RICE UNIVERSITY

Secure Implementation, Network Cost Sharing and Oligopolistic Price Discrimination

by

Rajnish Kumar

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE Doctor of Philosophy

APPROVED, THESIS COMMITTEE:

Dr. Anna Bogomolnaia, Associate Professor of Economics

Dr. Simon Grant, Lay Family Professor of Economics

Dr. Herve Moulin (Chair), George A. Peterkin Professor of Economics

Dr. Amit P. Pazgal, Jones School Distinguished Associate Professor of Management, Associate Professor of Marketing

HOUSTON, TEXAS
JUNE 2010
ABSTRACT

Secure Implementation, Network Cost Sharing and Ologopolistic Price Discrimination

Rajnish Kumar

In chapter 1, we consider the possibility of Secure Implementation in Production Economies beyond the result provided in the Saijo et al. (2007) paper. We find a large class of SCFs to be securely implementable. The serial SCF and the widely studied Fixed Path SCFs which contains serial SCF as a special case are all special cases of our function.

In chapter 2, which is a version of my work with Ruben Juarez, we consider the problem of sharing the cost of a network formed by choice of paths of agents to connect their demand nodes. Motivated by the inefficiency, instability and huge informational requirements of the widely used Shapley (Sh) cost sharing rules, we look for mechanisms in a setting of minimal informational requirement which
overcome the said shortcoming. We characterize a class of such mechanisms under different notions of robust implementations. We also discover that voluntary participation is possible in this setup with no more inefficiency than that of Sh.

In chapter 3, which is a version of my work with Levent Kutlu, we consider the aspect of price discrimination under oligopolistic setting. The environment has two stages of the game. In first stage the firms fight on the quantity they want to put in the market and then in the second stage they decide how to distribute that quantity among the buyers with different valuations. We characterize the unique NE of this game. The firms which ends up with higher quantity in the first stage sells to all the buyers whereas the smaller firm sells some of the high end buyers.
Acknowledgements

First and foremost I offer my sincere gratitude to my advisor or my academic father, Dr. Herve Moulin, without whom this work would not have existed. He has been my inspiration, guidance and support throughout the work of the thesis. On the one extreme, supporting me with his knowledge and patience and on the other, giving me the freedom to work on my own way, I just could not wish for a better advisor.

I owe my deep gratitude to my Micro teachers, Dr. Simon Grant, Dr. Anna Bogomolnaia and Dr. Geoffroy de Clippel for teaching me the way to think which I have been and will be using while solving problems. I am grateful to my teachers Dr. James N. Brown, Dr. Mahmoud El-Gamal, Dr. Juan Cordoba, Dr. Yoosoon Chang, Dr. Peter Hartley, Dr. Robin C. Sickles, Dr. Ronald Soligo and Dr. Marc P. Dudey for teaching me various fields in Economics.

I want to thank my co-authors Dr. Ruben Juarez and Dr. Levent Kutlu for the wonderful working experience I had with them. I thank my classmates Yongok Choi, Trang Dinh, Sinan Ertemel, Jiaqi Hao, Xiao He, Ekaterina Magakova and Michael Naaman for making the experience at Rice wonderful. I thank our program coordinator Altha Rodgers for all her helps throughout my graduate program.
also thank all my friends at Rice and in Houston for making Houston a home for me.

It is an honor for me to thank my teacher Dr. Arunava Sen for being the motivation to pursue my research in the field of Social Choice Theory and also continuously giving suggestion for the research. Various discussions with Dr. Manipushpak Mitra, Dr. Dipjyoti Majumdar, Dr. Anirban Kar, Dr. Justin Leroux, Dr. Gaurab Aryal and Dr. Siyang Xiong have been extremely helpful in the chapter 1 and chapter 2 of the thesis.

I thank my brothers Manoj Kumar and Manjul Kumar, my sister Vandana Sharma and my mother for their unconditional support throughout. Finally, I thank my father Late Ramagya Thakur who was the most influential teacher in shaping my life and I dedicate this thesis to him.
Contents

ABSTRACT  ii

Acknowledgements  iv

List of Figures  vii

Chapter 1. Secure Implementation in Production Economies 1

1.1. Introduction 1

1.2. Secure implementability 10

1.3. Serial Cost Sharing Methods 12

1.4. Serial SCF and generalized serial SCF 20

1.5. Secure implementability of Generalized Serial Mechanisms and the fixed path mechanisms 24

A Proofs 27

References 36

Chapter 2. Implementing Efficient Graphs in Connection Networks 39

2.1. Introduction 39

2.2. The model 45

2.3. Main result 53
2.4. Individually rational mechanisms 56
2.5. Conclusions 58
2.6. Proofs 59

References 87

Chapter 3. Capacity Constraint, Price Discrimination and Oligopoly 89

3.1. Introduction 89
3.2. The Model and Results 91
3.3. Cournot with price discrimination 113

References 115
List of Figures

1.1 Equilibria of the pivotal mechanism 5
1.2 Equilibria of the second-price auction 7
1.3 Fixed path method in two agent case 17
1.4 Serial SCF in two agent case 21

2.1 Symmetric networks with a common source and two sinks 40
2.2 Upper bound on PoS of Sh in undirected graphs for 2 agent case 61
2.3 Lower bound on PoS of Sh in 2 agent case 62
2.4 Inefficiency under smaller total cost 70
2.5 Inefficiency under larger total cost 70
2.6 Efficiency amounts to separability 72
2.7 PNI implies strong monotonicity 76
2.8 Worst case example 85
CHAPTER 1

Secure Implementation in Production Economies

1.1. Introduction

We consider the standard implementation problem where an outcome has to be chosen from a set of alternatives depending upon the characteristics (e.g., preferences) of the agents in the society. The rule which chooses this outcome based on the true preference profile\(^1\) (or any other such characteristic\(^2\)) of the agents is called a Social Choice Function (SCF). The problem of implementing this rule arises because the above said "characteristics of the agents" may be private information of these agents and it may not be in their best interest to reveal these true characteristics if they know how the outcome is going to be chosen based on their reports.

To achieve the goal of implementing a SCF it may be the case that the agents are directly asked to report their preferences or they may be asked to indulge in an indirect process where they interact under certain rules. In both the cases the institution which is used creates a game amongst the agents. These institutions are called mechanisms or game forms. The case where agents are required to report their preferences directly and the outcome is chosen according the SCF is called

---

\(^1\)A preference profile is a set of preferences – one for each agent.

\(^2\)In our framework, the characteristics of the agents we are considering are their preferences. But, more generally it can be the agents' endowments, the agents' abilities (e.g. production technology) etc.
direct mechanism. The other one is called indirect mechanism. Strategyproofness of direct mechanisms is a requirement on the mechanism that truth telling by each agent leads to a most favorable outcome for that agent, no matter what the other agents are reporting. In other words, truth-telling is a dominant strategy equilibrium under the mechanism if the mechanism is strategyproof. It seems natural that players will reveal the truth if it is dominant strategy to do so. However, the performance of strategyproof mechanisms in achieving socially desired goals has been in question for a long time. On the one hand, a sequence of experiments show that pivotal mechanisms\(^3\) fail to get the truth telling as a unique outcome (see Attiyeh et al. [2], Kawagoe and Mori [16], etc.). On the other hand, there are experiments which show that true valuation is not revealed by the subjects in second price auction\(^4\) experiments (see Kagel et al. [13], Kagel and Levin [14]). Some experimentalists argue that the subjects who don't play their dominant strategy must be confused by the complexity of the mechanisms where the dominant strategy may not be that clear. But neither epistemic(Aumann and Brandenburger [1]) nor evolutive(Hurwicz [12], Smith [27]) models of game theory provide unambiguous support for the elimination of weakly dominated strategies. In fact the only prediction that is supported in these models is that the outcome must be a Nash Equilibrium (NE).

---

\(^3\)Pivotal mechanisms are strategyproof mechanisms in the problem of provision of public goods

\(^4\)Second price auction is another example of strategyproof mechanism where the highest bidder gets the object and pays the highest losers bid. Others pay nothing.
This leads us to think of two problems associated with strategyproof mechanisms. First, truth-telling may not be an agent’s unique dominant strategy and using wrong dominant strategy may lead to wrong outcomes. Second, there can be NE other than the dominant strategy equilibrium which lead to wrong outcomes. To see this problem, consider a simple example of pivotal mechanism for two players. Suppose there is a costless public project to be undertaken if and only if the sum of the (reported) valuations of the project by the two agents is non negative. It is well known since Clarke [4] that the transfers according to pivotal mechanism\(^5\) induce truth telling as a dominant strategy equilibrium. It is fairly easy to see that no one can gain by reporting anything other than the true value irrespective of what the other is reporting. However, the true profile is not the only NE. As a matter of fact, as is demonstrated in figure 1.1 below, almost half of the two dimensional Euclidian space constitute the set of NE. Here, the axes represent the type (valuation) space of the agents and since the pivotal mechanism is a direct mechanism they also represent the strategy space of the agents. Notice that the area which correspond to the set of NE (the shaded area) has two regions. In the first region (which is shaded green), the corresponding outcome is socially desired. However, there exists another region (which is shaded yellow) of the similar size where the NE leads to outcome which is not socially desired.

\(^5\)A transfer \(t_i\) to agent \(i\) according to pivotal mechanism in this environment will be equal to the \(-v_j\) if agent \(i\) is pivotal i.e. absence of agent \(i\) would have changed the decision of undertaking the social project. Here, \(v_k\) is the valuation of agent \(k\) for the public project. In other words, If the presence of an agent alters the outcome in her favor, she must compensate the others for their (revealed) welfare loss.
Secure implementation (Saijo et al. [25]) is one way to remedy this problem faced by the strategyproof mechanisms. Secure implementation of a SCF requires the existence of a mechanism under which there is a dominant strategy equilibrium which leads to the socially desired outcome and all the NE under the mechanism also lead to the socially desired outcome. This mode of implementation has been tested on data and has been found to be performing significantly better than strategyproof mechanisms under the presence of multiple NEs (Cason et al. [5]). This nice property of secure implementation doesn’t come without costs. In many environments there doesn’t exist non trivial SCFs which are securely implementable. Following are some of the examples.

Consider a public good provision problem where the good must be provided if and only if the sum of valuations is non-negative. We have just seen above that the pivotal mechanism doesn’t securely implement this SCF. Notice that this SCF is efficient i.e. it maximizes the social surplus. It has been shown in theorem 7 in (Saijo et al. [25]) that there doesn’t exist any surplus maximizing SCF which can be securely implementable\(^6\). This negative result of incompatibility between surplus-maximizing and secure implementation in the quasilinear environment\(^7\) with discrete social decision is further illustrated by the second price auction where

\(^6\)This result is valid even when the consider multivalued Social Choice Correspondences (SCC) in place of SCFs.

\(^7\)Quasilinear preferences are represented by utility function which is additive and linear in one commodity called money.
a large set of NE correspond to the non-surplus-maximizing outcome. To see this point, consider a two player example where the valuation for the object to be auctioned are \( \theta_1 \) and \( \theta_2 \) by agent 1 and agent 2 respectively. Suppose, \( \theta_1 > \theta_2 > 0 \).

In order to maximize the total surplus, it must be the case that object is allocated to agent 1. But, as we see in figure 1.2 below, the set of NE is quite large. The lower right set of NE correspond to the surplus maximizing outcome whereas the upper left set of NE end up allocating the object to agent 2.

![Equilibria of the pivotal mechanism](image)

Figure 1.1. Equilibria of the pivotal mechanism

Another environment is where the social decision is a continuous variable but there are no transfers involved. Consider a \textit{single-peaked voting environment} where the set of alternatives is \( A = [0, 1] \) and set of possible preferences are those that

\(^8\text{Surplus maximization here means that the private good to be auctioned must be allocated to the agent with the highest valuation.}\)
are continuous and single-peaked\(^9\) on \(A\). In such an environment one can find nice SCFs which are strategyproof such as the \textit{median voter rule}\(^{10}\). Median voter rule enjoys nice properties like pareto-efficiency, non-dictatorship, non-bossiness apart from strategyproofness (in fact, group-strategyproofness). However, this rule is not securely implementable. Similarly, other well known-known rules, such as the one which picks the smallest of the peaks, are not securely implementable. As a matter of fact, it has been shown in theorem 8 of (Saijo et al. [25]) that only rules which are securely implementable in this environment are the dictatorial rules. If we relax the rule and allow it be multivalued then we can get non-dictatorial rules but they can not be pareto-efficient.

There are more recent negative results. Bochet and Sakai [3] show that in allotment economies the securely implementable rules are either efficient (priority rules) or symmetric (equal division) but not both. They also show that in "uniform rule" bad Nash Equilibria can be avoided but for that we need to allow for preplay talk among the players which is a different set up of implementation in itself. Fujinaka and Wakayama [11] show that in an economy with indivisible objects and money, the only securely implementable rules are the constant rules.

The above examples show how difficult it is to find securely implementable rules which have other nice properties. However, there are environments where

---

\(^9\)Single-peaked preferences requires the existence of a point \(p(u_i)\) for each \(i\) called the peak of agent \(i\) with preference \(u_i\), such that \(u_i\) is strictly increasing before \(p(u_i)\) and strictly decreasing after \(p(u_i)\).

\(^{10}\)Median voter rule picks the median of \(\{p(u_i)\}_{i \in N}\) given a profile \(u\).
securely implementable rules do exist. For example, such environments are found in quasilinear setup where the social decision is a continuous variable. It has been shown in Saijo et al. [25] that serial SCF (Moulin and Shenker [17]) is securely implementable in the one input one output production economy with convex cost technology. We look for other possible SCFs which are securely implementable in such environments. We find out that it is not just the serial SCF which is securely implementable but a class of SCFs called generalized serial SCFs (GSS) defined in (Shenker [26]) are also securely implementable when the technology has convex cost.
The generalized serial SCFs are described for production economies with smooth production technologies. By smooth production technology we mean that the feasible consumption bundle must lie on some smooth manifold in $\mathbb{R}^m$. Each agent reports his utility function which is defined over $\mathbb{R}^m$ and is non-decreasing, continuous, locally non-satiated and quasi-concave. Then, the mechanism allocates the set of feasible bundles corresponding to the unique NE of an underlying game. The GSS are more general than other generalizations of Serial Mechanism whose incentive properties have been studied in the literature. Among these are the Fixed Path Methods (FPM) where the share of total cost paid by an agent is decided by a path in $\mathbb{R}^n_+$ where $n$ is the number of agents. This path does not depend on the demand vector. We find out that under some assumptions on the cost functions and on the preferences (which guarantee the desired incentive properties of such methods), the FPMs are in fact special cases of GSS and thus all such FPMs are securely implementable. However, there are GSS which cannot be represented as Fixed Path Methods. We conjecture that if we require the mechanism to be non-constant, symmetric (anonymous) and smooth then GSS are the only mechanisms which are securely implementable.

At this point, it is very important to note the intuition why the serial SCF (or more generally the GSS) have such nice incentive property of secure implementability whereas, as we will discuss later, the SCF corresponding to other well known cost sharing rules like the Aumann-Shapley rule (which is the proportional
rule in homogeneous goods case) does not share this feature\(^{11}\). In the latter, by changing the report an agent can affect the outcome for all the agents simultaneously. In particular, that agent’s report changes the outcomes of such agents whose report in turn can change his outcome. This severe nature of externality in such SCF violates the acyclicity condition necessary for the combination of non-bossiness and strategyproofness (see Satterthwaite and Sonnenschein [29]) of the SCF which in turn is necessary for secure implementability. Under the serial SCF, on the contrary, the protection of lower demanders\(^{12}\) from the demands of the higher demanders makes the externality one sided which is not that severe. More precisely, a change in the report of low demander changes the outcomes for all the high demanders whereas small change in the report of high demanders doesn’t affect the outcome for the lower demanders.

The rest of the paper is arranged as follows. In section 2 we precisely introduce the notion of secure implementability and give one proposition which characterizes the securely implementable SCFs. In section 3 we define the serial cost sharing method and introduce some generalizations considered in literature with special focus on the ones whose strategic properties have been studied. In section 4 we define serial SCF and GSS and in section 5 we present two of our main results. The main proofs are gathered in the appendix A.

\(^{11}\)The SCF corresponding to the Aumann-Shapley rule is not even strategyproof

\(^{12}\)By low demander in homogeneous case we mean an agent who gets smaller share of the output and pays lower level of input as the final outcome of the SCF. In heterogeneous case, all the generalizations of serial mechanism rank the agents in an order based on different criteria.
1.2. Secure implementability

We consider an arbitrary set of alternatives $A$ and a finite set of agents $N = \{1, 2, \ldots, n\}$, where $n \geq 2$. Typical agents are represented by alphabets $i$, $j$ etc. The preference relation of agent $i$ over the set $A$ is represented by utility function $u_i$. The set of admissible utility functions for agent $i$ is denoted by $U_i$. The cartesian product of $U_1, U_2, \ldots, U_n$ is represented by $U$ i.e. $U \equiv \times_{i \in N} U_i$. A typical element of $U$ is a utility profile $u = (u_1, \ldots, u_n)$ which is an n-tuple of utility functions— one for each agent. A social choice function (SCF) $f : U \rightarrow A$, is a function that associates with every $u \in U$ a unique alternative $f(u)$ in $A$. A mechanism (or a game form) $g : S \rightarrow A$ is a function that assigns to every $s \in S$ a unique element of $A$, where $S \equiv \times_{i \in N} S_i$ and $S_i$ is the strategy space of agent $i$.

**Definition 1.1.** The mechanism $g$ is called a direct revelation mechanism associated with the SCF $f$ if $S_i = U_i$ for all $i \in N$ and $g(u) = f(u)$ for all $u \in U$.

Some times we may refer a direct revelation mechanism as the SCF if no confusion arises. When the strategies of agents $j \neq i$ is fixed at $s_{-i} \equiv (s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$, agent $i$ can induce certain outcomes by choosing strategies from the set $S_i$. The set of such outcomes denoted by $g(S_i, s_{-i})$ is called the attainable set or the **opportunity set** of agent $i$ at $s_{-i}$. More formally, $g(S_i, s_{-i}) \equiv \{b \in A \mid \exists s_i \in S_i \, \text{s.t.} \, g(s_i, s_{-i}) = b\}$. The set of alternatives that agent $i$ with utility $u_i$ ranks weakly below the alternative $a \in A$ is called the weak lower contour set for agent $i$ with utility $u_i$ at $a$ and is denoted by $L(a, u_i)$. More formally, $L(a, u_i) \equiv \{b \in A \mid u_i(a) \geq u_i(b)\}$. Given the
mechanism \( g : S \rightarrow A \), the strategy profile \( s \in S \) is a Nash Equilibrium (NE) of \( g \) at \( u \in U \) if \( \forall i \in N \), \( g(S_i, s_{-i}) \subseteq L(g(s), u_i) \). Let's denote by \( N^g(u) \) the set of Nash equilibria of \( g \) at \( u \).

**Definition 1.2.** The mechanism \( g \) implements \( f \) in Nash equilibria if for all \( u \in U \), (i) \( \exists s \in N^g(u) \) st. \( g(s) = f(u) \) and (ii) \( \forall s \in N^g(u), g(s) = f(u) \).

The SCF \( f \) is Nash implementable if there exists a mechanism that implements \( f \) in Nash equilibria. Given the mechanism \( g : S \rightarrow A \), the strategy profile \( s \in S \) is a Dominant strategy Equilibrium of \( g \) at \( u \in U \) if \( \forall i \in N, \forall s_{-i} \in S_{-i}, g(S_i, s_{-i}) \subseteq L(g(s_i, s_{-i}), u_i) \). Let's denote by \( D^g(u) \) the set of dominant strategy equilibria of \( g \) at \( u \).

**Definition 1.3.** The mechanism \( g \) implements \( f \) in Dominant Strategy equilibria if for all \( u \in U \), (i) \( \exists s \in D^g(u) \) st. \( g(s) = f(u) \) and (ii) \( \forall s \in D^g(u), g(s) = f(u) \).

The SCF \( f \) is Dominant Strategy implementable if there exists a mechanism that implements \( f \) in Dominant Strategy equilibria. We now introduce formally the concept of secure implementation which requires the existence of a mechanism which implements the SCF in Nash equilibria as well as in Dominant Strategy equilibria.

