_{y, z_i}^{-1}) M_{x_a^\epsilon, y}^{-1} (C_{y, z_i}, M_{x_b^\delta, y} C_{y, z_i}^{-1}) M_{x_b^\delta, y}^{-1}.$$

By relations from C1.1, the elements $(C_{y, z_i} M_{x_a^\epsilon, y} C_{y, z_i}^{-1})$ and $M_{x_b^\delta, y}^{-1}$ commute, and the elements $(C_{y, z_i}, M_{x_b^\delta, y} C_{y, z_i}^{-1})$ and $M_{x_a^\epsilon, y}^{-1}$ commute. By relations from R1.1, the elements $M_{x_a^\epsilon, y}^{-1}$

and $M_{x_b^\delta, y}^{-1}$ commute, and the elements $(C_{y, z_i} M_{x_a^\epsilon, y} C_{y, z_i}^{-1})$ and $(C_{y, z_i}, M_{x_b^\delta, y} C_{y, z_i}^{-1})$ commute. Working in Γ , this allows us to show:

$$[C_{y, z_i}, M_{x_a^\epsilon, y}][C_{y, z_i}, M_{x_b^\delta, y}] = [C_{y, z_i}, M_{x_b^\delta, y}][C_{y, z_i}, M_{x_a^\epsilon, y}],$$

meaning that the seventh expression from the last column of the table is trivial in Γ . The last two expressions are completely analogous, using R1.2 and C1.2, and R1.3 and C1.3 instead of R1.1 and C1.1, respectively.

Case 3. *Relations from N3.*

One of the relations from N3 is of the form $M_{x_a^{-1}, x_b}^{-1} M_{x_b^{-1}, x_a} M_{x_a, x_b} = I_a P_{a, b}$, for $1 \leq a, b \leq n$ and $a \neq b$. We rearrange this as

$$M_{x_b^{-1}, x_a} M_{x_a, x_b} = M_{x_a^{-1}, x_b} I_a P_{a, b}.$$

For the rest of this case, let $g = M_{x_b^{-1}, x_a} M_{x_a, x_b}$ and let $h = M_{x_a^{-1}, x_b} I_a P_{a, b}$, so that the relation in question is $g = h$. Support considerations imply that all elements of S_K are fixed in $F(S_K)$ by this the action of relation except possibly for those of the form $M_{x_a, y}$, $M_{x_b, y}$, C_{y, x_a} or C_{y, x_b} for $y \in Y$. Direct computation shows that $f(g, C_{y, x_a}) = f(h, C_{y, x_a})$ and $f(g, C_{y, x_b}) = f(h, C_{y, x_b})$, as elements of $F(S_K)$; only two possibilities for s are left. In the case $s = M_{x_a, y}$, we derive $f(g, s) = f(h, s)$ in Γ :

$$\begin{aligned} f(g, s) &= C_{y, x_b}^{-1} (C_{y, x_a} M_{x_a, y} C_{y, x_a}^{-1}) M_{x_b^{-1}, y}^{-1} C_{y, x_b} && \text{(computation in } F(S_K)) \\ &= C_{y, x_b}^{-1} M_{x_a^{-1}, y}^{-1} M_{x_b^{-1}, y}^{-1} C_{y, x_b} && \text{(relation R5)} \\ &= (C_{y, x_b}^{-1} M_{x_b^{-1}, y}^{-1}) M_{x_a^{-1}, y}^{-1} C_{y, x_b} && \text{(relation R1.1)} \\ &= (M_{x_b, y} C_{y, x_b}^{-1}) M_{x_a^{-1}, y}^{-1} C_{y, x_b} && \text{(relation R5)} \\ &= f(h, s) && \text{(computation in } F(S_K)). \end{aligned}$$

If $s = M_{x_b, y}$, we compute:

$$\begin{aligned} f(g, s) &= (M_{x_b, y} C_{y, x_b}^{-1} M_{x_b^{-1}, y}^{-1}) C_{y, x_a} M_{x_a, y}^{-1} C_{y, x_a}^{-1} C_{y, x_b} && \text{(computation in } F(S_K)) \\ &= C_{y, x_b}^{-1} (C_{y, x_a} M_{x_a, y}^{-1} C_{y, x_a}^{-1}) C_{y, x_b} && \text{(relation R5)} \\ &= C_{y, x_b}^{-1} M_{x_a^{-1}, y}^{-1} C_{y, x_b} && \text{(relation R5)} \\ &= f(h, s) && \text{(computation in } F(S_K)). \end{aligned}$$

The derivations for the other type of relation in N3 are similar.

