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Geometry along grafting rays in Teichmüller space

by

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Abstract

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In this work, we investigate the mid-range behavior of geometry along a grafting ray in Teichmüller space. The main technique is to describe the hyperbolic metric σ_t at a point along the grafting ray in terms of a conformal factor g_t times the Thurston (grafted) metric and study solutions to the linearized Liouville equation. We give a formula that describes, at any point on a grafting ray, the change in length of a sum of distinguished curves in terms of the hyperbolic geometry at the point. We then make precise the idea that once the length of the grafting locus is small, local behavior of the geometry for grafting on a general manifold is like that of grafting on a cylinder. Finally, we prove that the sum of lengths of is eventually monotone decreasing along grafting rays.

Acknowledgments

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I owe my deepest gratitude to my husband, David. The words "thank you" are insufficient for managing life when my mind has been otherwise occupied, coping with uncapped pens and scattered stacks of loose paper, and (most importantly) believing in me when I did not believe in myself.

Dedication

For David.

Without you, and the promise that you kept, this dream may have remained precisely that - a dream.

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Chapter 1

Introduction

In this paper, we examine a natural geometric deformation of hyperbolic structures, *grafting*. Let S be a compact, orientable surface of genus $g \geq 2$. Then a *complex structure* X on S is a maximal atlas of charts from S to $\mathbb{C}$ with biholomorphic transition functions. A *complex projective structure* (or $\mathbb{C}P^1$-structure) Z on S is a maximal atlas of charts from open subsets of S to $\mathbb{C}P^1$ that are restrictions of Möbius maps. Of primary interest is the Teichmüller space $\mathcal{T}(S)$ and the space of $\mathbb{C}P^1$-structures $\mathcal{P}(S)$. These are isotopy classes of complex and $\mathbb{C}P^1$-structures, respectively. In particular, because every Möbius map is holomorphic, every $\mathbb{C}P^1$-structure is in fact a *complex* structure, and this underlying structure makes S a Riemann surface. This defines the *forgetful projection map* $\pi : \mathcal{P}(S) \to \mathcal{T}(S)$.

There are two parameterizations of $\mathcal{P}(S)$. The first, essentially complex-analytic in nature, associates to every $Z \in \mathcal{P}(S)$ a holomorophic quadratic differential via the Schwarzian derivative of the developing map. The second, due to Thurston, shows that $\mathcal{P}(S)$ is homeomorphic to $\mathcal{ML}(S)\times\mathcal{P}(S)$, where $\mathcal{ML}(S)$ is the space of *measured geodesic laminations*. This homeomorphism $Gr : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{P}(S)$ is called the *projective grafting map.* The composition $gr = \pi \circ Gr : \mathcal{ML}(S)$ x $\mathcal{T}(S) \to \mathcal{T}(S)$ is called the *conformal grafting map.* The simplest picture of conformal grafting is as follows: choose a hyperbolic metric $X \in \mathcal{T}(S)$ and γ a simple closed X-geodesic. Then the *grafting of* X *by* $t\gamma$ is obtained by replacing γ by the flat cylinder $[\frac{-t}{2}, \frac{t}{2}] \times \gamma$. This gives a new conformal structure, whic we denote by $gr_{t\gamma}X$. Via uniformization, there is a new hyperbolic metric in the conformal class of $gr_{t\gamma}X$. Fixing $\gamma \in \mathcal{ML}(S)$, we call the set of conformal structures $\{gr_{t\gamma}X : t \in \mathbb{R}\}$ a *grafting ray.* We are interested in the evolution of geometric invariants of the hyperbolic surfaces along grafting rays in Teichmüller space.

In first chapter, we develop the background material and sketch Thurston's proof that the projective grafting map $Gr : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{P}(S)$ is a homeomorphism. The second chapter is a review of current research involving grafting. We begin the third chapter by introducing a technique for studying geometry at *any* point along a grafting ray generated by a simple closed geodesic, assuming that the geodesic in the homotopy class of the grafting locus does not escape the grafting cylinder. In section 3.2, we derive a formula that expresses the change in two geometric values in terms of the geometry of the hyperbolic surface in the conformal class of the grafted surface. More specifically, for any curve δ on the surface, denote by $\ell(t, \delta)$ the length of the δ with respect to the hyperbolic metric in the conformal class of $gr_{(t+t_0)\gamma}X$. Then the formula of section 3.2 can be paraphrased to say the following:

Theorem 4.1. *Let γ be a separating curve on S. Choose $t_0 \in \mathbb{R}^+$. Let σ_0 be the hyperbolic metric in the conformal class of $gr_{t_0\gamma}X$. Then if γ_0 is the hyperbolic geodesic in the homotopy class of γ, and α_0^+ and α_0^- are curves that are parallel to γ_0 with respect to σ_0, then*

$$-2\frac{\ell(0,\alpha_0^+)}{\ell}\left.\frac{d}{dt}\right|_{t=0}\ell(t,\alpha_0^+) - 2\frac{\ell(0,\alpha_0^-)}{\ell}\left.\frac{d}{dt}\right|_{t=0}\ell(t,\alpha_0^-) + 4\left.\frac{d}{dt}\right|_{t=0}\ell(t,\gamma_t)$$

is given in terms of t_0, the Euclidean length of the grafting cylinder, the conformal factor g, where $\sigma_0 = g(z)|dz|^2$, distance of α_0^+ and α_0^- from γ_0, and the average angle between ξ=constant, a curve of constant σ_0 (hyperbolic) geodesic curvature, and x=constant, a curve of constant Euclidean geodesic curvature.

In section 3.3, we observe that if the surface S is actually a cylinder and γ is the core geodesic for a hyperbolic metric X on S, then the geometry along the grafting ray is particularly simple to understand. We restate the formula given in section 3.2 in such a way that shows the defect of the grafting operation on the general surface from being like that of grafting on a cylinder. Finally, in section 3.4, we use Fourier analysis and properties of harmonic functions to estimate terms in the revised formula of section 3.3. This gives

Theorem 4.2. *For ℓ small enough and α^+ and α^- sufficiently close to the geodesic γ_0,*

$$-2\frac{\ell(0,\alpha^+)}{\ell}\left.\frac{d}{dt}\right|_{t=0}\ell(t,\alpha^+) - 2\frac{\ell(0,\alpha^-)}{\ell}\left.\frac{d}{dt}\right|_{t=0}\ell(t,\alpha^-) + 4\left.\frac{d}{dt}\right|_{t=0}\ell(t,\gamma_t) < 0 \quad (1.1)$$

Chapter 2

Background

2.1 Klein's perspective

In his fifth lecture of the colloquium following the 1893 Congress of Mathematicians held during the World's Fair Auxiliary in Chicago, Professor Felix Klein presented a geometric interpretation of the solution to *hypergeometric functions* [Kle10]. A hypergeometric function is a solution to the hypergeometric differential equation

$$
\begin{aligned}
0 = \frac{dw^2}{dz^2} + [&\frac{1-\alpha'-\alpha''}{z-a}(a-b)(a-c) + \frac{1-\beta'-\beta''}{z-b}(b-c)(b-a) \\
&+ \frac{1-\nu'-\nu''}{z-c}(c-a)(c-b)]\frac{dw}{dz} + [\frac{\alpha'\alpha''(a-b)(a-c)}{z-a} \\
&+ \frac{\beta'\beta''(b-c)(b-a)}{z-b} + \frac{\nu'\nu''(c-a)(c-b)}{z-c}][\frac{w}{(z-a)(z-b)(z-c}]
\end{aligned}
$$

where $z = a, b, c \in \mathbb{C}$ are three singular points and $\alpha', \alpha'', \beta', \beta'', \nu', \nu''$ are the exponents associated with a, b, and c respectively. Klein remarks that if ω_1 is a particular solution and ω_2 is a general solution to the above second order linear differential

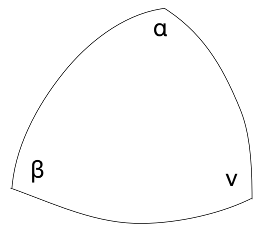

Figure 2.1: Triangle bounded by circular arcs with angles α, β, ν.

equation, then the ratio

$$\eta(z) = \frac{\omega_1}{\omega_2}(z)$$

satisfies a third order differential equation involving the *Schwarzian derivative*, which we will explore in depth in a later chapter. More importantly, he notes that $\mathbb{H}$, the upper half plane, with $a, b, c \in \mathbb{R}$, is taken conformally by each branch of the solution η to a triangular area "bounded by three circular arcs" with specified angles $\pi\alpha, \pi\beta, \pi\nu$ where $\alpha = \alpha' - \alpha''$ (and similarly for β and ν). In order to *analytically* classify solutions η, Klein proceeds to *geometrically* classify the types of triangles that can be obtained. Klein's classification of triangles is as follows: given constants α, β, ν, there is no restriction on the sum of the constants, so there is no guarantee that the triangle be either acute or convex. Stereographically project the plane containing the triangle onto the sphere, preserving angles. Then all other triangles can be obtained via two operations that Klein calls *lateral attachment* and *polar attachment.* Polar attachment is the process by which one side, say bc defines a circle that encloses abc and the area of this circle is added to abc. The angle at a is increased by 2π, so a becomes a branch point. In the case of lateral attachment, the side bc defines a circle

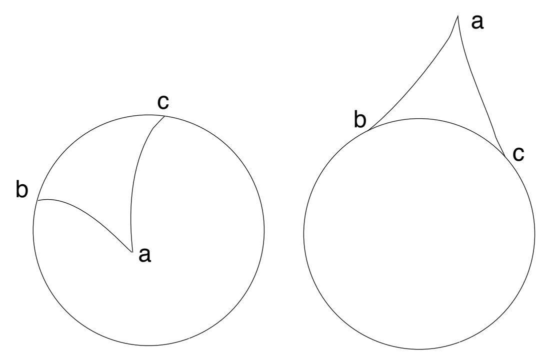

Figure 2.2: Polar attachment on the left; lateral attachment on the right.

exterior to *abc*, and the area of this enclosing circle is added to *abc*. The angles at b and c are each increased by π. Klein's result is the following theorem:

Theorem 2.1. *Suppose the* α, β, γ *are constants. Then there are two types of triangles that can be obtained:*

- *If none of the constants are greater than the sum of the other two, i.e.,*

$$|\alpha| \leq |\beta| + |\nu|$$

$$|\beta| \leq |\alpha| + |\nu|$$

$$|\nu| \leq |\alpha| + |\beta|$$

  then the triangle is obtained by lateral attachment on any or all three sides.

- *If* $|\alpha| \geq |\beta| + |\nu|$ *with* α *being the constant corresponding to the angle at* a*, then this triangle is formed by polar attachment along the opposite side, bc, and possibly lateral attachment along the other two sides.*

As we embark on an investigation into Teichmüller theory and complex projective structures via grafting, we keep in mind several themes from Klein's lecture on hypergeometric functions. First, the interpolation between the analytic and geometric viewpoint which helped Klein classify solutions will guide much of our discussion. Second, Klein's discovery that "grafting" pieces of the sphere onto a fundamental domain generates an parameterization of solutions via the Schwarzain derivative is the first example of our grafting construction.

2.2 Fundamentals: Teichmüller space and complex projective structures

2.2.1 (X, G)-structures

We begin by setting some notation and making some preliminary definitions, using [Gol88] as a guide.

Definition 2.2. Let X be a manifold, M be a manifold of the same dimension as X, and G be a Lie group acting transitively on X. Then an (X, G)-*atlas on* M is a pair (U, Φ) of an open cover U with coordinate charts $\Phi = \{\phi_i : U_i \to X\}$ such that the transition functions are restrictions of elements of G. An (X, G)-*structure on* M is a maximal (X, G)-atlas on M.

We begin by giving several examples of (X, G)-structures and specifying geometric and analytic concepts that "make sense" in each setting:

Structure	X	G	What makes sense?
C^0	$\mathbb{R}^n$	homeomorphisms	continuous functions
C^∞	$\mathbb{R}^n$	C^∞- diffeomorphisms	calculus
conformal	$\mathbb{C}$	angle preserving	angles
flat	$\mathbb{E}^n$	Euclidean isometries	Euclidean geometry
complex	$\mathbb{C}$	biholomorphisms	complex analysis
complex projective	$\mathbb{C}P^1$	Möbius	round circles, cx analysis, straight lines

2.2.2 Complex and complex projective structures

The preliminary objects in our study will be *complex structures* and *complex projective structures.*

Definition 2.3. Let S be a fixed closed Riemann surface of genus $g \geq 2$. Then the pair (R, f) of a second closed Riemann surface of genus g and an orientation-preserving diffeomorphism $f : R \to S$ is a complex structure on S. We say that two complex structures (R, f) and (R', g) on S are equivalent if there exists a biholomorphic mapping $h : R \to R'$ such that $g \circ f^{-1}$ is homotopic to h. Then $\mathcal{T}(S)$, the *Teichmüller space of* S, is the set of equivalence classes of complex structures.

Remark 2.4. Since the group of biholomorphisms is equivalent to group of conformal transformations of $\mathbb{C}$, we equivalently define $\mathcal{T}(S)$ to be the set of conformal classes of Riemannian metrics on $\mathcal{T}(S)$. In particular, two Riemannian metrics g and g' are in the same equivalence class if $g = \lambda g'$ for some positive function λ. We note that this equivalence of complex and conformal structures is unique to dimension 2.

Remark 2.5. As a consequence of the uniformization theorem, a surface with constant curvature is locally isometric to either the Euclidean plane, the sphere, or the upper half plane [HF92]. Thus if X is one of these model spaces and G is its isometry group, then a constant-curvature metric on a surface S is precisely an (X, G)-structure on S. Thus we can also view $\mathcal{T}(S)$ as the deformation space of hyperbolic metrics on S. This is the point of view which will be most relevant in the sequel.

Remark 2.6. By a classical theorem of Fricke, Teichmüller space is homeomorphic to a ball of (real) dimension $6g-6$. See Chapter 2 of [YI92] for an excellent exposition of this fact.

Definition 2.7. A *complex projective structure* Z on S is a maximal atlas of charts into $\mathbb{C}P^1$ with transition functions that are restrictions of Möbius transformations [Dum09]. Two complex projective structures Z_1 and Z_2 are said to be *isomorphic* if there exists an orientation-preserving diffeomorphism between them that pulls projective charts of the target back to the source. The structures Z_1 and Z_2 are *marked isomorphic* if this diffeomorphism is in fact homotopic to the identity. Let $\mathcal{P}(S)$ denote the space of marked isomorphism classes of complex projective structures on S.

There is a second approach to visualizing complex projective structures:

Definition 2.8. Given $Z \in \mathcal{P}(S)$, let $\tilde{Z}$ denote the lift of Z to the universal cover. Then one obtains a (dev, hol) pair in the following way: the map $dev : \tilde{S} \to \mathbb{C}P^1$ is a complex projective diffeomorphism from the universal cover onto its image with respect to the $\mathbb{C}P^1$-structures on $\tilde{S}$ and $\mathbb{C}P^1$. As $\pi_1(S)$ acts on $\tilde{S}$ via deck transformations, there exists $hol : \pi_1(S) \to PGL(2,\mathbb{C})$ such that dev is hol-equivariant, i.e.,

$$dev(\gamma \cdot x) = hol(\gamma) dev(x)$$

for all $\gamma \in \pi_1(S)$. Further, hol is unique up to conjugation. We claim that to every complex projective structure we can actually associate a unique (dev, hol) pair. In particular, given a (dev, hol) pair, pull back the canonical $\mathbb{C}P^1$-structure on $\mathbb{C}P^1$ via the hol-invariant map dev. Since $\pi_1(S)$ preserves this structure on $\tilde{S}$, it descends to a projective structure on S, and dev is the developing map of this structure.

Now two spaces are in play: $\mathcal{T}(S)$, equivalence classes of complex structures, and $\mathcal{P}(S)$, equivalence classes of complex projective structures. A fundamental fact relates the two spaces: because Möbius functions are holomorphic, every projective structure $Z \in \mathcal{P}(S)$ has an underlying complex structure. The *forgetful projection map*

$$\pi : \mathcal{P}(S) \to \mathcal{T}(S)$$

sends every $\mathbb{C}P^1$-structure to its underlying complex structure. This map is in fact surjective. To see this, note that as a consequence of the uniformization theorem, if $X \in \mathcal{T}(S)$, there exists a Fuchsian group Γ_X so that $X = \mathbb{H}/\Gamma_X$. As noted above, the upper half plane $\mathbb{H} \subset \mathbb{C}P^1$ and Γ_X is a group of Möbius transformations, so $\mathbb{H}/\Gamma_X$ has a natural projective structure, which we call the *standard Fuchsian structure.* Thus, given an X in $\mathcal{T}(S)$, there is at least one element in the preimage of X under π, namely $\mathbb{H}/\Gamma_X$.

2.3 Parameterization of $\mathcal{P}(S)$

In this section, we closely follow Dumas [Dum09]. Klein found that solutions to a hypergeometric differential equation are parameterized by differential equations involving the Schwarzian derivative, but are more concretely understood via the geometric operation of "attaching" a sphere. In a similar way, we will see that an analytic parameterization via holomorphic quadratic differentials and a geometric parameterization via *grafting* work together to provide a more comprehensive picture of $\mathcal{P}(S)$.

2.3.1 Analytic parameterization

Definition 2.9. Let $\Omega \subset \mathbb{C}$ be a connected open set. Let $f : \Omega \to \mathbb{C}P^1$ be a (locally injective) holomorphic map. Then the *Schwarzian derivative* of f is given by the following holomorphic quadratic differential:

$$S(f) := \left(\left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2 \right) dz^2$$

The local injectivity is obviously necessary to ensure that $f'(z) \neq 0$ so that $S(f)$ makes sense. We will now investigate several properties of $S(f)$ that will be important for our discussion.

Definition 2.10. Let $g : \Omega \to \mathbb{C}P^1$ be a locally injective holomorphic map and let f be a locally injective holomorphic map from $g(\Omega) \subset \mathbb{C} \subset \mathbb{C}P^1 \to \mathbb{C}P^1$. Then $S(f)$ satisfies the *cocycle property.* That is:

$$S(f \circ g) = g^* S(f) + S(g)$$

This is checked with a tedious application of the chain rule. We note that the condition $g^*S(f)$ actually translates to $[g'(z)]^2 S(f)(g(z))$, and this is where the holomorphic quadratic differential nature of the Schwarzian derivative is evident. Heuristically, the importance of the Schwarzian derivative is that it measures the extent to which a holomorphic function deviates from being a Möbius transformation. Namely, let

$$PSL_2(\mathbb{C}) = \{A \in GL_2(\mathbb{C}) : \det A = +1\}$$

denote the group of Möbius transformations. If $A \in PSL_2(\mathbb{C})$, with $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then A corresponds to a map $f : \mathbb{C} \to \mathbb{C}P^1$ given by $f(z) = \frac{az+b}{cz+d}$.

Proposition 2.11. If $A \in PSL_2(\mathbb{C})$ then $S(A) \equiv 0$. Further, if $S(f) = 0$ then f is the restriction of a Möbius transformation to a domain in $\mathbb{C}P^1$.

Proof. If f is the map corresponding to $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then we compute f' and f'' and substitute into the equation for the Schwarzian.

$$\begin{aligned} S(A) &= (\frac{-2c}{cz+d})' - \frac{1}{2}(\frac{-2c}{cz+d})^2 \\ &= 2c^2(cz+d)^{-2} - \frac{1}{2}(4c^2(cz+d)^{-2} = 0 \end{aligned}$$

Conversely, if $S(f) = 0$, then

$$(\log(f'))'' = \frac{1}{2}[(\log f')']^2$$

Solving this differential equation, conclude that f is the restriction of a Möbius transformation. □

Remark 2.12. A second, related property of Möbius transformations is that they are circle-preserving. With this in mind, one can imagine that a transformation that is "close" to being Möbius would send a disc to something that is "close" to a disc, or a *quasidisc*. From this point of view, the Schwarizan derivative measures the deformation of an infinitesimal disc under the given transformation.

Remark 2.13. The cocycle property and the vanishing of the Schwarzian for Möbius maps combine to give the invariance of the Schwarzian derivative under composition with elements of $PSL_2(\mathbb{C})$. In particular, if $A \in PSL_2(\mathbb{C})$ and g is holomorphic, then by the cocycle property

$$S(A \circ g) = g^* S(A) + S(g) = S(g)$$

where the last equality follows because $S(A) = 0$.

The goal is to use the Schwarzian derivative to define a parameterization of a fiber $P(X) = \pi^{-1}(X)$ of the complex projective structures over a point $X \in \mathcal{T}(S)$. Our parameterization will be a map $P(X) \to Q(X)$, where $Q(X)$ is the space of holomorphic quadratic differentials on $X \in \mathcal{T}(S)$. The space $Q(X)$ is a natural object for the identification, as by a theorem of Ahlfors [Ahl61], the holomorphic cotangent space to $\mathcal{T}(S)$ at X is $Q(X)$.

To accomplish this goal, we make use of the second point of view for $\mathbb{C}P^1$ structures. Let $Z = (dev, hol) \in P(X)$. Then the developing map can be thought of as a meromorphic function $\mathbb{H} \to \mathbb{C}P^1$. Thus the Schwarzian derivative of the developing map, $S(dev)$, is a holomorphic quadratic function on $\mathbb{H}$. Further, the holomorphic quadratic differential $S(dev)$ satisfies:

$$S(dev) = S(A_\gamma \circ dev) = S(dev \circ \gamma) = \gamma * S(dev) + S(\gamma) = \gamma * S(dev)$$

where the first and last equalities hold because of $PSL_2(\mathbb{C})$ invariance under the Schwarzian, and the second holds due to the equivariance condition. Hence $S(dev)$ descends to a holomorphic quadratic differential on X, which we denote by ϕ_{dev}. This defines a map

$$S : P(X) \to Q(X)$$

$$Z \mapsto \phi_{dev}$$

To show surjectivity of this map, we construct its inverse. Given ϕ a holomorphic function on a contractible open set $\Omega \subset \mathbb{C}$, consider the differential equation

$$u''(z) + \frac{1}{2}\phi(z)u(z) = 0 \tag{2.1}$$

By ODE theory, this has a two-dimensional vector space of holomorphic solutions. Choosing u_1 and u_2 as a basis, one can check that $w = \frac{u_1}{u_2}$ satisfies $S(w) = \phi(z)dz^2$. (This makes sense because the Wronskian is nonzero so that u_1 and u_2 never vanish simultaneously). The value $S(w)$ is also independent of choice of basis, because a change of basis amounts to changing w by a Möbius transformation. To construct the inverse to $S : P(X) \to Q(X)$, given $\phi \in Q(X)$, lift ϕ to $\tilde{\phi}$ on the universal cover $\tilde{S}$. Then solve the ODE above, obtaining the ratio $f_\phi = \frac{u_1}{u_2} : \mathbb{H} - \mathbb{C}P^1$ of the two solutions, which satisfies $S(f_\phi) = \tilde{\phi}$. The function f_ϕ will be the developing map and it remains to construct the appropriate holonomy. To do this, first note that for all $\gamma \in \pi_1(S)$,

$$S(f_\phi \circ \gamma) = \gamma * \tilde{\phi} = \tilde{\phi} = S(f_\phi)$$

i.e., $f_\phi \circ \gamma$ and f_ϕ differ by some element of $PSL_2(\mathbb{C})$. Call this element A_γ. Then define the holonomy map $hol_\phi : \pi_1(S) \to PSL_2(\mathbb{C})$ by $\gamma \mapsto A_\gamma$. Hence, given $\phi \in$

$Q(X)$ we have defined a projective structure $Z_\phi = (f_\phi, hol_\phi)$ that maps to ϕ under the map S.

