SOME REMARKS ON THE AUTOMORPHISM GROUPS OF COMPLEX SPACES

by Wilhelm Kaup

In these notes we want to generalize some known results in connection with the following problem:

(**L**). *Let X be a complex space*[1] *and* $\mathrm{Aut}(X)$ *the group of all complex-analytic automorphisms of X endowed with the compact-open topology. Under what conditions on X is* $\mathrm{Aut}(X)$ *a Lie group?*

1. It has been known for a long time that $\mathrm{Aut}(R)$ is a Lie group of dimension $\leqq 6$ for every connected Riemann surface R. Namely, if $\tilde{R}$ is the universal covering space of R, Γ the corresponding group of deck-transformations and $N(\Gamma) = \{\gamma \in \mathrm{Aut}(\tilde{R}) : \gamma\Gamma = \Gamma\gamma\}$ the normalizer of Γ, then $\mathrm{Aut}(\tilde{R})$ is a Lie group of dimension 6, 4, or 3 (according to whether $\tilde{R}$ is equivalent to the sphere, the complex line $\boldsymbol{C}$, or the unit disc) and $\mathrm{Aut}(R)$ is topologically isomorphic to the Lie group $N(\Gamma)/\Gamma$. Since for every complex space X the group $\mathrm{Aut}(X)$ is in a natural way isomorphic to a closed subgroup of $\mathrm{Aut}(\hat{X})$ ($\hat{X} :=$ normalization of X) we have: $\mathrm{Aut}(X)$ *is a Lie group for every irreducible complex space X of dimension one.*

In higher dimensions this is no longer true. Consider for instance the group $\mathrm{Aut}(\boldsymbol{C}^2)$ which contains all transformations of the form

$$(z, w) \to (z, w + f(z))$$

where f is an arbitrary holomorphic function on $\boldsymbol{C}$. But it is known by a theorem of Arens [*1*] that $\mathrm{Aut}(X)$ is a topological group for every complex space X.

A first general result was obtained in 1935 by H. Cartan [*4*]. He proved by applying Montel's theorem:

(**C**). $\mathrm{Aut}(D)$ *is a Lie group of dimension* $\leqq n(n+2)$ *for every bounded domain* $D \subset \boldsymbol{C}^n$.

This result has been generalized by the author [*10*] to the case that D is an irreducible BK-complete[2] complex space of dimension n. A further generalization is the following:

Theorem 1. *Let X be an irreducible complex space, $K \subset X$ a compact subset, and Y a covering space*[3] *of $X - K$. Then, if Y is BK-complete,* $\mathrm{Aut}(X)$ *is a Lie group.*

Theorem 1 for the special case $Y = X - K$ had been obtained already by Fujimoto [*5*]. Here we want to give a simple proof of Theorem 1 only by using methods from [*10*] and [*11*].

Note that for every Riemann surface R there exists a compact subset $K \subset R$ and a *BK*-complete covering space of $R - K$. The same is true for every compact complex space R. In the compact case however there is known a little bit more: Let X be a complex space and $G \subset \mathrm{Aut}(X)$ a subgroup with the induced topology from $\mathrm{Aut}(X)$. We say, "*G is a complex Lie group in a natural way*" if there exists a complex Lie group structure on G (compatible with the topology on G) such that the mapping $\phi: G \times X \to X$ defined by $\phi(g,x) = g(x)$ is holomorphic. Using holomorphic vector fields it can be seen easily [*9*] that—if $G \subset \mathrm{Aut}(X)$ is a complex Lie group in a natural way—the corresponding Lie group structure is uniquely determined. More generally, it satisfies the following universal property: For every complex Lie group H and every homomorphism $\alpha: H \to G$ the following conditions are equivalent: (i) α is holomorphic, (ii) the mapping $H \times X \to X$ defined by $(h,x) \to \alpha(h)x$ is holomorphic. The following result is due to Bochner and Montgomery [*2*]:

(BM). *For every compact complex manifold M the group* $\mathrm{Aut}(M)$ *is a complex Lie group in a natural way.*

Kerner [*12*] as well as Gunning [*7*] generalized this theorem to compact complex spaces and in [11] a further generalization was given: *Let X be a complex space, $K \subset X$ a compact subset and $G \subset \mathrm{Aut}(X)$ a subgroup with $G(K) = X$. Then the centralizer $Z = \{h \in \mathrm{Aut}(X): hg = gh$ for every $g \in G\}$ of G in* $\mathrm{Aut}(X)$ *is a complex Lie group in a natural way.* Here we present the following generalization of (*BM*):