Case 4. *Relations from N5 and Q4.1.*

Any relation in R_Q from N5 or Q4.1 can be written as

$$M_{x_c^\delta, x_b}^\epsilon M_{x_b^\epsilon, v} M_{x_c^\delta, v} = M_{x_b^\epsilon, v} M_{x_c^\delta, x_b}^\epsilon$$

for some $\epsilon, \delta = \pm 1$, some $v \in X \cup Z$ and distinct $1 \leq b, c \leq n$ with $v \neq x_b, x_c$. Actually, to write a relation from Q4.1 in this form, we must modify it with a relation from Q1.1; however, since we have already verified that those relations act trivially on Γ , this makes no difference. For the rest of this case, let $g_1 = M_{x_b^\epsilon, v} M_{x_c^\delta, v}$, $g_2 = M_{x_c^\delta, x_b}^\epsilon$ and $h = M_{x_b^\epsilon, v} M_{x_c^\delta, x_b}^\epsilon$, so that the relation reads $g_2 g_1 = h$. Lemma 5.10 implies that this relation fixes all elements of S_K with the following possible exceptions: $M_{v, y}$, $M_{v^{-1}, y}$ (if $v \in X$), $C_{v, y}$ (if $v \in Z$), $M_{x_b^\epsilon, y}$, $M_{x_b^{-\epsilon}, y}$, $M_{x_c^\delta, y}$, C_{y, x_b} and C_{y, x_c} . Further computation shows that $f(g_2 g_1, s) = f(h, s)$ in $F(S_K)$ for s equal to $M_{v^{-1}, y}$, $M_{x_b^\epsilon, y}$, $M_{x_b^{-\epsilon}, y}$, $M_{x_c^\delta, y}$, C_{y, x_b} or C_{y, x_c} . We now check the remaining two possibilities for s . If $v \in X$ and $s = M_{v, y}$, we derive $f(g_2 g_1, s) = f(h, s)$ in Γ :

$$\begin{aligned}
f(g_2 g_1, s) &= f(g_2, M_{v, y} C_{y, v}^{-1} M_{x_b^\epsilon, v}^{-1} M_{x_c^\delta, y}^{-1} C_{y, v}) && \text{(computation in } F(S_K)) \\
&= f(g_2, M_{v, y} C_{y, v}^{-1} M_{x_c^\delta, y}^{-1} M_{x_b^\epsilon, y}^{-1} C_{y, v}) && \text{(relation R1.1)} \\
&= M_{v, y} C_{y, v}^{-1} M_{x_b^\epsilon, y}^{-1} C_{y, v} && \text{(computation in } F(S_K)) \\
&= f(h, s) && \text{(computation in } F(S_K)).
\end{aligned}$$

If $v \in Z$ and $s = C_{v, y}$:

$$\begin{aligned}
f(g_2 g_1, s) &= f(g_2, C_{v, y} [M_{x_b^\epsilon, y}, C_{y, v}^{-1}] [M_{x_c^\delta, y}, C_{y, v}^{-1}]) && \text{(computation in } F(S_K)) \\
&= f(g_2, C_{v, y} [M_{x_b^\epsilon, y} M_{x_c^\delta, y}, C_{y, v}^{-1}]) && \text{(relations R1.1 and C1.1)} \\
&= C_{v, y} [M_{x_b^\epsilon, y}, C_{y, v}^{-1}] && \text{(computation in } F(S_K)) \\
&= f(h, s) && \text{(computation in } F(S_K)).
\end{aligned}$$

Case 5. *Relations from Q4.1'.*

This case is parallel to Case 4. Any relation in R_Q from Q4.1' can be written as

$$C_{z, x}^{-\epsilon} M_{x^\epsilon, v} C_{z, x}^\epsilon = C_{z, v} M_{x^\epsilon, v}$$

for some $\epsilon = \pm 1$, $x \in X$, $v \in X \cup Z$ and $z \in Z$ with $v \neq x, z$. Again, we note that we have modified this relation using a relation from Q1.1, but since these act trivially this is allowable. By Lemma 5.10, the only cases where we might have $f(h^{-1} g_2 g_1, s)$ not equal to s are if s equals $M_{v, y}$, $M_{v^{-1}, y}$ (if $v \in X$), $C_{v, y}$ (if $v \in Z$), $M_{x^\epsilon, y}$, $M_{x^{-\epsilon}, y}$, $C_{y, x}$, $C_{z, y}$ or $C_{y, z}$. Application of the definition of f shows that $f(g_2 g_1, s) = f(h, s)$ as elements of $F(S_K)$ unless s is $M_{v, y}$ or $C_{v, y}$. The computations to show that these are fixed in Γ by this relation are similar to the corresponding ones in Case 4.

