RICE UNIVERSITY

Limits of Minimal Surfaces with Increasing Genus

by

Soomin Kim

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

Doctor of Philosophy

APPROVED, THESIS COMMITTEE:

Michael Wolf, Professor, Chair
Mathematics

Robert M. Hardt, W. L. Moody Professor
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ABSTRACT

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Minimal Surfaces are surfaces which locally minimize area. These surfaces are well-known as mathematical idealizations of soap films, one area of the calculus of variations which applies to geometric modeling. This thesis is devoted to the classification of minimal surfaces, specifically limits of minimal surfaces with increasing genus. In this paper, we will see that a particular well-known family of minimal surfaces, indexed by increasing genus, has a limit, and, further, that limit is nearly a well-known example. This is the first nontrivial example of a limit being taken of a family of minimal surfaces of increasing topological complexity. As a classification result, this would limit the set of possible minimal surfaces, as we would see that new surfaces would not be created through the taking of limits of existing families of surfaces in this way.
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Chapter 1

Introduction

In this chapter, we give the definition of a minimal surface and some examples of minimal surfaces. We also state the main theorem of this thesis and sketch the proof.

1.1 Minimal Surfaces

Minimal surfaces are surfaces which locally minimize area. As mathematical idealizations of soap films, minimal surfaces have many equivalent definitions. We introduce one of the most common.

Definition 1.1.1. A surface is a minimal surface if it has zero mean curvature, $H \equiv 0$.

That is, $H = 0$ at every point. The mean curvature $H$ at a point is the average of the maximal and minimal curvatures.

Definition 1.1.2. A surface is complete if there is no path of finite length which leaves every compact set.

Classical examples of complete minimal surfaces are the plane, the catenoid, the helicoid, Scherk’s surface, Enneper’s surface, and Riemann’s minimal surface. In
this thesis, we will focus on two classical minimal surfaces. In 1835, Scherk found a singly periodic surface that we might think of as a minimal desingularization of two intersecting planes in $\mathbb{R}^3$. Part of the surface is illustrated in figure 1.1. This surface

![Scherk's singly periodic surface](image1)

Figure 1.1. Scherk's singly periodic surface

is a periodic surface so it goes infinitely to the north and south. Also the four ends in the quotient extend infinitely. Figure 1.1 is a picture of part of the surface.

Next, Enneper's surface, found in 1869, is a complete minimal surface with finite total curvature. Note that this surface is not embedded. In 1982, Chen-Gackstatter built a genus one and a genus two version of Enneper's surface ([CG82]). Later, Do Espirito-Santo built a genus three version ([DES94]). Then Sato proved the existence of genus $n > 1$ version of Enneper's surface for any finite $n$ ([Sat96]). Weber and Wolf also independently proved the existence of these surfaces of arbitrary genus ([WW02]). However, the surfaces are not known to be unique for genus $n \geq 3$. For
more pictures of these surfaces, see [Tha95].

Figure 1.3. Generalized Enneper’s surface of genus 4

**Notation 1.1.1.** We will refer to these surfaces as the **generalized Enneper’s surface of genus** $n$, and write $E(n)$.

Note that for any $n$, $E(n)$ has reflectional symmetries about two vertical planes and rotational symmetries about two horizontal straight lines. This thesis is on an infinite genus version of Enneper’s surface $E(\infty)$. We may have some natural questions.

*Does a normalized subsequence of $\{E(n)\}$ converge to a minimal surface?*

*So do we have the generalized Enneper’s surface of infinite genus? Or does $E(n)$ degenerate as $n \to \infty$?*

*If a normalized subsequence of $\{E(n)\}$ converges, what does the limit surface $E(\infty)$ look like?*

Our main theorem in this thesis states that a normalized subsequence of $\{E(n)\}$ converges in an appropriate sense. And the limit relates to Scherk’s surface.

**Main Theorem** (Theorem 4.1.2). Let $E(n)$ be the generalized Enneper’s surface of genus $n$. Then a normalized subsequence of $\{E(n)\}$ converges, uniformly on compacta.
To prove the main theorem, we first show the following theorem on the convergence of a normalized subsequence of coordinate domains \( \{ \Omega_g(n) \} \) in \( \mathbb{C} \) for one quarter of \( E(n) \).

**Theorem** (Theorem 4.1.1). Let \( E(n) \) be the generalized Enneper's surface of genus \( n \), and let \( P_i(n) \) be the \( i \)th period from the center of \( E(n) \). Then for any fixed \( i = 1, 2, ..., n - 1 \), there exists \( 0 < C_i < \infty \) so that

\[
\frac{1}{C_i} < \frac{|P_i(n)|}{|P_{i+1}(n)|} < C_i
\]

with \( C_i \) independent of \( n \). Therefore, a normalized subsequence of coordinate domains \( \{ \Omega_g(n) \} \) in \( \mathbb{C} \) for one quarter of \( E(n) \) converges, uniformly on compacta.

Actually, the \( i \)th period \( P_i(n) \) can be expressed by \( P_i(n) = \int_{y_{i-1}(n)}^{y_i(n)} g dh \). Here \( g \), \( dh \), and the points \( y_i(n) \) in \( \mathbb{R} \) are from the flat structure for \( E(n) \) (see chapter 2).

The following theorem gives information on the points \( y_i(n) \) in \( \mathbb{R} \) which makes the proof of theorem 4.1.1 relatively easy.

**Theorem** (Theorem 3.6.1). Let \( E(n) \) be the generalized Enneper's surface of genus \( n \), and let

\[
V(n) = (-y_n(n), ..., -y_2(n), -y_1(n), 0, y_1(n), y_2(n), ..., y_n(n)) \in \mathbb{R}^{2n+1},
\]

where \( 0 < y_1(n) < y_2(n) < ... < y_n(n) \in \mathbb{R} = \partial \mathbb{H} \), be the preimage of the \((2n+1)\) vertices of the zigzag boundary \( Z_g(n) = \partial \Omega_g(n) \) of a coordinate domain \( \Omega_g(n) \) under the Schwarz-Christoffel map \( \psi_g(n) : \mathbb{H} \to \Omega_g(n) \). Then for any fixed \( i = 1, 2, ..., n - 1 \), there exists \( 0 < c_i < \infty \) so that

\[
\frac{1}{c_i} < \frac{y_i(n) - y_{i-1}(n)}{y_{i+1}(n) - y_i(n)} < c_i
\]
with \(c_i\) independent of \(n\). Therefore, a normalized subsequence of \(\{V(n)\}\) converges, uniformly on finite subsets of indices.

### 1.2 Sketch of the Proof of the Main Theorem

The proof of the theorem 3.6.1 is completed through three propositions 3.3.1, 3.4.1, and 3.5.1. The steps 1-3 below are in the proof of proposition 3.3.1, which is a main step toward theorem 3.6.1. Then with the information from theorem 3.6.1, we prove theorem 4.1.1 and then the main theorem 4.1.2.

**Step 1. Integral estimates for periods**

Our first goal is to prove proposition 3.3.1.

**Proposition 3.3.1.** For any fixed \(i = 1, 2, \ldots, n - 1\), if \(y_i(n) - y_{i-1}(n)\) are uniformly stable in the genus \(n\), then \(y_{i+1}(n) - y_i(n)\) cannot tend to 0 as \(n \to \infty\).

For fixed \(i\), let \(y_{i+1}(n) - y_i(n) = \varepsilon_i(n)\). First, we obtain integral estimates for the length \(|P_i(n)| = \left|\int_{y_{i-1}(n)}^{y_i(n)} \frac{g}{\zeta} \, dh\right|\) of the \(i\)th period from the center of \(E(n)\), and for the length \(|P_{i+1}(n)| = \left|\int_{y_{i}(n)}^{y_{i+1}(n)} \frac{g}{\zeta} \, dh\right|\) of the \((i + 1)\)st period. Then, we observe that the ratio \(R(n) = \frac{\left|\int_{y_{i+1}(n)}^{y_i(n)} \frac{g}{\zeta} \, dh\right|}{\left|\int_{y_i(n)}^{y_{i+1}(n)} \frac{g}{\zeta} \, dh\right|}\) is an analytic function of \(\varepsilon_i(n)\). Let us write

\[
R(n) = \frac{\left|\int_{y_{i+1}(n)}^{y_i(n)} \frac{g}{\zeta} \, dh\right|}{\left|\int_{y_i(n)}^{y_{i+1}(n)} \frac{g}{\zeta} \, dh\right|} = \omega_{i,0}(n) \left(1 + \omega_{i,1}(n)\varepsilon_i(n) + \cdots\right). \tag{1.1}
\]

On the other hand, we compute that

\[
R(n) = \frac{\left|\int_{y_{i-1}(n)}^{y_i(n)} \frac{g}{\zeta} \, dh\right|}{\left|\int_{y_{i-1}(n)}^{y_i(n)} \frac{g}{\zeta} \, dh\right|} = \omega_{i,0}(n) \left(1 + \omega_{i,1}(n)\varepsilon_i(n) + \omega_{i,2}(n)\varepsilon_i(n) \ln \varepsilon_i(n) + \cdots\right). \tag{1.2}
\]
where \( \omega_{i,2}(n) \neq 0 \) independent of \( n \). Here, \( R(n) = \frac{\int_{\mu(n)}^{\nu(n)} gdh}{\int_{\mu(n)}^{\nu(n)} \frac{1}{2} \ dh} \) is not an analytic function of \( \varepsilon_i(n) \) in a neighborhood of 0, since it contains the term \( \varepsilon_i(n) \ln \varepsilon_i(n) \) (lemma 3.3.1).

This difference in expansions for \( R(n) \) with respect to \( \varepsilon_i(n) \) is crucial in our argument, since the right sides of the equations (1.1) and (1.2) are expansions of the same function \( R(n) \) (this condition will be called the period problem 3.2.1 in section 3.2).

**Step 2. Bounds for coefficients**

Now, we need to compare (1.1) with (1.2). Letting \( \varepsilon_i(n) \to 0 \), we first observe that \( v_{i,0}(n) = \omega_{i,0}(n) \). Moreover, for \( \varepsilon_i(n) \) small and fixed \( n \),

\[
1 + v_{i,1}(n)\varepsilon_i(n) = 1 + \omega_{i,1}(n)\varepsilon_i(n) + \omega_{i,2}(n)\varepsilon_i(n) \ln \varepsilon_i(n) + o(\varepsilon_i(n) \ln \varepsilon_i(n)).
\]

More simply,

\[
v_{i,1}(n) - \omega_{i,1}(n) = \omega_{i,2}(n) \ln \varepsilon_i(n) + o(\ln \varepsilon_i(n)), \tag{1.3}
\]

where \( \omega_{i,2}(n) \neq 0 \) independent of \( n \). If we assume that

\[
\varepsilon_i(n) \to 0
\]

as the genus \( n \to \infty \), then on the right side of (1.3), we have that

\[
\ln \varepsilon_i(n) \to -\infty.
\]

The only thing we need to check is the uniform boundedness in \( n \) of the coefficients, \( v_{i,1}(n) - \omega_{i,1}(n) \) and \( \omega_{i,2}(n)(\neq 0) \), in the equation (1.3). For this, we need explicit expressions for the periods and then of (1.1), (1.2), and (1.3).

**Step 3. Explicit forms for periods from differential equations and propo-**
sition 3.3.1

By using useful differential equations and considering the solutions (corollary 5.2.2), we find explicit expressions of periods of $E(n)$ (lemma 3.3.2) and a strong relationship among the coefficients $\nu_{i,1}(n), \omega_{i,1}(n),$ and $\omega_{i,2}(n)$. Also by comparing these periods with the ones from integral estimates in step 1, we see that $\nu_{i,1}(n) - \omega_{i,1}(n)$ and $\omega_{i,2}(n) (\neq 0)$ are uniformly bounded if $y_i(n) - y_{i-1}(n)$ is uniformly stable in $n$ (lemma 3.3.3).

Therefore, if we assume that $\varepsilon_i(n) \to 0$, then the equation (1.3) does not hold so that we arrive at a contradiction to having the same $R(n)$ from two different expansions (1.1) and (1.2). That is, we fail to satisfy the period problem as $n \to \infty$. This implies that

**Proposition 3.3.1.** For any fixed $i = 1, 2, \ldots, n - 1$, if $y_i(n) - y_{i-1}(n)$ are uniformly stable in the genus $n$, then $y_{i+1}(n) - y_i(n)$ cannot tend to 0 as $n \to \infty$.

**Step 4. Proposition 3.4.1**

In addition to proposition 3.3.1, to complete the uniform stability of $y_{i+1}(n) - y_i(n)$, we also need to show that for any fixed $i = 1, 2, \ldots, n - 1$, if $y_i(n) - y_{i-1}(n)$ are uniformly stable in the genus $n$, then $y_{i+1}(n) - y_i(n)$ cannot tend to $\infty$ as $n \to \infty$.

We show this first for any $i \geq 2$ in section 3.4:

**Proposition 3.4.1.** For any fixed $i = 2, 3, \ldots, n - 1$, if $y_i(n) - y_{i-1}(n)$ are uniformly stable in the genus $n$, then $y_{i+1}(n) - y_i(n)$ cannot tend to $\infty$ as $n \to \infty$.

By rescaling, this is the same as showing that for any fixed $i = 2, 3, \ldots, n - 1$, if $y_{i+1}(n) - y_i(n)$ are uniformly stable in $n$, then $y_i(n) - y_{i-1}(n)$ cannot tend to 0 as $n \to \infty$. 
Step 5. Proposition 3.5.1

Note that proposition 3.4.1 holds only for \(i \geq 2\), not for the \(i = 1\)st case. In order to complete theorem 3.6.1, we show the \(i = 1\)st case in section 3.5:

**Proposition 3.5.1.** If \(y_1(n) - y_0(n) = y_1(n) - 0 = y_1(n)\) are uniformly stable in the genus \(n\), then \(y_2(n) - y_1(n)\) cannot tend to \(\infty\) as \(n \to \infty\).

We will show this also by rescaling: if the \(y_2(n) - y_1(n)\) are uniformly stable in \(n\), then \(y_1(n)\) cannot tend to 0 as \(n \to \infty\). This \(i = 1\)st case is different from the other cases for \(i \geq 2\) in proposition 3.4.1. If \(y_1(n) - 0 = y_1(n)\) tends to 0 as the genus \(n \to \infty\), then \(0 - (-y_1(n)) = y_1(n)\) on the other side from 0 also tends to 0. Therefore, two adjacent lengths tend to 0 together.

As in the proof of proposition 3.3.1 and 3.4.1, we will find \(R(n)\) in two different ways and compare them. Let \(y_1(n) = \varepsilon(n)\). Then we observe that \(R(n) = \frac{|f^{(n)}_0 \varepsilon(n) gdh|}{|f^{(n)}_0 \varepsilon(n) \frac{1}{2} dh|} < o(\varepsilon(n))\). On the other hand, we can find a strictly positive lower bound of \(R(n)\), by comparing with \(R(n) = \frac{|f^{(n)}_0 \varepsilon(n) gdh|}{|f^{(n)}_0 = \varepsilon(n) \frac{1}{2} dh|}\) and using Harnack's inequality. If we assume that \(\varepsilon(n) \to 0\), then we arrive at a contradiction to having \(R(n)\). This implies proposition 3.5.1.

Step 6. Theorem 3.6.1 and theorem 4.1.1

By combining three propositions 3.3.1, 3.4.1, and 3.5.1, we have theorem 3.6.1:

**Theorem 3.6.1.** For any fixed \(i = 1, 2, \ldots, n - 1\), the ratio \(\frac{y_i(n) - y_{i-1}(n)}{y_{i+1}(n) - y_i(n)}\) is comparable to a constant independently of \(n\).

Now, we are ready to prove theorem 4.1.1. Since we already have the very useful property concerning the points \(y_k(n)\) from theorem 3.6.1, it is relatively easy to show that the ratio of periods \(\frac{|P_k(n)|}{|P_{k+1}(n)|} = \left| \frac{f^{y_k(n)}_{y_{k+1}(n)} gdh}{f^{y_{k+1}(n)}_{y_k(n)} gdh} \right|\) does not degenerate as \(n \to \infty\). Using Harnack’s inequality again, we complete theorem 4.1.1:
Theorem 4.1.1. For any fixed \( i = 1, 2, \ldots, n-1 \), the ratio of periods \( \frac{|P_i(n)|}{|P_{i+1}(n)|} \) is comparable to a constant independently of \( n \). Therefore, a normalized subsequence of coordinate domains \( \{\Omega_g(n)\} \) in \( \mathbb{C} \) converges, uniformly on compacta.

Step 7. Main theorem 4.1.1

From theorem 4.1.1, we have the convergence of a normalized subsequence of coordinate domains \( \{\Omega_g(n)\} \), say to \( \Omega_g(\infty) \). By applying Caratheodory Kernel Theorem which relates domain convergence and map convergence, we see the convergence of a subsequence of the Schwarz-Christoffel maps \( \{\psi_g(n) : \mathbb{H} \rightarrow \Omega_g(n)\} \), say to \( \psi_g(\infty) : \mathbb{H} \rightarrow \Omega_g(\infty) \). Also for conjugate case, we have the convergence of a subsequence of the Schwarz-Christoffel maps \( \{\psi_1(n) : \mathbb{H} \rightarrow \Omega_1(n)\} \), say to \( \psi_1(\infty) : \mathbb{H} \rightarrow \Omega_1(\infty) \). Therefore, consider the composition map

\[
\psi_1(\infty) \circ \psi_g(\infty)^{-1} : \Omega_g(\infty) \rightarrow \Omega_1(\infty).
\]

This says that there is a conformal map from \( \Omega_g(\infty) \) to \( \Omega_1(\infty) \) which satisfies the period problem (see chapter 2). This proves the main theorem

Theorem 4.1.2. A normalized subsequence of \( \{E(n)\} \) converges to a minimal surface of infinite genus.

In chapter 2, we introduce a basic tool to study minimal surfaces: the translation of problems concerning minimal surfaces to problems on plane geometry. This is done with zigzags for minimal surfaces and the Schwarz-Christoffel map. In chapter 3, we will show proposition 3.3.1 to be a crucial result to prove the main theorem. The main tools that we use to prove this proposition are differential equations for the periods of minimal surfaces. We use results on differential equations in this chapter although we will see the details in chapter 5. Also we will prove proposition 3.4.1 and
proposition 3.5.1 as the complements of proposition 3.3.1 to complete theorem 3.6.1. Eventually in chapter 4, we prove the main theorem, and give a conjecture on what the limit surface in our main theorem exactly is. In chapter 5, we will prove the useful results on differential equations relevant to minimal surfaces which we use in section 3.3.
Chapter 2

Background

A useful feature in the study of minimal surfaces is that surface problems can be translated into problems of plane geometry. In this chapter, we give information on the Weierstrass representation, zigzags, and the Schwarz-Christoffel map.

2.1 The Weierstrass Representation

In this section, we introduce a very well-known fact in the theory of minimal surfaces.

Definition 2.1.1. Let $M$ be a Riemann surface. Let $g$ be a holomorphic function on $M$, and $dh$ be a holomorphic form on $M$. Then the Weierstrass representation gives coordinate functions of a minimal surface $\phi : M \rightarrow \mathbb{R}^3$ which are harmonic and hence possess locally holomorphic extensions:

$$z \mapsto \text{Re} \int_{\phi} \left( \frac{1}{2} \left( \frac{1}{g} - g \right) dh, \frac{i}{2} \left( \frac{1}{g} + g \right) dh, dh \right).$$

Here, $g$ is the Gauss map, and $dh$ is the complexified height differential (not necessarily exact, despite the notation).

Here, we need to check whether this expression is well-defined. For well-definedness,
we require that
\[ \text{Re} \int_{\gamma} \left( \frac{1}{2} \left( \frac{i}{g} - g \right) dh, \frac{i}{2} \left( \frac{1}{g} + g \right) dh, \; dh \right) = 0 \]
for all cycles \( \gamma \) on \( M \), because analytic continuation around a cycle must leave the map unchanged. Thus the forms \( gdh \) and \( \frac{i}{g} dh \) are basic data for the Weierstrass representation, and so we study the flat singular metrics \( |gdh| \) and \( \left| \frac{i}{g} dh \right| \). For details, see [Wol05]. This translation to plane geometry makes some minimal surface problems surprisingly simple. For example, for the well-definedness problem above:

**Period problem 2.1.1.** The minimal surface existence condition reduces to the condition
\[ \int_{\gamma} gdh = \int_{\gamma} \frac{1}{g} dh \]
and
\[ \text{Re} \int_{\gamma} dh = 0 \]
for all cycles \( \gamma \) on \( M \). This is called the period problem.

In the rest of this background chapter, we will focus on these two basic Weierstrass data, \( gdh \) and \( \frac{i}{g} dh \), of the generalized Enneper's surface \( E(n) \) of genus \( n \), and finally we will see that the period problem for \( E(n) \) can be rephrased in a much simpler way.

**Remark 2.1.1.** We will see that for \( E(n) \), the second condition, \( \text{Re} \int_{\gamma} dh = 0 \) for all cycles \( \gamma \) on \( M \), is automatically satisfied, because \( dh \) is exact in this case. Therefore, we consider only the first condition from this point forward.

### 2.2 Zigzags for \( E(n) \): Translation to Plane Geometry

In this section, we will see that for the generalized Enneper's surface \( E(n) \) of genus \( n \), the mapping behavior of the complex coordinate maps, \( z \mapsto \int_{z}^{z} gdh \) and \( z \mapsto \int_{z}^{z} \frac{i}{g} dh \),...
\[ \int_{-\frac{1}{2}}^{\frac{1}{2}} dh, \] ends up being pleasant. First, let us restrict our attention to one quarter of \( E(n) \) due to its reflectional symmetries about two vertical planes. See figure 2.1.

![Figure 2.1. One quarter of \( E(4) \)](image)

This one quarter of \( E(n) \) is conformally the upper half plane \( \mathbb{H} \). Note that this one quarter is also symmetric about the horizontal straight line in the middle. Let \( \{p_i(n)\}, \quad i = -n, \ldots, -1, 0, 1, \ldots, n, \) be a set of the north and south poles of holes of \( E(n) \). For \( E(4) \), we have nine poles as in figure 2.1. The boundary of one quarter of \( E(n) \) consists of curves connecting these \((2n + 1)\) points \( p_i(n) \). From now on, we will focus on this boundary with the points \( p_i(n) \). Also for the period problem for \( E(n) \), we will focus only on the curves connecting the points \( p_i(n) = -n, \ldots, -1, 0, 1, \ldots, n \), along the boundary of one quarter of \( E(n) \), instead of all cycles \( \gamma \) on \( E(n) \). In addition, since the boundary of one quarter of \( E(n) \) is symmetric about the center, we will restrict our attention to curves connecting the points \( p_i(n), \quad i = 0, 1, \ldots, n \).

**Notation 2.2.1.** Let \( \gamma_i(n), \quad i = 1, 2, \ldots, n, \) be the curve connecting a point \( p_{i-1}(n) \) with the next point \( p_i(n) \) along the boundary of one quarter of \( E(n) \). We call the

\[ \int_{\gamma_i(n)} gdh, \quad i = 1, 2, \ldots, n, \] the *ith period from the center of \( E(n) \), and denote it by

\[ P_i(n) = \int_{\gamma_i(n)} gdh. \]

The following is the period problem for \( E(n) \).
Period problem 2.2.1. Let $E(n)$ be the generalized Enneper's surface of genus $n$.

The period problem for $E(n)$ reduces to the condition

$$
\int_{\gamma_i(n)} gh = \int_{\gamma_i(n)} \frac{1}{g} dh
$$

for all $\gamma_i(n)$, $i = 1, 2, ..., n$.

Later, we will rephrase this period problem again. For further argument, let us find the images of two basic maps $\int_{*}^{z} gh$ and $\int_{*}^{\frac{1}{g}} dh$ for one quarter of $E(n)$. By observing the gauss map $g$ and the height function $h$ along each $\gamma_i(n)$ on the boundary, we can check that the values of $\int_{\gamma(n)} gh$ have either real or imaginary numbers in turn. Therefore, we see that the complex coordinate map, $\varphi_g(n) : z \mapsto \int_{*}^{z} gh$, of one quarter of $E(n)$ has the image $\Omega_g(n)$ in $\mathbb{C}$ with a zigzag shaped boundary as in figure 2.2. This picture should be rotated by $\frac{\pi}{4}$ so that the each edge of the zigzag shaped boundary lies in the real or pure imaginary direction. The domain $\Omega_g(n)$ is bounded by an arc composed of $(2n + 2)$ horizontal and vertical line segments (with $(2n + 1)$ vertices) in turn. Therefore, the edges of the arc automatically have angles alternating between $\frac{\pi}{2}$ and $-\frac{\pi}{2}$.

Notation 2.2.2. We call such an arc a zigzag, and denote it by $Z_g(n)(= \partial \Omega_g(n))$. 

![Figure 2.2. The domain $\Omega_g(4)$ in $\mathbb{C}$ with the zigzag shaped boundary $Z_g(4)$](image)
Notice that a zigzag $Z_g(n)$ for one quarter of $E(n)$ is also symmetric about its center, corresponding to the symmetry of the boundary of one quarter of $E(n)$ about its center $p_0(n)$. The $i$th vertex from the center of $Z_g(n)$ is the image of the $i$th point $p_i(n)$ from the center of $E(n)$, and the $i$th edge from the center of $Z_g(n)$ is the image of the $i$th curve $\gamma_i(n)$ from the center of $E(n)$.

**Remark 2.2.1.** The $i$th period $P_i(n) = \int_{\gamma_i(n)} g dh$ from the center of $E(n)$ is the $i$th edge from the center of a zigzag $Z_g(n)$ in the plane geometry.

The conjugate coordinate map, $\varphi_\perp : z \mapsto \int_z^z \frac{1}{g} dh$ also has an image $\Omega_\perp(n)$ with the zigzag shaped boundary whose each edge lies in the real or pure imaginary direction. Note that while we have information on the exact angles of a zigzag, there is no information on the exact lengths of the line segments of a zigzag, because we do not know the exact change of the shape of $E(n)$ when we add handles. Therefore, we can have many candidate zigzags $Z_g(n)$ for $\int g dh$ of $E(n)$, but we are interested only in a zigzag $Z_g(n)$ which is a solution of the period problem, i.e. $\int_{\gamma_i(n)} g dh = \overline{\int_{\gamma_i(n)} \frac{1}{g} dh}$ for all $\gamma_i(n), i = 1, 2, \ldots, n$, on $E(n)$. In other words, we will check whether the $i$th edge from the center of a zigzag $Z_g(n)$ is the conjugate to the $i$th edge from the center of the conjugate zigzag $Z_\perp(n)$. Moreover, since both $\int_{\gamma_i(n)} g dh$ and $\overline{\int_{\gamma_i(n)} \frac{1}{g} dh}$ are either real or pure imaginary numbers, it is enough to check that $|\int_{\gamma_i(n)} g dh| = |\int_{\gamma_i(n)} \frac{1}{g} dh|$. In other words, the $i$th edge from the center of a zigzag $Z_g(n)$ has to have the same length as the $i$th edge from the center of the conjugate zigzag $Z_\perp(n)$. The following is the rephrased period problem for $E(n)$.

**Period problem 2.2.2.** Let $E(n)$ be the generalized Enneper's surface of genus $n$. In order for $E(n)$ to exist, we have

$$\left| \int_{\gamma_i(n)} g dh \right| = \left| \int_{\gamma_i(n)} \frac{1}{g} dh \right|$$

for all $\gamma_i(n), i = 1, 2, \ldots, n$. 
In the next section, we will approach the period problem with a more detailed expression.

2.3 The Schwarz-Christoffel Map

To solve the period problem, we need an exact expression for the edges of a zigzag for $E(n)$. Notice that the domain $\Omega_g(n)$ in $\mathbb{C}$ with a zigzag boundary is a polygon whose one point is $\infty$. Therefore, we can find a conformal mapping of the upper half plane $\mathbb{H}$ onto the polygon. Recall that the Schwarz-Christoffel map gives a conformal mapping of the upper half plane $\mathbb{H}$ onto a general polygon ([Hil62] §17.6). The map

$$
\begin{array}{ccc}
\mathbb{H} & \xrightarrow{\psi_g} & \Omega_g(n) \\
- y_{i-1}(4) & \rightarrow & y_{i}(4) \\
y_{i}(4) & \rightarrow & y_{i+1}(4) \\
y_{i+1}(4) & \rightarrow & \Omega_g(n) \\
\end{array}
$$

Figure 2.3. The Schwarz-Christoffel map $\psi_g(4) : \mathbb{H} \rightarrow \Omega_g(n)$ for $E(4)$

is determined by the angles and lengths of sides of a polygon. Our zigzag as the boundary of a polygon $\Omega_g(n)$ in $\mathbb{C}$ has angles alternating between $\frac{\pi}{2}$ and $-\frac{\pi}{2}$, and also has the $(2n+1)$ vertices. The Schwarz-Christoffel map $\psi_g$ of $\mathbb{H}$ to $\Omega_g(n) \subset \mathbb{C}$ can be expressed by

$$
\psi_g(n) : \zeta \mapsto \int_{*}^{\zeta} S(n) \frac{(z^2 - y_1(n)^2)^{\frac{1}{2}}(z^2 - y_3(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}{z^{\frac{1}{2}}(z^2 - y_2(n)^2)^{\frac{1}{2}}(z^2 - y_4(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_n(n)^2)^{\frac{1}{2}}} \, dz,
$$

where $S(n)$ is a scaling factor. Without loss of generality, let $n$ be an even integer. Here, the exponents $\pm \frac{1}{2}$ are from the angles $\pm \frac{\pi}{2}$ of a zigzag. Also, for each $i$, the point $y_i(n) \in \mathbb{R} = \partial \mathbb{H}$ corresponds to the $i^{th}$ vertex from the center of a zigzag $Z_g(n)(= \partial \Omega_g(n))$, and the location of the points $y_i(n)$ in the real axis is determined.
by the lengths of edges of a zigzag. Let

$$V(n) = (-y_n(n), ..., -y_2(n), -y_1(n), 0, y_1(n), y_2(n), ..., y_n(n)) \in \mathbb{R}^{2n+1},$$

where $0 < y_1(n) < y_2(n) < ... < y_n(n) \in \mathbb{R} = \partial \mathbb{H}$, be the preimage of the $(2n+1)$ vertices of a zigzag $Z_g(n) = \partial \mathcal{O}_g(n)$ under the Schwarz-Christoffel map. Then, the $ith$ period $P_i(n) = \int_{\gamma_i(n)} gdh$ of $E(n)$ which is the $ith$ edge of a zigzag $Z_g(n)$ can be expressed by

$$\int_{\gamma_i(n)} gdh = \int_{y_{i-1}(n)}^{y_i(n)} S(n) \frac{(z^2 - y_1(n)^2)^{\frac{1}{2}}(z^2 - y_2(n)^2)^{\frac{1}{2}}...(z^2 - y_{i-1}(n)^2)^{\frac{1}{2}}}{z^{\frac{1}{2}}(z^2 - y_2(n)^2)^{\frac{1}{2}}(z^2 - y_4(n)^2)^{\frac{1}{2}}...(z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}} dz.$$

Here, $S(n)$ is a scaling factor. For the expression of the conjugate period $\int_{\gamma_i(n)} \frac{1}{g} dh$, we use the same preimages of the vertices of a zigzag for $\int_{\gamma_i(n)} gdh$:

$$\int_{\gamma_i(n)} \frac{1}{g} dh = \int_{y_{i-1}(n)}^{y_i(n)} S^*(n) \frac{z^{\frac{1}{2}}(z^2 - y_2(n)^2)^{\frac{1}{2}}(z^2 - y_4(n)^2)^{\frac{1}{2}}...(z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}{(z^2 - y_1(n)^2)^{\frac{1}{2}}(z^2 - y_3(n)^2)^{\frac{1}{2}}...(z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}} dz,$$

where $S^*(n)$ is a scaling factor.

