

# MATHEMATICAL MODELS OF PHYSICAL SYSTEMS

*by Kenneth J. Cox*  
*Introduction*

This paper will examine the role of mathematical models in obtaining useful information concerning physical systems. In order to place the use of models into a reasonable perspective, an outline of logical approaches frequently utilized to solve scientific and engineering problems will be examined. Methods used to construct and verify mathematical models will then be discussed. Following this, two specific examples related to the author's field of work will be presented. This paper will be concluded with an examination of the capabilities and limitations of mathematical models in solving real world problems.

Prior to outlining solution procedures, examples of what information is desired from physical systems will be listed to emphasize that most methods have application in a number of disciplines.

Transfer of information over significant distances is a purpose of a communication system. A general objective of an engineering study in this field would be to design equipment to accomplish efficiently this transfer. Criteria or figures of merit for the response of such a system should include speed of response and fidelity of output. The operating environment of the system must be defined in order to evaluate properly the adequacy of any design.

The purpose of examining a biological process such as a waste water treatment system might be to predict future trends, and to provide information for corrective action. A useful result could be to control effectively levels of pollution inputs.

The diagnostic problem of a man with a physical illness is similar to the problems listed above. The first step consists of identifying the characteristics of the problem, and then in applying therapy to obtain a form of corrective action. An inherent portion of this solution is to establish cause and effect relationships in the physical system being examined.

A flight control system is developed to stabilize a spacecraft (dynamical plant) and to perform required maneuvers. A typical objective of a design study would be to synthesize controller parameters to obtain satisfactory

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response to specified inputs while operating in a particular environment. Pressure, temperature, and vibration effects will influence most physical systems to some degree, and realistic disturbance inputs may significantly affect the system output.

From the examples cited, most solutions to physical systems may be logically divided into the following categories:

- 1) An identification and formulation phase (where environmental conditions, constraints, and criteria are established).
- 2) A mathematical representation phase.
- 3) A mathematical solution phase.
- 4) An interpretation phase.

#### *Logical Approaches to Obtain Problem Solutions*

One of the more underestimated aspects of solving scientific problems is the formulation phase. Any real world problem that can be described in mathematical terms has progressed significantly toward a satisfactory solution. Today, many university graduates are extremely weak in the ability to formulate meaningful problems. However, this is not surprising, since this talent is generally obtained through experience. Some engineers never learn to penetrate the "formulation barrier." These persons believe that given an exact mathematical formulation (by a nebulous someone else), the real problem is to obtain a numerical solution.

After having established the objectives of any scientific investigation, the next phase is to develop mathematical models to represent physical or dynamical systems. Development of good mathematical models to represent processes is a difficult phase in any analysis or synthesis. This is a significant challenge because unavoidable idealizations are inherent in the modeling of real systems. It should be stated with emphasis that all subsequent study is influenced by decisions made in establishing a specific mathematical representation.

Two general methods are utilized to obtain representations of real systems. The first relies upon physical laws to obtain mathematically derived dynamical equations. An example is Newton's Law of Mechanics. The second method utilizes experimentally observed responses obtained for specified inputs. In general, this information is processed to establish cause and effect relationships.

The complexity of a mathematical model required for a given problem is largely dependent upon the information desired. In many instances, solutions to several simple models may provide information superior to the solution obtained from a single complex model. This piece-wise approach, where solutions are restricted within stated limits, may provide valuable insight into the characteristics of a process.

After formulating a scientific problem, and establishing a mathematical

model, the next step is to obtain a mathematical solution. Various distinct approaches are usually available as solution methods. Techniques associated with analog and digital computers are available as well as direct calculations. For a complicated scientific investigation, consideration should be given to obtaining several levels of solutions in a simultaneous manner. Detailed computer solutions, together with simplified hand calculations complement each other in obtaining insight into dynamical relationships.

An interpretation phase is required whenever mathematical solutions become available. Identification of solution patterns together with cause and effect relationships is desired. A useful result of this interpretation phase might be to simplify the problem by retaining only the essential properties in order to conduct additional extended studies. Another result might be to determine dynamical correlations so that responses may be predicted for additional inputs and/or conditions other than actually solved. This extrapolation of dynamical behavior is a valuable objective.

#### *Verification of Mathematical Models*

The previous section outlined logical approaches in obtaining solutions for practical problems. This section will examine methods used to verify that assumed mathematical models are reasonably related to the real world.