2.3.2 Grafting parameterization

Even though we now have successfully defined a parameterization of $P(X)$ to holomorphic quadratic differentials via the Schwarzian derivative, much like Klein and his hypergeometric functions, we have little sense of how the geometry of the complex projective structures in a fiber over X in $\mathcal{T}(S)$ are related. Our current goal, then, is to give a geometric description of all complex projective structures. The natural course of action, then, is to look for the simplest pieces and understand how to glue them together to create an arbitrary complex projective structure. Our building blocks will be cylinders, and the construction will be called *grafting*. To motivate the projective construction, we start with the conformal (complex) definition.

Definition 2.14. Let $X \in \mathcal{T}(S)$. Here, think of $\mathcal{T}(S)$ a conformal classes of metrics on S with a unique distinguished representative of each class, the constant curvature -1 (hyperbolic) metric. Choose any simple closed hyperbolic geodesic γ on X. Grafting is the operation of replacing γ by a Euclidean cylinder $\gamma \times [-\frac{t}{2}, \frac{t}{2}]$. The new surface obtained is called the *grafting of* X *by* $t\gamma$ and is denoted $gr_{t\gamma}X$.

The natural metric on $gr_{t\gamma}X$ is the *Thurston metric*, given by

$$\sigma_t = \begin{cases} |dz|^2 & \text{on the cylinder} \\ \sigma_0 = X & \text{on the complement of the cylinder} \end{cases}$$

This metric is certainly smooth away from the boundary of the grafting cylinder, and at the boundary it is $C^{1,1}$. To see this, note that near the boundary of the

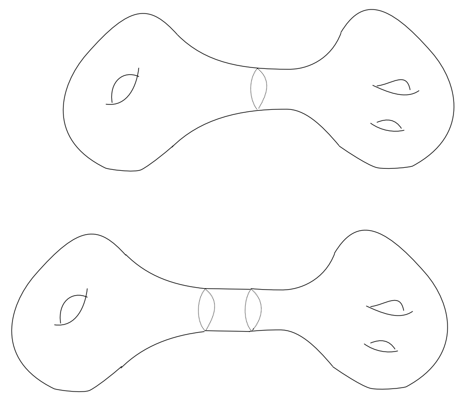

Figure 2.3: Inserting a Euclidean annulus in the grafting construction.

grafting cylinder, the hyperbolic metric σ is approximately $\cosh^2 w|dw|^2$. Then at $w = 0$, $\cosh^2 w$ has order one contact with the flat metric $|dw|^2$. The jump in second derivative is a direct result in the difference of curvature. This new class of conformal (complex) structures includes a hyperbolic metric. Thus we think of grafting as a map

$$gr : \mathcal{S} \times \mathbb{R}^+ \times \mathcal{T}(S) \to \mathcal{T}(S)$$

where $\mathcal{S}$ is the set of all simple closed curves on S, and in particular, $\mathcal{T}(S)$ is the set of hyperbolic metrics on S. We will call this map/construction *conformal grafting* when it is necessary to distinguish it from our next construction: *projective grafting.* To describe projective grafting, we must first understand the projective equivalent of a Euclidean cylinder. Again, choose $X \in \mathcal{T}(S)$ and $\gamma \in \mathcal{S}$. Let $\ell = \ell(\gamma, X)$ be the X-length of the unique hyperbolic geodesic in the homotopy class of γ. Giving X the standard Fuchsian structure, we note that the holonomy about γ is conjugate to $z \mapsto e^{\ell} z$. Next, choose $t \in (0, 2\pi)$. We shall denote by $\tilde{A}_t$ the sector of the complex

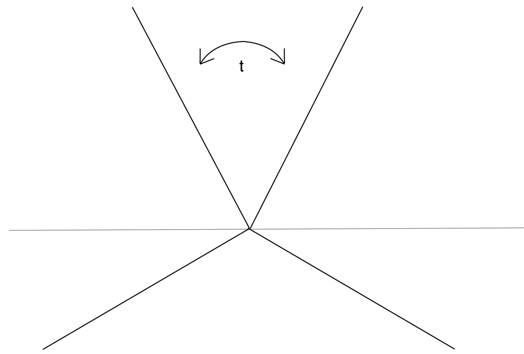

Figure 2.4: The region $\tilde{A}_t$

plane of angle t with vertex at the origin:

$$\tilde{A}_t := \{(r, \theta) : |\theta - \frac{\pi}{2}| < \frac{t}{2}\}$$

Then the quotient of $\tilde{A}_t$ by $\langle z \mapsto e^{\ell} z \rangle$ gives us the projective analogue we are looking for: a *projective t-annulus* $A_t = \tilde{A}_t / \langle z \mapsto e^{\ell} z \rangle$. In particular, forgetting about the projective structure that A_t inherits naturally as a quotient of a projective structure, the quotient A_t is isomorphic as a Riemann surface to a Euclidean cylinder of length t. Additionally, because Möbius transformations are circle-preserving, the image of $\tilde{A}_t$ under a general Möbius element is the intersection of two round discs with angle t.

To obtain a model of projective grafting, at each lift $\tilde{\gamma}$ of γ in the universal cover $\tilde{X}$, insert a copy of the sector $\tilde{A}_t$. Then apply Möbius transformations to $\tilde{A}_t$ and $\tilde{X} - \tilde{\gamma}$ so that the regions fit together. This gives a new projective structure $Gr_{t\gamma}X$ which corresponds to a natural projective structure on $gr_{t\gamma}X$: glue the standard Fuchsian structure on X to A_t (and these match at the boundaries because of the holonomy condition!). Also, $Gr_{t\gamma}X$ corresponds to a decomposition of the universal cover into

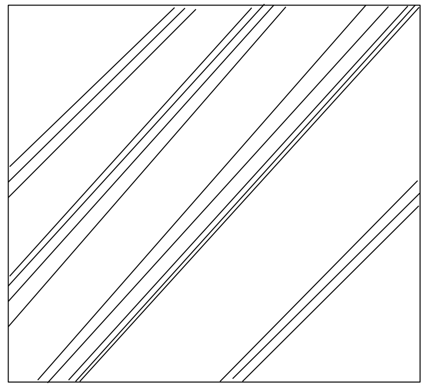

Figure 2.5: Locally, a geodesic lamination can look like a cantor set on the surface.

sectors of angle t and the complement of the sectors, which are regions bounded by circular arcs. We thus have the projective grafting map

$$Gr : \mathcal{S} \times \mathbb{R}^{+} \times \mathcal{T}(S) \to \mathcal{P}(S)$$

Remark 2.15. One glaring issue to take care of is that projective grafting has so far been defined only for $t \in (0, 2\pi)$, but the map above permits all $t \in \mathbb{R}^{+}$. This was done primarily for convenience; for $t \geq 2\pi$, $\tilde{A}_t$ can be thought of as a sector that "wraps around" the punctured plane $\mathbb{C}^*$ an appropriate number of times.

Remark 2.16. A second issue to address is that up to this point we have considered grafting only along *simple closed curves*, but this can be generalized as well to *measured geodesic laminations*. We follow [Bon96] and [Bon97] here.

Definition 2.17. Fix $X \in \mathcal{T}(S)$, thinking of X as hyperbolic (negatively curved) metric. Then an X-*geodesic lamination* of S is a partial foliation of S by X-geodesics, namely a closed subset λ of S decomposed into a union of simple disjoint geodesics called *leaves* which do not hit the boundary transversely.

Remark 2.18. We note that if X' is any other negatively curved metric, then for every leaf of the X'-geodesic lamination λ' there is a homotopy to a leaf of the X-geodesic lamination λ. This gives a natural correspondence between X' and X-geodesic laminations. To consider geodesic laminations which are in some sense truly distinct, we will define a *geodesic lamination* as an equivalence class of pairs (λ, X) where $(\lambda', X') \sim (\lambda, X)$ if λ' is the X'-geodesic lamination obtained from λ via homotopy.

The geodesic lamination will be the generalization of $\mathcal{S}$; the *transverse measure* we endow it with will play the role of $\mathbb{R}^+$.

Definition 2.19. A *transverse measure* μ on a geodesic lamination $\lambda = (\lambda, X)$ is a measure defined on each arc k transverse to λ so that if k_1 and k_2 are homotopic, $\mu(k_1) = \mu(k_2)$. By a measure, we mean a positive Radon measure.

Definition 2.20. A transverse measure μ has *full support* if the support of the measure it induces on each transverse arc k is precisely $k \cap \lambda$.

Definition 2.21. A *measured geodesic lamination* is a pair $(\lambda = (\lambda, X), \mu)$ of a geodesic lamination and a full support transverse measure.

Example 2.22. Choose λ to be a simple closed geodesic for X. Let μ be defined to be the Dirac mass of weight t on each arc k transverse to λ. Thus if $A \subset k$, then the mass of A is given by t times the cardinality of $A \cap \lambda$.

Definition 2.23. Let $\mathcal{ML}(\mathcal{S})$ denote the space of geodesic measured laminations on S. We note for future reference that $\mathcal{ML}(S)$ was shown by Thurston [**?**, Thurston]o be a piecewise-linear manifold of homeomorphic to $\mathbb{R}^{6g-6}$. See also [AFP79].

Now as weighted simple closed curves are dense in $\mathcal{ML}(S)$, we have a uniquely

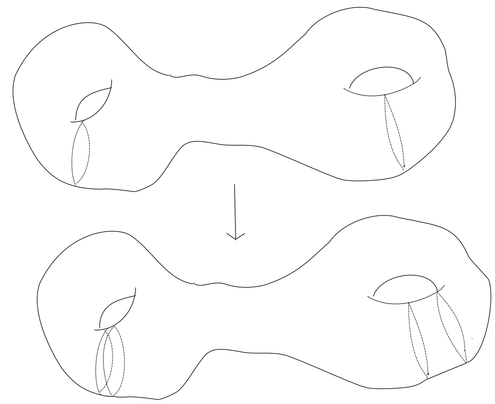

Figure 2.6: Grafting along a weighted multicurve.

defined continuous extension of grafting from $Gr : \mathcal{S} \times \mathbb{R}^+ \times \mathcal{T}(S) \to \mathcal{P}(S)$ to $Gr : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{P}(S)$. Since $gr = \pi \circ Gr$, we also have an extension $gr : \mathcal{ML}(S) \times \mathcal{T}(S) \to \mathcal{T}(S)$. To understand this extension, note that if $\gamma = \sum_{i=1}^{n} c_i \gamma_i$ is a weighted multicurve, then $gr_\gamma X$ is obtained by simultaneously replacing each γ_i with the cylinder of the appropriate length. For a general lamination $\lambda \in \mathcal{ML}(S)$, the grafted surface $gr_\lambda X$ is the Riemann surface obtained by a thickening of the lamination in a manner determined by the transverse measure μ.

At this point, the projective grafting map is merely a means of deforming one complex projective structure to get a second complex projective structure. The goal is to construct the inverse of the grafting map $Gr : \mathcal{ML}(S) \to \mathcal{P}(S)$. This requires a perspective that displays the interaction between the hyperbolic and projective structures, called *bending*. In particular, projective structures can be seen as hyperbolic structures bent in $\mathbb{C}P^1$, and grafting is the record of bending inscribed on the sphere at infinity. In this section, we closely follow Tanigawa [Tan97] .

Bending

Before we can define bending, we need to develop some vocabulary to describe the behavior of the Thurston metric on particular regions of the universal cover.

Definition 2.24. Let $Z \in \mathcal{P}(S)$ and let $\tilde{Z}$ denote the lift to the universal cover. Let ρ_Δ denote the canonical hyperbolic metric on the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. Then for any vector v at $z \in Z$, define the length of v by

$$t(v) = \inf \rho_\Delta(f^* v)$$

where the infimum is taken over all projective immersions $f : \Delta \to Z$. This is called the *Thurston metric.*

Definition 2.25. As we noted earlier, round circles or *discs* make sense in $\mathbb{C}P^1$ manifolds. Let Δ denote the unit disc. Thus, given $z \in \tilde{Z}$ there is a unique extremal mapping with respect to the Thurston metric, $f : \Delta \to \tilde{Z}$ containing z; define $D_z := f(\Delta)$ and call it the *maximal disc for* $z \in \tilde{Z}$.

Definition 2.26. A *frontier point* of D_z is a point $\omega \in \partial D_z$ such that if D_z is identified with $\mathbb{H}^2$ the upper half plane via f, then f can be extended as a projective map beyond ω. A point ζ is an *ideal boundary point* if f cannot be extended past ζ. Let $\partial_\infty D_z$ be the set of all ideal boundary points on ∂D_z.

Definition 2.27. Let $C(\partial_\infty D_z)$ denote the convex hull of $\partial_\infty D_z$ (taken with respect to the hyperbolic metric on $\mathbb{H}$).

Remark 2.28. The definition of a maximal disc implies that $z \in C(\partial_\infty D_z)$ and also that there are at least 2 ideal boundary points (or else the mapping f isn't ex-

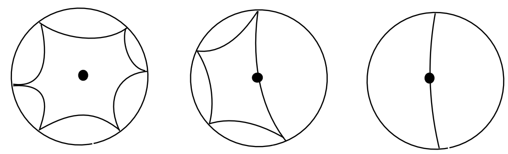

Figure 2.7: Three possibilities for $C(\partial_\infty D_z)$

tremal with respect to the Thurston metric). Then there are three possible cases for $C(\partial_\infty D_z)$:

- $\partial_\infty D_z$ contains at least three points and z is in the frontier of $C(\partial_\infty D_z)$. In this case, the flat and hyperbolic metrics agree at z.
- $\partial_\infty D_z$ contains at least three points and z is in the interior of $C(\partial_\infty D_z)$. In the setting described above, the Thurston metric near z is given by the hyperbolic metric $|dz|/Im(z)$.
- $\partial_\infty D_z$ contains exactly two points. Then $z \in C(\partial_\infty D_z)$ which is a line. Thinking of 0 and $\infty \in \mathbb{H}^2$ as the ideal boundary points, z is on the imaginary axis. Then the Thurston metric near z is given by the flat metric $|dz|/|z|$.

Example 2.29. Let $\tilde{Z}$ be the union of two discs, D and D'. Thinking of them in the upper half plane, if t is the angle between the two discs, then $\tilde{A}_t$, the sector at 0 we defined previously, is the region bounded by the two rays, and D and D' are half spaces with boundaries orthogonal to the rays. The Thurston metric is hyperbolic on D and D' and is flat on $\tilde{A}_t$. In other words, this is an example of grafting by t.

Example 2.30. We want to describe how the projective structure on a disc D in the Riemann sphere is given by a hyperbolic surface. Let $CH(D)$ denote the convex

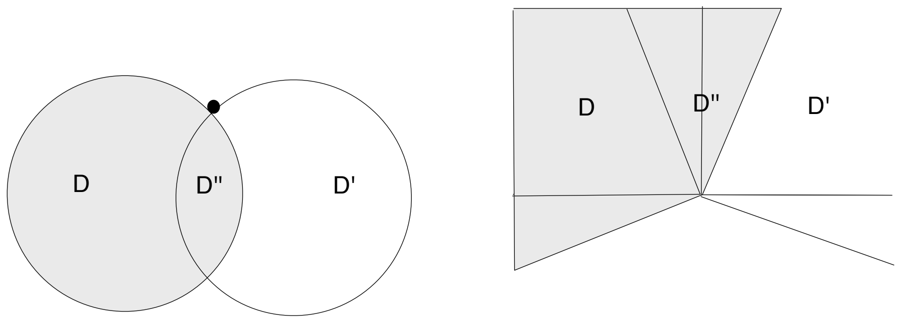

Figure 2.8: The Thurston metric is flat on the intersection D''. This picture is precisely the image for grafting.

hull of D in $\mathbb{H}^3$. The hyperbolic surface we will use is $CH(\partial D)$, defined to be the hyperbolic plane that is the boundary of the hyperbolic half space $CH(D)$. Given $z \in D$, there is a unique horosphere at z which is tangent to $CH(\partial D)$. The map $z \mapsto$ the tangent point is called the *nearest-point projection*. Now on the one hand, D is a disc and has a hyperbolic structure, and the nearest point projection sends this hyperbolic structure to $CH(\partial D)$. On the other hand, D is a subset of $\mathbb{C}P^1$, so the hyperbolic structure on D coincides with the projective structure. Thus in this case the *projective structure on* D is given by the *hyperbolic structure on* $CH(\partial D)$ via the nearest-point projection map. Given any complex projective structure Z, every $z \in \tilde{Z}$ is contained in a disc, so this fact is extremely important.

Definition 2.31. We describe how to *bend a hyperbolic surface along* ℓ and how this results in a new projective structure. As above, let D be a disc in $\mathbb{C}P^1$ and let ℓ be a geodesic line on $CH(\partial D)$. Then ℓ separates $CH(\partial D)$ into a left and right side, denoted by L and R respectively. Choose $\theta \in (0, \pi)$ and rotate R along ℓ by angle

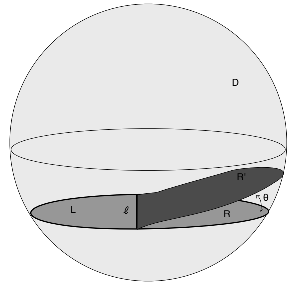

Figure 2.9: Bending the hyperbolic surface $CH(\partial D)$ along ℓ.

θ. Denote the image by R'. The result is a *pleated plane*, namely $L \cup R'$. Now this determines a new projective structure formed in the following manner: the boundary R' traces out part of a circle on $\mathbb{C}P^1$. Let $\partial D'$ denote this unique circle that contains R'. Let D' be the disc in $\mathbb{C}P^1$ with boundary $\partial D'$ and $D \subset D \cup D'$. Thus $D \cup D'$ with the projective structure inherited from $\mathbb{C}P^1$ is the new projective structure obtained via bending.

Now the question is: given a projective structure, how is the pleated plane obtained? Since all universal covers are unions of intersections of round circles, it suffices to think about our example earlier: $\tilde{Z} = D \cup D'$. For each $z \in \tilde{Z}$, take D_z and consider it as a disc in $\mathbb{C}P^1$. Then really we have two maximal discs - D and D'. On $\mathbb{C}P^1$, they intersect in a sector S and the hyperbolic planes containing their boundaries intersect in a hyperbolic line ℓ. The nearest point projection sends $D - S$ to a portion L of the hyperbolic plane for D and $D' - S$ is sent to a portion R' of the hyperbolic plane for D'. S is sent to ℓ.

What we have shown is that *bending* $CH(\partial D)$ along a curve γ is equivalent to *grafting* a flat part S into the hyperbolic structure D. Thus the procedure of obtaining a pleated plane from a projective structure is the inverse of the grafting map. For details we refer the reader to [Dum09, Tan97, KT92]

Concluding remarks

In hindsight, the fact that holomorphic quadratic differentials and geometric measured laminations are used to parameterize the same space is not surprising:

Theorem 2.32 (Hubbard, Masur [HM79])**.** *Let* $X \in \mathcal{T}(S)$*. Then* $Q(X)$ *is homeomorphic to* $\mathcal{ML}(\mathcal{S})$*, i.e., every measured lamination is realized by a unique holomorphic quadratic differential on* X*.*

Second, we pause to remark that Klein's geometric description of the solutions of hypergeometric functions is probably the first description of a grafting-like construction. At the end of Klein's lecture is written, "I hope that many more interesting results will be obtained in the future by such geometrical methods." In the next section, we discuss some further "interesting results" obtained via grafting.

Chapter 3

Grafting applications

This chapter marks a turn in our exploration. In the preceding sections, we have mainly focused on the projective grafting map. We now focus on the conformal grafting map, looking both at the λ-grafting map and the X-grafting map

$$gr_\lambda \cdot : \mathcal{T}(S) \to \mathcal{T}(S)$$

$$gr.X : \mathcal{ML}(S) \to \mathcal{T}(S)$$

3.1 Polar Coordinates for Teichmüller space

Given that λ-grafting defines a map from Teichmüller space to itself, it is natural to consider the injectivity and surjectivity of the map. McMullen [McM98] investigates this question via earthquake maps in the case of 1-dimensional Teichmüller space. An *earthquake* is a composition of grafting and a second geometric operation, *twisting*. Given $t \in \mathbb{R}$, λ a simple closed curve on a Riemann surface with hyperbolic metric X, $tw_{t\lambda}$, the twist along λ by t is obtained by cutting X along λ, twisting distance t to

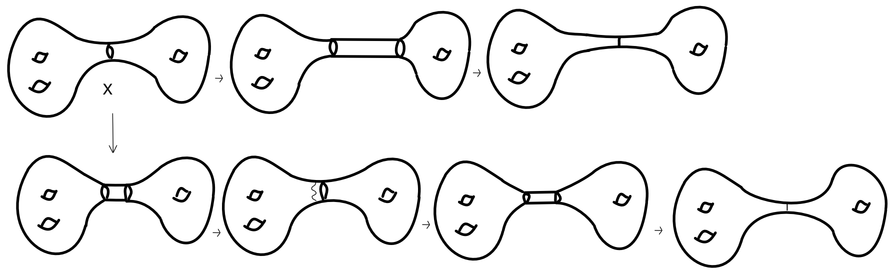

Figure 3.1: Grafting by $t + s$ along the top line versus iterated grafting.

the right, and re-gluing. A major difference between the twist map and the grafting map is that the twist map is a flow:

$$tw_{(s+t)\lambda}X = tw_{s\lambda}(tw_{t\lambda}X)$$

but in general the grafting map is not:

$$gr_{(s+t)\lambda}X \neq gr_{s\lambda}(gr_{t\lambda}X)$$

Intuitively, grafting fails to be a flow for the following reason: to obtain the surface $gr_{s\lambda}(gr_{t\lambda}X)$ from the surface $gr_{t\lambda}X$, one must locate the geodesic representative γ_t to insert the flat cylinder of length s. In particular, γ_t is not necessarily the same as the original geodesic γ. On the other hand, the surface $gr_{(s+t)\lambda}X$ is obtained solely by grafting along the geodesic γ, so the two operations have different geometric results as indicated in figure. Define the (complex) earthquake map by

$$eq_{(a+ib)\lambda}X = gr_{b\lambda}tw_{a\lambda}X$$

McMullen proves that the earthquake map is a holomorphic bijection from a connected open component of the complex plane to the one-dimensional Teichmüller space, with the positive grafting ray $\{gr_{t\lambda}X\}$ contained in the set $\{Y \in \mathcal{T}(S) : \ell_\lambda Y \leq \ell_\lambda X\}$. The holomorphicity of the earthquake map and the fact that twisting is a flow is then used to prove that the grafting map $gr_{\lambda}\cdot : \mathcal{T}(S) \to \mathcal{T}(S)$ is a homeomorphism. While this argument provides a nice picture for the 1-dimensional Teichmüller space, it did not easily extend to higher dimensional Teichmüller spaces.

The proof that the λ-grafting map is homeomorphic for arbitrary dimensional Teichmüller space was given by Scannell and Wolf [SW02] and is more geometric-analytic in nature. It studies the infinitesimal prescribed curvature equation for the grafted (Thurston) metric and the Jacobi equation for the infinitesimal variation of the geodesic boundaries of the grafting cylinders as a means of showing that the kernel of the λ-grafting map is empty. This is enough to show that the λ-grafting map is a real-analytic diffeomorphism, using McMullen's [McM98] result that grafting is real-analytic, Tanigawa's [Tan97] result that grafting is proper, and the fact that Teichmüller space is homeomorphic to a ball. The geometric-analytic approach taken is motivation for the considerations in a later chapter.

