Differential Operators on Curves

by Thomas Bloom

The formalism given by Grothendieck [2, Ch. 16] for differential operators on schemes can be immediately applied to analytic spaces [1, 5]. The object of this note is to describe the germs of differential operators at a singular point on a complex curve at which the curve is irreducible.

§1. Let $X$ be a complex curve irreducible at a singular point $p \in X$. Near $p$, the normalization of $X$ is a 1-1 analytic map $f: \Delta \to X$ of the form $t \mapsto (f_1(t), \ldots, f_r(t))$. Here $\Delta$ is the unit disc in the plane and we assume $f(0) = p$, $X$ is locally embedded in $\mathbb{C}^r$ and $p$ is the origin of $\mathbb{C}^r$. The functions $(f_i(t))_{i=1, \ldots, r}$ each vanish at $0 \in \Delta$, say to order $n_i$.

Furthermore, the subring $R$ of germs at $0 \in \Delta$ of analytic functions which are induced by functions from $X$ is of finite codimension (as a $\mathbb{C}$-vector space) in the ring of germs of analytic functions at $0 \in \Delta$.

We will denote by $N$ the minimal integer with the property that if $g$ vanishes at $0$ to order $\geq N$, then $g \in R$.

Differential operators on $X$ lift to differential operators on $\Delta$ with meromorphic coefficients.

There is, in fact a natural 1-1 correspondence between germs at $p \in X$ of analytic differential operators on $X$ and germs at $0 \in \Delta$ of differential operators with meromorphic coefficients which preserve $R$.

Now, let $D$ be the germ at $0 \in \Delta$ of a differential operator with meromorphic coefficients (henceforth abbreviated m.d.o) and write $D$ in the form

$$D = \sum a_{ij} t^i \frac{d^j}{dt^j} \text{ with } a_{ij} \in \mathbb{C}.$$  

1.1 Definition [3]. The strength of $D$ is defined as $\sup_{a_{ij} \neq 0} (i - j)$ and will be denoted by $\text{str}(D)$. Thus $\text{str}(D)$ is an integer, $\text{str}(D_1 \circ D_2) = \text{str}(D_1) + \text{str}(D_2)$ and $\text{str}(D_1 + D_2) \leq \max(\text{str}(D_1), \text{str}(D_2))$.

1.2 Remarks. If $\text{str}(D) \leq -N$ then $D$ preserves $R$. 

We will say that $D$ is of homogeneous strength if $i - j$ has a fixed value for all non-zero terms in the above expansion for $D$.

§2. To study differential operators at $p \in \mathbb{X}$ we will study the equivalent problem of m.d.o.'s which preserve $R$. We will first consider the case where $f_1(t), \ldots, f_i(t)$ are monomials in $t$. $\mathbb{X}$ is thus weighted homogeneous at $p$.

We let $S = \{m \in \mathbb{Z} \mid m = \Sigma_{i=1}^{n} n_im_i \text{ where } n_i \text{ is as above and the } m_i \text{ are integers } \geq 0\}$. The ring $R$ thus consists of all convergent power series of the form $\Sigma_{s \in S} a_s t^s$ where $a_s \in \mathbb{C}$.

2.1. Remarks. Let $D$ be a m.d.o which preserves $R$. Since $g \in R$ if and only if each monomial in the Taylor series for $g \in R$, each term in the expansion of $D$ as a sum of operators of homogeneous strength preserves $R$.

For $\mu$ an integer we let $Z(\mu, R) = \{\alpha \mid \alpha \text{ is an integer } > 0, t^\alpha \in R, \text{ but } t^{\alpha - \mu} \notin R\}$.

2.2. Lemma. If $\mu \in S$ then $\text{card}(Z(\mu, R)) = \mu - 1$.

If $\mu \notin S$, $\mu > 0$ then $\text{card}(Z(\mu, R)) \geq \mu$.

