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Simulation of Radiation Belt Electron Diffusion

by

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ABSTRACT

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This thesis presents theoretical and numerical studies of the radial diffusion of relativistic radiation belt electrons. The research has been focused particularly on the radiation belt phase space density profile, and radial diffusion due to particle drift resonance with ULF waves.

Observations have shown a strong connection between magnetospheric ULF oscillations and electron flux enhancements. I investigate radial diffusion coefficients based on theoretical analysis of particle diffusion in ULF perturbation electric and magnetic fields. The analytical diffusion coefficients consist of two terms: a symmetric term and an asymmetric term. The symmetric term agrees with earlier works, and the asymmetric terms are new. Both terms show good agreement with numerical test particle simulations. The asymmetric terms have higher $L$ dependence, which indicates they might be more important at higher $L$-shells or at times when the magnetospheric field is highly asymmetric.

A numerical radial diffusion model has been developed which can take into account dynamic boundary locations and values, plus effects of losses and sources. Several test cases are considered to study the effects of different diffusion coefficients, internal sources, external sources, and loss.
A method of converting observational particle flux to phase space density is also presented. Identifying the source and loss processes using observational data is currently one of the key issues for understanding and modeling radiation belt dynamics. We present a new measurement technique which utilizes two GOES satellites located at different local times to calculate the radial gradient of phase space density at geostationary locations. The result shows positive gradient at geomagnetic quiet periods.

To further study the high energy electron transport, especially the ULF related acceleration during storm times, I use the numerical radial diffusion model for the September 24-26, 1998 storm and compare the results with an MHD test particle simulation. The diffusion result using ULF-wave diffusion coefficients and a time-dependent outer-boundary condition can reproduce the main features of the MHD-particle results quite well. Using wave driven diffusion coefficients gives better results than using power law or $Kp$-dependent diffusion coefficients.
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Chapter 1

An Introduction to the Magnetosphere and the Radiation Belt

1.1 Introduction

The start of human knowledge of space can be traced back to 2000 years B.C. when the earliest recordings of the aurora phenomena were made. People's understanding of space remained superstitious until the 17th century, when scientists began to relate aurora to the earth's magnetic field and to matter from the sun. With the continuing accumulation of knowledge of the Earth's space environment, in 1958 the discovery of the belts of energetic charged particles in the Earth's magnetic field can be viewed as a milestone of magnetospheric physics [Walt, 1994]. Since then, the accelerating pace of space exploration has provided space physics with wider observational coverage (ground based, geostationary orbit, low-Earth orbit, polar orbit) and more comprehensive measurements (particle integral flux, differential flux, electric and magnetic field). At the same time, increasing manned and unmanned space activities also demand better space environment models from space science. Space weather or more specifically, the study of the high-energy electron population in the radiation belts is currently one of the most interesting and important topics in space science.

1.2 The Sun-Earth connection and magnetic storms
1.2.1 The solar wind and the Earth’s magnetosphere

The solar wind is fully-ionized plasma consisting of electrons and ions moving outward from the sun. As the solar wind flows, it carries with it the interplanetary magnetic field (IMF) which is ‘frozen’ into the solar wind plasma. The rotation of the Sun gives the magnetic field a spiral form. When the solar wind approaches the Earth its typical number density is around 3~5 cm$^{-3}$, the velocity is around 300-400 km/s at quiet times, and it can be as high as 900 km/s at storm times [Kivelson and Russell, 1995].

The Earth’s magnetosphere is the region surrounding the Earth that is dominated by the terrestrial magnetic field. The Earth’s internal magnetic field is generated by currents in the Earth’s rotating core. Within a few Earth radii ($1R_E = 6400$km) of the Earth’s surface the magnetic field is approximately a dipole field. The solar wind flow is supersonic (with Mach number ~4) and it forms a shock called the bow shock in front of the barrier presented by the geomagnetic field. The outer boundary of the magnetosphere is called the magnetopause, and the region of plasma between the shock and the magnetopause is called the magnetosheath. The magnetosheath contains a turbulent flow of hot plasma with densities higher than the solar wind. The IMF plasma compresses the magnetosphere on the dayside, flows around the magnetic barrier, and wraps the field lines into the tail region at the night side.