Given that the λ-grafting map is a diffeomorphism, a natural question to consider is whether its counterpart, the X-grafting map, is a diffeomorphism as well. As Dumas and Wolf [DW08] note, this question is made more complex by the fact that $\mathcal{ML}(S)$, the domain of the map, is a piecewise linear (i.e., not smooth) space. Using the complex linearity of the complex earthquake map and *shearing coordinates* for Teichmüller space, Dumas and Wolf prove that X-grafting map is in fact a *bitangentiable* homeomorphism, where this weaker form of differentiablity corresponds precisely to the lack of a natural differentiable structure on $\mathcal{ML}(S)$. To be specific: a *tangentiable* map between open sets in $\mathbb{R}^n$ is a map where one-sided directional derivatives are everywhere defined. A tangentiable manifold is a manifold with tangentiable transition functions, in particular, every piecewise-linear manifold is tangentiable. Thus $\mathcal{ML}(S)$ is tangentiable. Further, differentiable maps are trivially tangentiable, so we can also say that $\mathcal{T}(S)$ is tangentiable. Finally, a map between tangentiable manifolds is said to be *bitangentiable* if the map and its inverse are tangentiable and the tangent maps are everywhere homeomorphisms.

The Scannell-Wolf and Wolf-Dumas theorem together give the following: *grafting gives polar coordinates for Teichmüller space.* In particular, by fixing a given $X \in \mathcal{T}(S)$, we can think of Teichmüller space as parameterized by grafting rays $gr_{t\lambda}X$ for $\lambda \in \mathcal{ML}(S)$. Grafting rays are the main object of study in what follows.

3.2 Recent developments

Much of the current research has focused on using asymptotic analysis of the geometry along a grafting ray to investigate the relationship between a grafting ray and other distinguished paths in Teichmüller space. A particularly natural choice of path to study is a *Teichmüller geodesic.* Teichmüller's metric on Teichmüller space is given by the following: let X and Y be points in $\mathcal{T}(S)$. Then

$$d_{\mathcal{T}(S)}(X, Y) = \frac{1}{2}\log(K)$$

where K is the smallest possible dilatation for a quasiconformal map between the two points [Ahl06]. (Such a map exists by Teichmüller's theorem; see [Ber60] and [Leh87] for a particularly clear exposition.) In particular, a Teichmüller geodesic is defined by the stretching and contracting of the foliation induced by a holomorphic quadratic differential on the surface. Several authors have worked to make explicit the idea that for large t, grafting by $t\lambda$ is like a Teichmüller deformation with horizontal foliation λ. In particular, Diaz and Kim [DK11] have shown that if $\lambda = \sum c_i\gamma_i$ is a weighted multicurve, then the grafting ray $gr_{t\lambda}$ converges to $[c_i\gamma_i]$ in the Thurston boundary of $\mathcal{T}(S)$, and throughout its travels remains within a bounded distance of a Teichmüller geodesic. This bound is dependent only on the initial lengths of the curves in the grafting locus.

Choi, Dumas, and Rafi [YC] later strengthened this result to show that grafting rays starting at a point X which is ϵ-thick are Teichmüller quasi-geodesics, i.e., the grafting ray $\{gr_{t\lambda}X\}$, is within a bounded distance from the Teichmüller ray determined by (X, λ), and the bound depends only on the injectivity radius. Gupta [Gup11] makes a finer but less uniform comparison, showing that if λ is a multicurve, then there exists $Y \in \mathcal{T}(S)$ such that the Teichmüller distance between $\{gr_{t\lambda}X\}$ and the Teichmüller ray determined by (Y, λ) goes to zero.

Work that is primarily focused on the grafting ray, albeit still with an eye toward the behavior at infinity, is the question of iterated grafting. We noted before that unlike twisting, grafting is not a a flow:

$$gr_{s\gamma}(gr_{t\gamma}X) \neq gr_{(s+t)\gamma}X$$

Hensel [Hen11] gives quantitative control over how much grafting deviates from a flow. In particular, he shows that if the grafting locus is "short enough" with respect to the hyperbolic metric, then grafting behaves "almost" like a flow. An insight gathered from the proof is that if the grafting locus is short enough, the topology has little effect in uniformization, so that the distance between the middle of the grafting cylinder and the new hyperbolic geodesic is bounded, and well contained inside the standard hyperbolic collar obtained via the collar lemma.

A common theme in the above results is an understanding of the geometry of hyperbolic surface obtained via grafting, often relative to a holomorphic quadratic differential or in comparison to the projective geometry of a grafted surface. As we remarked when summarizing some of Hensel's results, the behavior of the grafting locus under uniformization is essential. Hensel and others provide estimates that bound from above and below the length of the new hyperbolic geodesic in terms of the Euclidean length of the grafting cylinder and the initial length of the grafting locus. But a natural question to ask is: how does the length of the grafting locus *change* as we proceed along the grafting ray? What geometric components are the driving forces in changing the length?

This is not an entirely new question. Let $X \in \mathcal{T}(S)$ be a hyperbolic metric on a closed surface S of genus $g \geq 2$. Let $\ell(\gamma, X)$ denote the X-length of the X-geodesic representative of γ. McMullen considers the geometry of grafting in [McM98]. In particular, for any $t \in \mathbb{R}$ and any $\lambda \in \mathcal{T}(S)$, he shows

$$\ell(\lambda, gr_{t\lambda}X) < \ell(\lambda, X)$$

To see this, first consider the definitions of projective and hyperbolic length. Given a complex projective structure Z on S, the *projective* length of a vector v is defined to be the infimum of the hyperbolic length of vectors $w \in T\mathbb{H}^2$ such that there exists a *projective* map $f : \mathbb{H} \to Z$ satisfying $f(v) = w$. The *hyperbolic* length of v is the infimum of the same quantity, but taken over *holomorphic* map $f : \mathbb{H} \to Z$ satisfying $f(v) = w$ [DK11]. Since holomorphic maps are projective maps, in general, the hyperbolic length is less than or equal to the projective length. Letting Th_t denote the Thurston metric underlying the hyperbolic metric $gr_{t\gamma}X$, we write $\ell(\cdot, gr_{t\gamma}X) \leq \ell(\cdot, Th_t)$ to express this fact. Now let $X \in \mathcal{T}(S)$, α and $t\gamma \in \mathcal{ML}(S)$. Applying this discussion,

$$\ell(\alpha, gr_{t\gamma}X) < \ell(\alpha, Th_t)$$

But then $\ell(\alpha, Th_t)$ is bounded above by the length of α before grafting, i.e., $\ell(\alpha, X)$ plus the additional length needed to cross the grafting cylinder. This is given by $i(\alpha, t\gamma) = t \cdot i(\alpha, \gamma)$, where $i(\cdot, \cdot)$ is the Thurston intersection form. Extending to measured laminations by continuity, we have for any curve α and β

$$\ell(\alpha, gr_{t\beta}X) < \ell(\alpha, X) + t \cdot i(\alpha, \beta)$$

In particular, then, for $\beta = \alpha$, we have $i(\alpha, \alpha) = 0$, giving $\ell(\alpha, gr_{t\alpha}X) < \ell(\alpha, X)$, as desired. While this information is helpful, it doesn't give much information about the *change* in length of the grafting locus along a grafting ray. The two main theorems in the literature are

Theorem 3.1 (McMullen [McM98]). *Let α be any simple closed curve in $\mathcal{ML}(S)$ and $X \in \mathcal{T}(S)$. Let $\ell_\alpha X$ be the geodesic length function on $\mathcal{T}(S)$. Then*

$$\left.\frac{d}{dt}\right|_{t=0} gr_{t\alpha}X = -\nabla \ell_\alpha X$$

where the gradient is taken with respect to the Weil-Petersson (WP) metric on $\mathcal{T}(S)$.

In other words, at the start of the grafting ray, grafting shortens curves as quickly as possible in the WP sense.

Theorem 3.2 (Dumas-Wolf [DW08]). *Let $X \in \mathcal{T}(S)$ and γ any simple closed hyperbolic geodesic on X. Then the hyperbolic length of the geodesic representative of γ along the grafting length is $\frac{\pi\ell(\gamma,X)}{t} + O(t^{-2})$ and for t sufficiently large,*

$$\frac{d}{dt}\ell(\gamma, gr_{t\gamma}X) = \frac{\pi\ell(\gamma, X)}{t^2} + O(t^{-3})$$

as $t \to \infty$, and where implicit constants depend on X and γ.

McMullen's theorem is proven by appealing to the holomorphicity of earthquake maps, so that

$$\frac{d}{dt}gr_{t\gamma}X = -i\frac{d}{dt}tw_{t\gamma}X$$

Now the Weil-Petersson metric is Kähler [Ahl61], so that the gradient of a function is i times the Hamiltonian vector field that it generates. But by a theorem of Wolpert [Wol81] the Hamiltonian flow generated by $-\ell_\alpha$ with respect to the WP symplectic form is the Fenchel-Nielsen rightward twist for α. Combining this with the two facts above gives the result. The Dumas-Wolf result is obtained by estimating the variation of extremal length of the grafting cylinder along grafting rays and showing that a certain sub-annulus in the lift comprises most of the area of the grafting cylinder. The following is an attempt to close the gap: McMullen and Dumas-Wolf give indication of the behavior of the length of the grafting locus at the start of the grafting ray and asymptotically. We give a formula for the derivative of length at any point along the grafting ray in terms of the hyperbolic geometry at the point. In so doing, we utilize techniques of geometric analysis which up to this point have not been explored in the context of the variation of geometry along grafting rays.

Chapter 4

A surface geometry approach to grafting

4.1 Introduction

There are two goals for this chapter. First, we give a detailed description of some tools one can utilize to study the change in the geometry at any point along a grafting ray generated by a simple closed geodesic. As simple closed geodesics are dense in the space of measured laminations, understanding this case is key to developing an understanding of the general case. At the end of this first section, we derive a formula that relates the geometry of the hyperbolic surface in the conformal class of $gr_{t_0\gamma}X$ to the change in the geometry along the grafting ray. The objective of the second section is to simplify the formula obtained in the first section, and prove a more precise version of Theorem 4.1 stated in the introduction:

Theorem 4.1. *Let γ be a separating curve on S. Choose $t_0 \in \mathbb{R}^+$. Let σ_0 be the hyperbolic metric in the conformal class of $gr_{t_0\gamma}X$. Then if γ_0 is the hyperbolic geodesic in the homotopy class of γ and α^+ and α^- are curves that are parallel to γ_0 with respect to σ_0, then*

$$-2\frac{\ell(0,\alpha^+)}{\ell}\left.\frac{d}{dt}\right|_{t=0}\ell(t,\alpha^+) - 2\frac{\ell(0,\alpha^-)}{\ell}\left.\frac{d}{dt}\right|_{t=0}\ell(t,\alpha^-) + 4\left.\frac{d}{dt}\right|_{t=0}\ell(t,\gamma_t)$$

is given in terms of t_0, the Euclidean length of the grafting cylinder, the conformal factor g, where $\sigma_0 = g(z)|dz|^2$, distance of α^+ and α^- from γ_0, and the average angle between ξ=constant, a curve of constant σ_0 (hyperbolic) geodesic curvature, and x=constant, a curve of constant Euclidean geodesic curvature.

We will give more details on the methods used to accomplish this at the beginning of the second section. In the third section, we revise the statement of Theorem 4.1. This revised version will be used in the fourth section, where we prove Theorem 4.2.

Theorem 4.2. *For ℓ small enough and α^+ and α^- sufficiently close to the geodesic γ_0,*

$$-2\frac{\ell(0,\alpha^+)}{\ell}\left.\frac{d}{dt}\right|_{t=0}\ell(t,\alpha^+) - 2\frac{\ell(0,\alpha^-)}{\ell}\left.\frac{d}{dt}\right|_{t=0}\ell(t,\alpha^-) + 4\left.\frac{d}{dt}\right|_{t=0}\ell(t,\gamma_t) < 0 \quad (4.1)$$

The main idea of the proof of Theorem 4.2 is to make precise the idea near the middle of the grafting cylinder, curves of constant hyperbolic geodesic curvature are, on average, very close to Euclidean geodesics.

4.2 Introduction to technique

In this section, we develop the necessary tools to derive

Proposition 4.13. *Let γ be a separating curve on S. Let $X \in \mathcal{T}(S)$ and let σ_0 be the hyperbolic metric in the conformal class of $gr_{t_0\gamma}X$. Then for α^+ and α^- and curves parallel to γ_0,*

$$
\begin{aligned}
& -2\ell \sec \ell\xi^{\alpha^\pm} \cdot 2 \left.\frac{d}{dt}\right|_{t=0} \int_{\alpha^\pm} ds_t + 4\ell \cdot 2 \left.\frac{d}{dt}\right|_{t=0} \int_{\gamma_0} ds_t - 2\ell |\tan \ell\xi^{\alpha^\pm}| \iint_{comp^\pm} \frac{\dot{G}}{g}\, \mathrm{d}A_0 \\
&= -2\ell \sec \ell\xi^{\alpha^\pm} \cdot 2\dot{c} \int_{\alpha^\pm} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\, ds_0 + 4\ell \cdot 2\dot{c} \int_{\gamma_0} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\, ds_0 \\
&+ 2\dot{c}\ell |\tan \ell\xi^{\alpha^\pm}| \iint_{graft \cap comp^\pm} (\log g)_{xx}\, \mathrm{d}x\mathrm{d}y + 2\dot{c}\ell \iint_{graft \cap \alpha - collar} |\tan \ell\xi| (\log g)_{xx}\, \mathrm{d}x\mathrm{d}y
\end{aligned}
$$

We begin by describing two natural choices for local coordinates on a surface at any point on a grafting ray. The first subsection ends with a description of the relationship between the coordinates. The second subsection is devoted to explicitly stating formulas for the derivative of length of a curve on a surface along the grafting ray. In the third subsection, we derive the *prescribed curvature equation.* In the last section, we motivate the derivation of the formula in Proposition 4.13 via two examples. Finally, we derive the formula in the above proposition.

To set the stage: let S be a compact, orientable surface of genus $g \geq 2$. Let $X = \sigma|dz|^2 \in \mathcal{T}(S)$. Assume that γ is a separating geodesic for X. The deformation of the hyperbolic geometry at a point $gr_{t_0\gamma}X$ along the grafting ray $\{gr_{(t+t_0)\gamma}X\}$ for $t, t_0 > 0$ is the subject of the following study. Recall, a natural metric for a grafted surface is a piecewise defined metric that is Euclidean on the grafting cylinder and hyperbolic on the complement, called the Thurston metric.

Definition 4.3. Let Th_t denote the Thurston metric for a surface with grafting cylinder of length $t_0 + t$.

$$Th_t = \begin{cases} |dz_t|^2 & \text{on the cylinder} \\ \sigma|dz|^2 & \text{on the complement of the cylinder} \end{cases}$$

where $z_t = (x_t, y_t)$ are Euclidean coordinates and z is a conformal coordinate for X.

Definition 4.4. Let σ_t be the constant curvature $-$ (hyperbolic) metric in the conformal class of Th_t. We will often call σ_0 the *time-0 hyperbolic metric.*

Definition 4.5. Define the function g_t to be the conformal factor $\frac{\sigma_t}{Th_t}$. In other words,

$$\sigma_t = \begin{cases} g_t(z_t)|dz_t|^2 & \text{on the cylinder} \\ g_t(z)\sigma|dz|^2 & \text{on the complement of the cylinder} \end{cases}$$

Remark 4.6. Recall that the Thurston metric is smooth away from the boundaries of the grafting cylinder and $C^{1,1}$ at the boundaries. On the other hand, the hyperbolic metric should be smooth everywhere. Thus the function g_t will also be $C^{1,1}$ at the boundaries of the grafting cylinder.

4.2.1 Representing the metrics along the grafting ray: two approaches.

Because the focus is on the point $gr_{t_0\gamma}X \in \mathcal{T}(S)$, we want to think of the conformal factors $\frac{\sigma_t}{|dz_t|^2}$ and $\frac{Th_t}{|dz_t|^2}$ as functions on $gr_{t_0\gamma}X$. In the first approach, we emphasize the geometric operation of grafting by tracing the stretch of the Euclidean cylinder along a grafting ray. In the second approach, we emphasize the unique σ_0-geodesic in the homotopy class of the grafting locus.

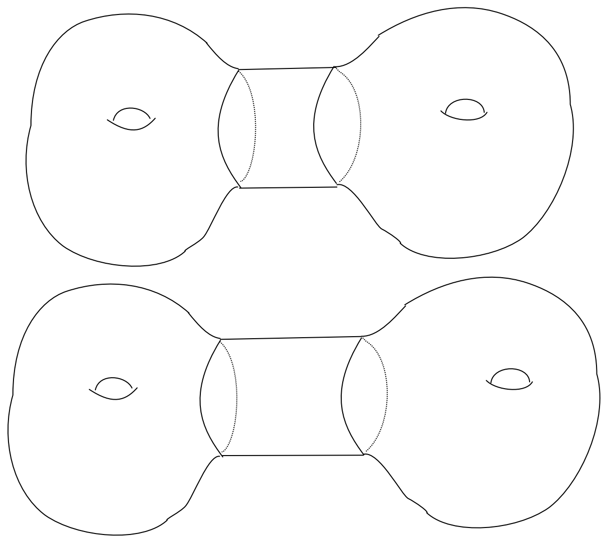

Figure 4.1: The surface with cylinder of length $t+t_0$ (bottom) differs from that of length t_0 (top) by a linear stretch along the cylinder.

First approach: grafting coordinates

Endowing $gr_{t_0\gamma}X$ with the Thurston metric, the cylinder of length $t+t_0$ is simply a linear stretch in the x-direction of the cylinder of length t_0. Thus the coordinates z_t can be thought of as a family of functions of the $z_0 = z = (x, y)$ Euclidean coordinates for the length t_0 grafting cylinder. I.e.,

$$z_t = (x_t(x,y), y_t(x,y) = (c_t x, y)$$

where $c_0 = 1$ and $c_t = \frac{t}{t+2t_0}$ is what we'll call the *stretch constant.* Then compute

$$|dz_t|^2 = dx_t^2 + dy_t^2 = d(c_t x)^2 + dy^2 = c_t^2 dx^2 + dy^2$$

and rewrite the expressions for the Thurston and hyperbolic metrics as follows:

$$Th_t = \begin{cases} c_t^2 dx^2 + dy^2 & \text{on the cylinder} \\ \sigma|dz|^2 & \text{on the complement of the cylinder} \end{cases}$$

$$\sigma_t = \begin{cases} g_t(z_t(x,y))[c_t^2 dx^2 + dy^2] & \text{on the cylinder} \\ g_t(z)\sigma|dz|^2 & \text{on the complement of the cylinder} \end{cases}$$

The moral of this first story is that the $z = (x, y)$ grafting coordinates are orthogonal – although not orthonormal – on the grafting cylinder for all the grafted metrics, so understanding how the geometry changes along a grafting ray corresponds to understanding the stretch c_t, how the conformal factors $g_t(z_t)$ are changing, and the "jump" in g_t at the boundary. However, the geometry of the hyperbolic metric $g(z)|dz|^2$ is somewhat obscured in the process.

Second approach: hyperbolic cylinder coordinates

To remedy the above ignorance of the time-0 hyperbolic geometry, (i.e., the geometry corresponding the hyperbolic metric in the conformal class of the grafted surface $gr_{t_0\gamma}X$), we choose geometrically friendly coordinates.

Assumption 4.7. We make two assumptions in all that follows. First, we assume that the hyperbolic geodesic γ_0 in the homotopy class of the grafting locus is strictly contained in the image of the grafting cylinder under uniformization. This ensures that there exists a hyperbolic collar about the grafting locus that is not disjoint from the grafting cylinder. We remark that by work of Hensel [Hen11] this is a reasonable assumption, especially for curves that are short. Second, we assume that γ_0, when thought of as a curve in the grafting cylinder, is a graph over a Euclidean geodesic.

In particular, choose a holomorphic coordinate $\zeta(z) = \xi(z) + i\eta(z)$ on the surface, adapted to the hyperbolic collar about the grafting locus (i.e, $\xi = 0$ is the curve γ_0).

Then

$$\sigma_0 = \begin{cases} g_0(z(x,y))|dz|^2 \frac{|d\zeta|}{|dz|}^2 = g_0(z)|d\zeta|^2 & \text{on the cylinder} \\ g_0(z)\sigma|dz|^2|\frac{|d\zeta|}{|dz|}^2 = g_0(z)\sigma|d\zeta|^2 & \text{on the complement of the cylinder} \end{cases}$$

Choosing ζ to be adapted to the hyperbolic collar means that if γ_0 is the the unique σ_0-geodesic in the homotopy class of the grafting locus, then σ_0 restricted to the standard hyperbolic collar (i.e., the hyperbolic collar given by the collar lemma) is the hyperbolic metric on a cylinder. In particular, a hyperbolic cylinder is invariant under rotation in the η direction, so the metric depends only on ξ. Thus

$$(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}) \log \sigma_0 = \frac{\partial^2}{\partial\xi^2} \log \sigma_0 = 2\sigma_0$$

Solving for σ_0, we obtain:

$$ds_0^2 = \ell^2 \sec^2 \ell\xi |d\zeta|^2 \tag{4.2}$$

where ℓ is the the σ_0-length of γ_0 and $u = \int_0^{\xi^u} \sec \ell\xi d\xi$ gives the hyperbolic distance from γ_0. We do not assume that the standard hyperbolic collar *contains* the grafting cylinder. As above, we will need to study the time-t metric in the ξ and η coordinates. σ_t is given by

$$ds_t^2 = \mathcal{E}_t(\xi,\eta)d\xi^2 + 2\mathcal{F}_t(\xi,\eta)d\xi d\eta + 2\mathcal{G}_t(\xi,\eta)d\eta^2$$

where

$$\mathcal{E}_t(\xi,\eta) = \langle \frac{\partial}{\partial\xi}, \frac{\partial}{\partial\xi} \rangle_t$$
$$\mathcal{F}_t(\xi,\eta) = \langle \frac{\partial}{\partial\xi}, \frac{\partial}{\partial\eta} \rangle_t$$
$$\mathcal{G}_t(\xi,\eta)_t = \langle \frac{\partial}{\partial\eta}, \frac{\partial}{\partial\eta} \rangle_t$$

The complication here, of course, is that the hyperbolic coordinates (ξ, η) are almost certainly not orthogonal in the time-t metrics, so that it is unclear how to fix a gauge to study the change in geometry. However, the coordinates have been chosen so that at time $t = 0$

$$\mathcal{E} = \mathcal{G} = \ell^2 \sec^2 \ell\xi = \frac{g(z)}{|\frac{d\zeta}{dz}|^2} \text{ and } \mathcal{F} = 0$$

Uniting the two coordinate systems.