Case 6. *Relations from Q4.2.*

Any relation in R_Q from Q4.2 can be written as

$$M_{x^\delta, z} M_{x^\delta, v}^\epsilon C_{z, v}^\epsilon = M_{x^\delta, v}^\epsilon C_{z, v}^\epsilon M_{x^\delta, z}$$

for $\delta, \epsilon = \pm 1$, $x \in X$, $v \in X \cup Z$ and $z \in Z$ with $v \neq x, z$. For the rest of this case let $g = M_{x^\delta, z}$ and $h = M_{x^\delta, v}^\epsilon C_{z, v}^\epsilon$, so that the relation reads $gh = hg$. By Lemma 5.10, the

only cases where we might have $f(gh, s)$ not equal to $f(hg, s)$ are where s equals $M_{v^\epsilon, y}$, $M_{v^{-\epsilon}, y}$ (if $v \in X$), $C_{v, y}$ (if $v \in Z$), $M_{x^\delta, y}$, $C_{y, x}$, $C_{z, y}$ or $C_{y, z}$ for some $y \in Y$. Computing f by the definition in $F(S_K)$ shows that $f(gh, s) = f(hg, s)$ as elements of $F(S_K)$ in all these cases except when s is $M_{v^{-\epsilon}, y}$ or $C_{v, y}$. If $v \in X$ and $s = M_{v^{-\epsilon}, y}$, we compute:

$$\begin{aligned}
f(gh, s) &= f(g, M_{v^{-\epsilon}, y} M_{x^\delta, y} C_{z, y}) && \text{(computation in } F(S_K)) \\
&= f(g, M_{v^{-\epsilon}, y} C_{z, y} M_{x^\delta, y}) && \text{(relation R1.2)} \\
&= M_{v^{-\epsilon}, y} C_{z, y} M_{x^\delta, y} && \text{(computation in } F(S_K)) \\
&= M_{v^{-\epsilon}, y} M_{x^\delta, y} C_{z, y} && \text{(relation R1.2)} \\
&= f(hg, s) && \text{(computation in } F(S_K)).
\end{aligned}$$

If $v \in Z$ and $s = C_{v, y}$, we compute:

$$\begin{aligned}
f(gh, s) &= f(g, C_{v, y} [M_{x^\delta, y}, C_{y, v}^{-\epsilon}] [C_{z, y}, C_{y, v}^{-\epsilon}]) && \text{(computation in } F(S_K)) \\
&= f(g, C_{v, y} [C_{z, y} M_{x^\delta, y}, C_{y, v}^{-\epsilon}]) && \text{(relations C1.2 and R1.2)} \\
&= C_{v, y} [C_{z, y} M_{x^\delta, y}, C_{y, v}^{-\epsilon}] && \text{(computation in } F(S_K)) \\
&= C_{v, y} [M_{x^\delta, y}, C_{y, v}^{-\epsilon}] [C_{z, y}, C_{y, v}^{-\epsilon}] && \text{(relations C1.2 and R1.2)} \\
&= f(hg, s) && \text{(computation in } F(S_K)).
\end{aligned}$$

Case 7. *Relations from Q4.2'.*

This case is completely analogous to Case 6. Any relation in R_Q from Q4.2' can be written as:

$$C_{z_j, z_i} C_{z_j, v}^\epsilon C_{z_i, v}^\epsilon = C_{z_j, v}^\epsilon C_{z_i, v}^\epsilon C_{z_j, z_i}$$

for $\epsilon = \pm 1$, $z_i, z_j \in Z$ with $z_i \neq z_j$ and $v \in X \cup Z$ with $v \neq z_i, z_j$. By Lemma 5.10, the only generators in S_K that might not be fixed by this relation are $M_{v^\epsilon, y}$, $M_{v^{-\epsilon}, y}$ (if $v \in X$), $C_{v, y}$ (if $v \in Z$), $C_{z_i, y}$, C_{y, z_i} , $C_{z_j, y}$ and C_{y, z_j} , for $y \in Y$. Direct computation in $F(S_K)$ shows that these are all fixed except possibly $M_{v^{-\epsilon}, y}$ and $C_{v, y}$. Derivations that show that this relation fixes these elements in Γ are parallel to the corresponding derivations in Case 6, using relations C1.3 and R1.3 instead of C1.2 and R1.2.