The interaction of the solar wind and the earth’s magnetic field is complex and still not fully understood, but with over 30 years of study and valuable observations from satellites, the principal features of the magnetosphere have been identified. Figure 1-1 illustrates major plasma regions and current systems.
The closed field lines of the inner magnetosphere trap the plasmas of two main populations: the cold and dense plasmasphere plasma and the energetic ring current particles. The plasmasphere particles co-rotate with the Earth's magnetic field. The ring current particles drift around the earth due to the gradient of the Earth's magnetic field. The high-energy (≥500keV) particles in the ring current region are called the radiation belt particles.

The main product of solar wind-magnetosphere coupling is the large-scale dawn-dusk convection electric field. Another important magnetospheric electric field is the co-
rotation electric field. The drift paths of ring current particles are affected by these electric fields. The drift paths of the radiation belt particles are dominated by the magnetic drifts.

The solar wind is the driver of most activity in the Earth’s magnetosphere. For example, disturbances on the surface of the Sun such as coronal mass ejections (CME) sometimes can trigger significant changes of the magnetosphere configuration such as storms and sub-storms.

1.2.2 Magnetic storms

A geomagnetic storm is the large-scale disturbance in the Earth’s magnetosphere resulting from a prolonged coupling between the solar wind and the Earth’s magnetosphere. Magnetic storms are usually triggered by the arrival of an interplanetary shock followed by a period of southward IMF. A large number of energetic particles are injected from the magnetotail into the ring current, and the strengthened ring current causes the drop of the northward equatorial magnetic field strength, and also causes the decrease of the $Dst$ value. The $Dst$ index is defined as the average of measurements of the horizontal component of the magnetic field by a series of mid-latitude and equatorial magnetometers with the quiet-time contribution subtracted. The $Dst$ index gives a measure of the storm-time ring current energy. Figure 1-2 shows the $Dst$ index during a typical magnetic storm.
Figure 1-2 Dst measurement during the September 24-26 1998 storm

Most magnetic storms can be divided into three well-defined phases based on the $Dst$ index. A storm begins with the sudden storm commencement (SSC), where the shock associated with increased solar wind velocity compresses the magnetosphere and increases the magnetic field strength measured by $Dst$. Immediately following this increase is a much larger decrease in $Dst$ in a period of several hours to one day. This period is called the main phase. During the main phase, the ring current in the inner magnetosphere increases. The final phase is the recovery phase, which lasts from about one day to several days. In this period the $Dst$ value recovers gradually back to pre-storm levels as the ring current decays.

1.3 Radiation belt and high energy electrons

As mentioned earlier, the radiation belts are energetic ions and electrons trapped by the Earth's magnetic field. Figure 1-3 is a schematic plot of the radiation belts. The low altitude boundary of the radiation belt is about 200-1000 km, due to collisions between
the trapped particles and the atmosphere. The outer boundary is close to the magnetopause where the solar wind induced variations remove particles from trapped motion. The electron belt is divided into an inner belt and an outer belt by a slot region between equatorial distances of about 2-2.5 $R_E$ from the earth. The slot region is formed due to pitch angle scattering of the electrons into the loss cone. In this thesis we focus primarily on the dynamics of the outer zone radiation belt electrons. The outer zone MeV electrons are highly variable and the inner zone electrons are relatively stable (except in extreme events that can fill the slot region and increase inner zone fluxes).

![Diagram of the inner and outer Radiation Belt]

Figure 1-3 The inner and outer Radiation Belt [Kivelson and Russell, 1995]

These currently unpredictable relativistic electron flux enhancements are potentially hazardous during space missions, causing radiation damage to spacecraft instrumentation and presenting a radiation hazard to astronauts [Baker, et al., 1994; Baker, et al., 1998; Space Studies Board, 2000; Gussenhoven, et al., 1991].

