ON BOUNDED SYMMETRIC DOMAINS

by Max Koecher

§1. Let $\mathfrak{V}_0$ be a finite dimensional real vector space and let $\mathfrak{V}$ be its complexification. The vector spaces $\mathfrak{V}_0$ and $\mathfrak{V}$ are equipped with the natural topologies.

Let D be a non empty open subset of $\mathfrak{V}$. The algebra that is given by the vector space of all holomorphic mappings $h: D \to \mathfrak{V}$ together with the product $(h, k) \to [h, k]$ defined by

$$[h,k](z) = \frac{\partial h(z)}{\partial z} k(z) - \frac{\partial k(z)}{\partial z} h(z), \qquad z \in D,$$

will be denoted by $\operatorname{Hol} D$. Then $\operatorname{Hol} D$ is a (infinite dimensional) complex Lie algebra.

Suppose that $f: D \to D'$ is a biholomorphic map where D and D' are non empty open subsets of $\mathfrak{V}$. Then we define a map ∇_f: $\operatorname{Hol} D \to \operatorname{Hol} D'$ by

$$(\nabla_f h)(z) = \left(\frac{\partial f^{-1}(z)}{\partial z}\right)^{-1} h(f^{-1}(z)), \qquad z \in D'.$$

Hence ∇_f is an isomorphism of the Lie algebras.

Denote by $\operatorname{Bih} D$ the group of all biholomorphic mappings of D onto itself. One observes that

$$\nabla: \operatorname{Bih} D \to \operatorname{Aut} \operatorname{Hol} D, \quad f \to \nabla_f,$$

becomes a monomorphism of the groups.

Let $\operatorname{Pol} \mathfrak{V}$ be the vector space of all polynomial mappings $p: \mathfrak{V} \to \mathfrak{V}$. Then $\operatorname{Pol} \mathfrak{V}$ becomes a subalgebra of the Lie algebra $\operatorname{Hol} \mathfrak{V}$ and one can prove that $\operatorname{Pol} \mathfrak{V}$ is a simple Lie algebra.

We are going to state some results about

(i) a one-to-one correspondence between bounded symmetric domains D and a certain class of subalgebras of $\operatorname{Pol} \mathfrak{V}$;

(ii) an explicit description of all bounded symmetric domains by means of some algebraic pairings of real vector spaces.

Furthermore, detailed information about the group Bih D can be obtained. (For the proofs and more details see: M. Koecher, *An elementary approach to bounded symmetric domains*, Lecture notes, Rice University, Houston, Texas, 1969).

§2. Let $\mathfrak{W}$ be a vector space of finite dimension over an arbitrary field. We consider a bilinear map $\Box: \mathfrak{W} \times \mathfrak{W} \to \operatorname{End}\mathfrak{W}$, $(a,b) \to a \Box b$, and denote by $\mathfrak{T}$ the subspace of $\operatorname{End}\mathfrak{W}$ spanned by $a \Box b$ where $a, b \in \mathfrak{W}$. Suppose that

(P.1) the *trace form* σ given by $\sigma(a,b) = \operatorname{trace}(a \Box b + b \Box a)$ is non degenerate.

Then the adjoint of $T \in \operatorname{End}\mathfrak{W}$ with respect to σ is denoted by T^*. Suppose further

(P.2) $$(a \Box b)c = (c \Box b)a,$$

(P.3) $$[T, a \Box b] = Ta \Box b - a \Box T^*b \text{ where } T \in \mathfrak{T},$$

(P.4) $$(a \Box b)^* = b \Box a.$$

A map $\Box: \mathfrak{W} \times \mathfrak{W} \to \operatorname{End}\mathfrak{W}$ satisfying the conditions (P.1) to (P.4) is called a *pairing* of $\mathfrak{W}$. If $\mathfrak{W}$ is a real vector space and if the trace form σ is positive definite then the pairing is said to be *positive definite.*

§3. *Suppose now that* $\Box: \mathfrak{V}_0 \times \mathfrak{V}_0 \to \operatorname{End}\mathfrak{V}_0$ *is a positive definite pairing*. Then $\Box$ can be extended to a pairing of $\mathfrak{V}$ and of $\mathfrak{V}^{\boldsymbol{R}}$ where $\mathfrak{V}^{\boldsymbol{R}}$ means the set $\mathfrak{V}$ considered as a vector space over $\boldsymbol{R}$. The spaces spanned by the pairings are denoted by $\mathfrak{T}_0$, $\mathfrak{T}$ and $\mathfrak{T}^{\boldsymbol{R}}$. Moreover, the map $(a,b) \to \sigma(a, \bar{b})$ becomes hermitian positive definite. For $T \in \operatorname{End}\mathfrak{V}$ we write $T > 0$ whenever $\bar{T}^* = T$ and $\sigma(Ta, \bar{a}) > 0$ for $0 \neq a \in \mathfrak{V}$.