A form of testing and experimental observations are necessary to establish the validity of a mathematical model. In addition to verification, another objective of testing is to develop an improved model for a given physical system. Techniques associated with testing strongly influence the validity of the information derived. An initial decision is required to establish which variables should be measured. Sampling and data processing techniques generally influence results. In many cases, sophisticated extrapolation is required when the testing environment is different than the design environment of the system. As an example, how should one test inertial instruments in an earth "g" environment in order to establish performance capabilities in a space or zero "g" environment?

Testing will not automatically insure the development of an improved mathematical model. Physical appreciation and understanding is central to an intelligent interpretive analysis of testing data.

#### *Specific Examples*

Two examples associated with spacecraft dynamical systems will now be presented. For each case, complete solutions have not been obtained at this time, and the examples therefore represent unresolved problems.

The first example is concerned with establishing a low gravity propellant "sloshing" model for a given spacecraft configuration. During parts of the lunar mission, passive (spin) stabilization is required to achieve thermal constraint conditions while minimizing the power consumed aboard the spacecraft. Thus, a minimum active control with thrusters and operative control electronics is desired. However, the spin rate allowed is limited by communication constraint requirements. A low gravity propellant sloshing model is required in order to establish whether this restricted mode of control is feasible. Two mathematical concepts in the development of models are illustrated in Figure 1. The first model

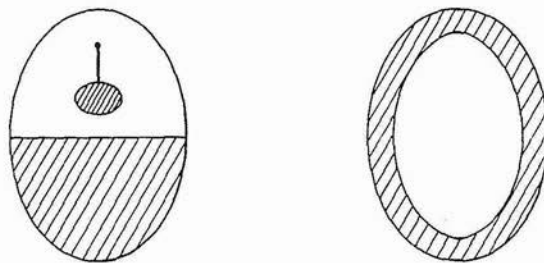


FIG. 1 - REPRESENTATIONS OF SLOSH MASS IN LOW GRAVITY ENVIRONMENT

assumes a fixed fluid mass, and a movable mass constrained to move as a pendulum. The second model assumes that all the fluid will be positioned around the tank walls due to surface tension. Control studies to evaluate the interaction between the propellant sloshing and the spacecraft rigid body have been conducted for the pendulum type mathematical model. However, the results were considered to be unreasonably conservative. Thus, the outstanding problem may be stated simply as: How does one obtain test data in order to construct a reasonable mathematical model for low gravity propellant sloshing?

A second example is concerned with establishing the adequacy of a landing radar system for the lunar landing mission phase. A reasonable performance model for this sensing system is desired for overall mission design studies. A simplified spacecraft representation in this thrusting phase is given in Figure 2. The performance of the radar sensors (pointing down) is influenced by (1) thrust plume effects; (2) reflectivity properties of the lunar surface; and (3) particle motion under thrusting conditions. In addition, the plume receives a "ground effect" at the termination phase of touchdown. The problem of how to test to obtain a reasonable radar performance model is a difficult one. Inherent uncertainties in the

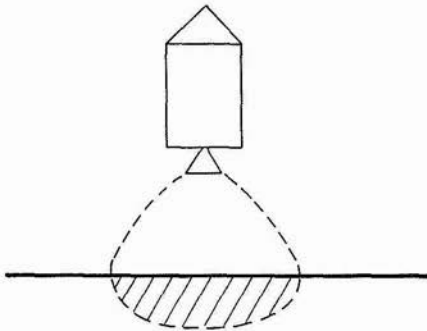


FIG. 2 - REPRESENTATION OF SPACECRAFT UNDER A THRUSTING LANDING CONDITION

properties of the lunar surface make this an even more challenging problem.

#### *Conclusions*

In concluding this brief discussion on the role of mathematical models in solving real world problems, several general remarks are pertinent.

Practical methods for synthesis and analysis of complex physical systems are developed over a period of time under expected iterative conditions. Thus, most problems are simply not completely resolved by the first attempt.

A potential danger in attempting to obtain realistic solutions is to restrict the constraints and criteria defined in the formulation phase in order to fit a specific mathematical format. Frequently, solutions are obtained to the problem formulation we know how to solve instead of the more difficult (but more realistic) problem formulations.

As a final remark, it is important to restate that there is no substitute for understanding and thought in the process of obtaining meaningful solutions to physical problems. A potential danger always exists that mathematical models developed for specific problems may be misused by applying the information out of context.