A number of approaches to radiation belt modeling have been proposed. One fundamental method is the direct simulation of test particle motion in the electric and
magnetic fields obtained from global MHD simulations (more details in Chapter 6). Such "MHD-particle" simulations are able to provide global pictures of the dynamic evolution of radiation belt flux over a variety of time scales and geomagnetic conditions [Elkington, et al., 2003; Hudson, et al., 1998]. Results from an MHD-particle simulation of the September 24-26, 1998 geomagnetic storm are shown in Figure 1-4. This storm is characterized by high solar wind speed and an order-of-magnitude increase in outer zone energetic electron fluxes. The storm sudden commencement led to a limited electron injection and energization, however the time scale of bulk acceleration suggests another acceleration mechanism was active.
Figure 1-4  Electron flux profiles of September 24, 1998 storm.

Flux is plotted in color vs energy and radial distance at local midnight. Grey contours indicate lines of constant first adiabatic invariant. [Elkington, 2000]

The flux plotted is measured at local midnight from the MHD-particle simulation. Grey contours indicate lines of constant first adiabatic invariant. The three panels are at three different times: pre-storm, main phase and recovery phase. The simulation results suggest diffusive evolution of electrons, conserving the first adiabatic invariant, over the course of the storm, but more work is needed to verify this (see Chapter 6).

Recent work has shown that large amplitude Pc5 waves are correlated with electron events and there is increasing evidence that the Pc5 ULF oscillations (in the 1.5 to 10
mHz frequency range) play a fundamental role in storm-time particle dynamics over periods of hours and longer [Baker, et al., 1998; Elkington, et al., 1999; , 2003; Rostoker, et al., 1998]. In keeping with these results, a probable mechanism for the diffusive evolution in the MHD-particle simulation is the drift-resonant interaction of MeV electrons and ULF waves. Local acceleration / cyclotron resonances with VLF waves is also important [Horne, et al., 2003; Summers, et al., 2002; Summers, et al., 1998].

The diffusive transport of relativistic electrons in the radiation belts can be modeled by a general Fokker-Planck equation which describes the evolution of phase space density in the coordinate space of the three adiabatic invariants [Beutier and Boscher, 1995; Bourdarie, et al., 1997; Schulz and Lanzerotti, 1974]. When the first two adiabatic invariants are conserved, this general equation reduces to the radial diffusion equation. Radial diffusion calculations with semi-empirical diffusion coefficients have been used to model MeV electron fluxes at synchronous orbit [Li, et al., 2001]. Recent analysis of particle drift resonance with ULF waves in [Elkington, et al., 2003] found electron resonance acceleration due to toroidal and poloidal MHD waves and resonance modes in the asymmetric background magnetic field.

1.4 Outline of this Thesis

The radiation belts are able to maintain their population while particles are lost into the slot regions and the atmosphere at low altitude, so there must be sources and transport to maintain the belt and to re-populate the belts after depletions. The major possible sources are lower energy electrons in the magnetosphere which may be accelerated to MeV energies, and particles from substorm injection. Electrons have been observed to be
accelerated to MeV energies during magnetic storms. A number of transport or acceleration mechanisms have been proposed including high frequency wave cycrotron resonance acceleration (local heating) [Horne, et al., 2003; Summers, et al., 2002; Summers, et al., 1998], and radial diffusion that brings in electrons from outer regions and increase their energy.

In this work I use phase space density as a function of adiabatic invariants to investigate non-adiabatic transport processes, and use a radial diffusion model based on electron drift resonant in ULF waves to study particle transport and acceleration in the radiation belts. Chapter 2 I derived ULF wave driven radial diffusion coefficients which are different from previous works. In Chapter 3 I described a newly-developed numerical radial diffusion code and used it to study phase space density evolution due to different idealized sources and loses. L-Shell and phase space density calculation methods are discussed in Chapter 4, including the conversion between phase space density and observational data (particle flux). Using these tools, and flux observations from satellites, we introduce a new method for determining the phase space density gradient at geosynchronous orbit in Chapter 5. In Chapter 6 a radial diffusion calculation of a real event (the September 24-26, 1998 magnetic storm) is compared to a MHD-particle simulation. Chapter 7 gives a summary and presents the main conclusions of the thesis.
Chapter 2