**Remark 2.3.1.** Using the Schwarz-Christoffel map, we now have an exact expression for edges (periods) of a zigzag. However, since we do not have information on the exact lengths of zigzag edges, we do not have information on the points $y_i(n), i = 1, 2, ..., n$.

In the next chapter, we will find a property on these $y_i(n)$ in $V(n)$ as the genus $n \to \infty$ (theorem 3.6.1), and in the following chapter, we will provide a positive result on periods of $E(n)$ as the genus $n \to \infty$ (theorem 4.1.1).

**Notation 2.3.1.** We will use the following notations interchangeably

$$\int_{\gamma_i(n)} gdh = \int_{y_{i-1}(n)}^{y_i(n)} gdh.$$
Chapter 3

Convergence of $V(n)$

The main purpose of this chapter is to show theorem 3.6.1 which is a crucial fact to prove the next theorem 4.1.1.

3.1 Motivation

Consider the flat structure of the generalized Enneper’s surface of genus $n$ from the previous chapter. Let $E(n)$ be the generalized Enneper’s surface of genus $n$. For each genus $n$, choose a coordinate domain $\Omega_g(n) \subset \mathbb{C}$ for one quarter of $E(n)$ which is a solution of the period problem (then the conjugate domain $\Omega_{\frac{1}{2}}(n) \subset \mathbb{C}$ is automatically determined). Note that the domain $\Omega_g(n)$ is determined by its symmetric zigzag boundary $Z_g(n) = \partial \Omega_g(n)$ with the $(2n + 1)$ vertices. Let us consider the Schwarz-Christoffel map which takes $\mathbb{H}$ onto $\Omega_g(n)$:

$$ \psi_g(n) : \mathbb{H} \rightarrow \Omega_g(n) \subset \mathbb{C}. $$

Let

$$ V(n) = (-y_n(n), ..., -y_2(n), -y_1(n), 0, y_1(n), y_2(n), ..., y_n(n)) \in \mathbb{R}^{2n+1}, $$
where \( y_1(n) < y_2(n) < \ldots < y_n(n) \in \mathbb{R} = \partial \mathbb{H} \), be the preimage of the \((2n + 1)\) vertices of the zigzag boundary \( Z_g(n) = \partial \Omega_g(n) \) under the Schwarz-Christoffel map \( \psi_g(n) \). Under our setting, \( V(n) \) is also the preimage of the \((2n + 1)\) vertices of the zigzag boundary \( Z_\hat{g}(n) = \partial \Omega_\hat{g}(n) \) of the conjugate domain \( \Omega_\hat{g}(n) \) under the Schwarz-Christoffel map \( \psi_\hat{g}(n) : \mathbb{H} \rightarrow \Omega_\hat{g}(n) \subset \mathbb{C} \).

\[
\begin{array}{cccccc}
- y_1(t) & 0 & y_1(t) & \psi_\hat{g} & - y_1(t) & 0 & y_1(t) \\
- y_2(t) & 0 & y_2(t) & \psi_\hat{g} & - y_2(t) & 0 & y_2(t) \\
- y_3(t) & 0 & y_3(t) & \psi_\hat{g} & - y_3(t) & 0 & y_3(t)
\end{array}
\]

Figure 3.1. The Schwarz-Christoffel map for \( E(n), n = 1, 2, 3 \)

Our interest is to see whether any edge (period) of the zigzag \( Z_g(n) \) can tend to 0 as the genus \( n \) increases. To see this, we are first interested in seeing whether the distance between any points \( y_i(n) \) in \( V(n) \) tends to 0 as the genus \( n \rightarrow \infty \). Our goal in this chapter is to show the following theorem.

**Theorem 3.6.1.** For any fixed \( i = 1, 2, \ldots, n - 1 \), there exists \( 0 < c_i < \infty \) so that

\[
\frac{1}{c_i} < \frac{y_i(n) - y_{i-1}(n)}{y_{i+1}(n) - y_i(n)} < c_i
\]

with \( c_i \) independent of \( n \). Therefore, a normalized subsequence of \( \{V(n)\} \) converges, uniformly on finite subsets of indices.

This theorem says that the ratio \( \frac{y_i(n) - y_{i-1}(n)}{y_{i+1}(n) - y_i(n)} \) of two adjacent distances among the points \( y_i(n) \) in \( V(n) \) is uniformly comparable in the genus \( n \). In other words, the
points $y_i(n)$ in $V(n)$ are distributed uniformly. We will prove theorem 3.6.1 through three propositions 3.3.1, 3.4.1, and 3.5.1. At this point, we define the notation we will often use.

**Notation 3.1.1.** We write

$$a_n \asymp 1$$

if there exists a constant $0 < c < \infty$ independent of $n$ so that $\frac{1}{c} < |a_n| < c$.

### 3.2 The Modified Period Problem

For the argument of chapter 3, it is convenient to modify the period problem 2.2.2 for $E(n)$ into the period problem 3.2.1 for $E(n)$, although we will come back to the period problem 2.2.2 in chapter 4. As we saw in section 2.3, the Schwarz-Christoffel map gives us the exact expression for periods of $E(n)$. Recall the $i$th period of $E(n)$

$$\int_{\gamma_i(n)} g dh = \int_{\gamma_{i-1}(n)}^{\gamma_i(n)} S(n) \left( z^2 - y_1(n)^2 \right)^{\frac{1}{2}} \left( z^2 - y_2(n)^2 \right)^{\frac{1}{2}} \ldots \left( z^2 - y_{n-1}(n)^2 \right)^{\frac{1}{2}} \frac{dz}{z^\frac{1}{2}(z^2 - y_1(n)^2)^{\frac{1}{2}}(z^2 - y_2(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}$$

where $S(n)$ is a scaling factor. Then the $i$th conjugate period of $E(n)$ is

$$\int_{\gamma_i(n)} \frac{1}{g} dh = \int_{\gamma_{i-1}(n)}^{\gamma_i(n)} S^*(n) \left( z^2 - y_1(n)^2 \right)^{\frac{1}{2}} \left( z^2 - y_2(n)^2 \right)^{\frac{1}{2}} \ldots \left( z^2 - y_{n-1}(n)^2 \right)^{\frac{1}{2}} \frac{dz}{(z^2 - y_1(n)^2)^{\frac{1}{2}}(z^2 - y_2(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}$$

where $S^*(n)$ is the scaling factor so that for all $\gamma_i(n), i = 1, 2, \ldots, n$,

$$\left| \int_{\gamma_i(n)} g dh \right| = \left| \int_{\gamma_i(n)} \frac{1}{g} dh \right|$$

(period problem 2.2.2 for $E(n)$). However, through all of chapter 3, we will ignore the scaling factors $S(n)$ and $S^*(n)$: let

$$S(n) = S^*(n) = 1.$$
Then $\int f_{\gamma_1(n)} \frac{g}{h} dh$ and $\int f_{\gamma_2(n)} \frac{1}{g} dh$ are no longer the same. Therefore, in this case, for the period problem we will compare the ratios $\frac{\int f_{\gamma_i(n)} \frac{g}{h} dh}{\int f_{\gamma_i(n)} \frac{1}{g} dh}$, for $i = 1, 2, ..., n$, instead. The following is our new modified period problem for $E(n)$ only for chapter 3:

**Period problem 3.2.1.** Let $E(n)$ be the generalized Enneper's surface of genus $n$. Set

$$dh = dz,$$

and

$$g = \frac{(z^2 - y_1(n)^2)^\frac{1}{2}(z^2 - y_3(n)^2)^\frac{1}{2}...(z^2 - y_{n-1}(n)^2)^\frac{1}{2}}{z^\frac{1}{2}(z^2 - y_2(n)^2)^\frac{1}{2}(z^2 - y_4(n)^2)^\frac{1}{2}...(z^2 - y_n(n)^2)^\frac{1}{2}}.$$

Then in order for $E(n)$ to exist, there must exist a constant $R(n)$ such that

$$R(n) = \frac{\int f_{\gamma_1(n)} \frac{g}{h} dh}{\int f_{\gamma_2(n)} \frac{1}{g} dh} = \frac{\int f_{\gamma_3(n)} \frac{g}{h} dh}{\int f_{\gamma_4(n)} \frac{1}{g} dh} = ... = \frac{\int f_{\gamma_n(n)} \frac{g}{h} dh}{\int f_{\gamma_1(n)} \frac{1}{g} dh} = \frac{\int f_{\gamma_n(n)} \frac{g}{h} dh}{\int f_{\gamma_n(n)} \frac{1}{g} dh}.$$

or

$$R(n) = \frac{\int_{0}^{y_1(n)} \frac{g}{h} dh}{\int_{0}^{y_1(n)} \frac{1}{g} dh} = \frac{\int_{y_1(n)}^{y_2(n)} \frac{g}{h} dh}{\int_{y_1(n)}^{y_2(n)} \frac{1}{g} dh} = ... = \frac{\int_{y_{n-1}(n)}^{y_n(n)} \frac{g}{h} dh}{\int_{y_{n-1}(n)}^{y_n(n)} \frac{1}{g} dh} = \frac{\int_{y_{n-1}(n)}^{y_n(n)} \frac{g}{h} dh}{\int_{y_{n-1}(n)}^{y_n(n)} \frac{1}{g} dh}.$$

**Remark 3.2.1.** In effect, $R(n) = \frac{S^n(n)}{S(n)}$.

### 3.3 Nondegeneration of the Points in $V(n)$, Part I

In this section, we will show proposition 3.3.1 which is the main step towards theorem 3.6.1. The main tools to prove proposition 3.3.1 are differential equations for periods for $E(n)$. We will use the result of corollary 5.2.2 on relevant differential equations, although we will prove it in chapter 5.

Let $E(n)$ be the generalized Enneper’s surface of genus $n$. For each genus $n$, choose a coordinate domain $\Omega_g(n) \subset \mathbb{C}$ for one quarter of $E(n)$ which is a solution of the
period problem (then the conjugate domain $\Omega_{\mathcal{g}}(n) \subset \mathbb{C}$ is automatically determined).

Note that the domain $\Omega_{\mathcal{g}}(n)$ is determined by its symmetric zigzag boundary $Z_{\mathcal{g}}(n) = \partial \Omega_{\mathcal{g}}(n)$ with the $(2n+1)$ vertices. Consider the Schwarz-Christoffel map which maps $\mathbb{H}$ onto $\Omega_{\mathcal{g}}(n)$:

$$\psi_{\mathcal{g}}(n) : \mathbb{H} \to \Omega_{\mathcal{g}}(n) \subset \mathbb{C} \quad \text{by}$$

$$\zeta \mapsto \int_{*}^{\zeta} \frac{(z^2 - y_1(n)^2)^{\frac{1}{2}}(z^2 - y_3(n)^2)^{\frac{1}{2}}... (z^2 - y_{i-1}(n)^2)^{\frac{1}{2}}(z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}{z^3(z^2 - y_2(n)^2)^{\frac{1}{2}}(z^2 - y_4(n)^2)^{\frac{1}{2}}... (z^2 - y_{i+1}(n)^2)^{\frac{1}{2}}... (z^2 - y_{n}(n)^2)^{\frac{1}{2}}} \, dz.$$  

Here,

$$V(n) = (-y_n(n), \ldots, -y_2(n), -y_1(n), 0, y_1(n), y_2(n), \ldots, y_n(n)) \in \mathbb{R}^{2n+1},$$

where $y_1(n) < y_2(n) < \ldots < y_n(n) \in \mathbb{R} = \partial \mathbb{H}$, is the preimage of the $(2n+1)$ vertices of the zigzag boundary $Z_{\mathcal{g}}(n) = \partial \Omega_{\mathcal{g}}(n)$ of the domain $\Omega_{\mathcal{g}}(n)$ under the Schwarz-Christoffel map $\psi_{\mathcal{g}}(n)$.

**Proposition 3.3.1.** For any fixed $i = 1, 2, \ldots, n-1$, if $y_i(n) - y_{i-1}(n) \asymp 1$, then $y_{i+1}(n) - y_i(n) \geq \varepsilon_i$ for some $\varepsilon_i > 0$ independent of $n$.

This proposition 3.3.1 says that for any fixed $i = 1, 2, \ldots, n-1$, if $y_i(n) - y_{i-1}(n)$ is uniformly stable in the genus $n$, then $y_{i+1}(n) - y_i(n)$ cannot tend to 0 as the genus $n \to \infty$.

**Proof.** For each finite genus $n$, choose a coordinate domain $\Omega_{\mathcal{g}}(n)$ for $E(n)$ which is a solution of the period problem, i.e. there exists a quantity $R(n)$ in the period problem 3.2.1 for $E(n)$. Let

$$V(n) = (-y_n(n), \ldots, -y_2(n), -1, 0, 1, y_2(n), \ldots, y_n(n)) \in \mathbb{R}^{2n+1},$$

where $1 < y_2(n) < \ldots < y_n(n) \in \mathbb{R} = \partial \mathbb{H}$, be the preimage of the $(2n+1)$ vertices of the zigzag boundary $Z_{\mathcal{g}}(n) = \partial \Omega_{\mathcal{g}}(n)$ of the domain $\Omega_{\mathcal{g}}(n)$ under the Schwarz-
Christoffel map $\psi_g(n) : \mathbb{H} \to \Omega_g(n) \subset \mathbb{C}$. Notice that we normalized the first point $y_1(n) = 1$ for all $n$ without loss of generality. Fix some $i$ and let

$$y_{i+1}(n) = y_i(n) + \varepsilon_i(n).$$

Assume that we can find a subsequence of coordinate domains $\{\Omega_g(n)\}$ such that in $V(n)$ corresponding to $\Omega_g(n)$,

$$\varepsilon_i(n) \to 0 \quad (3.1)$$

as the genus $n \to \infty$, while

$$y_i(n) - y_{i-1}(n) \asymp 1. \quad (3.2)$$

Under the assumption, we will arrive at a contradiction to satisfying the period problem as the genus $n \to \infty$: we will show that we do not have a quantity $R(n)$ in the modified period problem 3.2.1 for $E(n)$ as the genus $n \to \infty$, by comparing the ratios $R(n) = \frac{\int_{y_i(n)}^{y_{i+1}(n)} gdh}{\int_{y_i(n)}^{y_{i+1}(n)} \frac{1}{g} dh}$ and $R(n) = \frac{\int_{y_{i-1}(n)}^{y_i(n)} gdh}{\int_{y_{i-1}(n)}^{y_i(n)} \frac{1}{g} dh}$. The key observation will be that

$$R(n) = \frac{\int_{y_i(n)}^{y_{i+1}(n)} gdh}{\int_{y_i(n)}^{y_{i+1}(n)} \frac{1}{g} dh}$$

is an analytic function of $\varepsilon_i(n)$, but $R(n) = \frac{\int_{y_{i-1}(n)}^{y_i(n)} gdh}{\int_{y_{i-1}(n)}^{y_i(n)} \frac{1}{g} dh}$ is not in a neighborhood of 0. We will complete the proof of this proposition through five steps. In steps 1 and 2, we will obtain the integral estimates for periods of $E(n)$, and in step 3, we will obtain the differential equation approach for periods of $E(n)$.

Without loss of generality, let $i$ be an even integer.

**Step 1.** Integral estimates for the $(i + 1)$st period $\int_{y_i(n)}^{y_{i+1}(n)} gdh$ and the conjugate period $\int_{y_i(n)}^{y_{i+1}(n)} \frac{1}{g} dh$. First, we will compute the length $\left| \int_{y_i(n)}^{y_{i+1}(n)} gdh \right|$ of the $(i + 1)$st period.

$$\left| \int_{y_i(n)}^{y_{i+1}(n)} gdh \right| = \left| \int_{y_i(n)}^{y_i(n) + \varepsilon_i(n)} \frac{(z^2 - 1)^{\frac{1}{2}} \ldots (z^2 - y_i(n)^2)^{\frac{1}{2}} (z^2 - [y_i(n) + \varepsilon_i(n)]^2)^{\frac{1}{2}} \ldots}{z^\frac{1}{2} (z^2 - y_i(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{i+2}(n)^2)^{\frac{1}{2}} \ldots} \frac{dz}{(z^2 - y_{i-1}(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{i}(n)^2)^{\frac{1}{2}} \ldots} \right|$$
\[ = \Psi(\varepsilon_i(n)) \int_{y_i(n)}^{y_{i+1}(n)} \left( \frac{y_i(n) + \varepsilon_i(n) - z}{z - y_i(n)} \right)^{\frac{1}{2}} dz. \]

This step follows from the Mean Value Property; note that \( \Psi(\varepsilon_i(n)) \) is an analytic function of \( \varepsilon_i(n) \). Next, by substituting \( z - y_i(n) = \varepsilon_i(n)t \), we have

\[
\left| \int_{y_i(n)}^{y_{i+1}(n)} \frac{1}{g} \, dh \right| = \Psi(\varepsilon_i(n))\varepsilon_i(n) \int_0^1 \left( \frac{1 - t}{t} \right)^{\frac{1}{2}} dt, \tag{3.3} \]

where \( \Psi(\varepsilon_i(n)) \) is an analytic function of \( \varepsilon_i(n) \). Next, compute the length of the conjugate period

\[
\left| \int_{y_i(n)}^{y_{i+1}(n)} \frac{1}{g} \, dh \right| = \left| \int_{y_i(n)}^{y_{i+1}(n)+\varepsilon_i(n)} \frac{z^{\frac{1}{2}}(z^2 - y_2(n)^2)^{\frac{1}{2}} \cdots (z^2 - y_i(n)^2)^{\frac{1}{2}} (z^2 - y_{i+2}(n)^2)^{\frac{1}{2}} \cdots}{(z^2 - y_{i-1}(n)^2)^{\frac{1}{2}} \cdots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}} dz \right|
\times \left( z - y_i(n) \right)^{\frac{1}{2}} \int_{y_i(n)}^{y_{i+1}(n)+\varepsilon_i(n)} \left( \frac{z - y_i(n)}{y_i(n) + \varepsilon_i(n) - z} \right)^{\frac{1}{2}} dz.
\]

Here, \( \Psi^*(\varepsilon_i(n)) \) is also an analytic function of \( \varepsilon_i(n) \). Next, by substituting \( z - y_i(n) = \varepsilon_i(n)t \), we have

\[
\left| \int_{y_i(n)}^{y_{i+1}(n)} \frac{1}{g} \, dh \right| = \Psi^*(\varepsilon_i(n))\varepsilon_i(n) \int_0^1 \left( \frac{t}{1 - t} \right)^{\frac{1}{2}} dt, \tag{3.4} \]

\( \Psi(\varepsilon_i(n)) \) is an analytic function of \( \varepsilon_i(n) \).

Now, we are ready to see the ratio of (3.3) and (3.4). Note that in (3.3) and (3.4), we have \( \int_0^1 \left( \frac{t}{1 - t} \right)^{\frac{1}{2}} dt = \int_0^1 \left( \frac{1 - t}{t} \right)^{\frac{1}{2}} dt \). Therefore, we conclude that the ratio

\[
R(n) = \frac{\int_{y_i(n)}^{y_{i+1}(n)} \frac{1}{g} \, dh}{\int_{y_i(n)}^{y_{i+1}(n)} \frac{1}{g} \, dh} = \frac{\left| \int_{y_i(n)}^{y_{i+1}(n)+\varepsilon_i(n)} \frac{1}{g} \, dh \right|}{\left| \int_{y_i(n)}^{y_{i+1}(n)+\varepsilon_i(n)} \frac{1}{g} \, dh \right|} = \frac{\Psi(\varepsilon_i(n))}{\Psi^*(\varepsilon_i(n))} \tag{3.5} \]

is an analytic function of \( \varepsilon_i(n) \) independent of \( n \). In the next step, we will compute
\[ R(n) = \frac{\int_{\gamma_i^{-1}(n)}^{\gamma_i(n)} g \, dh}{\int_{\gamma_i^{-1}(n)}^{\gamma_i(n)} \frac{1}{g} \, dh}. \]

**Step 2.** Integral estimates for the \( i \)th period \( \int_{\gamma_i^{-1}(n)}^{\gamma_i(n)} g \, dh \) and the conjugate period \( \int_{\gamma_i^{-1}(n)}^{\gamma_i(n)} \frac{1}{g} \, dh \) and bounds for their coefficients. In this step, we will compute the \( i \)th period and the conjugate period. We state lemma 3.3.1 which gives us the expansions of the \( i \)th period and the conjugate period and also bounds for the leading coefficients in the expansions (lemma 3.3.1 (1)). Using the lemma, we will obtain (in the equation (3.10)) a new expression for the quantity \( R(n) \) which we defined in (3.5). We will then compare that expression with (3.5)

**Lemma 3.3.1.** Let \( i \geq 1 \) be any even integer and for fixed \( i \), let \( y_{i+1}(n) = y_i(n) + \epsilon_i(n) \). If \( y_i(n) - y_{i-1}(n) \approx 1 \), then the \( i \)th period and the conjugate period for \( E(n) \) can be expressed by the following expansions in \( \epsilon_i(n) \):

\[
\int_{\gamma_i^{-1}(n)}^{\gamma_i(n)} g \, dh = \Phi(\epsilon_i(n)) \left\{ b_{i,0}(n) + \lambda_{i,1}(n)\epsilon_i(n) + \lambda_{i,2}(n)\epsilon_i(n)^2 + \ldots \right\} + a_{i,0}(n)\epsilon_i(n) \ln \epsilon_i(n),
\]

\[
\int_{\gamma_i^{-1}(n)}^{\gamma_i(n)} \frac{1}{g} \, dh = \Phi^*(\epsilon_i(n)) \left\{ b_{i,0}^*(n) + \lambda_{i,1}^*(n)\epsilon_i(n) + \lambda_{i,2}^*(n)\epsilon_i(n)^2 + \ldots \right\} + a_{i,0}^*(n)\epsilon_i(n) \ln \epsilon_i(n),
\]

where

1. \( b_{i,0}(n), a_{i,0}(n), b_{i,0}^*(n), \) and \( a_{i,0}^*(n) \) are all \( \approx 1 \),

2. \( b_{i,0}(n) > 0, a_{i,0}(n) < 0, b_{i,0}^*(n) > 0, \) and \( a_{i,0}^*(n) > 0 \),

3. \( \lambda_{i,k}(n) \) and \( \lambda_{i,k}^*(n), k = 1, 2, \ldots, \) are uniformly bounded in \( n \),

4. \( \Phi(\epsilon_i(n)) \) and \( \Phi^*(\epsilon_i(n)) \) are analytic functions of \( \epsilon_i(n) \).

For an odd \( i \geq 1 \), we have the same result with the exception that the signs of \( a_{i,0}(n) \) and \( a_{i,0}^*(n) \) reverse: \( a_{i,0}(n) > 0, \) and \( a_{i,0}^*(n) < 0. \)
Remark 3.3.1. What is important here are the uniform bounds on the coefficients listed in (1) and their signs listed in (2). Notice in (2) that $a_{i,0}(n)$ and $a^*_{i,0}(n)$ have different signs, while $b_{i,0}(n)$ and $b^*_{i,0}(n)$ have same ones. Therefore,

$$\left(\frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a^*_{i,0}(n)}{b^*_{i,0}(n)}\right) \neq 0$$

(3.8)

independent of $n$. Moreover, by (1) we have

$$\left(\frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a^*_{i,0}(n)}{b^*_{i,0}(n)}\right) \approx 1.$$  \hspace{1cm} (3.9)

This is a crucial fact behind the proof of proposition 3.3.1.

Remark 3.3.2. We will not be interested in the size of $\Phi(\varepsilon_i(n))$ and $\Phi^*(\varepsilon_i(n))$ in the sequel.

Before we prove lemma 3.3.1, we will continue our argument on the quantity $R(n)$. Using the expansions in lemma 3.3.1, we now have a new expression for the quantity $R(n)$ which we defined in (3.5). From (3.6) and (3.7) in lemma 3.3.1, we obtain

$$R(n) = \left|\frac{\int_{y_{i-1}(n)}^{y_i(n)} gdh}{\int_{y_{i-1}(n)}^{y_i(n)} \frac{1}{g} dh}\right|$$

$$\Phi(\varepsilon_i(n)) \left\{ b_{i,0}(n) + \lambda_{i,1}(n)\varepsilon_i(n) + \ldots + a_{i,0}(n)\varepsilon_i(n)\ln\varepsilon_i(n) \right\}$$

$$\Phi^*(\varepsilon_i(n)) \left\{ b^*_{i,0}(n) + \lambda^*_{i,1}(n)\varepsilon_i(n) + \ldots + a^*_{i,0}(n)\varepsilon_i(n)\ln\varepsilon_i(n) \right\}$$

$$\Phi(\varepsilon_i(n)) b_{i,0}(n) \left( 1 + \left(\frac{\lambda_{i,1}(n)}{b_{i,0}(n)} - \frac{\lambda^*_{i,1}(n)}{b^*_{i,0}(n)}\right) \varepsilon_i(n) + \ldots \right)$$

$$+ \left(\frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a^*_{i,0}(n)}{b^*_{i,0}(n)}\right) \varepsilon_i(n)\ln\varepsilon_i(n) + \ldots \right\},$$

(3.10)

where \(\left(\frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a^*_{i,0}(n)}{b^*_{i,0}(n)}\right) \neq 0\) independent of $n$. Note that $\Phi(\varepsilon_i(n))$ and $\Phi^*(\varepsilon_i(n))$ are still analytic functions of $\varepsilon_i(n)$.

Remark 3.3.3. While we concluded in (3.5) that $R(n)$ is analytic in $\varepsilon_i(n)$, we now see
from (3.10) that it is not an analytic function of \( \varepsilon_i(n) \) in a neighborhood of 0, since the coefficient of the \( \varepsilon_i(n) \ln \varepsilon_i(n) \) term, \( \frac{a_{i,0}^{(n)}}{b_{i,0}^{(n)}} - \frac{a_{i,0}^{(n)}}{b_{i,0}^{(n)}} \), is not zero independent of \( n \). This difference is crucial in our argument, because (3.5) and (3.10) are expansions of the same function \( R(n) \) (the modified period problem 3.2.1 for \( E(n) \)).

To compare \( R(n) \) in (3.5) and (3.10), we need their explicit expressions. In addition, if we can find a relationship among the coefficients in (3.5) and (3.10), then it will allow us to relate them. We will continue this argument using differential equations in the next step 3. We finish step 2 with the proof of lemma 3.3.1.