Proof. Consider the congruence classes of integers mod $\mu$. In each such congruence class $\equiv 0$ there is at least one $\alpha \in Z(\mu, R)$. If $\mu \in S$ there is, by the semi-group property of $S$, precisely one such $\alpha$ in each non-zero congruence class and none in the zero congruence class. Thus if $\mu \in S$, $\text{card}(Z(\mu, R)) = \mu - 1$. If $\mu \notin S$ there is at least one $\alpha \in Z(\mu, R)$ with $\alpha$ in the zero congruence class. Thus if $\mu \notin S$, $\text{card}(Z(\mu, R)) \geq \mu$.

2.3. Lemma. The minimal order of a m.d.o of homogeneous strength $\mu$ and no terms of order zero which preserves $R$ is $\text{card}(Z(\mu, R)) + 1$. Such an operator is unique up to a scalar multiple.

Proof. An m.d.o of order $k$, homogeneous strength $\mu$ and no term of order zero can be written uniquely in the form

$$D = \sum_{i=1}^{k} a_i t^{i-\mu} \frac{d^i}{dt^i} \text{ with } a_i \in \mathbb{C}. $$

Now if $D$ is to preserve $R$ we must have $D(t^\alpha) = 0$ for all monomials $t^\alpha \in R$ for which $t^{\alpha - \mu} \notin R$, that is, for all $\alpha \in Z(\mu, R)$. These conditions impose certain linear relations on the coefficients $a_i$ of the form

$$a_1 \alpha + a_2 \alpha (\alpha - 1) + \cdots + a_k \alpha (\alpha - 1) \cdots (\alpha - (k-1)) = 0.$$ 

If $k = \text{card}(Z(\mu, R))$, the square matrix with rows
Thus, the minimal value of \( k \) which ensures a non-zero solution to the above linear homogeneous equations is \( k = \text{card}(Z(\mu, R)) + 1 \).

2.4 Corollary. No m.d.o preserving \( R \) has a meromorphic leading coefficient when written as a sum of operators of decreasing order.

**Proof.** If \( D \) is an m.d.o of homogeneous strength \( \mu > 0 \) and has a meromorphic leading coefficient, then the order of \( D \) must be \( < \mu \). This contradicts Lemmas 2.3 and 2.4. If \( \mu \leq 0 \), then clearly no coefficient is meromorphic.

§3. We now turn to the case where \( f_1(t), \ldots, f_k(t) \) are not necessarily monomials. We denote by \( \bar{R} \) the ring formed by the initial terms in the Taylor expansion of elements in \( R \). We denote by \( \bar{S} \) the corresponding semi-group.

3.1. Lemma. Let \( D \) be a m.d.o of strength \( \mu \) which preserves \( R \). Then \( D_\mu \), the terms in \( D \) of homogeneous strength \( \mu \), preserves \( \bar{R} \).

**Proof.** For \( g \in R \) with initial term \( \bar{g} \), the initial term of \( D(g) \) is \( D_\mu(\bar{g}) \).

3.2 Theorem. Let \( D \) be a m.d.o which preserves \( R \) and denote its order by \( \text{ord}(D) \). Then \( \text{ord}(D) \geq \text{str}(D) \).

**Proof.** This follows from lemmas 2.2, 2.3, and 3.1.

3.3 Theorem. Let \( P_\mu \) be a m.d.o of homogeneous strength \( \mu \) which preserves \( \bar{R} \). A sufficient condition that there exists a m.d.o \( P' \) of strength \( < \mu \) and order \( \leq m \) such that \( P_\mu + P' \) preserves \( R \) is that \( \text{card}(Z(j, \bar{R})) < m \) for \( j = \mu - 1, \mu - 2, \ldots, -N + 1 \).

**Proof.** As a preliminary step choose a \( C \)-basis for \( R \bmod t^N \) as follows: The basis consists of polynomials \( v_1, \ldots, v_i \) with initial terms \( \bar{v}_1, \ldots, \bar{v}_i \) and such that the monomial \( \bar{v}_i \) does not occur in the expression for \( v_j \) \((j \neq i)\). Given a convergent power series \( h \), let \( H = \text{sum of the monomials in the expansion of } h \) which are in \( R \bmod t^N \).