Radial Diffusion Coefficients for Radiation Belt Transport

2.1 Energetic electron motion in the inner magnetosphere

In this section I briefly summarize well-known results from the theory of guiding center motion, adiabatic invariants and diffusive transport. The fundamental equation that describes the motion of a particle of charge $q$ and mass $m$ in magnetic and electric fields $B$ and $E$ is the Lorentz force equation:

$$\frac{dp}{dt} = q (v \times B + E)$$ (2-1)

Here, in the usual notation, $v$ is the particle velocity and $p=mv$ is the relativistic momentum, where $\gamma = \left(1 - v^2 / c^2 \right)^{-1/2}$ is the Lorentz factor. The particle motion can be decomposed into components parallel and perpendicular to the magnetic field. When the gyroradius of the particle is small compared to the scale length of the electric and magnetic fields (i.e., $\rho / L \ll 1$), the perpendicular motion consists of rapid gyromotion around the 'guiding center' and the motion of the guiding center itself [Northrop, 1963]. The parallel motion consists of an $E_n$ force and the so-called mirror force which repels particles from strong-field regions.
The drifting of a particle’s guiding center can be caused by the presence of an electric field, the gradient of the magnetic field, or the curvature of the magnetic field. The velocity can be decomposed into the following components:

(1) The electric drift term $V_E = \frac{E \times B}{B^2}$, in the direction perpendicular to both $E$ and $B$.

(2) Particles with parallel velocity $v_\parallel \neq 0$ will experience curvature drift. When $\nabla \times B = 0$, the curvature drift velocity is $V_c = \frac{\gamma m v_\parallel^2}{q B^3} (B \times \nabla B)$.

(3) And for particles that have velocity component $v_\perp$ perpendicular to the direction of magnetic field the gradient drift $\nabla B$ velocity is $V_G = \frac{\gamma m v_\perp^2}{2 q B^3} (B \times \nabla B)$.

The total perpendicular guiding center drift velocity is the sum of all these components: $V_D = V_E + V_C + V_G$.

The drift and mirror motions are essential for understanding the trapped motion of particle in the Earth’s magnetic field in the inner magnetosphere. An energetic trapped particle gyrates around the geomagnetic field line, at the same time it bounces parallel along the field line between mirror points, and it drifts around the Earth by moving across magnetic field lines. These three types of motion are illustrated in Figure 2-1.
Figure 2-1 Charged particle motion in the inner magnetosphere [Space Weather at Rice University]

Figure 2-2 shows the trajectories of equatorially-mirroring particles with zero electric field (neglecting Shabansky orbits, in which the trajectories of equatorially-mirroring particles near the magnetopause can jump to high magnitude latitudes across the dayside [Shabansky, 1971]). For equatorial mirroring particles $v_{\parallel} = 0$, so $V_c = 0$. If the electric field $E$ is zero then the particle motion is only gradient drift. Note that the gradient drift is in a direction perpendicular to $B$ and $\nabla B$. Hence this drift will carry particles along a contour of constant $B$. This characteristic is useful when tracing the guiding center drift paths of particles near the Earth’s magnetic equatorial plane. An example of the equatorial const-$B$ contours of Earth’s magnetic field is shown in Figure 2-2 (based on the Tsyganenko 2001 magnetic field model [Tsyganenko, 2002a; Tsyganenko, 2002b]).
Figure 2-2 Constant magnetic field contour generated using Tsyganenko 01 model

The trajectories are asymmetric, generally further away from the Earth on the dayside than the night side. This is due to the asymmetry of the background magnetic field, the dayside magnetosphere is compressed by the solar wind. In this plot, the dayside magnetopause is located at roughly $10 \, R_E$, and the last closed particle drifting orbit is roughly at $9 \, R_E$ at noon-side, and $7 \, R_E$ at midnight.

2.2 Three adiabatic invariants

For trapped electrons, the gyro, bounce and drift motions have three widely-separated time scales, and associated with each of these motions is an adiabatic invariant. Generally, an adiabatic invariant is the integral of canonical momentum over a periodic
orbit. Slow variations of the external fields will not change the value of corresponding adiabatic invariant.