In what follows, we will move freely between the two coordinate systems, as some computations are more natural in one than the other. The relationship between the grafting coordinates and the hyperbolic coordinates is best encapsulated in the following:

$$\langle , \rangle_t = g_t(z_t(x,y)\langle , \rangle_{\mathrm{Euc}(t)} X_{\mathrm{graft}} + g_t(z)\langle , \rangle_\sigma X_{\mathrm{comp}}$$

Then

$$\begin{aligned}
\mathcal{E}_t &= \langle \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi} \rangle_t \\
&= g_t(z_t(x,y))\langle \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi} \rangle_{\mathrm{Euc}(t)} X_{\mathrm{graft}} + g_t(z)\langle \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi} \rangle_\sigma X_{\mathrm{comp}} \\
&= g_t(z_t(x,y))\langle \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y}, \frac{\partial x}{\partial \xi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi}\frac{\partial}{\partial y} \rangle_{\mathrm{Euc(t)}} X_{\mathrm{graft}} + g_t(z)\langle \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi} \rangle_\sigma X_{\mathrm{comp}} \\
&= g_t(z_t(x,y))[(\frac{\partial x}{\partial \xi})^2 \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \rangle_{\mathrm{Euc(t)}} X_{\mathrm{graft}} + (\frac{\partial y}{\partial \xi})^2 \langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \rangle_{\mathrm{Euc(t)}} X_{\mathrm{graft}}] + g_t(z)\langle \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi} \rangle_\sigma X_{\mathrm{comp}} \\
&= g_t(z_t(x,y))[(\frac{\partial x}{\partial \xi})^2 c_t^2 + (\frac{\partial y}{\partial \xi})^2] X_{\mathrm{graft}} + g_t(z)\langle \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi} \rangle_\sigma X_{\mathrm{comp}} \qquad (4.3)
\end{aligned}$$

Similar calculations for $\mathcal{F}_t$ and $\mathcal{G}_t$ can be made. In particular

$$\mathcal{G}_t = g_t(z_t(x,y))[(\frac{\partial x}{\partial \eta})^2 c_t^2 + (\frac{\partial y}{\partial \eta})^2] X_{\mathrm{graft}} + g_t(z)\langle \frac{\partial}{\partial \eta}, \frac{\partial}{\partial \eta} \rangle_\sigma X_{\mathrm{comp}} \qquad (4.4)$$

Additionally, it will at times be more natural to consider the derivatives of the hyperbolic coordinates in the grafting directions. As ζ is holomorphic, ξ and η are harmonic. Using the harmonicity, one computes that the derivatives are related as follows:

$$\begin{aligned}
\frac{\partial x}{\partial \xi} &= \frac{1}{|\frac{d\zeta}{dz}|^2}\frac{\partial \eta}{\partial y} = \frac{1}{|\frac{d\zeta}{dz}|^2}\frac{\partial \xi}{\partial x} \\
\frac{\partial x}{\partial \eta} &= \frac{1}{|\frac{d\zeta}{dz}|^2}\frac{-\partial \xi}{\partial y} = \frac{1}{|\frac{d\zeta}{dz}|^2}\frac{\partial \eta}{\partial x} \\
\frac{\partial y}{\partial \xi} &= \frac{1}{|\frac{d\zeta}{dz}|^2}\frac{-\partial \eta}{\partial x} = \frac{1}{|\frac{d\zeta}{dz}|^2}\frac{\partial \xi}{\partial y} \\
\frac{\partial y}{\partial \eta} &= \frac{1}{|\frac{d\zeta}{dz}|^2}\frac{\partial \xi}{\partial x} = \frac{1}{|\frac{d\zeta}{dz}|^2}\frac{\partial \eta}{\partial y}
\end{aligned} \tag{4.5}$$

We shall refer to these relationships as necessary in the sequel.

4.2.2 Length

Let γ_t denote the σ_t geodesic in the homotopy class of γ, the grafting locus. By a theorem in hyperbolic geometry, the geodesic γ_t both exists and is unique. Then if $\ell(t, \gamma_t)$ denotes the σ_t-length of γ_t, we have

$$\left.\frac{d}{dt}\right|_{t=0} \ell(t, \gamma_t) = \left.\frac{d}{dt}\right|_{t=0} \ell(t, \gamma_0) + \left.\frac{d}{dt}\right|_{t=0} \ell(0, \gamma_t)$$

But γ_0 is in the family $\{\gamma_t\}$, and as a geodesic, is length-minimizing. Thus

$$\left.\frac{d}{dt}\right|_{t=0} \ell(0, \gamma_t) = 0$$

so that the quantity of interest is simply

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} \ell(t,\gamma_t) &= \left.\frac{d}{dt}\right|_{t=0} \ell(t,\gamma_0) \\ &= \left.\frac{d}{dt}\right|_{t=0} \int_{\gamma_0} ds_t \\ &= \int_{\gamma_0} \left.\frac{d}{dt}\right|_{t=0} ds_t\end{aligned}$$

By the assumption 4.7 that the geodesic in the homotopy class of the grafting locus is contained in the grafting cylinder, a coordinate expression for $\left.\frac{d}{dt}\right|_{t=0} \ell(t,\gamma_0)$ is given by looking at the change in the metric restricted to the grafting cylinder. For convenience of notation, write $g_t = g_t(z_t(x,y))$, $\dot{G} = \left.\frac{d}{dt}\right|_{t=0} g_t(z_t(x,y))$, and

$$\dot{c} = \left.\frac{d}{dt}\right|_{t=0} c_t = \left.\frac{d}{dt}\right|_{t=0} \frac{t}{t+2t_0} = \frac{1}{2t_0}$$

As an X-grafting ray is tangentiable, these derivatives make sense. Then for a curve contained in the grafting cylinder,

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} ds_t &= \left.\frac{d}{dt}\right|_{t=0} \sqrt{g_t[c_t^2 dx^2 + dy^2]} \\ &= \frac{1}{2}\frac{\dot{G}}{g}\, ds_0 + \dot{c}\frac{dx^2}{dx^2+dy^2}\, ds_0 \end{aligned} \tag{4.6}$$

In other words, the derivative of the length of the hyperbolic geodesic in the homotopy class of the grafting locus depends on the ratio of the change in the conformal factor to the original metric (i.e, $\frac{\dot{G}}{g}$) and the stretch factor times the square length of the projection of the time-0 geodesic in the direction of the stretch ($\dot{c}\frac{dx^2}{dx^2+dy^2}$). Thus understanding the average of $\frac{\dot{G}}{g}$ on γ_0 is of paramount importance.

To see the derivative of length expressed in the hyperbolic coordinates, recall the definition of $\mathcal{E}_t$ (4.3) and $\mathcal{G}_t$ (4.4).

$$\begin{aligned}\dot{\mathcal{E}} &= \frac{d}{dt}\bigg|_{t=0} g_t(z_t(x,y))[(\frac{\partial x}{\partial \xi})^2\, c_t^2 + (\frac{\partial y}{\partial \xi})^2] X_{\text{graft}} + g_t(z)\langle \frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi}\rangle_\sigma X_{\text{comp}} \\ &= \dot{G}[(\frac{\partial x}{\partial \xi})^2 + (\frac{\partial y}{\partial \xi})^2] X_{\text{graft}} + 2\dot{c} g[(\frac{\partial x}{\partial \xi})^2] X_{\text{graft}} + \dot{g}\frac{\partial}{\partial \xi}\rangle_\sigma X_{\text{comp}}\end{aligned}$$

Using the identities (4.5),

$$\frac{\dot{\mathcal{E}}}{\mathcal{E}} = [\frac{\dot{G}}{g} + 2\dot{c}\frac{\xi_x^2}{|\frac{d\zeta}{dz}|^2}] X_{\text{graft}} + \frac{\dot{g}}{g} X_{\text{comp}} \tag{4.7}$$

A similar calculation gives

$$\frac{\dot{\mathcal{G}}}{\mathcal{G}} = [\frac{\dot{G}}{g} + 2\dot{c}\frac{\eta_x^2}{|\frac{d\zeta}{dz}|^2}] X_{\text{graft}} + \frac{\dot{g}}{g} X_{\text{comp}} \tag{4.8}$$

For any curve α that is an ξ=constant curve in the grafting cylinder, substitute (4.8) into (4.6) to find,

$$\begin{aligned}\frac{d}{dt}\bigg|_{t=0} \int_\alpha ds_t &= \int_\alpha \frac{1}{2}\frac{\dot{G}}{g} + \dot{c}\frac{\xi_x^2 d\xi^2 + 2\xi_x\eta_x d\xi d\eta + \xi_y^2 d\eta^2}{|\frac{d\zeta}{dz}|^2}\, ds_0 \\ &= \int_\alpha \frac{1}{2}\frac{\dot{G}}{g} + \dot{c}\frac{\eta_x^2}{|\frac{d\zeta}{dz}|^2} ds_0 \\ &= \int_\alpha \frac{1}{2}\frac{\dot{\mathcal{G}}}{\mathcal{G}}\, ds_0\end{aligned}$$

We record for completeness that

$$\int_\alpha \frac{\dot{\mathcal{E}}}{\mathcal{E}} ds_0 = 2\,\frac{d}{dt}\bigg|_{t=0}\int_\alpha ds_t + 2\dot{c}\int_\alpha \frac{\xi_x^2 - \eta_x^2}{|\frac{d\zeta}{dz}|^2}\, ds_0$$

4.2.3 Prescribed curvature equation.

In this section we derive the linearized prescribed curvature equation,

Formula 4.10. *Let g_t be the conformal factors $\frac{\sigma_t}{Th_t}$. Let $\dot{G}$ denote the derivative $\frac{d}{dt}\big|_{t=0} g_t(z_t)$. Writing $g = g_0$, we have*

$$(\frac{1}{g}\Delta - 2)(\frac{\dot{G}}{g}) = \frac{2\dot{c}}{g}(\log g)_{xx} X_{grafting\ cylinder}$$

We begin with some definitions.

Definition 4.8. Let f be the conformal factor for a conformal metric. Then the *Liouville equation* is$\Delta \log f(z) = -2K$ where Δ is the Laplace-Beltrami operator for the metric and K is the Gaussian curvature. This is also called the *prescribed curvature equation* and is a conformal invariant [FG00].

Remark 4.9. The Laplace-Beltrami operator for the time-t hyperbolic metric along the grafting ray is given by the following pair of equations

$$\Delta_t = \begin{cases} \frac{1}{g_t(z_t(z))}(\frac{\partial^2}{\partial x_t^2} + \frac{\partial^2}{\partial y^2}) & \text{on the grafting cylinder} \\ \frac{1}{g_t(z)}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} & \text{on the complement of the grafting cylinder} \end{cases}$$

To compute the time-t prescribed curvature equation for the conformal factors g_t, recall that the metrics σ_t have constant curvature $K = -1$. Applying the definition of the Laplace-Beltrami operator, we have

$$2 = \begin{cases} \frac{1}{g_t(z_t(z))}(\frac{\partial^2}{\partial x_t^2} + \frac{\partial^2}{\partial y^2}) \log g_t(z_t(z)) & \text{on the grafting cylinder} \\ \frac{1}{g_t(z)}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \log g_t(z) & \text{on the complement} \end{cases}$$

To derive 4.10, we need to study the variation of the metric, i.e., differentiate both equations at $t = 0$. For the derivative of the conformal factors, we write $\dot{G} = \frac{d}{dt}\big|_{t=0} g_t(z_t(z))$ and $\dot{g} = \frac{d}{dt}\big|_{t=0} g_t(z)$.

$$0 = \begin{cases} -\frac{\dot{G}}{g^2}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \log g + \frac{1}{g}\frac{d}{dt}\big|_{t=0}(\frac{\partial^2}{\partial x_t^2} + \frac{\partial^2}{\partial y^2}) \log g + \frac{1}{g}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\frac{\dot{G}}{g} & \text{on the cylinder} \\ -\frac{\dot{g}}{g^2}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \log g + \frac{1}{g}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})(\frac{\dot{g}}{g}) & \text{on the complement} \end{cases}$$

Using that the $t = 0$ prescribed curvature equation is

$$\frac{1}{g}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \log g = 2$$

on the whole surface (as $z_0 = z$), it is possible to simplify the linearized prescribed curvature equation:

$$0 = \begin{cases} -2\frac{\dot{G}}{g} + \frac{1}{g}\,\frac{d}{dt}\Big|_{t=0}(\frac{\partial^2}{\partial x_t^2} + \frac{\partial^2}{\partial y^2}) \log g + \frac{1}{g}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\frac{\dot{G}}{g} & \text{on the cylinder} \\ -2\frac{\dot{g}}{g} + \frac{1}{g}(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\frac{\dot{g}}{g} & \text{on the complement} \end{cases} \tag{4.9}$$

The next piece to consider is

$$\frac{d}{dt}\Big|_{t=0}(\frac{\partial^2}{\partial x_t^2} + \frac{\partial^2}{\partial y^2})$$

Certainly, the time derivative of the y space derivative is zero, as it is independent of t. (Grafting is a linear stretch in only the x-direction). For ease of notation, write $x_t = \phi(t, x)$. Then

$$\begin{aligned} \partial/\partial\phi(t,x) &= \frac{\partial/\partial x}{\partial\phi(x,t)/\partial x} \\ &= (\phi(t,x)_x)^{-1}\partial/\partial x \\ \partial^2/\partial\phi(t,x)^2 &= (\xi(t,x)_x)^{-1}\partial/\partial x((\phi(t,x)_x)^{-1}\partial/\partial x) \\ &= \partial/\partial x(\phi(t,x)_x\partial/\partial x) + (\phi(t,x)_x)^{-2}\partial^2/\partial x^2 \end{aligned}$$

Now because $\partial/\partial x(\phi(t,x)) = c(t)$ (a constant depending on t, not x), we have that $\partial/\partial x(\phi(t,x)_x) = \partial/\partial x(c(t)) = 0$. Thus

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0}\partial^2/\partial\phi(t,x)^2 &= \frac{d}{dt}|_{t=0}(\phi(t,x)_x)^{-2}\partial^2/\partial x^2 \\ &= -2c(0)^{-3}\dot{c}\,\partial^2/\partial x^2 \\ &= -2\dot{c}\,\partial^2/\partial x^2 \end{aligned} \tag{4.10}$$

The last step follows because the time-0 stretch map ξ is the identity map. Writing $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and substituting equation (4.10) into equation (4.9), we have:

$$0 = \begin{cases} -2\frac{\dot{G}}{g} - \dot{c}(\frac{\partial^2}{\partial x^2}\log g) + \frac{1}{g}\Delta\frac{\dot{G}}{g} & \text{on the cylinder} \\ -2\frac{\dot{g}}{g} + \frac{1}{g}\Delta\frac{\dot{g}}{g} & \text{on the complement} \end{cases}$$

Abusing notation slightly, writing $\dot{g} = \dot{G}$, gives

Formula 4.10.

$$(\frac{1}{g}\Delta - 2)(\frac{\dot{G}}{g}) = \frac{2\dot{c}}{g}(\log g)_{xx} X_{\text{grafting cylinder}}$$

The study of solutions $\frac{\dot{G}}{g}$ to equation (4.10) drives the remainder of this section.

4.2.4 Applying Formula 4.10

In this section, we derive Proposition 4.13. To motivate the derivation, we consider two examples.

Example 4.11. We note that if g is invariant under rotation in the η direction then the prescribed curvature equation for g on the grafting cylinder is

$$\frac{1}{g}\Delta \log g = 2 \Leftrightarrow \frac{\partial^2}{\partial \xi^2}\log g = 2g \Leftrightarrow \frac{\partial^2}{\partial x^2}\log g - 2g = 0$$

One can check that in fact $\sec^2 \ell\xi$ satisfies this equation, which is what we used earlier (4.2) to get the hyperbolic metric on the grafting cylinder. Further, if $g = \ell^2 \sec^2 \ell\xi |\frac{d\zeta}{dz}|^2$, one can check that a second element in the kernel of the operator $\Delta_g - 2$ is $\tan \ell\xi$:

$$\Delta \tan \ell\xi = (2\ell^2 \sec^2 \ell\xi)(\xi_x^2 + \xi_y^2)(\tan \ell\xi) + (\sec^2 \ell\xi)(\xi_{xx} + \xi_{yy})$$

Then, using that ξ is harmonic and $|\frac{d\zeta}{dz}|^2 = \xi_x^2 + \xi_y^2$,

$$\Delta \tan \ell\xi = 2g \, \tan \ell\xi$$

Further, if ξ^u is curve of distance u from γ_0, and $u = \int_0^{\xi^u} \sec \ell\xi d\xi$, then

$$\begin{aligned}
\sinh(u) &= \frac{1}{2}[e^u - e^{-u}] \\
&= \frac{1}{2}[\exp(\log(\sec \ell\xi^u + \tan \ell\xi^u)) - \exp(-\log(\sec \ell\xi^u + \tan \ell\xi^u))] \\
&= \frac{1}{2}[\sec \ell\xi^u + \tan \ell\xi^u - \frac{1}{\sec \ell\xi^u + \tan \ell\xi^u}]
\end{aligned}$$

Simplifying,

$$\begin{aligned}
&= \frac{1}{2}\frac{\sec \ell\xi^u + \tan \ell\xi^u}{\sec \ell\xi^u + \tan \ell\xi^u} 2\tan(\ell\xi^u) \\
&= \tan(\ell\xi^u)
\end{aligned}$$

Similarly, $\cosh u = \sec \ell\xi^u$. Then $\sinh u$ and $\cosh u$ are also in the kernel of the operator $\frac{1}{g}\Delta - 2$.

Example 4.12. The following fact is the catalyst for the main investigation. Let dA_0 be the hyperbolic area form and let ω be any analytic function. Suppose

$$(\Delta_g - 2)\omega = \Psi$$

Then

$$\iint_S \Psi \frac{\dot{G}}{g} \, \mathrm{d}A_0 = \iint_S (\Delta_g - 2)(\omega) \, \frac{\dot{G}}{g} \, \mathrm{d}A_0 \tag{4.11}$$

and because $\Delta_g - 2$ is a self-adjoint operator on H^1,

$$\begin{aligned}
\iint_S \Psi \frac{\dot{G}}{g} \, \mathrm{d}A_0 &= \iint \omega \, (\Delta_g - 2)^{-1} (\frac{\dot{G}}{g}) \, \mathrm{d}A_0 &\quad (4.12) \\
&= \iint_S \omega \, \frac{2\dot{c}}{g} (\log g)_{xx} X_{\text{graft}} \, \mathrm{d}A_0 &\quad (4.13)
\end{aligned}$$

where the last line follows from Formula 4.10. In other words, if Ψ times $\frac{\dot{G}}{g}$ is the derivative of a geometric quantity, equation 4.13 relates this derivative to the time-0 hyperbolic geometry. In particular, let α be any curve parallel to γ_0 and let δ_α denote a delta-function on α. Suppose that for some function ω

$$\delta_\alpha = \Psi = (\frac{1}{g}\Delta - 2)(\omega)$$

Now looking at the left-hand side of 4.13

$$\iint_S \Psi \frac{\dot{G}}{g}\, dA_0 = \iint_S \delta_\alpha (\frac{\dot{G}}{g})\, \mathrm{d}A_0$$

Using that the hyperbolic metric is conformal to the Euclidean metric with conformal factor g,

$$\iint_S \Psi \frac{\dot{G}}{g}\, dA_0 = \iint \delta_\alpha (\frac{\dot{G}}{g} g)\, \mathrm{d}x\mathrm{d}y$$

Applying the δ-function,

$$\iint_S \Psi \frac{\dot{G}}{g}\, dA_0 = \int_\alpha \frac{\dot{G}}{g} g\, ds_E$$
$$\iint_S \Psi \frac{\dot{G}}{g}\, dA_0 = \int_\alpha \frac{\dot{G}}{g} \sqrt{g}\sqrt{g}\, ds_E$$

Again, the Euclidean and hyperbolic line elements differ by a conformal factor, namely $\sqrt{g}\, ds_E = ds_0$. Substituting this into the previous line,

$$\iint_S \Psi \frac{\dot{G}}{g}\, dA_0 = \int_\alpha \frac{\dot{G}}{g} \sqrt{g}\, ds_0$$

Now if δ is contained in the standard hyperbolic collar, then the hyperbolic metric restricted to the standard hyperbolic collar is $\ell^2 \sec^2 \ell\xi$. The assumption that α is an ξ=constant curve implies that $\sqrt{g}(\alpha) = \ell \sec \ell \xi^\alpha = \ell \cosh u$ (where as before u is the hyperbolic distance of α from γ_0) is constant, and we can pull it out of the integral:

$$\iint_S \Psi \frac{\dot{G}}{g}\, dA_0 = \sqrt{g}(\alpha) \int_\alpha \frac{\dot{G}}{g}\, ds_0 \tag{4.14}$$

Now substituting in equation(4.6) for the derivative of length, (writing $\frac{d}{dt}\Big|_{t=0} \int_\alpha ds_t$ for the derivative of the hyperbolic t-length of α), we find

$$\iint_S \Psi \frac{\dot{G}}{g}\, dA_0 = 2\sqrt{g}(\alpha)\, \frac{d}{dt}\Big|_{t=0} \int_\alpha ds_t - 2\dot{c}\sqrt{g}(\alpha) \int_\alpha \frac{\xi_y^2}{\xi_x^2 + \xi_y^2}\, ds_0 \tag{4.15}$$

Substituting the identity (4.14) into equation (4.13),

$$2\sqrt{g}(\alpha)\, \frac{d}{dt}\Big|_{t=0} \int_\alpha ds_t - 2\dot{c}\sqrt{g}(\alpha) \int_\alpha \frac{\xi_y^2}{\xi_x^2 + \xi_y^2}\, ds_0 = \iint_S \omega\, \frac{2\dot{c}}{g} (\log g)_{xx} X_{\text{graft}}\, \mathrm{d}A_0$$

Moving $-2\dot{c}\sqrt{g}(\alpha) \int_\alpha \frac{\xi_y^2}{\xi_x^2+\xi_y^2}\, ds_0$ to the right-hand side, we have a formula for the derivative of length of a curve parallel to the hyperbolic geodesic in terms of ω, time-0 conformal factor g, $\dot{c} = \frac{1}{2t_0}$, and the cosine of the angle between α and a Euclidean geodesic:

$$2\sqrt{g}(\alpha)\, \frac{d}{dt}\Big|_{t=0} \int_\alpha ds_t = 2\dot{c}\sqrt{g}(\alpha) \int_\alpha \frac{\xi_y^2}{\xi_x^2 + \xi_y^2}\, ds_0 + \iint_S \omega\, \frac{2\dot{c}}{g} (\log g)_{xx} X_{\text{graft}}\, \mathrm{d}A_0$$

The above example suggests that to isolate a single curve - say γ_0 - to study its behavior, we construct a function ω such that $(\Delta_g - 2)(\omega) = \delta_{\gamma_0}$ where δ_{γ_0} is a delta-function along that curve. To avoid introducing lots of complicated terms that may obscure the geometric meaning, we'll look for a relatively "simple" function - say, a function built out of pieces that are in the kernel of the operator. So near γ_0, it makes sense to consider the function $\omega = |\tan \ell\xi| = |\sinh u|$. As it is currently defined, the function ω only vanishes under $\Delta_g - 2$ on the hyperbolic collar neighborhood of γ_0. To define a function that makes sense on all of S, we first assume that γ is a *separating curve* for S, i.e., cutting the surface S along γ produces two surfaces, each with γ as the single boundary component. Then choose curves α_+ and α_- that are contained in the hyperbolic collar, parallel to γ_0, and bound a smaller cylinder - the α-collar - with core geodesic γ_0. In particular, the $\alpha^\pm$ are ξ=constant curves. Later, we will fix the the location of these curves, but for the time being, the only assumption we make (to simplify some calculations later) is that the $\alpha^\pm$ do not intersect the boundary of the grafting cylinder, although they may be as close as we wish. Let $\xi^{\alpha^\pm}$ represent the corresponding coordinates, i.e., $|\tan \ell\xi^{\alpha^+}| = |\sinh u^\pm|$ is a constant. In particular, α^+ and α^- may not be equidistant from γ_0, so u^+ is not necessarily equal to u^-. Denote by comp$^\pm$ the complementary portions of the surface to the α-collar, bounded by α_+ and α_- respectively. Then define

$$\omega = |\tan \ell\xi| X_{\alpha-\mathrm{collar}} + |\tan \ell\xi^{\alpha^+}| X_{\mathrm{comp}^+} + |\tan \ell\xi^{\alpha^-}| X_{\mathrm{comp}^-}$$

If we had not assumed that γ_0 were separating, then if $\tan \ell\xi^{\alpha^+} \neq -\tan \ell\xi^{\alpha^-}$, the function ω would not be well-defined. A routine but tedious calculation gives

$$\Psi := (\Delta_g - 2)(\omega) = \frac{2}{\ell}(-\delta_{\alpha^+} + 2\delta_{\gamma_0} - \delta_{\alpha^{-1}}) - 2(|\tan \ell\xi^{\alpha^+}| X_{\mathrm{comp}^+} + |\tan \ell\xi^{\alpha^-}| X_{\mathrm{comp}^-})$$

Now we proceed as in the example above:

$$\iint_S \omega \frac{2\dot{c}}{g} (\log g)_{xx} X_{\mathrm{graft}} \, \mathrm{d}A_0 = \iint_S \Psi \frac{\dot{g}}{g} \, \mathrm{d}A_0$$

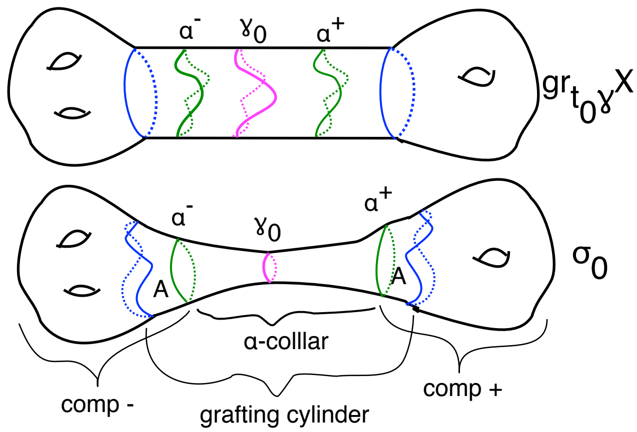

Figure 4.2: The boundary curves, $\alpha^{\pm}$, and the geodesic as imagined on the grafted (top image) and hyperbolic (bottom image) surfaces.