Theorem 2. *Let M be a compact complex manifold of pure dimension n and $A \subset M$ an analytic subset of dimension $d \leqq n/2 - 1$. Then* $\mathrm{Aut}(M - A)$ *is a complex Lie group in a natural way and the restriction mapping $\{g \in \mathrm{Aut}(M): gA = A\} \to \mathrm{Aut}(M - A)$ is biholomorphic.*

Finally we generalize a result of [*10*] (saying that $\mathrm{Aut}(X)$ is a Lie group for every irreducible *BK*-complete X) to the following:

Theorem 3. *Let X be an irreducible complex space of dimension n and Y a K-complete covering space of X. If there exist n independent*[4] *bounded holomorphic functions on Y, then* $\mathrm{Aut}(X)$ *is a Lie group.*

2. In this part we want to give the proofs to Theorems 1–3. Let X, Y be complex spaces, Y' the one point compactification of Y (as a topological space) and $\mathscr{C}(X, Y')$ the set of all continuous mappings $X \to Y'$ endowed with the compact-open topology. The set $\mathrm{Hol}(X, Y)$ of all holomorphic mappings $X \to Y$ is in a natural way a subspace of $\mathscr{C}(X, Y')$ and we have as a result of [11]: *If Y admits a BK-complete covering space then* $\mathrm{Hol}(X, Y)$ *is relatively compact in* $\mathscr{C}(X, Y')$. We need further the following well-known result of Bochner and Montgomery [3]: *Assume X is an irreducible complex space. Then every locally compact subgroup* $G \subset \mathrm{Aut}(X)$ *is a Lie group.*

Proof of Theorem 1. Since $K \subset X$ is compact there exists a natural number r and for $k=1,\cdots,r$ there exist open subsets $U_k \subset\subset V_k \subset\subset W_k \subset\subset X$ such that the following is true: (i) $K \subset U := \bigcup_{k=1}^{r} U_k$, (ii) every $\bar{W}_k$ has a BK-complete neighborhood. Put $V := \bigcup_{k=1}^{r} V_k$, $L := \{g \in \mathrm{Aut}(X): g(K) \subset U$ and $g(\bar{V}_k) \subset W_k\}$, and $M := L \cap L^{-1}$. Since $\mathrm{Aut}(X)$ is a topological group we can find a symmetric neighborhood N of the identity $1 \in \mathrm{Aut}(X)$ with $NN \subset M$. Assume that (g_n) is a sequence in M. Then for every k the sequence $(g_n | V_k)$ has a convergent subsequence in $\mathrm{Hol}(V_k, X)$ and therefore $(g_n | V)$ has a convergent subsequence in $\mathrm{Hol}(V, X)$. On the other hand, we have $g_n^{-1}(K) \subset U$ for every n (or equivalently $g_n(X - U) \subset X - K$). Therefore the sequence $(g_n | X - U)$ has a convergent subsequence in $\mathscr{C}(X - U, X')$. Since $\mathscr{C}(X, X')$ has countable topology we have finally $M \subset\subset \mathscr{C}(X, X')$. Let us now consider a sequence (h_n) in N. Then (eventually by taking a subsequence) we may assume that there exist elements $f, g, h, f_m \in \mathscr{C}(X, X')$ such that $h = \lim(h_n)$, $g = \lim(h_n^{-1})$, $f_m = \lim_n(h_m^{-1} h_n)$, and $f = \lim(f_m)$. Obviously we may assume that K is not empty and therefore the open set $Q = h^{-1}(X)$ is not empty. For every $x \in Q$ and $m > 0$ we have $f_m(x) = h_m^{-1}(h(x))$ and hence $f(x) = gh(x) = x$. Since X is irreducible it follows $f = 1 \in \mathrm{Aut}(X)$. Now assume that there exists a sequence (x_n) in Q with $y = \lim(x_n) \in X - Q$. Then $h(y) \notin X$ and because of $f_m(x_n) = h_m^{-1}(h(x_n))$ we also have $f_m(y) \notin X$ for every m. But this is a contradiction to $\lim f_m(y) = y$ and it follows $Q = X$. As a limit of holomorphic mappings h is holomorphic, i.e., $h \in \mathrm{Hol}(X, X)$ and $gh = 1 \in \mathrm{Aut}(X)$. In the same way it follows $g \in \mathrm{Hol}(X, X)$ and hence $h \in \mathrm{Aut}(X)$. Therefore $\bar{N} \subset \mathrm{Aut}(X)$ is a compact neighborhood of $1 \in \mathrm{Aut}(X)$, i.e., $\mathrm{Aut}(X)$ is locally compact and hence a Lie group. Q.E.D.