The pairing $\Box: \mathfrak{V} \times \mathfrak{V} \to \operatorname{End}\mathfrak{V}$ induces a subset $\mathfrak{Q}$ of $\operatorname{Pol}\mathfrak{V}$ as follows: We identify the endomorphisms of $\mathfrak{V}$ and the homogeneous linear polynomials of $\operatorname{Pol}\mathfrak{V}$ and we define the homogeneous polynomials p_b of degree two by $p_b(z) = (z \Box b)z$ where $b \in \mathfrak{V}$. Then the subspace

$$\mathfrak{Q} = \mathfrak{Q}_\Box = \mathfrak{V}^{\boldsymbol{R}} + \mathfrak{T}^{\boldsymbol{R}} + \widetilde{\mathfrak{V}}^{\boldsymbol{R}}, \quad \widetilde{\mathfrak{V}}^{\boldsymbol{R}} = \{p_b : b \in \mathfrak{V}^{\boldsymbol{R}}\},$$

becomes a subalgebra of the Lie algebra $(\operatorname{Pol}\mathfrak{V})^{\boldsymbol{R}}$.

Theorem 1. *The real Lie algebra* $\mathfrak{Q}$ *is semisimple.*

Writing the elements of $\mathfrak{Q}$ as $q = a + T + p_b$, $a, b \in \mathfrak{V}^{\boldsymbol{R}}$ and $T \in \mathfrak{T}$ we define a linear transformation Θ of $\mathfrak{Q}$ by

$$\Theta(a + T + p_b) = b - T^* + p_a.$$

A verification shows that Θ is an automorphism of $\mathfrak{Q}$ of period 2. Furthermore define $\bar{q}$ by $\bar{q}(z) = \overline{q(\bar{z})}$ and Θ_0 by

$$\Theta_0(a + T + p_b) = -a + T - p_b.$$

Finally put

$$\Theta_+ q = \Theta \bar{q}, \quad \Theta_- q = \Theta_0 \Theta_+ q, \quad q \in \mathfrak{Q}.$$

By a verification we see that Θ_0 and $\Theta_\pm$ are again automorphisms of $\mathfrak{Q}$ of period 2.

We are interested in the group $\mathrm{Aut}\,\mathfrak{Q}$ and its subgroups

$$\mathrm{Aut}(\mathfrak{Q}, \Theta_\pm) = \{\Phi : \Phi \in \mathrm{Aut}\,\mathfrak{Q},\ \Phi\Theta_\pm = \Theta_\pm \Phi\}.$$

Clearly, these groups are real linear algebraic groups.

Theorem 2. a) $\mathrm{Aut}(\mathfrak{Q}, \Theta_+)$ *and* $\mathrm{Aut}(\mathfrak{Q}, \Theta_-)$ *are semisimple and* $\mathrm{Aut}(\mathfrak{Q}, \Theta_-)$ *is maximal compact in* $\mathrm{Aut}\,\mathfrak{Q}$.

b) *The complexifications of the Lie algebras of* $\mathrm{Aut}(\mathfrak{Q}, \Theta_+)$ *and of* $\mathrm{Aut}(\mathfrak{Q}, \Theta_-)$ *are both isomorphic to* $\mathfrak{Q}$ *(considered as complex Lie algebra).*

§4. For $a, b \in \mathfrak{V}$ we define a linear transformation $B(a,b)$ of $\mathfrak{V}$ by

$$B(a,b)c = c + (a \,\Box\, b)c + \tfrac{1}{4}(a \,\Box\, [(b \,\Box\, c)b])a.$$

An observation yields $\overline{[B(a,b)]^*} = B(\bar{b}, \bar{a})$. Let $Z = Z_\Box$ be the connected component containing 0 of the set of z in $\mathfrak{V}$ such that $B(z, -\bar{z}) > 0$.

Theorem 3. a) *Z is a bounded symmetric domain.*

b) *There exists a subgroup G of* $\mathrm{Bih}\,Z$ *such that $f \to \nabla_f$ becomes an isomorphism of G onto the identity component of* $\mathrm{Aut}(\mathfrak{Q}, \Theta_+)$.

c) *G acts transitively on Z and the index of G in* $\mathrm{Bih}\,Z$ *is finite.*

d) *The subgroup of G that fixes 0 is linear.*

e) *The elements of* $\mathrm{Bih}\,Z$ *are birational.*

The linear transformation $B(a,b)$ can be used for an explicit construction of the elements of $\mathrm{Bih}\,Z$. Furthermore the Bergman kernel κ of Z is given by

$$\kappa(z) = [\det B(z, -\bar{z})]^{-1}.$$

Finally one obtains all bounded symmetric domains by this construction:

Theorem 4. *Let D be a bounded symmetric domain in the complex vector space $\mathfrak{V}$. Then there exists a real form $\mathfrak{V}_0$ of $\mathfrak{V}$ and a positive definite pairing* $\Box : \mathfrak{V}_0 \times \mathfrak{V}_0 \to \mathrm{End}\,\mathfrak{V}_0$ *such that D and $Z_\Box$ are biholomorphically equivalent.*

UNIVERSITY OF MUNICH and RICE UNIVERSITY