Substituting and simplifying,

$$= -\frac{2}{\ell}\int_0^1 \frac{\dot{G}}{g}\ell^2\sec^2\ell\xi^{\alpha^+}\,d\eta + \frac{4}{\ell}\int_0^1 \frac{\dot{G}}{g}\ell^2\sec^2\ell\xi^{\gamma_0}\,d\eta - \frac{2}{\ell}\int_0^1 \frac{\dot{G}}{g}\ell^2\sec^2\ell\xi^{\alpha^-}\,d\eta$$
$$- 2|\tan\ell\xi^{\alpha^+}|\iint_{\mathrm{comp}^+}\frac{\dot{G}}{g}\mathrm{d}A_0 - 2|\tan\ell\xi^{\alpha^-}|\iint_{\mathrm{comp}^-}\frac{\dot{G}}{g}\mathrm{d}A_0$$

multiplying through by ℓ, using the expression (4.6) for $\frac{d}{dt}\big|_{t=0} ds_t$ and separating terms not involving a time derivative of the metric to one side, grouping together the sum of terms involving α^+ and α^- for simplicity of presentation, we obtain

$$- 2\ell\sec\ell\xi^{\alpha^{\pm}}\cdot 2\,\frac{d}{dt}\bigg|_{t=0}\int_{\alpha^{\pm}} ds_t + 4\ell\cdot 2\,\frac{d}{dt}\bigg|_{t=0}\int_{\gamma_0} ds_t - 2\ell|\tan\ell\xi^{\alpha^{\pm}}|\iint_{\mathrm{comp}^{\pm}}\frac{\dot{G}}{g}\,\mathrm{d}A_0$$
$$= -2\ell\sec\ell\xi^{\alpha^{\pm}}\cdot 2\dot{c}\int_{\alpha^{\pm}}\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\,ds_0 + 4\ell\cdot 2\dot{c}\int_{\gamma 0}\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\,ds_0$$
$$+ 2\dot{c}\ell|\tan\ell\xi^{\alpha^{\pm}}|\iint_{\mathrm{graft}\cap\mathrm{comp}^{\pm}}(\log g)_{xx}\,\mathrm{d}x\mathrm{d}y + 2\dot{c}\ell\iint_{\mathrm{graft}\cap\alpha-\mathrm{collar}}|\tan\ell\xi|(\log g)_{xx}\,\mathrm{d}x\mathrm{d}y$$

We pause for a moment to note the geometry embedded in the above equation. The left-hand side is a sum of derivatives of length, plus a term related to the derivative of area of the complement of the α-collar. More specifically,

$$\begin{aligned}\iint_{\text{comp}^{\pm}} \frac{\dot{G}}{g}\,\mathrm{d}A_0 &= \iint_{\text{graft}\cap\text{comp}^{\pm}} \frac{\dot{G}}{g}\,\mathrm{d}A_0 + \iint_{\text{comp}^{\pm}-\text{graft}} \frac{\dot{G}}{g}\,\mathrm{d}A_0 \\ &= \frac{d}{dt}\Big|_{t=0} \iint_{\text{graft}\cap\text{comp}^{\pm}} \mathrm{d}A_t - \dot{c}\iint_{\text{graft}\cap\text{comp}^{\pm}} \mathrm{d}A_0 + \frac{d}{dt}\Big|_{t=0}\iint_{\text{comp}^{\pm}-\text{graft}} \mathrm{d}A_t \\ &= \frac{d}{dt}\Big|_{t=0}\iint_{\text{comp}^{\pm}} \mathrm{d}A_t - \dot{c}\iint_{\text{graft}\cap\text{comp}^{\pm}} \mathrm{d}A_0\end{aligned}$$

While keeping the geometry in mind is important, it will be mechanically easier for us to leave the expression in terms of $\frac{\dot{G}}{g}$. On the right hand side, the terms $\int \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\,ds_0$ express the cosine of the average angle between an ξ=constant curve and an x=constant curve. In other words, the closer that ξ and y are to being orthogonal - ie., the closer hyperbolic curves of constant curvature are to matching curves of Euclidean constant curvature - the closer these terms are to zero. The remaining two integrals will be manipulated significantly in the sequel so that their geometric contribution is clarified. To summarize:

Proposition 4.13. Let γ be a separating curve on S. Let $X \in \mathcal{T}(S)$ and let σ_0 be the hyperbolic metric in the conformal class of $gr_{t_0\gamma}X$. Then for α^+ and α^- and curves parallel to γ_0,

$$\underbrace{-2\ell\sec\ell\xi^{\alpha^{\pm}}\cdot 2\,\frac{d}{dt}\Big|_{t=0}\int_{\alpha^{\pm}} ds_t}_{\text{derivative of length}} + 4\ell\cdot 2\,\underbrace{\frac{d}{dt}\Big|_{t=0}\int_{\gamma_0} ds_t}_{\text{derivative of length}} - 2\ell|\tan\ell\xi^{\alpha^{\pm}}|\underbrace{\iint_{\text{comp}^{\pm}}\frac{\dot{G}}{g}\,\mathrm{d}A_0}_{\text{derivative of area}}$$

$$= \underbrace{-2\ell\sec\ell\xi^{\alpha^{\pm}}\cdot 2\dot{c}\int_{\alpha^{\pm}}\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\,ds_0 + 4\ell\cdot 2\dot{c}\int_{\gamma_0}\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\,ds_0}_{\text{angle between hyp. and Euc. curves}}$$

$$+ 2\dot{c}\ell|\tan\ell\xi^{\alpha^{\pm}}|\underbrace{\iint_{\text{graft}\cap\text{comp}^{\pm}}(\log g)_{xx}\,\mathrm{d}x\mathrm{d}y}_{\text{approximate area}} + 2\dot{c}\ell\iint_{\text{graft}\cap\alpha-\text{collar}}|\tan\ell\xi|(\log g)_{xx}\,\mathrm{d}x\mathrm{d}y$$

Remark 4.14. Let $gr_{\infty\gamma}X$ denote the conformal structure obtained by grafting a semi-infinite cylinder along γ. In other words, $gr_{\infty\gamma}X$ is the conformal limit of the grafting ray $gr_{t\gamma}X$. This conformal class has some hyperbolic invariants that can be considered as part of the geometric data for grafting. In particular, we can identify the limiting positions of a parallel curve α and the boundary of the grafting cylinder, and the geodesic curvature of these curves. In light of this observation, the terms on the right hand side of Proposition 4.13 are part of the geometric data for grafting.

Remark 4.15. Of note in the above formula is that every term except for one - the derivative of area term - is concentrated on the grafting cylinder and the α-collar. We imagine that the metric outside the grafting cylinder and α-collar has been impacted by the grafting operation and want to capture this in terms of the geometry on the collar or near the boundary of the collar. This is the objective of the next section.

4.3 A geometric formula

The overall goal of this section is to prove Theorem 4.1. Four steps are required. The first step is to choose a special family of curves with fixed length, α_t. After giving a few remarks on geometric properties of this family, we use elementary differential geometry to compute the first variation of length of α. This is given in terms of $\int_\alpha \dot{\alpha}^1\, ds_0$, the average normal variation of $\dot{\alpha}^1$ over α. The second step is to compute this average variational vector field. In the third step, we relate $\int_\alpha \dot{\alpha}^1\, ds_0$ to the change in area term $\iint_{\text{comp}^+} \frac{\dot{G}}{g} dA_0$ found in Proposition 4.13. This is accomplished by applying Stokes' theorem and relating $\frac{\dot{\mathcal{G}}}{\mathcal{G}}$ to $\frac{\dot{G}}{g}$ via the prescribed curvature equation.

In the fourth step, we combine the results of the first three steps to write $\iint_{\mathrm{comp}^\pm} \frac{\dot{G}}{g}\, dA_0$ in terms of $\frac{d}{dt}\big|_{t=0} \ell(t, \alpha^\pm)$. Substituting this into Proposition 4.13 and simplifying gives the main formula.

4.3.1 First variation of length

A special family of curves

Begin by noting the fact that $\alpha^\pm$ is parallel to γ_0. A quick calculation [Lip69] shows that $\alpha^\pm$ has constant (scalar) geodesic curvature

$$\kappa_0(\alpha^\pm) = \frac{\tan \ell\xi^{\alpha^\pm}}{\sec \ell\xi^{\alpha^\pm}}$$

Note that $u = \int \sec \ell\xi d\xi = \ln|\tan \ell\xi + \sec \ell\xi|$ is the distance of a curve α from γ_0. One can check via definitions of cosh and sinh that

$$\cosh u = \sec \ell\xi$$

$$\sinh u = \tan \ell\xi$$

and hence $\kappa_0(\alpha^\pm) = \frac{\cosh u^\pm}{\sinh u^\pm} = \tanh u^\pm$ where $u^\pm$ is the distance of $\alpha^\pm$ from γ_0. Now consider a family of hyperbolic cylinders with ℓ_t the length of the core geodesic γ_t. One may define as above, u_t to be the distance of curves from γ_t with respect to the t-hyperbolic metric. There are two natural families of curves which include the parallel curves $\alpha^\pm$. The first choice is a family of curves δ_t so that with respect to the t-metric, δ_t has the same geodesic curvature as α. However, if we imagine grafting a cylinder about its core curve, as the cylinder lengthens, a curve δ_t of fixed geodesic curvature gets closer to the core geodesic. This "collapsing" effect of curves of fixed geodesic curvature means that studying the variation of these curves does not provide good information about the variation of the geometry near the boundary of the grafting cylinder. To remedy this, we turn to the second natural choice of a family of curves.

Definition 4.16 ($\alpha_t^{\pm}$). Define a family $\alpha_t^{\pm}$ of curves on S of fixed length that are parallel to γ_t. In other words, for each of $\alpha^{\pm}$ (omitting $\pm$ for ease of notation)

$$\ell \cosh u = \ell_t(\alpha_t) = \ell_t \cosh u_t$$

for some $u < \sinh(\frac{1}{\sinh(\ell^{-1})})$, appropriate u_t, and $\ell_0 = \ell$. The bound on u ensures that α is actually contained in the hyperbolic collar by the standard collar lemma [Kee74].

Remark 4.17. In the case of grafting a cylinder, one finds that as the length of the cylinder increases, a curve of fixed length moves further from the core geodesic. In particular, choosing $\alpha^{\pm}$ to be close to the boundary of the grafting cylinder means that the family cannot collapse into the core geodesic.

Remark 4.18. We must be careful about choosing u so that this family exists. In particular, u must be large enough so that $\ell_t \leq \ell \cosh u$ for t in an ϵ-neighborhood of t_0 but also must be small enough so that the family u_t doesn't escape the $t + t_0$ hyperbolic collar.

We now consider more specific geometric properties of the families $\alpha_t^{\pm}$. For a family of curves of fixed geodesic length, each of which is parallel to a hyperbolic geodesic, the derivative of length of the curves is easily expressed in terms of ℓ, the length of the core geodesic at time 0, the distance u of α from the core geodesic, and the variation of u_t from γ_t.

$$\left.\frac{d}{dt}\right|_{t=0} \ell_t \cosh u_t^{\pm} = 0 \Rightarrow \dot{\ell} \cosh u^{\pm} + \dot{u}^{\pm} \ell \sinh u^{\pm} = 0$$

This shows that the variation of $\alpha_t^{\pm}$ from γ_t is given with respect to the derivative of length of the core geodesic:

$$\dot{u^{\pm}} = -\frac{\dot{\ell}}{\ell}\frac{\cosh u^{\pm}}{\sinh u^{\pm}}$$

Additionally, a curve parallel to a hyperbolic geodesic has constant geodesic curvature. In particular, the geodesic curvature of $\alpha_t^{\pm}$ is in fact independent of the the length ℓ_t of the core geodesic and it depends solely on the distance of the curve from the geodesic:

$$\kappa_t(\alpha_t^{\pm}) = \tanh u_t^{\pm} \Rightarrow \frac{d}{dt}\bigg|_{t=0} \kappa_t(\alpha_t^{\pm}) = \frac{\dot{u}^{\pm}}{\cosh^2 u^{\pm}} \tag{4.16}$$

As the general goal is to understand lengths, combine the above two equations to understand the derivative of geodesic curvature solely in terms of derivative of length ℓ of the core geodesic and the distance of the curves $\alpha^{\pm}$ from the core time-0 geodesic:

$$\Rightarrow \frac{d}{dt}\bigg|_{t=0} \kappa_t(\alpha_t^{\pm}) = \frac{\dot{u}}{\cosh^2 u} = -\frac{\dot{\ell}}{\ell}\frac{\cosh u^{\pm}}{\sinh u^{\pm}}\frac{1}{\cosh^2 u^{\pm}} = \frac{-\dot{\ell}}{\ell}\frac{1}{\cosh u^{\pm}\sinh u^{\pm}}$$

This fact will be explicitly invoked later. To summarize, we have selected a pair of families of curves $\alpha_t^{\pm}$ so that the variation of $\alpha_t^{\pm}$ with respect to the time-t metrics are well understood. It is natural to try to understand what controls the variation of length of $\alpha^{\pm}$ with respect to the time-t hyperbolic metrics.

Variation of length of α

Recalling the above formula, one unknown term is the derivative of length of $\alpha^{\pm}$. But we also know that (leaving off the $\pm$ for simplicity) for our family of curves α_t,

$$0 = \frac{d}{dt}\bigg|_{t=0} \int_{\alpha_t} ds_t = \frac{d}{dt}\bigg|_{t=0} \int_{\alpha} ds_t + \frac{d}{dt}\bigg|_{t=0} \int_{\alpha_t} ds_t \tag{4.17}$$

$$\frac{d}{dt}\bigg|_{t=0} \int_{\alpha} ds_t = -\frac{d}{dt}\bigg|_{t=0} \int_{\alpha_t} ds_t \tag{4.18}$$

Now we consider the last term above: the first variation of length of α. Think of α_t as curves parameterized by $\tau \in [0,1]$ with $\alpha_t(0) = \alpha_t(1)$ identified. (Secretly, τ is just a unit speed parametrization of η). Denote by ${}^0\nabla = \nabla$ the covariant derivative for the hyperbolic metric σ_0. Then using facts from elementary differential geometry,

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} \int_{\alpha} ds_t &= \frac{d}{dt}\bigg|_{t=0} \int_0^1 \langle \frac{\partial \alpha}{\partial \tau}, \frac{\partial \alpha}{\partial \tau} \rangle_0 \, d\tau \\
&= \int_0^1 \frac{1}{2} \langle \frac{\partial \alpha}{\partial \tau}, \frac{\partial \alpha}{\partial \tau} \rangle_0^{-1/2} 2 \langle \nabla_{\alpha'(\tau)} \frac{\partial \alpha}{\partial \tau}, \frac{\partial \alpha}{\partial \tau} \rangle_0 \, d\tau \\
&= \int_0^1 \langle \frac{\partial \alpha}{\partial \tau}, \frac{\partial \alpha}{\partial \tau} \rangle_0^{-1/2} \{ \frac{d}{d\tau} \langle \frac{\partial \alpha}{\partial \tau}, \frac{\partial \alpha}{\partial t} \rangle_0 - \langle \nabla_{\alpha'(\tau)} \frac{\partial \alpha}{\partial \tau}, \frac{\partial \alpha}{\partial t} \rangle_0 \} \, d\tau
\end{aligned}$$

using that the endpoints of α_t are identified, the first term above vanishes and letting $'$(prime) denote space derivatives and $\dot{}$ (dot) denote time derivatives, we have

$$\frac{d}{dt}\bigg|_{t=0} \int_{\alpha} ds_t = -\int_0^1 \langle \frac{\partial \alpha}{\partial \tau}, \frac{\partial \alpha}{\partial \tau} \rangle_0^{-1/2} \langle \nabla_{\alpha'(\tau)} \alpha'(\tau), \dot{\alpha} \rangle_0 \, d\tau \tag{4.19}$$

Now we choose to think of $\alpha_t(\tau)$ as a family of curves *on* the time-0 hyperbolic surface. In other words, using our geometrically advantageous holomorphic coordinate $\zeta = \xi + i\eta$,

$$\alpha_t(\tau) = \alpha_t^1(\tau) \frac{\partial}{\partial \xi} + \alpha_t^2(\tau) \frac{\partial}{\partial \eta}$$

In particular, we find

$$\dot{\alpha} = \dot{\alpha}^1 \frac{\partial}{\partial \xi} + \dot{\alpha}^2 \frac{\partial}{\partial \eta}$$

as $\alpha_0 = \alpha$ is defined to be an ξ=constant curve, $\alpha^1 \equiv 0$. Additionally assuming that α is parameterized with constant speed α^2_τ, one computes the geodesic curvature vector

$$\nabla_{\alpha'(\tau)} \frac{\partial \alpha}{d\tau} = \frac{||\alpha'||_0^2}{||n||_0} \alpha^2_\tau \tanh u \frac{\partial}{\partial \xi}$$

and substituting we find

$$\begin{aligned}
\frac{d}{dt}\bigg|_{t=0} \int_\alpha ds_t &= -\int_0^1 \langle \frac{\partial \alpha}{\partial \tau}, \frac{\partial \alpha}{\partial \tau} \rangle_0^{-1/2} \langle \frac{||\alpha'||_0^2}{||n||_0} \alpha^2_\tau \tanh u \frac{\partial}{\partial \xi}, \dot{\alpha}^1 \frac{\partial}{\partial \xi} + \dot{\alpha}^2 \frac{\partial}{\partial \eta} \rangle_0 \, d\tau \\
&= \tanh u \, \ell \cosh u \int_0^1 \dot{\alpha}^1 \langle \alpha^2_\tau \frac{\partial}{\partial \xi}, \alpha^2_\tau \frac{\partial}{\partial \xi} \rangle_0^{1/2} \, d\tau \\
&= \ell \sinh u \int_\alpha \dot{\alpha} \, ds_0
\end{aligned}$$

equivalently,

$$= \ell \tan \ell \xi^\alpha \int_\alpha \dot{\alpha} \, ds_0$$

In other words, understanding the variation of length is equivalent to understanding the average of the variational vector field $\dot{\alpha^1}$ on α. Computing this variational field is our next task.

4.3.2 Variational vector field

Consider the geodesic curvature vector for one of the families α_t:

$${}^t\nabla_{\frac{\alpha'_t}{||\alpha'_t||_t}} \frac{\alpha'_t}{||\alpha'_t||_t} = \kappa_t(\alpha_t) \frac{n_t}{||n_t||_t}$$

where ${}^t\nabla$ and $||\cdot||_t$ the covariant derivative and length, respectively, taken with respect to the t-hyperbolic metric, and κ_t is the scalar geodesic curvature. Then differentiating with respect to t and focusing only on the normal component of the variation,

$$\frac{d}{dt}\bigg|_{t=0} \kappa_t(\alpha_t)\frac{-\alpha^2_{t,\tau}}{||n_t||_t} = \frac{d}{dt}\bigg|_{t=0} ||\alpha'_t||_t^{-2}\Gamma^1_{22}(\alpha^2_\tau)^2 + ||\alpha'_0||^{-2}\{\frac{d}{dt}\bigg|_{t=0} \alpha^k_{t,\tau\tau} +^t \Gamma^k_{ij}\alpha^i_{t,\tau}\alpha^j_{t,\tau}\}$$

where ${}^t\Gamma^k_{ij}$ are Christoffel symbols for the time-t metric written with respect to ξ, η-coordinates. Further $\alpha^1_\tau = 0$ so that $(i,j) = (2,2)$ is the only choice yielding nonzero entries. Thus we have

$$\frac{d}{dt}\bigg|_{t=0} \kappa_t(\alpha_t)\frac{-\alpha^2_{t,\tau}}{||n_t||_t} = \frac{d}{dt}\bigg|_{t=0} ||\alpha'_t||_t^{-2}\Gamma^1_{22}(\alpha^2_\tau)^2$$
$$+ ||\alpha'_0||^{-2}\{\dot\alpha^1_{\tau\tau} + \dot\Gamma^1_{22}(\alpha^2_\tau)^2 + \partial_m\Gamma^1_{22}\dot\alpha^m(\alpha^2_\tau)^2 + 2\Gamma^1_{22}\dot\alpha^2_\tau\alpha^2_\tau\}$$

But $\partial_2\Gamma^1_{22} = 0$ (as the metric is independent of η) and so the equation becomes

$$\frac{d}{dt}\bigg|_{t=0} \kappa_t(\alpha_t)\frac{-\alpha^2_{t,\tau}}{||n_t||_t} = \frac{d}{dt}\bigg|_{t=0} ||\alpha'_t||_t^{-2}\Gamma^1_{22}(\alpha^2_\tau)^2$$
$$+ ||\alpha'_0||^{-2}\{\dot\alpha^1_{\tau\tau} + \dot\Gamma^1_{22}(\alpha^2_\tau)^2 + \partial_1\Gamma^1_{22}\dot\alpha^1(\alpha^2_\tau)^2 + 2\Gamma^1_{22}\dot\alpha^2_\tau\alpha^2_\tau\}$$

Now integrate the equation $d\tau$ (again, τ is just a unit-speed parameterization of η, so this is secretly $d\eta$). Then because the derivative of length of the curves in the family is zero and Γ^1_{22} is constant on α, the first term on the right vanishes. Similarly, the integrals of $\dot\alpha^1_{\tau\tau}$ and $\dot\alpha^2_\tau$ vanish. Our equation thus becomes

$$\int_0^1 \frac{d}{dt}\bigg|_{t=0} \kappa_t(\alpha_t)\frac{-\alpha^2_{t,\tau}}{||n_t||_t}d\tau = ||\gamma'_0||_0^{-2}\int_0^1 \dot\Gamma^{22}_1(\alpha^2_\tau)^2 + \partial_1\Gamma^1_{22}\dot\alpha^1(\alpha^2_\tau)^2 d\tau$$
$$\int_0^1 ||\alpha'_0||_0^2(\dot\Gamma^1_{22}(\alpha^2_\tau)^2 + \partial\Gamma^1_{22}\dot\alpha(\alpha^2_\tau)^2 d\tau = \frac{d}{dt}\bigg|_{t=0}\int_0^1 \kappa_t(\alpha_t)\frac{-\alpha^2_\tau}{||n||_0}d\tau$$

Using that $ds^2_t = \mathcal{E}_t d\xi^2 + 2\mathcal{F}_t d\xi d\eta + 2\mathcal{G}_t d\eta^2$ and the expression for ${}^t\Gamma^1_{22}$ in terms of the metric tensor, one finds

$$\dot\Gamma^1_{22} = \frac{\dot{\mathcal{F}}_\eta}{\mathcal{E}} - 2\frac{\dot{\mathcal{G}}_\xi}{2\mathcal{E}} + \ell\tan\ell\xi\frac{\dot{\mathcal{E}}}{\mathcal{E}}$$

Substituting this into the above equation, doing some basic calculus, and exchanging τ for η, we obtain

$$\int_0^1 \frac{\partial}{\partial \xi}(\frac{\dot{\mathcal{G}}}{\mathcal{G}})\,\mathrm{d}\eta + \ell \tan \ell \xi^\alpha \int_0^1 \frac{\dot{\mathcal{E}}}{\mathcal{E}} - \frac{\dot{\mathcal{G}}}{\mathcal{G}}\,\mathrm{d}\eta + \ell \sec \ell \xi^\alpha \left.\frac{d}{dt}\right|_{t=0} \int_0^1 \kappa_t \mathrm{d}\eta = \ell \sec \ell \xi^\alpha \int_\alpha \dot{\alpha}^1 ds_0 \tag{4.20}$$

Using 4.7 and 4.8 we compute $\frac{\dot{\mathcal{E}}}{\mathcal{E}} - \frac{\dot{\mathcal{G}}}{\mathcal{G}} = 2\dot{c}\frac{\xi_x^2 - \xi_y^2}{|\frac{d\zeta}{dz}|^2}$. Now substitute this into 4.20 above:

$$\int_0^1 \frac{\partial}{\partial \xi}(\frac{\dot{\mathcal{G}}}{\mathcal{G}})\,\mathrm{d}\eta + \ell \tan \ell \xi^\alpha \int_0^1 2\dot{c}\frac{\xi_x^2 - \xi_y^2}{|\frac{d\zeta}{dz}|^2}\,\mathrm{d}\eta + \ell \sec \ell \xi^\alpha \left.\frac{d}{dt}\right|_{t=0} \int_0^1 \kappa_t \mathrm{d}\eta = \ell \sec \ell \xi^\alpha \int_\alpha \dot{\alpha}^1 ds_0 \tag{4.21}$$

The derivative of geodesic curvature, $\left.\frac{d}{dt}\right|_{t=0} \int_0^1 \kappa_t \mathrm{d}\eta$, is also known in terms of $\frac{G}{g}$ and the time-0 metric by equation (4.16). However, for simplicity we leave the expression as it is. Our next task is to relate $\int_0^1 \frac{-1}{2}(\frac{\dot{\mathcal{G}}}{\mathcal{G}})_\xi d\eta$ to an integral over the a piece of the surface bounded by α.