**Proof of lemma 3.3.1.** We first show the expansion (3.6).

\[
\left| \int_{y_{i-1}(n)}^{y_i(n)} g dh \right| = \left| \int_{y_{i-1}(n)}^{y_i(n)} \frac{(z^2 - 1)^{\frac{1}{2}} \cdots (z^2 - y_{i-1}(n)^2)^{\frac{1}{2}} (z^2 - [y_i(n) + \varepsilon_i(n)]^2)^{\frac{1}{2}} \cdots}{z^2 - y_{2}(n)^2)^{\frac{1}{2}} \cdots (z^2 - y_{i}(n)^2)^{\frac{1}{2}} \cdots (z^2 - y_{n}(n)^2)^{\frac{1}{2}} \cdots} \right| dz \\
= \Phi(\varepsilon_i(n)) \int_{y_{i-1}(n)}^{y_i(n)} \left( \frac{(z - y_{i-1}(n))(y_i(n) + \varepsilon_i(n) - z)}{y_i(n) - z} \right)^{\frac{1}{2}} dz
\]

This step follows from the Mean Value Property; note that \( \Phi(\varepsilon_i(n)) \) is an analytic function of \( \varepsilon_i(n) \). We continue the computation, dividing the domain into two parts.

\[
= \Phi(\varepsilon_i(n)) \left\{ \int_{y_{i-1}(n)}^{y_{i-1}(n)+y_{i}(n)} \left( \frac{(z - y_{i-1}(n))(y_i(n) + \varepsilon_i(n) - z)}{y_i(n) - z} \right)^{\frac{1}{2}} dz \\
+ \int_{y_{i-1}(n)+y_{i}(n)}^{y_i(n)} \left( \frac{(z - y_{i-1}(n))(y_i(n) + \varepsilon_i(n) - z)}{y_i(n) - z} \right)^{\frac{1}{2}} dz \right\} \\
= \Phi(\varepsilon_i(n)) \left\{ C_i(\varepsilon_i(n)) \int_{y_{i-1}(n)}^{y_{i-1}(n)+y_{i}(n)} (z - y_{i-1}(n))^{\frac{1}{2}} dz \\
+ D_i(n) \int_{y_{i-1}(n)+y_{i}(n)}^{y_i(n)} \left( \frac{y_i(n) + \varepsilon_i(n) - z}{y_i(n) - z} \right)^{\frac{1}{2}} dz \right\}.
\]
This also follows from the Mean Value Property. Here,

\[
C_i(\varepsilon_i(n)) = \left( \frac{y_i(n) + \varepsilon_i(n) - z_*}{y_i(n) - z_*} \right)^{\frac{1}{2}} = 1 + \frac{1}{2(y_i(n) - z_*)} \varepsilon_i(n) + ..., \]

where \( z_* \) is some number such that \( y_{i-1}(n) < z_* < \frac{y_i(n) + y_{i-1}(n)}{2} \). Since \( \frac{1}{2(y_i(n) - y_{i-1}(n))} < \frac{1}{2(y_i(n) - z_*)} < \frac{1}{y_i(n) - y_{i-1}(n)} \), let

\[
C_i(\varepsilon_i(n)) = 1 + \frac{c_i(n)}{y_i(n) - y_{i-1}(n)} \varepsilon_i(n) + ..., \tag{3.12}
\]

where \( \frac{1}{2} < c_i(n) < 1 \). And

\[
D_i(n) = (z_{**} - y_{i-1}(n))^{\frac{1}{2}},
\]

where \( z_{**} \) is some number such that \( \frac{y_{i-1}(n) + y_i(n)}{2} < z_{**} < y_i(n) \). Since \( \sqrt{\frac{y_i(n) - y_{i-1}(n)}{2}} < \sqrt{z_{**} - y_{i-1}(n)} < \sqrt{y_i(n) - y_{i-1}(n)} \), let

\[
D_i(n) = d_i(n) \sqrt{y_i(n) - y_{i-1}(n)}. \tag{3.13}
\]

where \( \frac{1}{\sqrt{2}} < d_i(n) < 1 \). We now continue our computation (3.11). Setting \( z - y_{i-1}(n) = w \) and \( y_i(n) - z = t \),

\[
= \Phi(\varepsilon_i(n)) \left\{ C_i(\varepsilon_i(n)) \int_0^{\frac{y_i(n) - y_{i-1}(n)}{2}} w^\frac{1}{2} dw + D_i(n) \int_0^{\frac{y_i(n) - y_{i-1}(n)}{2}} \left( \frac{t + \varepsilon_i(n)}{t} \right)^{\frac{1}{2}} dt \right\}
\]

\[
= \Phi(\varepsilon_i(n)) \left\{ C_i(\varepsilon_i(n)) \int_0^{\frac{y_i(n) - y_{i-1}(n)}{2}} w^\frac{1}{2} dw + 2D_i(n) \int_0^{\frac{y_i(n) - y_{i-1}(n)}{2}} \sqrt{u^2 + \varepsilon_i(n)} du \right\}
\]

\[
= \Phi(\varepsilon_i(n)) \left\{ C_i(\varepsilon_i(n)) \left[ \frac{2}{3} w^{\frac{3}{2}} \right]_0^{\frac{y_i(n) - y_{i-1}(n)}{2}} \right. \\
+ D_i(n) \left[ u \sqrt{u^2 + \varepsilon_i(n)} + \varepsilon_i(n) \ln \left( u + \sqrt{u^2 + \varepsilon_i(n)} \right) \right]_0^{\frac{y_i(n) - y_{i-1}(n)}{2}} \right\}
\]
\begin{align*}
&= \Phi(\varepsilon_i(n)) \left\{ C_i(\varepsilon_i(n)) \left[ \frac{2}{3} \left( \frac{y_i(n) - y_{i-1}(n)}{2} \right)^{\frac{3}{2}} \right] \\
&+ D_i(n) \left[ \sqrt{\frac{y_i(n) - y_{i-1}(n)}{2}} - \sqrt{\frac{y_i(n) - y_{i-1}(n)}{2}} + \varepsilon_i(n) \right] \\
&+ \varepsilon_i(n) \ln \left( \sqrt{\frac{y_i(n) - y_{i-1}(n)}{2}} + \sqrt{\frac{y_i(n) - y_{i-1}(n)}{2}} + \varepsilon_i(n) \right) - \frac{1}{2} \varepsilon_i(n) \ln \varepsilon_i(n) \right\} \\
&= \Phi(\varepsilon_i(n)) \left\{ C_i(\varepsilon_i(n)) \frac{\sqrt{2}}{6} (y_i(n) - y_{i-1}(n))^{\frac{3}{2}} \\
&+ D_i(n) \left[ \frac{y_i(n) - y_{i-1}(n)}{2} + \frac{1}{2} \varepsilon_i(n) + \ldots \right] \\
&+ \varepsilon_i(n) \left( \ln \sqrt{2 (y_i(n) - y_{i-1}(n))} + \frac{1}{2 (y_i(n) - y_{i-1}(n))} \varepsilon_i(n) + \ldots \right) - \frac{1}{2} \varepsilon_i(n) \ln \varepsilon_i(n) \right\} \\
&= \Phi(\varepsilon_i(n)) \left\{ \left( b_{i,0}(n) + \lambda_{i,1}(n) \varepsilon_i(n) + \lambda_{i,2}(n) \varepsilon_i(n)^2 + \ldots \right) + a_{i,0}(n) \varepsilon_i(n) \ln \varepsilon_i(n) \right\}.
\end{align*}

(3.14)

Here, the coefficients \( b_{i,0}(n) \), \( a_{i,0}(n) \) and \( \lambda_{i,k}(n) \), \( k = 1, 2, \ldots \), are all estimated in terms of \( y_i(n) - y_{i-1}(n) \). For example, by applying (3.12) and (3.13) to (3.14),

\[ b_{i,0}(n) = \frac{\sqrt{2}}{6} (y_i(n) - y_{i-1}(n))^{\frac{3}{2}} + D_i(n) \frac{y_i(n) - y_{i-1}(n)}{2} \]
\[ = \left( \frac{\sqrt{2}}{6} + \frac{d_i(n)}{2} \right) (y_i(n) - y_{i-1}(n))^{\frac{3}{2}}, \]

\[ a_{i,0}(n) = -\frac{D_i(n)}{2} = -\frac{d_i(n)}{2} \sqrt{y_i(n) - y_{i-1}(n)}, \]

\[ \lambda_{i,1}(n) = \left[ \frac{\sqrt{2}}{6} c_i(n) + d_i(n) \left( \frac{1}{2} + \frac{1}{2} \ln 2 (y_i(n) - y_{i-1}(n)) \right) \right] \sqrt{y_i(n) - y_{i-1}(n)}, \]

where \( \frac{1}{2} < c_i(n) < 1 \) and \( \frac{1}{\sqrt{2}} < d_i(n) < 1 \). Since we assumed that \( y_i(n) - y_{i-1}(n) \simeq 1 \),
all coefficients $b_{i,0}(n), a_{i,0}(n)$, and $\lambda_{i,k}(n), k = 1, 2, \ldots$, are uniformly bounded in $n$. In particular, $b_{i,0}(n)$ and $a_{i,0}(n)$ are $\asymp 1$.

Next, to show the $i$th conjugate period (3.7), note

$$
\left| \int_{y_{i-1}(n)}^{y_{i}(n)} \frac{1}{g} \, dz \right| = \left| \int_{y_{i-1}(n)}^{y_{i}(n)} \frac{z^{1/2} \left( z^2 - y_2(n)^2 \right)^{1/2} \cdots \left( z^2 - y_{i-1}(n)^2 \right)^{1/2} \left( z^2 - y_{i+1}(n)^2 \right)^{1/2} \cdots \frac{\left( z^2 - y_{i}(n)^2 \right)^{1/2}}{\left( z^2 - y_{i-1}(n)^2 \right)^{1/2}} \, dz} \right|
$$

$$
= \Phi^*(\varepsilon_i(n)) \int_{y_{i-1}(n)}^{y_{i}(n)} \left( \frac{y_i(n) - z}{(z - y_{i-1}(n))(y_i(n) + \varepsilon_i(n) - z)} \right)^{1/2} \, dz.
$$

Here, $\Phi^*(\varepsilon_i(n))$ is an analytic function of $\varepsilon_i(n)$.

$$
= \Phi^*(\varepsilon_i(n)) \left\{ \int_{y_{i-1}(n)}^{\frac{y_{i-1}(n) + y_i(n)}{2}} \left( \frac{y_i(n) - z}{(z - y_{i-1}(n))(y_i(n) + \varepsilon_i(n) - z)} \right)^{1/2} \, dz 
\right. \
+ \int_{\frac{y_{i-1}(n) + y_i(n)}{2}}^{y_i(n)} \left( \frac{y_i(n) - z}{(z - y_{i-1}(n))(y_i(n) + \varepsilon_i(n) - z)} \right)^{1/2} \, dz \right\}
$$

$$
= \Phi^*(\varepsilon_i(n)) \left\{ C^*_i(n) \int_{y_{i-1}(n)}^{\frac{y_{i-1}(n) + y_i(n)}{2}} \left( \frac{1}{z - y_{i-1}(n)} \right)^{1/2} \, dz 
\right. \
+ D^*_i(n) \int_{\frac{y_{i-1}(n) + y_i(n)}{2}}^{y_i(n)} \left( \frac{y_i(n) - z}{(y_i(n) + \varepsilon_i(n) - z)} \right)^{1/2} \, dz \right\}.
$$

Here,

$$
C^*_i(\varepsilon_i(n)) = \left( \frac{y_i(n) - z^*_i}{y_i(n) - z^*_i + \varepsilon_i(n)} \right)^{1/2} = 1 - \frac{1}{2(y_i(n) - z^*_i)} \varepsilon_i(n) + \ldots,
$$

where $z^*_i$ is some number such that $y_{i-1}(n) < z^*_i < \frac{y_{i-1}(n) + y_i(n)}{2}$. Since $\frac{1}{2(y_i(n) - y_{i-1}(n))} < \frac{1}{2(y_i(n) - z^*_i)} < \frac{1}{y_i(n) - y_{i-1}(n)}$, let

$$
C^*_i(\varepsilon_i(n)) = 1 + \frac{c^*_i(n)}{y_i(n) - y_{i-1}(n)} \varepsilon_i(n) + \ldots,
$$

(3.16)
where \(-1 < c^*_i(n) < -\frac{1}{2}\). And

\[
D^*_i(n) = \left( \frac{1}{z^*_i - y_{i-1}(n)} \right)^{\frac{1}{2}},
\]

where \(z^*_i\) is some number such that \(\frac{y_{i-1}(n) + y_i(n)}{2} < z^*_i < y_i(n)\). Since \(\sqrt{\frac{1}{y_i(n) - y_{i-1}(n)}} < \sqrt{\frac{2}{y_i(n) - y_{i-1}(n)}}\), let

\[
D^*_i(n) = \frac{d^*_i(n)}{\sqrt{y_i(n) - y_{i-1}(n)}}, \tag{3.17}
\]

where \(1 < d^*_i(n) < \sqrt{2}\). We continue our computation (3.15).

\[
= \Phi^*(\varepsilon_i(n)) \left\{ C^*_i(\varepsilon_i(n)) \int_0^{\frac{y_i(n) - y_{i-1}(n)}{2}} \left( \frac{1}{w} \right)^{\frac{1}{2}} dw + D^*_i(n) \int_0^{\frac{y_i(n) - y_{i-1}(n)}{2}} \left( \frac{t}{t + \varepsilon_i(n)} \right)^{\frac{1}{2}} dt \right\}
\]

\[
= \Phi^*(\varepsilon_i(n)) \left\{ C^*_i(\varepsilon_i(n)) \int_0^{\frac{y_i(n) - y_{i-1}(n)}{2}} \left( \frac{1}{w} \right)^{\frac{1}{2}} dw + 2D^*_i(n) \int_0^{\frac{y_i(n) - y_{i-1}(n)}{2}} \frac{u^2}{\sqrt{u^2 + \varepsilon_i(n)}} du \right\}
\]

\[
= \Phi^*(\varepsilon_i(n)) \left\{ C^*_i(\varepsilon_i(n)) \left[ 2u^{\frac{1}{2}} \right]_0^{\frac{y_i(n) - y_{i-1}(n)}{2}} + D^*_i(n) \left[ u^2 + \varepsilon_i(n) \right]^{\frac{y_i(n) - y_{i-1}(n)}{2}} \right\}
\]

\[
= \Phi^*(\varepsilon_i(n)) \left\{ C^*_i(\varepsilon_i(n)) \sqrt{2(y_i(n) - y_{i-1}(n))} 
+ D^*_i(n) \left[ \sqrt{\frac{y_i(n) - y_{i-1}(n)}{2}} \right] + \varepsilon_i(n) 
- \varepsilon_i(n) \ln \left( \frac{\sqrt{y_i(n) - y_{i-1}(n)} + \varepsilon_i(n)}{2} + \frac{1}{2} \varepsilon_i(n) \ln \varepsilon_i(n) \right) \right\}
\]

\[
= \Phi^*(\varepsilon_i(n)) \left\{ C^*_i(\varepsilon_i(n)) \sqrt{2(y_i(n) - y_{i-1}(n))} 
+ D^*_i(n) \left[ \left( \frac{y_i(n) - y_{i-1}(n)}{2} + \frac{1}{2} \varepsilon_i(n) + ... \right) 
- \varepsilon_i(n) \left( \ln \sqrt{2(y_i(n) - y_{i-1}(n))} + \frac{1}{2(y_i(n) - y_{i-1}(n))} \varepsilon_i(n) + ... \right) + \frac{1}{2} \varepsilon_i(n) \ln \varepsilon_i(n) \right] \right\}
\]
\[
\begin{align*}
\Phi^*(\varepsilon_i(n)) & \left\{ C^*_i(\varepsilon_i(n)) \sqrt{2(y_i(n) - y_{i-1}(n))} \\
+ D^*_i(n) \left[ \frac{y_i(n) - y_{i-1}(n)}{2} + \left( \frac{1}{2} - \frac{1}{2} \ln \left[ 2(y_i(n) - y_{i-1}(n)) \right] \right) \varepsilon_i(n) + \ldots \\
+ \frac{1}{2} \varepsilon_i(n) \ln \varepsilon_i(n) \right] \right\} \\
= \Phi^*(\varepsilon_i(n)) \left\{ (b^*_{i,0}(n) + \lambda^*_{i,1}(n)\varepsilon_i(n) + \lambda^*_{i,2}(n)\varepsilon_i(n)^2 + \ldots) + a^*_{i,0}(n)\varepsilon_i(n) \ln \varepsilon_i(n) \right\}.
\end{align*}
\]

(3.18)

Here, the coefficients \( b^*_{i,0}(n), a^*_{i,0}(n) \) and \( \lambda^*_{i,k}(n), k = 1, 2, \ldots \), are all estimated in terms of \( y_i(n) - y_{i-1}(n) \). For example, by applying (3.16) and (3.17) to (3.18),

\[
\begin{align*}
b^*_{i,0}(n) &= \sqrt{2(y_i(n) - y_{i-1}(n))} + D^*_i(n) \frac{y_i(n) - y_{i-1}(n)}{2} \\
&= \left( \sqrt{2} + \frac{d^*_i(n)}{2} \right) \sqrt{y_i(n) - y_{i-1}(n)},
\end{align*}
\]

\[
\begin{align*}
a^*_{i,0}(n) &= \frac{D^*_i(n)}{2} = \frac{d^*_i(n)}{2} \frac{1}{\sqrt{y_i(n) - y_{i-1}(n)}},
\end{align*}
\]

\[
\lambda^*_{i,1}(n) = \left[ \sqrt{2}c^*_i(n) + d^*_i(n) \left( \frac{1}{2} - \frac{1}{2} \ln \left[ 2(y_i(n) - y_{i-1}(n)) \right] \right) \right] \frac{1}{\sqrt{y_i(n) - y_{i-1}(n)}},
\]

where \(-1 < c^*_i(n) < -\frac{1}{2} \) and \( 1 < d^*_i(n) < \sqrt{2} \). Since we assumed that \( y_i(n) - y_{i-1}(n) \approx 1 \), all coefficients \( b^*_{i,0}(n), a^*_{i,0}(n) \) and \( \lambda^*_{i,k}(n), k = 1, 2, \ldots \), are uniformly bounded in \( n \). Especially \( b^*_{i,0}(n) \) and \( a^*_{i,0}(n) \) are \( \approx 1 \). This completes the proof of lemma 3.3.1. □

**Step 3. Explicit forms for periods for \( E(n) \) and relationships of coefficients.** In step 1, we had integral estimates (3.3) and (3.4) for the \((i + 1)st\) period \( |\int_{y_i(n)}^{y_{i+1}(n)} gdh| \) and the conjugate period \( |\int_{y_{i-1}(n)}^{y_i(n)} \frac{1}{g}dh| \). Also in step 2, we had integral estimates (3.6) and (3.7) for the \( ith\) period \( |\int_{y_{i-1}(n)}^{y_i(n)} gdh| \) and the conjugate period \( |\int_{y_i(n)}^{y_{i+1}(n)} \frac{1}{g}dh| \). In the following lemma 3.3.2, we will see the explicit expressions of these four periods using differential equations. We will also see the strong relationship among the coefficients. Lemma 3.3.2 will use the result of corollary 5.2.2 on the relevant differential equations, although we will prove it in chapter 5. From lemma 3.3.2, we will find the explicit
expressions of $R(n) = \frac{\int_{y_{i-1}(n)}^{y_{i+1}(n)} g dh}{\int_{y_{i-1}(n)}^{y_{i+1}(n)} \frac{1}{g} dh}$ in (3.5) and $R(n) = \frac{\int_{y_{i-1}(n)}^{y_{i}(n)} gdh}{\int_{y_{i-1}(n)}^{y_{i}(n)} \frac{1}{g} dh}$ in (3.10) rephrased as (3.23) and (3.24). Finally, we will compare them.

**Lemma 3.3.2.** For any fixed $i = 1, 2, ..., n-1$, let $y_{i+1}(n) = y_{i}(n) + \varepsilon_{i}(n)$. Then the following periods of $E(n)$ have the form:

\[
\left| \int_{y_{i}(n)}^{y_{i+1}(n)} g dh \right| = K_{i}(n)\varepsilon_{i}(n) \left( a_{i,0}(n) + a_{i,1}(n)\varepsilon_{i}(n) + a_{i,2}(n)\varepsilon_{i}(n)^{2} + ... \right), \quad (3.19)
\]

\[
\left| \int_{y_{i}(n)}^{y_{i+1}(n)} \frac{1}{g} dh \right| = K_{i}^{*}(n)\varepsilon_{i}(n) \left( a_{i,0}^{*}(n) + a_{i,1}^{*}(n)\varepsilon_{i}(n) + a_{i,2}^{*}(n)\varepsilon_{i}(n)^{2} + ... \right), \quad (3.20)
\]

\[
\left| \int_{y_{i-1}(n)}^{y_{i}(n)} g dh \right| = L_{i}(n) \left\{ b_{i,0}(n) + b_{i,1}(n)\varepsilon_{i}(n) + b_{i,2}(n)\varepsilon_{i}(n)^{2} + ... \right. \quad (3.21)
\]

\[+\varepsilon_{i}(n)\ln \varepsilon_{i}(n) \left( a_{i,0}(n) + a_{i,1}(n)\varepsilon_{i}(n) + a_{i,2}(n)\varepsilon_{i}(n)^{2} + ... \right) \} , \]

\[
\left| \int_{y_{i-1}(n)}^{y_{i}(n)} \frac{1}{g} dh \right| = L_{i}^{*}(n) \left\{ b_{i,0}^{*}(n) + b_{i,1}^{*}(n)\varepsilon_{i}(n) + b_{i,2}^{*}(n)\varepsilon_{i}(n)^{2} + ... \right. \quad (3.22)
\]

\[+\varepsilon_{i}(n)\ln \varepsilon_{i}(n) \left( a_{i,0}^{*}(n) + a_{i,1}^{*}(n)\varepsilon_{i}(n) + a_{i,2}^{*}(n)\varepsilon_{i}(n)^{2} + ... \right) \} .
\]

Here, the coefficients $K_{i}(n)$, $L_{i}(n)$, $K_{i}^{*}(n)$, and $L_{i}^{*}(n)$, as well as $a_{i,k}(n)$, $b_{i,k}(n)$, $a_{i,k}^{*}(n)$, and $b_{i,k}^{*}(n)$, $k = 0, 1, 2, ...$, are not functions of $\varepsilon_{i}(n)$. In particular, $a_{i,0}(n)$, $b_{i,0}(n)$, $a_{i,0}^{*}(n)$, and $b_{i,0}^{*}(n) \neq 0$.

**Remark 3.3.4.** It is crucially important that in (3.19) and (3.21), we have the same $a_{i,k}(n)$, $k = 0, 1, 2, ...$. Also in (3.20) and (3.22), we have the same $a_{i,k}^{*}(n)$, $k = 0, 1, 2, ...$. This strong relationship among the coefficients allows us to relate $R(n) = \frac{\int_{y_{i-1}(n)}^{y_{i+1}(n)} g dh}{\int_{y_{i-1}(n)}^{y_{i+1}(n)} \frac{1}{g} dh}$ to $R(n) = \frac{\int_{y_{i-1}(n)}^{y_{i+1}(n)} g dh}{\int_{y_{i-1}(n)}^{y_{i+1}(n)} \frac{1}{g} dh}$.

**Remark 3.3.5.** If we assume $y_{i}(n) - y_{i-1}(n) \approx 1$ as in previous lemma 3.3.1, then in (3.21) and (3.22) we can choose the same $a_{i,0}(n), b_{i,0}(n), a_{i,0}^{*}(n)$, and $b_{i,0}^{*}(n)$ as in lemma 3.3.1 by adjusting $L_{i}(n)$ and $L_{i}^{*}(n)$ (and then $K_{i}(n)$ and $K_{i}^{*}(n)$) so that $a_{i,0}(n), b_{i,0}(n), a_{i,0}^{*}(n)$, and $b_{i,0}^{*}(n) \approx 1$, and also $\left( \frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a_{i,0}^{*}(n)}{b_{i,0}^{*}(n)} \right) \neq 0$. Therefore, if
\[ y_i(n) - y_{i-1}(n) \approx 1, \text{ then} \]
\[
\left( \frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a_{i,1}(n)}{b_{i,1}(n)} \right) \approx 1.
\]

Before we prove lemma 3.3.2, we will continue our argument on the quantity \( R(n) \) in (3.5) and (3.10). From lemma 3.3.2 which gives us the explicit forms of periods and their relationships, we now see the explicit forms of \( R(n) \). First, from (3.19) and (3.20) we have the explicit form of \( R(n) \) in (3.5).

\[
R(n) = \left. \frac{f_{y_{i+1}(n)}}{f_{y_{i}(n)}} \right|_0^1 - \frac{1}{y_{i}(n)} \frac{1}{y_{i+1}(n)} \frac{a_{i,0}(n) + a_{i,1}(n) + a_{i,2}(n) + ...}{b_{i,0}(n) + b_{i,1}(n) + b_{i,2}(n) + ...} \\
= \frac{K_i(n)\varepsilon_i(n)}{K_i^*(n)\varepsilon_i(n)} \left\{ \frac{a_{i,0}(n)}{a_{i,0}(n)} - \frac{a_{i,1}(n)}{a_{i,0}(n)} \right\} \varepsilon_i(n) + ... \right\}.
\] (3.23)

Second, from (3.21) and (3.22) we have the explicit form of \( R(n) \) in (3.10).

\[
R(n) = \left. \frac{f_{y_{i+1}(n)}}{f_{y_{i}(n)}} \right|_0^1 - \frac{1}{y_{i}(n)} \frac{1}{y_{i+1}(n)} \frac{L_i(n)\varepsilon_i(n) + ... + \varepsilon_i(n) \ln \varepsilon_i(n) [a_{i,0}(n) + a_{i,1}(n) + ...]}{L_i^*(n)\varepsilon_i(n) + ... + \varepsilon_i(n) \ln \varepsilon_i(n) [a_{i,0}(n) + a_{i,1}(n) + ...]} \\
= \frac{L_i(n)\varepsilon_i(n)}{L_i^*(n)\varepsilon_i(n)} \left\{ 1 + \left( \frac{a_{i,0}(n)}{a_{i,0}(n)} - \frac{a_{i,1}(n)}{a_{i,0}(n)} \right) \varepsilon_i(n) + ... \right\}.
\] (3.24)

We remind the reader that the right hand sides of (3.23) and (3.24) are expansions of the same function \( R(n) \) (the modified period problem 3.2.1 for \( E(n) \)). Recall our assumption (3.1):

\[ \varepsilon_i(n) \to 0 \]

as the genus \( n \to \infty \). Letting \( \varepsilon_i(n) \to 0 \), we first observe that

\[ \frac{K_i(n)a_{i,0}(n)}{K_i^*(n)a_{i,0}^*(n)} = \frac{L_i(n)b_{i,0}(n)}{L_i^*(n)b_{i,0}^*(n)} \]
in (3.23) and (3.24). Moreover, for small \( \varepsilon_i(n) \) and fixed \( n \),

\[
1 + \left( \frac{a_{i,1}(n)}{a_{i,0}(n)} - \frac{a^*_i(n)}{a^*_i(0)} \right) \varepsilon_i(n) + ... \\
= 1 + \left( \frac{b_{i,1}(n)}{b_{i,0}(n)} - \frac{b^*_i(n)}{b^*_i(0)} \right) \varepsilon_i(n) + ... + \left( \frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a^*_i(n)}{b^*_i(0)} \right) \varepsilon_i(n) \ln \varepsilon_i(n) + ...
\]

More simply, for small \( \varepsilon_i(n) \) we have

\[
\left( \frac{a_{i,1}(n)}{a_{i,0}(n)} - \frac{a^*_i(n)}{a^*_i(0)} \right) - \left( \frac{b_{i,1}(n)}{b_{i,0}(n)} - \frac{b^*_i(n)}{b^*_i(0)} \right) = \left( \frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a^*_i(n)}{b^*_i(0)} \right) \ln \varepsilon_i(n) + o(\ln \varepsilon_i(n)).
\]

(3.25)

Therefore, the period problem 3.2.1 for \( E(n) \) under our assumption reduces to checking to see whether \( E(n) \) satisfies the equation (3.25) even as the genus \( n \to \infty \). The key observation here is that we have the term of \( \ln \varepsilon_i(n) \) on the right side of the equation. Under our assumption that \( \varepsilon_i(n) \to 0 \), we have

\[
\ln \varepsilon_i(n) \to -\infty.
\]

The only thing we need to check is the comparability of coefficients of the equation (3.25) as \( n \to \infty \). If we see the uniform comparability of coefficients \( \left( \frac{a_{i,1}(n)}{a_{i,0}(n)} - \frac{a^*_i(n)}{a^*_i(0)} \right) - \left( \frac{b_{i,1}(n)}{b_{i,0}(n)} - \frac{b^*_i(n)}{b^*_i(0)} \right) \) and \( \left( \frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a^*_i(n)}{b^*_i(0)} \right) \) of the equation (3.25), then we will have the desired contradiction. We will continue this argument on the uniform comparability of coefficients in the next step 4. We finish step 3 with the proof of lemma 3.3.2.

**Proof of lemma 3.3.2.** To prove lemma 3.3.2, we state corollary 5.2.2 here on relevant differential equations and prove it in chapter 5.