**Definition 1.4.** The mechanism \( g \) securely implements the SCF \( f \) if for all \( u \in U \), (i) \( \exists s \in D^g(u) \) st. \( g(s) = f(u) \) and (ii) \( \forall s \in N^g(u), g(s) = f(u) \).
The SCF $f$ is *securely implementable* (SI) if there exists a mechanism that securely implements $f$. Strategyproofness is a requirement on a SCF that truth telling by the agents is a dominant strategy under the direct revelation mechanism. More formally, the SCF $f$ satisfies *strategy proofness* (SP) if, $\forall u \in U, \forall i \in N, \forall \tilde{u}_i \in U_i, u_i(f(u)) \geq u_i(f(\tilde{u}_i, u_{-i}))$. Another technical property on the SCF, introduced in Saijo e.t. al. [25], which together with strategyproofness characterizes secure implementability is called *rectangularity property* and is defined as following. The SCF $f$ satisfies the *rectangularity property* (RP) if for all $u, \tilde{u} \in U$, if $u_i(f(\tilde{u}_i, \tilde{u}_{-i})) = u_i(f(u_i, \tilde{u}_{-i}))$ for all $i \in N$, then $f(\tilde{u}) = f(u)$. The following characterization due to Saijo e.t. al. [25] will be used in one of our main results.

**Proposition 1.1.** (Saijo e.t. al. [25]): A SCF $f$ is Securely Implementable if and only if $f$ satisfies Strategyproofness and Rectangularity Property.

### 1.3. Serial Cost Sharing Methods

Serial cost sharing method was first introduced for an environment where the goods demanded by the agents are homogeneous or, in other words, the agents demand various quantities of the same good. Since our purpose here is to extend this method to more general settings, we will define the problem in an environment where each agent $i \in N$ demands $q_i \in [0, q_{\text{max}}] \subset R_+$ quantity\(^{13}\) a personalized\(^{14}\)

\(^{13}q_{\text{max}}\) can be $\infty$.

\(^{14}\)In some of the more general models e.g., [17], [15], each agent may demand quantities of some or all of the goods.
good \( i \). Thus \( q_i \), the \( i \)th component of vector \( q \in R^N_+ \), can be thought of as the demand for good \( i \) as well as the demand of agent \( i \). The cost of serving these demands is \( C(q) \), which must be divided among the agents; the cost share of agent \( i \) is given by \( x_i(q; C) \). The preferences of agent \( i \) is defined over \( R^2 \) which is continuous, increasing in \( q_i \), decreasing in \( x_i \) and the upper contour set is convex\(^{15}\).

Let a concave utility function \( u_i(q_i, x_i) \) represent the preference of agent \( i \). Recall that for the homogeneous goods case \( C(q) = C(q_N) \) where, \( q_N = \sum_{i \in N} q_i \). Here the serial cost sharing method is defined as follows. Consider, Without loss of generality \( q_1 \leq q_2 \leq \ldots \leq q_n \). Define, \( q^i = (q_1, q_2, \ldots, q_{i-1}, q_i, q_{i+1}, \ldots, q_n) \) then,

\[
(1) \quad x_i(q; C) = \frac{C(q^i)}{n + 1 - i} - \sum_{k=1}^{i-1} \frac{C(q^k)}{(n + 1 - k)(n - k)}
\]

This method works as follows. Agent 1, with the lowest demand \( q_1 \) pays \( 1/n \) th of the cost of \( nq_1 \). Agent 2, with the second lowest demand pays agent 1’s cost share, plus \( 1/(n - 1) \)th of the incremental cost from \( nq_1 \) to \( q_1 + (n - 1)q_2 \). Agent 3, with the next lowest demand pays agent 2’s cost share, plus \( 1/(n - 2) \) th of the incremental cost from \( q_1 + (n - 1)q_2 \) to \( q_1 + q_2 + (n - 2)q_3 \). And so on. This method is characterized by "anonymity" and "invariance of the cost share of low demanders by a change in the demand of high demanders". The demand game

\(^{15}\)An special case which is widely studied in this framework is the preference which is quasilinear in \( x_i \) and concave in \( q_i \).
generated by this method is as follows. Each agent has a strategy (demand) space which is $R_+$ and his cost share as a function of the demand profile is computed by (1). The payoff is given by the utility function defined above. It should be noted that the serial cost sharing method (1) is defined for any arbitrary cost function. However, if we assume the cost function to be strictly\textsuperscript{16} convex (increasing marginal costs), then this demand game has very strong strategic properties. In this demand game the NE is unique, robust to coalitional deviations and the only rationalizable strategy profile. Moreover, this NE is the unique outcome of adaptive learning (Milgrom and Roberts, [19]).

Given the nice strategic and equity properties that the serial method enjoys in homogeneous good setting, it is natural to look for the extension for the rule in more general settings. In particular, a natural question is what the counterpart of the serial method in heterogeneous good (multidimensional) case will be. Among the various approaches to extend the Serial Mechanism to the case of heterogeneous goods, it is a general consensus (Koplin [15], Koster et al. [17], Sprumont [28], Friedman [7], Friedman and Moulin [9] etc.) that the mechanism must coincide with the Serial Mechanism in the homogeneous case. This property is referred to as serial extension. But, the task of extending the serial mechanism to heterogeneous goods case is not an easy one as was first demonstrated by Koplin [15]. He shows using a nice counterexample that serial extension is not compatible with

---

\textsuperscript{16}strictness is not needed if the preferences of the agents are strictly convex.
other desired properties namely consistency (he calls it direct aggregation invariance), scale invariance and additivity each of which is compelling in its own sense. Consistency is the requirement from the cost sharing method that the cost shares be invariant if we relabel the commodities. Scale invariance requires that the units in which the goods are measured does not affect the cost shares. Additivity is a decentralizability axiom which says that if we can separate the production into different processes, then we should be able to apply the same cost sharing method in each process and the still get the same cost shares. Therefore, knowing that we can not be too demanding with respect to serial extension there have been different approaches to pin down the class of methods which carry on the properties of serial method for homogeneous methods to the heterogeneous goods environment. These approaches can be broadly categorized into two groups— one which focusses on axiomatic approach (Koster et al. [17], Sprumont [28]) and the other which is concerned about the strategic properties (Friedman [7], Friedman and Moulin [9], Friedman [8]).

Since we are more interested in the strategic properties, we analyze the second approach. Friedman [7] studies the strategic properties of these methods which we describe in the next paragraph and finds out that these do enjoy nice strategic properties similar to serial cost sharing in homogeneous goods case. He finds out
that the game induced by such methods is solvable by iterative elimination of overwhelmed strategies\textsuperscript{17} introduced in Friedman & Shenker [10]\textsuperscript{18}.

This natural extension of the serial method (1) to the heterogeneous case, where $C(q)$ is an arbitrary non-decreasing and continuously differentiable function of its $n$ variables, which was introduced in Friedman and Moulin [9], is defined as follows. Consider a path\textsuperscript{19} $\gamma^{SC}$ from 0 to $q$ given by $\gamma^{SC}(t; q) = (te) \land q$, for $t \geq 0$, where $(p \land q)_i = \min\{p_i, q_i\}$ and $e = (1, 1, ..., 1)$ is the unit vector in $R^N$. This path essentially follows the diagonal of the n-dimensional positive orthant till its coordinates are smaller than all the coordinates of the demand vector $q$. As soon as it meets the demand of some agent, it starts following the projection of the diagonal in the hyperplane where that coordinate is fixed at the demand in that coordinate and so on. Given such a path $\gamma^{SC}$ the cost sharing mechanism is given by,

\begin{equation}
    x^{SC}_i(q; C) = \int_0^\infty \partial_i C(\gamma^{SC}(t; q)) d\gamma^{SC}_i(t; q)
\end{equation}

Here, $\partial_i C(p)$ is the partial derivative of $C$ with respect to $p_i$ evaluated at $p$.

It is clear from (1) that the path relevant to an agent is independent of higher demands. Thus, the cost shares of agents are unaffected by small changes in the

\textsuperscript{17}A strategy $s_i$ for agent $i$ is overwhelmed by strategy $\bar{s}_i$ with respect to $S_{-i}$ if the best that agent $i$ can get over $S_{-i}$ by playing $s_i$ is worse than the worst that he gets by playing $\bar{s}_i$.

\textsuperscript{18}Notice that this is stronger property than solvability in elimination of dominated strategies.

\textsuperscript{19}SC in the symbol underlines the point that this path corresponds to the generalization of Serial Cost (SC) sharing rule.
demands of higher demanding agents. Therefore, the externality is one sided (and thus, acyclic). Intuitively, due to this reason this mechanism enjoys nice strategic properties that we will see in Theorem 1.1. Moreover, due the same reason, the nice strategic properties are preserved if the \( \gamma^{SC} \) is replaced by any arbitrary continuous non-decreasing path \( \phi(t; C) \wedge q \), where \( \phi \) satisfies the following properties. For fixed \( C \), \( \phi \) is non-decreasing and continuous in \( t \) with \( \phi(0; C) = 0 \) and \( \lim_{t \to \infty} \phi_i(t; C) > q^{\max} \) for all \( i \). See figure 1.3 below for an example of such a \( \phi \).

![Figure 1.3. Fixed path method in two agent case](image)

This liberty of choosing the \( \phi \) gives rise to a huge class of cost sharing methods called Fixed Path Methods (FPM). There is a FPM corresponding to each fixed
path $\phi$ which can be defined as follows

$$x^\phi_i(q; C) = \int_0^\infty \partial_i C(\phi(t; C) \wedge q)d(\phi_i(t; C) \wedge q_i)$$

These are called fixed path methods because the path $\phi$, which does not depend on $q$ and thus, are in a sense fixed, uniquely defines a method. One example of a fixed path is the path which follows the the edges of the rectangle $[0,q]$ in some predecided order and this leads to the incremental methods. Notice that when cost function is symmetric or when $\phi$ is independent of the cost function then the only symmetric FPM is the Friedman-Moulin method (1) defined by the path which is the diagonal of the positive orthant. Leroux [18] provides a justification of non-symmetric paths. However, symmetry is trivially satisfied when the cost function is not symmetric and we allow $\phi$ to be a function of $C$. This gives rise to a huge class of symmetric methods. Clearly, we will be sacrificing additivity in most of the cases but we can recover scale invariance and even stronger properties like ordinality. Ordinality is a stronger requirement than scale invariance. Scale invariance requires that the cost shares should be invariant to linear transformation of the demand profile whereas ordinality requires that it should be invariant to any arbitrary monotonic transformations, possibly non linear. The path which most closely follows the spirit of serial method is the path which defines Moulin-Shenker ordinal method discussed in Sprumont [28]. This path which we will call $\phi^{MS}$ is defined by the solution of the following differential equation

$$d\phi_i^{MS}(t; C)/dt = 1/\partial_i C(\phi^{MS}(t; C))$$
satisfying the boundary condition $\phi^{MS}(0; C) = 0$. This path has the property that on any point on the path the incremental cost generated by a small move along the path is shared equally among the agents not fully served. Other examples of FPMs can be generated by applying a FPM to any suitably normalized problem e.g. applying FPM to axially normalized problem (Friedman [7]). One seemingly natural FPM thus generated discussed in Friedman [7] is the use of diagonal path after axial normalization of the problem.

Given the set of agents $N$, utility profile $u = \{u_i\}_{i \in N}$, a cost function $C$, a fixed path method $x^\phi(\cdot; \cdot)$ induces a cost sharing game $\Gamma(x, u)$. These induced games have variety of strategic properties: uniqueness of NE, Strong Equilibria, uniqueness of set of rationalizable outcomes and convergence of adaptive learners. Friedman [7] shows these properties for fixed path methods by showing that the induced games are O-Solvable which in turn implies all these properties.

**Theorem 1.1.** (Friedman [7]): Assume that the marginal cost $(\partial_i C(q))$ is strictly increasing in all variables, $x^\phi(\cdot; \cdot)$ is a fixed path method and that preferences, $u_i(q_i, x_i)$ are increasing in $q_i$, decreasing in $x_i$, and concave. Then the induced game is O-solvable.

It should be noted that there can be paths which depends on $q$ and we can use such paths to define "path methods" in a similar fashion as (3). One prominent
example of such a path method is the Aumann-Shapley method where the path is the ray joining the origin to the demand vector, thus for each demand there corresponds a path. More precisely, the path which generates the Aumann-Shapley method is given by $\phi^{AS}(t; C)(q) = tq$. We notice that this path is not a fixed path and the demand game generated by this method does not share the appealing strategic properties that is enjoyed by the FPMs. The Aumann-Shapley method in the homogeneous goods case is the proportional method. It has been shown in Watts [30] (see also Moulin [20] for detailed analyses) that uniqueness of NE is not guaranteed in the proportional demand games for general convex preferences and a sufficient condition has been shown to be the binormality of preferences. Moreover, as we will discuss in next section, even when the NE is unique this method doesn’t share the strategic properties of that of the FPMs. Intuitively, this happens because a change in the demand by any agent changes the cost shares of all the agents. For more on such path methods and axiomatic characterization of methods generated by paths and more generally by convex combinations of paths please refer to Friedman and Moulin [9].

1.4. Serial SCF and generalized serial SCF

We mentioned in the last section that if the production technology has increasing marginal costs and the preferences are convex then the serial rule (1) defined in the homogeneous goods case induces a game which admits a unique NE. A serial
social choice function (SCF) for a fixed cost function $C$ associates this unique NE allocation to the preference profile generating this game.

Figure 1.4 above demonstrates the Serial mechanism (SCF) in two individual and two good economy where one good $x$ is the input (x-axis) and the other good $q$ is output (y-axis). The production technology is decreasing returns to scale i.e. the cost function is convex. The blue curve is $c(q)$, red one is $c(2q)/2$ and the black one coincides with the red one till point A and then goes parallel to blue curve. More precisely, the black curve has two parts. The part below A is the locus of points that are $1/2$ of some point on the blue curve. The part above A is the locus of points whose vector sum to the point A belongs to the blue curve. The high valuation agent (H) is the agent whose MRS is higher for the output with
respect to the input. The other agent is the low valuation agent (L). The agents are required to report their utility functions and allocation is assigned according to the Serial cost sharing rule. More details on the algorithm to implement the serial SCF can be found in Moulin and Shenker [17]. The purpose of bringing the 2X2 case of Serial SCF here is that the Generalized Serial SCF is defined very closely in the spirit of Serial mechanism here. The three conditions below in the definition of generalized serial functions are linked to the following three observation in the above picture.

1) The opportunity set of L remains unaffected by change in the preferences of H as long as H has higher valuation than L.

2) Owing to the convexity of production function and the preferences, there is a unique maximizer point A for L on his opportunity set given H and also B for H on her opportunity set given L.

3) Owing to no kinks in red and black curve at A, A remains the optimum point for L even after small changes in preference by H.

1.4.1. Generalized Serial Mechanism (SCF):

The generalized serial SCF is defined for an economy with n agents and m goods where n and m are greater than 1. Production technology \( P \) is a \( m - p \) dimensional smooth manifold which represents a technology where out of m-goods, p are inputs and m-p are outputs. Set of alternatives \( A \) is the set of allocation to the agents in

...
$N$ which is feasible under $P$. More formally, $A \equiv \{x \in R_{+}^{m} | (\sum_{i=1}^{n} x_i) \in P\}$. One example of such set of alternatives where $m = 2$ and $p = 1$ is the set of allocations for the two agents in the above example which add up the a point which lie on the blue curve in figure 4. The set of admissible utility functions $U_i$ for agent $i$ contains the functions $u_i : R^m \rightarrow R$ that are continuous, non-decreasing in all dimensions, locally non satiated and quasi-concave. Linear utilities of agent $i$ is the subset $L$ of $U_i$ which isomorphic to $R_+^m - \{0^m\}$.

**Definition 1.5** (Generalized Serial Function (Shenker) [26]). Consider a function $a : R_{+}^{m} \rightarrow A$ st., $\forall z \in R_{+}^{n}$ and $\forall \lambda \in L$:

1. $z_i \leq z_j \Rightarrow a_i(z) = a_i(z_{-j}, s_j), \forall s_j \in [z_i, \infty],$
2. $\lambda.a_i(z_{-i}, s_i) \text{ has a unique maximizer } \overline{s}_i, \forall i$
3. If $\overline{s}_i$ is the unique maximizer of $\lambda.a_i(z_{-i}, s_i)$ then $\overline{s}_i$ is also the unique maximizer of $\lambda.a_i(z'_{-i}, s_i)$ \forall $z'$ st., $\forall j \neq i, MIN[z'_j, z_j] < \overline{s}_i \Rightarrow z'_j = z_j.$

Such a function "a" is called a "generalized serial function"

Let's denote by $F$ the set of all generalized serial functions. For a given utility profile $u$ a function $a \in F$ induces the normal form game $\Gamma(a; u) \equiv (N, \forall i; S_i = R_+, \{u_i(a_i(\cdot))\}_{i \in N})$ where $N$ is the set of players, $R_+$ is the strategy space for all players and the payoff function for player $i$ is given by $u_i \circ a_i(\cdot)$. Such games possess unique NE.

**Lemma 1.1.** : $\forall u \in U, \forall a \in F; \Gamma(a; u)$ has a unique NE.
Proof. The proof consists of two steps. In first step it is shown that there can not be more than one NE and then an explicit algorithm is given to construct a NE. A formal proof can be found in Appendix A.1 below.

Definition 1.6 (Generalized Serial Mechanism). \( \zeta^a \) is a generalized serial mechanism \( \text{(GSS)} \) associated with \( a \in F \) if \( \zeta^a(u) = a(z) \) where \( z \) is the unique NE of \( \Gamma(a;u) \).

Let's denote by \( G \) the set of all generalized serial mechanisms.

1.5. Secure implementability of Generalized Serial Mechanisms and the fixed path mechanisms

Now we are ready to present our main result which encompasses the result of Saijo et al. [25]

Theorem 1.2. Any generalized serial mechanism \( \text{(GSS)} \) is securely implementable.

Proof. We show the secure implementability of GSS by showing that the GSS are strategyproof and that they satisfy the rectangularity property. Then by proposition 1 the desired result follows. Please refer to the Appendix A.2 below for a complete proof.

Now we define a class of social choice functions called fixed path social choice functions based on fixed path cost sharing rules. Let's assume the conditions on
the cost function and the preferences that were used in theorem 1.1. Then from the theorem 1.1 we know that there will be unique NE in the game $\Gamma(x^\phi, u)$ induced by the the cost sharing rule $x^\phi$ based on the fixed path $\phi$.

**Definition 1.7.** A fixed path social choice function $\xi^{x^\phi}$ associates the allocation corresponding to the unique NE of the game $\Gamma(x^\phi, u)$ to the preference profile $u$.

The following theorem states that all such fixed path SCF are securely implementable.

**Theorem 1.3.** Under the assumptions of theorem 1.1, a fixed path social choice function $\xi^{x^\phi}$ is a special case of generalized serial social choice function and thus are securely implementable.

**Proof.** The proof consists of explicitly constructing a generalized serial function "a" for every fixed path social choice function $\xi^{x^\phi}$. We use two lemmas for proving the desired properties of such "a". Please refer to Appendix A.3 below for a comprehensive proof. □

At this time we would like to emphasize that the SCFs corresponding to path methods other than fixed path methods may not be securely implementable. One such method as we discussed above is the Aumann-Shapley method which corresponds to the proportional method in the homogeneous goods case. To ensure the uniqueness of NE in the demand game lets consider linear utilities (which are obviously binormal) given by $u_i(q_i, x_i) = b_i q_i - x_i$ and convex cost technology given
by $c(y) = y^2/2$. Proportional cost shares are given by $x_i^{pr}(q) = \frac{q_i}{q_N}c(q_N)$, where $q_N = \sum_{i \in N} q_i$. Let's define the proportional SCF $\xi^{pr}$ which associates to every utility profile $u$ the unique NE of the demand game $\Gamma(x^{pr}, u)$. We notice that this SCF is not securely implementable. As a matter of fact these are not even strategyproof. To see this consider a two agent situation. Let the linear utilities of agents 1 and 2 be defined by the parameters $b_1$ and $b_2$. Then, whenever $b_i$'s are close enough to ensure the active participation of both agents, the unique NE demand profile $(q_1^*, q_2^*)$ is given by $q_i^* = \frac{4}{9}(b_i - \frac{b_j}{2})$ and the equilibrium cost shares turn out to be $x_i = \frac{4}{9}(b_i - \frac{b_j}{2}) + \frac{2}{9}b_j(b_i - \frac{b_j}{2})$, $i, j \in \{1, 2\}$. Therefore, the optimal report $\tilde{b}_i^*$ of agent $i$ with true parameter $b_i$ is given by $\tilde{b}_i^* = \frac{3}{2}b_i + \frac{5}{4}b_j$ where $b_j$ is the report of agent $j$. Clearly, there are profitable manipulation of reports by agents. In particular, suppose $b_1 = b_2 = b$ and agent 1 reports truthfully then the optimal report of agent 2 is $\frac{11}{4}b$.

