ALGEBRAIC CYCLES AND HOLOMORPHIC CONVEXITY*

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1. *Recollection of Known Facts*

Let X be a complex manifold, $D \subset\subset X$ a relatively compact open subset of X with smooth boundary. We actually will assume

$$D = \{x \in X \mid \phi(x) < 0\}$$

where ϕ is a C^∞ function on X and $d\phi \neq 0$ on $\partial D = \bar{D} - D$.

Let $\mathscr{H}(D) = H^0(D, \mathscr{O})$ the space of holomorphic functions on D and let $z_0 \in \partial D$. We will say that $f \in \mathscr{H}(D)$ is (holomorphically) *extendible over* z_0 if there exists a neighborhood $V(z_0)$ of z_0 and $\hat{f} \in \mathscr{H}(D \cup V(z_0))$ with $\hat{f}\,|_D = f$.

The domain D is called a domain of holomorphy (a good domain of holomorphy) if for each $z_0 \in \partial D\, \exists f \in \mathscr{H}(D)$ not extendible over z_0 (and $\mathscr{H}(D)$ separate points).

The Theorem of E. E. Levi asserts that if D is a domain of holomorphy then the hermitian form

$$(*)\quad \mathscr{L}(\phi)_{T_{z_0}} = \begin{cases} \mathscr{L}(\phi) \equiv \sum \left(\dfrac{\partial^2 \phi}{\partial z_\alpha \partial \bar{z}_\beta}\right)_{z_0} u^\alpha \bar{u}^\beta \\ \text{restricted to} \\ \sum \left(\dfrac{\partial \phi}{\partial z_0}\right)_{z_0} u^\alpha = 0 \end{cases} \qquad (z_\alpha \textit{ local coordinates at } z_0)$$

has no negative eigenvalue.

The Levi problem asks if condition (*) is sufficient to ensure that D is a domain of holomorphy.

An example of Grauert shows that this is not the case unless $\mathscr{L}(\phi)\big|_{T_{z_0}}$ is non degenerate. He has proved

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Theorem of Grauert [4]. *If D satisfies* (*) *and* $\mathscr{L}(\phi)\big|_{T_{z_0}}$ *is non degenerate then D is a domain of holomorphy.*

(Moreover if ϕ can be chosen so that $\mathscr{L}(\phi) > 0$ on all of the D then D is a good domain of holomorphy.)

2. *Extension to Higher Cohomology Groups*

Let $\mathscr{F}$ be a locally free sheaf on X, for example, $\mathscr{F} = \mathcal{O}$ or $\mathscr{F} = \Omega^d$ the sheaf of holomorphic d-forms.

We will call D a d-domain of holomorphy (for $\mathscr{F}$) if $\forall z_0 \in \partial D$ $\exists \xi \in H^d(D,\mathscr{F})$ which is not extendible over z_0.; (The notion of extendibility being as that for H^o.)

Levi Theorem [1]. *If D is a d-domain of holomorphy (for a locally free sheaf $\mathscr{F}$) then* $\forall z_0 \in \partial D$

$$(^*)_d \quad \begin{array}{ll} \text{in } \mathscr{L}(\phi)\big|_{T_{z_0}} & \textit{number of negative eigenvalues } \leqq d \\ \text{in } \mathscr{L}(\phi)\big|_{T_{z_0}} & \textit{number of positive eigenvalues } \leqq n-d-1. \end{array}$$

One can consider the d-Levi-problem and as before the Grauert example shows that for it to be solvable we have to assume that $\forall z_0 \in \partial D$ $\mathscr{L}(\phi)\big|_{T_{z_0}}$ is non degenerate. Actually then one can prove the following:

Theorem [1]. *If D satisfies* $(^*)_d$ *and* $\mathscr{L}(\phi)\big|_{T_{z_0}}$ *is non degenerate* $\forall z_0 \in D$ *then D is a d-domain of holomorphy for any locally free sheaf.*

A domain D satisfying the hypothesis of the theorem will be called *strictly d-convex.*

If ϕ can be chosen so that $\mathscr{L}(\phi)$ has at least n-d positive eigenvalues on D then D will be called *strictly d-complete.*

Remarks. We cannot talk about good d-domains of holomorphy since a cohomology class cannot be evaluated at a point.

It is this gap that has somehow motivated the considerations of the next section.

3. *Holomorphic Convexity and Cycles*

(a) We recall that D is called *holomorphically convex* if for any divergent sequence $\{x_\nu\} \subset D$ we can find $f \in \mathscr{H}(D)$ with

$$\sup\left|f(x_\nu)\right| = \infty.$$

Moreover if $\mathscr{H}(D)$ separate points on D then D is called *holomorphically complete.*

Let us consider the following space

$$C_0^+(D) = \{ \sum n_i p_i \mid n_i \text{ integer } \geqq 0, \text{ almost all } n_i = 0, \ p_i \in D\}$$
$$= D \cup D^{(2)} \cup D^{(3)} \cup \cdots$$

where the union is disjoint union and $D^{(s)} = s$-fold symmetric product. The space $C_0^+(D)$ is an analytic space and we have a natural linear map:

$$\rho_0 : H^o(D, \mathcal{O}) \to H^o(C_0^+(D), \mathcal{O})$$
$$\rho_0(f)(\sum n_i p_i) = \sum n_i f(p_i).$$

Note that holomorphic convexity (completeness) of D implies holomorphic convexity (completeness) of $C_0^+(D)$ and these properties are already exhibited by $\operatorname{im} \rho_0$.