**Corollary 5.2.2.** Let

\[
g = \frac{(z^2 - 1)^{\frac{1}{2}}(z^2 - y_3^2)^{\frac{1}{2}}(z^2 - y_5^2)^{\frac{1}{2}}... (z^2 - y_{n-1}^2)^{\frac{1}{2}}}{z^{\frac{1}{2}}(z^2 - y_3^2)^{\frac{1}{2}}(z^2 - y_4^2)^{\frac{1}{2}}(z^2 - y_5^2)^{\frac{1}{2}}... (z^2 - y_n^2)^{\frac{1}{2}}}.
\]
where $y_i$ are real numbers, and let

$$y_{i+1} = y_i + \varepsilon_i$$

for some $i = 1, 2, \ldots, n - 1$. Let $i$ be an even integer. Then for any $\gamma = [y_j, y_{j+1}]$, $j = 0, 1, \ldots, n - 1$,

(a) $h = \int_{\gamma} g \, dz$ satisfies the following differential equation

$$0 = \left( \frac{1}{2} + \sum_{k=1 \atop k \neq i, i+1}^{n} (-1)^{k+1} \right) h$$

$$- \varepsilon_i(2y_i + \varepsilon_i) \left( \frac{1}{2(y_i + \varepsilon_i)} + \sum_{k=1 \atop k \neq i, i+1}^{n} (-1)^{k} \frac{y_i + \varepsilon_i}{y_k^2 - (y_i + \varepsilon_i)^2} \right) \frac{\partial h}{\partial \varepsilon_i}$$

$$+ \varepsilon_i(2y_i + \varepsilon_i) \frac{\partial^2 h}{\partial \varepsilon_i^2} + \sum_{k=1 \atop k \neq i, i+1}^{n} \frac{y_k(y_k^2 - y_i^2)}{(y_i + \varepsilon_i)^2 - y_k^2} \frac{\partial h}{\partial y_k},$$

and the equation has the solution in the form of either

$$h = \varepsilon_i \left( a_{i,0} + a_{i,1}\varepsilon_i + a_{i,2}\varepsilon_i^2 + \ldots \right) \quad \text{or}$$

$$h = (b_{i,0} + b_{i,1}\varepsilon_i + b_{i,2}\varepsilon_i^2 + \ldots) + \varepsilon_i \ln \varepsilon_i \left( a_{i,0} + a_{i,1}\varepsilon_i + a_{i,2}\varepsilon_i^2 + \ldots \right).$$

(b) $l = \int_{\gamma} \frac{1}{\delta} \, dz$ satisfies the following differential equation

$$0 = \left( \frac{3}{2} + \sum_{k=1 \atop k \neq i, i+1}^{n} (-1)^{k} \right) l$$

$$- \varepsilon_i(2y_i + \varepsilon_i) \left( \frac{1}{2(y_i + \varepsilon_i)} + \sum_{k=1 \atop k \neq i, i+1}^{n} (-1)^{k} \frac{y_i + \varepsilon_i}{y_k^2 - (y_i + \varepsilon_i)^2} \right) \frac{\partial l}{\partial \varepsilon_i}$$
and the equation has the solution in the form of either

\[ l = \varepsilon_i \left( a_{i,0}^* + a_{i,1}^* \varepsilon_i + a_{i,2}^* \varepsilon_i^2 + \ldots \right) \quad \text{or} \]

\[ l = \left( b_{i,0}^* + b_{i,1}^* \varepsilon_i + b_{i,2}^* \varepsilon_i^2 + \ldots \right) + \varepsilon_i \ln \varepsilon_i \left( a_{i,0}^* + a_{i,1}^* \varepsilon_i + a_{i,2}^* \varepsilon_i^2 + \ldots \right). \]

Here, \( a_{i,0}, b_{i,0}, a_{i,0}^*, \) and \( b_{i,0}^* \) are \( \neq 0, \) and the coefficients \( a_{i,k}, b_{i,k}, a_{i,k}^*, \) and \( b_{i,k}^*, \)
\( k = 0, 1, \ldots, \) are not functions of \( \varepsilon_i. \) The same argument holds for odd \( i. \)

From corollary 5.2.2 (a), we know that a period of \( E(n) \) has the form of either

\[ \int_{y_{j-1}(n)}^{y_j(n)} gdh = \varepsilon_i(n) \left( a_{i,0}(n) + a_{i,1}(n) \varepsilon_i(n) + a_{i,2}(n) \varepsilon_i(n)^2 + \ldots \right) \quad (3.26) \]

or \[ \int_{y_{j-1}(n)}^{y_j(n)} gdh = b_{i,0}(n) + b_{i,1}(n) \varepsilon_i(n) + b_{i,2}(n) \varepsilon_i(n)^2 + \ldots \quad (3.27) \]

\[ + \varepsilon_i(n) \ln \varepsilon_i(n) \left( a_{i,0}(n) + a_{i,1}(n) \varepsilon_i(n) + a_{i,2}(n) \varepsilon_i(n)^2 + \ldots \right), \]

for any \( j = 1, 2, \ldots, n. \) However, we need the length \( |\int gdh| \) of the period. Notice that the function

\[ g = \frac{(z^2 - 1)^{\frac{1}{2}}(z^2 - y_3(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}{z^{\frac{1}{2}}(z^2 - y_2(n)^2)^{\frac{1}{2}}(z^2 - y_4(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_n(n)^2)^{\frac{1}{2}}} \]

has exponents only of \( \frac{1}{2} \) and \( -\frac{1}{2}. \) Therefore, for any \( j = 1, 2, \ldots, n, \)

\[ \left| \int_{y_{j-1}(n)}^{y_j(n)} gdh \right| = \int_{y_{j-1}(n)}^{y_j(n)} gdh \quad \text{or} \quad \left| \int_{y_{j-1}(n)}^{y_j(n)} gdh \right| = \pm i \int_{y_{j-1}(n)}^{y_j(n)} gdh. \]

Hence, \( \left| \int_{y_{j-1}(n)}^{y_j(n)} gdh \right| \) has the same form as \( \int_{y_{j-1}(n)}^{y_j(n)} gdh. \) From the previous integral estimate (3.3) in step 1, we can check that the length \( |\int_{y_{i+1}(n)}^{y_i(n)} gdh| \) of the \((i+1)st\)
period must have the first form (3.26). Therefore, we have the desired result in (3.19) in lemma 3.3.2. Also from integral estimate (3.6) in step 2, the length \( \left| \int_{y_i(n)}^{y_{i-1}(n)} g dh \right| \) of the \( i \)th period must have the second form (3.27). This is the desired result (3.21) in lemma 3.3.2.

Similarly, from corollary 5.2.2 (b), we know that the conjugate period \( \int \frac{1}{g} dh \) has the form of either

\[
\int_{y_j(n)}^{y_{j-1}(n)} \frac{1}{g} dh = \varepsilon_i(n) \left( a^*_{i,0}(n) + a^*_{i,1}(n)\varepsilon_i(n) + a^*_{i,2}(n)\varepsilon_i(n)^2 + \ldots \right)
\]

or

\[
\int_{y_j(n)}^{y_{j-1}(n)} \frac{1}{g} dh = b^*_{i,0}(n) + b^*_{i,1}(n)\varepsilon_i(n) + b^*_{i,2}(n)\varepsilon_i(n)^2 + \ldots
\]

\[
+ \varepsilon_i^*(n) \ln \varepsilon_i(n) \left( a^*_{i,0}(n) + a^*_{i,1}(n)\varepsilon_i(n) + a^*_{i,2}(n)\varepsilon_i(n)^2 + \ldots \right),
\]

for any \( j = 1, 2, \ldots, n \). Then by checking the previous integral estimates (3.4) and (3.7) in step 1 and 2, we obtain the desired results (3.20) and (3.22) in lemma 3.3.2. This completes the proof of lemma 3.3.2.

\[\square\]

**Step 4.** Uniform comparability of coefficients in the equation (3.25). Recall the equation (3.25) from the previous step: for small \( \varepsilon_i(n) \),

\[
\left( \frac{a_{i,1}(n)}{a_{i,0}(n)} - \frac{a^*_{i,1}(n)}{a^*_{i,0}(n)} \right) - \left( \frac{b_{i,1}(n)}{b_{i,0}(n)} - \frac{b^*_{i,1}(n)}{b^*_{i,0}(n)} \right) = \left( \frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a^*_{i,0}(n)}{b^*_{i,0}(n)} \right) \ln \varepsilon_i(n) + o(\ln \varepsilon_i(n))
\]

in the equations in lemma 3.3.2. The key observation was that we have the term of \( \ln \varepsilon_i(n) \) which goes to \(-\infty\) under the assumption (3.1) that \( \varepsilon_i(n) \to 0 \). In this step, we prove lemma 3.3.3 which gives the property of the uniform comparability of coefficients in the equation (3.25) under the assumption (3.2) that \( y_i(n) - y_{i-1}(n) \approx 1 \).

**Lemma 3.3.3.** In lemma 3.3.2, if \( y_i(n) - y_{i-1}(n) \approx 1 \), then we can choose coefficients
so that

\[
\left( \frac{a_{i,1}(n)}{a_{i,0}(n)} - \frac{a_{i,1}^*(n)}{a_{i,0}^*(n)} \right) - \left( \frac{b_{i,1}(n)}{b_{i,0}(n)} - \frac{b_{i,1}^*(n)}{b_{i,0}^*(n)} \right) \quad \text{and} \quad \left( \frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a_{i,0}^*(n)}{b_{i,0}^*(n)} \right) \neq 0
\]

are uniformly bounded in the genus \( n \).

**Proof of lemma 3.3.3.** In lemma 3.3.2, as we saw in remark 3.3.5, if \( y_i(n) - y_{i-1}(n) \approx 1 \), then we can choose the same \( a_{i,0}(n), b_{i,0}(n), a_{i,0}^*(n), \) and \( b_{i,0}^*(n) \) as in lemma 3.3.1 by adjusting \( K_i(n), L_i(n), K_i^*(n), \) and \( L_i^*(n) \) so that \( a_{i,0}(n), b_{i,0}(n), a_{i,0}^*(n), \) and \( b_{i,0}^*(n) \approx 1 \) with \( \left( \frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a_{i,0}^*(n)}{b_{i,0}^*(n)} \right) \neq 0 \). Then first, it is obvious that

\[
\left( \frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a_{i,0}^*(n)}{b_{i,0}^*(n)} \right) \approx 1.
\]

Second, with the same \( a_{i,0}(n), b_{i,0}(n), a_{i,0}^*(n), \) and \( b_{i,0}^*(n) \), we will check the uniform boundedness of \( \left( \frac{a_{i,1}(n)}{a_{i,0}(n)} - \frac{a_{i,1}^*(n)}{a_{i,0}^*(n)} \right) - \left( \frac{b_{i,1}(n)}{b_{i,0}(n)} - \frac{b_{i,1}^*(n)}{b_{i,0}^*(n)} \right) \). Our only interest now is in the coefficients \( a_{i,1}(n), b_{i,1}(n), a_{i,1}^*(n), \) and \( b_{i,1}^*(n) \). For this, we need to compare the expressions for \( \left| \int_{y_{i-1}(n)}^{y_i(n)} gdh \right| \) and \( \left| \int_{y_{i-1}(n)}^{y_i(n)} \frac{1}{g} dh \right| \) in (3.21) and (3.22) of lemma 3.3.2, from the differential equations, with the expansions for \( \left| \int_{y_{i-1}(n)}^{y_i(n)} gdh \right| \) and \( \left| \int_{y_{i-1}(n)}^{y_i(n)} \frac{1}{g} dh \right| \) in (3.6) and (3.7) of lemma 3.3.1, from the integral estimates. First, let us write (3.21) and (3.22), from differential equations, again with the newly adjusted \( a_{i,0}(n), b_{i,0}(n), a_{i,0}^*(n), \) and \( b_{i,0}^*(n) \) under the assumption \( y_i(n) - y_{i-1}(n) \approx 1 \):

\[
\left| \int_{y_{i-1}(n)}^{y_i(n)} gdh \right| = L_i(n) \left\{ b_{i,0}(n) + b_{i,1}(n)\varepsilon(n) + b_{i,2}(n)\varepsilon(n)^2 + \ldots \right\} + \varepsilon(n) \ln \varepsilon(n) \left( a_{i,0}(n) + a_{i,1}(n)\varepsilon(n) + a_{i,2}(n)\varepsilon(n)^2 + \ldots \right), \quad (3.21)
\]

\[
\left| \int_{y_{i-1}(n)}^{y_i(n)} \frac{1}{g} dh \right| = L_i^*(n) \left\{ b_{i,0}^*(n) + b_{i,1}^*(n)\varepsilon(n) + b_{i,2}^*(n)\varepsilon(n)^2 + \ldots \right\} + \varepsilon(n) \ln \varepsilon(n) \left( a_{i,0}^*(n) + a_{i,1}^*(n)\varepsilon(n) + a_{i,2}^*(n)\varepsilon(n)^2 + \ldots \right). \quad (3.22)
\]
Here, \(a_{i,0}(n), b_{i,0}(n), a_{i,0}^*(n),\) and \(b_{i,0}^*(n)\) × 1 with \(\frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a_{i,0}^*(n)}{b_{i,0}^*(n)}\) \(\neq 0.\) Also recall (3.6) and (3.7), from the integral estimates:

\[
\left| \int_{y_{i-1}(n)}^{y_i(n)} gdh \right| = \Phi(\varepsilon_i(n)) \left\{ b_{i,0}(n) + \lambda_{i,1}(n)\varepsilon_i(n) + \lambda_{i,2}(n)\varepsilon_i(n)^2 + \ldots \right. \]  \(+ a_{i,0}(n)\varepsilon_i(n) \ln \varepsilon_i(n) \}, \tag{3.6}
\]

\[
\left| \int_{y_{i-1}(n)}^{y_i(n)} \frac{1}{g} dh \right| = \Phi^*(\varepsilon_i(n)) \left\{ b_{i,0}^*(n) + \lambda_{i,1}^*(n)\varepsilon_i(n) + \lambda_{i,2}^*(n)\varepsilon_i(n)^2 + \ldots \right. \]
\[+ a_{i,0}^*(n)\varepsilon_i(n) \ln \varepsilon_i(n) \}. \tag{3.7}
\]

Here, \(a_{i,0}(n), b_{i,0}(n), a_{i,0}^*(n),\) and \(b_{i,0}^*(n)\) are the same as above. Also, \(\lambda_{i,1}(n)\) and \(\lambda_{i,1}^*(n)\) are \(\asymp 1\) by lemma 3.3.1 (3). However, note that \(\Phi(\varepsilon_i(n))\) and \(\Phi^*(\varepsilon_i(n))\) are still analytic functions of \(\varepsilon_i(n)\). Since (3.21) must be the same as (3.6), and since (3.22) must be the same as (3.7), let

\[
\Phi(\varepsilon_i(n)) = L_i(n) \left\{ 1 + \tau_{i,1}(n)\varepsilon_i(n) + \tau_{i,2}(n)\varepsilon_i(n)^2 + \ldots \right\}, \tag{3.28}
\]

\[
\Phi^*(\varepsilon_i(n)) = L_i^*(n) \left\{ 1 + \tau_{i,1}^*(n)\varepsilon_i(n) + \tau_{i,2}^*(n)\varepsilon_i(n)^2 + \ldots \right\}, \tag{3.29}
\]

where \(L_i(n)\) and \(L_i^*(n)\) are the same as in (3.21) and (3.22), and \(\tau_{i,k}(n)\) and \(\tau_{i,k}^*(n)\), \(k = 1, 2, \ldots,\) are real numbers.

**Remark 3.3.6.** \(\tau_{i,1}(n)\) and \(\tau_{i,1}^*(n)\) can grow arbitrarily large as \(n \to \infty,\) but we will see that this is inconsequential in our argument.

By substituting (3.28) and (3.29) to (3.6) and (3.7), we have

\[
\left| \int_{y_{i-1}(n)}^{y_i(n)} gdh \right| = L_i(n) \left\{ b_{i,0}(n) + [\lambda_{i,1}(n) + \tau_{i,1}(n)] b_{i,0}(n) \varepsilon_i(n) + \ldots \right.
\]
\[+ \varepsilon_i(n) \ln \varepsilon_i(n) (a_{i,0}(n) + \tau_{i,1}(n)a_{i,0}(n)\varepsilon_i(n) + \ldots) \}, \]

\[
\left| \int_{y_{i-1}(n)}^{y_i(n)} \frac{1}{g} dh \right| = L_i^*(n) \left\{ b_{i,0}^*(n) + [\lambda_{i,1}^*(n) + \tau_{i,1}^*(n)] b_{i,0}^*(n) \varepsilon_i(n) + \ldots \right.
\]
\[ \pm \varepsilon_i(n) \ln \varepsilon_i(n) \left( a_{i,0}^*(n) + \tau_{i,1}^*(n) a_{i,0}^*(n) \varepsilon_i(n) + \ldots \right) \} . \]

Now, by comparing these with coefficients in (3.21) and (3.22), we obtain information on the coefficients \( a_{i,1}(n), b_{i,1}(n), a_{i,1}^*(n), \) and \( b_{i,1}^*(n) \) as the following:

\[
\begin{align*}
    b_{i,1}(n) &= \lambda_{i,1}(n) + \tau_{i,1}(n) b_{i,0}(n), \\
    b_{i,1}^*(n) &= \lambda_{i,1}^*(n) + \tau_{i,1}^*(n) b_{i,0}^*(n), \\
    a_{i,1}(n) &= \tau_{i,1}(n) a_{i,0}(n), \quad (3.21) \\
    a_{i,1}^*(n) &= \tau_{i,1}^*(n) a_{i,0}^*(n) .
\end{align*}
\]

Therefore,

\[
\begin{align*}
    \frac{a_{i,1}(n)}{a_{i,0}(n)} - \frac{a_{i,1}^*(n)}{a_{i,0}^*(n)} &= \frac{\tau_{i,1}(n) a_{i,0}(n)}{a_{i,0}(n)} - \frac{\tau_{i,1}^*(n) a_{i,0}^*(n)}{a_{i,0}^*(n)} \\
    &= \tau_{i,1}(n) - \tau_{i,1}^*(n), \\
    \frac{b_{i,1}(n)}{b_{i,0}(n)} - \frac{b_{i,1}^*(n)}{b_{i,0}^*(n)} &= \frac{\lambda_{i,1}(n) + \tau_{i,1}(n) b_{i,0}(n)}{b_{i,0}(n)} - \frac{\lambda_{i,1}^*(n) + \tau_{i,1}^*(n) b_{i,0}^*(n)}{b_{i,0}^*(n)} \\
    &= \frac{\lambda_{i,1}(n)}{b_{i,0}(n)} - \frac{\lambda_{i,1}^*(n)}{b_{i,0}^*(n)} + \tau_{i,1}(n) - \tau_{i,1}^*(n).
\end{align*}
\]

Hence,

\[
\begin{align*}
    \left( \frac{a_{i,1}(n)}{a_{i,0}(n)} - \frac{a_{i,1}^*(n)}{a_{i,0}^*(n)} \right) - \left( \frac{b_{i,1}(n)}{b_{i,0}(n)} - \frac{b_{i,1}^*(n)}{b_{i,0}^*(n)} \right) &= \frac{\lambda_{i,1}(n)}{b_{i,0}(n)} - \frac{\lambda_{i,1}(n)}{b_{i,0}(n)}. 
\end{align*}
\]

Since \( b_{i,0}(n), \lambda_{i,1}(n), b_{i,0}^*(n), \) and \( \lambda_{i,1}^*(n) \) are uniformly bounded in \( n \), we see that \( \frac{\lambda_{i,1}(n)}{b_{i,0}(n)} - \frac{\lambda_{i,1}(n)}{b_{i,0}(n)} \) is uniformly bounded in \( n \). Therefore, we conclude that

\[
\begin{align*}
    \left( \frac{a_{i,1}(n)}{a_{i,0}(n)} - \frac{a_{i,1}^*(n)}{a_{i,0}^*(n)} \right) - \left( \frac{b_{i,1}(n)}{b_{i,0}(n)} - \frac{b_{i,1}^*(n)}{b_{i,0}^*(n)} \right) \simeq 1.
\end{align*}
\]

This completes the proof of lemma 3.3.3. \( \square \)
Step 5. Conclusion. Finally, in this step we will finish the proof of proposition 3.3.1. Recall the equation (3.25): for small $\varepsilon_i(n)$,

$$
\left( \frac{a_{i,1}(n)}{a_{i,0}(n)} - \frac{a_{i,1}^*(n)}{a_{i,0}^*(n)} \right) - \left( \frac{b_{i,1}(n)}{b_{i,0}(n)} - \frac{b_{i,1}^*(n)}{b_{i,0}^*(n)} \right) = \left( \frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a_{i,0}^*(n)}{b_{i,0}^*(n)} \right) \ln \varepsilon_i(n) + o(\ln \varepsilon_i(n))
$$

in the equations in lemma 3.3.2. This equation came from the modified period problem 3.2.1, by comparing $R(n) = \left( \int_{\gamma(n)} \frac{g dh}{f_{\gamma(n)}} \right)$ in (3.23) and $R(n) = \left( \int_{\gamma(n)} \frac{g dh}{f_{\gamma(n)}} \right)$ in (3.24). We just proved in lemma 3.3.3 in step 4 that if $y_i(n) - y_{i-1}(n) \asymp 1$, then we can choose coefficients so that the coefficients $\left( \frac{a_{i,1}(n)}{a_{i,0}(n)} - \frac{a_{i,1}^*(n)}{a_{i,0}^*(n)} \right)$ and $\left( \frac{b_{i,0}^*(n)}{b_{i,0}(n)} - \frac{b_{i,0}^*(n)}{b_{i,0}^*(n)} \right)$ on the left and right sides of the equation (3.25) are uniformly bounded in the genus $n$. Therefore, the equation (3.25) does not hold as $\varepsilon_i(n) \to 0$, because $\ln \varepsilon_i(n) \to -\infty$. In other words, under the assumption that $\varepsilon_i(n) \to 0$, we do not have the same $R(n)$ from two different expansions in (3.23) and in (3.24). Therefore, we conclude that for each even $i \geq 1$, if $y_i(n) - y_{i-1}(n) \asymp 1$, then there exists $\varepsilon_i > 0$ such that

$$
\varepsilon_i = \inf_n \varepsilon_i(n).
$$

Hence, for each even $i \geq 1$, if $y_i(n) - y_{i-1}(n) \asymp 1$, then $y_{i+1}(n) - y_i(n) = \varepsilon_i(n) \geq \varepsilon_i$ for some $\varepsilon_i > 0$ independent of $n$.

Note that at the beginning, we let $i$ be an even integer. We also need to see whether the same argument holds for each odd $i$. However, we saw in lemma 3.3.1 in step 2 that for odd $i$, we have the same result with only the exception that the signs of $a_{i,0}(n)$ and $a_{i,0}^*(n)$ reverse. Therefore, we have the same result for each odd $i$ through steps 3-5. So we have finally proved proposition 3.3.1. \qed
3.4 Nondegeneration of the Points in $V(n)$, Part II

In proposition 3.3.1 in the previous section, we saw that for any fixed $i = 1, 2, ..., n-1$, if $y_i(n) - y_{i-1}(n)$ are uniformly stable in the genus $n$, then $y_{i+1}(n) - y_i(n)$ cannot tend to 0 as $n \to \infty$. However, to complete the uniform stability of $y_{i+1}(n) - y_i(n)$, we also need to show that if $y_i(n) - y_{i-1}(n)$ are uniformly stable in the genus $n$, then $y_{i+1}(n) - y_i(n)$ cannot tend to $\infty$ as $n \to \infty$, either. We will show this for any $i \geq 2$ in proposition 3.4.1 in this section and for the $i = 1$st case in proposition 3.5.1 in the following section.

**Proposition 3.4.1.** For any fixed $i = 2, 3, ..., n-1$, if $y_i(n) - y_{i-1}(n) \gg 1$, then $y_{i+1}(n) - y_i(n) \leq \eta_i$ for some $\eta_i < \infty$ independent of $n$.

**Proof.** By rescaling, this is the same as showing that for any fixed $i = 2, 3, ..., n-1$, if $y_{i+1}(n) - y_i(n) \gg 1$, then $y_i(n) - y_{i-1}(n) \geq \varepsilon_{i-1}$ for some $\varepsilon_{i-1} > 0$ independent of $n$ instead. Moreover, in order to use the information of proposition 3.3.1, we will show that for any $i = 1, 2, ..., n-2$, if $y_{i+2}(n) - y_{i+1}(n) \gg 1$, then $y_{i+1}(n) - y_i(n) \geq \varepsilon_i$ for some $\varepsilon_i > 0$ independent of $n$. Here, we will compare $R(n) = \int_{y_i(n)}^{y_{i+1}(n)} g dh$ and $R(n) = \int_{y_{i+1}(n)}^{y_{i+2}(n)} g dh$, instead of $R(n) = \int_{y_i(n)}^{y_{i+1}(n)} \frac{g}{\beta} dh$ and $R(n) = \int_{y_{i-1}(n)}^{y_i(n)} \frac{g}{\beta} dh$ in proposition 3.3.1, for the period problem 3.2.1.

For each finite genus $n$, choose a coordinate domain $\Omega_g(n)$ for $E(n)$ which is a solution of the period problem, i.e. there exists a quantity $R(n)$ in the period problem 3.2.1 for $E(n)$. Note that the domain $\Omega_g(n)$ is determined by its symmetric zigzag boundary $Z_g(n) = \partial \Omega_g(n)$ with the $(2n + 1)$ vertices. Let

$$V(n) = (-y_n(n), ..., -y_2(n), -1, 0, 1, y_2(n), ..., y_n(n)) \in \mathbb{R}^{2n+1},$$

where $1 < y_2(n) < ... < y_n(n) \in \mathbb{R} = \partial \mathbb{H}$, be the preimage of the $(2n + 1)$ vertices of the zigzag boundary $Z_g(n) = \partial \Omega_g(n)$ of the domain $\Omega_g(n)$ under the Schwarz-
Christoffel map $\psi_g(n) \mathbb{H} \to \Omega_g(n) \subset \mathbb{C}$. Notice that we normalized the first point $y_1(n) = 1$ for all $n$ without loss of generality. Fix some $i$ and let

$$y_{i+1}(n) = y_i(n) + \varepsilon_i(n).$$

Assume that we can find a subsequence of coordinate domains $\{\Omega_g(n)\}$ such that in $V(n)$ corresponding to $\Omega_g(n)$,

$$\varepsilon_i(n) \to 0$$

as the genus $n \to \infty$, while

$$y_{i+2}(n) - y_{i+1}(n) \approx 1,$$

instead of $y_i(n) - y_{i-1}(n) \approx 1$ in (3.2) in proposition 3.3.1. Without loss of generality, let $i$ be an even integer. Then we have the same result as in step 1 of proposition 3.3.1: $R(n) = \int_{y_i(n)}^{y_{i+1}(n)} \frac{g}{\varepsilon_i(n)} dh$ is an analytic function of $\varepsilon_i(n)$ independent of $n$. Next, we will state a lemma 3.4.1 as a corollary of lemma 3.3.1 in step 2 of proposition 3.3.1.

Notice that we have switched the signs of $a_{i,0}(n)$ and $a_{i,0}^*(n)$ from lemma 3.3.1.

**Lemma 3.4.1** (Corollary of lemma 3.3.1). Let $i \geq 1$ be any even integer and for fixed $i$, let $y_{i+1}(n) = y_i(n) + \varepsilon_i(n)$. If $y_{i+2}(n) - y_{i+1}(n) \approx 1$, then the $(i + 2)$nd period and the conjugate period for $E(n)$ can be expressed by the following expansions in $\varepsilon_i(n)$:

$$\int_{y_{i+1}(n)}^{y_{i+2}(n)} g dh = \Phi(\varepsilon_i(n)) \left\{ b_{i,0}(n) + \lambda_{i,1}(n) \varepsilon_i(n) + \lambda_{i,2}(n) \varepsilon_i(n)^2 + \ldots \right. + a_{i,0}(n) \varepsilon_i(n) \ln \varepsilon_i(n) \right\},$$

$$\int_{y_{i+1}(n)}^{y_{i+2}(n)} \frac{1}{g} dh = \Phi^*(\varepsilon_i(n)) \left\{ b_{i,0}^*(n) + \lambda_{i,1}^*(n) \varepsilon_i(n) + \lambda_{i,2}^*(n) \varepsilon_i(n)^2 + \ldots \right. + a_{i,0}^*(n) \varepsilon_i(n) \ln \varepsilon_i(n) \right\},$$
where

1. \( b_{i,0}(n), a_{i,0}(n), b_{i,0}^*(n), \) and \( a_{i,0}^*(n) \) are all \( \leq 1 \),

2. \( b_{i,0}(n) > 0, a_{i,0}(n) > 0, b_{i,0}^*(n) > 0, \) and \( a_{i,0}^*(n) < 0, \)

3. \( \lambda_{i,k}(n) \) and \( \lambda_{i,k}^*(n), k = 1, 2, ..., \) are uniformly bounded in \( n \),

4. \( \Phi(\varepsilon_i(n)) \) and \( \Phi^*(\varepsilon_i(n)) \) are analytic functions of \( \varepsilon_i(n) \).

For an odd \( i \geq 1 \), we have the same result with the exception that the signs of \( a_{i,0}(n) \) and \( a_{i,0}^*(n) \) reverse: \( a_{i,0}(n) < 0, \) and \( a_{i,0}^*(n) > 0. \)

**Remark 3.4.1.** This result is the same as lemma 3.3.1 with only the exception of the switched signs of \( a_{i,0}(n) \) and \( a_{i,0}^*(n) \). Therefore, we also have

\[
\left( \frac{a_{i,0}(n)}{b_{i,0}(n)} - \frac{a_{i,0}^*(n)}{b_{i,0}^*(n)} \right) \propto 1
\]

which is a crucial fact behind the proof of proposition 3.4.1.

We skip the proof of lemma 3.4.1. We will continue the proof of proposition 3.4.1. By lemma 3.4.1, we see that \( R(n) = \frac{\int^{\pi_{i+2}(n)}_{\pi_{i+1}(n)} g dh}{\int^{\pi_{i+2}(n)}_{\pi_{i+1}(n)} \frac{1}{g} dh} \) is not an analytic function of \( \varepsilon_i(n) \) in a neighborhood of 0. Then by the same argument as in proposition 3.3.1, through steps 3-5, we have a contradiction to having a quantity \( R(n) \) as \( n \to \infty \) in the modified period problem 3.2.1. This completes the proof of proposition 3.4.1. \( \square \)

**Remark 3.4.2.** Note that proposition 3.4.1 is only for \( i \geq 2, \) not \( i = 1. \) In order to complete theorem 3.6.1, besides proposition 3.3.1 and 3.4.1 we have to show that proposition 3.4.1 holds also when \( i = 1. \) We will see this argument in the following section.
3.5 Nondegeneration of the Initial Points in $V(n)$

In proposition 3.3.1, we saw that for any fixed $i = 1, 2, ..., n-1$, if $y_i(n) - y_{i-1}(n) \asymp 1$, then $y_{i+1}(n) - y_i(n) \geq \varepsilon_i$ for some $\varepsilon_i > 0$ independent of $n$. In proposition 3.4.1, we saw that for any $i = 2, 3, ..., n-1$, if $y_i(n) - y_{i-1}(n) \asymp 1$, then $y_{i+1}(n) - y_i(n) \leq \eta_i$ for some $\eta_i < \infty$ independent of $n$. However, note that proposition 3.4.1 is only for $i \geq 2$, not for $i = 1$. Thus, when $y_1(n) - y_0(n) \asymp 1$, we still do not know if $y_2(n) - y_1(n) = y_2(n) - 1$ is uniformly bounded as $n \to \infty$. In this section, we consider this case. We will prove proposition 3.5.1 which is the complement of proposition 3.4.1 for the $i = 1$ case.