We see that the FPMs are special case of GSS. However, there are GSS which can not be represented by FPM. One trivial example is a constant SCF. Therefore we conclude that GSS are more general than FPMs and have nice strategic properties.

We conclude by the following conjecture which we leave for future work.

**Conjecture 1.1.** Every smooth, nonconstant, anonymous and securely implementable scf is an element of $G$. 

\section*{A Proofs}

\subsection*{A.1 Proof of lemma 1.1}

Step 1- Given any $u \in U$ and any $a \in F$; $\Gamma(a; u)$ can not have more than one NE.

Proof: Let $z$ and $z'$ be two distinct NE with $z = (z_1, z_2, z_3, \ldots, z_n)$ and $z' = (z'_1, z'_2, z'_3, \ldots, z'_n)$.

There must exist an element $i$ such that $z_i \neq z'_i$ and $\min \{z_j, z'_j\} < \min \{z_i, z'_i\} \implies z_j = z'_j$.

Without loss of generality, say $z'_i < z_i$. But then $z'_i = \arg\max_{s \in [0,1]} u_i(a_i(s, z_i)) = z_i$ which is a contradiction.

Step 2- Given any $u \in U$ and any $a(z) \in F$; The following algorithm generates a profile $z$ which is a (the) NE of $\Gamma(a; u)$.

Algorithm:

1) Set $z=(1,1,\ldots,1)$.

2) Define $s_i^1 = \arg\max_{s \in [0,1]} u_i(a_i(s, z_{-i}))$, $\forall i$.

3) Without loss of generality, let $s_i^1 = \min \{s_i^1\}$. Set $z_i = s_i^1$ and leave the other elements of $z$ unchanged.

4) Define $s_i^2 = \arg\max_{s \in [0,1]} u_i(a_i(s, z_{-i}))$, $\forall i$. 

5) Without loss of generality, let \( s_2^2 = \min\{s_i^1\} \). Set \( z_2 = s_2^2 \) and leave the other elements of \( z \) unchanged.

6) Repeat the process to update \( z_3, z_4, \ldots, z_n \).

Claim: The profile \( z \) obtained by the above algorithm is a NE of \( \Gamma(a; u) \).

Proof:
Claim 1. If \( s_i^1 \leq s_{i+1}^{i+1} \) for all \( i = 1, 2, \ldots, n-1 \), then \( z \) is a NE.

Proof: Straightforward from property 3 and the way \( z \) has been constructed.

Claim 2. \( s_i^1 \leq s_{i+1}^{i+1} \) for all \( i = 1, 2, \ldots, n-1 \)

Proof:

Part 1. \( s_1^1 \leq s_2^2 \). This holds because, \( s_1^2 = s_1^1 = z_1 \) (because 1 is solving the same optimization exercise) and \( s_2^2 < s_1^1 \implies s_2^2 = s_2^1 \) which contradicts the definition of \( s_1^1 \).

Part 2. If \( s_i^1 \leq s_{i+1}^{i+1} \) for all \( i < k \) then \( s_k^k \leq s_{k+1}^{k+1} \).

Proof: Notice first that \( s_l^k = z_l = s_l^l \) for all \( l < k \). This is true because of condition 3. Now \( s_{k+1}^{k+1} < s_k^k \implies s_{k+1}^{k+1} = s_{k+1}^k \) which contradicts the definition of \( s_k^k \). ■

A.2 Proof of theorem 1.2
Strategyproofness of GSS follows from Theorem 7.2.3 in Dasgupta et al. [6], given our domain of preferences being monotonically closed and the fact that GSS is a single valued Nash Implementable SCF.

We will prove the Rectangularity Property:

\[ \forall u, \tilde{u} \in U \; \{ u_i(\zeta^a(\tilde{u})) = u_i(\zeta^a(u_i, \tilde{u}_{-i})) \; \forall i \in N \implies \zeta^a(\tilde{u}) = \zeta^a(u) \} \]

Proof:

Fix an arbitrary pair of utility profiles \( u, \tilde{u} \in U \)

Let \( u_i(\zeta^a(\tilde{u})) = u_i(\zeta^a(u_i, \tilde{u}_{-i})) \; \forall i \in N \)

Define, \( NE(\Gamma(a; u_i, \tilde{u}_{-i})) = \tilde{z}^i; \; NE(\Gamma(a; \tilde{u})) = \tilde{z} \; ; \; NE(\Gamma(a; u)) = z. \) (Notice the notation; \( \tilde{z}^i \) is a vector and \( \tilde{z}_i \) is the i'\text{th} component of the vector \( \tilde{z} \). For example, \( \tilde{z}^i \) is the k'th component of \( \tilde{z}^i \).)

Step1: \( \tilde{z}^i = \tilde{z}, \; \forall i \in N. \)

Proof: Let \( \tilde{z}^i \neq \tilde{z} \) for some \( i. \)

Now, we must have an element \( k \) st. \( \tilde{z}^i_k \neq \tilde{z}_k \) and \( \min\{\tilde{z}^i_j, \tilde{z}_j\} < \min\{\tilde{z}_k, \tilde{z}_k\} \implies \tilde{z}^j_k = \tilde{z}_j. \)

Case1: \( k \neq i \)

Without loss of generality, say, \( \tilde{z}^i_k < \tilde{z}_k. \)

\[ \tilde{z}^i_k = \arg\max_{s \in [0,1]} \tilde{u}_k(a_k(s, \tilde{z}^i_{-k})) = \arg\max_{s \in [0,1]} \tilde{u}_k(a_k(s, \tilde{z}_{-k})) = \tilde{z}_k \text{ which is a contradiction.} \]

Case2: \( k = i \)
Here there are two relevant cases,

Case 2.1: $\tilde{z}_i^1 < \tilde{z}_i$

Then we have,

$$\tilde{z}_i = \operatorname*{argmax}_{a \in [0,1]} u_i(a_i(s, \tilde{z}_i)) = \operatorname*{argmax}_{a \in [0,1]} u_i(a_i(s, \tilde{z}_i))$$

From property 1 in the definition of $a$, we must have the following

$$a_i(\tilde{z}_i^1, \tilde{z}_{-i}) = a_i(\tilde{z}_i, \tilde{z}_{-i})$$

or, $a_i(\tilde{z}_i) = a_i(\tilde{z}_i^1, \tilde{z}_{-i})$

$$\implies u_i(a_i(\tilde{z}_i)) = u_i(a_i(\tilde{z}_i^1, \tilde{z}_{-i}))$$

We also know, $u_i(\zeta^a(\bar{u})) = u_i(\zeta^a(u_i, \bar{u}_{-i})) \forall i \in N$

or, $u_i(a(\bar{z})) = u_i(a(\tilde{z}_i)), \forall i \in N.$

or, $u_i(a_i(\tilde{z})) = u_i(a_i(\tilde{z}_i)), \forall i \in N.$

Therefore, $u_i(a_i(\tilde{z})) = u_i(a_i(\tilde{z}_i^1, \tilde{z}_{-i}))$

In other words, $u_i(a_i(\tilde{z}_i, \tilde{z}_{-i})) = u_i(a_i(\tilde{z}_i^1, \tilde{z}_{-i}))$

But then, $\tilde{z}_i = \tilde{z}_i^1$ because $\tilde{z}_i^1$ is unique maximizer of $u_i(a_i(s, \tilde{z}_i))$ and $u_i(a_i(s, \tilde{z}_{-i})).$

Case 2.2: $\tilde{z}_i^1 > \tilde{z}_i$

From property 1 in the definition of $a$, we must have the following
\[ a_i(z_i, \tilde{z}_{-i}) = a_i(\tilde{z}_i, \tilde{z}_{-i}) \]

or, \[ a_i(\tilde{z}) = a_i(\tilde{z}_i, \tilde{z}_{-i}) \]

\[ \implies u_i(a_i(\tilde{z})) = u_i(a_i(\tilde{z}_i, \tilde{z}_{-i})) \]

We also know, \[ u_i(a_i(\tilde{z})) = u_i(a_i(\tilde{z}^i)), \forall i \in N. \]

Therefore we have, \[ u_i(a_i(\tilde{z}^i)) = u_i(a_i(\tilde{z}_i, \tilde{z}_{-i}^i)) \]

But then, \[ \tilde{z}_i^i = \tilde{z}_i \] because \( \tilde{z}_i^i \) is unique maximizer of \( u_i(a_i(s, \tilde{z}_{-i}^i)) \) and \( u_i(a_i(s, \tilde{z}_{-i}^i)) \).

\[ \square \]

Notice, the above step establishes the following property:
\[ \hat{u}_i(a_i(s, \tilde{z}_{-i})) \] and \( u_i(a_i(s, \tilde{z}_{-i})) \) both are maximized at \( \tilde{z}_i = \tilde{z}_i^i \) for all \( i \in N \) —

Step 2:
\[ \zeta^a(\tilde{u}) = \zeta^a(u) \]

Proof:

Proving \( a(z) = a(\tilde{z}) \) should be enough since, by definition \( \zeta^a(\tilde{u}) = \zeta^a(u) \iff a(z) = a(\tilde{z}) \).

In fact, we will prove a stronger property, namely, \( z = \tilde{z} \).

Suppose not and let \( z \neq \tilde{z} \).

Now, we must have an element \( k \) st. \( z_k \neq \tilde{z}_k \) and \( \min\{z_j, \tilde{z}_j\} < \min\{z_k, \tilde{z}_k\} \implies z_j = \tilde{z}_j \).
There can be two cases,

Case 1. \( z_k > \tilde{z}_k \)

Then we get the following expression, where first and fourth equalities are from definition, second follows from the \( \blacklozenge \) and third is due to the property 3 in the definition of function "a"

\[
\tilde{z}_k = \arg\max_{s \in [0,1]} \tilde{u}_k(a_k(s, \tilde{z}_k)) = \arg\max_{s \in [0,1]} u_k(a_k(s, \tilde{z}_k)) = \arg\max_{s \in [0,1]} u_k(a_k(s, z_k)) = z_k
\]

and we reach a contradiction.

Case 2. \( z_k < \tilde{z}_k \)

Then we get the following expression, where first and fourth equalities are from definition, third follows from the \( \blacklozenge \) and second is due to the property 3 in the definition of function "a"

\[
z_k = \arg\max_{s \in [0,1]} u_k(a_k(s, z_k)) = \arg\max_{s \in [0,1]} u_k(a_k(s, \tilde{z}_k)) = \arg\max_{s \in [0,1]} \tilde{u}_k(a_k(s, \tilde{z}_k)) = \tilde{z}_k
\]

and we hit another contradiction to conclude the proof.

\[\blacksquare\]

A.3 Proof of theorem 1.3

We first present two lemmas which will be the key to the proof of theorem 3 below.
Lemma 1.2. (Lemma 1 in Friedman [7]): Assume that marginal cost is strictly increasing in all variables and that $x_i^\phi(\cdot; \cdot)$ is a fixed path method. Define $z_i(q_i) = \min\{t | \phi_i(t) \geq q_i\}$. Then:

(a) $x_i^\phi(q; C)$ is strictly increasing and strictly convex in $q_i$.
(b) $x_i^\phi(q; C)$ is non decreasing in $q_j$ for all $j \neq i$.
(c) For all $q$ and $\hat{q}_j$ such that both $z_j(q_j)$ and $z_j(\hat{q}_j) \geq z_i(q_i)$ then $x_i^\phi(q; C) = x_i^\phi(q_{-j}; \hat{q}_j; C)$.

Lemma 1.3. (Lemma 2 in Moulin & Shenker [17]): Let $h_1(X), h_2(X)$ be two increasing and strictly convex functions from $\mathbb{R}^+$ onto itself that coincide up to $\lambda_0$:

$$h_1(\lambda) = h_2(\lambda) \quad \text{for all } \lambda, \quad 0 \leq \lambda \leq \lambda_0$$

Then for every utility function $u_i$ in $U_i$, the (unique) maximizers of $u_i(h_k(\lambda), \lambda)$ on $\mathbb{R}_+$, denoted by $\lambda_k$, $k = 1, 2$ are on the same side of $\lambda_0$:

$$\lambda_1 \geq \lambda_0 \iff \lambda_2 \geq \lambda_0, \quad \lambda_1 = \lambda_0 \iff \lambda_2 = \lambda_0.$$

Proof of theorem:

Fix a cost function $C$ satisfying the assumptions of theorem 1.1. Let the domain of utility functions representing the preferences satisfying the assumptions
be \( U \). Let the the set of alternatives be \( A \equiv \{(q, x) : q \in [0, q_{\text{max}}]^N, x \in R^N_+ \) and \( \sum_{i \in N} x_i = C(q) \}. Consider a fixed path \( \phi \) and the associated fixed path social choice function \( \xi^{\phi} : U \rightarrow A \) which allocates the outcome corresponding to the unique NE of \( \Gamma(x^\phi, u) \) to the preference profile \( u \). Consider \( D_i = \{0\} \cup \{t \in R_+ | \phi'_i(t) \text{ is positive}\}^{21} \) and \( z_i(q_i) = \min[t | \phi_i(t) \geq q_i] \) (see figure 3 above for such an example of \( z_i \)). We claim that a function \( a : \times_{i \in N} D_i \rightarrow A \) which is defined as follows is a generalized serial function and the associated generalized serial SCF \( \zeta^a = \xi^{z^\phi} \). Let \( a_i(z) = (q_i(z), x_i^\phi(q(z))) \) for all \( i \), where \( q_i(z) = \phi_i(z_i) \) and \( q(z) = (\phi_1(z_1), \phi_2(z_2), ..., \phi_n(z_n)) \). We will now prove the following three properties of \( a \) using lemma 1.2 and lemma 1.3 above and the assumption on preferences.

\[
\forall z \in \times_{i \in N} D_i \text{ and } \forall \lambda \in L : \\
(1) z_i \leq z_j \Rightarrow a_i(z) = a_i(z_{-j}, s_j), \forall s_j \in [z_i, \infty], \\
(2) \lambda. a_i(z_{-i}, s_i) \text{ has a unique maximizer } \overline{s}_i, \forall i \\
(3) \text{If } \overline{s}_i \text{ is the unique maximizer of } \lambda. a_i(z_{-i}, s_i) \text{ then } \overline{s}_i \text{ is also the unique maximizer of } \lambda. a_i(z'_{-i}, s_i) \forall z' \text{ st., } \forall j \neq i, \text{MIN}[z'_j, z_j] < \overline{s}_i \Rightarrow z'_j = z_j
\]

First thing to notice is that even though the domain of "a" is not the same as in the original definition, the properties of "a" is retained exactly. This is so because for all \( i \), the \( D_i \) is order-isomorphic to \( R_+ \) given \( D_i \) is concatenation of open-closed intervals with "0" included. Now we will show the above three properties one by one. To see that property 1 is true, notice that \( z_i \) uniquely defines \( q_i(z) = \phi_i(z_i) \) which is independent of \( z_{-i} \). Also, part (c) of the lemma 1.2 implies that

---

21 \( \phi'_i(t) \) is the left hand derivative of \( \phi_i \) at \( t \).
\[ x_i^\phi(q(z)) = x_i^\phi(q(\tilde{z})) \text{ for all } z', \forall j \neq i, MIN[z_j', z_j] < s_i \rightarrow z_j' = z_j. \]

Property 2 is a consequence of part (a) of lemma 1.1 and the linearity of preferences. We first notice that strict convexity of \( x_i^\phi(q_i; C) \) in \( q_i \) and linearity of preferences which are increasing in \( q_i \) and decreasing in \( x_i \) ensures a unique maximizer \( q_i^* \). But then, there will be a unique \( z_i^* \) for this \( q_i^* \) by the definition of \( z_i \). Property 3 is a bit more subtle and the proof is as follows. Consider two points \( z \) and \( z' \) in \( x_i \in N D_i \).

Consider a coordinate \( i \). Let, \( \forall j \neq i, MIN[z_j', z_j] < s_i \rightarrow z_j' = z_j. \) Consider \( \lambda \in \{R_+ \times R_- \}/\{0\} \). Let \( \bar{s}_i = \arg \max_{s_i \in D_i} \lambda \cdot a_i(z_{-i}, s_i) \) and \( \bar{s}_i = \arg \max_{s_i \in D_i} \lambda \cdot a_i(z_{-i}, s_i). \)

Let's call \( \{a_j(z_{-i}, \bar{s}_i)\}_{j \in N} = \{(\bar{q}_j, \bar{x}_j)\}_{j \in N} \) and \( \{a_j(z_{-i}, \bar{s}_i)\}_{j \in N} = \{(\bar{q}_j, \bar{x}_j)\}_{j \in N} \).

From part (c) of lemma 1.2 we know \( x_i^\phi(q_{-i}, q_i; C) \) & \( x_i^\phi(q_{-i}, q_i; C) \) coincide for all \( q_i \in [0, \tilde{q}_i]. \) Also, we know from part (b) of lemma 2 that \( x_i^\phi(q_{-i}, \cdot; C) \) & \( x_i^\phi(q_{-i}, \cdot; C) \) both are strictly convex in \( q_i \). By the definition of \( \bar{s}_i \) it follows that \( \bar{q}_i = \arg \max_{q_i \in [0, q_{\max}]} \lambda \cdot (q_i, x_i^\phi(q_{-i}, q_i; C)). \) But then from lemma 1.3 we must have \( \bar{q}_i = \bar{q}_i. \)

Finally to conclude the proof we notice, \( q_i \) being one to one function of \( z_i \) implies that \( \bar{s}_i = \bar{s}_i. \) Also the way we have defined \( \zeta^a \) and \( \xi^{a^*} \), they coincide.
References


CHAPTER 2

Implementing Efficient Graphs in Connection Networks

2.1. Introduction

We consider the problem of sharing the cost of a congestion-free network which meets the connection demands of a set of agents. The agents simultaneously choose a path in the network connecting the demand nodes of the agents, and a mechanism splits the total cost of the network formed among the participants. This type of problem arises in many contexts ranging from water distribution systems, road networks, telecommunications services and multicast transmission to large computer networks such as Internet.

The Shapley Mechanism ([3]), which divides the cost of every edge equally among its users, has become focal in this setup. Even though Shapley looks a natural mechanism in this setting, there are serious problems associated with it which we discuss as following. First, this method may provide wrong incentives to the players and they may end up choosing an inefficient graph in equilibrium. Indeed, consider the network in figure 2.1 right. The equilibrium under the Shapley mechanism is \((st_1, st_2)\) which has a total cost equal to 2, whereas the efficient connection network has cost equal to \(\frac{3}{2} + \epsilon\). Even the best equilibrium can be as costly
Figure 2.1. Symmetric networks with a common source and two sinks

as $H(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$ times the cost of optimal graph, where $k$ is the number of users ([2]). Next issue with Shapley mechanism is its asymmetry at equilibrium. Even though the mechanism is symmetric, at equilibrium it may charge different amounts to agents who are in exactly symmetric situations before the choice of paths by the agents. To see this problem, consider the symmetric network for two agents with common sources and two sinks depicted in the left panel of figure 2.1. Here, the Nash equilibria of the Shapley mechanism are $(st_1, st_1t_2)$ or $(st_2t_1, st_2)$. Thus agents pay either $(\frac{1}{2}, 1-e)$ or $(1-e, \frac{1}{2})$ depending on the equilibrium. Hence, even though the network is symmetric, agents pay different costs at equilibrium under the Shapley mechanism. This example also points to the multiplicity of equilibria and thus the problem of equilibrium selection. Next major concern with
this mechanism is that they are not continuous in the network structure. The mechanism is very discontinuous and hence unstable: the two networks in figure 2.1 can be arbitrarily close under any measure, whereas the equilibriums will be arbitrarily different under the Shapley mechanism. Continuity is also desirable since unavoidable measurement errors in practical life may lead to very unfair outcomes.