The canonical momentum of a relativistic charged particle in the magnetic field is

\[ \mathbf{P} = m\gamma v + q\mathbf{A}, \]  

(2-2)

where \( \mathbf{A} \) is the vector potential.

By integrating \( \mathbf{P} \) over the particle’s gyro orbit it can be shown that the action integral

\[ J_1 \equiv \frac{1}{2\pi} \oint [\mathbf{p} + q\mathbf{A}] \cdot d\mathbf{l} \]  

is equal to \( \frac{p_{\perp}^2}{2qB} \). It is conventional to define the first adiabatic invariant

\[ M = \frac{q}{m} J_1 = \frac{p_{\perp}^2}{2mB}. \]  

(2-3)

The first adiabatic invariant is conserved when the variation in the magnetic and electric field is much slower than the gyro period, and the spatial change of the field is of a scale much larger than the gyro radius.

The conservation of the first adiabatic invariant implies that by moving a particle from a region of weaker magnetic field to a region of stronger magnetic field, the particle will gain energy in order to keep \( \frac{p_{\perp}^2}{B} \) constant.

The second adiabatic invariant, commonly written as \( J \), is the integral of the particle bounce motion between the two mirror points: \( J_2 = \oint p_{\perp} \, ds \) (since \( \oint \mathbf{A} \cdot d\mathbf{l} \) is zero for motion along the field line). A more convenient quantity is

\[ K = \frac{J}{2\sqrt{2m_0\mu}} = \int (B_m - B)^{\frac{1}{2}} \, ds \]  

[Roederer, 1970].
The third type of periodic motion is the particle's longitudinal drift around the Earth. The integral over a drift trajectory is

\[ J_3 = \oint (p + qA) \cdot dl = q\oint (\nabla \times A) \cdot dl = q\oint B \cdot ds, \]

(2-4)

where \( \Phi \) is the magnetic flux enclosed by the drift path and in the drift integral \( p \cdot dl \) is negligible compared to \( qA \cdot dl \) by a factor of order \( \rho/L_0 \). Since \( q \) is constant, \( \Phi \) is also an adiabatic invariant.

The three adiabatic invariants constitute an optimal coordinate system for describing the dynamics of the geo-magnetically trapped. If one or more of the adiabatic invariants are broken (for example, by variations of electric and magnetic fields) a particle will be transported to a new value of the invariant. The drift period is larger than the bounce period which is larger than the gyro period. When the time scale of field fluctuation is close to the drift period, the third adiabatic invariant will be broken. While the first and second adiabatic invariants are still conserved.

### 2.3 Diffusion equation

The particle distribution function in the 6-dimensional position-momentum phase space \( (q, p) \) is called phase space density (PSD). In the coordinate space of the 3 adiabatic invariants, we can write \( f = f(J_1, J_2, J_3, t) \), or \( f(M, K, \Phi, t) \). Note that this representation contains no phase information. If the particles are uniformly distributed in all phase angles, such phase information is not needed. Also, the PSD can be averaged over all phase angles. Spacecraft instruments typically measure differential flux at a
given energy $\varepsilon$ at a given position $\mathbf{r}$, $j(\varepsilon, \mathbf{r}, t)$, rather than PSD as a function of the adiabatic invariants. The relations of PSD to differential flux will be discussed in Chapter 4.

If the three adiabatic invariants are all conserved, then Liouville's theorem implies that the distribution function $f(M, J, \Phi)$ will remain unchanged. Alternatively, particle transport due to the breaking of one or more of the adiabatic invariants can be represented by a diffusion equation of the form [Haerendel, 1968]

$$\frac{\partial f}{\partial t} + \sum_i \frac{\partial}{\partial J_i} [\Gamma_i f] = \sum_{i,j} \frac{\partial}{\partial J_i} \left[ D_{ij} \frac{\partial f}{\partial J_j} \right] - \text{loss} + \text{source},$$  

where here $f$ is the PSD averaged over gyro, bounce and drift angles, $J_i$ is the $i$th adiabatic invariant, with $i = 1, 2, 3$, $\Gamma_i$ and $D_{ij}$ are drag and diffusion transport coefficients, and possible loss and source terms are included generically. As mentioned above, $J_1, J_2, J_3$ are proportional to $M$, $K$, and $\Phi$ [Schulz, 1996], and for equatorially-mirroring particles $K = 0$.