**Proposition 3.5.1.** If $y_1(n) - y_0(n) \asymp 1$, then $y_2(n) - y_1(n) < \eta_1$ for some $\eta_1 \leq \infty$ independent of $n$.

**Remark 3.5.1.** As in the proof of proposition 3.4.1, by rescaling this is the same as showing that if $y_2(n) - y_1(n) \asymp 1$, then $y_1(n) - y_0(n) \geq \varepsilon_0$ for some $\varepsilon_0 > 0$ independent of $n$ instead. Note that this $i = 1$ case is different from the other cases for $i \geq 2$ in proposition 3.4.1. If $y_1(n) - y_0(n) = y_1(n)$ tends to 0 as the genus $n \to \infty$, then $y_0(n) - (-y_1(n)) = y_1(n)$ on the other side from 0 also tends to 0. Therefore, two adjacent lengths tend to 0 together.

**Proof.** By rescaling, we will show that if $y_2(n) - y_1(n) \asymp 1$, then $y_1(n) - y_0(n) \geq \varepsilon_0$ for some $\varepsilon_0 > 0$ independent of $n$ instead.

For each finite genus $n$, choose a coordinate domain $\Omega_\varphi(n)$ for $E(n)$ which is a solution of the period problem, i.e. there exists a quantity $R(n)$ in the period problem 3.2.1 for $E(n)$. Let

$$V(n) = (-y_6(n), ..., -y_8(n), -1, -\varepsilon(n), 0, \varepsilon(n), 1, y_3(n), ..., y_n(n)) \in \mathbb{R}^{2n+1},$$

where $0 < \varepsilon(n) < 1 < y_3(n) < ... < y_n(n) \in \mathbb{R} = \partial \mathbb{H}$, be the preimage of the
\((2n + 1)\) vertices of the zigzag boundary \(Z_g(n) = \partial \Omega_g(n)\) of the domain \(\Omega_g(n)\) under the Schwarz-Christoffel map \(\psi_g(n) : \mathbb{H} \to \Omega_g(n) \subset \mathbb{C}\). Notice that we normalized the first point \(y_2(n) = 1\) for all \(n\) without loss of generality. Here are properties of this setting:

1. By proposition 3.3.1, \(\varepsilon(n)\) does not tend to 1. That is,

\[
y_2(n) - y_1(n) = 1 - \varepsilon(n) > \varepsilon_1
\]

for some \(\varepsilon_1 > 0\) independent of \(n\),

2. By proposition 3.3.1 and 3.4.1, \(\frac{y_i(n) - y_{i-1}(n)}{y_{i+1}(n) - y_i(n)} \asymp 1\) for any fixed \(i = 2, 3, ..., n - 1\),

3. By 1 and 2, for any fixed \(i = 2, 3, ..., n - 1\),

\[
y_i(n) - y_{i-1}(n) \asymp 1.
\]

Assume that we can find a subsequence of coordinate domains \(\{\Omega_g(n)\}\) such that in \(V(n)\) corresponding to \(\Omega_g(n)\),

\[
\varepsilon(n) \longrightarrow 0
\]

Under this assumption, we will find a contradiction to satisfying the period problem as the genus \(n \to \infty\): we will show that we do not have a quantity \(R(n)\) in the modified period problem 3.2.1 for \(E(n)\) as the genus \(n \to \infty\), by comparing the ratios \(R(n) = \left| \frac{\int_0^{\psi(n)} \frac{g}{\sqrt{1 - \frac{1}{2}y^2}} dh}{\int_0^{\psi(n)} \frac{1}{\sqrt{1 - \frac{1}{2}y^2}} dh} \right|\) and \(R(n) = \left| \frac{\int_0^{\psi(n)} \frac{g}{\sqrt{1 - \frac{1}{2}y^2}} dh}{\int_0^{\psi(n)} \frac{1}{\sqrt{1 - \frac{1}{2}y^2}} dh} \right|\). The key observation will be that

\[
R(n) = \left| \frac{\int_0^{\psi(n)} \frac{g}{\sqrt{1 - \frac{1}{2}y^2}} dh}{\int_0^{\psi(n)} \frac{1}{\sqrt{1 - \frac{1}{2}y^2}} dh} \right| < o(\varepsilon(n)).
\]

Therefore, if we find a strictly positive lower bound of \(R(n)\), then we will arrive at the desired contradiction. First, we will consider the case with even genus \(n\). Recall the \(ith\) period of \(E(n)\)

\[
\int_{y_{i-1}(n)}^{\psi(n)} \frac{g}{\sqrt{1 - \frac{1}{2}y^2}} dh = \int_{y_{i-1}(n)}^{\psi(n)} \frac{(z^2 - \varepsilon(n)^2)^{\frac{1}{2}}(z^2 - y_3(n)^2)^{\frac{1}{2}}(z^2 - y_5(n)^2)^{\frac{1}{2}}... (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}} dz.
\]
Let
\[ F_n(z) = \frac{(z^2 - y_3(n)^2)^\frac{1}{2}(z^2 - y_5(n)^2)^\frac{1}{2} \cdots (z^2 - y_n(n)^2)^\frac{1}{2}}{(z^2 - y_4(n)^2)^\frac{1}{2}(z^2 - y_6(n)^2)^\frac{1}{2} \cdots (z^2 - y_{n-1}(n)^2)^\frac{1}{2}}. \] (3.32)

This is a product of an even number of factors \((z^2 - y_i(n)^2)^{\frac{1}{2}}\) for \(i = 3, 4, \ldots, n\), so that \(F_n(z)\) has a positive value for \(z \in [0, 1]\). We will complete the proof of this proposition through four steps.

**Step 1.** Here, we will compute an upper bound of \(R(n)\) using \(R(n) = \frac{\int_{\gamma} F_n(z) dh}{\int_{\gamma} |g| dh}\). First, for an upper bound of the length \(\int_{\gamma} |g| dh\) of the first period in the numerator,

\[
\left| \int_{\gamma} F_n(z) g dh \right| = \left| \int_{\gamma} \frac{(z^2 - \varepsilon(n)^2)^{\frac{1}{2}}(z^2 - y_3(n)^2)^{\frac{1}{2}}(z^2 - y_5(n)^2)^{\frac{1}{2}} \cdots (z^2 - y_n(n)^2)^{\frac{1}{2}}}{z^\frac{1}{2}(z^2 - 1)^{\frac{1}{2}}(z^2 - y_4(n)^2)^{\frac{1}{2}}(z^2 - y_6(n)^2)^{\frac{1}{2}} \cdots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}} dz \right|
\]

\[
= \left| \int_{\gamma} \frac{(z^2 - \varepsilon(n)^2)^{\frac{1}{2}}}{z^\frac{1}{2}(z^2 - 1)^{\frac{1}{2}}} F_n(z) dz \right|
\]

\[
< \left( \sup_{z \in [0, \varepsilon(n)]} F_n(z) \right) \int_{\gamma} \frac{(z^2 - \varepsilon(n)^2)^{\frac{1}{2}}}{z^\frac{1}{2}(1 - z^2)^{\frac{1}{2}}} dz
\]

\[
< \left( \sup_{z \in [0, \varepsilon(n)]} F_n(z) \right) \frac{1}{(1 - \varepsilon(n)^2)^{\frac{1}{2}}} \int_{\gamma} \left( \frac{(z^2 - z)^{\frac{1}{2}}}{z} \right) dz
\]

\[
= \left( \sup_{z \in [0, \varepsilon(n)]} F_n(z) \right) \frac{\varepsilon(n)^{\frac{3}{2}}}{(1 - \varepsilon(n)^2)^{\frac{1}{2}}} \int_{0}^{1} \left( \frac{1 - t^2}{t} \right)^{\frac{1}{2}} dt
\]

by \(z = \varepsilon(n)t\). Since \(1 + t < 2\) when \(t \in [0, 1]\), we conclude that

\[
\left| \int_{\gamma} F_n(z) g dh \right| < \left( \sup_{z \in [0, \varepsilon(n)]} F_n(z) \right) \frac{\varepsilon(n)^{\frac{3}{2}}}{(1 - \varepsilon(n)^2)^{\frac{1}{2}}} \sqrt{2} \int_{0}^{1} \left( \frac{1 - t}{t} \right)^{\frac{1}{2}} dt. \] (3.33)

Next, for a lower bound of the length \(\int_{\gamma} |g| dh\) of the first conjugate period in the denominator of \(R(n)\),

\[
\left| \int_{\gamma} \frac{1}{g} \right| = \left| \int_{\gamma} \frac{(z^2 - \varepsilon(n)^2)^{\frac{1}{2}}(z^2 - y_3(n)^2)^{\frac{1}{2}}(z^2 - y_5(n)^2)^{\frac{1}{2}} \cdots (z^2 - y_n(n)^2)^{\frac{1}{2}}}{(z^2 - \varepsilon(n)^2)^{\frac{1}{2}}(z^2 - y_4(n)^2)^{\frac{1}{2}}(z^2 - y_6(n)^2)^{\frac{1}{2}} \cdots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}} dz \right|
\]

\[
= \left| \int_{\gamma} \frac{(z^2 - \varepsilon(n)^2)^{\frac{1}{2}}}{(z^2 - \varepsilon(n)^2)^{\frac{1}{2}}} \frac{1}{F_n(z)} dz \right|
\]

\[
= \left| \int_{\gamma} \frac{1}{F_n(z)} dz \right|
\]
\[
> \left( \inf_{z \in [0, \varepsilon(n)]} \frac{1}{F_n(z)} \right) \int_0^{\varepsilon(n)} \frac{z^{\frac{1}{2}}(1 - z^2)^{\frac{1}{4}}}{(\varepsilon(n)^2 - z^2)^{\frac{3}{2}}} dz \\
> \left( \sup_{z \in [0, \varepsilon(n)]} \frac{1}{F_n(z)} \right) (1 - \varepsilon(n)^2)^{\frac{1}{2}} \int_0^{\varepsilon(n)} \left( \frac{z}{\varepsilon(n)^2 - z^2} \right)^{\frac{1}{2}} dz \\
> \left( \sup_{z \in [0, \varepsilon(n)]} \frac{1}{F_n(z)} \right) \varepsilon(n)^{\frac{1}{2}} (1 - \varepsilon(n)^2)^{\frac{1}{2}} \int_0^1 \left( \frac{t}{1 - t^2} \right)^{\frac{1}{2}} dt.
\]

Since \(1 + t < 2\) when \(t \in [0, 1]\), we conclude that

\[
\left| \int_0^{\varepsilon(n)} \frac{1}{g} dh \right| > \left( \sup_{z \in [0, \varepsilon(n)]} \frac{1}{F_n(z)} \right) \varepsilon(n)^{\frac{1}{2}} (1 - \varepsilon(n)^2)^{\frac{1}{2}} \frac{1}{\sqrt{2}} \int_0^1 \left( \frac{t}{1 - t} \right)^{\frac{1}{2}} dt. \tag{3.34}
\]

Now, we are ready to see an upper bound of the ratio \(R(n)\) of (3.33) and (3.34).

Note that in (3.33) and (3.34), we have \(\int_0^1 \left( \frac{1-t}{t} \right)^{\frac{1}{2}} dt = \int_0^1 \left( \frac{t}{1-t} \right)^{\frac{1}{2}} dt.\) Therefore, we have the following upper bound of \(R(n)\).

\[
R(n) = \left| \int_0^{\varepsilon(n)} g dh \right| \frac{\int_0^{\varepsilon(n)} \frac{1}{g} dh}{\int_0^{\varepsilon(n)} \frac{1}{g} dh} < \left( \sup_{z \in [0, \varepsilon(n)]} F_n(z) \right)^2 \frac{2\varepsilon(n)}{1 - \varepsilon(n)^2}. \tag{3.35}
\]

Actually, \(0 < F_n(z) < 1\) when \(z \in [0, \varepsilon(n)]\). Therefore, we have an upper bound of \(R(n)\) dominated by \(\varepsilon(n)\).

**Step 2.** In this step, we will compute a lower bound of \(R(n)\) using \(R(n) = \left| \int_{\varepsilon(n)}^1 g dh \right|\). For a lower bound of the length \(\int_{\varepsilon(n)}^1 g dh\) of the second period in the numerator,

\[
\left| \int_{\varepsilon(n)}^1 g dh \right| = \left| \int_{\varepsilon(n)}^1 \frac{(z^2 - \varepsilon(n)^2)^{\frac{1}{4}}(z^2 - y_3(n)^2)^{\frac{1}{4}}(z^2 - y_5(n)^2)^{\frac{1}{4}} \ldots (z^2 - y_{n-1}(n)^2)^{\frac{1}{4}}}{z^{\frac{1}{2}}(z^2 - 1)^{\frac{1}{4}}F_n(z) dz} \right|
\]

\[
= \left| \int_{\varepsilon(n)}^1 \frac{(z^2 - \varepsilon(n)^2)^{\frac{1}{4}}}{z^{\frac{1}{2}}(z^2 - 1)^{\frac{1}{4}}F_n(z) dz} \right|
\]

\[
> \left( \inf_{z \in [\varepsilon(n), 1]} F_n(z) \right) \int_{\varepsilon(n)}^1 \frac{(z^2 - \varepsilon(n)^2)^{\frac{1}{4}}}{z^{\frac{1}{2}}(1 - z^2)^{\frac{3}{4}} dz}.
\]
Let
\[ a(n) = \int_{\varepsilon(n)}^{1} \frac{(z^2 - \varepsilon(n)^2)^{\frac{1}{2}}}{z^{\frac{1}{2}}(1 - z^2)^{\frac{1}{2}}} \, dz. \]

Since we know from (3.30) that \(1 - \varepsilon(n) > \varepsilon_1\) for some \(\varepsilon_1 > 0\) independent of \(n\), the domain of the integral is stable. Therefore, we have that
\[ a(n) \asymp 1. \]

In other words, there exists \(0 < c < \infty\) such that \(\frac{1}{c} < a(n) < c\) independent of \(n\). Hence, we conclude that
\[ \left| \int_{\varepsilon(n)}^{1} g \, dh \right| > \left( \inf_{z \in [\varepsilon(n), 1]} F_n(z) \right) a(n), \quad (3.36) \]

where \(a(n) \asymp 1\). Next, for an upper bound of the length \(\int_{\varepsilon(n)}^{1} g \, dh\) of the second conjugate period in the denominator of \(R(n)\),
\[
\left| \int_{\varepsilon(n)}^{1} \frac{1}{g} \, dh \right| = \left| \int_{\varepsilon(n)}^{1} \frac{\frac{1}{2} (z^2 - 1)^{\frac{1}{2}} (z^2 - y_4(n)^2)^{\frac{1}{2}} (z^2 - y_6(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_n(n)^2)^{\frac{1}{2}}}{\left( z - \varepsilon(n)^2 \right)^{\frac{1}{2}} F_n(z)} \, dz \right|
\]
\[
= \left| \int_{\varepsilon(n)}^{1} \frac{\frac{1}{2} (z^2 - 1)^{\frac{1}{2}}}{\left( z - \varepsilon(n)^2 \right)^{\frac{1}{2}} F_n(z)} \, dz \right|
\]
\[
< \left( \sup_{z \in [\varepsilon(n), 1]} \frac{1}{F_n(z)} \right) \left( \inf_{z \in [\varepsilon(n), 1]} \frac{\frac{1}{2} (1 - z^2)^{\frac{1}{2}}}{\left( z^2 - \varepsilon(n)^2 \right)^{\frac{1}{2}}} \right) \int_{\varepsilon(n)}^{1} \frac{\frac{1}{2} (1 - z^2)^{\frac{1}{2}}}{\left( z^2 - \varepsilon(n)^2 \right)^{\frac{1}{2}}} \, dz
\]
\[
= \left( \inf_{z \in [\varepsilon(n), 1]} \frac{1}{F_n(z)} \right) \int_{\varepsilon(n)}^{1} \frac{\frac{1}{2} (1 - z^2)^{\frac{1}{2}}}{\left( z^2 - \varepsilon(n)^2 \right)^{\frac{1}{2}}} \, dz.
\]

Let
\[ a^*(n) = \int_{\varepsilon(n)}^{1} \frac{\frac{1}{2} (1 - z^2)^{\frac{1}{2}}}{\left( z^2 - \varepsilon(n)^2 \right)^{\frac{1}{2}}} \, dz. \]

Since we know from (3.30) that \(1 - \varepsilon(n) > \varepsilon_1\) for some \(\varepsilon_1 > 0\) independent of \(n\), the
domain of the integral is stable. Therefore, we have that

\[ a^*(n) \approx 1. \]

Hence, we conclude that

\[
\left| \int_{\varepsilon(n)}^{1} \frac{1}{g} dh \right| < \left( \frac{1}{\inf_{z \in [\varepsilon(n), 1]} F_n(z)} \right) a^*(n),
\]

(3.37)

where \( a^*(n) \approx 1 \) independent of \( n \). Finally, from (3.36) and (3.37), we obtain a lower bound of \( R(n) \)

\[
R(n) = \left| \int_{\varepsilon(n)}^{1} g dh \right| > \left( \inf_{z \in [\varepsilon(n), 1]} F_n(z) \right)^2 \frac{a(n)}{a^*(n)},
\]

(3.38)

where \( \frac{a(n)}{a^*(n)} \approx 1 \). Actually, \( 0 < F_n(z) < 1 \) when \( z \in [\varepsilon(n), 1] \).

**Step 3.** Here, we write the upper bound of \( R(n) \) in (3.35) and the lower bound of \( R(n) \) in (3.38) together.

\[
\left( \inf_{z \in [\varepsilon(n), 1]} F_n(z) \right)^2 \frac{a(n)}{a^*(n)} < R(n) < \left( \sup_{z \in [0, \varepsilon(n)]} F_n(z) \right)^2 \frac{2\varepsilon(n)}{1 - \varepsilon(n)^2}.
\]

(3.39)

Therefore, we have the following inequality

\[
\left( \inf_{z \in [\varepsilon(n), 1]} F_n(z) \right)^2 \frac{a(n)}{a^*(n)} < \left( \sup_{z \in [0, \varepsilon(n)]} F_n(z) \right)^2 \frac{2\varepsilon(n)}{1 - \varepsilon(n)^2},
\]

i.e.

\[
\left( \frac{\inf_{z \in [\varepsilon(n), 1]} F_n(z)}{\sup_{z \in [0, \varepsilon(n)]} F_n(z)} \right)^2 \frac{a(n)}{a^*(n)} < \frac{2\varepsilon(n)}{1 - \varepsilon(n)^2},
\]

(3.40)

where \( \frac{a(n)}{a^*(n)} \approx 1 \).

**Remark 3.5.2.** Note that we assumed that \( \varepsilon(n) \to 0 \) as \( n \to \infty \), but \( \frac{a(n)}{a^*(n)} \approx 1 \).
independent of \( n \). Therefore, if we find a good bound for \( \left( \inf_{z \in [\varepsilon(n), 1]} \frac{F_n(z)}{\sup_{z \in [0, \varepsilon(n)]} F_n(z)} \right) \), we will have the desired contradiction. In this step, we will focus on the properties of the function \( F_n(z) \) near \([0, 1] \). First, let us recall Harnack’s inequality which will give us a useful result.

**Theorem 3.5.1** (Harnack [GT83] 2.3). Let \( u \) be a positive harmonic function in \( \Omega \subset \mathbb{R}^n \). Then for any bounded subdomain \( \Omega' \subset \subset \Omega \), we have

\[
\frac{\sup_{\Omega'} u}{\inf_{\Omega'} u} < M
\]

where \( M = M(n, \Omega, \Omega') \) depends on \( n \), \( \Omega \), and \( \Omega' \).

**Remark 3.5.3.** The constant \( M \) depends on the domain \( \Omega \) and the subdomain \( \Omega' \), but not on function \( u \).

We skip the proof of the Harnack’s theorem here. Now, we will prove the following lemma using Harnack’s inequality.

**Lemma 3.5.1.** For any \( z \in [0, \varepsilon(n)] \) and \( w \in [\varepsilon(n), 1] \),

\[
\frac{1}{M} < \frac{F_n(w)}{F_n(z)} < M
\]

where \( M \) is a constant independent of \( n \).

**Proof of lemma 3.5.1.** Recall the function \( F_n(z) \)

\[
F_n(z) = \frac{(z^2 - y_3(n)^2)^{\frac{1}{2}}(z^2 - y_5(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}{(z^2 - y_4(n)^2)^{\frac{1}{2}}(z^2 - y_6(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_n(n)^2)^{\frac{1}{2}}},
\]

The function \( F_n(z) \) is a holomorphic function with no zeros and no poles in a neighborhood of \([0, 1]\) in \( \mathbb{C} \), and also \( F_n(z) \) is positive on \([0, 1]\). Therefore, if \( \text{Re} F_n(z) > 0 \) on some domain \( \Omega \subset \mathbb{C} \) including \([0, 1]\), we can apply Harnack’s inequality which will
give us a good relationship between \( \inf_{z \in [0,1]} F_n(z) \) and \( \sup_{z \in [0,1]} F_n(z) \).

First, we need to find a domain \( \Omega \) including \([0, 1]\) on which \( \Re F_n(z) > 0 \). For this, since we know from (3.31) that \( y_3(n) - 1 \preceq 1 \), for each \( n \) choose a midpoint

\[
m = \frac{y_3(n) - 1}{2}
\]

between 1 and \( y_3(n) \). Then let \( \Omega \) be the rectangle with the vertices at \((m, m), (-m, m), (-m, -m), \) and \((m, -m)\) as in figure 3.2. To have a positive real part

\[
\text{Figure 3.2. The domain } \Omega
\]

on \( \Omega \), the function \( F_n(z) \) must satisfy

\[
-\frac{\pi}{2} < \arg F_n(z) < \frac{\pi}{2}
\]

for all \( z \in \Omega \). Let us look at the \( \arg F_n(z) \).

\[
\arg F_n(z) = \arg \left( \frac{z^2 - y_3(n)^2)^{1/2} (z^2 - y_5(n)^2)^{1/2} \ldots (z^2 - y_{n-1}(n)^2)^{1/2}}{(z^2 - y_4(n)^2)^{1/2} (z^2 - y_6(n)^2)^{1/2} \ldots (z^2 - y_n(n)^2)^{1/2}} \right)
\]

\[
= \sum_{k=3}^{n} (-1)^{k+1} \arg(z^2 - y_k(n)^2)^{1/2}
\]

\[
= \frac{1}{2} \sum_{k=3}^{n} (-1)^{k+1} [\arg(z + y_k(n)) + \arg(z - y_k(n))].
\]
Let

\[ \theta_k^+ = \arg(z + y_k(n)) \]
\[ \theta_k^- = \arg(z - y_k(n)) . \]

Then,

\[ \arg F_n(z) = \frac{1}{2} \sum_{k=3}^{n} (-1)^{k+1} (\theta_k^+ + \theta_k^-) . \]  
(3.41)

**Remark 3.5.4.** Notice that \( \arg F_n(z) \) is an alternating series. Therefore, we will be interested in the monotonicity of \( (\theta_k^+ + \theta_k^-) \) in \( k \).

To check that \(-\frac{\pi}{2} < \arg F_n(z) < \frac{\pi}{2}\) for all \( z \in \Omega \), we divide this domain into six subdomains \( \Omega_1, \Omega_2, ..., \Omega_6 \) as in figure 3.2. Since \( F_n(z) \) is a symmetric function, we will consider the only three domains \( \Omega_1, \Omega_2, \) and \( \Omega_3 \).

**Case 1.** Let \( \Omega_1 = \{(x, y) \in \mathbb{C} \mid 0 < x \leq m, 0 \leq y \leq m \} \). For \( z \in \Omega_1 \),

![Figure 3.3. The subdomain \( \Omega_1 \)](image)

\[ 0 < \theta_k^+ < \frac{\pi}{2} \text{ and } \frac{\pi}{2} < \theta_k^- < \pi \text{ for any } k = 3, 4, ..., n. \] First, we need to find a better bound of \( (\theta_k^+ + \theta_k^-) \). Consider the triangle with vertices at \( z, -y_k(n), \) and \( y_k(n) \). When \( z \in \Omega_1 \), the length of the left side \( (z, -y_k(n)) \) is bigger than the length of the right side \( (z, y_k(n)) \). Thus, the opposite angle \( \pi - \theta_k^- \) to \( (z, -y_k(n)) \) is bigger than the opposite angle \( \theta_k^+ \) to \( (z, y_k(n)) \). That is,
\[ \theta^+_k < \pi - \theta^-_k. \]

Therefore,

\[ \frac{\pi}{2} < \theta^+_k + \theta^-_k < \pi, \]

for any \( k = 3, 4, \ldots, n \). Next, since \( F_n(z) \) in (3.41) is an alternating series, we need to compare \((\theta^+_k + \theta^-_k)\) with \((\theta^+_{k+1} + \theta^-_{k+1})\). For this, let us first compare \(\tan(\theta^+_k + \theta^-_k)\) with \(\tan(\theta^+_{k+1} + \theta^-_{k+1})\).

\[
\tan(\theta^+_k + \theta^-_k) = \frac{\tan \theta^+_k + \tan \theta^-_k}{1 - \tan \theta^+_k \tan \theta^-_k} = \frac{\frac{Imz}{y_k(n) + Rez} - \frac{Imz}{y_k(n) - Rez}}{1 - \frac{Imz}{y_k(n) + Rez} \frac{Imz}{y_k(n) - Rez}} = -\frac{2(Rez)(Imz)}{y_k(n)^2 - (Rez)^2 + (Imz)^2}.
\]

From this equation, we see that for \( z \in \Omega_1 \),

\[ \tan(\theta^+_k + \theta^-_k) < \tan(\theta^+_{k+1} + \theta^-_{k+1}). \]

That is, \(\tan(\theta^+_k + \theta^-_k)\) is a strictly increasing function in \( k \). Since \( \frac{\pi}{2} < \theta^+_k + \theta^-_k < \pi \) and \( \tan x \) is a strictly increasing function on \((\frac{\pi}{2}, \pi)\), we conclude that \(\theta^+_k + \theta^-_k\) is a strictly increasing function in \( k \). Finally in (3.41), we have that \(\arg F_n(z)\) is an alternating increasing series, and \( \frac{\pi}{2} < \theta^+_k + \theta^-_k < \pi \) for any \( k = 3, 4, \ldots, n \). Therefore,

\[ -\pi < \sum_{k=3}^{n} (-1)^{k+1} (\theta^+_k + \theta^-_k) < 0, \]

and thus,

\[ -\frac{\pi}{2} < \arg F_n(z) < 0. \]

Therefore, for \( z \in \Omega_1 \), \( \text{Re} F_n(z) > 0 \).
Case 2. Let $\Omega_2 = \{(x, y) \in \mathbb{C} \mid x = 0, 0 \leq y \leq m\}$. For $z \in \Omega_2$, $\theta_k^+ + \theta_k^- = \pi$ for any $k = 3, 4, ..., n$. In this case, in (3.41), $\arg F_n(z) = 0$ since $n$ is even. Therefore, for $z \in \Omega_2$, $\text{Re} F_n(z) > 0$.

Case 3. Let $\Omega_2 = \{(x, y) \in \mathbb{C} \mid -m \leq x < 0, 0 \leq y \leq m\}$. For $z \in \Omega_3$, $0 < \theta_k^+ < \frac{\pi}{2}$ and $\frac{\pi}{2} < \theta_k^- < \pi$ for any $k = 3, 4, ..., n$. This argument is similar to the one for $z \in \Omega_1$.

When $z \in \Omega_3$, \[ (z, -y_k(n)) < (z, y_k(n)), \]
\[ \theta_k^+ > \pi - \theta_k^-. \]
Therefore, we have a better lower bound:

$$\pi < \theta_k^+ + \theta_k^- < \frac{3}{2}\pi,$$

In other words,

$$-\frac{\pi}{2} < \theta_k^+ + \theta_k^- < -\pi,$$

for any $k = 3, 4, ..., n$. From an argument similar to for $z \in \Omega_1$, in (3.41), we have that $\arg F_n(z)$ is an alternating increasing series, and $-\frac{\pi}{2} < \theta_k^+ + \theta_k^- < -\pi$ for any $k = 3, 4, ..., n$. Therefore,

$$0 < \sum_{k=3}^{n} (-1)^{k+1} (\theta_k^+ + \theta_k^-) < \pi,$$

and thus,

$$0 < \arg F_n(z) < \frac{\pi}{2}.$$

Therefore, for $z \in \Omega_3$, $ReF_n(z) > 0$.