Finally, we notice that the amount of information needed for Shapley mechanism may not be practical in many settings. The Shapley mechanism needs as input the paths chosen by each agent. This information can be out of reach in many settings. Consider for instance the network of roads in a state, district or a country to be financed by the users of the roads. The procurement of the information on exact paths used by the drivers needs the compulsory installment of GPS (Global Positioning System) in all the vehicles and the data to be stored and updated by a central taxing authority. Due to privacy issues this may not be possible politically (see for example [10]). However, tax based on the number of miles driven can be implemented without raising that much privacy concerns. Road maintenance taxes, based on the miles driven by every user have been used in pilot programs in Oregon since January 2009, and other states like Ohio, Pennsylvania, Colorado, Florida, Rhode Island, Minnesota and Texas are considering them (see [5, 6, 7, 8, 9, 10]). This kind of environment requires mechanisms where the input is the total cost of the paths used by the agents rather than the path itself. Moreover, in spite of the information on the paths being available, it may sometimes be desirable to use just the total costs of the paths rather than
the paths itself. Consider, for instance a big or highly dynamic network structure, where agents join and leave the network continuously. It may be impractical to change the formulae of our mechanism every time the network changes. One such example is sharing the cost of a telephone network or Internet where the agreement is generally monthly but there are agents coming in and leaving the network continuously. Notice that, long distance calls being charged the same makes sense irrespective of number of users who share the edges\(^1\). There are normative concerns too for charging the agents who may not be responsible for their links not being shared by a lot users. Examples are electricity/water supply or postal services to remote villages.

This type of setting demands a new framework which is easy to implement in such settings where the inputs of the mechanism are only the total cost of the agents demand and the total cost of the network formed. This type of problem resembles the classic bankruptcy problem (also referred in the literature as rationing or taxation problem), where a given amount of resources (e.g., money) must be divided among beneficiaries with unequal claims on the resources (see [18] [20] for detailed surveys about the problem).

\(^1\)The choice of path is not a strategy for the telephone user and thus the setting is not exactly the same but the cost-sharing method has a similar motivation, namely its simpler than charging every caller differently based on the path used.
2.1.1. Overview of the results

We propose mechanisms which implement the efficient graph in a centralized communication network. Our definition of implementation is weaker than that of full implementation. More precisely, we say that an outcome is implemented by a mechanism when that outcome is a Nash Equilibrium (NE) in the game induced by the mechanism. We also provide an equilibrium selection rule when multiplicity of equilibria exists. We require the implemented graph to Pareto dominate any other graph which is an equilibrium under that mechanism, whenever possible. The main contribution of the paper is the characterization of mechanisms which implements the efficient graph under such robust equilibria. It turns out that the mechanisms monotonic in total cost, which admits efficient graph as equilibrium and Pareto dominating other equilibrium graphs, also admits efficient graph as a strong equilibrium (Theorem 2.1). We also give a characterization of the average cost mechanism (AC) ([17] [13]) which divides the total cost of the network equally among its participants (Theorem 2.2).

The main downplay of AC is that it does not meet individual rationality (IR, also referred in the literature as voluntary participation): agents demanding cheap links may pay more than the cost of their demands, thus they may subsidize agents who demand expensive links. We show that there is no efficient rule that is compatible with IR (Theorem 2.3). However, we find out that the egalitarian rule (EG), a rule reminiscent to the AC that meets IR, always possesses a pure strategy
NE and satisfies IR. EG is optimum across all rules meeting IR under the Price of Stability measure (PoS),\(^2\) the traditional inefficiency measure used in this literature (see [3]). EG is no more wasteful than the Shapley mechanism. It has a price if stability equal to \(H(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}\), where \(k\) is the number of agents in the network\(^3\) (Theorem 2.3ii). This is remarkable since, as we have discussed before, EG requires much less information than Sh. Finally, the proportional method, a seemingly natural method in this framework, also admits a pure strategy NE but is far more inefficient than the egalitarian rule (Theorem 2.3iii).

2.1.2. Related literature

The performance of Sh have been widely studied in recent literature. [3] studies the equilibrium behavior of separable mechanisms, a class of decentralized mechanisms that divides the cost of each edge among its users. The PoS of separable mechanisms with linear cost-sharing function is at least \(H(k)\) (which is \(O(\log k)\)), where \(k\) is the number of agents [3]. \(H(k)\) is also also the upper bound on PoS(Sh) in general graphs [2], thus Sh is optimal among linear separable mechanisms. PoS(Sh) is achieved in directed graphs. If the graph is undirected, PoS(Sh) is lower than \(H(k)\). [1] finds a new upper bound of \(O(\log \log k)\) when the graph is single source.

\(^2\)PoS is computed by finding the maximum of the ratio of the best Nash equilibrium and the efficient graph over all problems.

\(^3\)An alternative measure is the price of anarchy (PoA). PoA is computed similarly to PoS, but using the worst Nash equilibrium instead of the best. EG and Sh are also equally inefficient under PoA. Both rules have a PoA equal to \(k\). However, PoA is not informative since any other symmetric rule has the same PoA.
and there are no steiner nodes. [11] finds a new upper bound of $O(\log k / \log \log k)$ for single source networks when steiner nodes are allowed. [3] shows that the upper bound in two player case with single source is $\frac{4}{3}$. As it turns out, the PoS is even when there are two sources and two sinks. We provide a proof of this claim (claim 2.1) in the section 2.6.1.

2.2. The model

We fix the number of agents $\bar{K} = \{1, 2, \ldots, k\}$. A network cost-sharing problem is a tuple $N = \langle G, K \rangle$, where $G = (V, E)$ is a network which is a directed or undirected such that each edge $e \in E$ has a non negative cost $c_e$. $K = \{\{s_1, t_1\}, \{s_2, t_2\}, \ldots, \{s_k, t_k\}\}$, where $\{s_i, t_i\} \in 2^V$ for all $i \in \bar{K}$, is the set of sources and sinks that agents want to connect. When there is no confusion, we also denote $K = \bar{K}$ the set of agents. Let the set of all graphs be $G$, and the set of all network cost-sharing problems be denoted by $N$.

Given a problem $N \in N$, a strategy for agent $i$ is a path $P_i \subseteq E$ which connects $s_i$ to $t_i$. Let the set of paths connecting $s_i$ to $t_i$ be $\Pi_i(N)$. Let $\Pi(N) \equiv \times_{i \in \bar{K}} \Pi_i(N)$ is the set of strategy profiles of all agents in network $N$. $P = \{P_i\}_{i=1}^k \in \Pi(N)$ will be used to denote a strategy profile of the agents. When there is no confusion we denote $\Pi_i(N)$ and $\Pi(N)$ simply as $\Pi_i$ and $\Pi$ respectively. Let $G_P = (\bigcup_{i \in \bar{K}} P_i)$, the network formed by the choice of paths by different agents. Let $C(P) = \sum_{e \in G_P} (c_e)$ the cost of the graph formed by strategies $P$. 
Let $\mathcal{N} = \bigcup_{N \in \mathcal{N}} P(N) \times N$ the union of all problems with their respective strategies.

**Definition 2.1.** A cost-sharing mechanism is a mapping $\varphi : \mathcal{N} \to \mathbb{R}^+_k$ such that

$$\sum_{i \in K} \varphi_i(P, N) = C(P)$$

for all $(P, N) \in \mathcal{N}$.

A cost-sharing mechanism assigns non-negative cost-shares to the users of the network based on their demands such that the total cost of the network formed is exactly collected.

**Example 2.1.**

- The Shapley mechanism, $\text{Sh}$, divides the cost of every link equally across it users, that is $\text{Sh}_i(P, N) = \sum_{e \in P_i} \frac{u(e, P)}{U(e, P)}$ for all $i \in \bar{K}$, where $U(e, P)$ is the number of users of link $e$ in the strategy profile $P$.

- The proportional to stand-alone mechanism, $\eta^r$, divides the cost of the network in proportion to every user’s stand-alone cost. That is, $\eta_i^r(P, N) = \frac{SA_i(N)}{\sum_{j=1}^{k} SA_j(N)} C(P)$ for all $i \in \bar{K}$, where $SA_i(N) = \min_{P_i \in \Pi(N)} C(P_i)$ is the stand alone of agent $i$ in network $N$.

- The Average cost mechanism $\text{AC}$ divides the cost of the network formed equally across all users. That is $\text{AC}_i(P, N) = \frac{C(P)}{k}$ for all $i \in \bar{K}$.

The Shapley mechanism is a separable mechanism, that is it divides the cost of every link only across its users, and adds those costs for all links in the network formed. Alternative separable mechanisms can be constructed by considering
different cost-sharing rules for the links, for instance by giving priority across all users. Nevertheless, Sh is the optimal mechanism (using the price of stability measure, see below) across all separable mechanism ([3]). Sh can be computed in polynomial time.

On the other hand, \( \eta^{pr} \) divides the cost of the network in proportion to the stand-alone of the agents. Since the stand-alone of every agent has to be computed for every network, this mechanism uses the full information of the network.

AC divides the cost of the network formed equally across the users of the network. It is the most egalitarian rule, reminiscent to the classic head tax rule where the size of the demands of the agents is not important, only the size of the total cost of the network formed. AC uses less information than Sh or \( \eta^{pr} \), since only the total cost of the network formed and the number of agents is needed to compute the cost-sharing allocation. There is no need to know the stand-alone of the agents, or the users of certain links. As such, its computation complexity is minimal.

**Definition 2.2.** A cost-sharing mechanism \( \varphi \) is network independent if for any two problems \( N = \langle G, K \rangle \) and \( N' = \langle G', K' \rangle \) and strategies \( P \in P(N) \) and \( P' \in P(N') \) such that \( C(P_i) = C(P'_i) \) for all \( i \in K \) and \( C(P) = C(P') \): \( \varphi(P, N) = \varphi(P', N') \).

Network independence captures those mechanisms that only depend on the cost of the network being formed and the cost of the demands of the agents. Neither
Sh nor \eta^p are network independent. On the other hand, AC only uses the total cost of the network and the number of users, thus it is network independent. More complex network independent mechanism are discussed below.

Let \( S^k = \{(c; y) \in \mathbb{R}_+ \times \mathbb{R}_+^k | \max_i y_i \leq c \leq \sum_i y_i \} \).

**Lemma 2.1.** A cost-sharing mechanism \( \varphi \) is network independent if and only if there is a unique function \( \xi : S^k \to \mathbb{R}_+^k \) such that \( \sum_i \xi_i(c; y) = c \) for all \( (c; y) \in S^k \), and

\[
\varphi(P, N) = \xi(C(P); C(P_1), \ldots, C(P_k))
\]

for all problems \((P, N) \in \mathcal{N}\).

**Proof.** The sufficient part is obvious. We prove the necessity only.

First, for any \((\tilde{c}; \tilde{y}) \in S^k\) we construct the network \( \tilde{N}(\tilde{c}; y) \) as follows. Assume without loss of generality that \( \tilde{y}_1 \geq \tilde{y}_2 \geq \cdots \geq \tilde{y}_k \). Choose \( i, i \in \{1, \ldots, k\} \) such that:

\[
\tilde{y}_1 + \tilde{y}_2 + \cdots + \tilde{y}_i \leq c < \tilde{y}_1 + \tilde{y}_2 + \cdots + \tilde{y}_{i+1}.
\]

Let \( \tilde{N}(c; y) \) be a linear network such that every agent has a unique strategy. All agents 1 to \( i \) have demand \( \tilde{y}_i \) that do not intersect. Agent \( i + 1 \) has demand \( \tilde{y}_{i+1} \) such that a segment of length \( c - (\tilde{y}_1 + \tilde{y}_2 + \cdots + \tilde{y}_i) \) does not intersect the other agents, and \( \tilde{y}_1 + \tilde{y}_2 + \cdots + \tilde{y}_{i+1} - c \) intersects the other agents. Agent \( j, j > i + 1 \) has demand \( \tilde{y}_j \) contained on the demands of the agents \( \{1, \ldots, i + 1\} \).
Clearly, the unique strategy of agent \( k \) in \( N(c; y) \) is \( y_k \), and the network formed by all strategies has cost \( c \). Define \( \xi : S^k \to \mathbb{R}^k_+ \) as \( \xi(c; y) = \varphi(N(c; y)) \).

Second, consider any arbitrary network \( N = \langle G, K \rangle \) and a set of demands \( P \). On one hand, notice that \( C(P) \geq C(P_i) \) for every agent \( i \), since \( P_i \subseteq P \). On the other hand, notice that \( C(P) \leq C(P_1) + \cdots + C(P_k) \), since \( P \subseteq P_1 \cup P_2 \cup \cdots \cup P_k \).

Let \( y_i = C(P_i) \) and \( c = C(P) \). Then \( (c; y) \in S^k \). By network independence: \( \varphi(P, N) = \varphi(N(c; y)) = \xi(c; y) \). The uniqueness of \( \xi \) follows because it is well defined on \( S^k \).

Notice a network independent mechanism is reduced to the function \( \xi \) that is similar to a taxation (rationing, bankruptcy) solution ([20, 18]). Since we only work on mechanisms that are network independent, we refer without loss of generality to the function \( \xi \) as a mechanism. We describe below some desirable properties on the function \( \xi \).

**Definition 2.3.** A mechanism is continuous if the function \( \xi : S^k \to \mathbb{R}^k_+ \) is a continuous function with the Euclidean distance.

Continuous mechanisms capture the fact that small perturbations on the demand or cost of the network should not change the total allocation of the cost. All the network independent mechanism described in this paper meet continuity. Continuity is used on all the result without referring to it.
Given a problem \( N = \langle G, K \rangle \), we say \( P^* \) is an efficient graph if \( P^* \in \arg \min_{P \in \Pi(N)} C(P) \). That is, \( P^* \) is a graph that connects all the agents at a minimal cost.

Given the problem \( N = \langle G, K \rangle \), the mechanism \( \xi \) induces the following non-cooperative game \( \Gamma^\xi(N) \equiv <\tilde{K}, \{\Pi_i(N)\}_{i \in \tilde{K}}, \{\xi_i\}_{i \in \tilde{K}} \rangle \), where the representation of the game is the standard representation of game in normal form. Namely, \( \tilde{K} = \{1, \ldots, k\} \) is the set of players, \( \Pi_i(N) \) is the strategy space of player \( i \), and \( \xi_i \) is the (negative of) payoff function of player \( i \) which maps a strategy profile to real numbers.

\( P \) is a Nash Equilibrium (NE) of \( \Gamma^\xi(N) \), if \( P_i \in \arg \min_{\tilde{P}_i \in \Pi_i(N)} \xi_i(\tilde{P}_i, P_{-i}) \) for all \( i \). Let

\[
NE(\Gamma^\xi(N)) \equiv \{ P \in \Pi(N) | P \text{ is a Nash Equilibrium of } \Gamma^\xi(N) \}
\]

be the set of Nash equilibriums of the game \( \Gamma^\xi(N) \).

We say that \( \xi \) (weakly) implements \( P \), if \( P \in NE(\Gamma^\xi(N)) \).

**Definition 2.4.** The mechanism \( \xi \) is efficient (EFF) if it implements an efficient graph for any problem \( N \), that is \( P^* \in NE(\Gamma^\xi(N)) \) for some efficient graph \( P^* \).

The definition of efficiency just requires an efficient graph to be selected as a Nash equilibrium. This does not preclude other equilibriums to be selected.
Notice AC is efficient. Indeed, at any strategy profile $P^*$ that implements an efficient graph every agent is paying $\frac{C(P^*)}{k}$. If an agent $i$ deviates from $P^*$ to $\tilde{P}_i$, then he will pay $\frac{C(P_i, P^*_{-i})}{k}$. Clearly, $\frac{C(P_i, P^*_{-i})}{k} \geq \frac{C(P^*)}{k}$ by the optimality of $P^*$.

**Definition 2.5.** The mechanism $\xi$ Pareto Nash Implements (PNI) an efficient graph if for any problem $N$, it implements an efficient graph and that graph Pareto dominates any other equilibrium. That is, for any problem $N$:

- There is an efficient graph $P^*$ such that $P^* \in NE(\Gamma^\xi(N))$, and
- For any other $P \in NE(\Gamma^\xi(N)) : \xi(P^*) \leq \xi(P)$.

PNI is a very robust property that guarantees the efficient allocation is selected even when multiplicity of equilibria arise. In the case of multiplicity of equilibria, PNI guarantees that all agents would prefer the efficient graph to any other equilibrium. Hence, multiplicity of equilibria is not an issue.

In particular, this guarantees that whenever there are multiplicity of equilibria such that agent $i$ prefers equilibrium $P^i$ to $P^j$, and agent $j$ prefers equilibrium $P^j$ to $P^i$, there should exist another equilibrium $P^*$ (the efficient equilibrium) such that agent $i$ prefers equilibrium $P^*$ to $P^i$ and agent $j$ also prefers equilibrium $P^*$ to $P^j$.

The AC mechanism is also PNI. Indeed, at the efficient graph $P^*$, this equilibrium would Pareto dominate any other equilibrium $\tilde{P}$ since $\frac{C(P^*)}{k} \leq \frac{C(\tilde{P})}{k}$. 
Definition 2.6. The mechanism \( \xi \) Strongly Nash Implements (SNI) an efficient graph if for any problem \( N \) it implements an efficient graph in strong Nash equilibrium. That is for any problem \( N \),

- There is an efficient graph \( P^* \) such that \( P^* \in NE(\Gamma^\xi(N)) \), and
- for any group of agents \( S \subset \{1, \ldots, k\} \), and \( P \in \Pi(N) \) such that \( P_{-S} = P^*_{-S} \), if \( \xi_i(P) > \xi_i(P^*) \) for some \( i \in S \), then \( \xi_j(P) < \xi_j(P^*) \) for some \( j \in S \).

Under SNI there is no group of agents who can coordinate paths and weakly improve all of them, and at least one agent in the group strictly improve. In particular, this is similar to the Strong Nash equilibrium and to the literature on group strategyproofness ([12, 15]).

On the other hand, SNI is stronger than weakly group strategyproof, where profitable deviations are such that all agents strictly gain. We provide an example below that shows that this property is not enough to derive the main theorem.

The AC mechanism is also SNI. Indeed, at any deviation \( \tilde{P}_S \) of the group of agent \( S \) from the efficient graph \( P^* \), it should be that \( \frac{C(P^*)}{k} \leq \frac{C(\tilde{P}_S, P^*)}{k_{NS}} \) for all \( i \in S \). Hence no agent in \( S \) would strictly improve by deviating.

Definition 2.7. There is an efficient graph \( P^* \) such that \( P^* \in NE(\Gamma^\xi(N)) \), and
- for any group of agents \( S \subset \{1, \ldots, k\} \), and \( P \in \Pi(N) \) such that \( P_{-S} = P^*_{-S} \), if \( \xi_i(P) > \xi_i(P^*) \) for some \( i \in S \), then \( \xi_j(P) < \xi_j(P^*) \) for some \( j \in S \).

- The mechanism is demand monotonic (DM) if for all feasible problems \( (c; y), (c; \tilde{y}) \in S^k \) such that \( y_{-i} = \tilde{y}_{-i} \) and \( y_i < \tilde{y}_i \) : \( \xi_i(c; y) \leq \xi_i(c; \tilde{y}) \).
The mechanism is strongly demand monotonic (SDM) if for all feasible problems \((c; y), (c; \tilde{y}) \in S^k\) such that \(y_i = \tilde{y}_i\) and \(y_i < \tilde{y}_i\): \(\xi_{-i}(c; y) \geq \xi_{-i}(c; \tilde{y})\).

Demand monotonicity is a weak property that requires that whenever the demand of the agent increases, everything else fixed, his payment should not decrease. Notice that does not preclude the payment of other agents would not change. Under SDM, the increase on the demand of one agent does not increase the payment of other agents. In particular, notice that SDM implies DM since all the agent's payments have to add up to a constant.

AC is clearly strongly demand monotonic since \(AC(c; y) = AC(c; \tilde{y})\). Thus the increase of the demand of one agent does not change the payments of the other agents.

2.3. Main result

We now turn to the main result of the paper. We characterize the mechanisms that meet the efficiency properties discussed above.

**Theorem 2.1.** Assume there are three or more agents, then the following statements are equivalent for the mechanism \(\xi\):

1. \(\xi\) is EFF and SM.
2. \(\xi\) PNI the efficient graph.
3. \(\xi\) SNI the efficient graph.
(4) There is a monotonic function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+^k$ such that $\sum_i f_i(c) = c$ and for all feasible problems $(c; y)$, $\xi(c; y) = f(c)$.