Say \( H = \sum_{i=1}^{l} c_i \bar{v}_i \). Then \( h \in R \) if and only if \( h = \sum_{i=1}^{l} c_i v_i \bmod t^N \).

We will construct the required operator \( P' \) inductively as follows. Having chosen operators of homogeneous strength \( P_{\mu + 1}, \ldots, P_{r+1} \) choose \( P_r \) so
that it satisfies the following conditions: For each \( \alpha \in \mathbb{Z}(r, \overline{R}) \) take \( g \in R \) with its initial term \( \bar{g} \) a monomial of degree \( \alpha \). Let \( G = \text{sum of the monomials in } (P_{\mu} + \cdots, P_{r+1})(g) \) which are also in \( \overline{R} \mod t^N \) and are of order \( < \alpha - r \).

Say \( G = \sum_{i=1}^{l} d_i \bar{v}_i \). Let \( \lambda t^{\alpha-r} \) be the monomial of order \( \alpha - r \) in \( \sum_{i=1}^{l} d_i \bar{v}_i \) (possibly \( \lambda = 0 \)). \( P_r \) is chosen so that \( P_r(g_1) + P_{r+1}(g_2) + \cdots, P_n(g_{r+1-n}) = \lambda t^{\alpha-r} \) where \( g_1 + g_2 + \cdots \) is the Taylor expansion of \( g \) in terms of order \( \alpha, \alpha + 1, \cdots \) etc. Thus for each \( \alpha \in \mathbb{Z}(r, \overline{R}) \) one linear relation is imposed on the coefficients of \( P_r \). Since \( \text{card} \mathbb{Z}(r, \overline{R}) < N+1 \) for \( \mu > r \geq -N+1 \) by hypothesis, there is an operator of order \( \leq m \) and homogeneous strength \( r \) satisfying the above equations (as in lemma 2.3). The above procedure is repeated until \( r = -N+1 \). Then \( P' = \sum_{i=-n}^{N+1} P_i \) has the required properties.

3.4. Corollary. There is a m.d.o of strength \( m \) and order \( m \) preserving \( R \) if \( m \) is sufficiently large.

Proof. Let \( M = \sup_{N \geq j > -N+1}(\text{card} \mathbb{Z}(j, \overline{R}) + 1) \). For \( j \geq N, j \in \mathbb{S} \); so \( \text{card} \mathbb{Z}(j, \overline{R}) = j - 1 \). Now consider any \( m \geq M \). \( M \geq N \) so \( m \in \mathbb{S} \). Thus, by Lemmas 2.2 and 2.3 there is a m.d.o of order \( m \) and strength \( m \) preserving \( \overline{R} \). Now, \( \text{card} \mathbb{Z}(j, \overline{R}) < m \) for \( j = m - 1, \cdots, -N+1 \) so, applying Theorem 3.3, we obtain the required m.d.o.

Let \( A \) be an analytic ring with unique maximal ideal \( \mathcal{A} \). Under composition, the differential operators on \( A \) form a non-commutative ring, denoted \( \text{Diff}(A) \) which is filtered by order. We denote by \( \text{GrDiff}(A) \) the associated graded ring which is commutative. Given \( D \in \text{Diff}(A) \) we denote by \( \bar{D} \) the homogeneous element it induces in \( \text{GrDiff}(A) \).

3.5. Corollary. Given a differential operator \( D \in \text{Diff}(R) \), there exists an integer \( s \) such that \( \bar{D}^s \in \mathcal{A} \text{GrDiff}(R) \) if and only if \( \text{ord}(D) > \text{str}(D) \).

Proof. Let \( D \) be of strength \( \mu \) and order \( > \mu \). \( D \) may be written in the form \( D = \sum_{k=0}^{\infty} h_k(d^k/dt^k) \) where \( h_k \) is analytic and \( h_k(0) = 0 \). Now \( D^M \) (\( M \) is the integer introduced in Cor. 3.4) is an operator with leading term \( (h_k)^M(d^{km}/dt^{km}) \) and there is, by Corollary 3.4, an m.d.o \( P \) preserving \( R \) of order \( kM \) and strength \( kM \). Now \( h_{kM} \in \mathcal{A} \), so there exists an element \( h \in \mathcal{A} \) such that \( \text{ord}(D^M - hP) < kM \). Thus \( \bar{D}^M = \bar{h}\bar{P} \) in \( \text{GrDiff}(A) \).