Finally, we conclude that for $z \in \Omega$, $-\frac{\pi}{2} < \arg F_n(z) < \frac{\pi}{2}$ and then $ReF_n(z) > 0$. Therefore, $ReF_n(z)$ is a positive harmonic function on $\Omega$. By applying Harnack’s inequality to $ReF_n(z)$ on $\Omega$, we obtain the following: on the subdomain $\Omega' = [0, 1] \subset \Omega$,

$$\frac{\sup_{z \in [0,1]} ReF_n(z)}{\inf_{z \in [0,1]} ReF_n(z)} = \frac{\sup_{z \in [0,1]} F_n(z)}{\inf_{z \in [0,1]} F_n(z)} < M$$

where the constant $M$ depends on $\Omega$ and $\Omega'$, but not on $F_n$. Thus, For any $z \in [0, \varepsilon(n)],$

$$F_n(z) \leq \sup_{z \in [-1,1]} F_n(z) < M \inf_{z \in [-1,1]} F_n(z)$$
\[ < M \inf_{z \in [\varepsilon(n), 1]} F_n(z) \]
\[ \leq MF_n(w), \]

for any \( w \in [\varepsilon(n), 1] \). Also, for any \( w \in [\varepsilon(n), 1] \),

\[ F_n(w) \leq \sup_{z \in [-1, 1]} F_n(z) \]
\[ < M \inf_{z \in [-1, 1]} F_n(z) \]
\[ < M \inf_{z \in [0, \varepsilon(n)]} F_n(z) \]
\[ \leq MF_n(z), \]

for any \( z \in [0, \varepsilon(n)] \). Therefore, we conclude that

\[ \frac{1}{M} < \frac{F_n(w)}{F_n(z)} < M \]

for any \( z \in [0, \varepsilon(n)] \) and \( w \in [\varepsilon(n), 1] \) independent of \( F_n \). This completes the proof of lemma 3.5.1.

\[ \square \]

**Step 4.** We continue the proof of proposition 3.5.1. Recall the inequality in (3.40)

\[ \left( \frac{\inf_{z \in [\varepsilon(n), 1]} F_n(z)}{\sup_{z \in [0, \varepsilon(n)]} F_n(z)} \right)^2 \frac{a(n)}{a^*(n)} \leq \frac{2\varepsilon(n)}{1 - \varepsilon(n)^2}, \]

where \( \frac{a(n)}{a^*(n)} \approx 1 \). Remember that this inequality came from the period problem 3.2.1, by comparing \( R(n) = \frac{\int_{\varepsilon(n)} \rho dh}{\int_0^{\varepsilon(n)} \frac{1}{\beta} dh} \) in (3.35) and \( R(n) = \frac{\int_{\varepsilon(n)} \rho dh}{\int_0^{\varepsilon(n)} \frac{1}{\beta} dh} \) in (3.38). Now by lemma 3.5.1, we have

\[ \frac{1}{M^2} \frac{a(n)}{a^*(n)} \leq \frac{2\varepsilon(n)}{1 - \varepsilon(n)^2}. \]

Since \( \frac{a(n)}{a^*(n)} \approx 1 \), and since \( M \) is a fixed constant, this inequality does not hold as
\( \varepsilon(n) \to 0 \). In other words, under the assumption that \( \varepsilon(n) \to 0 \), we do not have same \( R(n) \) from (3.35) and (3.38). Therefore, we conclude that there exists \( \varepsilon_0 > 0 \) such that

\[
\varepsilon_0 = \inf_n \varepsilon(n).
\]

Note that at the beginning, we let \( n \) be an even genus. For odd genus \( n \), we have a similar argument. In this case, the \( i \)th period is

\[
P_i(n) = \int_{\gamma_i(n)} gdh
= \int_{\gamma_{i-1}(n)} (z^2 - \varepsilon(n)^2)^{\frac{1}{2}} (z^2 - y_5(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n-2}(n)^2)^{\frac{1}{2}} (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}
\cdot \frac{dz}{z^{\frac{1}{2}} (z^2 - 1)^{\frac{1}{2}} (z^2 - y_4(n)^2)^{\frac{1}{2}} (z^2 - y_6(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}
\]

Let

\[
F_n(z) = \frac{(z^2 - y_5(n)^2)^{\frac{1}{2}} (z^2 - y_6(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n-2}(n)^2)^{\frac{1}{2}}}{(z^2 - y_4(n)^2)^{\frac{1}{2}} (z^2 - y_6(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}
\]

Notice that this product does not contain the final factor \( (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}} \). This is still a product of the even numbers \( (z^2 - y_i(n)^2)^{\frac{1}{2}} \) for \( i = 3, 4, \ldots, n-1 \), so that \( F_n(z) \) still has a positive value for \( z \in [0,1] \) and satisfies lemma 3.5.1. Then we can easily check that for odd genus \( n \), we have

\[
\frac{\sqrt{y_n(n)^2 - 1}}{1 - y_n(n)^2} \left( \inf_{z \in [\varepsilon(n),1]} F_n(z) \right)^2 \frac{a(n)}{a^*(n)} < R(n) < \frac{\sqrt{y_n(n)^2}}{1 - \varepsilon(n)^2} \left( \sup_{z \in [0,\varepsilon(n)]} F_n(z) \right)^2 \frac{2\varepsilon(n)}{1 - \varepsilon(n)^2}.
\]

i.e.

\[
(y_n(n)^2 - 1) \left( \inf_{z \in [\varepsilon(n),1]} F_n(z) \right)^2 \frac{a(n)}{a^*(n)} < R(n) < y_n(n)^2 \left( \sup_{z \in [0,\varepsilon(n)]} F_n(z) \right)^2 \frac{2\varepsilon(n)}{1 - \varepsilon(n)^2}.
\]
where \( \frac{a(n)}{a^*(n)} \approx 1 \), instead of (3.39) for even genus \( n \). Therefore, we have

\[
y_n(n)^2 - 1 \left( \inf_{z \in \varepsilon(n) \cdot [1]} \frac{F_n(z)}{\sup_{z \in [0, \varepsilon(n)]} F_n(z)} \right)^2 \frac{a(n)}{a^*(n)} < \frac{2 \varepsilon(n)}{1 - \varepsilon(n)^2}.
\]

where \( \frac{a(n)}{a^*(n)} \approx 1 \), instead of (3.40) for even genus \( n \). Also by applying lemma 3.5.1 (but with a different \( M \)),

\[
y_n(n)^2 - 1 \cdot \frac{1}{M^2} \frac{a(n)}{a^*(n)} < \frac{2 \varepsilon(n)}{1 - \varepsilon(n)^2}.
\] (3.42)

where \( \frac{a(n)}{a^*(n)} \approx 1 \). Let us look at

\[
y_n(n)^2 - 1 = 1 - \frac{1}{y_n(n)^2} > 1 - \frac{1}{y_3(n)^2}.
\]

we know from (3.31) that \( y_3(n) - 1 > \varepsilon_2 \) for some \( \varepsilon_2 > 0 \) independent of \( n \). Therefore,

\[
y_n(n)^2 - 1 \cdot \frac{1}{y_n(n)^2} > 1 - \frac{1}{(1 + \varepsilon_2)^2}
\]

independent of \( n \). Thus, for odd genus \( n \), the left side of the inequality (3.42) still has a strictly positive lower bound, and the inequality does not hold as \( \varepsilon(n) \to 0 \). Therefore, there exists \( \varepsilon_0 > 0 \) such that

\[
\varepsilon_0 = \inf \varepsilon(n).
\]

This completes the proof of proposition 3.5.1. \( \square \)
3.6 Convergence of $V(n)$

Finally, from the previous three propositions, we will prove theorem 3.6.1 which gives information on the points $y_i(n)$ in $V(n)$.

Let $E(n)$ be the generalized Enneper's surface of genus $n$. For each genus $n$, choose a coordinate domain $\Omega_g(n) \subset \mathbb{C}$ for one quarter of $E(n)$ which is a solution of the period problem (then the conjugate domain $\Omega_g(n) \subset \mathbb{C}$ is automatically determined). Note that the domain $\Omega_g(n)$ is determined by its symmetric zigzag boundary $Z_g(n) = \partial \Omega_g(n)$ with the $(2n + 1)$ vertices. Let us consider the Schwarz-Christoffel map which takes $\mathbb{H}$ onto $\Omega_g(n)$:

$$\psi_g(n) : \mathbb{H} \rightarrow \Omega_g(n) \subset \mathbb{C}.$$ 

Let

$$V(n) = (-y_n(n), ..., -y_2(n), -y_1(n), 0, y_1(n), y_2(n), ..., y_n(n)) \in \mathbb{R}^{2n+1},$$

where $y_1(n) < y_2(n) < ... < y_n(n) \in \mathbb{R} = \partial \mathbb{H}$, be the preimage of the $(2n + 1)$ vertices of the zigzag boundary $Z_g(n) = \partial \Omega_g(n)$ under the Schwarz-Christoffel map $\psi_g(n)$. Under our setting, $V(n)$ is also the preimage of the $(2n + 1)$ vertices of the zigzag boundary $Z_g(n) = \partial \Omega_1(n)$ of the conjugate domain $\Omega_1(n)$ under the Schwarz-Christoffel map $\psi_1(n) : \mathbb{H} \rightarrow \Omega_1(n) \subset \mathbb{C}$. We show that the points $y_i(n)$ in $V(n)$ are distributed uniformly; in particular, we prove

**Theorem 3.6.1.** For any fixed $i = 1, 2, ..., n - 1$, there exists $0 < c_i < \infty$ so that

$$\frac{1}{c_i} < \frac{y_i(n) - y_{i-1}(n)}{y_{i+1}(n) - y_i(n)} < c_i$$

with $c_i$ independent of $n$. Therefore, a normalized subsequence of $\{V(n)\}$ converges, uniformly on finite subsets of indices.
Proof. Proposition 3.3.1 says that for any fixed \( i = 1, 2, \ldots, n-1 \), if \( y_i(n) - y_{i-1}(n) \asymp 1 \), then \( y_{i+1}(n) - y_i(n) \geq \varepsilon_i \) for some \( \varepsilon_i > 0 \) independent of \( n \). Therefore, we have

\[
\frac{y_i(n) - y_{i-1}(n)}{y_{i+1}(n) - y_i(n)} < \beta_i
\]

with a constant \( \beta_i < \infty \) independent of \( n \). Also by proposition 3.4.1 and 3.5.1, we know that for any fixed \( i = 1, 2, \ldots, n-1 \), if \( y_i(n) - y_{i-1}(n) \asymp 1 \), then \( y_{i+1}(n) - y_i(n) \leq \eta_i \) for some \( \eta_i < \infty \) independent of \( n \). Therefore,

\[
\alpha_i < \frac{y_i(n) - y_{i-1}(n)}{y_{i+1}(n) - y_i(n)}
\]

with a constant \( \alpha_i > 0 \) independent of \( n \). Therefore, for any fixed \( i = 1, 2, \ldots, n-1 \), there exists \( 0 < c_i < \infty \) so that

\[
\frac{1}{c_i} < \frac{y_i(n) - y_{i-1}(n)}{y_{i+1}(n) - y_i(n)} < c_i
\]

with \( c_i \) independent of \( n \). We have proved the first part of theorem 3.6.1.

Now, with this fact on distances among points \( y_i(n) \) in \( V(n) \), we can say that a normalized subsequence of \( \{V(n)\} \) converges as the genus \( n \to \infty \) in the following sense. Let

\[
V(n) = (-y_n(n), \ldots, -y_2(n), -1, 0, 1, y_2(n), \ldots, y_n(n)) \in \mathbb{R}^{2n+1},
\]

where \( 1 < y_2(n) < \ldots < y_n(n) \in \mathbb{R} \). Notice that we normalized the first point \( y_1(n) = 1 \) for all \( n \). Then by (3.43) we just proved, we have that

\[
y_i(n) - y_{i-1}(n) \asymp 1
\]

for any fixed \( i = 1, 2, \ldots, n \). In other words, any distance between two adjacent points
in $V(n)$ is uniformly comparable in $n$, after normalization of the first point $y_1(n) = 1$. For any fixed index $m$, consider the projection map

$$V(n) = (-y_n(n), \ldots, -y_m(n), \ldots, -y_2(n), -1, 0, 1, y_2(n), \ldots, y_m(n), \ldots, y_n(n)) \in \mathbb{R}^{2n+1}$$

$$\mapsto V_m(n) = (-y_m(n), \ldots, -y_2(n), -1, 0, 1, y_2(n), \ldots, y_m(n)) \in \mathbb{R}^{2m+1}.$$

Then a normalized subsequence of $\{V_m(n)\}$ converges as $n \to \infty$ in the topology of $\mathbb{R}^{2m+1}$. Hence, we say that a normalized subsequence of $\{V(n)\}$ converges, uniformly on finite subsets of indices. This completes the proof of theorem 3.6.1.
Chapter 4

Convergence of $E(n)$

In this chapter, we show the main theorem 4.1.2 on convergence of the generalized Enneper’s surfaces $E(n)$ of genus $n$, and give a conjecture on the limit surface.

**Theorem 4.1.2.** Let $E(n)$ be the generalized Enneper’s surface of genus $n$. Then a normalized subsequence of $\{E(n)\}$ converges, uniformly on compacta.

To show this main theorem, we need several steps: Let $E(n)$ be the generalized Enneper’s surface of genus $n$. For each genus $n$, choose a coordinate domain $\Omega_g(n) \subset \mathbb{C}$ for one quarter of $E(n)$ which is a solution of the period problem (then the conjugate domain $\Omega_{\bar{g}}(n) \subset \mathbb{C}$ is automatically determined). Note that the domain $\Omega_g(n)$ is determined by its symmetric zigzag boundary $Z_g(n) = \partial \Omega_g(n)$ with $(2n+1)$ vertices. Let us consider the Schwarz-Christoffel map which takes $\mathbb{H}$ onto $\Omega_g(n)$:

$$\psi_g(n) : \mathbb{H} \to \Omega_g(n) \subset \mathbb{C}.$$ 

Let

$$V(n) = (-y_n(n), ..., -y_2(n), -y_1(n), 0, y_1(n), y_2(n), ..., y_n(n)) \in \mathbb{R}^{2n+1},$$

where $y_1(n) < y_2(n) < \ldots < y_n(n) \in \mathbb{R} = \partial \mathbb{H}$, be the preimage of the $(2n+1)$
vertices of the zigzag boundary \( Z_g(n) = \partial \Omega_g(n) \) under the Schwarz-Christoffel map \( \psi_g(n) \). Under our setting, \( V(n) \) is also the preimage of the \((2n+1)\) vertices of the zigzag boundary \( Z_1(n) = \partial \Omega_1(n) \) of the conjugate domain \( \Omega_1(n) \) under the Schwarz-Christoffel map \( \psi_1(n) : \mathbb{H} \to \Omega_1(n) \subset \mathbb{C} \).

1. (Theorem 3.6.1) At first, in section 3.6, we saw the crucial fact that a normalized subsequence of \( \{V(n)\} \) converges, uniformly on finite subsets of indices.

2. (Theorem 4.1.1) Now, using this fact, we will show that a normalized subsequence of coordinate domains \( \{\Omega_g(n)\} \) in \( \mathbb{C} \) converges, uniformly on compacta, say to \( \Omega_g(\infty) \). Then it is obviously true that a normalized subsequence of the conjugate domains \( \{\Omega_1(n)\} \) in \( \mathbb{C} \) also converges, uniformly on compacta, say to \( \Omega_1(\infty) \).

3. (Theorem 4.1.2) Then we will show that a subsequence of the Schwarz-Christoffel maps \( \{\psi_g(n) : \mathbb{H} \to \Omega_g(n)\} \) converges in the sense of Caratheodory, say to \( \psi_g(\infty) : \mathbb{H} \to \Omega_g(\infty) \). Then a subsequence of the Schwarz-Christoffel maps \( \{\psi_1(n) : \mathbb{H} \to \Omega_1(n)\} \) for the conjugate domains also converges, say to \( \psi_1(\infty) : \mathbb{H} \to \Omega_1(\infty) \). Therefore, consider the composition map

\[
\psi_1(\infty) \circ \psi_g(\infty)^{-1} : \Omega_g(\infty) \to \Omega_1(\infty).
\]

This says that there is a conformal map from \( \Omega_g(\infty) \) to \( \Omega_1(\infty) \) which satisfies the period problem. Hence, we conclude that a normalized subsequence of \( \{E(n)\} \) converges, uniformly on compacta.

### 4.1 Convergence of \( E(n) \)

Our goal in this section is to show our main theorem. To show the main theorem 4.1.2, we first need to show that a normalized subsequence of coordinate domains
\{\Omega_g(n)\} in \mathbb{C} converges, uniformly on compacta. However, since the domain \(\Omega_g(n)\) is uniquely determined by its zigzag boundary \(Z_g(n) = \partial \Omega_g(n)\), we need to see only the convergence of a normalized subsequence of zigzags \(\{Z_g(n)\}\). In the following theorem, we will obtain important information on the lengths of the edges of a zigzag boundary \(Z_g(n)\). In theorem 3.6.1, we showed that for any fixed \(i\), the ratio \(\frac{y_i(n) - y_{i-1}(n)}{y_{i+1}(n) - y_i(n)}\) of two adjacent distances among points \(y_i(n)\) in \(V(n)\) is uniformly comparable in \(n\). This makes our problem relatively easy. Here, we will show in the following theorem 4.1.1 that for any fixed \(i\), the ratio \(\frac{|P_i(n)|}{|P_{i+1}(n)|} = \frac{\int_{y_i(n)}^{y_{i+1}(n)} gdh}{\int_{y_i(n)}^{y_{i+1}(n)} gdh} = \frac{\int_{y_i(n)}^{y_{i+1}(n)} gdh}{\int_{y_i(n)}^{y_{i+1}(n)} gdh}\) of lengths of two adjacent edges of \(Z_g(n)\) is uniformly comparable in \(n\).

**Theorem 4.1.1.** Let \(E(n)\) be the generalized Enneper's surface of genus \(n\), and let \(P_i(n)\) be the \(i\)th period from the center of \(E(n)\). Then for any fixed \(i = 1, 2, ..., n - 1\), there exists \(0 < C_i < \infty\) so that

\[
\frac{1}{C_i} < \frac{|P_i(n)|}{|P_{i+1}(n)|} < C_i
\]

with \(C_i\) independent of \(n\). Therefore, a normalized subsequence of coordinate domains \(\{\Omega_g(n)\}\) in \(\mathbb{C}\) for one quarter of \(E(n)\) converges, uniformly on compacta.

Remember that the \(i\)th period \(P_i(n)\) from the center of \(E(n)\) is the \(i\)th edge from the center of a zigzag boundary \(Z_g(n) = \partial \Omega_g(n)\) of a coordinate domain \(\Omega_g(n)\) in \(\mathbb{C}\) for one quarter of \(E(n)\) (and of \(Z_{\frac{1}{g}}(n) = \partial \Omega_{\frac{1}{g}}(n)\)). Therefore, the equation (4.1) says that the ratio of lengths of two adjacent edges of \(Z_g(n)\) (and \(Z_{\frac{1}{g}}(n)\)) is uniformly comparable in \(n\).

**Proof.** For each genus \(n\), choose a coordinate domain \(\Omega_g(n) \subset \mathbb{C}\) for one quarter of \(E(n)\) which is a solution of the period problem. Then consider the Schwarz-Christoffel map which takes \(\mathbb{H}\) onto \(\Omega_g(n)\):

\[
\psi_g(n) : \mathbb{H} \to \Omega_g(n) \subset \mathbb{C}
\]

by
\[ \zeta \mapsto \int_{*} S(n) \frac{(z^2 - 1)^{\frac{1}{2}}(z^2 - y_3(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_i(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}{z^{\frac{1}{2}}(z^2 - y_2(n)^2)^{\frac{1}{2}}(z^2 - y_4(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n+1}(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_n(n)^2)^{\frac{1}{2}}} \, dz, \]

where \( S(n) \) is a scaling factor (section 2.3). Here,

\[ V(n) = (-y_n(n), \ldots, -y_2(n), -1, 0, 1, y_2(n), \ldots, y_n(n)) \in \mathbb{R}^{2n+1}, \]

where \( 1 < y_2(n) < \ldots < y_n(n) \in \mathbb{R} = \partial \mathbb{H} \), is the preimage of the \((2n+1)\) vertices of the zigzag boundary \( Z_g(n) = \partial \Omega_g(n) \) of the domain \( \Omega_g(n) \) under the Schwarz-Christoffel map \( \psi_g(n) \). Note that since we fixed the first point \( y_1(n) = 1 \) for all \( n \), by applying the previous theorem 3.6.1 we have that

\[ y_k(n) - y_{k-1}(n) \geq 1 \quad (4.2) \]

for any fixed \( k = 1, 2, \ldots, n \). In other words, any distance between two adjacent points in \( V(n) \) is uniformly comparable in \( n \), after normalization of the first point \( y_1(n) = 1 \) for all \( n \). First, we will consider the case with even genus \( n \). Recall the \( i \)th period \( P_i(n) \) and the \((i + 1)\)st period \( P_{i+1}(n) \) of \( E(n) \)

\[
P_i(n) = \int_{\gamma_i(n)} gdh
= \int_{y_{i-1}(n)}^{y_i(n)} S(n) \frac{(z^2 - 1)^{\frac{1}{2}}(z^2 - y_3(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_i(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}{z^{\frac{1}{2}}(z^2 - y_2(n)^2)^{\frac{1}{2}}(z^2 - y_4(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n+1}(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_n(n)^2)^{\frac{1}{2}}} \, dz,
\]

\[
P_{i+1}(n) = \int_{\gamma_{i+1}(n)} gdh
= \int_{y_i(n)}^{y_{i+1}(n)} S(n) \frac{(z^2 - 1)^{\frac{1}{2}}(z^2 - y_3(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_i(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}{z^{\frac{1}{2}}(z^2 - y_2(n)^2)^{\frac{1}{2}}(z^2 - y_4(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n+1}(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_n(n)^2)^{\frac{1}{2}}} \, dz,
\]
where $S(n)$ is a scaling factor. Without loss of generality, let $i$ be an odd integer. Let

$$
F_{n,i}(z) = \frac{(z^2 - y_{i+2}(n))^\frac{1}{2}(z^2 - y_{i+4}(n))^\frac{1}{2} \cdots (z^2 - y_{n-1}(n))^\frac{1}{2}}{(z^2 - y_{i+3}(n))^\frac{1}{2}(z^2 - y_{i+5}(n))^\frac{1}{2} \cdots (z^2 - y_{n}(n))^\frac{1}{2}}.
$$

This is a product of an even number of factors $(z^2 - y_k(n))^\frac{1}{2}$ for $k = i + 2, i + 3, \ldots, n$, so that $F_{n,i}(z)$ has a positive value for $z \in [y_{i-1}(n), y_{i+1}(n)]$.

First, we will see bounds for $|P_i(n)|$. For a lower bound of $|P_i(n)|$,

$$
|P_i(n)| > \left( \inf_{z \in [y_{i-1}(n), y_i(n)]} F_{n,i}(z) \right) \times \int_{y_{i-1}(n)}^{y_i(n)} S(n) \frac{(z^2 - 1)^\frac{1}{2}(z^2 - y_3(n))^\frac{1}{2} \cdots (z^2 - y_i(n))^\frac{1}{2}}{z^\frac{3}{2}(z^2 - y_2(n))^\frac{1}{2}(z^2 - y_4(n))^\frac{1}{2} \cdots (z^2 - y_{i+1}(n))^\frac{1}{2}} dz.
$$

Let

$$
a_i(n) = \int_{y_{i-1}(n)}^{y_i(n)} \frac{(z^2 - 1)^\frac{1}{2}(z^2 - y_3(n))^\frac{1}{2} \cdots (z^2 - y_i(n))^\frac{1}{2}}{z^\frac{3}{2}(z^2 - y_2(n))^\frac{1}{2}(z^2 - y_4(n))^\frac{1}{2} \cdots (z^2 - y_{i+1}(n))^\frac{1}{2}} dz.
$$

Since the integrand of $a_i(n)$ is a finite product, and since we know from (4.2) that $y_k(n) - y_{k-1}(n) \approx 1$ for any fixed $k = 1, 2, \ldots, n$, we have that

$$
a_i(n) \approx 1.
$$

In other words, there exists $0 < c_i < \infty$ such that $\frac{1}{c_i} < a_i(n) < c_i$ independent of $n$.

Therefore, we conclude that

$$
|P_i(n)| > \left( \inf_{z \in [y_{i-1}(n), y_i(n)]} F_{n,i}(z) \right) a_i(n) S(n), \quad (4.3)
$$

where $a_i(n) \approx 1$. Similarly, for a upper bound of $|P_i(n)|$, we have

$$
|P_i(n)| < \left( \sup_{z \in [y_{i-1}(n), y_i(n)]} F_{n,i}(z) \right) a_i(n) S(n), \quad (4.4)
$$

where $a_i(n) \approx 1$. 

Now, we will see bounds for $|P_{i+1}(n)|$ in the similar manner. For a lower bound of $|P_{i+1}(n)|$,

$$|P_{i+1}(n)| > \left( \inf_{z \in [y_i(n), y_{i+1}(n)]} F_{n,i}(z) \right) \times \int_{y_i(n)}^{y_{i+1}(n)} S(n) \frac{(z^2 - 1) \cdots (z^2 - y_i(n)^2) \cdots (z^2 - y_{i+1}(n)^2)}{z^{\frac{1}{2}} (z^2 - y_2(n)^2) \cdots (z^2 - y_{i+1}(n)^2)} \, dz.$$

Let

$$a_{i+1}(n) = \int_{y_i(n)}^{y_{i+1}(n)} \frac{(z^2 - 1) \cdots (z^2 - y_i(n)^2) \cdots (z^2 - y_{i+1}(n)^2)}{z^{\frac{1}{2}} (z^2 - y_2(n)^2) \cdots (z^2 - y_{i+1}(n)^2)} \, dz.$$

Since the integrand of $a_{i+1}(n)$ is a finite product, and since we know from (4.2) that $y_k(n) - y_{k-1}(n) \sim 1$ for any fixed $k = 1, 2, ..., n$, we have that

$$a_{i+1}(n) \sim 1.$$

Therefore, we conclude that

$$|P_{i+1}(n)| > \left( \inf_{z \in [y_i(n), y_{i+1}(n)]} F_{n,i}(z) \right) a_{i+1}(n) S(n), \tag{4.5}$$

where $a_{i+1}(n) \sim 1$. Similarly, for a upper bound of $|P_{i+1}(n)|$, we have

$$|P_{i+1}(n)| < \left( \sup_{z \in [y_i(n), y_{i+1}(n)]} F_{n,i}(z) \right) a_{i+1}(n) S(n), \tag{4.6}$$

where $a_{i+1}(n) \sim 1$.

Here, by writing (4.3), (4.4), (4.5), and (4.6) together, we obtain that

$$\left( \inf_{z \in [y_i(n), y_{i+1}(n)]} \frac{F_{n,i}(z)}{F_{n,i}(z)} \right) a_i(n) < \frac{|P_i(n)|}{|P_{i+1}(n)|} < \left( \sup_{z \in [y_i(n), y_{i+1}(n)]} \frac{F_{n,i}(z)}{F_{n,i}(z)} \right) a_{i+1}(n) \tag{4.7}$$
where \( \frac{a_i(n)}{a_{i+1}(n)} \gg 1 \). At this point, we will see a general relationship between a value \( F_{n,i}(z) \) for any \( z \in [y_{i-1}(n), y_i(n)] \) and a value \( F_{n,i}(w) \) for any \( w \in [y_i(n), y_{i+1}(n)] \).

What follows is a corollary of lemma 3.5.1 in the proof of proposition 3.5.1.

**Lemma 4.1.1** (Corollary of lemma 3.5.1). For any \( z \in [y_{i-1}(n), y_i(n)] \) and \( w \in [y_i(n), y_{i+1}(n)] \),
\[
\frac{1}{M_i} < \frac{F_{n,i}(z)}{F_{n,i}(w)} < M_i
\]

where \( M_i \) is a constant independent of \( n \).

**Proof of 4.1.1.** This proof follows the exact same steps as in lemma 3.5.1. Here, \( F_{n,i}(z) \) is a holomorphic function with no zeros and no poles on a neighborhood of \([y_{i-1}(n), y_{i+1}(n)]\) in \( \mathbb{C} \). To apply Harnack’s inequality, we will first find some domain \( \Omega_i \subset \mathbb{C} \) including \([y_{i-1}(n), y_{i+1}(n)]\) on which \( \text{Re} F_{n,i}(z) > 0 \). Let \( \Omega_i \subset \mathbb{C} \) be the rectangle with vertices at \((m, m), (-m, m), (-m, -m), \) and \((m, -m)\), where \( m \) is a midpoint between \( y_{i+1}(n) \) and \( y_{i-2}(n) \):
\[
m = \frac{y_{i+2}(n) - y_{i+1}(n)}{2}.
\]

Remember from (4.2) that \( y_k(n) - y_{k-1}(n) \gg 1 \) for any fixed \( k = 1, 2, ..., n \).

Then we can check that \( \text{Re} F_{n,i}(z) > 0 \) on the domain \( \Omega_i \) through the same argument as in lemma 3.5.1. Therefore, \( \text{Re} F_{n,i}(z) \) is a positive harmonic function on \( \Omega_i \).