The mechanisms characterized by theorem 2.1 are demand independent, that is the cost-share of every agent do not depend on whether the agents are demanding cheap or expensive links, instead they only depend on the total cost of the network formed. The average cost mechanism, generated by $f(c) = (\xi, \ldots, \xi)$, is the only mechanism in this class that treat equal agents equally.

The above statements are independent. Indeed, consider the mechanism

$$\tilde{\xi}(c; y) = (\min\{y_3, \frac{c}{k}\}, \frac{2c}{k} - \min\{y_3, \frac{c}{k}\}, \frac{c}{k}, \ldots, \frac{c}{k}).$$

First notice that $\tilde{\xi}$ implements the efficient graph because at the efficient graph agents $\{3, \ldots, k\}$ do not have the incentive to deviate since by doing so their payment is going to increase. On the other hand, agents $\{1, 2\}$ do not have any incentive to deviate from the efficient equilibrium since the functions $\min\{y_3, \frac{c}{k}\}$ and $\frac{2c}{k} - \min\{y_3, \frac{c}{k}\}$ are weakly monotonic in the total cost of the network and do not depend on their report.

$\tilde{f}$ is also an example of a mechanism that is not SNI, but agents cannot strictly improve by coordinating. Hence the mechanisms characterized by Theorem 2.1 are not weakly group strategyproof.
2.3.1. Efficient mechanisms for two agents

The example above shows that for three or more agents, EFF is not enough to characterize the demand independent rules. On the other hand, this property is enough when there are two agents. The property is an immediate consequence of a separability lemma described below.

**Proposition 2.1.** Assume there are two agents, $K = \{1, 2\}$. A mechanism is efficient if and only if there is a monotonic function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+^2$ such that $f_1(c) + f_2(c) = c$ and for all feasible problems $(c; y)$, $\xi(c; y) = f(c)$.

2.3.2. Equal treatment of equals

**Definition 2.8.** The mechanism satisfies equal treatment of equals (ETE) if for all agents $i, j$ and $(c; y) \in N^K$ such that $y_i = y_j : \xi_i(c; y) = \xi_j(c; y)$.

ETE is the standard property of equal responsibility for the cost of the good. Equal agents with the same demand should be allocated the same cost. There is a large class of solutions that meet ETE. We describe in section 2.4 alternative rules that meet ETE, like the Proportional and Egalitarian solution.

**Theorem 2.2.** A mechanism is EFF and ETE if and only if it is AC.

Notice this proposition is not directly implied by theorem 2.1, since we do not need Strong Monotonicity. Instead, it is a separability lemma discussed in section 5.
2.4. Individually rational mechanisms

Definition 2.9. The mechanism is individually rational if for all \((c; y) \in S^k\):
\[
\xi_i(c; y) \leq y_i \text{ for all } i.
\]

Individually rational mechanisms rule out cross-subsidies, that is no agent should pay more than the cost of their demand.

Notice neither \(AC\) nor any mechanism discussed in theorem 2.1 meet individual rationality. Therefore, the traditional incompatibility of efficiency, budget balance and individual rationality also holds in this problem.\(^4\)

On the other hand, there is a large class of individually rational mechanisms that are network independent: most of the mechanisms discussed in the rationing/bankruptcy literature meet IR, see for instance [20, 18].

Definition 2.10. • The proportional mechanism (PR): \(PR_i(c; y) = \frac{y_i}{y_1 + \cdots + y_k} c\)

• The egalitarian mechanism (EG): \(EG_i(c; y) = \min\{y_i, \lambda\}\) where \(\lambda\) solves \(\sum_i \min\{y_i, \lambda\} = c\).

\(^4\)Nevertheless, this incompatibility only holds since we consider Network Independent mechanisms. If we remove network independence then there is large class of mechanisms that always implement the efficient network and at the same time meet individual rationality. For instance, consider the proportional to stand-alone mechanism \(p_{PR}\) discussed above. \(p_{PR}\) is individually rational because no agent pays more than his stand alone, which in turns is less than his demand. On the other hand, \(p_{PR}\) implements the efficient allocation because the cost-share of every agent is in proportion to the cost of the network, therefore any deviation of the efficient graph that increases the total cost of the network formed would increase the cost share of all the agents.
PR and EG are the traditional and most compelling mechanisms in the cost-sharing literature. PR divides the cost in proportion to their demands. On the other hand, EG divides the cost equally across the agents subject to no agent paying more than their demand.

Contrary to the traditional analysis of this problem. The games induced by PR and EG are not potential games, therefore the previous potential techniques used in the analysis of this problems do not work anymore. We do not know if any mechanism (induced by a rationing method) always has a pure strategy Nash equilibrium. Nevertheless, we show below that PR and EG always have a pure strategy Nash equilibrium and provide algorithms to compute them.

Lemma 2.2. PR and EG always admit a pure strategy Nash equilibrium.

Even if the existence of equilibrium of other individually rational mechanisms is unknown, no mechanism would be more efficient than EG.

Theorem 2.3. i. There is no mechanism that is individually rational and EFF. Any individually rational mechanism has a PoS at least $H(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$.

ii. The PoS of EG is $H(k)$.

iii. The PoS of PR is of order $k$.

Since the Shapley mechanism has a price of stability equal to $H(k)$, then EG is as inefficient Shapley. No other individually rational mechanism can be more
efficient than $EG$ and Shapley. On the other hand, the traditional proportional mechanism is extremely inefficient, since its price of stability is bounded by $k$, its maximal loss approaches that in the limit.

2.5. Conclusions

This paper provides a new perspective to the problem of cost-sharing in networks. In particular, we provide new concepts of implementation and characterize the class of mechanisms that meet three robust definitions of efficiency. The average cost mechanism is the benchmark mechanism characterized by this paper.

It is also shown that efficiency and individual rationality are not compatible. The egalitarian mechanism is optimal across the mechanisms that are individually rational. $EG$ is not efficient, but is an optimal mechanism across all individually rational mechanisms using the price of stability measure. We also show that $EG$ outperforms the Shapley mechanism on the grounds of efficiency, stability and fairness.

We do not know if $EG$ is the unique optimum mechanism within the individually rational mechanisms, but conjecture this is true. We know that other mechanisms, like the proportional mechanism, are much more inefficient than $EG$. The difficulty we encounter in tackling this question is that even for simple rationing mechanisms, there is no general technique to evaluate whether or not these mechanisms even have a pure strategy Nash equilibrium.
Finally, we conjecture that our main characterization theorem discussed above on Pareto-Efficient implementation can be extended to a more general class of mechanisms that only depend on the path chosen by the agents and the network formed (notice this class contains the Shapley mechanism and all traditional separable mechanisms).

2.6. Proofs

2.6.1. The claim in the "Related literature" section

Before stating the claim we define potential games which will be used in the proof of the claim.

**Definition 2.11.** A game is said to be a **potential game** if there exists a function $P : S_1 \times S_2 \times \ldots \times S_k \rightarrow R$ such that for every $1 \leq i \leq k$, for every $x_i, x'_i \in S_i$, and for every $x_{-i} \in S_{-i}$ it holds that $u_i(x_i, x_{-i}) > u_i(x'_i, x_{-i}) \iff P(x_i, x_{-i}) > P(x'_i, x_{-i})$. If it holds also that $u_i(x_i, x_{-i}) - u_i(x'_i, x_{-i}) = P(x_i, x_{-i}) - P(x'_i, x_{-i})$, then the game is said to be an **exact potential game**.

The game induced by the Sh is an exact potential game where the potential function $P$ is the following

\[
P(x) = \sum_{e \in E} c_e H(n_x(e))
\]
where, (i) $n_x(e)$ is the number of players whose strategy contains edge $e \in E$ under the strategy profile $x$ and (ii) $H(k) = 1 + 1/2 + 1/3 + \ldots + 1/k$.

**Claim 2.1.** The upper bound on PoS of Shapley mechanism in general undirected graphs is $4/3$ when there are two players.

**Proof.** Let $\{s_1, t_1\}$ and $\{s_2, t_2\}$ be the demand nodes of player 1 and 2 respectively. Notice first, that in any optimal graph or in any graph which is a NE, the shared section of paths of the agents is an interval. Otherwise, there will be cycles contradicting the optimality of the graph or giving at least one of the players a strictly profitable deviation under the Shapley rule. Let the cost structure of the optimal graph be the following. $x_3$ is the cost of the shared path. $x_1$ & $x_2$ respectively are the costs of the paths connecting $s_1$ & $s_2$ to the shared path. $\dot{x}_1$ & $\dot{x}_2$ respectively are the costs of the paths connecting $t_1$ & $t_2$ to the shared path. Similarly, the costs of the respective paths in the equilibrium graph achieved by the best response dynamics (which always converges) starting from the optimal graph are $y_1, y_2, y_3, \dot{y}_1, \dot{y}_2$. See figure 2.2 below (the dotted graph is the NE from br dynamics from the optimal graph and the solid graph is the optimal graph). □

Since the NE is achieved from the br dynamics, the potential of the NE must be no more than that of the optimal. Therefore, we have the following inequality

\[ y_1 + y_2 + 3y_3/2 + \dot{y}_1 + \dot{y}_2 \leq x_1 + x_2 + 3x_3/2 + \dot{x}_1 + \dot{x}_2 \]  

(1)

Since the dotted graph is a NE, we have the following inequality
Figure 2.2. Upper bound on PoS of Sh in undirected graphs for 2 agent case

\[ y_1 \leq x_1 + x_2 + y_2 / 2 \]  \hspace{1cm} (2)

\[ y_2 \leq x_1 + x_2 + y_1 / 2 \]  \hspace{1cm} (3)

(2) + (3) \implies

\[ y_1 + y_2 \leq 4(x_1 + x_2) \]  \hspace{1cm} (4)

Similarly,

\[ \dot{y}_1 + \dot{y}_2 \leq 4(\dot{x}_1 + \dot{x}_2) \]  \hspace{1cm} (5)

(4) + (5) \implies

\[ y_1 + y_2 + \dot{y}_1 + \dot{y}_2 \leq 4(\dot{x}_1 + \dot{x}_2 + x_1 + x_2) \]  \hspace{1cm} (6)

(6) \times 1/3 + (1) \times 8/3 \implies

\[ 3(y_1 + y_2 + \dot{y}_1 + \dot{y}_2) + 4y_3 \leq 4(\dot{x}_1 + \dot{x}_2 + x_1 + x_2 + x_3) \]
\[ 3(y_1 + y_2 + y_1 + y_2 + y_3) \leq 4(\hat{x}_1 + \hat{x}_2 + x_1 + x_2 + x_3) \]

The bound is tight is demonstrated by the known example in figure 2.3 below.

Figure 2.3. Lower bound on PoS of Sh in 2 agent case

Here both the agents 1 and 2 want to connect to the same source and they end up using the edges (st_1) and (st_2) as the only NE and thus making the total connection cost go to 2. Note that the minimum cost of connection is \(3/2+\epsilon\). Thus in the limit the PoS is \(4/3\).

2.6.2. Proof of Lemma 2.2

2.6.2.1. Existence of equilibrium for PR.
Proof. We prove a stronger property which is that the best response (br) dynamics (one agent at a time) of any arbitrary fixed ordering of agents converges to a NE, no matter from where we start the br dynamics. Suppose on contrary, that for some fixed ordering of agents the br dynamics from some point "s" does not converge. This means that there is a cycle of a finite length \( l - s(1) \rightarrow s(2) \rightarrow s(3) \rightarrow \ldots \ldots \rightarrow s(l) \rightarrow s(1) \). Say, without loss of generality, this cycle includes deviations by the set of agents \( M = \{1, 2, \ldots, m\} \subseteq K \). The strategy of agents in \( K/M \) is fixed at \( s^{-M} \). Notice that \( l \) is at least as big as \( 2m \). This is so because after the \( l \) best responses we arrive at the original strategy profile i.e., \( s(1) \). Since, every agent in \( M \) is a part of the cycle which in turn means that they change their strategy at least once. Therefore, it must be the case that every agent in \( M \) takes its turn at least twice so that they reach the original profile i.e., \( s(1) \). Let's assume that agent \( i \in M \) takes its turn in the br dynamics \( n_i > 1 \) number of times so that \( \sum_{i \in M} n_i = l \). Let the strategies played by the agent \( i \) in the cycle be \( s^{i;1}, s^{i;2}, \ldots, s^{i;n_i}, s^{i;1} \) and so on. Let's call the agent who takes his turn of br in the movement from \( s_t \) to \( s_{t+1} \) as agent \( a_t \). Therefore, \( s(1) = (s^{1;1}, s^{2;1}, \ldots, s^{n_i;1}, s^{-M}) \), \( s(2) = (s^{a_1;2}, s_{-a_1}(1)) \), \( s(3) = (s^{a_2;2}, s_{-a_2}(2)) \), \( \ldots \ldots \ldots \ldots \), \( s(l-1) = (s^{a_{l-1};n_{a_l-1}}, s_{-a_{l-1}}(l-2)) \), \( s(l) = (s^{a_l;n_{a_l}}, s_{-a_l}(l-1)) \). Here, we use the standard notation where \( s_{-i}(t) \) represents the strategy profile of \( K \setminus \{i\} \) fixed at that in \( s(t) \). We abuse the notation and say that the cost of \( s^{p\bar{p}} \) is equal to \( s^{p\bar{p}} \). Here the cost of the network formed by the strategy profile \( s(i) = C(G_{s(i)}) \). Now, \( \xi_j^{s(i)}(C(G_{s(i)}); s(i)) = s^{j;p} A_i \) where \( A_i \) is fixed for any particular \( s(i) \) and \( s^{j;p} \) represents the strategy of agent \( j \) in \( s(i) \).
The fixed $A_i$ for an $s(i)$ is the ratio of $C(G_{s(i)})$ to the sum of the costs of individual paths in $s(i)$.

Now every step of the cycle corresponds to an inequality which we will present as following:

Step 1: $s(1) \rightarrow s(2) \implies$

\[(1) \quad s^{a_1;2} \times A_2 < s^{a_1;1} \times A_1\]

Step 2: $s(2) \rightarrow s(3) \implies$

\[(2) \quad s^{a_2;2} \times A_3 < s^{a_2;1} \times A_2\]

Step 3: $s(3) \rightarrow s(4) \implies$

\[(3) \quad s^{a_3;t} \times A_4 < s^{a_3;t-1} \times A_3; t = \begin{cases} 3 \text{ if } a_3 = a_1 \\ 2 \text{ otherwise} \end{cases}\]

Step p: $s(p) \rightarrow s(p + 1) \implies$
Step I: \( s_t \rightarrow s_1 \) \( \implies \)

If we multiply the systems (2), (3), ..., (1) together, then everything else cancels out and we are left with \( s_{a1}^2 \times A_2 > s_{a1}^1 \times A_1 \) which contradicts the inequality (1). Therefore, we conclude that there can not be any cycle no matter what ordering of agents and what initial point we follow for the best response dynamics. 

2.6.2.2. Existence of equilibrium for EG and \( POS(EG) = H(k) \).

Proof. We prove by induction on the number of players that EG has an equilibrium and the \( POS(EG) = H(k) \).

The base of induction is one player. This case is trivially true for one player since the game is just an optimization exercise and any optimal graph, which is a cheapest path of connecting her demand nodes, is a NE .

\(^5\)Notice, we can do that since everything here is positive.
We now assume that for all networks with number of agents \( m < n \), with an efficient graph \( G^*_m \) there exists a NE which costs no more than \( H(m) \ast C(G^*_m) \). We claim that for a network with \( n \) agents, with an efficient graph \( G^*_n \) there exists a NE of cost no more than \( H(n) \ast C(G^*_n) \). Let's call the set \( N \equiv \{1, 2, 3, \ldots, n\} \).

Let's start from the efficient graph \( G^*_n \). Now, under \( \xi^\text{uni}(G^*_n) \) there can be two cases. Either there exist an agent \( t \) s.t. \( \xi^\text{uni}_t(G^*_n) \leq \frac{G^*_n}{n} \) or \( \xi^\text{uni}_t(G^*_n) = \frac{G^*_n}{n} \), \( \forall j \in N \).

If it is the the first case then pick the agent with the lowest cost share\(^7\) and call this agent, "agent i". If it is the second case then there can be two cases. Either there exist an agent who has a profitable deviation or there doesn't exist such an agent. If there doesn't exist such an agent then our claim is trivially true since \( G^*_n \) is a NE. If such agents exist then pick one of them and call her "agent i".

Now, ask the agent \( i \) to take her best response. There can be two cases—either \( \xi^\text{uni}_i(G^*_n) = \lambda(y) = \frac{C(G^*_n)}{n} \) or \( \xi^\text{uni}_i(G^*_n) = y_i < \frac{C(G^*_n)}{n} \). In both the cases, the only way agent \( i \) has a profitable best deviation is when she moves to a cheapest path \( P^*_i \) (which is also called the stand alone of agent \( i \) ) connecting her demand nodes s.t.,

\[
(6.1) \quad C(P^*_i) < \xi^\text{uni}_i(G^*_n) \leq \frac{C(G^*_n)}{n}
\]

\(^6\)\( \xi^\text{uni}(G) \) is defined in the obvious way, where \( d(G) = C(G) \) and \( y_i = C(P_i) \) where \( P_i \) is the path chosen by agent \( i \).

\(^7\)In fact we can pick any agent with the cost share less than \( \frac{C(G^*_n)}{n} \). It doesn’t matter for the proof.
Now, there can be two cases.

**Case 1:** There exists such a cheapest path $P_i^*$ and agent $i$ moves to $P_i^*$.

In this case, the new network has a cost $\hat{C}$ s.t., $C(G_n^*) \leq \hat{C} \leq C(G_n^*) + C(P_i^*)$.

Let's consider an efficient graph $G_{-i}^*$ for connecting all the agents in $N/\{i\}$. Notice that since there are less nodes to be connected and all edges are still available, we have the following inequality

\begin{equation}
C(G_{-i}^*) \leq C(G_n^*)
\end{equation}

Now, ignoring agent $i$ there will be a network game with the the player set $N/\{i\}$. From the induction hypothesis it follows that there exists a graph configuration $G_{-i}^{NE}$ which is a NE of this game and

\begin{equation}
C(G_{-i}^{NE}) \leq H(n - 1) * C(G_{-i}^*)
\end{equation}

We claim that if we add player $i$ to the set $N/\{i\}$ then the configuration $\tilde{G}_n$, where $i$ is playing $P_i^*$ and $N/\{i\}$ are fixed at the configuration $G_{-i}^{NE}$, is a NE of the game amongst the player set $N$. Let's denote the demand profile in $\tilde{G}_n$ and $G_{-i}^{NE}$ by $y$ and $y_{-i}$ respectively. First notice that the optimality of $G_n^*$ implies that $C(\tilde{G}_n) \geq C(G_n^*)$. But, this means that agent $i$ who is paying $C(P_i^*) < \frac{C(G_n^*)}{n} \leq \frac{C(\tilde{G}_n)}{n}$ does not have any profitable deviation. This is so because $P_i^*$ is the cheapest path to
connect her demand nodes and it is impossible to bring the $\lambda(y)$ below $\frac{C(G^*_n)}{n}$ given the optimality of $G^*_n$. Let's think about the players in $N/\{i\}$ under $\tilde{G}_n$. By the addition of player $i$ in the network, the strategy space of players in $N/\{i\}$ remain unchanged. Only thing which may change is the cost shares of the agents. For all $j \neq i$, $\xi_{ij}(G^N_{-i}) \geq \xi_{ij}(\tilde{G}_n)$. This happens so because $\lambda(y) \leq \lambda(y_{-i})$. Thus, the agents whose cost shares were below $\lambda(y)$ remain unaffected. Also, they will not have any profitable deviation since they are already paying their stand alone costs and by introduction of new player $i$ their strategy set remains unchanged and thus their stand alone remains unchanged. For the agents whose cost shares were above $\lambda(y)$, their stand alone must be above $\lambda(y)$. Therefore, the only deviation $\hat{y}_j$ which is profitable to such an agent $j$ is the one which brings the $\lambda(y_{-j}, \hat{y}_j)$ below $\lambda(y)$. But, such a deviation would have been profitable in the game with the player set $N/\{i\}$ under the configuration $G^N_{-i}$ contradicting $G^N_{-i}$ being a NE. Thus we have shown that $\tilde{G}_n$ is a NE. Only thing which remains to be shown is that $C(\tilde{G}_n) \leq H(n) * C(G^*_n)$. Since $\tilde{G}_n$ is the union of the edges of $G^N_{-i}$ and $P_i^*$ we must have
\( C(\tilde{G}_n) \leq C(G^{NE}_{-i}) + C(P_i^*) \tag{2.2} \)

\[ \leq H(n - 1) \ast C(G^*_{-i}) + C(P_i^*) \tag{2.3} \]

\[ \leq H(n - 1) \ast C(G^*_n) + C(P_i^*) \tag{2.4} \]

\[ \leq H(n - 1) \ast C(G^*_n) + \frac{C(G^*_n)}{n} \tag{2.5} \]

\[ = H(n) \ast C(G^*_n) \tag{2.6} \]

Here the second, third and fourth inequalities comes from (6.3), (6.2) and (6.1) respectively.

