A REMARK ON THE PAPER OF M. SCHLESSINGER

by David Mumford

In the conference itself, I spoke on a theorem asserting the existence of "semi-stable" reductions for analytic families of varieties over a disc, smooth outside the origin. This talk turned out to be difficult to transcribe into a paper of moderate size and instead will be incorporated into the notes of a seminar which I am running together with G. Kempf, B. Saint-Donat, and Tai, which we will publish in the Springer Lecture Notes.

Here I would like to add a footnote to Schlessinger's calculations of versal deformations. He studied the situation: $V = \text{complex } n + 1\text{-dimensional vector space; } \mathbf{P}(V) = n\text{-dimensional projective space of } 1\text{-dimensional subspaces of } V; \ Y \subset \mathbf{P}(V) \text{ a smooth } r\text{-dimensional variety, } r \geq 1; C \subset V \text{ the cone over } Y.$

Let $L = \mathcal{O}_Y(1).$ Assume:

$$H^0(\mathbf{P}(V), \mathcal{O}_{\mathbf{P}(V)}(k)) \to H^0(Y, L^k) \text{ is surjective, } k \geq 1$$

(We may also assume by replacing $\mathbf{P}(V)$ by a linear space that it is an isomorphism for $k = 1$). Then he proved:

a) There is a natural injection of functors:

$$\tilde{\mathcal{F}} = \left\{\text{Deformations} \right\}/\text{projective of } Y \text{ in } \mathbf{P}(V) \left/ \text{automorphisms} \right. \to \left\{\text{Deformations} \right\}/\text{of } C$$

b) $T^1_C$ has a natural graded structure

$$T^1_C = \bigoplus_{k=-\infty}^{+\infty} (T^1_C)_k$$

such that $(T^1_C)_0 \cong \text{image of Zariski tangent space to } \tilde{\mathcal{F}},$

c) If $(T^1_C)_k = (0)$ for $k \neq 0$, then $\tilde{\mathcal{F}}$ is isomorphic to the functor of deformations of $C$, i.e., all deformations of $C$ remain conical.

d) If $r \geq 2$ and $L$ is sufficiently ample on $Y$, then the condition in (c) is satisfied.

What I would like to show here is:

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d') If \( r = 1 \), \( L \) is sufficiently ample on \( Y \) and \( Y \) has genus \( \geq 2 \) and is not hyperelliptic, then again the condition in (c) is satisfied. This gives:

**Corollary.** There exist normal singularities of surfaces with no non-singular deformation!

To prove (d'), we let \( U = C - (0) \) and use the exact sequences:

\[
\begin{align*}
\Gamma(V, \theta_{\nu}) & \xrightarrow{\alpha} \Gamma(C, N_C) \xrightarrow{\beta} T^{\dagger}_C \rightarrow 0 \\
\Gamma(V - (0), \theta_{\nu}) & \xrightarrow{\gamma} \Gamma(U, N_C).
\end{align*}
\]

Now \( C^* \) acts in a natural way on both \( \theta_{\nu} \) and \( N_C \), and if \( \pi: V - (0) \rightarrow \mathbb{P}(V) \) is the projection, then both \( \pi_* \theta_{\nu} \) and \( \pi_* N_C \) decompose into direct sums of their eigenspaces for the various characters of \( C^* \). Moreover, the \( C^* \) invariant sections are:

\[
(\pi_* \theta_{\nu})^{C^*} \cong \mathcal{O}_{\mathbb{P}(V)}(1) \otimes_C V
\]

and \( \alpha \) induces the map \( \alpha' = \gamma \circ \beta \)

\[
\alpha': \mathcal{O}_{\mathbb{P}(V)}(1) \otimes_C V \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}(V)} \rightarrow N_Y
\]

(\( \beta \) = standard map).

Thus we get:

\[
\begin{align*}
\Gamma(V - (0), \theta_{\nu}) & \xrightarrow{\alpha} \Gamma(U, N_C) \\
\bigoplus_{v = -\infty}^{+\infty} \Gamma(\mathbb{P}(V), \mathcal{O}(v+1) \otimes_C V) & \xrightarrow{\alpha'_v} \bigoplus_{v = -\infty}^{+\infty} \Gamma(Y, N_Y(v)).
\end{align*}
\]

So if

\[
(T^1_C)_v = \text{coker}[\Gamma(\mathbb{P}^*, \mathcal{O}(v+1) \otimes_C V) \xrightarrow{\alpha'_v} \Gamma(Y, N_Y(v))],
\]

then \( T^1_C = \bigoplus_{v = -\infty}^{+\infty} (T^1_C)_v \). We must compute these groups.