(b) Let us assume that X is a projective algebraic manifold. Let

$$C_d^+(D) = \{ \sum n_i A_i \mid n_i \text{ integers} \geqq 0, \text{ almost all } 0, A_i \text{ irreducible compact analytic subset of } D, \ \dim_{\boldsymbol{C}} A_i = d\}.$$

It is known that $C_d^+(D)$ has a natural weakly normal structure of a locally algebraic variety. (Weakly normal means that continuous functions holomorphic at non singular points are holomorphic.)

Example 1. $D = X = P_m(\boldsymbol{C})$, $d = n-1$ then

$$C_{n-1}^+(P_n(\boldsymbol{C})) = P_n(\boldsymbol{C}) \cup P_{\binom{n+2}{2}-1}(\boldsymbol{C}) \cup \cdots$$

Example 2. $D = P_n(\boldsymbol{C}) - \text{pt.}$, $d = n-1$ then

$$C_{n-1}^+(D) = \boldsymbol{C}^n \cup \boldsymbol{C}^{\binom{n+2}{2}-1} \cup \cdots$$

Let us take as analogue of $H^o(D, \mathcal{O})$ the space $H^d(D, \Omega^d)$. Given $\xi \in H^d(D, \Omega^d)$, this is represented by a C^∞ form ϕ^{dd} of type (d, d), $\bar{\partial}\phi^{dd} = 0$ given up to the addition of a $\bar{\partial}$ of a C^∞ form μ of type $(d, d-1)$.

By a theorem of Lelong $\forall c \in C_d^+(D)$, $\int_c \phi^{dd}$ exists and moreover $\int_c \bar{\partial}\mu^{dd-1} = 0$, thus

$$\xi(c) = \int_c \phi^{dd}$$

is a well-defined function on $C_d^+(D)$.

Theorem [2]. 1) *The function* $\xi(c)$ *is holomorphic so that we get a linear map*

$$\rho_0 : H^d(D,\Omega^d) \to H^0(C_d^+(D),\mathcal{O}).$$

2) *If D is strictly d-convex then for any divergent sequence $\{c_\nu\}$ in $C_d^+(D)$ we can find $\xi \in H^d(D,\Omega^d)$ such that*

$$\sup|\xi(c_\nu)| = \infty.$$

3) *If D is strictly d-complete then given $c_1, c_2 \in C_d^+(D)$, $c_1 \neq c_2$ exists $\xi \in H^d(D,\Omega^d)$ with*

$$\xi(c_1) \neq \xi(c_2).$$

This theorem tells us therefore that if D is strictly d-convex then $C_d^+(D)$ is holomorphically convex and if moreover D is d-complete then $C_d^+(D)$ is holomorphically complete.

(c) These results can be extended to manifolds X without boundary when these are union of increasing sequences of strictly d-convex (d-complete) domains which satisfy a certain Runge condition.

In particular if X is obtained by removing from a compact projective algebraic manifold Z a non singular subvariety Y of codimension $d+1$ which is a complete intersection, then X has the properties of strictly d-complete domains.

The map

$$\rho_0 : H^d(X,\Omega^d) \to H^o(C_d^+(X),\mathcal{O})$$

already produces infinitely many holomorphic functions on $C_d^+(X)$; but in general it is not injective.

To describe the kernel of ρ_0 it turns out necessary to discard as negligible all finite dimensional vector spaces (which is a "class" Φ of abelian groups in the sense of Serre [5]). Then one can prove the following:

Theorem [3]. *For X of the type described above, the sequence:*

$$H^d(X,\Omega^{d-1}) \overset{d}{\to} H^d(X,\Omega^d) \overset{\rho_0}{\to} H^o(C_d^+(X),\mathcal{O})$$

is Φ-exact (i.e., $\operatorname{im} d \cap \ker \rho_0$ *is of finite codimension in* $\ker\rho_0$ *and* $\operatorname{im} d$).

The above theorem clearly suggests that we have started our considerations with the wrong type of groups. The most appropriate appear to be the Aeppli group

$$V^{dd}(X) = \frac{\operatorname{Ker}\{A^{dd}(X) \overset{\partial\bar\partial}{\to} A^{d+1\,d+1}(X)\}}{\bar\partial A^{d\,d-1}(X) + \partial A^{d-1\,d}(X)}$$

($A^{rs}(X) = C^\infty(r,s)$ forms on X). Indeed then the integration gives a map

$$\hat{\rho}_0 : V^{dd}(X) \to H^o(C_d^+(X), \mathscr{H}),$$

$\mathscr{H}$ being the sheaf of germs of pluriharmonic (complex valued) functions. Then for a X as above $\hat{\rho}_0$ is a Φ-injective map [3].

REFERENCES

[1] ANDREOTTI, A. AND F. NORGUET, Problème de Levi et convexité holomorphe pour les classes de cohomologie, Ann. Scuola Nor. Sup. Pisa **20** (1966), 197–241.

[2] ———, La convexité holomorphe dans l'espace des cycles d'une variété algébrique, Ann. Scuola Nor. Sup. Pisa **21** (1967), 31–82.

[3] ———, Cycles of algebraic manifolds and Aeppli cohomology, Ann. Scuola Nor. Sup. Pisa (to appear).

[4] GRAUERT, H., On Levi's problem and imbeddings of real-analytic manifolds, Ann. of Math. **68** (1958), 450–472.

[5] SERRE, J.-P., Groupes d'homotopie et classes de groupes abeliens, Ann. of Math. **58** (1953), 258–294.

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