**Case 2:** There doesn’t exist a br deviation for agent \( i \). But, from the way we have chosen our agent \( i \), this means that \( y_i \) under \( G^*_n \) is less than \( \frac{C(G^*_n)}{n} \). Then we call the existing choice of the path by agent \( i \) as \( P_i^* \) and everything else follows exactly as in the case 1 above. \( \square \)

### 2.6.3. Preliminary Lemmas

**Definition 2.12.** The mechanism is monotonic in cost if for all feasible problems \( (c; y), (c'; y) \in \mathcal{N}^K \) such that \( c < c' \): \( \xi(c; y) \leq \xi(c'; y) \).

**Lemma 2.3.** If the mechanism \( \xi \) is efficient then it is monotonic in total cost.
Proof. Consider two feasible problems \((c; y)\) and \((\hat{c}; y)\), where \(\hat{c} > c\) and \((\hat{c} - c) < \min_{i\in K}\{y_i\}\). Suppose, there exists an agent \(i\) and an efficient \(\xi\) such that \(\xi_i(\hat{c}; y) < \xi_i(c; y)\). Then we can have a network configuration which will contradict the efficiency of \(\xi\). Consider a network where, agents \(j \neq i\) have just one strategy each \(P_j\) which costs \(y_j\). Agent \(i\) has two strategies \(P_i\) and \(P'_i\) both of which cost \(y_i\).
but $P_i$ makes the total cost of the network $c$ and $P_i'$ makes the total cost go up to $c$. To see what kind network will generate these problems, consider the following two cases. Case 1: $c \leq \sum_{j \neq i} y_j$. In this case we can have a configuration as shown in figure 2.4. Here, the demands of agents in $K \setminus \{i\}$ is contained in the interval $a \rightarrow b$ which costs $c$. This is possible since when $c = \sum_{j \neq i} y_j$, we can have $a \rightarrow b$ as the concatenation of the demand links of the agents $j \neq i$. When $c < \sum_{j \neq i} y_j$, we can have the demand links overlapping e.g., when $\max_{j \neq i} y_j = c$, then $a \rightarrow b$ is the demand link of the biggest demander and all other demands overlap with his. $P_i = s_i \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow t_i$ and $P_i' = s_i \rightarrow v_2 \rightarrow v_3 \rightarrow t_i$. All the costly links of $P_i$ is contained in $\{ \cup P_j \}$ whereas there are links of cost $c' - c$ which are not contained in $\{ \cup P_j \}$ under $P_i'$. Again, this is possible since $c'$ and $c$ are close enough to guarantee that for all $i$ we can have such paths. Case 2: $\sum_{j \in K} y_j > c > \sum_{j \neq i} y_j$. In this case we can have a configuration as shown in figure 2.5. Here, the interval $a \rightarrow b$ is the concatenation of the demand links of agents in $K \setminus \{i\}$. Thus $|a \rightarrow b| = \sum_{j \neq i} y_j$, $|s_i \rightarrow a| = c - \sum_{j \neq i} y_j$, $|a \rightarrow d| = c' - c$. $|s_i \rightarrow a \rightarrow d| = |s_i \rightarrow a' \rightarrow d| = c' - \sum_{j \neq i} y_j$. $P_i = s_i \rightarrow a \rightarrow d \rightarrow t_i$ and $P_i' = s_i \rightarrow a' \rightarrow d \rightarrow t_i$. Notice that it may be the case that $t_i = b$. Now clearly in both the cases, $i$ will have a profitable deviation from the efficient graph of cost $c$ thus contradicting the efficiency of $\xi$. Thus we have shown that efficient $\xi$ must be monotonic in total cost in some open neighborhood of $c$ for all $c$. Therefore, we can extend the argument to conclude that $\xi$ must be monotonic in total cost in general. □
Figure 2.6. Efficiency amounts to separability

**Lemma 2.4** (Separability Lemma). *If the mechanism $\xi$ is efficient then $\Rightarrow$ $\xi(C; y) = (\xi_1(C; y_1), \xi_2(C; y_2), \ldots, \xi_k(C; y_k))$. That is, any efficient mechanism is separable and assigns the costs shares to the agents independently of their demand.*

**Proof.** If we prove that for any feasible problems $(c; y)$ and $(c; \tilde{y}_j, y_{-j})$, any continuous and efficient $\xi$ must have $\xi_i(c; y) = \xi_i(c; \tilde{y}_i, y_{-i})$ then we are done. Consider a feasible problem $(c; y)$. Consider a graph as shown in Figure 2.6 which generates this problem. The sources and sinks of agents $j \neq i$ lie on the the ray $a \to b$ according the demand profile, i.e., the agent with the highest demand covers most of the span on $a \to b$ and so on. Thus, an agent $j \neq i$ has one strategy which generates the demand $y_j$. Agent $i$ has two strategies—either connect $s_i - t_i$ through $v_1$ or through $v_2$. The demands of agent $i$ when connecting through $v_1$ and $v_2$ are
\(\tilde{y}_i\) and \(y_i\) respectively. Now, the total cost when \(i\) uses \(v_1\) and \(v_2\) are respectively "\(c + \epsilon\)" and "\(c\)". Notice, by moving the position of \(v_2\) and arranging the demand links of the agents \(j \neq i\), we can generate all the feasible problems \((c; y_i, y_{-i})\). Also, by moving the position of \(v_1\) and arranging the demand links of the agents \(j \neq i\), we can generate all the feasible problems \((c + \epsilon; y_i, y_{-i})\). Consider an efficient \(\xi\) which is continuous. Efficiency of \(\xi\) requires the following inequality

\[
(2.7) \quad \xi_i(c; y_i, y_{-i}) \leq \xi_i(c + \epsilon; \tilde{y}_i, y_{-i})
\]

Using continuity we get

\[
(2.8) \quad \xi_i(c; y_i, y_{-i}) \leq \xi_i(c; \tilde{y}_i, y_{-i})
\]

Similarly, switching the position of \(v_1\) and \(v_2\) and using continuity again we get

\[
(2.9) \quad \xi_i(c; y_i, y_{-i}) \geq \xi_i(c; \tilde{y}_i, y_{-i})
\]

Thus, we conclude that \(\xi_i(c; y_i, y_{-i}) = \xi_i(c; \tilde{y}_i, y_{-i})\) for all feasible problems \((c; y_i, y_{-i})\) and \((c; \tilde{y}_i, y_{-i})\).  \(\square\)
2.6.4. Proof of Proposition 2.1

Consider a problem \((c; y_1, y_2) \in S^2\).

By separability lemma: \(\xi_1(c; y_1, y_2) = \xi_1(c; c, y_2)\).

By budget balance: \(\xi_2(c; y_1, y_2) = \xi_2(c; c, y_2)\). Thus, \(\xi(c; y_1, y_2) = \xi(c; c, y_2)\).

By separability lemma: \(\xi_2(c; c, y_2) = \xi_2(c; c, c)\).

By budget balance: \(\xi_1(c; c, y_2) = \xi_1(c; c, c)\). Thus, \(\xi(c; c, y_2) = \xi(c; c, c)\).

Hence \(\xi(c; y_1, y_2) = \xi(c; c, c)\).

2.6.5. Proof of Theorem 2.1

2.6.5.1. 1. \(\implies 4..\)

Proof. Consider a continuous \(\xi\) which is efficient and strongly monotonic. Consider two arbitrary feasible problems \((c; y)\) and \((c; \bar{y})\). We will prove that \(\xi(c; y) = \xi(c; \bar{y}) = f(c)\). The monotonicity of \(f\) comes from lemma 1. Let \(a = \frac{1}{k} \sum_{i \in K} y_i\) and \(\bar{a} = \frac{1}{k} \sum_{i \in K} \bar{y}_i\). Assume without loss of generality that \(y_1 \leq y_2 \leq y_3 \leq \ldots \leq y_k\) and \(\bar{y}_1 \leq \bar{y}_2 \leq \bar{y}_3 \leq \ldots \leq \bar{y}_k\).

Step 1: \(\xi(c; y) = \xi(c; a, a, \ldots, a)\) and \(\xi(c; \bar{y}) = \xi(c; \bar{a}, \bar{a}, \ldots, \bar{a})\)

Proof:

Consider the following problems: \(P_0 = (c; y), P_1 = (c; a, y_2, y_3, \ldots y_k), P_2 = (c; a, a, y_3, y_4, \ldots, y_k), \ldots, P_k = (c; a, a, \ldots, a)\). Notice first that feasibility of \(P_0\) implies the feasibility of \(P_1, P_2, \ldots, P_k\). This is true because maximum of the demand profile doesn’t go above \(y_k\) in all these problems and sum of the individual demands
is always at least \( k \cdot a = \sum_{i \in K} y_i \). Similarly, if we define the counterpart problems \( \tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_k \) where \( \tilde{P}_i = (c; \tilde{a}, \tilde{a}, \ldots, \tilde{a}, \tilde{y}_{i+1}, \tilde{y}_{i+2}, \ldots, \tilde{y}_{k-1}, \tilde{y}_k) \), then again all of them will be feasible.

Now, due to the separability lemma (lemma 2) we must have \( \xi_1(P_0) = \xi_1(P_1) \). But then, strong monotonicity and budget balancedness implies \( \xi_{-1}(P_0) = \xi_{-1}(P_1) \). Thus, we have \( \xi(P_0) = \xi(P_1) \). Using the same argument we have \( \xi(P_i) = \xi(P_{i+1}) \) and \( \xi(\tilde{P}_i) = \xi(\tilde{P}_{i+1}) \) for all \( 0 \leq i \leq k - 1 \). Thus, we have \( \xi(P_0) = \xi(P_k) \) and \( \xi(\tilde{P}_0) = \xi(\tilde{P}_k) \) as desired.

Step 2: \( \xi(c; a, a, \ldots, a) = \xi(c; \tilde{a}, \tilde{a}, \ldots, \tilde{a}) \)

Proof:

Notice first that feasibility of \( (c; a, a, \ldots, a) \) & \( \xi(c; \tilde{a}, \tilde{a}, \ldots, \tilde{a}) \) implies that any problem \( (c; \tilde{a}) \) where some of the \( \hat{a}_i = a \) and other \( \hat{a}_i = \tilde{a} \) is also feasible. Now, lemma 2 implies \( \xi_1(c; a, \tilde{a}, \ldots, \tilde{a}) = \xi_1(c; \tilde{a}, a, \ldots, \tilde{a}) \). Now, there can be three cases- \( a < \tilde{a}, a > \tilde{a} \) or \( a = \tilde{a} \). In the first two cases strong monotonicity and budget balancedness implies \( \xi_{-1}(c; a, \tilde{a}, \ldots, \tilde{a}) = \xi_{-1}(c; \tilde{a}, a, \ldots, \tilde{a}) \) and we get \( \xi(c; a, \tilde{a}, \ldots, \tilde{a}) = \xi(c; \tilde{a}, a, \ldots, \tilde{a}) \). The third case trivially implies \( \xi(c; a, \tilde{a}, \ldots, \tilde{a}) = \xi(c; \tilde{a}, \tilde{a}, \ldots, \tilde{a}) \) since its the same problem so the solution must be the same. Similarly, we get

\[ \xi(c; \tilde{a}, \tilde{a}, \ldots, \tilde{a}) = \xi(c; a, a, \ldots, a) = \xi(c; a, a, \tilde{a}, \ldots, \tilde{a}) = \ldots = \xi(c; a, a, \ldots, a) \]

2. \( \implies \) 1.
Proof. We know that $\xi$ PNI efficient graph implies $\xi$ is efficient. We will prove that $\xi$ PNI efficient graph implies $\xi$ is strongly monotonic. Consider a $\xi$ which PNI efficient graph and a feasible problem $(c; y)$ and assume without loss of generality that $y_1 < y_2 < \ldots < y_k$. Now, consider a graph as shown in figure 2.7 below.

![Graph diagram](image)

Figure 2.7. PNI implies strong monotonicity

Here every agent has two strategies—either use the path in the solid graph or use that in the dotted graph. Let’s call the solid graph as "**" and the dotted graph as "*". Let "*" be a small perturbation of "**" as following. The cost of path of an agent $j \neq i$ in both the graphs is $y_j$. The cost of path of agent $i$ in "**" and "*" are $y_i$ and $\tilde{y}_i$ where $\tilde{y}_i$ is in a neighborhood of $y_i$ and $\tilde{y}_i > y_i$ and $|\tilde{y}_i - y_i| < \min_{j,k \in K} |y_j - y_k|$. This restriction guarantees the ranking to be preserved in the perturbed problem. Let the total cost of "**" and "*" be "$c - \epsilon$" and "$c$". 

---

8The case of weak inequality will follow from the assumption of continuity on our method.
respectively. First we will show that this graph generates all feasible problems
\((c; y)\). This happens if and only if the following system has a solution:

\[
x_1 + a_1 = y_1 \\
x_2 + a_2 + a_1 = y_2 \\
x_3 + a_3 + a_2 + a_1 = y_3 \\
\vdots \\
x_k + a_k + a_{k-1} + \ldots + a_1 = y_k \\
\sum_{i=1}^{k} x_i + \sum_{i=1}^{k} a_i = c \\
\forall i \in K; x_i, a_i \geq 0
\]

We use Farka’s Lemma to prove that this system has a solution:

From the Farka’s lemma we know that \(Ax = b; x \geq 0\) has a solution if and only
if \(A^Tz \geq 0; b^Tz < 0\) doesn’t have a solution.

Here, the \((k + 1) \times (2k)\) matrix \(A\), vector \(x\) and vector \(b\) are defined as follows:

\[
A = \begin{bmatrix}
1 & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots \\
0 & 1 & 0 & \ldots & 1 & 1 & 0 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
1 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots \\
\end{bmatrix}
\]
\[
x = \begin{bmatrix} x_1 & x_2 & \ldots & x_k & a_1 & a_2 & \ldots & a_k \end{bmatrix}^T
\]
\[
b = \begin{bmatrix} y_1 & y_2 & \ldots & y_k & c \end{bmatrix}^T
\]

\[A^T z \geq 0; \ b^T z < 0 \text{ gives the following } (2k + 1) \text{ inequalities;}
\]

(1) \[z_1 + z_2 + \ldots + z_{k+1} \geq 0\]

(2) \[z_2 + z_3 + \ldots + z_{k+1} \geq 0\]

(\ldots) \quad \vdots

(k) \[z_k + z_{k+1} \geq 0\]

(k+1) \[z_1 + z_{k+1} \geq 0\]
\[(k+2)\quad z_2 + z_{k+1} \geq 0\]

\[
(2k)\quad z_k + z_{k+1} \geq 0
\]

\[(2k+1)\quad y_1z_1 + y_2z_2 + \ldots + y_kz_k + cz_{k+1} < 0\]

Now, do the following operation on the first \(k\) inequalities: \(y_1 \times (1) + (y_2 - y_1) \times (2) + \ldots + (y_k - y_{k-1}) \times (k)\), to get,

\[(2k+2)\quad y_1z_1 + y_2z_2 + \ldots + y_kz_k + y_kz_{k+1} \geq 0\]

Now, for the inequalities (2k+1) and (2k+2) to be compatible, it must be the case that \(z_{k+1} < 0\). Let, this be the case and let (2k+2) and (2k+1) hold. Then, (2k+1) implies:
This is true because feasibility requires $\sum_{i \in K} y_i \geq c$. Now, if we do the following operation on inequalities $(k+1)$ through $(2k)$: $y_1 \times (k + 1) + y_2 \times (k + 2) + .... + y_n \times (2k)$, then we get,

\[(2k+4) \quad y_1z_1 + y_2z_2 + ..... + y_kz_k + \left( \sum_{i \in K} y_i \right)z_{k+1} \geq 0\]

which contradicts $(2k+3)$ to give us the desired result.

We now prove the strong monotonicity of $\xi$. Clearly, the efficiency of $\xi$ implies that "*" is a NE but since "*" is a perturbation of "**", we will have "*" as a NE for the perturbation small enough. The fact that $\xi$ PNI the efficient graph implies the following inequality

$$\xi(c - c; y_i, y_{-i}) \leq \xi(c; \tilde{y}_i, y_{-i})$$

Using continuity we get,

$$\xi(c; y_i, y_{-i}) \leq \xi(c; \tilde{y}_i, y_{-i})$$
Now consider a perturbation where every thing is exactly the same but "***"

costs "c + ε". Using the same argument of Pareto Nash implementability and
continuity we get

\[ \xi(c; y_i, y_{-i}) \geq \xi(c; \tilde{y}_i, y_{-i}) \]

Thus we conclude that \( \xi(c; y_i, y_{-i}) = \xi(c; \tilde{y}_i, y_{-i}) \) for \( \tilde{y}_i \) in an open neighborhood

of \( y_i \). But we can, repeatedly using the open neighborhood argument, show that
this is true for any arbitrary \( y_i \) and \( \tilde{y}_i \) as long as \( (c; y_i, y_{-i}) \) and \( (c; \tilde{y}_i, y_{-i}) \) are both
feasible.

3. \( \implies \) 4.

Consider a continuous \( \xi \) which implements the efficient graph in Strong NE.
Consider two arbitrary feasible problems \((c; y)\) and \((c; \tilde{y})\). We will prove that
\( \xi(c; y) = \xi(c; \tilde{y}) = f(c) \). The monotonicity of \( f \) comes from lemma 1. Let \( a = \frac{1}{k} \sum_{i \in K} y_i \) and \( \bar{a} = \frac{1}{k} \sum_{i \in K} \tilde{y}_i \). Assume without loss of generality that \( y_1 \leq y_2 \leq y_3 \leq \ldots \leq y_k \) and \( \tilde{y}_1 \leq \tilde{y}_2 \leq \tilde{y}_3 \leq \ldots \leq \tilde{y}_k \).

**Step 1:** \( \xi(c; y) = \xi(c; a, a, \ldots, a) \) and \( \xi(c; \tilde{y}) = \xi(c; a, a, \ldots, a) \)

Proof:

Consider the following problems: \( P_0 = (c; y) \), \( P_1 = (c; a, y_2, y_3, \ldots y_k) \), \( P_2 = (c; a, a, y_3, y_4, \ldots, y_k) \), \ldots \, P_k = (c; a, a, \ldots, a) \). Notice first that feasibility of \( P_0 \)
implies the feasibility of \( P_1, P_2, \ldots, P_k \). This is true because maximum of the demand
profile doesn’t go above \( y_k \) in all these problems and sum of the individual demands
is always at least \( k \cdot a = \sum_{i \in K} y_i \). Similarly, if we define the counterpart problems \( \tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_k \) where \( \tilde{P}_i = (c; \tilde{a}, \tilde{a}, \ldots, \tilde{a}, \tilde{y}_{i+1}, \tilde{y}_{i+2}, \ldots, \tilde{y}_{k-1}, \tilde{y}_k) \), then again all of them will be feasible.