Conversely, if \( \bar{D}^s \in \mathcal{A} \text{GrDiff}(A) \), there is an m.d.o \( P \) with \( \text{ord}(P) = s \text{ord}(D) \) such that \( \text{ord}(D^s - hP) < \text{ord}(P) \) for some \( h \in \mathcal{A} \). Say \( D^s = hP + P' \) where \( \text{ord}(P') < \text{ord}(D^s) \). Now \( \text{str}(D^s) \leq \max(\text{str}(hP), \text{str}(P')) \). But \( \text{str}(hP) < \text{str}(P) \leq \text{ord}(P) = \text{ord}(D^s) \).
In either case, \( \text{str}(D^0) < \text{ord}(D^0) \), so
\[
\text{str}(D) < \text{ord}(D).
\]

Thus, we may associate to the ring \( R \) a semi-group \( S' = \{ \mu \in \mathbb{Z} \mid \text{there exists a m.d.o } D \text{ preserving } R \text{ and such that } \text{ord}(D) = \text{str}(D) = \mu \} \). \( S' \) is, of course, intrinsic to the singularity of \( X \) at \( p \) since the above corollary gives an intrinsic characterization of those operators \( D \) such that \( \text{ord}(D) = \text{str}(D) \). In fact, \( S' \) is the semi-group of integers which are the degrees of the homogeneous elements of the \( \mathbb{C} \)-algebra \( \text{GrDiff}(R) / \sqrt{\mathbb{M} \text{GrDiff}(R)} \). Now \( \text{GrDiff}(R) / \sqrt{\mathbb{M} \text{GrDiff}(R)} \) is a one-dimensional Noetherian ring.

The relations in the ring may be described as follows: Let \( a \) and \( b \) be homogeneous elements of degrees \( \alpha, \beta \) respectively. Let \( \gamma = \text{g.c.d.}(\alpha, \beta) \). Then there exists non-zero \( \lambda_1, \lambda_2 \in \mathbb{C} \) such that
\[
\lambda_1 a^{\alpha/\gamma} = \lambda_2 b^{\beta/\gamma}.
\]
and all relations are generated by ones of the above form. Thus the ring is completely characterized by \( S' \).

For the weighted homogeneous curves of section 2, \( S' = S \); but, as example 4.2 shows, \( S' \neq S \) in general.

**3.6 Corollary.** \( \text{GrDiff}(R) \) and \( \text{Diff}(R) \) are finitely generated \( R \)-algebras.

**Proof.** It is a standard algebraic argument that if \( \text{GrDiff}(R) \) is finitely generated then \( \text{Diff}(R) \) is finitely generated as a left or right \( R \)-algebra. We will prove that \( \text{GrDiff}(R) \) is finitely generated, in fact, generated by operators of order \( < 2M \) where \( M \) is the integer introduced in corollary 3.4.

Since \( \text{card}(\mathbb{Z}(j, R)) < M \) for \( j = M, M - 1, \ldots, M - (N - 1) \), there exist (following the procedure of Lemma 2.3) m.d.o's \( P_0, \ldots, P_{N-1} \) preserving \( R \), of order \( M \) and such that \( P_i \) is of homogeneous strength \( M - i \) for \( i = 0, \ldots, N - 1 \). Applying Theorem 3.3 there exist m.d.o's \( Q_0, \ldots, Q_{N-1} \) preserving \( R \), of order \( M \) and such that \( \text{str}(Q_i) = M - i \) for \( i = 0, \ldots, N - 1 \).

