This note is a brief exposition of recent results. No proofs are offered, except for some very simple examples.

Let $M$ be a compact Hausdorff space and $C(M)$ the algebra of all continuous complex valued functions on $M$. $C(M)$ is a Banach algebra under the uniform norm

$$\| f \| = \sup_{p \in M} |f(p)| .$$

A function algebra on $M$ is a closed subalgebra $A$ of $C(M)$ containing the identity and separating the points of $M$. The latter condition means that if $p$ and $q$ are distinct points of $M$ there exists a function $f$ in $A$ such that $f(p) \neq f(q)$.  

A classical example is obtained when $M$ is a compact set in the plane and $A$ is the function algebra generated by $z$, the usual complex coordinate. Here the description of $A$ is the problem of uniform approximation on $M$ by polynomials. If $M$ does not separate the plane, then Mergelyan [5] has shown that $A$ is the algebra of all continuous functions on $M$ which are holomorphic at interior points. If we replace $z$ by a more general continuous function $f$ on $M$, then $f$ must be injective in order for the algebra $A$ it generates to separate the points of $M$. Thus $f$ is a homeomorphism, and it transports the new problem back to the previous one with $M$ replaced by $f(M)$. Since $f(M)$ does not separate the plane if this is true of $M$, $f$ induces an isometric isomorphism of $A$ with the algebra of continuous functions on $f(M)$ which are holomorphic at interior points.

It should be mentioned that $M$ does not separate the plane if and only if the maximal ideal space $M_A$ of the Banach algebra $A$ contains no other complex homomorphisms than those arising from evaluation at points of $M$. When this occurs for a function algebra $A$ we say $M_A = M$.

In fact, the condition $M_A = M$ is usually assumed for the results below, and we should review some basic facts about $M_A$. We shall discuss algebras $A$ generated by a set $F$ of continuous complex valued functions separating...
the points of $M$. Thus $A$ consists of all uniform limits of polynomials in the functions of $F$. In case $F = \{f_1, \ldots, f_n\}$ is finite, the functions $f_j$ are the coordinates of a homeomorphism also denoted $F: M \to \mathbb{C}^n$. By means of this map the description of $A$ becomes equivalent to the problem of characterizing those continuous functions on $F(M)$ which can be uniformly approximated by polynomials in $z_1, \ldots, z_n$, the usual coordinates in $\mathbb{C}^n$. The condition that $M_A = M$ is equivalent \cite{7} to requiring that $F(M)$ be \textit{polynomially convex}. A compact set $K$ in $\mathbb{C}^n$ enjoys this property if for each point $z \notin K$ there exists a polynomial $p$ in $n$ variables such that

$$|p(z)| > \sup_{w \in K} |p(w)|.$$  

An application of the maximum principle shows that when $n = 1$ this reduces to Mergelyan's condition that $\mathbb{C} - K$ be connected. However, a useful characterization of this property is presently unavailable in higher dimensions. In general, $M_A$ can be identified \cite{7} with the smallest polynomially convex set containing $F(M)$, called its polynomial hull.

Returning to compact sets $M$ in the plane, let us consider the algebra $A$ generated by a set $F = \{f, g\}$ of two functions. We suppose that $M$ is a closed disk centered at 0. This problem was also treated by Mergelyan \cite{5}, who found as a consequence of his result above that if $g = z$ and $f$ is a real valued function none of whose level sets separate the plane, then $A$ consists of all continuous functions on $M$ which are holomorphic at the interior points of each level set of $f$. In particular, if no level set has interior points, then $A = C(M)$.

A different tack was taken by Wermer \cite{8} who considered $z$ and a continuously differentiable complex valued function $f$. He showed that the set

$$E = \{\zeta \in M: \overline{\partial}f(\zeta) = 0\}$$

plays an important role in the structure of $A$ ($\overline{\partial}f(\zeta) = 0$ means that $f$ satisfies the Cauchy-Riemann equations at $\zeta$). In fact, under the hypothesis that $M_A = M$, he found that $A$ is the set of all continuous functions $f$ on $M$ whose restriction $\big|E$ to $E$ is in the uniform closure $R(E)$ on $E$ of the rational functions with poles outside $E$. Since much is known about the algebra $R(E)$ this amounts to a very explicit description of $A$. For example, if $E$ has plane measure zero, then a theorem of Hartogs and Rosenthal \cite{3} yields $R(E) = C(E)$, so that $A = C(M)$. If $E$ has only finitely many complementary components in the plane, then $R(E)$ is all continuous functions on $E$ which are holomorphic at interior points \cite{5}. This yields a corresponding description of $A$. 

Wermer's assumption that $M_A = M$ can be shown to be necessary, since the maximal ideal space of $R(E)$ is always $E$.

We consider a simple example in which $f(z) = |z|^2$. Here $E = \{0\}$ so that Wermer's conclusion would amount to $A = C(M)$. But $f$ is constant on each circle $C$ concentric with the origin, so each function in $A$ is the uniform limit on $C$ of polynomials in $z$. This property is clearly not possessed by every continuous function, so we do not have $A = C(M)$. In this case $F(M) = \{(z, w): w = |z|^2 \leq 1\}$, which is not polynomially convex since at each point of the form $(0, r^2)$ with $0 < r \leq 1$ the maximum principle applied to a polynomial $p$ considered as a function of the first variable shows that

$$|p(0, r^2)| \leq \sup_{|z| = r} |p(z, r^2)|.$$

In fact, the polynomial hull of $F(M)$ is just its ordinary convex hull.