4.3.3 Relating the variational formula to the complement of the α-collar

The main observation to make is that the curves on the boundary of the α-collar are also boundary curves for the complement of the α-collar in the surface. Consider, in to start, the curve α^+. It is a natural boundary for the region comp$^+$. Then by Stokes' theorem,

$$-\frac{1}{2}\int_{\alpha^+} \frac{\partial}{\partial \xi}(\frac{\dot{\mathcal{G}}}{\mathcal{G}})\mathrm{d}\eta = \frac{1}{2}\iint_{\text{comp}^+} \Delta_\zeta(\frac{\dot{\mathcal{G}}}{\mathcal{G}})\,\mathrm{d}\xi\mathrm{d}\eta \tag{4.22}$$

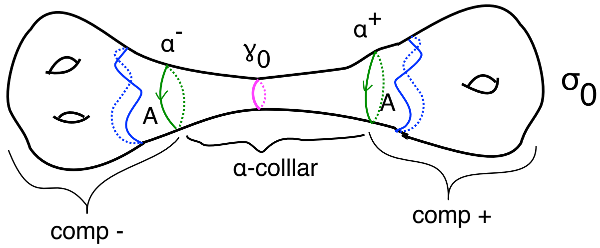

Figure 4.3: α^+ is a natural boundary for comp$^+$.

But we can actually compute the term on the right hand side:

$$\Delta_\zeta(\frac{\dot{\mathcal{G}}}{\mathcal{G}}) = \Delta_\zeta(\frac{\dot{G}}{g} + 2\dot{c}\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}X_{\text{graft}})$$

$$\Delta_\zeta(\frac{\dot{\mathcal{G}}}{\mathcal{G}}) = \frac{1}{|\frac{d\zeta}{dz}|^2}\Delta_E(\frac{\dot{G}}{g}) + \Delta_\zeta(2\dot{c}\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}X_{\text{graft}})$$

Solving the linearized prescribed curvature equation (4.10) for $\Delta_E(\frac{\dot{G}}{g})$ and substituting,

$$\Delta_\zeta(\frac{\dot{\mathcal{G}}}{\mathcal{G}}) = \frac{1}{|\frac{d\zeta}{dz}|^2}(2\frac{\dot{G}}{g} + 2\dot{c}(\log g)_{xx}X_{\text{graft}}) + \Delta_\zeta(2\dot{c}\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}X_{\text{graft}}) \tag{4.23}$$

Now replacing $\Delta_\zeta(\frac{\dot{\mathcal{G}}}{\mathcal{G}})$ in equation (4.22) above by the identity in (4.23),

$$\begin{aligned}
-\frac{1}{2}\int_{\alpha^+}(\frac{\dot{\mathcal{G}}}{\mathcal{G}})_\xi d\eta &= \frac{1}{2}\iint_{\text{comp}^+}\Delta_\zeta(\frac{\dot{\mathcal{G}}}{\mathcal{G}})\,d\xi d\eta \\
&= \frac{1}{2}\iint_{\text{comp}^+} 2\frac{\dot{G}}{g} + 2\dot{c}(\log g)_{xx}X_{\text{graft}}\,\mathrm{d}x\mathrm{d}y \\
&+ \frac{1}{2}\iint_{\text{comp}^+}\Delta_\zeta(2\dot{c}\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}X_{\text{graft}})\,\mathrm{d}\xi\mathrm{d}\eta \\
&= \iint_{\text{comp}^+}\frac{\dot{G}}{g}\,dA_0 + \dot{c}\iint_{\text{comp}^+\cap\text{graft}}(\log g)_{xx}\mathrm{d}x\mathrm{d}y \\
&+ \dot{c}\iint_{\text{comp}^+\cap\text{graft}}\Delta_\zeta\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\mathrm{d}\xi\mathrm{d}\eta
\end{aligned} \tag{4.24}$$

In the last step of the above equation, we used the identity

$$\iint_{\text{comp}^+} \Delta_\zeta\left(\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} X_{\text{graft}}\right) \mathrm{d}\xi\mathrm{d}\eta = \iint_{\text{comp}^+\cap\text{graft}} \Delta_\zeta \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \mathrm{d}\xi\mathrm{d}\eta$$

To get this, distribute the Laplacian,

$$\iint_{\text{comp}^+} \Delta_\zeta\left(\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} X_{\text{graft}}\right) \mathrm{d}\xi\mathrm{d}\eta = \iint_{\text{comp}^+} \left(\Delta_\zeta \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\right) X_{\text{graft}} + \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \Delta_\zeta X_{\text{graft}} + \nabla \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \cdot \nabla X_{\text{graft}} \, \mathrm{d}\xi\mathrm{d}\eta \tag{4.25}$$

Integrate the second term on the right-hand side by parts. This integration gives

$$\iint_{\text{comp}^+} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \Delta_\zeta X_{\text{graft}} \mathrm{d}\xi\mathrm{d}\eta = -\iint_{\text{comp}^+} \nabla \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \cdot \nabla X_{\text{graft}} \, \mathrm{d}\xi\mathrm{d}\eta + \int_{\alpha^+} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \nabla X_{\text{graft}} \cdot \mathbf{n} \, \mathrm{d}\eta$$

The first term $-\iint_{\text{comp}} \nabla \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \cdot \nabla X_{\text{graft}} \, \mathrm{d}\xi\mathrm{d}\eta$ above cancels naturally with the last term in (4.25). Noting that ∇X_{graft} is zero except at the boundary of the grafting cylinder and recalling our assumption that $\alpha^\pm$ do not intersect the boundary, the last term $\int_{\alpha^+} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \nabla X_{\text{graft}} \cdot \mathbf{n} \, \mathrm{d}\eta$ vanishes as well. Substituting the above work (4.24) into (4.20) for the variation vector field for α^+, we obtain the following

Proposition 4.19. For a family α_t^+ of curves of fixed length with α^+ disjoint from the boundary of the grafting cylinder,

$$\begin{aligned}\ell \sec \ell \xi^{\alpha^+} \int_{\alpha^+} \dot{\alpha}^1 ds_0 = & \iint_{\text{comp}^+} \frac{\dot{G}}{g} \, dA_0 + \dot{c} \iint_{\text{graft}\cap\text{comp}^+} (\log g)_{xx} \, \mathrm{d}x\mathrm{d}y \\ & + \dot{c} \iint_{\text{graft}\cap\text{comp}^+} \Delta_\zeta \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \\ & + \ell \tan \ell \xi^{\alpha^+} \int_0^1 2\dot{c} \frac{\xi_x^2 - \xi_y^2}{|\frac{d\zeta}{dz}|^2} \, \mathrm{d}\eta + \ell \sec \ell \xi^{\alpha^+} \left.\frac{d}{dt}\right|_{t=0} \int_0^1 \kappa_t \mathrm{d}\eta\end{aligned}$$

Using that the orientation outward pointing normal vector for the region complementary to α^- points in the opposite direction as that of α^+, we obtain the corresponding proposition for the α^- variational equation.

Proposition 4.20.

$$\ell \sec \ell \xi^{\alpha^-} \int_{\alpha^-} \dot{\alpha}^1 ds_0 = -\iint_{\text{comp}^-} \frac{\dot{G}}{g}\, dA_0 - \dot{c} \iint_{\text{graft}\cap\text{comp}^-} (\log g)_{xx}\, \mathrm{d}x\mathrm{d}y$$
$$- \dot{c} \iint_{\text{graft}\cap\text{comp}^-} \Delta_\zeta \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}$$
$$+ \ell \tan \ell \xi^{\alpha^-} \int_0^1 2\dot{c} \frac{\xi_x^2 - \xi_y^2}{|\frac{d\zeta}{dz}|^2}\, \mathrm{d}\eta + \ell \sec \ell \xi^{\alpha^-} \left.\frac{d}{dt}\right|_{t=0} \int_0^1 \kappa_t \mathrm{d}\eta$$

Again, we pause to consider the geometry embodied in the above formula. The variational vector field depends on the change in area of the surface bounded by α, the time-0 area of the area outside the α-collar but inside the grafting cylinder, the time-0 length of α, the derivative of geodesic curvature of α_t (and consequently, the derivative of length of the core geodesic), and the average angle between curves of constant *hyperbolic* geodesic curvature and constant *Euclidean* geodesic curvature.

4.3.4 A general formula

In this section, we derive the main equation:

Theorem 4.21. *4.1 Let S be any surface, γ a separating curve, and $t_0 \in \mathbb{R}$. Then for any curves $\alpha^\pm$ parallel to γ_0 that have a family α_t with $\ell(t, \alpha_t) = 0$, the changes in length along the grafting ray are expressed:*

$$-2\ell \sec \ell \xi^{\alpha^\pm} \left.\frac{d}{dt}\right|_{t=0} \ell(t, \alpha^\pm) + 4\ell \left.\frac{d}{dt}\right|_{t=0} \ell(t, \gamma_t)$$
$$= -2\dot{c}\ell \{ |\tan \ell \xi^{\alpha^\pm}| \iint_{graft\cap comp^\pm} \Delta_E \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\, \mathrm{d}x\mathrm{d}y - \iint_{\alpha-collar} |\tan \ell \xi| (\log g)_{xx} \mathrm{d}x\mathrm{d}y$$
$$+ 2\ell \int_{\alpha^\pm} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} d\eta - 4\ell \int_{\gamma_0} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} d\eta + 2\ell \tan^2 \ell \xi^{\alpha^\pm} \int_{\alpha^\pm} \frac{\xi_x^2}{|\frac{d\zeta}{dz}|^2} d\eta \}$$

Remark 4.22. This equation is not ideal in the sense that the derivative of length of the core geodesic requires as input the derivative of lengths of some complementary curves. However, one can understand the presence of this term in the following way: if the curves $\alpha^{\pm}$ are fairly far away from the geodesic γ_0, then they hold all the information about the change in the geometry outside the grafting cylinder. Additionally, the only two requirements we have placed on the curves $\alpha^{\pm}$ is that they are disjoint from the boundary of the grafting cylinder and that they permit this family α_t.

The proof is to simplify the equation from Proposition4.13 using equations we have derived in the last three sections.

Proof. We begin by recalling 4.13:

$$
\begin{aligned}
& -2\ell \sec \ell\xi^{\alpha^{\pm}} \cdot 2 \left.\frac{d}{dt}\right|_{t=0} \int_{\alpha^{\pm}} ds_t + 4\ell \cdot 2 \left.\frac{d}{dt}\right|_{t=0} \int_{\gamma_0} ds_t - 2\ell |\tan \ell\xi^{\alpha^{\pm}}| \iint_{\text{comp}^{\pm}} \frac{\dot{G}}{g}\, \mathrm{d}A_0 \\
&= -2\ell \sec \ell\xi^{\alpha^{\pm}} \cdot 2\dot{c} \int_{\alpha^{\pm}} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\, ds_0 + 4\ell \cdot 2\dot{c} \int_{\gamma_0} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\, ds_0 \\
&+ 2\dot{c}\ell |\tan \ell\xi^{\alpha^{\pm}}| \iint_{\text{graft}\cap\text{comp}^{\pm}} (\log g)_{xx}\, \mathrm{d}x\mathrm{d}y + 2\dot{c}\ell \iint_{\text{graft}\cap\alpha-\text{collar}} |\tan \ell\xi| (\log g)_{xx}\, \mathrm{d}x\mathrm{d}y
\end{aligned}
$$

The goal is to replace the third term in the first line, $\iint_{\text{comp}^{\pm}} \frac{\dot{G}}{g} dA_0$, with terms that are focused on the grafting cylinder. Manipulating the equations from propositions 4.19 and 4.20 in the previous section, we have an identity for $\iint_{\text{comp}^{\pm}} \frac{\dot{G}}{g} dA_0$ that depends on the average variational vector field of α:

$$
\begin{aligned}
-2\ell |\tan \ell\xi^{\alpha^{\pm}}| \iint_{\text{comp}^{\pm}} \frac{\dot{G}}{g} dA_0 &= -2\ell^2 \sec \ell\xi^{\alpha^{\pm}} \tan \ell\xi^{\alpha^{\pm}} \int_{\alpha^{\pm}} \dot{\alpha}\, ds_0 \\
&+ 2\ell \tan \ell\xi \{\pm\dot{c} \iint_{\text{graft}\cap\text{comp}^{\pm}} (\log g)_{xx} \mathrm{d}x\mathrm{d}y \pm \dot{c} \iint_{\text{graft}\cap\text{comp}^{\pm}} \Delta_{\zeta} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \} \\
&+ 2\ell^2 \tan^2 \ell\xi^{\alpha^{\pm}} \int_0^1 2\dot{c} \frac{\xi_x^2 - \xi_y^2}{|\frac{d\zeta}{dz}|^2}\, \mathrm{d}\eta + 2\ell^2 \sec \ell\xi^{\alpha^{\pm}} \tan \ell\xi^{\alpha^{\pm}} \qquad (4.26)
\end{aligned}
$$

The first step is to recall that the assumption that the family α_t satisfies $\frac{d}{dt}\big|_{t=0}\,\ell(t,\alpha_t) = 0$ and apply the equation for the first variation of length (4.19).

$$\begin{aligned} 0 &= 2\ell \sec \ell\xi^{\alpha^\pm}\, \frac{d}{dt}\bigg|_{t=0} \ell(t,\alpha_t^\pm) \\ &= 2\ell \sec \ell\xi^{\alpha^\pm}\, \frac{d}{dt}\bigg|_{t=0} \ell(t,\alpha^\pm) + 2\ell \sec \ell\xi^{\alpha^\pm}\, \frac{d}{dt}\bigg|_{t=0} \ell(0,\alpha_t^\pm) \end{aligned}$$

Moving one term to the left-hand side of the equation,

$$2\ell \sec \ell\xi^{\alpha^\pm}\, \frac{d}{dt}\bigg|_{t=0} \ell(t,\alpha^\pm) = -2\ell \sec \ell\xi^{\alpha^\pm}\, \frac{d}{dt}\bigg|_{t=0} \ell(0,\alpha_t^\pm)$$

Using (4.19),

$$2\ell \sec \ell\xi^{\alpha^\pm}\, \frac{d}{dt}\bigg|_{t=0} \ell(t,\alpha^\pm) = -2\ell^2 \sec \ell\xi^{\alpha^\pm} \tan \ell\xi^{\alpha^\pm} \int_{\alpha^\pm} \dot{\alpha}\, ds_0 \tag{4.27}$$

In other words, the derivative of length of the parallel curves α with respect to the hyperbolic metrics is given by the negative of the first variation of length. Now apply this identity (4.27) to the above equation (4.26), to write the derivative of area term $\iint_{\text{comp}^\pm} \frac{\dot{G}}{g}\, dA_0$ in terms of the derivative of length:

$$\begin{aligned} -2\ell|\tan \ell\xi^{\alpha^\pm}| \iint_{\text{comp}^\pm} \frac{\dot{G}}{g} dA_0 &= 2\ell \sec \ell\xi^{\alpha^\pm}\, \frac{d}{dt}\bigg|_{t=0} \ell(t,\alpha^\pm) \\ &+ 2\ell \tan \ell\xi\{\pm\dot{c} \iint_{\text{graft}\cap\text{comp}^\pm} (\log g)_{xx} \mathrm{d}x\mathrm{d}y \pm \dot{c} \iint_{\text{graft}\cap\text{comp}^\pm} \Delta_\zeta \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\} \\ &+ 2\ell^2 \tan^2 \ell\xi^{\alpha^\pm} \int_0^1 2\dot{c}\frac{\xi_x^2 - \xi_y^2}{|\frac{d\zeta}{dz}|^2}\, \mathrm{d}\eta + 2\ell^2 \sec \ell\xi^{\alpha^\pm} \tan \ell\xi^{\alpha^\pm}\, \frac{d}{dt}\bigg|_{t=0} \int_0^1 \kappa_t \mathrm{d}\eta \end{aligned} \tag{4.28}$$

This is the identity we will substitute into 4.13 derived from the prescribed curvature equation. Specifically,

$$
\begin{aligned}
&- 2\ell \sec \ell\xi^{\alpha^\pm} \cdot 2\, \frac{d}{dt}\bigg|_{t=0} \int_{\alpha^\pm} ds_t + 4\ell \cdot 2\, \frac{d}{dt}\bigg|_{t=0} \int_{\gamma_0} ds_t + (\text{substitute } 4.28) \\
&= -2\ell \sec \ell\xi^{\alpha^\pm} \cdot 2\dot{c} \int_{\alpha^\pm} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\, ds_0 + 4\ell \cdot 2\dot{c} \int_{\gamma_0} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\, ds_0 \\
&+ 2\dot{c}\ell |\tan \ell\xi^{\alpha^\pm}| \iint_{\text{graft}\cap\text{comp}^\pm} (\log g)_{xx}\, \mathrm{d}x\mathrm{d}y + 2\dot{c}\ell \iint_{\text{graft}\cap\alpha-\text{collar}} |\tan \ell\xi| (\log g)_{xx}\, \mathrm{d}x\mathrm{d}y
\end{aligned}
$$

Moving terms from (4.28) that are independent of the change in metric to the right hand side, we obtain

$$
\begin{aligned}
&- 2\ell \sec \ell\xi^{\alpha^\pm}\, \frac{d}{dt}\bigg|_{t=0} \ell(t, \alpha^\pm) + 8\ell\, \frac{d}{dt}\bigg|_{t=0} \ell(t, \gamma_t) + 2\ell^2 \sec \ell\xi^{\alpha^\pm} \tan \ell\xi^{\alpha^\pm}\, \frac{d}{dt}\bigg|_{t=0} \int_0^1 \kappa_t \mathrm{d}\eta \\
&= -2\ell\dot{c} |\tan \ell\xi^{\alpha^\pm}| \iint_{\text{graft}\cap\text{comp}^\pm} (\log g)_{xx} + \Delta_E \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\, \mathrm{d}x\mathrm{d}y - 2\ell^2 \tan^2 \ell\xi^{\alpha^\pm} \int_{\alpha^\pm} 2\dot{c}\Big(\frac{\xi_x^2 - \xi_y^2}{|\frac{d\zeta}{dz}|^2}\Big) d\eta \\
&+ 2\ell\dot{c} |\tan \ell\xi^{\alpha^\pm}| \iint_{\text{graft}\cap\text{comp}^\pm} (\log g)_{xx} \mathrm{d}x\mathrm{d}y + 2\ell\dot{c} \iint_{\alpha-\text{collar}} |\tan \ell\xi| (\log g)_{xx} \mathrm{d}x\mathrm{d}y \\
&- 2\ell \sec \ell\xi^{\alpha^\pm} \int_{\alpha^\pm} 2\dot{c} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} ds_0 + 4\ell \int_{\gamma_0} 2\dot{c} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} ds_0
\end{aligned}
\tag{4.29}
$$

Cancel the first with fourth term on the right-hand side. Then using that $ds_0 = \ell \sec \ell\xi d\eta$ in tandem with the identity $\tan^2 \ell\xi + 1 = \sec^2 \ell\xi$, we obtain

$$
\begin{aligned}
&- 2\ell \sec \ell\xi^{\alpha^\pm}\, \frac{d}{dt}\bigg|_{t=0} \ell(t, \alpha^\pm) + 8\ell\, \frac{d}{dt}\bigg|_{t=0} \ell(t, \gamma_t) + 2\ell^2 \sec \ell\xi^{\alpha^\pm} \tan \ell\xi^{\alpha^\pm}\, \frac{d}{dt}\bigg|_{t=0} \int_0^1 \kappa_t \mathrm{d}\eta \\
&= -2\dot{c}\ell \{ |\tan \ell\xi^{\alpha^\pm}| \iint_{\text{graft}\cap\text{comp}^\pm} \Delta_E \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\, \mathrm{d}x\mathrm{d}y - \iint_{\alpha-\text{collar}} |\tan \ell\xi| (\log g)_{xx} \mathrm{d}x\mathrm{d}y \\
&+ 2\ell \int_{\alpha^\pm} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} d\eta - 4\ell \int_{\gamma_0} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} d\eta + 2\ell \tan^2 \ell\xi^{\alpha^\pm} \int_{\alpha^\pm} \frac{\xi_x^2}{|\frac{d\zeta}{dz}|^2} d\eta \}
\end{aligned}
\tag{4.30}
$$