Now, let \( T \in \text{Diff}(R) \) be of order \( \geq 2M \) and suppose \( \text{ord}(T) - \text{str}(T) \equiv K \mod N \). Then, take (by 3.4) \( T' \) a m.d.o preserving \( R \) such that \( \text{ord}(T') = \text{str}(T') = \text{ord}(T) - M \). Consider \( Q_K \circ T' \). Now \( \text{ord}(Q_K \circ T') = \text{ord}(T) \) and \( \text{str}(T - Q_K \circ T') \equiv 0 \mod N \). Thus there exists \( h \in R \) such that \( \text{ord}(T - hQ_K \circ T') < \text{ord}(T) \) so that \( hQ_K \circ T' = \bar{T} \) in \( \text{GrDiff}(R) \).

If \( \text{ord}(T') \geq 2M \), we repeat the above procedure. It is clear, thus, that operators of order \( < 2M \) generate \( \text{GrDiff}(R) \) as an \( R \)-algebra.
4. We will illustrate by two examples.

4.1 \( f_1(t) = t^2, f_2(t) = t^3 \). Here \( R = \{ \sum_{i=0}^{n} a_i t^i | a_1 = 0 \} \), \( X = \{ x, y \in \mathbb{C}^2 | x^3 = y^2 \} \).

The operators

\[
D_1 = \frac{d^2}{dt^2} - \frac{2}{i} \frac{d}{dt} \quad \text{and} \quad D_2 = \frac{d^3}{dt^3} - \frac{3}{i} \frac{d^2}{dt^2} + \frac{3}{i^2} \frac{d}{dt}
\]

preserve \( R \). They represent tangent vectors which are a basis for the Zariski tangent space to \( X \) at \((0,0)\).

Since \( \text{ord}(D_1) = \text{str}(D_1) = 2 \), and \( \text{ord}(D_2) = \text{str}(D_2) = 3 \), they generate

\[
\frac{\text{GrDiff}(R)}{\sqrt{\mathcal{M}}}\text{GrDiff}(R).
\]

4.2. \( f_1(t) = t^4, f_2(t) = t^5 + t^6 \). Here \( N = 12 \).

As a special \( \mathbb{C} \) basis of \( R \mod t^{12} \) (as in Lemma 3.3) we have \( v_1 = t^4, v_2 = t^5 + t^6, v_3 = t^8, v_4 = t^9 - 2t^{11}, v_5 = t^{10} + 2t^{11} \). \( \bar{R} \) is the ring associated to \( t^4, t^5 \).

There is no m.d.o preserving \( R \) of strength 4 and order 4. For suppose \( P \) were such a m.d.o and \( P = P_4 + P_3 + \ldots \) its expansion into terms of homogeneous strength. \( P_4 \) would preserve \( \bar{R} \) and would be unique up to a constant multiple. \( P_3 \) must satisfy the equations

\[
P_3(t^4) = 0
\]
\[
P_3(t^5) + P_4(t^6) = 0
\]
\[
tP_4(t^9) = P_3(t^5)
\]
\[
2P_4(t^{11}) + P_5(t^{10}) = 0
\]
\[
tP_4(t^{14}) = 2P_3(t^{14})
\]

These are five linearly independent equations for the coefficients of \( P_3 \), so it must be of order \( \geq 5 \).

In fact, the minimal order operator \( P \) such that \( P \) preserves \( R \) and \( P(t^4) \) is nonzero at zero is of order 5. Thus \( P \) is an operator of minimal order which represents a tangent vector; however, \( \text{ord}(P) > \text{str}(P) \).

4.3 Kantor [4] and Stutz [6] studied differential operators on certain analytic spaces of dimension \( > 1 \). A detailed study of differential operators on the curves \( x^b - y^a = 0 \) was made by Jaffe [3]. With regard to the specific questions discussed in this note we point out that \( A \) is the local
ring at $(0,0,0)$ of analytic functions on the \( \{ x, y, z \in \mathbb{C}^3 \mid xy^2 - z^2 = 0 \} \), then \( \text{GrDiff}(A) \) is not a finitely generated \( A \)-algebra whereas \( \text{Diff}(A) \) is a finitely generated algebra.

Note: It has been brought to the author's notice that similar results to the ones in this note, in particular Corollary 3.6, have been announced by Jean-Pierre Vigué, \textit{C. R. Acad. Sc. Paris}, t. 274 (March 1972), 895–899.

REFERENCES