Wermer's conclusion is easily expressed as the conjunction of the two statements

(1) $A$ contains the ideal of continuous functions which vanish on $E$;
(2) $R(E) = \{f \mid E : f \in A\}$.

His view suggests the consideration of two continuously differentiable functions $f$ and $g$, and the set

$$E = \{\xi \in M : df \wedge dg(\xi) = 0\}.$$

This definition of $E$ reduces to the one above if $g = z$, since $\overline{\partial}f = 0$ if and only if $df$ is a multiple of $dz$.

In this more general situation statement (1) makes sense and is true [1]. However it is necessary to replace statement (2) by a description involving $f$ and $g$. A possible one is suggested below. Of course, if either $f$ or $g$ is a diffeomorphism the problem is reduced to Wermer's case in an obvious manner. On the other hand, the absence of such a global coordinate causes severe difficulties.

But let us be bold and generalize considerably, to the case where $M$ is a compact subset of a real continuously differentiable manifold of dimension $n$, $F$ a set of continuously differentiable complex valued functions on $M$, and $A$ the function algebra generated by $F$. Here we set

$$E = \{p \in M : df_1 \wedge \cdots \wedge df_n(p) = 0 \text{ for all } n\text{-tuples } f_1, \ldots, f_n \text{ of functions in } F\}.$$

Then statement (1) is true if $M_A = M$ and
(a) \( M \) is contained in a sufficiently differentiable manifold of arbitrary dimension embedded in \( \mathbb{C}^n \), and \( E = \emptyset \) (so that \( A = C(M) \)) [6], and

(b) \( M \) is contained in a real-analytic manifold of arbitrary dimension, and \( F \) is an arbitrary set of real-analytic functions [2].

As an example where statement (1) can easily be verified, let \( M = \{(x, y, t) : x^2 + y^2 + t^2 \leq 1\} \), \( f(x, y, t) = x + iy = z \), \( g(x, y, t) = t \overline{z} \), and \( h(x, y, t) = t \). If \( F = \{f, g, h\} \), it is easily seen that \( F(M) \) is polynolmially convex in \( \mathbb{C}^3 \), and also that \( E = \{(x, y, t) : t = 0\} \). \( A \) contains the set of functions \( fh, g, \) and \( h \), which is closed under complex conjugation, separates the points of \( M - E \), and has no common zero there. An application of the Stone-Weierstrass theorem shows that \( A \) verifies statement (1).

To obtain a generalization of Wermer’s conclusions, let us consider the set \( H(F) \) of all continuously differentiable functions \( f \) for which at each point \( p \) in \( M \) there exist \( f_1, \ldots, f_n \) in \( F \) and complex numbers \( \lambda_1, \ldots, \lambda_n \) such that

\[
df = \lambda_1 df_1 + \cdots + \lambda_n df_n,
\]

where \( df \) denotes the differential of \( f \) at \( p \). In Wermer’s example and the one above, \( H(F) \) consists of all functions which satisfy the Cauchy-Riemann equations on \( E \). When \( F \) is finite it transports \( H(F) \) over onto the set of continuously differentiable functions on \( F(M) \) which satisfy the induced or “tangential” Cauchy-Riemann equations on \( F(M) \).

It is clear that \( A \) is contained in the uniform closure \( \overline{H(F)} \) of \( H(F) \), since any polynomial in the functions of \( F \) is in \( H(F) \). A standard conjecture is that if \( \overline{M}_A = M \), then \( A \) equals \( \overline{H(F)} \). This is verified in Wermer’s case [8] and in the three-dimensional example above. It is also true if \( M \) happens to be a polynomially convex real submanifold of \( \mathbb{C}^n \) which is a Reinhardt set, meaning that together with each of its points \( z \), \( M \) contains the orbit of \( z \) under the natural action of the \( n \)-torus on \( \mathbb{C}^n \). It will be observed that these are very sharp restrictions on \( M \), but the result is not completely trivial even for this case.

The proof, as pointed out by H. Rossi, is a straightforward extension to higher dimensions of a method which can be used for \( n = 1 \), when \( M \) is a closed disk. Stokes’s theorem is used to show that the multivariate Fourier coefficients of a function \( f \) in \( H(F) \) on different tori are related as if \( f \) were holomorphic in a neighborhood of \( M \), and to show that these coefficients vanish at any multi-index which has a negative integer in it. These properties mean that there is a power series in \( z_1, \ldots, z_n \) whose restriction to any torus \( T \) in \( M \) is the Fourier series for \( f \) on \( T \). The uniform continuity of \( f \) on \( M \)
is used to show that the Cesaro means of this power series converge uniformly to $f$ on $M$. This type of argument was used in $C^a$ originally by K. de Leeuw [4].

REFERENCES


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