Case 8. *Relations from Q5.*

Any relation in R_Q from Q5 can be written as

$$M_{x^{-\epsilon}, z} C_{z, x}^\epsilon = C_{z, x}^\epsilon M_{x^\epsilon, z}^{-1}$$

for $\epsilon = \pm 1$, $x \in X$ and $z \in Z$. The only generators in S_K that might not be fixed by this relation are of the form $M_{x^\epsilon, y}$, $M_{x^{-\epsilon}, y}$, $C_{y, z}$, $C_{z, y}$ or $C_{y, z}$ for $y \in Y$. Straightforward computation shows that $C_{y, x}$ and $C_{y, z}$ are fixed as elements of $F(S_K)$. Let $g = M_{x^{-\epsilon}, z} C_{z, x}^\epsilon$

and let $h = C_{z,x}^\epsilon M_{x^\epsilon,z}^{-1}$. If $s = M_{x^\epsilon,y}$:

$$\begin{aligned}
f(g, s) &= (M_{x^\epsilon,y} C_{y,x}^{-\epsilon} M_{x^{-\epsilon},y}) C_{y,z} M_{x^{-\epsilon},y}^{-1} C_{z,y}^{-1} C_{y,z}^{-1} C_{y,x}^\epsilon && \text{(computation in } F(S_K)) \\
&= C_{y,x}^{-\epsilon} C_{y,z} M_{x^{-\epsilon},y}^{-1} C_{z,y}^{-1} C_{y,z}^{-1} C_{y,x}^\epsilon && \text{(relation R5)} \\
&= C_{y,x}^{-\epsilon} C_{y,z} (C_{y,x}^\epsilon M_{x^\epsilon,y} C_{y,x}^{-\epsilon}) C_{z,y}^{-1} C_{y,z}^{-1} C_{y,x}^\epsilon && \text{(relation R5)} \\
&= f(h, s) && \text{(computation in } F(S_K)).
\end{aligned}$$

If $s = M_{x^{-\epsilon},y}$:

$$\begin{aligned}
f(g, s) &= (C_{y,z}^{-1} M_{x^{-\epsilon},y} C_{y,z}) (C_{z,y} M_{x^{-\epsilon},y}) (C_{y,z}^{-1} M_{x^{-\epsilon},y}^{-1} C_{y,z}) && \text{(computation in } F(S_K)) \\
&= C_{z,y} M_{x^{-\epsilon},y} && \text{(relation C2.1)} \\
&= M_{x^{-\epsilon},y} C_{z,y} && \text{(relation R1.2)} \\
&= f(h, s) && \text{(computation in } F(S_K)).
\end{aligned}$$

If $s = C_{z,y}$:

$$\begin{aligned}
f(h, s) &= C_{y,x}^{-\epsilon} C_{z,y} [(C_{y,x}^\epsilon M_{x^\epsilon,y} C_{y,x}^{-\epsilon}) C_{z,y}^{-1}, C_{y,z}] C_{y,x}^\epsilon && \text{(computation in } F(S_K)) \\
&= C_{y,x}^{-\epsilon} C_{z,y} [M_{x^{-\epsilon},y}^{-1} C_{z,y}^{-1}, C_{y,z}] C_{y,x}^\epsilon && \text{(relation R5)} \\
&= C_{y,x}^{-\epsilon} (C_{z,y} M_{x^{-\epsilon},y}^{-1} C_{z,y}^{-1}) C_{y,z} C_{z,y} M_{x^{-\epsilon},y} C_{y,z}^{-1} C_{y,x}^\epsilon \\
&= C_{y,x}^{-\epsilon} M_{x^{-\epsilon},y}^{-1} C_{y,z} C_{z,y} M_{x^{-\epsilon},y} C_{y,z}^{-1} C_{y,x}^\epsilon && \text{(relation R1.2)} \\
&= C_{y,x}^{-\epsilon} C_{y,z} (C_{y,z}^{-1} M_{x^{-\epsilon},y}^{-1} C_{y,z}) (C_{z,y} M_{x^{-\epsilon},y}) C_{y,z}^{-1} C_{y,x}^\epsilon \\
&= C_{y,x}^{-\epsilon} C_{y,z} (C_{z,y} M_{x^{-\epsilon},y}) (C_{y,z}^{-1} M_{x^{-\epsilon},y}^{-1} C_{y,z}) C_{y,z}^{-1} C_{y,x}^\epsilon && \text{(relation C2.1)} \\
&= C_{y,x}^{-\epsilon} C_{y,z} C_{z,y} [M_{x^{-\epsilon},y}, C_{y,z}^{-1}] C_{y,z}^{-1} C_{y,x}^\epsilon \\
&= f(g, s) && \text{(computation in } F(S_K)).
\end{aligned}$$