By applying Harnack’s inequality to \( F_{n,i}(z) \) on \( \Omega_i \), we obtain the following: on the subdomain \( \Omega'_i = [y_{i-1}(n), y_{i+1}(n)] \subset \Omega_i \),
\[
\sup_{z \in [y_{i-1}(n), y_{i+1}(n)]} \frac{\text{Re} F_{n,i}(z)}{\inf_{z \in [y_{i-1}(n), y_{i+1}(n)]} \text{Re} F_{n,i}(z)} = \frac{\sup_{z \in [y_{i-1}(n), y_{i+1}(n)]} F_{n,i}(z)}{\inf_{z \in [y_{i-1}(n), y_{i+1}(n)]} F_{n,i}(z)} < M_i
\]
where $M_i$ depends on $\Omega_i$ and $\Omega_i'$, but not on $F_{n,i}$. Therefore, for any $z \in [y_{i-1}(n), y_i(n)]$,

$$F_{n,i}(z) \leq \sup_{z \in [y_{i-1}(n), y_i(n)]} F_{n,i}(z)$$

$$< M_i \left( \inf_{z \in [y_i(n), y_{i+1}(n)]} F_{n,i}(z) \right)$$

$$< M_i \left( \inf_{z \in [y_i(n), y_{i+1}(n)]} F_{n,i}(z) \right)$$

$$\leq M_i F_{n,i}(w),$$

for any $w \in [y_i(n), y_{i+1}(n)]$. Also, for any $w \in [y_i(n), y_{i+1}(n)]$,

$$F_{n,i}(w) \leq \sup_{z \in [y_{i-1}(n), y_i(n)]} F_{n,i}(z)$$

$$< M_i \left( \inf_{z \in [y_i(n), y_{i+1}(n)]} F_{n,i}(z) \right)$$

$$< M_i \left( \inf_{z \in [y_i(n), y_{i+1}(n)]} F_{n,i}(z) \right)$$

$$\leq M_i F_{n,i}(z),$$

Figure 4.1. The domain $\Omega_i$
for any \( z \in [y_{i-1}(n), y_i(n)] \). Hence, we have

\[
\frac{1}{M_i} < \frac{F_{n,i}(z)}{F_{n,i}(w)} < M_i
\]

for any \( z \in [y_{i-1}(n), y_i(n)] \) and \( w \in [y_i(n), y_{i+1}(n)] \) independent of \( F_{n,i} \). This completes the proof of lemma 4.1.1. \qed

We continue the proof of theorem 4.1.1. Recall the inequality in (4.7)

\[
\left( \inf_{z \in [y_{i-1}(n), y_i(n)]} \frac{F_{n,i}(z)}{F_{n,i}(z)} \right) \frac{a_i(n)}{a_{i+1}(n)} < \frac{|P_i(n)|}{|P_{i+1}(n)|} < \left( \sup_{z \in [y_{i-1}(n), y_i(n)]} \frac{F_{n,i}(z)}{F_{n,i}(z)} \right) \frac{a_i(n)}{a_{i+1}(n)}
\]

where \( \frac{a_i(n)}{a_{i+1}(n)} \approx 1 \). Now, by applying lemma 4.1.1, we have

\[
\frac{1}{M_i} \frac{a_i(n)}{a_{i+1}(n)} < \frac{|P_i(n)|}{|P_{i+1}(n)|} < M_i \frac{a_i(n)}{a_{i+1}(n)}.
\]

Since \( \frac{a_i(n)}{a_{i+1}(n)} \approx 1 \), and since \( M_i \) is a fixed constant independent of \( n \), we have finally proved that for any fixed \( i = 1, 2, \ldots, n-1 \), there exists \( 0 < C_i < \infty \) so that

\[
\frac{1}{C_i} < \frac{|P_i(n)|}{|P_{i+1}(n)|} < C_i
\]

with \( C_i \) independent of \( n \).

Note that at the beginning, we let \( n \) be an even genus. For odd genus \( n \), we have a similar argument. In this case, the \( ith \) period \( P_i(n) \) and the \((i+1)st\) period \( P_{i+1}(n) \) are

\[
P_i(n) = \int_{\gamma_i(n)} gdh
\]

\[
= \int_{y_{i-1}(n)}^{y_i(n)} S(n) \frac{(z^2 - 1)^{\frac{1}{2}} \ldots (z^2 - y_i(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n-2}(n)^2)^{\frac{1}{2}} (z^2 - y_{n}(n)^2)^{\frac{1}{2}}}{z^{\frac{1}{2}} (z^2 - y_2(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{i+1}(n)^2)^{\frac{1}{2}} \ldots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}} \, dz,
\]
\[ P_{i+1}(n) = \int_{y_{i+1}(n)}^{y_i(n)} gdh \]

\[ = \int_{y_{i+1}(n)}^{y_i(n)} S(n) \frac{(z^2 - 1)^\frac{1}{3} \ldots (z^2 - y_i(n)^2)^\frac{1}{3} \ldots (z^2 - y_{i-2}(n)^2)^\frac{1}{3} (z^2 - y_{i+1}(n)^2)^\frac{1}{3} \ldots (z^2 - y_{n-1}(n)^2)^\frac{1}{3}}{z^\frac{1}{3} (z^2 - y_2(n)^2)^\frac{1}{3} \ldots (z^2 - y_{i+1}(n)^2)^\frac{1}{3} \ldots (z^2 - y_{n-1}(n)^2)^\frac{1}{3}} dz, \]

where \( S(n) \) is a scaling factor. Let

\[ F_{n,i}(z) = \frac{(z^2 - y_{i+2}(n)^2)^\frac{1}{3} (z^2 - y_{i+4}(n)^2)^\frac{1}{3} \ldots (z^2 - y_{n-2}(n)^2)^\frac{1}{3}}{(z^2 - y_{i+3}(n)^2)^\frac{1}{3} (z^2 - y_{i+5}(n)^2)^\frac{1}{3} \ldots (z^2 - y_{n-1}(n)^2)^\frac{1}{3}}. \]

Notice that this product does not contain the final factor \( (z^2 - y_{n}(n)^2)^\frac{1}{3} \). This is still a product of an even number of factors \( (z^2 - y_k(n)^2)^\frac{1}{3} \) for \( k = i + 2, i + 3, \ldots, n - 1 \), so that \( F_{n,i}(z) \) still has a positive value for \( z \in [y_{i-1}(n), y_{i+1}(n)] \) and satisfies lemma 4.1.1. Recall

\[ a_i(n) = \int_{y_{i-1}(n)}^{y_i(n)} \frac{(z^2 - 1)^\frac{1}{3} (z^2 - y_3(n)^2)^\frac{1}{3} \ldots (z^2 - y_i(n)^2)^\frac{1}{3}}{z^\frac{1}{3} (z^2 - y_2(n)^2)^\frac{1}{3} \ldots (z^2 - y_{i+1}(n)^2)^\frac{1}{3}} dz, \]

\[ a_{i+1}(n) = \int_{y_i(n)}^{y_{i+1}(n)} \frac{(z^2 - 1)^\frac{1}{3} (z^2 - y_3(n)^2)^\frac{1}{3} \ldots (z^2 - y_i(n)^2)^\frac{1}{3}}{z^\frac{1}{3} (z^2 - y_2(n)^2)^\frac{1}{3} \ldots (z^2 - y_{i+1}(n)^2)^\frac{1}{3}} dz. \]

Then we can easily check that for odd genus \( n \), we have

\[ |P_i(n)| > \left( \inf_{z \in [y_{i-1}(n), y_{i+1}(n)]} F_{n,i}(z) \right) (y_n(n)^2 - y_i(n)^2)^\frac{1}{3} a_i(n) S(n), \]

\[ |P_i(n)| < \left( \sup_{z \in [y_{i-1}(n), y_{i+1}(n)]} F_{n,i}(z) \right) (y_n(n)^2 - y_{i-1}(n)^2)^\frac{1}{3} a_i(n) S(n), \]

where \( a_i(n) \asymp 1 \), instead of (4.3) and (4.4) for even genus \( n \), and

\[ |P_{i+1}(n)| > \left( \inf_{z \in [y_{i}(n), y_{i+1}(n)]} F_{n,i}(z) \right) (y_n(n)^2 - y_{i+1}(n)^2)^\frac{1}{3} a_{i+1}(n) S(n), \]

\[ |P_{i+1}(n)| < \left( \sup_{z \in [y_{i}(n), y_{i+1}(n)]} F_{n,i}(z) \right) (y_n(n)^2 - y_i(n)^2)^\frac{1}{3} a_{i+1}(n) S(n), \]
where $a_{i+1}(n) \asymp 1$, instead of (4.5) and (4.6) for even genus $n$. Therefore, we have

$$\left( \frac{\inf_{z \in [y_{i-1}(n), y_{i}(n)]} F_{n,i}(z)}{\sup_{z \in [y_{i}(n), y_{i+1}(n)]} F_{n,i}(z)} \right) \frac{a_i(n)}{a_{i+1}(n)} < \frac{|P_i(n)|}{|P_{i+1}(n)|} < \left( \frac{y_n(n)^2 - y_{i-1}(n)^2}{y_n(n)^2 - y_{i+1}(n)^2} \right)^{\frac{1}{2}} \left( \frac{\sup_{z \in [y_{i-1}(n), y_{i}(n)]} F_{n,i}(z)}{\inf_{z \in [y_{i}(n), y_{i+1}(n)]} F_{n,i}(z)} \right) \frac{a_i(n)}{a_{i+1}(n)},$$

where $\frac{a_i(n)}{a_{i+1}(n)} \asymp 1$, instead of (4.7) for even genus $n$. Also by applying lemma 4.1.1 (but with a different $M_i$),

$$\frac{1}{M_i} \frac{a_i(n)}{a_{i+1}(n)} < \frac{|P_i(n)|}{|P_{i+1}(n)|} < \left( \frac{y_n(n)^2 - y_{i-1}(n)^2}{y_n(n)^2 - y_{i+1}(n)^2} \right)^{\frac{1}{2}} M_i \frac{a_i(n)}{a_{i+1}(n)},$$

where $\frac{a_i(n)}{a_{i+1}(n)} \asymp 1$ and $M_i$ is a fixed constant independent of $n$. Let us look at

$$\left( \frac{y_n(n)^2 - y_{i-1}(n)^2}{y_n(n)^2 - y_{i+1}(n)^2} \right)^{\frac{1}{2}} = \left( 1 + \frac{y_{i+1}(n)^2 - y_{i-1}(n)^2}{y_n(n)^2 - y_{i+1}(n)^2} \right)^{\frac{1}{2}}.$$

We know from (4.2) that $y_{i+1}(n)^2 - y_{i-1}(n)^2 \asymp 1$ in the numerator. Therefore, $\left( 1 + \frac{y_{i+1}(n)^2 - y_{i-1}(n)^2}{y_n(n)^2 - y_{i+1}(n)^2} \right)^{\frac{1}{2}}$ is bounded in the genus $n$, whether $y_n(n)$ in the denominator is bounded in $n$ or not. Therefore, for odd genus $n$, we still have the same result as for even genus $n$ that for any fixed $i = 1, 2, ..., n - 1$, there exists $0 < C_i < \infty$ so that

$$\frac{1}{C_i} < \frac{|P_i(n)|}{|P_{i+1}(n)|} < C_i$$

with $C_i$ independent of $n$. We have proved the first part of theorem 4.1.1.

Now, with this fact on lengths $|P_i(n)|$ of periods (lengths of edges of a zigzag $Z_\rho(n) = \partial \Omega_\rho(n)$), we can say that a normalized subsequence of zigzags $\{Z_\rho(n)\}$
converges uniformly on compacta. Look at the $ith$ edge of $Z_g(n) = \partial \Omega_g(n)$

\[
P_i(n) = \int_{\gamma_i(n)} gdh = \int_{y_i(n)}^{y_{i-1}(n)} S(n) \frac{(z^2 - 1)^{\frac{1}{2}}(z^2 - y_3(n)^2)^{\frac{1}{2}}...(z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}{z^{\frac{1}{2}}(z^2 - y_2(n)^2)^{\frac{1}{2}}(z^2 - y_4(n)^2)^{\frac{1}{2}}...(z^2 - y_n(n)^2)^{\frac{1}{2}}} \, dz,
\]

where $S(n)$ is a scaling factor. With the scaling factor, we normalize the first edge $P_1(n)$ of $Z_g(n)$: for each genus $n$, choose $S(n)$ so that

\[
|P_1(n)| = 1.
\]

Then by the equation (4.1) we just proved, we have that

\[
|P_i(n)| \approx 1
\]

for any fixed $i = 1, 2, ..., n$. In other words, any $ith$ edge $P_i(n)$ of a zigzag $Z_g(n)$ is uniformly comparable in $n$, after normalization of the first edge $P_1(n)$. Also assume that the center of a zigzag $Z_g(n)$ lies on $0 \in \mathbb{C}$ (or any fixed point in $\mathbb{C}$) for all $n$. Therefore, a normalized subsequence of \{Z_g(n)\} converges uniformly on compacta.

Since $Z_g(n) = \partial \Omega_g(n)$, we can say that a normalized subsequence of coordinate domains \{\Omega_g(n)\} converges uniformly on compacta, say to $\Omega_g(\infty)$. It is obviously true that a normalized subsequence of the conjugate domains \{\Omega_\bar{g}(n)\} also converges uniformly on compacta, say to $\Omega_\bar{g}(\infty)$. This completes the proof of theorem 4.1.1. 

Finally, we are ready to prove our main theorem on convergence of a normalized subsequence of the generalized Enneper's surfaces \{E(n)\}.

**Theorem 4.1.2.** Let $E(n)$ be the generalized Enneper's surface of genus $n$. Then a normalized subsequence of \{E(n)\} converges, uniformly on compacta.

**Proof.** From theorem 4.1.1, we know that a normalized subsequence of coordinate
domains \( \{ \Omega_g(n) \} \) in \( \mathbb{C} \) converges, uniformly on compacta, say to \( \Omega_g(\infty) \) (then a normalized subsequence of the conjugate domains \( \{ \Omega^*_g(n) \} \) in \( \mathbb{C} \) also converges, uniformly on compacta, say to \( \Omega^*_g(\infty) \)). From this fact, the proof of this theorem on convergence of a normalized subsequence of \( \{ E(n) \} \) is almost trivial. We will prove this theorem in two ways: one follows naturally from the proof of theorem 4.1.1 and the other uses Caratheodory Kernel Theorem. For the first approach, we will show that a subsequence of the Schwarz-Christoffel maps \( \{ \psi_g(n) : \mathbb{H} \to \Omega_g(n) \} \) converges, uniformly on compacta (then a subsequence of the Schwarz-Christoffel maps \( \{ \psi^*_g(n) : \mathbb{H} \to \Omega^*_g(n) \} \) also converges, uniformly on compacta). Recall the Schwarz-Christoffel map

\[
\psi_g(n) : \mathbb{H} \to \Omega_g(n) \subset \mathbb{C} \quad \text{by}
\]

\[
\zeta \mapsto \int_0^\zeta gdh = \int_0^\zeta S(n) \frac{(z^2 - 1)^{\frac{1}{2}}(z^2 - y_3(n)^2)^{\frac{1}{2}} \cdots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}{z^{\frac{1}{2}}(z^2 - y_2(n)^2)^{\frac{1}{2}}(z^2 - y_4(n)^2)^{\frac{1}{2}} \cdots (z^2 - y_n(n)^2)^{\frac{1}{2}}} \, dz,
\]

where \( S(n) \) is a scaling factor (section 2.3). For each \( n \), choose the scaling factor \( S(n) \) so that

\[
|P_1(n)| = \left| \int_0^1 gdh \right| = 1.
\]

Look at the 1-form

\[
gdh = S(n) \frac{(z^2 - 1)^{\frac{1}{2}}(z^2 - y_3(n)^2)^{\frac{1}{2}} \cdots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}{z^{\frac{1}{2}}(z^2 - y_2(n)^2)^{\frac{1}{2}}(z^2 - y_4(n)^2)^{\frac{1}{2}} \cdots (z^2 - y_n(n)^2)^{\frac{1}{2}}} \, dz.
\]

The proof of lemma 4.1.1, in the proof of theorem 4.1.1, tells us that a subsequence of

\[
\frac{(z^2 - 1)^{\frac{1}{2}}(z^2 - y_3(n)^2)^{\frac{1}{2}} \cdots (z^2 - y_{n-1}(n)^2)^{\frac{1}{2}}}{z^{\frac{1}{2}}(z^2 - y_2(n)^2)^{\frac{1}{2}}(z^2 - y_4(n)^2)^{\frac{1}{2}} \cdots (z^2 - y_n(n)^2)^{\frac{1}{2}}} \, dz
\]

converges in \( n \), uniformly on compacta, by Harnack's inequality. However, with the scaling factor \( S(n) \), a subsequence of the form \( gdh \) could converge to 0 or \( \infty \) as the
genus $n \to \infty$. However, since we chose $S(n)$ so that

$$|P_1(n)| = \left| \int_0^1 gdh \right| = 1,$$

for each genus $n$, a subsequence of the form $gdh$ does not converge to 0 nor $\infty$. Therefore, a subsequence of the 1-form $gdh$ converges. Hence, we conclude that a subsequence of the Schwarz-Christoffel maps $\{\psi_g(n) : \mathbb{H} \to \Omega_g(n)\}$ converges uniformly in $n$, say to

$$\psi_g(\infty) : \mathbb{H} \to \Omega_g(\infty) \subset \mathbb{C} \quad (4.8)$$

on any compact set. It is obviously true that a subsequence of the Schwarz-Christoffel maps $\{\psi_{1/2}(n) : \mathbb{H} \to \Omega_{1/2}(n)\}$ converges as $n \to \infty$, say to

$$\psi_{1/2}(\infty) : \mathbb{H} \to \Omega_{1/2}(\infty) \subset \mathbb{C}. \quad (4.9)$$

Now, consider the composition map of (4.8) and (4.9)

$$\psi_{1/2}(\infty) \circ \psi_g(\infty)^{-1} : \Omega_g(\infty) \to \Omega_{1/2}(\infty).$$

This says that there is a conformal map from $\Omega_g(\infty)$ to $\Omega_{1/2}(\infty)$ which satisfies the period problem. Therefore, we conclude that a normalized subsequence of $\{E(n)\}$ converges to a minimal surface of infinite genus.

Alternatively, we can also see the convergence of a normalized subsequence of $\{E(n)\}$ using Carathéodory Kernel Theorem which relates domain convergence and map convergence. We state a definition and the theorem without proof.

**Definition 4.1.1.** Let $w_0 \in \mathbb{C}$ be given and let $G_n$ be domains with $w_0 \in G_n \subset \mathbb{C}$. We say that

$$G_n \to G \quad \text{as} \quad n \to \infty \quad \text{with respect to} \quad w_0.$$
in the sense of kernel convergence if

1. either \( G = \{w_0\} \), or \( G \) is a domain \( \neq \mathbb{C} \) with \( w_0 \in \mathbb{C} \) such that some neighborhood of every \( w \in G \) lies in \( G_n \) for large \( n \);

2. for \( w \in \partial G \) there exist \( w_n \in \partial G_n \) such that \( w_n \to w \) as \( n \to \infty \).

**Theorem 4.1.3** (Carathéodory Kernel Theorem). Let \( f \) map \( \mathbb{D} \) conformally onto \( G_n \) with \( f_n(0) = w_0 \) and \( f_n'(0) > 0 \). If \( G = \{w_0\} \), let \( f(z) \equiv w_0 \), otherwise let \( f \) map \( \mathbb{D} \) conformally onto \( G \) with \( f(0) = w_0 \) and \( f(0) > 0 \). Then, as \( n \to \infty \),

\[
f_n \to f \quad \text{locally uniformly in } \mathbb{D} \iff G_n \to G \quad \text{with respect to } w_0.
\]

Let us come back to our problem on convergence of a subsequence of \( \{E(n)\} \). We already saw in theorem 4.1.1 that a normalized subsequence of coordinate domains \( \{\Omega_g(n)\} \) in \( \mathbb{C} \) converges, uniformly on compacta. Let \( \Omega_g(n_k) \) converges to \( \Omega_g(\infty) \) as \( n_k \to \infty \) on compacta. This convergence of domains satisfies definition 4.1.1 in the sense of kernel convergence. Let

\[
G_{n_k} = \Omega_g(n_k) \subset \mathbb{C},
\]

\[
G = \Omega_g(\infty) \subset \mathbb{C},
\]

and

\[
w_0 = 1 + i.
\]

Note that for all \( n \), the domain \( G_{n_k} = \Omega_g(n_k) \) contains \( w_0 = 1 + i \), since the center of the zigzag boundary \( Z_g(n_k) = \partial \Omega_g(n_k) \) lies on \( 0 \), and since the edges of the zigzag lies on real or imaginary directions (see section 2.2). Therefore, we can say that

\[
\Omega_g(n_k) \to \Omega_g(\infty) \quad \text{with respect to } 1 + i \quad (4.10)
\]
on compacta in the sense of kernel convergence. The only thing we have left to see now is whether we have maps $f_{n_k}$ which take $\mathbb{D}$ conformally onto $G_{n_k} = \Omega_g(n_k)$ with $f_{n_k}(0) = w_0 = 1 + i$ and $f'_{n_k}(0) > 0$. First, for each $n_k$ we have a surjective conformal Schwarz-Christoffel map from the upper half plane $\mathbb{H}$ to a domain $\Omega_g(n_k)$:

$$
\psi_g(n_k) : \mathbb{H} \to \Omega_g(n_k) \subset \mathbb{C}
$$

with $\psi'_g(n_k)(0) > 0$. Second, we also always can find a surjective conformal map from the unit disk $\mathbb{D}$ to the upper half plane $\mathbb{H}$:

$$
\phi_g(n_k) : \mathbb{D} \to \mathbb{H}
$$

with $\phi_g(n_k)(0) = \psi^{-1}_g(n_k)(w_0)$ and $\phi'_g(n_k)(0) > 0$. Then we have our desired map which takes $\mathbb{D}$ conformally onto $\Omega_g(n_k)$:

$$
\begin{equation}
\begin{aligned}
f_{n_k} &= \psi_g(n_k) \circ \phi_g(n_k) : \mathbb{D} \to \Omega_g(n_k) \\
\end{aligned}
\end{equation}
$$

(4.11)

with $f_{n_k}(0) = 1 + i$ and $f'_{n_k}(0) > 0$. By applying (4.10) and (4.11) to Caratheodory theorem 4.1.3, we conclude that there is a limit map

$$
\begin{equation}
\begin{aligned}
f : \mathbb{D} \to \Omega_g(\infty)
\end{aligned}
\end{equation}
$$

(4.12)

such that $f_{n_k} \to f$ locally uniformly in $\mathbb{D}$. It is obviously true that also for the conjugate case, there is a limit map

$$
\begin{equation}
\begin{aligned}
f^* : \mathbb{D} \to \Omega_{\overline{g}}(\infty).
\end{aligned}
\end{equation}
$$

(4.13)
Finally, consider the composition map of (4.12) and (4.13)

\[ f^* \circ f^{-1} : \Omega_g(\infty) \to \Omega_{\hat{g}}(\infty). \]

It says that there is a conformal map from \( \Omega_{\hat{g}}(\infty) \) to \( \Omega_g(\infty) \) which satisfies the period problem. Therefore, we conclude that a normalized subsequence of \( \{E(n)\} \) converges locally uniformly in \( n \) to a minimal surface of infinite genus. This completes the proof of our main theorem. \[ \square \]

### 4.2 Conjecture for the Limit Surface \( E(\infty) \)

We showed that a normalized subsequence of the generalized Enneper’s surfaces \( E(n) \) of genus \( n \) converges to a minimal surface of infinite genus. The next natural question would be what the limit surface looks like. We state our conjecture here on what the limit surface exactly is.

**Conjecture 4.2.1.** Let \( E(\infty) \) be the limit surface of the generalized Enneper’s surface \( E(n) \), and let \( P_i(\infty) \) be the \( i \)th period of \( E(\infty) \). Then \( |P_i(\infty)| = |P_j(\infty)| \) for all \( i \) and \( j \) so that the limit can only be Scherk’s singly periodic surface for \( \theta = \frac{\pi}{2} \).
Chapter 5

Relevant Differential Equations

Here we will see the details of the differential equations and their solutions for periods \( \int_\gamma g \, dh \) and \( \int_\gamma \frac{1}{g} \, dh \) in corollary 5.2.2 which we already used in step 3 in the proof of proposition 3.3.1.

5.1 Useful Differential Equations

Proposition 5.1.1. Let \( g = (z^2)^{\alpha_0}(z^2 - 1)^{\alpha_1}(z^2 - y_3^2)^{\alpha_2}(z^2 - y_3^2)^{\alpha_3}... (z^2 - y_n^2)^{\alpha_n} \), where \( y_k \) and \( \alpha_k \) are any complex numbers. Then \( g \) satisfies the following differential equation:

\[
\frac{\partial}{\partial z} A_p g = (1 + 2 \sum_{k=0}^{n} \alpha_k)g + \left( \frac{1 - 2y_p^2}{y_p} - \sum_{k=0}^{n} \frac{\alpha_k}{\alpha_p} \frac{y_p(y_p^2 - 1)}{y_p^2 - y_k^2} \right) \frac{\partial g}{\partial y_p} + \frac{y_p^2 - 1}{2\alpha_p} \frac{\partial^2 g}{\partial y_p^2} + \sum_{k=0}^{n} \frac{y_k(y_k^2 - 1)}{y_p^2 - y_k^2} \frac{\partial g}{\partial y_k},
\]

where \( A_p = \frac{z(z^2-1)}{z^2-y_p^2} \) for any \( p = 2, 3, ..., n \).

Remark 5.1.1. It is important that we do not have any \( z \) on the right side.
Proof. First, check the following:

$$\frac{\partial g}{\partial z} = 2z \left( \sum_{k=0}^{n} \frac{\alpha_k}{z^2 - y_k^2} \right) g.$$  \hspace{1cm} (5.2)

For any \( k = 0, 1, 2, ..., n, \)

$$\frac{\partial g}{\partial y_k} = \frac{-2\alpha_k y_k}{z^2 - y_k^2} g.$$  \hspace{1cm} (5.3)

$$\frac{\partial^2 g}{\partial y_p^2} = \left( \frac{-2\alpha_p}{z^2 - y_p^2} + \frac{4\alpha_p(\alpha_p - 1)y_p^2}{(z^2 - y_p^2)^2} \right) g = \frac{1}{y_p} \frac{\partial g}{\partial y_p} + \frac{4\alpha_p(\alpha_p - 1)y_p^2}{(z^2 - y_p^2)^2} g.$$  \hspace{1cm} (5.4)

$$A_p = \frac{z(z^2 - 1)}{z^2 - y_p^2}.$$  \hspace{1cm} (5.5)

$$\frac{\partial A_p}{\partial z} = \frac{z^4 + (1 - 3y_p^2)z^2 + y_p^2}{(z^2 - y_p^2)^2} = 1 + \frac{y_p^2}{z^2 - y_p^2} + \frac{2y_p^2(1 - y_p^2)}{(z^2 - y_p^2)^2}.$$  \hspace{1cm} (5.6)

This expansion in \( z^2 - y_p^2 \) is a crucial part in our argument.

$$zA_p = \frac{z^2(z^2 - 1)}{z^2 - y_p^2} = z^2 - y_p^2 + 2y_p^2 - 1 + \frac{y_p^2(y_p^2 - 1)}{z^2 - y_p^2}.$$  \hspace{1cm} (5.7)

From (5.7), we obtain (5.8) and (5.9):

$$\frac{zA_p}{z^2 - y_p^2} = 1 + \frac{2y_p^2 - 1}{z^2 - y_p^2} + \frac{y_p^2(y_p^2 - 1)}{(z^2 - y_p^2)^2}.$$  \hspace{1cm} (5.8)

$$\frac{zA_p}{z^2 - y_k^2} = \frac{zA_p - zA_k}{y_p^2 - y_k^2} = 1 + \frac{y_p^2(y_p^2 - 1)}{y_p^2 - y_k^2} \left( \frac{1}{z^2 - y_p^2} - \frac{y_p^2(y_p^2 - 1)}{y_p^2 - y_k^2} \right) \frac{1}{z^2 - y_k^2}.$$  \hspace{1cm} (5.9)

for \( k \neq p \). Here, the first step follows by the first step in (5.7), and the second step follows by the second step in (5.7).