Proof of Theorem 2. Put $Y = M - A$ and $G = \{g \in \mathrm{Aut}(M): gA = A\}$. By (BM) G is a complex Lie group in a natural way and the restriction mapping $\rho: G \to \mathrm{Aut}(Y)$ defined by $\rho(f) = f \mid Y$ is continuous and injective. Fix $g \in \mathrm{Aut}(Y)$ and consider the graph $Q = \{(x,y) \in Y \times Y : y = gx\}$ which is analytic in $Y \times Y$. Since $g: Y \to Y$ is proper, Q is also an analytic subset of $X \times X - A \times A$. Because $\dim(A \times A) \leqq n-2 < \dim(Q)$ we can apply the continuation theorem for analytic sets [*13*]: $S = \bar{Q}$ is an analytic subset of $X \times X$. Let $\pi: S \to X$ be defined by $\pi(x,y) = x$. Then $(S, \pi^{-1}(A), \pi, X, A)$ is a proper modification map in the sense of [*6*] and we have $\dim(\pi^{-1}(A)) \leqq n-2$, by [*6*] (Satz 4, p. 285); it follows that π is biholomorphic, i.e., S is the graph of an automorphism $\tilde{g} \in \mathrm{Aut}(M)$. Therefore ρ is also surjective. Both G and $\mathrm{Aut}(Y)$ are complete topological groups (see [*1*] or [*5*]) with countable topology. By the open mapping theorem (see, e.g., [*8*]) ρ is an isomorphism of topological groups. Now take the complex lie group structure on $\mathrm{Aut}(Y)$ that makes ρ biholomorphic. For this complex structure the canonical mapping $\phi: \mathrm{Aut}(Y) \times Y \to Y$ is holomorphic. Q.E.D.

It should be noted that $d \leqq n/2 - 1$ probably is not the best possible condition in Theorem 2. By using the same continuation argument for the graph as in the proof to Theorem 2 it is, for instance, easy to see: *For every compact Riemann surface R and every finite subset* $F \subset R$ *the group* $\mathrm{Aut}(R - F)$ *is a complex Lie group.*

Assume that X is a complex space, $A \subset X$ a nowhere dense analytic subset and $\mathscr{H}(X) = \mathrm{Hol}(X, \boldsymbol{C})$, the space of holomorphic functions on X. Then—as is well known—the restriction mapping $\rho: \mathscr{H}(X) \to \mathscr{H}(X - A)$ is a homeomorphism onto a closed subspace of $\mathscr{H}(X - A)$. The proof can be given along the following lines: Let us first assume that X is a domain in $\boldsymbol{C}^n$. Denote by S the set of singular points of A and consider the restriction mappings $\mathscr{H}(X) \overset{\rho''}{\to} \mathscr{H}(X - S) \overset{\rho'}{\to} \mathscr{H}(X - A)$. Then—using induction on $\dim(A)$—it is enough to show the statement for ρ' (since $\dim(S) < \dim(A)$). Because the problem is local in X we may therefore assume that X is the unit ball in $\boldsymbol{C}^n$ and that $A \subset \{(z_1, \cdots, z_n) \in \boldsymbol{C}^n : z_1 = 0\}$. But then every sequence in $\mathscr{H}(X)$ which is bounded in $X - A$ is also bounded in X. To prove the general case we may assume that X has pure dimension n and that there exist a domain $D \subset \boldsymbol{C}^n$ and a proper discrete holomorphic mapping $\tau: X \to D$. Then $B = \tau(A)$ is analytic in D and for every $f_m \in \mathscr{H}(X)$ there exists a polynomial $P_m \in \mathscr{H}(D)[\delta]$ with leading coefficient 1 and of minimal degree such that $P_m(f_m) = 0 \in \mathscr{H}(X)$. Furthermore if $(f_m \mid X - A)$ is a convergent sequence in $\mathscr{H}(X - A)$ then $(P_m \mid D - B)$ is convergent in $\mathscr{H}(D - B)[\delta]$ and hence (P_m) is a convergent sequence in $\mathscr{H}(D)[\delta]$. But then also (f_m) is a convergent sequence in $\mathscr{H}(X)$.

Here we need the following generalization of the above statement:

Lemma. *Let X, Y be complex spaces and $A \subset X$ a nowhere dense analytic subset. Assume that Y has a K-complete covering space. Then the restriction mapping $\rho: \mathrm{Hol}(X, Y) \to \mathrm{Hol}(X - A, Y)$ is a homeomorphism onto a (i.e., not closed) subspace of* $\mathrm{Hol}(X - A, Y)$.