Now, due to the separability lemma (lemma 2) we must have \( \xi_1(P_0) = \xi_1(P_1) \). Also, strong Nash implementability implies that \( \xi_{-1}(P_0) = \xi_{-1}(P_1) \). To see this, suppose that it is not the case and for some agent \( j \neq 1 \), we have \( \xi_j(P_0) \neq \xi_j(P_1) \). Assume without loss of generality that \( \xi_j(P_0) < \xi_j(P_1) \). This means \( \exists \tilde{j} \in K \setminus \{1, j\} \) s.t., \( \xi_{\tilde{j}}(P_0) > \xi_{\tilde{j}}(P_1) \), because of budget balancedness. Consider a network where all the agents 2, 3, ..., \( k \) have just one strategy which costs \( y_2, y_3, \ldots, y_k \) and agent 1 has two strategies, where one of them costs \( y_1 \) and the other costs \( a \). In both the cases, the total cost of the network is \( c \). Thus one of the configurations generates the problem \( P_0 \) and the other \( P_1 \). Now both the configurations of the network is efficient and therefore at least one of them must be a strong NE under \( \xi \). But clearly none of them is a strong NE. From \( P_1 \) the group \( \{1, \tilde{j}\} \) has a profitable deviation and from \( P_0 \) the group \( \{1, \tilde{j}\} \). Thus, we have \( \xi(P_0) = \xi(P_1) \). Using the same argument we have \( \xi(P_i) = \xi(P_{i+1}) \) and \( \xi(\tilde{P}_i) = \xi(\tilde{P}_{i+1}) \) for all \( 0 \leq i \leq k - 1 \). Thus, we have \( \xi(P_0) = \xi(P_k) \) and \( \xi(\tilde{P}_0) = \xi(\tilde{P}_k) \) as desired.

**Step 2:** \( \xi(c; a, a, \ldots, a) = \xi(c; \tilde{a}, \tilde{a}, \ldots, \tilde{a}) \)

Proof:

Notice first that feasibility of \( (c; a, a, \ldots, a) \) & \( \xi(c; \tilde{a}, \tilde{a}, \ldots, \tilde{a}) \) implies that any problem \( (c; \tilde{a}) \) where some of the \( \tilde{a}_i = a \) and other \( \tilde{a}_i = \tilde{a} \) is also feasible. Now,
lemma 2 implies $\xi_1(c; a, \bar{a}, ..., \bar{a}) = \xi_1(c; \bar{a}, \bar{a}, ..., \bar{a})$. And again, the strong Nash implementability implies $\xi_{-1}(c; a, \bar{a}, ..., \bar{a}) = \xi_{-1}(c; \bar{a}, \bar{a}, ..., \bar{a})$. The proof of this statement is analogous to the one in step 1. Thus we have $\xi(c; a, \bar{a}, ..., \bar{a}) = \xi(c; \bar{a}, \bar{a}, ..., \bar{a})$. Similarly, we get $\xi(c; \bar{a}, \bar{a}, ..., \bar{a}) = \xi(c; a, \bar{a}, ..., \bar{a}) = \xi(c; a, a, \bar{a}, ..., \bar{a}) = \xi(c; a, a, ..., \bar{a}) = \xi(c; a, a, ..., a)$.

The results "4. $\implies$ 1.", "4. $\implies$ 2" and "4. $\implies$ 3" are straightforward and the proof is omitted.

2.6.6. Proof of Theorem 2.2

**Proof.** The "if" part is clear. For, "only if" consider an arbitrary feasible problem $(c; y)$. Assume without loss of generality that $y_1 \geq y_2 \geq y_3 \geq ... \geq y_k$. Let $a = \frac{1}{k} \sum_{i=1}^{k} y_i$. Consider a problem $(c; a, a, ..., a)$ and suppose that $\xi$ is continuous, efficient and satisfies ETE. Notice, the feasibility of $(c; y)$ implies the feasibility of $(c; a, a, ..., a)$ and any other problem $(c; \hat{y})$ where $\hat{y}_i = y_i$ for all $i \in \{1, 2, ..., l\}$ and $\hat{y}_i = a$ for all $i \in \{l, l + 1, ..., k - 1, k\}$. Now, the ETE property of $\xi$ implies

\[
(2.10) \quad \xi(c; a, a, ..., a) = (c/k, c/k, ..., c/k)
\]

Using lemma 2 and applying ETE again we get,
Now again applying lemma 2, and ETE we have,

\[ \xi(c; y_1, y_2, a, a, \ldots, a) = (x_1, c/k, x, x, \ldots, x) \]

But if we change the ordering of 1 & 2 while arriving the above profile then we should have,

\[ \xi(c; y_1, y_2, a, a, \ldots, a) = (c/k, x_2, x, x, \ldots, x) \]

But since the ordering is immaterial so we must have \(x_1, x_2, x = c/k\). And thus we have,

\[ \xi(c; y_1, y_2, a, a, \ldots, a) = (c/k, c/k, \ldots, c/k) \]

Repeating the same argument, we conclude that \( \xi(c; y) = (c/k, c/k, \ldots, c/k) \) \qed

2.6.7. Proof of Theorem 2.3

2.6.7.1. Incompatibility of Efficiency and IR.
Proof. We show by an example that any individually rational cost sharing rule must have a PoS of at least $H(k)$. Consider a situation as shown in figure 2.8. Here, every agent $i$ has two strategies—either connect its demand nodes directly where the cost of the path is $1/i$ or connect through the path where link costs are $0$ and $1 + \epsilon$. Consider any arbitrary cost sharing method $\xi$ which satisfies individual rationality. We will show that the only equilibrium under such method will be where every agent is using their direct path to $t$. Suppose, this is not the case. This means there can be two cases. First case is where all the agents use a free link to $v$ and then the common link of cost $1 + \epsilon$ to $t$. But then at least one of the agents must be paying more than $1/k$. Let's assume that this agent is the $k$th agent in some configuration\(^9\) of the graph. Then he will have a profitable deviation.

\(^9\)It is important to note that just one such configuration is enough since PoS is measure of performance of the best NE in the worst case example.
to go to the direct link of cost $1/k$ under any individually rational rule. The other case to consider is when $s$ agents are using their direct link and $k-s$ agents are sharing the common link to "v". Then it follows from individual rationality of the $s$ agents that at least one of the agents in $k-s$ must be paying more than $1/(k-s)$. Notice, that in this case there exists an unused direct link, say $s_j \rightarrow t$, of cost $1/s_j$ which is at most $1/(k-s)$. Now in some configuration of the graph agent $j$ will be the agent who is paying the above said amount of more than $1/(k-s)$ and thus he would like to deviate. We have just shown that the only NE in some configuration of this example has a cost equal to $H(k)$ whereas the efficient graph has a cost equal to $1 + \epsilon$ where everyone uses a costless link to node $v$ and then the common link to $t$. \hfill \Box

2.6.7.2. Lower bound for PoS(PR).

\textbf{Proof.} Consider a situation as shown in figure 2.8. We show that the unique equilibrium of the proportional method is of order $k$. Let, the costs of links $s_i \rightarrow t$ be $x_i$ and the other things be exactly the same as in figure 2.8. Straightforward computations show that the $k$-th agent will deviate from the efficient graph of cost $1 + \epsilon$ if $x_k \leq \frac{1-k+\sqrt{(k-1)^2+4k(k-1)}}{2k}$. As $k$ grows, $x_k$ converges to the golden number $\frac{\sqrt{5}-1}{2}$ in contrast to $1/k$ for the uniform method which goes to zero. Also $x_{t-1} > x_t$ for all $t = 2, 3, .., k$ and $x_1 = 1$. Thus the lower bound on the PoS of proportional method is $\sum_{t=1}^{k} x_i$ which is of order $k$. \hfill \Box
References


87


CHAPTER 3

Capacity Constraint, Price Discrimination and Oligopoly

3.1. Introduction

The strategic interactions of the firms in industries have been analysed in many settings. The literature essentially has many strands originating from Cournot, Bertrand and Stackelberg. On the one hand, the outcome of the Cournot is more realistic, but on the other hand, the setup of price competition in Bertrand is more close to reality. The extremes of Cournot and Bertrand has been put together in the seminal paper by Kreps & Scheinkman [5] where capacity competition followed by price competition justifies the Cournot outcome. In many industries the existence of leaders and followers is a natural phenomenon. This is the source of another strand originating from Stackelberg [7]. Yet another dimension of firms' behavior when they have market power is that of price discrimination. Therefore, the coexistence of price discrimination with the price and quantity competition is a natural setup to analyse. This happens invariably in all industries with airline industry being a good example. To fix ideas, we will stick to the airline industry example for illustration. There is a class of recent literature focussing on this aspect [8, 4, 3, 6] (see [8] for a survey). Hazledine [3], considers the Cournot setup
with price discrimination. He finds out that the contrast from the single-price standard Cournot model is in the quantity produced in the market. He also finds out that the average price in the market is independent of the degree of price discrimination and thus the standard models' prediction is not misleading in terms of the average price. Kutlu [6] incorporates price discrimination in the Stackelberg model and finds a counterintuitive result that leader does not price-discriminate.

Our paper differs from the earlier works in that we analyse this situation as a two stage game. In the first stage, the firms compete on quantities that they will put in the market and in the second stage they will decide what fraction of the quantity they will sell to different group of buyers. In other words, in the second stage of competition for price discrimination there is a capacity constraint. For example, in the airline industry the valuation of the buyers is a function of the time when they are buying the tickets. The business travellers whose plans are generally last moment have less elastic demand whereas the tourists whose plans are almost always flexible have relatively more elastic demand. Thus different bins (groups) of buyers can be grouped based on the day they want to buy a particular airline seat. For example, higher bins consists of the likes of business travellers. In the first stage, when the firms enter the market, they buy certain number of planes thus the total number of seats are decided for the second stage of the game. They can not buy planes everyday but they can decide how to allocate the total number of tickets during a time frame. This critical assumption of the stages is missing in
the literature that we just reviewed and we hope that this will explain the missing results.

Our paper also differs from most other works e.g., [3, 6] in that we find results for general demand function rather than linear demand. We consider two firms for simplicity. The main finding of our paper is that in the second stage both the firms will be active in the higher bins. If there are $k$ bins then the firm with higher capacity will be active in all the $k$ bins. Also, the smaller firm will be active in top $t$ bins (proposition 3.1). Moreover, in the bins $1, 2, ..., t - 1$ it will match the quantity sold by the bigger firm (proposition 3.2). In proposition 3.2 we characterize the behavior of the firms upto finding $i$. Although the exact value of $i$ is not given in proposition 3.2, for linear demand case we solve for the unique $i$ using the corollaries 3.1 and 3.2. Finally, we show firms behavior in the benchmark Cournot case.

### 3.2. The Model and Results

Assume for simplicity that there are only two firms, $A$ and $B$, in the market. Let the constant marginal cost of these firms which is normalized to zero. We assume that each consumer buys at most one unit of the good. The firms know valuations of the consumers and can prevent resale of the good. They divide the consumers into bins according to their reservation prices. The price of the good for the $k^{th}$ bin is given by:
(3.1) \[ P^k = P(Q^k) \]

where \( q_A^i \) and \( q_B^i \) are the quantities sold in bin \( i \) by \( A \) and \( B \); \( Q^k \equiv \sum_{i=1}^{k} (q_A^i + q_B^i) \) is the total quantity sold in all bins from 1 to \( k \); and \( P \) is a twice continuously differentiable, strictly decreasing, and strictly concave inverse demand function that represents consumers’ valuations.

Assume that the total capacities of the firms are exogenously given by \( Q_A \) and \( Q_B \). Given these capacities firms are competing on the shares that they assign to each bin. Hence, firms choose \( s_A = (s_A^1, s_A^2, ..., s_A^{K-1}, s_A^K) \) and \( s_B = (s_B^1, s_B^2, ..., s_B^{K-1}, s_B^K) \) with \( \sum_{i=1}^{K} s_A^i = 1 \) and \( \sum_{i=1}^{K} s_B^i = 1 \) where \( q_A^i = Q_A s_A^i \) and \( q_B^i = Q_B s_B^i \).

Going back to our example of airline seats offered for a specific route, from now on we can think of the product ‘an airline seat’ and a seller ‘an airline’.

We provide the solution for the optimization problem of \( A \) and the \( B \)’s solution is the same. The optimization problem of the firms are given by:

(3.2) \[
\max \pi_A = Q_A \sum_{i=1}^{K} P^i s_A^i \\
\text{st } s_A^i \geq 0 \text{ and } \sum_{i=1}^{K} s_A^i = 1
\]
The Lagrangian for (3.2) is given by:

\begin{align*}
\mathcal{L}_A &= \pi_A + \mu_A \left( \sum_{j=1}^{K} s_j^A - 1 \right) \\
\end{align*}

Let \( \bar{\mu}_A = \frac{\mu_A}{Q_A} \). For any \( i = 1, 2, ..., K \) the Kuhn-Tucker conditions are given by:

\begin{align*}
(3.4) \quad 
 P^i + \sum_{k=1}^{K} \frac{\partial P^k}{\partial Q} \frac{\partial Q^k}{\partial s_A^k} s_A^k + \bar{\mu}_A &\leq 0 \\
(3.5) \quad 
 (P^i + \sum_{k=i}^{K} \frac{\partial P^k}{\partial Q} \frac{\partial Q^k}{\partial s_A^k} s_A^k + \bar{\mu}_A)s_A^k &= 0 \\
(3.6) \quad 
 \sum_{k=1}^{K} s_A^k &= 1 \\
(3.7) \quad 
 s_A^i &\geq 0
\end{align*}

**Proposition 3.1.** Assume that for some bin \( i \in \{1, 2, ..., K\} \) we have \( s_A^i = 0 \), then \( s_A^{i+1} = 0 \).

\footnote{Note that we are solving the problem of an active firm. Therefore it is assumed that \( Q_A > 0 \).}

\footnote{For notational simplicity we represent \( \frac{\partial P(Q)}{\partial Q} \) by \( \frac{\partial P^i}{\partial Q} \).}
Proof. Assume to get a contradiction that \( s_A^i = 0 \) and \( s_A^{i+1} > 0 \) for some \( i \in \{1, 2, ..., K - 1\} \). Then we have:

\[
P^i \leq -\sum_{k=i}^{K} \frac{\partial P^k}{\partial Q} \frac{\partial Q^k}{\partial s_A^k} s_A^k - \bar{\mu}_A = -\sum_{k=i+1}^{K} \frac{\partial P^{k+1}}{\partial Q} \frac{\partial Q^k}{\partial s_A^k} s_A^k - \bar{\mu}_A = P^{i+1}
\]

Here the inequality comes from the Kuhn-Tucker conditions; the first equality follows from our assumption that \( s_A^i = 0 \); and the second equality follows from the Kuhn-Tucker conditions given that \( s_A^{i+1} > 0 \). Hence, \( P^i \leq P^{i+1} \). But by the monotonicity of the demand \( P^i \geq P^{i+1} \) implying that \( P^i = P^{i+1} \). This in turn implies that there are \( K - 1 \) bins which is a contradiction. \( \square \)

The following proposition shows the behavior of the firms in all the bins for a general demand function. Even though we don’t have a closed form solution, for a specific demand function this proposition gives a recursive way to get the explicit solution. In one of the corollaries later we give an explicit solution for the linear demand case.

**Proposition 3.2.** Assume that \( Q_A \leq Q_B \). Let \( i \in \{1, 2, ..., K, K + 1\} \) be such that \( s_A^i = s_A^{i+1} = \ldots = s_A^K = s_A^{K+1} = 0 \) and \( s_A^j > 0 \) for all \( j < i \).\(^3\) The optimal shares for \( A \) and \( B \) are described as follows:

\(^3\)Even though there are \( K \) bins, we are using the index up to \( K + 1 \) in order to include the case where \( Q_A = Q_B \) or they are so close that \( A \) is active in all the bins. Hence, \( i = K + 1 \) means that \( s_A^i > 0 \) in all bins \( i = 1, 2, ..., K \).
Case I \((j < i - 2)\):

\[
P(2Q_A \sum_{k=1}^{j} s_A^k) - P(2Q_A \sum_{k=1}^{j+1} s_A^k) = -\frac{\partial P_j}{\partial Q} Q_A s_A^j
\]

(3.8)

\[
P(2Q_B \sum_{k=1}^{j} s_B^k) - P(2Q_B \sum_{k=1}^{j+1} s_B^k) = -\frac{\partial P_j}{\partial Q} Q_B s_B^j
\]

(3.9)

Case II \((j = i - 2)\):

\[
P(2Q_A \sum_{k=1}^{i-2} s_A^k) - P(Q_A(1 + \sum_{k=1}^{i-2} s_A^k) + Q_B s_B^{i-1}) = -\frac{\partial P^{i-2}}{\partial Q} Q_A s_A^{i-2}
\]

(3.10)

\[
P(2Q_B \sum_{k=1}^{i-2} s_B^k) - P(Q_A + Q_B \sum_{k=1}^{i-1} s_B^k) = -\frac{\partial P^{i-2}}{\partial Q} Q_B s_B^{i-2}
\]

(3.11)

Case III \((j > i - 2)\):

\[
s_A^j = 0 \text{ for } j > i - 1
\]

(3.12)

\[
s_A^{i-1} = 1 - \sum_{k=1}^{i-2} s_A^k
\]

(3.13)

\[
P(Q_A + Q_B \sum_{k=1}^{j} s_B^k) - P(Q_A + Q_B \sum_{k=1}^{j+1} s_B^k) = -\frac{\partial P_j}{\partial Q} Q_B s_B^j \text{ for } j < K
\]

(3.14)

\[
s_B^K = 1 - \sum_{k=1}^{K-1} s_B^k
\]

(3.15)
Proof. We will first prove the following lemmas which will be the key to the proof of the proposition.

**Lemma 3.1.** Assume that $Q_A \leq Q_B$. Let $i \in \{2, \ldots, K, K + 1\}$ be such that $s_A^i = s_A^{i+1} = s_A^{K+1} = 0$ and $s_A^j > 0$ for all $j < i$. Then for all $j \in \{1, 2, \ldots, K\}$ we have $s_B^j > 0$.

**Proof.** In order to prove the lemma, we consider two cases.

Case 1 ($i \leq K$): If $i \leq K$, then $s_B^K > 0$. Otherwise, there will not be $K$ bins which is a contradiction.