Now, we are ready to compute the left side of the equation (5.1). From (5.2) and (5.6),

$$\frac{\partial}{\partial z} A_p g = A_p \frac{\partial g}{\partial z} + \frac{\partial A_p}{\partial z} g.$$
\[= 2zA_p \left( \sum_{k=0}^{n} \frac{\alpha_k}{z^2 - y_k^2} \right) g + \left( 1 + \frac{1 - y_p^2}{z^2 - y_p^2} + \frac{2y_p^2(1 - y_p^2)}{(z^2 - y_p^2)^2} \right) g. \]  

(5.10)

We will check the first term on the right side of (5.10).

\[zA_p \left( \sum_{k=0}^{n} \frac{\alpha_k}{z^2 - y_k^2} \right) = zA_p \frac{\alpha_p}{z^2 - y_p^2} + zA_p \sum_{k=0 \atop k \neq p}^{n} \frac{\alpha_k}{z^2 - y_k^2} \]

By substituting (5.8) and (5.9),

\[= \alpha_p \left( 1 + \frac{2y_p^2 - 1}{z^2 - y_p^2} + \frac{y_p^2(y_p^2 - 1)}{(z^2 - y_p^2)^2} \right) + \sum_{k=0 \atop k \neq p}^{n} \alpha_k \left( 1 + \frac{y_p^2(y_p^2 - 1)}{y_p^2 - y_k^2} \frac{1}{z^2 - y_p^2} - \frac{y_k^2(y_k^2 - 1)}{y_k^2 - y_p^2} \frac{1}{z^2 - y_k^2} \right) . \]  

(5.11)

We continue to compute (5.10). Recall the equation (5.10).

\[\frac{\partial}{\partial z} A_p g = 2zA_p \left( \sum_{k=0}^{n} \frac{\alpha_k}{z^2 - y_k^2} \right) g + \left( 1 + \frac{1 - y_p^2}{z^2 - y_p^2} + \frac{2y_p^2(1 - y_p^2)}{(z^2 - y_p^2)^2} \right) g \]

By substituting (5.11),

\[= 2\alpha_p \left( 1 + \frac{2y_p^2 - 1}{z^2 - y_p^2} + \frac{y_p^2(y_p^2 - 1)}{(z^2 - y_p^2)^2} \right) g + \sum_{k=0 \atop k \neq p}^{n} \alpha_k \left( 1 + \frac{y_p^2(y_p^2 - 1)}{y_p^2 - y_k^2} \frac{1}{z^2 - y_p^2} - \frac{y_k^2(y_k^2 - 1)}{y_k^2 - y_p^2} \frac{1}{z^2 - y_k^2} \right) g \]

\[+ \left( 1 + \frac{1 - y_p^2}{z^2 - y_p^2} + \frac{2y_p^2(1 - y_p^2)}{(z^2 - y_p^2)^2} \right) g \]

By rearranging,

\[= (1 + 2 \sum_{k=0}^{n} \alpha_k) g + \left( 2\alpha_p \frac{2y_p^2 - 1}{z^2 - y_p^2} + \frac{1 - y_p^2}{z^2 - y_p^2} \right) g + \frac{2(\alpha_p - 1)y_p^2(y_p^2 - 1)}{(z^2 - y_p^2)^2} g \]
\[ +2 \sum_{k=0 \atop k \neq p}^{n} \alpha_k \frac{y_k^2(y_p^2 - 1)}{y_p^2 - y_k^2} \frac{1}{z^2 - y_p^2} g - 2 \sum_{k=0 \atop k \neq p}^{n} \alpha_k \frac{y_k^2}{y_p^2 - y_k^2} \frac{1}{z^2 - y_k^2} g \]

From (5.3) and (5.4),

\[ = (1 + 2 \sum_{k=0}^{n} \alpha_k)g + \left( \frac{1 - 2y_p^2}{y_p} + \frac{y_p^2 - 1}{2\alpha_p y_p} \right) \frac{\partial g}{\partial y_p} + \frac{y_p^2 - 1}{2\alpha_p} \left( \frac{\partial^2 g}{\partial y_p^2} - \frac{1}{y_p} \frac{\partial g}{\partial y_p} \right) \]

\[ - \sum_{k=0 \atop k \neq p}^{n} \frac{\alpha_k y_p(y_p^2 - 1)}{y_p^2 - y_k^2} \frac{\partial g}{\partial y_p} + \sum_{k=0 \atop k \neq p}^{n} \frac{y_k(y_k^2 - 1)}{y_p^2 - y_k^2} \frac{\partial g}{\partial y_k} \]

Here, \( \frac{y_p^2 - 1}{2\alpha_p y_p} \frac{\partial g}{\partial y_p} \) in the second term and the third term is canceled. Therefore,

\[ \frac{\partial}{\partial z} A_p g = (1 + 2 \sum_{k=0}^{n} \alpha_k)g + \left( \frac{1 - 2y_p^2}{y_p} - \sum_{k=0 \atop k \neq p}^{n} \frac{\alpha_k y_p(y_p^2 - 1)}{y_p^2 - y_k^2} \right) \frac{\partial g}{\partial y_p} + \frac{y_p^2 - 1}{2\alpha_p} \frac{\partial^2 g}{\partial y_p^2} \]

\[ + \sum_{k=0 \atop k \neq p}^{n} \frac{y_k(y_k^2 - 1)}{y_p^2 - y_k^2} \frac{\partial g}{\partial y_k} . \]

\[ \square \]

**Corollary 5.1.1.** If we choose \( \gamma \) such that \( \int_{\gamma} \frac{\partial}{\partial z} A_p g = 0 \) on the left side of the equation (5.1) in proposition 5.1.1, then \( h = \int_{\gamma} g \) satisfies the following differential equation:

\[ 0 = (1 + 2 \sum_{k=0}^{n} \alpha_k)h + \left( \frac{1 - 2y_p^2}{y_p} - \sum_{k=0 \atop k \neq p}^{n} \frac{\alpha_k y_p(y_p^2 - 1)}{y_p^2 - y_k^2} \right) \frac{\partial h}{\partial y_p} + \frac{y_p^2 - 1}{2\alpha_p} \frac{\partial^2 h}{\partial y_p^2} + \sum_{k=0 \atop k \neq p}^{n} \frac{y_k(y_k^2 - 1)}{y_p^2 - y_k^2} \frac{\partial h}{\partial y_k} . \]

The above equation holds for \( \gamma \) which is a curve from \( y_i \) to \( y_j \) for any \( i, j = 0, 1, \ldots, n \), not passing through the other points \( y_k, k \neq i, j \). We write this in the following corollary.

**Corollary 5.1.2.** Let \( g = (z^2)^{\alpha_0}(z^2 - 1)^{\alpha_1}(z^2 - y_1^2)^{\alpha_2}(z^2 - y_2^2)^{\alpha_3} \ldots (z^2 - y_n^2)^{\alpha_n} \), where \( y_k \) and \( \alpha_k \) are any complex numbers. Let \( \gamma \) be a curve from \( y_i \) to \( y_j \) for any \( i, j = 0, 1, \ldots, n \).
0, 1,..., n, not passing through the other points \(y_k, k \neq i, j\). Then \(h = \int_\gamma g\) satisfies the following differential equation:

\[
0 = (1+2\sum_{k=0} a_k)h + \left(1 - \frac{2y_p^2}{y_p} - \sum_{k=p}^{n} \frac{a_k y_p (y_k^2 - 1)}{2(\alpha_p y_p - y_k^2)} \right) \frac{\partial h}{\partial y_p} + \frac{y_p^2 - 1}{2} \frac{\partial^2 h}{\partial y_p^2} + \sum_{k=p}^{n} \frac{y_k (y_k^2 - 1) \partial h}{y_p^2 - y_k^2} \frac{\partial h}{\partial y_k}.
\]

Proof. Consider the Pochhammer loop around \(y_i\) and \(y_j\). Shrink the circles around \(y_i\) and \(y_j\) to \(y_i\) and \(y_j\). Then the integral \(\int_\gamma \frac{\partial}{\partial z} A_p g\) on the left side of the equation (5.1) goes to 0 over the circles.

\[\square\]

### 5.2 Application to Periods of \(E(n)\)

Now we are ready to apply corollary 5.1.2 to our period problem for the generalized Enneper’s surface \(E(n)\) of genus \(n\). In the following corollary 5.2.1, the function \(g\) is the same one which we discussed in proposition 3.5.1. Later, we will use this corollary to prove corollary 5.2.2 which uses the function \(g\) that we used in propositions 3.3.1 and 3.4.1.

**Corollary 5.2.1.** Let

\[
g = \frac{(z^2 - \varepsilon^2)^{\frac{1}{2}}(z^2 - y_2^2)^{\frac{1}{2}}(z^2 - y_3^2)^{\frac{1}{2}}... (z^2 - y_{n-1}^2)^{\frac{1}{2}}}{z^{\frac{1}{2}}(z^2 - 1)^{\frac{1}{2}}(z^2 - y_4^2)^{\frac{1}{2}}(z^2 - y_5^2)^{\frac{1}{2}}... (z^2 - y_n^2)^{\frac{1}{2}}},
\]

where \(y_j\) are real numbers. Then for any \(\gamma = [y_j, y_{j+1}], j = 0, 1, 2, ..., n - 1,\)

(a) \(h = \int_\gamma g dz\) satisfies the following differential equation

\[
0 = \left(\frac{1}{2} + \sum_{k=3}^{n} (-1)^{k+1}\right)h + (1 - \varepsilon^2) \frac{1}{2\varepsilon} + \sum_{k=3}^{n} (-1)^{k} \frac{\varepsilon}{y_k^2 - \varepsilon^2} \frac{\partial h}{\partial \varepsilon} \frac{\partial^2 h}{\partial \varepsilon^2} + \sum_{k=3}^{n} y_k(y_k^2 - 1) \frac{\partial h}{\partial y_k},
\]
and the equation has the solution in the form of either

\[ h = a_0 + a_2 \varepsilon^2 + a_4 \varepsilon^4 + a_6 \varepsilon^6 + \ldots \quad \text{or} \]

\[ h = \varepsilon^{\frac{3}{2}} (b_0 + b_2 \varepsilon^2 + b_4 \varepsilon^4 + b_6 \varepsilon^6 + \ldots) \]

(b) \[ l = \int_{\gamma_{\varepsilon}} \frac{1}{y} \, dz \] satisfies the following differential equation

\[
0 = \left( \frac{3}{2} + \sum_{k=3}^{n} (-1)^k \right) l + (1 - \varepsilon^2) \left( \frac{1}{2 \varepsilon} + \sum_{k=3}^{n} (-1)^k \frac{\varepsilon}{y_k^2 - \varepsilon^2} \right) \frac{\partial l}{\partial \varepsilon} \\
+ (1 - \varepsilon^2) \frac{\partial^2 l}{\partial \varepsilon^2} + \sum_{k=3}^{n} \frac{y_k(y_k^2 - 1)}{\varepsilon^2 - y_k^2} \frac{\partial l}{\partial y_k}.
\]

and the equation has the solution in the form of either

\[ l = a_0^* + a_2^* \varepsilon^2 + a_4^* \varepsilon^4 + a_6^* \varepsilon^6 + \ldots \quad \text{or} \]

\[ l = \varepsilon^{\frac{3}{2}} (b_0^* + b_2^* \varepsilon^2 + b_4^* \varepsilon^4 + b_6^* \varepsilon^6 + \ldots). \]

**Remark 5.2.1.** In equations (a) and (b), note that the signs of coefficients of \( \frac{\partial^2 h}{\partial \varepsilon^2} \) and \( \frac{\partial^2 l}{\partial \varepsilon^2} \) in the third terms are different. This difference gives us the difference in order of \( \varepsilon \) in solutions.

**Proof.** (a) Let \( y_2 = y = \varepsilon \) with \( \alpha_0 = \alpha_2 = -\frac{1}{2} \) in corollary 5.1.2. Here \( \alpha_0 = -\frac{1}{4} \), \( \alpha_1 = -\frac{1}{2}, \alpha_k = \frac{1}{2} \) for odd \( k \geq 3 \), and \( \alpha_k = -\frac{1}{2} \) for even \( k \geq 3 \). Then we have

\[
0 = \left( 1 + 2 \left( \frac{1}{4} - \frac{1}{2} + \frac{1}{2} \right) + \sum_{k=3}^{n} (-1)^{k+1} \right) h \\
+ \left( \frac{1 - 2 \varepsilon^2}{\varepsilon} + \frac{\varepsilon(\varepsilon^2 - 1)}{2 \varepsilon^2} + \frac{\varepsilon(\varepsilon^2 - 1)}{\varepsilon^2 - 1} + \sum_{k=3}^{n} (-1)^k \frac{\varepsilon(\varepsilon^2 - 1)}{\varepsilon^2 - y_k^2} \right) \frac{\partial h}{\partial \varepsilon} \\
+ (\varepsilon^2 - 1) \frac{\partial^2 h}{\partial \varepsilon^2} + \sum_{k=3}^{n} \frac{y_k(y_k^2 - 1)}{\varepsilon^2 - y_k^2} \frac{\partial h}{\partial y_k}.
\]
By simplifying this, we obtain the desired equation in (a).

\[
0 = \left( \frac{1}{2} + \sum_{k=3}^{n} (-1)^{k+1} \right) h + (1 - \varepsilon^2) \left( \frac{1}{2\varepsilon} + \sum_{k=3}^{n} (-1)^{k \frac{\varepsilon}{y_k^2 - \varepsilon^2}} \right) \frac{\partial h}{\partial \varepsilon} \\
- (1 - \varepsilon^2) \frac{\partial^2 h}{\partial \varepsilon^2} + \sum_{k=3}^{n} \frac{y_k(y_k^2 - 1)}{\varepsilon^2 - y_k^2} \frac{\partial h}{\partial y_k}.
\]

(5.12)

Now, we will find the solution. This equation is a homogeneous linear second-order equation. Let

\[
h = \sum_{m=0}^{\infty} a_m \varepsilon^{m+r},
\]

where \(a_0 \neq 0\) and \(a_m\) is not a function of \(\varepsilon\). Then

\[
\frac{\partial h}{\partial \varepsilon} = (m + r) \sum_{n=0}^{\infty} a_m \varepsilon^{m+r-1},
\]

\[
\frac{\partial^2 h}{\partial \varepsilon^2} = (m + r)(m + r - 1) \sum_{m=0}^{\infty} a_m \varepsilon^{m+r-2}.
\]

\[
\frac{\partial h}{\partial y_k} = \sum_{m=0}^{\infty} \frac{\partial a_m}{\partial y_k} \varepsilon^{m+r}.
\]

In the equation (5.12), the indicial equation (when \(m = 0\)) depends only on the terms with the lowest exponent of \(\varepsilon\): \(\frac{1}{2\varepsilon} \frac{\partial h}{\partial \varepsilon}\) and \(-\frac{\partial^2 h}{\partial \varepsilon^2}\). Thus, the indicial equation is

\[
\left( \frac{1}{2} r - r(r - 1) \right) a_0 \varepsilon^{n+r-2} = 0.
\]

For the details, see [Bra83]. Therefore, we have

\[
r = 0 \text{ or } \frac{3}{2}.
\]
By substituting these choices of $r$ into $h = \sum_{m=0}^{\infty} a_m \varepsilon^{m+r}$, we obtain

$$h = a_0 + a_2 \varepsilon^2 + a_4 \varepsilon^4 + a_6 \varepsilon^6 + \ldots; \quad \text{or}$$

$$h = \varepsilon^{\frac{3}{2}}(b_0 + b_2 \varepsilon^2 + b_4 \varepsilon^4 + b_6 \varepsilon^6 + \ldots).$$

(b) Let $y_p = y_2 = \varepsilon$ with $\alpha_p = \alpha_2 = -\frac{1}{2}$ in corollary 5.1.2. Here $\alpha_0 = \frac{1}{4}$, $\alpha_1 = \frac{1}{2}$, $\alpha_k = -\frac{1}{2}$ for odd $k \geq 3$, and $\alpha_k = \frac{1}{2}$ for even $k \geq 3$. Then we have

$$0 = \left(1 + 2\left(\frac{1}{4} + \frac{1}{2} - \frac{1}{2}\right) + \sum_{k=3}^{n} (-1)^k\right) l$$

$$+ \left(1 - \frac{2\varepsilon^2}{\varepsilon} + \frac{\varepsilon(\varepsilon^2 - 1)}{2\varepsilon^2} + \frac{\varepsilon(\varepsilon^2 - 1)}{\varepsilon^2 - 1} + \sum_{k=3}^{n} (-1)^k \frac{\varepsilon(\varepsilon^2 - 1)}{\varepsilon^2 - y_k^2}\right) \frac{\partial l}{\partial \varepsilon}$$

$$+ (1 - \varepsilon^2) \frac{\partial^2 l}{\partial \varepsilon^2} + \sum_{k=3}^{n} \frac{y_k(y_k^2 - 1)}{\varepsilon^2 - y_k^2} \frac{\partial l}{\partial y_k}.$$

By simplifying this, we obtain the desired equation in (b).

$$0 = \left(\frac{3}{2} + \sum_{k=3}^{n} (-1)^k\right) l + (1 - \varepsilon^2) \left(\frac{1}{2\varepsilon} + \sum_{k=3}^{n} (-1)^k \frac{\varepsilon}{y_k^2 - \varepsilon^2}\right) \frac{\partial l}{\partial \varepsilon}$$

$$+ (1 - \varepsilon^2) \frac{\partial^2 l}{\partial \varepsilon^2} + \sum_{k=3}^{n} \frac{y_k(y_k^2 - 1)}{\varepsilon^2 - y_k^2} \frac{\partial l}{\partial y_k}. \quad (5.13)$$

To find the solution, we have a similar argument as in (a). This equation is also a homogeneous linear second-order equation. Let

$$l = \sum_{m=0}^{\infty} a_m^* \varepsilon^{m+r},$$

where $a_0^* \neq 0$ and $a_m^*$ is not a function of $\varepsilon$. Then the indicial equation (when $m = 0$) of the equation (5.13) depends only on the terms with the lowest exponent of $\varepsilon : \frac{1}{2\varepsilon} \frac{\partial l}{\partial \varepsilon}$.
and \( \frac{\partial^2 y}{\partial x^2} \). Thus, the indicial equation is

\[
\left( \frac{1}{2} r + r(r - 1) \right) a_0^* \varepsilon^{m+r-2} = 0.
\]

Therefore, we have

\[
r = 0 \quad \text{or} \quad \frac{1}{2}.
\]

By substituting these choices of \( r \) into \( l = \sum_{m=0}^{\infty} a_m^* \varepsilon^{m+r} \),

\[
l = a_0^* + a_2^* \varepsilon^2 + a_4^* \varepsilon^4 + a_6^* \varepsilon^6 + \ldots \quad \text{or}
\]

\[
l = \varepsilon^{\frac{1}{2}} (b_0^* + b_2^* \varepsilon^2 + b_4^* \varepsilon^4 + b_6^* \varepsilon^6 + \ldots).
\]

\(\square\)

Finally, we will prove the next crucial result we used in lemma 3.3.2 in step 3 in the proof of proposition 3.3.1. This is important because this gives us the relationship between coefficients in the expansions of periods of \( E(n) \) in lemma 3.3.2.

**Corollary 5.2.2.** Let

\[
g = \frac{(z^2 - 1) \frac{1}{2} (z^2 - y_2^2) \frac{3}{2} (z^2 - y_3^2) \frac{5}{2} \ldots (z^2 - y_m^2) \frac{2m-1}{2}}{\varepsilon^{\frac{1}{2}} (z^2 - y_2^2) \frac{3}{2} (z^2 - y_3^2) \frac{5}{2} \ldots (z^2 - y_m^2) \frac{2m-1}{2}},
\]

where \( y_j \) are real numbers, and let

\[
y_{i+1} = y_i + \varepsilon_i
\]

for some \( i = 1, 2, \ldots, n - 1 \). Let \( i \) be an even integer. Then for any \( \gamma = [y_j, y_{j+1}] \), \( j = 0, 1, \ldots, n - 1 \),
(a) \( h = \int \gamma \, gdz \) satisfies the following differential equation

\[
0 = \left( \frac{1}{2} + \sum_{k=1 \atop k \neq i, i+1}^{n} (-1)^{k+1} \right) h
- \varepsilon_i (2y_i + \varepsilon_i) \left( \frac{1}{2(y_i + \varepsilon_i)} + \sum_{k=1 \atop k \neq i, i+1}^{n} (-1)^{k} \frac{y_i + \varepsilon_i}{y_k^2 - (y_i + \varepsilon_i)^2} \right) \frac{\partial h}{\partial \varepsilon_i}
+ \varepsilon_i (2y_i + \varepsilon_i) \frac{\partial^2 h}{\partial \varepsilon_i^2} + \sum_{k=1 \atop k \neq i, i+1}^{n} \frac{y_k(y_k^2 - y_i^2)}{(y_i + \varepsilon_i)^2 - y_k^2} \frac{\partial h}{\partial y_k},
\]

and the equation has the solution in the form of either

\[
h = \varepsilon_i \left( a_{i,0} + a_{i,1} \varepsilon_i + a_{i,2} \varepsilon_i^2 + \ldots \right) \quad \text{or} \quad h = (b_{i,0} + b_{i,1} \varepsilon_i + b_{i,2} \varepsilon_i^2 + \ldots) + \varepsilon_i \ln \varepsilon_i \left( a_{i,0} + a_{i,1} \varepsilon_i + a_{i,2} \varepsilon_i^2 + \ldots \right).
\]

(b) \( l = \int \gamma^2 \, dz \) satisfies the following differential equation

\[
0 = \left( \frac{3}{2} + \sum_{k=1 \atop k \neq i, i+1}^{n} (-1)^{k} \right) l
- \varepsilon_i (2y_i + \varepsilon_i) \left( \frac{1}{2(y_i + \varepsilon_i)} + \sum_{k=1 \atop k \neq i, i+1}^{n} (-1)^{k} \frac{y_i + \varepsilon_i}{y_k^2 - (y_i + \varepsilon_i)^2} \right) \frac{\partial l}{\partial \varepsilon_i}
- \varepsilon_i (2y_i + \varepsilon_i) \frac{\partial^2 l}{\partial \varepsilon_i^2} + \sum_{k=1 \atop k \neq i, i+1}^{n} \frac{y_k(y_k^2 - y_i^2)}{(y_i + \varepsilon_i)^2 - y_k^2} \frac{\partial l}{\partial y_k},
\]

and the equation has the solution in the form of either

\[
l = \varepsilon_i \left( a_{i,0}^* + a_{i,1}^* \varepsilon_i + a_{i,2}^* \varepsilon_i^2 + \ldots \right) \quad \text{or} \quad l = (b_{i,0}^* + b_{i,1}^* \varepsilon_i + b_{i,2}^* \varepsilon_i^2 + \ldots) + \varepsilon_i \ln \varepsilon_i \left( a_{i,0}^* + a_{i,1}^* \varepsilon_i + a_{i,2}^* \varepsilon_i^2 + \ldots \right).
\]
Here, \( a_{i,0}, b_{i,0}, a_{i,i}, \) and \( b_{i,i} \) are \( \neq 0, \) and the coefficients \( a_{i,k}, b_{i,k}, a_{i,k}, \) and \( b_{i,k}, \)
\( k = 0,1,\ldots, \) are not functions of \( \varepsilon_i. \) The same argument holds for odd \( i. \)

**Remark 5.2.2.** In the equations in (a) and (b), the second terms are \( o(\varepsilon_i). \) These are crucial parts to obtain \( \varepsilon_i \ln \varepsilon_i \) terms in the solutions.

**Remark 5.2.3.** Notice in (a) that the two solutions of the function \( h \) use the same \( a_{i,k}, k = 0,1,2,\ldots, \) This also holds in (b). This observation was crucial in the proof of proposition 3.3.1.

**Proof.** (a) We will use some trick to obtain the second term \( o(\varepsilon_i). \) Divide the numerator and the denominator of \( g \) by \( y_i. \) Then

\[
gdz = y_i^\alpha ((\frac{z}{y_i})^2 - (\frac{z}{y_i})^2)^{\frac{1}{2}}((\frac{z}{y_i})^2 - (\frac{y_i^2}{y_i})^2)^{\frac{1}{2}}\ldots((\frac{z}{y_i})^2 - (\frac{y_{n-1}^2}{y_i})^2)^{\frac{1}{2}}d\frac{z}{y_i}.
\]

Here, \( y_i^\alpha \) is not important in our argument. Let

\[
\frac{y_k}{y_i} = x_k \text{ and } \frac{z}{y_i} = t.
\]

Then

\[
gdz = y_i^{\alpha+1}(t^2 - x_1^2)^{\frac{1}{2}}(t^2 - x_2^2)^{\frac{1}{2}}\ldots(t^2 - x_{i+1}^2)^{\frac{1}{2}}\ldots(t^2 - x_{n-1}^2)^{\frac{1}{2}}dt.
\]

We can apply corollary 5.1.2 with \( y_p = x_{i+1} \) and \( \alpha_p = \alpha_{i+1} = \frac{1}{2}. \) Here \( \alpha_0 = -\frac{1}{4}, \)
\( \alpha_i = -\frac{1}{2}, \) \( \alpha_k = \frac{1}{2} \) for odd \( k \geq 1, \) and \( \alpha_k = -\frac{1}{2} \) for even \( k \geq 1. \) Or use corollary 5.2.1

thinking of \( \varepsilon_i \) as \( x_{i+1}. \) Then we have

\[
0 = \left(\frac{1}{2} + \sum_{k=1}^{n} (-1)^{k+1} \right) h
\]

\[
+ (1 - x_{i+1}^2) \left(\frac{1}{2x_{i+1}} - \sum_{k=1 \atop k \neq i+1}^{n} (-1)^{k} \frac{x_{i+1}}{x_{i+1}^2 - x_k^2}\right) \frac{\partial h}{\partial x_{i+1}}
\]
\[ + (x_{i+1}^2 - 1) \frac{\partial^2 h}{\partial x_{i+1}^2} + \sum_{\substack{k=1 \\ k \neq i, i+1}}^n \frac{x_k(x_k^2 - 1)}{x_{i+1}^2 - x_k^2} \frac{\partial h}{\partial x_k}. \]  

Remark 5.2.4. In the second term, \( 1 - x_{i+1}^2 \) will give us the term \( o(\varepsilon_i) \) later.

We want to write the equation (5.15) again in terms of \( y_k \). Since \( y_{i+1} = y_i + \varepsilon_i \),

\[ x_{i+1} = \frac{y_{i+1}}{y_i} = 1 + \frac{\varepsilon_i}{y_i}. \]  

Therefore,

\[ \frac{\partial h}{\partial x_{i+1}} = y_i \frac{\partial h}{\partial \varepsilon_i}, \]
\[ \frac{\partial^2 h}{\partial x_{i+1}^2} = y_i^2 \frac{\partial^2 h}{\partial \varepsilon_i^2}, \]
\[ \frac{\partial h}{\partial x_k} = y_i \frac{\partial h}{\partial y_k}. \]  

By substituting (5.14) and (5.17), the equation (5.15) becomes

\[ 0 = \left( \frac{1}{2} + \sum_{\substack{k=1 \\ k \neq i, i+1}}^n (-1)^{k+1} \right) h \]
\[ + (y_i^2 - y_{i+1}^2) \left( \frac{1}{2y_{i+1}} - \sum_{\substack{k=1 \\ k \neq i, i+1}}^n (-1)^{k} \frac{y_{i+1}^2}{y_i^2 - y_k^2} \right) \frac{\partial h}{\partial \varepsilon_i}, \]
\[ + (y_{i+1}^2 - y_i^2) \frac{\partial^2 h}{\partial \varepsilon_i^2} + \sum_{\substack{k=1 \\ k \neq i, i+1}}^n \frac{y_k^2(y_i^2 - y_k^2)}{y_{i+1}^2 - y_k^2} \frac{\partial h}{\partial y_k}. \]

From \( y_{i+1} = y_i + \varepsilon_i \) and \( y_{i+1}^2 - y_i^2 = \varepsilon_i(2y_i + \varepsilon_i) \), we obtain the desired equation

\[ 0 = \left( \frac{1}{2} + \sum_{\substack{k=1 \\ k \neq i, i+1}}^n (-1)^{k+1} \right) h \]
\[-\varepsilon_i(2y_i + \varepsilon_i) \left( \frac{1}{2(y_i + \varepsilon_i)} - \sum_{k=1 \atop k \neq i, i+1}^{n} (-1)^k \frac{y_i + \varepsilon_i}{(y_i + \varepsilon_i)^2 - y_k^2} \right) \frac{\partial h}{\partial \varepsilon_i} \]

\[+ \varepsilon_i(2y_i + \varepsilon_i) \frac{\partial^2 h}{\partial \varepsilon_i^2} + \sum_{k=1 \atop k \neq i, i+1}^{n} \frac{y_k^2(y_k^2 - y_i^2)}{(y_i + \varepsilon_i)^2 - y_k^2} \frac{\partial h}{\partial y_k} \] (5.18)

Now, we will find the solution. This equation is a homogeneous linear second-order equation. Let

\[h = \sum_{m=0}^{\infty} a_{i,m} \varepsilon_i^{m+r},\]

where \(a_0 \neq 0\) and \(a_m\) is not a function of \(\varepsilon_i\). Then

\[\frac{\partial h}{\partial \varepsilon_i} = (m + r) \sum_{m=0}^{\infty} a_{i,m} \varepsilon_i^{m+r-1}.\]

\[\frac{\partial^2 h}{\partial \varepsilon_i^2} = (m + r)(m + r - 1) \sum_{m=0}^{\infty} a_{i,m} \varepsilon_i^{m+r-2}.\]

\[\frac{\partial h}{\partial y_k} = \sum_{m=0}^{\infty} \frac{\partial a_{i,m}}{\partial y_k} \varepsilon_i^{m+r}.\]

In the equation (5.18), the indicial equation \((m = 0)\) depends only on the terms with the lowest exponent of \(\varepsilon_i\): \(\varepsilon_i 2y_i \frac{\partial^2 h}{\partial \varepsilon_i^2}\). Thus, the indicial equation is

\[2y_i r(r - 1)a_{i,0}\varepsilon_i^{m+r-1} = 0.\]

Therefore,

\[r = 0 \text{ or } 1.\]

By substituting \(r = 1\) to

\[h = \sum_{m=0}^{\infty} a_{i,m} \varepsilon_i^{m+r},\]

\[h = \varepsilon_i (a_{i,0} + a_{i,1} \varepsilon_i + a_{i,2} \varepsilon_i^2 + ...).\]
Another possible solution is

\[ h = \left( b_{i,0} + b_{i,1} \varepsilon_i + b_{i,2} \varepsilon_i^2 + \ldots \right) + \varepsilon_i \ln \varepsilon_i (a_{i,0} + a_{i,1} \varepsilon_i + a_{i,2} \varepsilon_i^2 + \ldots). \]

For the details, see [Bra83].

(b) A similar argument holds for \( l = \int \frac{1}{g} \). Use

\[
\frac{1}{g} dz = y_N^{\alpha+1} \frac{t^{\frac{1}{2}}(t^2 - x_2^2)^{\frac{1}{2}}(t^2 - x_3^2)^{\frac{1}{2}} \ldots (t^2 - x_{n-1}^2)^{\frac{1}{2}} dt.\]

For the solution, a similar argument applies. The equation in (b) is a homogeneous linear second-order equation. Let

\[
l = \sum_{m=0}^{\infty} a_{i,m}^* \varepsilon_i^{m-r},
\]

where \( a_0^* \neq 0 \) and \( a_{i,m}^* \) is not a function of \( \varepsilon_i \). Then the indicial equation (when \( m = 0 \)) of the equation depends only on the terms with the lowest exponent of \( \varepsilon_i : -2y_i \frac{\partial^2}{\partial \varepsilon_i^2} \).

Thus, the indicial equation is

\[-\varepsilon_i 2y_i r(r - 1)a_{i,0}^* \varepsilon_i^{m+r-2} = 0.\]

Therefore,

\[ r = 0 \quad \text{or} \quad 1.\]

By substituting \( r = 1 \) to \( l = \sum_{m=0}^{\infty} a_{i,m}^* \varepsilon_i^{m+r} \),

\[ l = \varepsilon_i (a_{i,0}^* + a_{i,1}^* \varepsilon_i + a_{i,2}^* \varepsilon_i^2 + \ldots).\]
Another possible solution is

\[ l = (b_{i,0}^* + b_{i,1}^* \varepsilon_i + b_{i,2}^* \varepsilon_i^2 + \ldots) + \varepsilon_i \ln \varepsilon_i (a_{i,0}^* + a_{i,1}^* \varepsilon_i + a_{i,2}^* \varepsilon_i^2 + \ldots). \]

This completes the proof of corollary 5.2.2. \qed
Bibliography