The final step is to use that the derivative of the time-t hyperbolic geodesic curvature of α_t has been computed in terms of the derivative of length of the core geodesic:

$$
\frac{d}{dt}\bigg|_{t=0} \int_0^1 \kappa_t \mathrm{d}\eta = \frac{-\dot{\ell}}{\ell} \frac{1}{\sec \ell\xi^{\alpha^\pm} \tan \ell\xi^{\alpha^\pm}} \tag{4.31}
$$

Since the coefficient of $\frac{d}{dt}\Big|_{t=0}\int_0^1 \kappa_t \mathrm{d}\eta$ in equation (4.30) and the denominator in the above derivative (4.31) cancel precisely, after substituting (4.31) into (4.30), we have the main formula given in Theorem 4.1:

$$
\begin{aligned}
&- 2\ell \sec \ell\xi^{\alpha^\pm} \frac{d}{dt}\Big|_{t=0} \ell(t,\alpha^\pm) + 4\ell \frac{d}{dt}\Big|_{t=0} \ell(t,\gamma_t) \\
&= -2\dot{c}\ell\{|\tan \ell\xi^{\alpha^\pm}| \iint_{\mathrm{graft}\cap\mathrm{comp}^\pm} \Delta_E \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \,\mathrm{d}x\mathrm{d}y - \iint_{\alpha-\mathrm{collar}} |\tan \ell\xi|(\log g)_{xx}\mathrm{d}x\mathrm{d}y \\
&+ 2\ell \int_{\alpha^\pm} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} d\eta - 4\ell \int_{\gamma_0} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} d\eta + 2\ell \tan^2 \ell\xi^{\alpha^\pm} \int_{\alpha^\pm} \frac{\xi_x^2}{|\frac{d\zeta}{dz}|^2} d\eta\}
\end{aligned}
$$

□

4.4 Observations

We consider the particularly simple case of the grafting operation on a cylinder, thought of as a genus zero surface with two boundary components. In this case, the conformal factor g maps the Euclidean curves that foliate the grafting cylinder to curves of hyperbolic geodesic curvature. In other words, $\xi = x$ and $\xi_y = 0$. We noted before that the hyperbolic metric on the cylinder is invariant under rotation in the y-direction, so in that case $2g = (\log g)_{xx}$. It is natural, then, to present the above formula from Theorem 4.1 in a way that shows the defect of the grafting operation on a general manifold from being like that of grafting on a cylinder. We start by noting which terms vanish and which terms persist in the cylinder case:

$$
\begin{aligned}
&- 2\ell \sec \ell \xi^{\alpha^\pm} \left.\frac{d}{dt}\right|_{t=0} \ell(t, \alpha^\pm) + 4\ell \left.\frac{d}{dt}\right|_{t=0} \ell(t, \gamma_t) \\
&= -2\dot{c}\ell\{\underbrace{2\ell \tan^2 \ell \xi^{\alpha^\pm} \int_{\alpha^\pm} \frac{\xi_x^2}{|\frac{d\zeta}{dz}|^2} d\eta - \iint_{\alpha-\text{collar}} |\tan \ell \xi| (\log g)_{xx} \mathrm{d}x\mathrm{d}y}_{\text{persists in cylinder case}} \\
&\underbrace{+ |\tan \ell \xi^{\alpha^\pm}| \iint_{\text{graft}\cap\text{comp}^\pm} \Delta_E \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \,\mathrm{d}x\mathrm{d}y + 2\ell \int_{\alpha^\pm} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} d\eta - 4\ell \int_{\gamma_0} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} d\eta\}}_{\text{vanishes in cylinder case}}
\end{aligned}
\tag{4.32}
$$

Now re-write the term $\frac{\xi_x^2}{|\frac{d\zeta}{dz}^2|}$ to represent how an x=constant curve is distorted from being a geodesic with respect to the hyperbolic metric:

$$
\frac{\xi_x^2}{|\frac{d\zeta}{dz}|^2} = 1 - \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \tag{4.33}
$$

Specifically, the term $\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}$ measures the cosine of the angle between a curve of hyperbolic geodesic curvature and an x=constant curve. Similarly, re-write the derivative of the conformal factor, $(\log g)_{xx}$,

$$
(\log g)_{xx} = 2g - (\log g)_{yy} = 2\ell^2 \sec^2 \ell\xi |\frac{d\zeta}{dz}|^2 - (\log g)_{yy} \tag{4.34}
$$

so that the term $(\log g)_{yy}$ measures the defect of the metric from being rotationally invariant with respect to y. Plugging 4.33 and (4.34) into equation ((4.32)) and integrating once by parts, we obtain

$$
\begin{aligned}
&- 2\ell \sec \ell \xi^{\alpha^\pm} \left.\frac{d}{dt}\right|_{t=0} \ell(t, \alpha^\pm) + 4\ell \left.\frac{d}{dt}\right|_{t=0} \ell(t, \gamma_t) \\
&= -2\dot{c}\ell\{\underbrace{\ell \tan^2 \ell \xi^{\alpha^+} + \ell \tan^2 \ell \xi^{\alpha^-}}_{\text{cylinder value}} + \underbrace{(2\ell - 2\ell \tan^2 \ell \xi^{\alpha^\pm}) \int_{\alpha^\pm} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \mathrm{d}\eta - 4\ell \int_{\gamma_0} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \mathrm{d}\eta}_{\text{angle defect}} \\
&+ \underbrace{\iint_{\alpha-\text{collar}} |\tan \ell \xi| (\log g)_{yy} \mathrm{d}x\mathrm{d}y}_{\text{angle defect}} + \underbrace{|\tan \ell \xi^{\alpha^\pm}| \iint_{\text{graft}\cap\text{comp}^\pm} \Delta_E \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \,\mathrm{d}x\mathrm{d}y}_{\text{symmetry defect}}
\end{aligned}
\tag{4.35}
$$

Note that it is clear that if the surface on which we perform the grafting operation behaves precisely like the cylinder, the above geometric quantity is negative. The next task will be to show that this is true in general.

4.5 An application

In this section, we apply Theorem 4.1 to prove the following theorem:

Theorem 4.23 (Lengths decrease.)**.** *For ℓ small enough and $\alpha^{\pm}$ sufficiently close to γ_0 (depending on ℓ) to γ_0,*

$$-2\sec\ell\xi^{\alpha^+}\left.\frac{d}{dt}\right|_{t=0}\ell(t,\alpha^+) - 2\sec\ell\xi^{\alpha^-}\left.\frac{d}{dt}\right|_{t=0}\ell(t,\alpha^-) + \left.\frac{d}{dt}\right|_{t=0}\ell(t,\gamma_t) < 0$$

The proof of the theorem proceeds in two parts. First, we will isolate three terms (which vanish in the cylinder case) and show that the sum of these terms is clearly positive in every general case. The main tool we use to do this is

Lemma 4.24. *The function $S(\xi) = \int_\xi \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} d\eta$ is convex in ξ.*

The objective of the second part of the proof is to show that the remaining terms (there will be three to consider) are small. The three terms we will study will either be integrals over ξ=constant curves or over regions bounded by ξ=constant curves. After showing that these terms are well-approximated by integrals over x=constant curves, we use methods of Fourier analysis to prove three lemmas that give the desired bounds. For convenience of the reader, we will summarize the proof again after we conclude the derivation of the necessary bounds.

4.5.1 Positive terms

To accomplish the first task, we need a lemma:

Lemma 4.24. The function $S(\xi) = \int_\xi \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} d\eta$ is convex in ξ.

Proof. Begin by noting that that the function $\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}$ is subharmonic. In particular

$$\Delta_E \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} = 2\frac{\xi_{yy}^2 + \xi_{xy}^2}{|\frac{d\zeta}{dz}|^2} + \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \frac{4\xi_x^2(\xi_{xy}^2 + \xi_{yy}^2)}{|\frac{d\zeta}{dz}|^4} \geq 0$$

Then Δ_ζ differs from Δ_E by a (positive) conformal factor, so we also have $\Delta_\zeta \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \geq 0$. Now expanding $\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}$ by Fourier modes as

$$\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} = s_0(\xi) + \sum_{n\neq 0} s_n(\xi) e^{2\pi i n \eta}$$

subharmonicity implies that each $s_n''(\xi) \geq 0$, so in particular $s_0''(\xi) \geq 0$. Now consider

$$\frac{\partial^2}{\partial \xi^2} \int_\xi \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} d\eta = \frac{\partial^2}{\partial \xi^2} \int_\xi s_0 + \sum s_n(\xi) e^{2\pi i n \eta} d\eta$$

since ξ is closed, all terms times $e^{2\pi i n \eta}$ vanish

$$= \frac{\partial^2}{\partial \xi^2} \int_\xi s_0 d\eta$$

$$= s_0''(\xi) \geq 0$$

proving that $S(\xi)$ is convex in ξ as desired. □

Corollary 4.25.

$$|\tan \ell \xi^{\alpha^\pm}| \iint_{graft \cap comp^\pm} \Delta_E \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \, \mathrm{d}x \mathrm{d}y \geq 0$$

Proof. This merely requires subharmonicity of the function $\frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}$. □

Corollary 4.26. *If α^+ and α^- are any pair of curves parallel to the core geodesic γ_0 then*

$$\int_{\alpha^+} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} d\eta + \int_{\alpha^-} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} d\eta \geq 2 \int_{\gamma_0} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} d\eta$$

Combining the corollaries 4.25 and 4.26 with equation (4.35), we obtain the following:

$$\begin{aligned}
&- 4\ell \sec \ell \xi^{\alpha^\pm} \left.\frac{d}{dt}\right|_{t=0} \ell(t, \alpha^\pm) + 4\ell \left.\frac{d}{dt}\right|_{t=0} \ell(t, \gamma_t) \\
&= -2\dot{c}\ell \{ \underbrace{\ell \tan^2 \ell \xi^{\alpha^+} + \ell \tan^2 \ell \xi^{\alpha^-}}_{\text{cylinder value -positive}} \\
&+ \underbrace{2\ell \int_{\alpha^\pm} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \mathrm{d}\eta - 4\ell \int_{\gamma_0} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \mathrm{d}\eta}_{\text{always positive}} \underbrace{-2\ell \tan^2 \ell \xi^{\alpha^\pm} \int_{\alpha^\pm} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \mathrm{d}\eta}_{\text{negative}} \\
&+ \underbrace{|\tan \ell \xi^{\alpha^\pm}| \iint_{\text{graft}\cap\text{comp}^\pm} \Delta_E \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \,\mathrm{d}x\mathrm{d}y}_{\text{always positive}} + \underbrace{\iint_{\alpha-\text{collar}} |\tan \ell \xi| (\log g)_{yy} \mathrm{d}x\mathrm{d}y}_{\text{sign unknown}} \qquad (4.36)
\end{aligned}$$

The grouping of terms with the description "cylinder value - positive" are the terms that persist in the cylinder case, and by virtue of being squares, are positive. The first grouping with the description "always positive" is positive by Corollary 4.26. The term in the third line described as "always positive" is positive by Corollary 4.25. This concludes the first part of the proof. The second part of the proof is to show that the terms that are negative or of unknown sign are small.

4.5.2 Small terms

We now begin the second task, showing that the sum

$$- 2\ell \tan^2 \ell \xi^{\alpha^\pm} \int_{\alpha^\pm} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \mathrm{d}\eta + \iint_{\alpha-\text{collar}} |\tan \ell \xi| (\log g)_{yy} \mathrm{d}x\mathrm{d}y \qquad (4.37)$$

is very small when ℓ is small enough and if the $\alpha^{\pm}$ are moved slightly toward the center of the grafting cylinder. To simplify the task, we write

$$(\log g)_{yy} = 2\ell^2 \sec^2 \ell\xi + 2\ell\xi_{yy} \tan \ell\xi + (\log |\frac{d\zeta}{dz}|^2)_{yy}$$

and note that the first term is clearly positive, so the term $\iint_{\alpha-\text{collar}} |\tan \ell\xi| 2\ell^2 \xi_y^2 \sec^2 \ell\xi \mathrm{d}x\mathrm{d}y$ can be grouped with the positive terms. Thus our goal is to show that the remaining terms are small in comparison to the positive terms that characterize the cylinder case. Specifically, we want to show the following:

Lemma 4.30. *Thinking of $\xi^{\alpha^{\pm}}$ as functions on the Euclidean cylinder, $\xi^{\alpha^{\pm}}$ obtains a maximum distance $|x|$ from $x = 0$. If this maximum value satisfies*

$$|x| \leq \frac{2L}{\sqrt{8}\pi}(\frac{(\xi_0^0)^2}{\sqrt{\tilde{A}}} - 1) + \frac{t_0}{2}$$

we have

$$2\ell \tan^2 \ell\xi^{\alpha^{\pm}} \int_{\alpha^{\pm}} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|}^2 < \frac{1}{2}\ell \tan^2 \ell\xi^{\alpha^{\pm}}$$

Lemma 4.32. *Thinking of $\xi^{\alpha^{\pm}}$ as functions on the Euclidean cylinder, $\xi^{\alpha^{\pm}}$ obtains a maximum distance $|x|$ from $x = 0$. If this maximum value satisfies*

$$|x| \leq (\frac{\sqrt{8}\pi}{2\tilde{B}L} - 1)\frac{L}{\sqrt{8}\pi} + \frac{t_0}{2}$$

then

$$\iint_{\alpha-collar} |\tan \ell\xi| 2\ell \tan \ell\xi \, \xi_{yy} \, \mathrm{d}x\mathrm{d}y| \leq \frac{1}{2}\ell \tan^2 \ell\xi^{\alpha^{\pm}}$$

Lemma 4.33. *For small enough ℓ such that there exists x satisfying*

$$C\left(\frac{L(2\ell - \sqrt{8}\pi^2)}{2L\ell^3 - 2C\ell\sqrt{8}\pi}\right) \leq |x|$$

The following bound holds:

$$\iint_{\alpha-collar} |\tan \ell\xi||(\log |\frac{d\zeta}{dz}|^2)_{yy}| \leq \frac{1}{2}\ell \tan^2 \ell\xi^{\alpha^{\pm}}$$

While each of the terms will require a slightly different trick, the main idea is the same: moving slightly in from the boundary of the grafting cylinder, the curves of hyperbolic constant geodesic curvature approximate (with exponentially increasing accuracy) curves of Euclidean constant geodesic curvature. In other words, as ξ=constant moves toward the center of the grafting cylider, ξ_x becomes very close to 1 and ξ_y gets very small. We make this more specific in the next two propositions. First, we argue that the average of ξ_x on an ξ=constant curve is nonzero. Then we show that the maximal variation of an ξ=constant curve from an x=constant curve decreases very fast as $x \to 0$.

Proposition 4.27. $\int_{x=x_0} \xi_x \, dy = \xi_0^0 \neq 0$

Proof. Let $\xi_x = \sum \xi_n(x) e^{2\pi i n y/L}$. Then as ξ is harmonic, ξ_x is harmonic, so for each n we have

$$(\xi_n(x))'' = \frac{4\pi^2 n^2}{L^2} \xi_n(x)$$

In particular, for $\xi_0''(x) = 0$ so that $\xi_0(x)$ is at most linear in x. In other words,

$$\xi_0(x) = \xi_0^0 + \xi_0^1 x$$

for some constants ξ_0^0, ξ_0^1. But we also know that

$$\frac{\partial}{\partial x} \int_{x=x_0} \xi_x \, dy = \int_{x=x_0} \xi_{xx} \, dy = \int_{x=x_0} -\xi_{yy} \, dy = 0$$

where the third equality uses that ξ is harmonic. Then in particular,

$$\begin{aligned} 0 = \frac{\partial}{\partial x}\int_x \xi_x \, dy &= \frac{\partial}{\partial x}\int_{x=} \sum \xi_n(x) e^{2\pi i n y/L} \, dy \\ &= \frac{\partial}{\partial x}\xi_0(x) \\ &= \frac{\partial}{\partial x}(\xi_0^0 + \xi_0^1 x) \\ &= \xi_0^1 \end{aligned}$$

Thus $\int_{x=x_0} \xi_x \, dy = \xi_0^0$ for every $x_0 \in [-\frac{t_0}{2}, \frac{t_0}{2}]$. The next claim is that in fact ξ_0^0 is nonzero. Suppose for the sake of contradiction that it vanishes. Then in particular

$$\frac{\partial}{\partial x}\int_{x=x_0} \xi \, dy = \int_{x=x_0} \xi_x \, dy = \xi_0^0 = 0$$

In other words, the average value of ξ over every x=constant curve is the same. Thus

$$\int_x \xi dy = A$$

for some constant A and every x in the grafting cylinder. Then by the mean value theorem for integrals, for every x_0 there exists some $y_0 \in [0, L]$ such that

$$\xi(x_0, y_0) = A$$

I.e, the curve generated by such points (x_0, y_0) in the hyperbolic cylinder is the preimage of ξ=constant curve of value A. Either A is nonzero or it is zero. Suppose A is nonzero. We have worked under the assumption (4.7) that the preimage γ_0^* of the geodesic γ_0 in the Euclidean cylinder is a graph over some x=constant curve. In particular, this means that γ_0^* and the curve generated by the points $\{(x_0, y_0)\}$ intersect transversely at some point. On the other hand, we know the map ξ is a

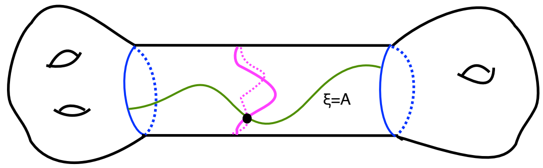

Figure 4.4: If $A \neq 0$ then the pre-image of the curve $\xi = A$ must intersect γ_0^*.

continuous surjection from the annulus to a connected component of the real line, and that the curves $\gamma_0 \equiv \xi = 0$ and $\xi = A \neq 0$ are parallel. Since the preimages intersect in at least one point, the images in the hyperbolic cylinder must intersect at least once. However, parallel curves that intersect must actually be the same curve, so $0 = \xi \equiv \gamma_0 \equiv \xi = A$. This then implies that $A = 0$, which is a contradiction. If $A = 0$, we can play the same game, switching the roles of γ_0 and the ξ=constant curve above. Hence ξ_0^0 must be nonzero. □

Proposition 4.28. The variation of an ξ=constant curve from an x=constant curve decays exponentially fast as the ξ=constant curve is moved toward the center of the cylinder. Specifically, for any $x = x_0$, and any $y_0, y_1 \in [0, L]$,

$$|\xi(x_0, y_0) - \xi(x_0, y_1)| \leq \sqrt{\tilde{A}} e^{\frac{\sqrt{8}\pi}{2L}(|x_0| - t_0/2)}$$

Proof. As we noted earlier, the intuition is that ξ=constant curves are "close" to being x=constant curves (as they actually agree in the cylinder case) so that ξ_x is close to 1 and consequently ξ is nearly perpendicular to y=constant curves, and ξ_y is very small. To consider the maximal variation of an ξ=constant curve from an

x=constant curve, select any $x_0 \in [\frac{-t_0}{2}, \frac{t_0}{2}]$ and any $y_1, y_2 \in [0, L]$. Then

$$\begin{aligned} |\xi(x_0, y_2) - \xi(x_0, y_1)| &= |\int_{y_1}^{y_2} \frac{\partial}{\partial y}\xi(x_0, y)\, dy| \\ &\leq \max_{y_1,y_2} \int_{y_1}^{y_2} |\xi_y| dy \\ &\leq \int_0^1 |\xi_y|\, dy \\ &= \int_0^1 (\xi_y^2)^{1/2}\, dy \\ &\leq \{\int_0^1 \xi_y^2\, dy\}^{1/2} \end{aligned}$$

Expanding

$$\xi_y = \sum a_n(x) e^{2\pi i n y/L} \tag{4.38}$$

by Fourier modes, we first note that $\int_{x=x_0} \xi_y = 0$ so that $a_0(x) \equiv 0$. Then by the same technique as above

$$\int_{x=x_0} \xi_y^2\, dy = \sum_{n\neq 0} a_n^2(x)$$

Now consider

$$\frac{\partial^2}{\partial x}(\sum_{n\neq 0} a_n(x)) = \sum_{n\neq 0} 2a_n''(x)a_n x + [a_n'(x)]^2 \tag{4.39}$$

since ξ_y is harmonic, each a_n satisfies $a_n''(x) = \frac{4\pi^2 n^2}{L^2} a_n(x)$, giving

$$= 2 \cdot \frac{4\pi^2 n^2}{L^2}[a_n(x)]^2 + [a_n'(x)]^2 \tag{4.40}$$

$$\geq \frac{8\pi^2}{L^2} \sum a_n(x)^2 \tag{4.41}$$

Now suppose that F is any function satisfying the equality in (4.41) with the same boundary conditions as above. Then by a maximum principle argument, $F \geq \sum_{n\neq 0} a_n(x)^2$.

We can take this one step further: if

$$\sum_{n\neq 0} a_n(\frac{t_0}{2}) \neq \sum_{n\neq 0} a_n(\frac{-t_0}{2})$$

set $\tilde{A} := \max\{a_n(\frac{t_0}{2}), a_n(\frac{-t_0}{2})\}$ and require F to be a function satisfying $F'' = \frac{8\pi^2}{L^2}$ with $F(\frac{t_0}{2}) = F(\frac{-t_0}{2}) = \tilde{A}$, then again $\sum_{n\neq 0} a_n^2(x) \leq F$. The point of this is that if one shows that F decays rapidly in x, so does the sum of interest, (4.38). But we know F is symmetric and satisfies the ODE $F'' = \frac{8\pi^2}{L^2}F$, so

$$\int_x \xi_y^2 \leq F = \tilde{A}\frac{\cosh(\frac{\sqrt{8}\pi}{L}x)}{\cosh(\frac{\sqrt{8}\pi}{L}\frac{t_0}{2})} \leq \tilde{A}e^{\frac{\sqrt{8}\pi}{L}(|x|-\frac{t_0}{2})}$$

Returning to the desired term, we have

$$|\xi(x_0,y_2) - \xi(x_0,y_1)| \leq \int_{x=x_0} |\xi_y|\, dy \leq \{\int_{x=x_0} \xi_y^2\, dy\}^{1/2} \leq \sqrt{\tilde{A}}e^{\frac{\sqrt{8}\pi}{2L}(|x|-t_0/2)}$$

□

Remark 4.29. This proposition implies that any estimates proved for integrals over x=constant curves can be used to show estimates for integrals over ξ=constant curves.

We apply the above two propositions to estimate the three terms.

Lemma 4.30. For $\alpha^{\pm}$ such that the maximum value $|x|$ attained by $\xi^{\alpha^{\pm}}$ satisfies

$$|x| \leq \frac{2L}{\sqrt{8}\pi}(\frac{(\xi_0^0)^2}{\sqrt{\tilde{A}}} - 1) + \frac{t_0}{2}$$

we have

$$2\ell\tan^2\ell\xi^{\alpha^{\pm}} \int_{\alpha^{\pm}} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|}^2 < \frac{1}{2}\ell\tan^2\ell\xi^{\alpha^{\pm}}$$

Proof. For the first term, note that the above proof shows

Proposition 4.31.

$$\int_{x=x_0} \xi_y^2\, dy \leq \tilde{A}\frac{\cosh(\sqrt{8}\pi x_0)}{\cosh(\sqrt{8}\pi\frac{t_0}{2})}$$

where $\tilde{A}$ is a finite constant.