Case 2 ($i = K + 1$): We prove by induction. From Proposition 3.1 it follows that $s_B^1 > 0$. Let’s assume that $s_B^j > 0$ for all $j < t$. We will show that $s_B^t > 0$. Assume not, i.e. $s_B^t = s_B^{t+1} = \ldots = s_B^K = 0$. From the Kuhn-Tucker conditions we know that for $j < t$:

\[
P^j + Q_A \sum_{k=j}^{K} \frac{\partial p_k}{\partial Q} s_A^k + \bar{\mu}_A = 0 \tag{3.16}
\]

\[
P^j + Q_B \sum_{k=j}^{K} \frac{\partial p_k}{\partial Q} s_B^k + \bar{\mu}_B = 0 \tag{3.17}
\]

Let $A_j = Q_A \sum_{k=j}^{K} \frac{\partial p_k}{\partial Q} s_A^k$ and $B_j = Q_B \sum_{k=j}^{K} \frac{\partial p_k}{\partial Q} s_B^k$. Therefore:
\[ P^j + A_j + \bar{\mu}_A = 0 \]  
\[ P^j + B_j + \bar{\mu}_B = 0 \]  
\[ P^t + A_t + \bar{\mu}_A = 0 \]  
\[ P^t + B_t + \bar{\mu}_B \leq 0 \]

Subtracting the equality (3.20) from the inequality (3.21) gives:

\[ B_t - A_t + \bar{\mu}_B - \bar{\mu}_A \leq 0 \]

From (3.18) and (3.19), we know that:

\[ \bar{\mu}_B - \bar{\mu}_A = A_j - B_j \]

Therefore, we have:

\[ B_t - A_t + A_j - B_j \leq 0 \]
\[ B_j - A_j + A_{j-1} - B_{j-1} = 0 \]

From equations (3.24) and (3.25), we have:
(3.26) 

\[-Q_B \frac{\partial P_{t-1}}{\partial Q} s_{t-1}^B + Q_A \frac{\partial P_{t-1}}{\partial Q} s_{t-1}^A \leq 0\]

\[-Q_B \frac{\partial P_{t-2}}{\partial Q} s_{t-2}^B + Q_A \frac{\partial P_{t-2}}{\partial Q} s_{t-2}^A = 0\]

\[\vdots\]

(3.27) 

\[-Q_B \frac{\partial P_{t}}{\partial Q} s_{t}^B + Q_A \frac{\partial P_{t}}{\partial Q} s_{t}^A = 0\]

From monotonicity of demand, we have \(\frac{\partial P_{t}}{\partial Q} < 0\). Therefore:

(3.28) 

\[Q_B s_{t-1}^B \leq Q_A s_{t-1}^A\]

Summing over bins 1, 2, ..., t - 1 we get:

(3.29) 

\[Q_B \sum_{k=1}^{t-1} s_k^B \leq Q_A \sum_{k=1}^{t-1} s_k^A\]

or

(3.30) 

\[Q_B \leq Q_A \sum_{k=1}^{t-1} s_k^A < Q_A\]

The strict inequality follows from the fact that A is active in all bins until bin K. This is a contradiction. \(\square\)
Lemma 3.2. Assume that \( Q_A \leq Q_B \). Let \( i \in \{3,\ldots, K, K + 1\} \) be such that \( s_A^i = s_A^{i+1} = \cdots = s_A^{K+1} = 0 \) and \( s_A^j > 0 \) for all \( j < i \). Then for \( j < i - 1 \) we have:

\[
q_A^j = q_B^j
\]

Proof. Note that by Lemma 3.1 we have \( s_B^j > 0 \) for \( j \in \{1, 2, \ldots, K\} \). Hence, for all \( j < i \) we have

\[
P^j = -\sum_{i=1}^{K} \frac{\partial P^j}{\partial Q} Q_A s_A^i - \bar{\mu}_A = -\sum_{i=1}^{K} \frac{\partial P^j}{\partial Q} Q_B s_B^i - \bar{\mu}_B.
\]

Hence,

\[
P^j - P^{j+1} = q_A^j = q_B^j.
\]

Now, we continue to prove the proposition. Note that for \( j < i - 1 \) we have:

\[
(P(Q^j) - P(Q^{j+1})) = (-Q_A \sum_{k=1}^{K} \frac{\partial P^k}{\partial Q} s_A^k - \bar{\mu}_A) - (-Q_A \sum_{k=1}^{K} \frac{\partial P^k}{\partial Q} s_A^k - \bar{\mu}_A)
\]

\[
= Q_A \left( \sum_{k=j}^{K} \frac{\partial P^k}{\partial Q} s_A^k - \sum_{k=j}^{K} \frac{\partial P^k}{\partial Q} s_A^k \right)
\]

Also, by Lemma 3.1 using the similar steps as above we get:

\[
P(Q^j) - P(Q^{j+1}) = -\frac{\partial P^j}{\partial Q} Q_B s_B^j
\]

For case I, we have \( j < i - 2 \). Therefore \( Q^j = Q_A \sum_{k=1}^{j} s_A^k + Q_B \sum_{k=1}^{j} s_B^k \). Since, \( j < i - 2 \) by Lemma 3.2 we have \( Q_A s_A^j = Q_B s_B^j \). Hence, \( Q^j = 2Q_A \sum_{k=1}^{j} s_A^k = 2Q_B \sum_{k=1}^{j} s_B^k \) and \( Q^{j+1} = 2Q_A \sum_{k=1}^{j+1} s_A^k = 2Q_B \sum_{k=1}^{j+1} s_B^k \).
For case II, we have $j = i - 2$. Therefore $Q^j = 2Q_A \sum_{k=1}^{i-2} s^k_A = 2Q_B \sum_{k=1}^{i-2} s^k_B$ and $Q^{j+1} = Q_A (1 + \sum_{k=1}^{i-2} s^k_A) + Q_B s_B^{i-1} = Q_A + Q_B \sum_{k=1}^{i-1} s^k_B$.

For case III, notice that $Q_A$ is exhausted after bin $i - 1$. For bin $i - 1$, $s_A^{i-1}$ is the residual share for $A$. By Lemma 3.1, $B$ is active in bins $i, i + 1, ..., K$, i.e. $s^i_B, s_B^{i+1}, ..., s^K_B > 0$. Therefore from the Kuhn-Tucker conditions for all $j = i, i + 1, ..., K - 1$ we have:

\begin{align*}
(3.36) & \quad P^j + Q_B \sum_{k=j}^{K} \frac{\partial P^k}{\partial Q} s_B^k + \mu_B = 0 \\
(3.37) & \quad P^{j+1} + Q_B \sum_{k=j+1}^{K} \frac{\partial P^k}{\partial Q} s_B^k + \mu_B = 0
\end{align*}

Hence, we have:

\begin{align*}
(3.38) & \quad P^j - P^{j+1} = -\frac{\partial P^j}{\partial Q} Q_B s_B^j \\
\text{or} \\
(3.39) & \quad P(Q_B \sum_{k=1}^{j} s_B^k + Q_A) - P(Q_B \sum_{k=1}^{j+1} s_B^k + Q_A) = -\frac{\partial P^j}{\partial Q} Q_B s_B^j
\end{align*}

Now, we describe the solution algorithm. From the cases above, we can recursively solve $s_B^j$ in terms of $s_B^1$ for $j \leq K$. Moreover, since we have $\sum_{k=1}^{K} s_B^k = 1,$
we can solve for $s_B^1$. Once we have the solution for $s_B$'s we can solve for $s_A$'s as follows. From Case I, we can recursively solve $s_A^j$ in terms of $s_A^1$ for $j < i - 1$. Since we have $s_A^{i-1} = 1 - \sum_{k=1}^{i-2} s_A^k$, we can solve for $s_A^{i-1}$ in terms of $s_A^1$ as well. In order to solve for $s_A^1$, we use Lemma 3.2. That is, given $s_B^1$ the solution for $s_A^1$ is given by:

\begin{equation}
    s_A^1 = \frac{Q_B}{Q_A} s_B^1
\end{equation}

Note that depending on the value of $i$ some of the cases disappear. Hence, the sequence of shares might start from Case II or Case III rather than Case I. Whenever $i > 3$ the solution algorithm starts from Case I; if $i = 3$, the the solution algorithm starts from Case II; and if $i = 2$, the the solution algorithm starts from Case III.

In Proposition 3.2 we described the conditions for equilibrium shares for a general demand function. Now, in the following proposition, we give more conditions which will help identifying $i$.

**Proposition 3.3.** For any $i = 2, 3, ..., K + 1$ we have:
Proof. First, we prove the inequality (3.41). From the Kuhn-Tucker conditions we know that:

\begin{align*}
\text{(3.43)} & \quad P_i^t + A_t + \bar{\mu}_A \leq 0 \\
\text{(3.44)} & \quad P_i^t + B_t + \bar{\mu}_B = 0 \\
\text{(3.45)} & \quad P_i^{t-1} + A_{t-1} + \bar{\mu}_A = 0 \\
\text{(3.46)} & \quad P_i^{t-1} + B_{t-1} + \bar{\mu}_B = 0
\end{align*}

where $A_j = Q_A \sum_{k=j}^{K} \frac{\partial P_k}{\partial Q} s_k^A$ and $B_j = Q_B \sum_{k=j}^{K} \frac{\partial P_k}{\partial Q} s_k^B$.

Then we have:

\begin{align*}
\text{(3.47)} & \quad P_i^t - P_i^{t-1} + A_t - A_{t-1} \leq 0 \\
\text{(3.48)} & \quad P_i^t - P_i^{t-1} + B_t - B_{t-1} = 0
\end{align*}

Hence:

\begin{align*}
\text{(3.49)} & \quad A_t - A_{t-1} \leq B_t - B_{t-1}
\end{align*}
or

\[ (3.50) \quad -\frac{\partial P^{i-1}}{\partial Q} Q_A s_A^{i-1} \leq -\frac{\partial P^{i-1}}{\partial Q} Q_B s_B^{i-1} \]

By monotonicity of the demand we know that \( \frac{\partial P^{i-1}}{\partial Q} < 0 \). Therefore we have:

\[ (3.51) \quad Q_A s_A^{i-1} \leq Q_B s_B^{i-1} \]

Now, we prove the inequality (3.42). From Proposition 3.2 we know that:

\[ (3.52) \quad P(2Q_A \sum_{k=1}^{i-2} s_A^k) - P(Q_A(1 + \sum_{k=1}^{i-2} s_A^k + Q_B s_B^{i-1})) = -\frac{\partial P^{i-2}}{\partial Q} Q_A s_A^{i-2} \]

or

\[ (3.53) \quad P(2Q_A \sum_{k=1}^{i-2} s_A^k) - P(2Q_A \sum_{k=1}^{i-2} s_A^k + Q_A s_A^{i-1} + Q_B s_B^{i-1}) = -\frac{\partial P^{i-2}}{\partial Q} Q_A s_A^{i-2} \]

Since \( Q_A s_A^{i-1} \leq Q_B s_B^{i-1} \) by monotonicity of the demand we have:

\[ (3.54) \quad P(2Q_A \sum_{k=1}^{i-2} s_A^k + Q_A s_A^{i-1} + Q_B s_B^{i-1}) \leq P(2Q_A \sum_{k=1}^{i-1} s_A^k) \]

Therefore:

\[ (3.55) \quad P(2Q_A \sum_{k=1}^{i-2} s_A^k) - P(2Q_A \sum_{k=1}^{i-1} s_A^k) \leq -\frac{\partial P^{i-2}}{\partial Q} Q_A s_A^{i-2} \]
In what follows we consider the linear demand case for expositional simplicity\(^4\).

**Corollary 3.1** (Corollary to Proposition 3.2). Assume that \( Q_A \leq Q_B \). Let \( i \in \{2, \ldots, K, K + 1\} \) be such that \( s_A^i = s_A^{i+1} = \ldots = s_A^K = 0 \) and \( s_A^j > 0 \) for all \( j < i \). Moreover, assume that the demand is linear given by:

\[
P^j = a - Q^j
\]

The optimal shares for A and B are described as follows:

**Case 1** (\( i = 2 \)):

\[
s_A^1 = 1
\]

\[
s_A^j = 0 \text{ if } j > 1
\]

\[
s_B^j = \frac{1}{K}
\]

**Case 2** (\( i \geq 3 \)):

**Case I** (\( j < i - 1 \)):

---

\(^4\)For a general demand function the equilibrium can be calculated in a similar fashion.
Case II ($j = i - 1$):

\begin{align*}
(3.62) \quad s_{A}^{i-1} & = 1 - (2 - \frac{1}{2i-3})s_{A}^{1} \\
(3.63) \quad s_{B}^{i-1} & = 2s_{B}^{1} - \frac{Q_{A}}{Q_{B}}
\end{align*}

Case III ($j > i - 1$):

\begin{align*}
(3.64) \quad s_{A}^{j} & = 0 \\
(3.65) \quad s_{B}^{j} & = 2s_{B}^{1} - \frac{Q_{A}}{Q_{B}}
\end{align*}

**Proof.** For Case 1, note that by definition of $i$ and Lemma 3.1 we have $s_{A}^{1} = 1$. From equations (3.14) and (3.15) we have:
(3.66) \((a - Q_A - Q_B \sum_{k=1}^{j} s_B^k) - (a - Q_A - Q_B \sum_{k=1}^{j+1} s_B^k) = Q_B s_B^j \) for \( j < K \)

(3.67) \( s_B^K = 1 - \sum_{k=1}^{K-1} s_B^k \)

Hence:

(3.68) \( s_B^{j+1} = s_B^j \) for \( j < K \)

(3.69) \( s_B^K = 1 - \sum_{k=1}^{K-1} s_B^k \)

This implies that:

(3.70) \( s_B^j = \frac{1}{K} \)

For Case 2, we only prove the \( i > 3 \) case. The \( i = 3 \) case is similar. For Case I, by equations (3.8) and (3.9) for any \( j < i - 2 \) we have:

(3.71) \( (a - 2Q_A \sum_{k=1}^{j} s_A^k) - (a - 2Q_A \sum_{k=1}^{j+1} s_A^k) = Q_A s_A^j \)

(3.72) \( (a - 2Q_B \sum_{k=1}^{j} s_B^k) - (a - 2Q_B \sum_{k=1}^{j+1} s_B^k) = Q_B s_B^j \)
Hence:

\[(3.73) \quad 2s_A^{j+1} = s_A^j \]
\[(3.74) \quad 2s_B^{j+1} = s_B^j \]

Hence, for any \(j < i - 1\) we have:

\[(3.75) \quad s_A^j = \frac{1}{2^{j-1}} s_A^1 \]
\[(3.76) \quad s_B^j = \frac{1}{2^{j-1}} s_B^1 \]

Equation (3.62) follows from equation (3.60) and the fact that \(s_A^{i-1} = 1 - \sum_{k=1}^{i-2} s_A^k\). Now, we derive equation (3.63). From equation (3.11) we know that:

\[(3.77) \quad (a - 2Q_B \sum_{k=1}^{i-2} s_B^k) - (a - Q_A - Q_B \sum_{k=1}^{i-1} s_B^k) = Q_B s_B^{i-2} \]
\[(3.78) \quad s_B^K = 1 - \sum_{k=1}^{K-1} s_B^k \]

Hence:
(3.79) \(-2q_B \sum_{k=1}^{i-2} s_B^k - (a - q_A - q_B \sum_{k=1}^{i-1} s_B^k) = q_B s_B^{i-2}\)

Hence, we have:

\[q_A + q_B s_B^{i-1} = q_B s_B^{i-2} + q_B \sum_{k=1}^{i-2} s_B^k\]

\[q_A + q_B s_B^{i-1} = q_B s_B^{i-3} + q_B \sum_{k=1}^{i-3} s_B^k\]

\[\vdots\]

\[q_A + q_B s_B^{i-1} = 2q_B s_B^1\]

This implies that:

(3.80) \(s_B^{i-1} = 2s_B^1 - \frac{q_A}{q_B}\)

Case III directly follows from equations (3.12), (3.14), and (3.80).

Corollary 3.2 (Corollary to Proposition 3.3). For any \(i = 2, 3, ..., K + 1\) we have:

(3.81) \(2s_A^{i-1} \leq s_A^{i-2}\)
Proof. By Proposition 3.3 we know that:

$$(3.82) \quad (a - 2QA \sum_{k=1}^{i-2} s_A^k) - (a - 2QA \sum_{k=1}^{i-1} s_A^k) \leq QS_A^{i-2}$$

Hence:

$$(3.83) \quad 2s_A^{i-1} \leq s_A^{i-2}$$

Now, by utilizing the results above we solve the linear demand case for an arbitrary $i$. The solution for $i = 2$ case is already given in Corollary 3.1. Hence, in what follows we assume that $i > 2$.

Using Case II in Proposition 3.2 we have:

$$(3.84) \quad QS_A^{i-1} + QS_B^{i-1} = QS_A^{i-2}$$

Also from Corollary 3.2 we have:

$$(3.85) \quad s_A^{i-1} \leq \frac{1}{2}s_A^{i-2}$$
Thus we get the following system which will characterize $i$:

\begin{align}
(3.86) \quad & y + x + 2x + 4x + \ldots + 2^{i-3}x = Q_A \\
(3.87) \quad & (K - i + 2)(x - y) + x + 2x + 4x + \ldots + 2^{i-3}x = Q_B \\
(3.88) \quad & 0 \leq y \leq \frac{x}{2} \\
(3.89) \quad & Q_A \leq Q_B
\end{align}

Letting $H_i = 2^{i-2} - 1$ and $K_i = K - i + 2$ we have:

\begin{align}
(3.90) \quad & y + H_i x = Q_A \\
(3.91) \quad & -K_i y + (K_i + H_i)x = Q_B \\
(3.92) \quad & 0 < y \leq \frac{x}{2} \\
(3.93) \quad & Q_A \leq Q_B
\end{align}

Solving for $x$ and $y$ we have:
\[(3.94) \quad x = \frac{\tilde{K}_i Q_A + Q_B}{\tilde{K}_i H_i + \tilde{K}_i + H_i}\]

\[(3.95) \quad y = \frac{\tilde{K}_i Q_A + H_i(Q_A - Q_B)}{\tilde{K}_i H_i + \tilde{K}_i + H_i}\]

From the inequality (3.92) we have:

\[(3.96) \quad \frac{1 + 2H_i}{\tilde{K}_i + 2H_i} \geq \frac{Q_A}{Q_B} > \frac{H_i}{\tilde{K}_i + H_i}\]

Let \(\theta_i = \frac{1+2H_i}{\tilde{K}_i+2H_i}\) and \(\lambda_i = \frac{H_i}{\tilde{K}_i+H_i}\). Now, we show the uniqueness of the equilibrium. Note that it is enough to show that \([[\theta_i, \lambda_i]]\) partitions \([0,1]\). This simply means that for any given \(\frac{Q_A}{Q_B}\) value, there will be one and only one corresponding set \([[\theta_i, \lambda_i]]\). This set identifies the \(i\) that gives the equilibrium. First, note that \(\frac{\partial \theta_i}{\partial i} \geq 0\) and \(\frac{\partial \lambda_i}{\partial i} \geq 0\). Moreover, we know that \(\theta_{K+1} = 1\) and \(\lambda_2 = 0\). Hence, if \(\lambda_i = \theta_{i-1}\) for any \(i = 2, 3, ..., K\), then \([[\theta_i, \lambda_i]]\) partitions \([0,1]\). We want to show that:

\[(3.97) \quad \theta_{i-1} = \frac{1 + 2H_{i-1}}{\tilde{K}_{i-1} + 2H_{i-1}} = \frac{H_i}{\tilde{K}_i + H_i} = \lambda_i\]

or
We conclude that for any given \( \frac{\partial A}{\partial b} \) there exists a unique equilibrium for the quantity choices of \( A \) and \( B \). The equilibrium is determined by the conditions from Corollary 3.1 and inequality system (3.96).

Now, we consider the profits of the firms as a function of \( i \). The profit of \( i \) is given by:
\[ \pi_A = Q_A \left[ \sum_{i=1}^{t-2} (a - 2Q_A \sum_{k=1}^{i-2} s_A^k) s_A^i + (a - 2Q_A \sum_{k=1}^{i-2} s_A^k - Q_A s_A^{i-1} - Q_B s_B^{i-1}) s_A^{i-1} \right] \]

\[ = Q_A \left[ a + \left\{ -2Q_A \sum_{i=1}^{t-2} \left( \sum_{k=1}^{i-2} s_A^k \right) s_A^i + (-2Q_A \sum_{k=1}^{i-2} s_A^k - Q_A s_A^{i-2}) s_A^{i-1} \right\} \right] \]

\[ = Q_A \left[ a + \left\{ -2Q_A \left[ (2 - \frac{1}{2i-3}) s_A^{i-1} \right] - 2Q_A \left[ 2 - \frac{1}{2i-3} + \frac{1}{2i-2} \right] \left( \frac{1}{2i-2} s_A^{i-1} \right) \right\} \right] \]

\[ = Q_A \left[ a - 2Q_A \left[ s_A^{i-1} \right]^2 \left\{ (2 - \frac{1}{2i-3})^2 + (2 - \frac{1}{2i-3} + \frac{1}{2i-2}) \left( \frac{1}{2i-2} \right) \right\} \right] \]

\[ = Q_A \left[ a - 2Q_A \left[ s_A^{i-1} \right]^2 \left\{ 4 - 32 \frac{1}{2i} + 64 \left( \frac{1}{2i} \right)^2 + 8 \frac{1}{2i} - 32 \left( \frac{1}{2i} \right)^2 + 16 \left( \frac{1}{2i} \right)^2 \right\} \right] \]

\[ = Q_A \left[ a - 2Q_A \left[ s_A^{i-1} \right]^2 \left\{ 4 - 24 \frac{1}{2i} + 48 \left( \frac{1}{2i} \right)^2 \right\} \right] \]

Note that \( \frac{\partial \pi_A}{\partial i} \geq 0 \).

### 3.3. Cournot with price discrimination

In this section we provide generalization of the benchmark Cournot competition model. Our generalization of the Cournot model is reminiscent of Hazledine's (2006) model. In this model, as in the former section, we assume that there are two firms, \( A \) and \( B \), in the market. We normalize the costs of the firms to zero. The firms divide the consumers into \( K \) bins according to their reservation prices.
The demand is assumed to be linear and given by equation (3.56). We assume that firms are playing a two-stage game where in the first stage they choose the capacities and in the second stage they simultaneously choose the shares that they assign to each bin.

The symmetric solution would imply that $Q = Q_A = Q_B$. This implies that both firms are using all available bins. By Corollary 3.1, we conclude that:

\begin{align*}
(3.105) & \quad s_A^j &= \frac{1}{2^{j-1}} s_A^1 \\
(3.106) & \quad s_B^j &= \frac{1}{2^{j-1}} s_B^1 \\
\end{align*}

or

\begin{align*}
(3.107) & \quad q_A^j = q_B^j = \frac{Q}{2^j - \frac{1}{2^{j-1}}} \\
\end{align*}

These quantities accord with the findings of Hazledine (2006).
References