This completes the proof of Proposition 5.5. \square

A Appendix : Simple closed curves and conjugacy classes in $\pi_1(\Sigma_g)$

In this appendix, we prove Lemma 1.1 from §1. Recall that it asserts that a collection c_1, \dots, c_k of conjugacy classes in $\pi_1(\Sigma_g)$ corresponds to a collection $\gamma_1, \dots, \gamma_k$ of homotopy classes of disjoint oriented nonnullhomotopic simple closed curves on Σ_g such that $\Sigma_g \setminus (\gamma_1 \cup \dots \cup \gamma_k)$ is connected if and only if there exists a standard basis $a_1, b_1, \dots, a_g, b_g$ for $\pi_1(\Sigma_g)$ with $c_i = [[a_i]]$ for $1 \leq i \leq k$.

Define

$$\mathcal{B} = \{([a_1], [b_1], \dots, [a_g], [b_g]) \mid a_1, b_1, \dots, a_g, b_g \text{ is a standard basis for } \pi_1(\Sigma_g)\}$$

and

$$\mathcal{X}_k = \{(\gamma_1, \dots, \gamma_k) \mid \gamma_1, \dots, \gamma_k \text{ are homotopy classes of oriented closed curves on } \Sigma_g\}.$$

Let $\mathcal{Y}_k \subset \mathcal{X}_k$ equal

$$\{(\gamma_1, \dots, \gamma_k) \in \mathcal{X}_k \mid \text{the } \gamma_i \text{ are distinct and simple and } \Sigma_g \setminus (\gamma_1 \cup \dots \cup \gamma_k) \text{ is connected}\}.$$

Remark. The fact that $\Sigma_g \setminus (\gamma_1 \cup \dots \cup \gamma_k)$ is connected implies that the γ_i are nonnullhomotopic.

There is a natural map $\rho : \mathcal{B} \rightarrow \mathcal{X}_k$ taking $(\llbracket a_1 \rrbracket, \llbracket b_1 \rrbracket, \dots, \llbracket a_g \rrbracket, \llbracket b_g \rrbracket) \in \mathcal{B}$ to the tuple of closed curves corresponding to $(\llbracket a_1 \rrbracket, \llbracket a_2 \rrbracket, \dots, \llbracket a_k \rrbracket)$, and our goal is to show that $\rho(\mathcal{B}) = \mathcal{Y}_k$.

The group $\text{Out}(\pi_1(\Sigma_g))$ acts on \mathcal{B} and the group $\text{Mod}(\Sigma_g)$ acts on \mathcal{X}_k . The Dehn-Nielsen-Baer theorem (see [11]) says that $\text{Mod}(\Sigma_g) \cong \text{Out}(\pi_1(\Sigma_g))$, and under this identification the map ρ is equivariant. Now, it is clear that $\text{Out}(\pi_1(\Sigma_g))$ acts transitively on \mathcal{B} . Moreover, the action of $\text{Mod}(\Sigma_g)$ on \mathcal{X}_k preserves \mathcal{Y}_k , and using the “change of coordinates” principle from [11, §1.3] one can show that $\text{Mod}(\Sigma_g)$ acts transitively on \mathcal{Y}_k . Since the image of ρ certainly contains at least one element of \mathcal{Y}_k , it follows that the image equals \mathcal{Y}_k , as desired.

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