The idea is to determine \( N_Y \) explicitly on \( Y \) without actually using the embedding of \( Y \) defined by \( L \). Consider in fact \( N^*_Y(1) \) via the dual of \( \alpha' \) as a subbundle of \( \mathcal{O}_Y \otimes_C V^* \)

\[
N^*_Y(1) \subset \mathcal{O}_{\mathbb{P}(V)}(1) |_Y \subset \mathcal{O}_Y \otimes_C V^*
\]

hence for every \( x \in Y \):
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\[ \left[ N^*_Y(1) \otimes \mathcal{O}_x/m_x \right] \subset \left[ \theta^*_P(V)(1) \otimes \mathcal{O}_x/m_x \right] \subset V^*. \]

It is easy to see that under these inclusions, if \( x' \in C \) lies over \( x \):

\[
\theta^*_P(V)(1) \otimes \mathcal{O}_x/m_x = \left\{ \text{space of linear forms } \lambda \text{ on } V \right\} \text{ such that } \lambda(x') = 0
\]

\[
N^*_Y(1) \otimes \mathcal{O}_x/m_x = \left\{ \text{space of linear forms } \lambda \text{ on } V \right\} \text{ such that } \lambda(x') = 0 \text{ and } I = 0 \text{ is tangent to } Y \text{ at } x
\]

But now by assumption:

\[
V^* \cong \Gamma(P(V), \mathcal{O}_{P(V)}(1)) \overset{\sim}{\longrightarrow} \Gamma(Y, L)
\]

and under this isomorphism, the linear forms \( \lambda \) such that \( I = 0 \) and is tangent to \( Y \) at \( x \) go over to the sections of \( L \) vanishing at \( x \) to 2nd order, i.e. \( \Gamma(Y, m_x^2 \cdot L) \). Now consider

\[
\Delta \subset Y \times Y \text{ with } p_1^*L(-2\Delta) \\
\downarrow p_2 \\
Y \text{ with } p_2_*[p_1^*L(-2\Delta)].
\]

Then it is easily seen that \( p_2_*[p_1^*L(-2\Delta)] \) is a locally free sheaf on \( Y \) and that

\[
p_2_*[p_1^*L(-2\Delta)] \otimes \mathcal{O}_x/m_x \cong \Gamma(Y \otimes \{y\}, p_1^*L(-2\Delta) \otimes \mathcal{O}_x/m_x)
\]

\[
\cong \Gamma(Y, m_x^2 \cdot L).
\]

Thus the two sub-bundles:

a) \( p_2_*[p_1^*L(-2\Delta)] \subset p_2_*[p_1^*L] = \Gamma(Y, L) \otimes \mathcal{O}_Y \)

b) \( N^*_Y(1) \subset V^* \otimes \mathcal{O}_Y \cong \Gamma(Y, L) \otimes \mathcal{O}_Y \)

are equal. Now assume \( r = 1 \), so that \( Y \) is a curve and \( \mathcal{O}(-2\Delta) \) is an invertible sheaf on \( Y \times Y \). Then by Serre duality for the morphism \( p_2 \), we can identify \( N_Y(-1) \) as a quotient of \( V \otimes \mathcal{O}_Y \) or \( \Gamma(Y, L) \otimes \mathcal{O}_Y \):

\[
\begin{array}{ccc}
V \otimes \mathcal{O}_Y & \xrightarrow{\ast(-1)} & N_Y(-1) \\
\cong & & \cong \\
\Gamma(Y, L) \otimes \mathcal{O}_Y & \longrightarrow & \text{Hom}(p_2, \ast[p_1^*L(-2\Delta)], \mathcal{O}_Y) \\
\cong & & \cong \\
R^1p_2_*\ast(\text{Hom}(p_1^*L, \Omega_Y \otimes \Omega_Y \otimes \Omega_Y)) & \rightarrow & R^1p_2_*\ast(\text{Hom}(p_1^*L(-2\Delta), \Omega_Y \otimes \Omega_Y \otimes \Omega_Y)) \\
\cong & & \cong \\
R^1p_2_*\ast(p_1^*(\Omega_Y \otimes L^{-1})) & \rightarrow & R^1p_2_*\ast(p_1^*(\Omega_Y \otimes L^{-1})(2\Delta)) \\
\end{array}
\]
We want to show that \((T^*_C)_v = (0)\) if \(v \neq 0\), i.e.,

\[
\Gamma(Y, R^1p_{2,*}[p_1^*(\Omega_Y \otimes L^{-1})] \otimes L^v) \to \Gamma(Y, R^1p_{2,*}[p_1^*(\Omega_Y \otimes L^{-1})(2\Delta)] \otimes L^v)
\]
is surjective if \(v \neq 1\). If deg \(L > 2g\), then \(p_{2,*}\) of the two sheaves in square brackets is zero, hence by the Leray spectral sequence for \(p_{2,*}\), the above map is the same as:

\[
H^1(Y \times Y, p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^*L^v)
\]

\[
\to H^1(Y \times Y, p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^*L^v \otimes \mathcal{O}(2\Delta)).
\]

We treat the surjectivity in three cases:

**Case I.** \(v \geq 2\): Consider the sheaf cokernel

\[
p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^*L^v \to p_1^*(\Omega_Y \otimes L^{-1}) \otimes p_2^*L^v \otimes \mathcal{O}(2\Delta) \to K_v \to 0.
\]

It is a sheaf of \(\mathcal{O}_{2\Delta}\)-modules so it lies in an exact sequence between \(\mathcal{O}_\Delta \cong \mathcal{O}_Y\)-modules

\[
0 \to (\mathcal{O}(\Delta) \otimes \mathcal{O}_\Delta) \otimes L_v^{-1} \otimes \Omega_Y \to K_v \to (\mathcal{O}(2\Delta) \otimes \mathcal{O}_\Delta) \otimes L_v^{-1} \otimes \Omega_Y \to 0
\]

\[
\cong L_v^{-1} \otimes (\Omega_Y)^{-1}.
\]

So if deg \(L > 4g - 4\), \(H^1(K_v) = (0)\) when \(v \geq 2\).

**Case II.** \(v = 0\): Consider the Leray spectral sequence for \(p_1\). Since we have assumed \(Y\) is not hyperelliptic

a) \(p_1, \mathcal{O}_{\frac{1}{2}X}(2\Delta) \cong p_1, \mathcal{O}_{\frac{1}{2}X}(2\Delta)\)

and

b) \(R^1p_1, \mathcal{O}_{\frac{1}{2}X}(\Delta)\) is a locally free sheaf \(\mathcal{E}\) of rank \(g - 2\). Now we have:

\[
0 \to H^1(Y, \Omega_Y \otimes L^{-1}) \to H^1(Y \times Y, p_1^*\Omega_Y \otimes L^{-1}) \to H^0(Y, \Omega_Y \otimes L^{-1} \otimes R^1p_1, \mathcal{O}_{\frac{1}{2}X}(2\Delta)) \to 0
\]

\[
\text{by (a)}
\]

\[
0 \to H^1(Y, \Omega_Y \otimes L^{-1} \otimes p_1, \mathcal{O}(2\Delta)) \to H^1(Y \times Y, p_1^*\Omega_Y \otimes L^{-1}(2\Delta)) \to H^0(Y, \Omega_Y \otimes L^{-1} \otimes \mathcal{E}) \to 0
\]

Note that \(\mathcal{E}\) does not depend on \(L\). So by (b) there is an integer \(n_0\) depending only on \(Y\) such that if deg \(L > n_0\), then \((\Omega_Y \otimes \mathcal{E}) \otimes L^{-1}\) has no sections.

**Case III:** \(v \leq -1\): Surjectivity in this case always follows from surjectivity when \(v = 0\). In fact, if we know that

\[
V \to \Gamma(Y, \mathcal{N}_Y \otimes L^{-1}) \to 0
\]

is surjective, I claim \(\Gamma(Y, \mathcal{N}_Y \otimes L^{-1}) = (0)\), \(v \geq 2\). If not, \(\mathcal{N}_Y \otimes L^{-2}\) has
a non-zero section $s$. Then for all $t \in \Gamma(Y, L) \cong V^*$, $t \otimes s$ is a non-zero section of $N_Y \otimes L^{-1}$. Thus we must get all sections of $N_Y \otimes L^{-1}$ in this way. But this means that all these sections are proportional, hence do not generate $N_Y \otimes L^{-1}$. But since

$$V \otimes \theta_Y \to N_Y \otimes L^{-1}$$

is surjective and $V \otimes \theta_Y$ is generated by its sections, so is $N_Y \otimes L^{-1}$. This is a contradiction, so $s = 0$.

This completes the proof of (d'). Finally two remarks:

(A) If you look at the case $Y = \mathbb{P}^1$, $L = \theta_{\mathbb{P}^1}(k)$, then $C$ = cone over the rational curve of degree $n$ in $\mathbb{P}^n$ and the sequences we have used enable us to compute $T^I_c$ easily. In fact it turns out that if $k \geq 3$,

$$(T^I_c)_l = (0), \quad \text{if } l \neq -1$$

$$\dim (T^I_c)_{-1} = 2k - 4.$$  

It seems most reasonable to conjecture that the versal deformation space of this $C$ is a non-singular $k - 1$-dimensional space but with a $0$-dimensional embedded component at the origin if $k \geq 4$.2

(B) What happens in the hyperelliptic case? If, for instance, $\pi: Y \to \mathbb{P}^1$ is the double covering and $L = \pi^* \theta_{\mathbb{P}^1}(k)$, then $C$ is itself a double covering of the rational cone considered in (A) which is known to have non-singular deformations. Do these lift to deformations of this $C$?

NOTES


2. H. Pinkham has recently proved that this is true if $k \geq 5$, but if $k = 4$, the versal deformation space has two components, a smooth 3-dimensional one and a smooth 1-dimensional one crossing normally at the origin! (Cf. "Deformations of cones with negative grading," J. of Algebra, to appear.)