Then

$$\int_{x=x_0} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2}\, dy \leq \max\{\xi_x^2+\xi_y^2\}^{-2}\int_{x=x_0}\xi_y^2\, dy$$

The main idea is that $\xi_x = \sum \xi_n(x)e^{2\pi i n y/L}$, where the nonzero term $\xi_n(x)$ must decay quickly into the collar, as each satisfies

$$\xi_n''(x) = \frac{4\pi^2 n^2}{L^2}\xi_n(x) > \frac{4\pi^2}{L^2}\xi_n(x)$$

Thus ξ_x is close to the value ξ_0^0 which is nonzero (and big) so that $\xi_x^2+\xi_y^2$ is always close to ξ_0^0. Thus

$$\int_{\alpha^\pm} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|}^2 \sim \frac{\tilde{A}}{\xi_0^2}\frac{\cosh(\sqrt{8}\pi x_0)}{\cosh(\sqrt{8}\pi\frac{t_0}{2})}$$

concluding the proof. □

Now consider the second term:

Lemma 4.32. Thinking of $\xi^{\alpha^\pm}$ as functions on the Euclidean cylinder, $\xi^{\alpha^\pm}$ obtains a maximum distance $|x|$ from $x=0$. If this maximum value satisfies

$$|x| \leq (\frac{\sqrt{8}\pi}{2\tilde{B}L}-1)\frac{L}{\sqrt{8}\pi}+\frac{t_0}{2}$$

then

$$\iint_{\alpha-\text{collar}} |\tan \ell\xi| 2\ell \tan \ell\xi\, \xi_{yy}\, \mathrm{d}x\mathrm{d}y| \leq \frac{1}{2}\ell \tan^2 \ell\xi^{\alpha^\pm}$$

Proof.

$$\iint_{\alpha-\text{collar}} |\tan \ell\xi| 2\ell \tan \ell\xi\, \xi_{yy}\, \mathrm{d}x\mathrm{d}y| \leq \iint_{\alpha-\text{collar}} 2\ell \tan^2 \ell\xi\, |\xi_{yy}| \mathrm{d}x\mathrm{d}y$$

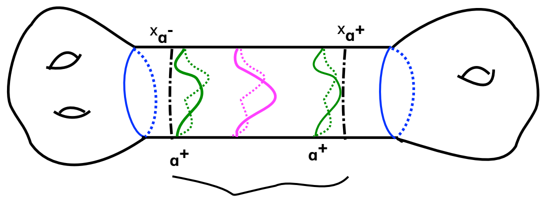

Figure 4.5: The α-rectangle is the smallest Euclidean sub-cylinder containing the α-collar.

Let $M = \max\{\ell \tan^2 \ell \xi^{\alpha^+}, \ell \tan^2 \ell \xi^{\alpha^-}\}$ then

$$\leq 2M \iint_{\alpha-\text{collar}} |\xi_{yy}| \mathrm{d}x\mathrm{d}y$$

now let $x_{\alpha^\pm}$ denote the largest values of x obtained by the functions $\xi^{\alpha^\pm}(z)$. Denote by α-rectangle the Euclidean cylinder that is bounded by $x = x_{\alpha^\pm}$. Then

$$\leq 2M \iint_{\alpha-\text{rectangle}} |\xi_{yy}| \mathrm{d}x\mathrm{d}y$$

$$\leq 2M \iint_{\alpha-\text{rectangle}} |\xi_{yy}| \mathrm{d}y\mathrm{d}x$$

On the other hand, we know

$$\int_{x=x_0} |\xi_{yy}|\, dy = \int_{x=x_0} [(\xi_{yy})^2]^{1/2}\, dy$$

By Cauchy-Schwarz inequality,

$$\leq \{\int_{x=x_0} \xi_{yy}^2\, dy\}^{1/2}$$

And since ξ_{yy} is harmonic, a similar argument as above gives us that

$$[\int_{x=x_0} |\xi_{yy}|\, dy]^2 \leq \tilde{B} \frac{\cosh(\frac{\sqrt{8}\pi}{L} x_0)}{\cosh(\frac{\sqrt{8}\pi}{L} \frac{t_0}{2})} \leq \tilde{B} e^{\frac{\sqrt{8}\pi}{L}(|x| - t_0/2)}$$

and hence

$$\int_{x=x_0} |\xi_{yy}|\, dy \leq \sqrt{\tilde{B}} e^{\frac{\sqrt{8}\pi}{2L}\pi(|x|-t_0/2)}$$

Substituting this estimate back into the integral, we obtain

$$\begin{aligned}\iint_{\alpha-\text{collar}} |\tan \ell\xi| 2\ell \tan \ell\xi\, \xi_{yy}\, \mathrm{d}x\mathrm{d}y| &\leq 2M \int \sqrt{\tilde{B}} e^{\frac{\sqrt{8}}{2L}\pi(|x|-t_0/2)} \mathrm{d}x \\ &= 2M\sqrt{\tilde{B}} \frac{2L}{\sqrt{8}\pi} (e^{\frac{\sqrt{8}\pi}{2}(|x_{\alpha^+}|-\frac{t_0}{2})} + e^{\frac{\sqrt{8}\pi}{2L}(|x_{\alpha^-}|-\frac{t_0}{2})} - 2)\end{aligned}$$

letting x_α be the max of $x_{\alpha^\pm}$,

$$\leq 2M\sqrt{\tilde{B}} \frac{L}{\sqrt{8}\pi} e^{\frac{\sqrt{8}\pi}{L}|(x_\alpha|-\frac{t_0}{2})}$$

This gives the desired rate of decay. □

The last term that we need to bound is

$$\iint_{\alpha-\text{collar}} |\tan \ell\xi| (\log |\frac{d\zeta}{dz}|^2)_{yy} \mathrm{d}x\mathrm{d}y \tag{4.42}$$

To accomplish this, we must work a little more carefully. The main idea is that on the α-collar, x is like ξ and on any x=constant curves, $(\log g)_{yy}$ vanishes. In other words, the actual value of the integral 4.42 is concentrated in a thin strip near boundary of the α-collar.

Lemma 4.33. For small enough ℓ such that there exists x satisfying

$$C\left(\frac{L(2\ell - \sqrt{8}\pi^2)}{2L\ell^3 - 2C\ell\sqrt{8}\pi}\right) \leq |x|$$

The following bound holds:

$$\iint_{\alpha-\text{collar}} |\tan \ell\xi| |(\log |\frac{d\zeta}{dz}|^2)_{yy}| \leq \frac{1}{2}\ell \tan^2 \ell\xi^{\alpha^\pm}$$

Proof. Keeping in mind that moving just a little way from the boundary of the grafting cylinder causes ξ_y to be small, assume to start that the integral 4.42 of interest is actually

$$\iint_{\alpha-\text{collar}} |\tan \ell x| (\log |\frac{d\zeta}{dz}|^2)_{yy} \mathrm{d}x\mathrm{d}y$$

Taking the absolute value, we find

$$= |\iint_{\alpha-\text{rectangle}} |\tan \ell x| (\log |\frac{d\zeta}{dz}|^2)_{yy} \mathrm{d}x\mathrm{d}y + \iint_{\alpha-\text{collar-rectangle}} |\tan \ell x| (\log |\frac{d\zeta}{dz}|^2)_{yy} \mathrm{d}x\mathrm{d}y|$$

where the α-rectangle is the largest subsurface of the α-collar bounded by x=constant curves. Let $x_0^{\pm}$ denote the boundaries of the α-rectangle.

$$\leq |\iint_{\alpha-\text{rectangle}} |\tan \ell x| (\log |\frac{d\zeta}{dz}|^2)_{yy} \mathrm{d}x\mathrm{d}y| + |\iint_{\alpha-\text{collar - rectangle}} |\tan \ell x| (\log |\frac{d\zeta}{dz}|^2)_{yy} \mathrm{d}x\mathrm{d}y|$$

$$= |\int |\tan \ell x| \{\int_x (\log |\frac{d\zeta}{dz}|^2)_{yy} \mathrm{d}y\} \mathrm{d}x| + |\iint_{\alpha-\text{collar - rectangle}} |\tan \ell x| (\log |\frac{d\zeta}{dz}|^2) \mathrm{d}x\mathrm{d}y|$$

Since $\int_x (\log |\frac{d\zeta}{dz}|^2)_{yy} \mathrm{d}y = 0$, the first term vanishes. Thus,

$$\iint_{\alpha-\text{collar}} |\tan \ell x| (\log |\frac{d\zeta}{dz}|^2)_{yy} \mathrm{d}x\mathrm{d}y \leq 0 + \iint_{\alpha-\text{collar - rectangle}} |\tan \ell x| \cdot |(\log |\frac{d\zeta}{dz}|^2)_{yy}| \mathrm{d}x\mathrm{d}y$$

Define $N_{\alpha^{\pm}}$ to be the two components in the complement of the α-rectangle in the α-collar. $N_{\alpha^{\pm}}$ has boundaries $x = x_0^{\pm}$ and $\xi = \xi^{\alpha^{\pm}}$. Now each $\xi^{\alpha^{\pm}}$ has some point $x_{\alpha}^{\pm}$ which is farthest from $x = 0$. Set $x^{\alpha} = \max\{x_{\alpha}^{+}, x_{\alpha}^{-}\}$. Then

$$\iint_{\alpha-\text{collar}} |\tan \ell x| (\log |\frac{d\zeta}{dz}|^2)_{yy} \mathrm{d}x\mathrm{d}y \leq |\tan \ell x^{\alpha}| \iint_{N_{\alpha}^{\pm}} |(\log |\frac{d\zeta}{dz}|^2)_{yy}| \mathrm{d}x\mathrm{d}y$$

Each of the $N_{\alpha^{\pm}}$ is contained in a Euclidean tube bounded by $x = x_0^{\pm}$ and $x = x_{\alpha}^{\pm}$. Call these tubes $T_{\alpha^{\pm}}$. Then the integral over the skinny neighborhood $N_{\alpha^{\pm}}$ is less than or equal to that over the skinny tube $T_{\alpha^{\pm}}$ that includes it:

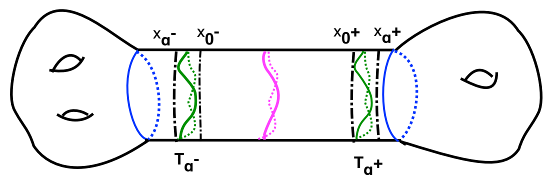

Figure 4.6: The skinny tubes $T_{\alpha^\pm}$ bound the wandering of the ξ=constant curves.

$$\iint_{\alpha-\text{collar}} |\tan \ell x| (\log |\frac{d\zeta}{dz}|^2)_{yy} \mathrm{d}x\mathrm{d}y \leq |\tan \ell x^\alpha| \iint_{T_{\alpha^\pm}} |(\log |\frac{d\zeta}{dz}|^2)_{yy}| \mathrm{d}x\mathrm{d}y \tag{4.43}$$

By Lemma 4.30 in the section, the area between the x-values of $x_0^\pm$ and $x_\alpha^\pm$ is at most $\sqrt{\tilde{A}} e^{\frac{\sqrt{8}\pi}{2L}(x_\alpha^\pm - t_0/2)}$. Substituting into the above equation (4.43),

$$\iint_{\alpha-\text{collar}} |\tan \ell x| (\log |\frac{d\zeta}{dz}|^2)_{yy} \mathrm{d}x\mathrm{d}y \leq |\tan \ell x^\alpha| \sqrt{\tilde{A}} e^{\frac{\sqrt{8}\pi}{2L}(x - t_0/2)} \max_{T_{\alpha^\pm}} |(\log |\frac{d\zeta}{dz}|^2)_{yy}| \tag{4.44}$$

Now $\log |\frac{d\zeta}{dz}|^2$ is harmonic (it's the log of the modulus of a holomorphic function) so in particular its partials are harmonic. Note that as long as z_0 is at least $\frac{L}{2}$ from the boundary of the grafting cylinder, the ball of radius $\frac{L}{2}$ is contained in the grafting cylinder. Using standard properties of harmonic functions,

$$\begin{aligned}
|(\log |\frac{d\zeta}{dz}|^2)_{yy}| &\leq \frac{4}{\pi} \int_{B_{\frac{L}{2}}(z_0)} (\log |\frac{d\zeta}{dz}|^2)_{yy} \mathrm{d}x\mathrm{d}y| \\
&\leq \frac{4}{\sqrt{\pi}} \{ \int_{B_{\frac{L}{2}}(z)} [(\log |\frac{d\zeta}{dz}|^2)_{yy}]^2 \mathrm{d}x\mathrm{d}y \}^{1/2} \\
&< \frac{4}{\sqrt{\pi}} \int_{|x-x_0| \leq \frac{L}{2}} \{ \int_{B_{\frac{L}{2}}(z)} [(\log |\frac{d\zeta}{dz}|^2)_{yy}]^2 \mathrm{d}x\mathrm{d}y \}^{1/2} \\
&= \frac{4}{\sqrt{\pi}} \{ \int_{x_0 - \frac{L}{2}}^{x_0 + \frac{L}{2}} \int_{x=w} [(\log |\frac{d\zeta}{dz}|^2)_{yy}]^2 \}^{1/2}
\end{aligned}$$

using the same trick as before,

$$\frac{4}{\sqrt{\pi}}\{\int_{x_0-\frac{L}{2}}^{x_0+\frac{L}{2}} \tilde{D}\frac{L}{\sqrt{8}\pi}e^{\frac{\sqrt{8}\pi}{L}(|x|-\frac{t_0}{2})}dy\}^{1/2}$$
$$\leq \frac{4}{\sqrt{\pi}}(\frac{\tilde{D}L}{\sqrt{8}\pi})^{1/2}e^{\frac{\sqrt{8}\pi}{2L}(|x_0|-\frac{t_0}{2})}$$

Now substituting back into (4.44), we have for values of ξ that stay at least $\frac{L}{2}$ away from the boundary,

$$|\iint_{\alpha-\text{collar}}|\tan \ell x|(\log|\frac{d\zeta}{dz}|^2)\mathrm{d}x\mathrm{d}y| \leq |\tan \ell x^\alpha|\sqrt{\tilde{A}}e^{\frac{\sqrt{8}\pi}{2L}(x-t_0/2)}\max_{T_{\alpha\pm}}|(\log|\frac{d\zeta}{dz}|^2)_{yy}|$$
$$\leq |\tan \ell x^\alpha|\sqrt{\tilde{A}}e^{\frac{\sqrt{8}\pi}{2L}(|x^\alpha|-t_0/2)}\cdot\frac{4}{\sqrt{\pi}}(\frac{\tilde{D}L}{\sqrt{8}\pi})^{1/2}e^{\frac{\sqrt{8}\pi}{2L}(|x^\alpha|-\frac{t_0}{2})}$$
$$= C|\tan \ell x^\alpha|e^{\frac{\sqrt{8}\pi}{L}(|x^\alpha|-\frac{t_0}{2})}$$

Now asymptotically, $\ell \sim \frac{\pi L}{t_0} + \mathcal{O}(t^{-2})$. Thus

$$e^{\frac{\sqrt{8}\pi}{L}(|x^\alpha|-\frac{t_0}{2})} \sim e^{\frac{\sqrt{8}\pi}{L}(|x^\alpha|-\frac{\pi L}{2\ell})}$$

Our goal is to compare the whole right hand side against $\ell\tan^2 \ell x$, so we need to show that $Ce^{\frac{\sqrt{8}\pi}{L}(|x^\alpha|-\frac{\pi L}{2\ell})}$ is small in comparison to $\ell \tan \ell x$. Using a Taylor series expansion, our desired inequality is:

$$Ce^{\frac{\sqrt{8}\pi}{L}(|x^\alpha|-\frac{\pi L}{2\ell})} \leq \ell^2|x^\alpha| \tag{4.45}$$

But (4.45) is true when

$$C\left(\frac{L(2\ell-\sqrt{8}\pi^2)}{2L\ell^3-2C\ell\sqrt{8}\pi}\right) \leq |x^\alpha|$$

So for small enough ℓ such that there exists x satisfying the above equation, we have the desired result. □

This concludes the estimates that we require.

4.5.3 Summary

To finish the proof, recall, the formula for the derivatives of length is given in (4.36):

$$
\begin{aligned}
& -2\ell \sec \ell \xi^{\alpha^{\pm}} \left.\frac{d}{dt}\right|_{t=0} \ell(t, \alpha^{\pm}) + 4\ell \left.\frac{d}{dt}\right|_{t=0} \ell(t, \gamma_t) \\
& = -2\dot{c}\ell\{\underbrace{\ell \tan^2 \ell \xi^{\alpha^+} + \ell \tan^2 \ell \xi^{\alpha^-}}_{\text{cylinder value -positive}} \\
& + \underbrace{2\ell \int_{\alpha^{\pm}} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \mathrm{d}\eta - 4\ell \int_{\gamma_0} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \mathrm{d}\eta}_{\text{always positive}} \underbrace{-2\ell \tan^2 \ell \xi^{\alpha^{\pm}} \int_{\alpha^{\pm}} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \mathrm{d}\eta}_{\text{negative}} \\
& + \underbrace{|\tan \ell \xi^{\alpha^{\pm}}| \iint_{\text{graft}\cap\text{comp}^{\pm}} \Delta_E \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \, \mathrm{d}x\mathrm{d}y}_{\text{always positive}} + \underbrace{\iint_{\alpha-\text{collar}} |\tan \ell \xi| 2\ell^2 \xi_y^2 \sec^2 \ell \xi \mathrm{d}x\mathrm{d}y}_{\text{positive}} \\
& + \underbrace{\iint_{\alpha-\text{collar}} |\tan \ell \xi| 2\ell \xi_{yy} \tan \ell \xi \mathrm{d}x\mathrm{d}y + \iint_{\alpha-\text{collar}} |\tan \ell \xi| (|\log|\frac{d\zeta}{dz}|^2)_{yy} \, \mathrm{d}x\mathrm{d}y}_{\text{sign unknown}}
\end{aligned}
$$

In the second part of the proof, we dealt with the negative terms and terms of unknown sign. In particular, Lemmas 4.30, 4.32, 4.33 together showed that for ℓ small enough and x close enough to the center of the grafting cylinder, the sum of the negative terms and the terms of unknown sign is strictly less than that of one of the positive terms:

$$
\begin{aligned}
2\ell \tan^2 \ell \xi^{\alpha^+} + \ell \tan^2 \ell \xi^{\alpha^-} &> \frac{3}{2}\{\ell \tan^2 \ell \xi^{\alpha^+} + \ell \tan^2 \ell \xi^{\alpha^-}\} \\
&> | -2\ell \tan^2 \ell \xi^{\alpha^{\pm}} \int_{\alpha^{\pm}} \frac{\xi_y^2}{|\frac{d\zeta}{dz}|^2} \mathrm{d}\eta \\
&+ \iint_{\alpha-\text{collar}} |\tan \ell \xi| 2\ell \xi_{yy} \tan \ell \xi \mathrm{d}x\mathrm{d}y \\
&+ \iint_{\alpha-\text{collar}} |\tan \ell \xi| (|\log|\frac{d\zeta}{dz}|^2)_{yy} \, \mathrm{d}x\mathrm{d}y| \qquad (4.46)
\end{aligned}
$$

This implies the sum of the terms in the parantheses of (4.36) is positive. Hence

$$-2\ell \sec \ell \xi^{\alpha^{\pm}} \left.\frac{d}{dt}\right|_{t=0} \ell(t, \alpha^{\pm}) + 4\ell \left.\frac{d}{dt}\right|_{t=0} \ell(t, \gamma_t) < -2\dot{c}\ell\{\text{positive}\} < 0$$

as desired.

Bibliography

[AFP79] F. Laudenback A. Fathi and V. Poenaru, *Travaux de thurston sur les surfaces*, no. 66-67, Astéreique, 1979. 20

[Ahl61] L.V. Ahlfors, *Some remarks on teichmuller's space of riemann surfaces*, The Annals of Mathematics, Second series **74** (1961), no. 1, 171–191. 14, 34

[Ahl06] ———, *Lectures on quasiconformal mappings*, University Lecture Series, vol. 38, American Mathematical Society, Providence, RI, 2006. 30

[Ber60] L. Bers, *Analytic functions*, ch. Quasiconformal mappings and Teichmüller's theorem, Princeton University Press, 1960. 30

[Bon96] F. Bonahon, *Shearing hyperbolic surfaces, bending pleated surfaces, and thurston's symplectic form*, Annales de la faculté des sciences de Toulouse **5** (1996), no. 2, 233–297. 19

[Bon97] ———, *Geodesic laminations with transverse holder distributions*, Annales scientifiques de l'EnS 4e serie **30** (1997), no. 2, 240. 19

[DK11] R. Diaz and I. Kim, *Asymptotic behavior of grafting rays*, Geometriae Dedicata **DOI 10.1007/s10711-011-9632-x** (2011). 31, 33

[Dum09] D. Dumas, *Handbook of teichmuller theory*, vol. 2, ch. 12, pp. 455–508, European Mathematical Society, 2009. 10, 12, 26

[DW08] D. Dumas and M. Wolf, *Projective structures, grafting, and measured laminations*, Geometry and Topology **12** (2008), no. 1, 351–386. 29, 34

[FG00] N. Lakic F.P. Gardiner, *Quasiconformal teichmüller theory*, Mathematical Surveys and Monographs, vol. 76, American Mathematical Society, 2000. 47

[Gol88] W. M. Goldman, *Geometric structures and varieties of representations*, Proceedings of Amer. Math. Soc. Summer Conference, 1988, pp. 169–198. 8

[Gup11] S. Gupta, *Asymptoticity of grafting and teichmuller rays i*, preprint, September 2011. 31

[Hen11] S. W. Hensel, *Iterated grafting and holonomy lifts of teichmuller space*, Geometriae Dedicata **155** (2011), no. 1, 31–67. 31, 41

[HF92] I.Kra H.M. Farkas, *Riemann surfaces, second edition*, Springer-Verlag, 1992. 10

[HM79] J. Hubbard and H.A. Masur, *Quadratic differentials and foliations*, Acta. Math. (1979), no. 142, 221–274. 26

[Kee74] L. Keen, *Collars on riemann surfaces*, Ann. of Math. Stud. **79** (1974), 263–268. 58

[Kle10] F. Klein, *Theory of functions and geometry*, Lectures on Mathematics [Evanston colloquium] (Providence, RI), American Mathematical Society, AMS Chelsea Publishing, 1910, pp. 33–40. 5

[KT92] Y. Kamishima and S.P. Tan, *Deformation spaces on geometric structures*, Aspects of low-dimensional manifolds (Kinokuniya, Tokyo), vol. 20, Adv. Studies Pure Math, 1992, pp. 263–299. 26

[Leh87] O. Lehto, *Univalent functions and teichmüller spaces*, Springer-Verlag, New York, 1987. 30

[Lip69] M. Lipschutz, *Schaum's ouline of differential geometry*, McGraw-Hill, 1969. 57

[McM98] C.T. McMullen, *Complex earthquakes and teichmuller theory*, Journal of the American Mathematical Society **11** (1998), no. 2, 283–320. 27, 29, 32, 33

[SW02] K. P. Scannell and M. Wolf, *The grafting map of teichmuller space*, Journal of the American Mathematical Society **15** (2002), no. 4, 893–927. 29

[Tan97] H. Tanigawa, *Grafting, harmonic maps, and projective structures on surfaces*, Journal of Differential Geometry **47** (1997), 399–419. 21, 26, 29

[Wol81] S. Wolpert, *An elementary formula for the fenchel-nielsen twist*, Comment. Math. Helvetici (1981), no. 56, 132–135. 34

[YC] K. Rafi Y.E. Choi, D. Dumas, *Grafting rays fellow travel teichmuller geodesics*. 31

[YI92] M. Taniguchi Y. Imayoshi, *An introduction to teichmüller spaces*, Springer-Verlag, 1992. 10