Proof. We may assume that X has countable topology and that Y is K-complete. Then consider a sequence (f_n) in $\mathrm{Hol}(X, Y)$ with $\lim (f_n \mid X - A) = f \mid X - A$ for some $f \in \mathrm{Hol}(X, Y)$. For every $a \in A$ there exist a BK-complete open neighborhood U of $f(a) \in Y$, a holomorphic mapping $\phi: Y \to \boldsymbol{C}^q$ for some q, and a neighborhood W of $\phi f(a) \in \boldsymbol{C}^q$ such that $\bar{U} \cap \phi^{-1}(W) \subset U$. But $\lim(\phi f_n) = \phi f$, i.e., for some connected neighborhood V of a and $n \geqq n_0$ it follows $\phi f_n(V) \subset W$ and $f_n(V) \cap U \neq \varnothing$, i.e., $f_n(V) \subset U$. But then we have $f = \lim(f_n)$.

Note that this lemma is not true for arbitrary Y. Consider for instance $\boldsymbol{C}^n$ in the standard way as the affine part of the projective space $\boldsymbol{P}_n$ and define $X = \boldsymbol{C}^n$, $A =$ the origin of $\boldsymbol{C}^n$, $Y = \boldsymbol{P}_n$, and $f_m \in \mathrm{Hol}(X, Y)$ by $f_m(z) = mz$ for every m.

Proof of Theorem 3. We may assume $X = Y$. Denote by B the set of all bounded holomorphic functions on X and put $M = \{x \in X : x$ is an isolated point of $\bigcap_{f \in B} f^{-1}f(x)\}$. Then $A = X - M$ is a nowhere dense analytic subset of X and $gM = M$ for every $g \in \mathrm{Aut}(X)$. M is BK-complete and therefore $\mathrm{Aut}(M)$ is a Lie group. By the lemma the restriction mapping $\rho: \mathrm{Aut}(X) \to \mathrm{Aut}(M)$ is a homeomorphism onto a subgroup of $\mathrm{Aut}(M)$. But both of the groups are complete, i.e., $\mathrm{Aut}(X)$ is isomorphic to a closed subgroup of $\mathrm{Aut}(M)$ and hence is a Lie group.

NOTES

1. Complex spaces are in the sense of Serre (= reduced complex spaces).
2. A complex space X is called BK-complete if there exists to every non discrete subset $N \subset X$ a bounded holomorphic function on X which is not constant on N.
3. By covering space we mean an unramified unlimited covering space.
4. Holomorphic functions $f_1, \cdots, f_r$ on a complex space X are called independent if at some point $x \in X$ the differentials $df_1, \cdots, df_r$ are linearly independent (or equivalently, if for some $y \in X$ the analytic set $\bigcap_{k=1}^{r} f_k^{-1} f_k(y)$ has codimension r in y).

REFERENCES

[1] Arens, R., Topologies for homeomorphism groups, Amer. J. Math. **68** (1946), 593–610.

[2] Bochner, S. and D. Montgomery, Groups on analytic manifolds, Ann. of Math. **48** (1947), 659–669.

[3] ———, Locally compact groups of differentiable transformations, Ann. of Math. **47** (1946), 639–653.

[4] CARTAN, H., Sur les groupes de transformations analytiques, Actualités Sci. Ind. **198** (1935).

[5] FUJIMOTO, H., On the holomorphic automorphism groups of complex spaces, Nagoya Math. J. **33** (1968), 85–106.

[6] GRAUERT, H. and R. REMMERT, Zur Theorie der Modifikationen I: Stetige und eigentliche Modifikationen komplexer Räume, Math. Ann. **129** (1955), 274–296.

[7] GUNNING, R. C., On Vitali's theorem for complex spaces with singularities, J. Math. Mech. **8** (1959), 133–141.

[8] HUSAIN, T., Introduction to Topological Groups, Philadelphia and London (1966).

[9] KAUP, W., Infinitesimale Transformationsgruppen komplexer Räume, Math. Ann. **160** (1965), 72–92.

[10] ———, Reelle Transformationsgruppen und invariante Metriken auf komplexen Räumen, Inventiones Math. **3** (1967), 43–70.

[11] ———, Hyperbolische komplexe Räume, Ann. Inst. Fourier **18** (1968), 303–330.

[12] KERNER, H., Über die Automorphismengruppen kompakter komplexer Räume, Arch. Math. **11** (1960), 282–288.

[13] REMMERT, R. and K. STEIN, Über die wesentlichen Singularitäten analytischer Mengen, Math. Ann. **126** (1953), 263–306.

UNIVERSITY OF NOTRE DAME and UNIVERSITY OF MÜNSTER