It is convenient to use the Roederer $L$ value instead of $\Phi$ as the third adiabatic invariant [Roederer, 1970]. This is defined as:

$$L = \frac{2\pi k_0}{R_E \Phi},$$  

where $k_0 = B_E R_E^3$ is the Earth dipole moment ($B_E = 0.32$ gauss, $R_E$ is the Earth radius). For a simple dipole magnetic field, $L$ reduces to $R / R_E$, the radial distance in units of Earth radius. For more realistic magnetic field models, the calculation of $L$ involves integrating the magnetic flux $\Phi$ enclosed by the particle's drift orbit. Then the
value of $L$ is generally not equal to $R/R_E$, and it varies with time depending on the geomagnetic conditions. In Chapter 4 I will give a more detailed discussion of the calculation of Roederer $L$ value and phase space density.

Assuming $M$ is conserved, $K = 0$ (equatorially-mirroring particles), and ignoring drag and non-stochastic processes, the diffusion equation is reduced to a pure radial diffusion equation:

$$\frac{\partial f}{\partial t} = L^2 \frac{\partial}{\partial L} \left[ \frac{D_{LL}}{L^2} \frac{\partial f}{\partial L} \right] - \text{loss + source}. \quad (2-7)$$

Here $f$ is the phase space density as a function of $M$, $K$, and $L$, and $D_{LL}$ is the radial diffusion coefficient given by $D_{LL} = \frac{\langle (\Delta L)^2 \rangle}{2\tau}$, the ensemble-averaged squared displacement in $L$, divided by an appropriate time (this definition will be discussed further in this chapter). We choose $f$ to have units of particle density per unit $\vec{x}, \vec{p}$ space.

The numerical method for solving this diffusion equation will be presented in Chapter 3. In Chapter 6, the simulation study of the September 1998 magnetic storm will solve this radial diffusion equation with ULF-wave-driven diffusion coefficients, and with time-dependent outer boundary locations and values.

The main objective of the remainder of this chapter is to derive analytical form of the radial diffusion coefficient $D_{LL}$.

### 2.4 Electric radial diffusion coefficients
Many previous works have tried to specify the radial diffusion coefficients, either analytically or empirically. Empirical methods estimate the diffusion coefficients by fitting with observational data [Brautigam and Albert, 2000; Selesnick, et al., 1997], and analytical methods derive the diffusion coefficients from the first principles of particle motion [Brizard and Chan, 2001; , 2004; Elkington, et al., 1999; , 2003; Fälthammar, 1965; , 1968; Schulz and Evitar, 1969; Schulz and Lanzerotti, 1974]. Empirical diffusion coefficients are normally functions of observable geomagnetic parameters such as $Kp$ index, solar wind parameters, $Dst$ index, etc, while analytical diffusion coefficients are physics-based and are directly related to the characteristics of the electric and magnetic fields, such as the wave’s power spectral density. The analytical diffusion coefficients are potentially better because they are a function of more fundamental quantities and their parameters contain more comprehensive information (e.g., a geomagnetic parameter like $Kp$ is a 3-hour averaged index, while power spectral density is function of time and radial distances or $L$ values).

The Pc5 ULF wave pulsations have been proposed [Rostoker, et al., 1998] as a likely driver of the acceleration of electrons in the Earth’s outer radiation belt. There is observational evidence that prolonged periods of enhanced Pc5 waves are associated with relativistic electron flux enhancements [Iles, 2001; Mathie and Mann, 2000; O'Brien, et al., 2001; Rostoker, et al., 1998]

Early work [Fälthammar, 1965; Schulz and Lanzerotti, 1974] studied radial diffusion by stochastic variations of the electric and magnetic fields in a symmetric dipole magnetic field. This yielded radial diffusion coefficients that are proportional to wave power spectral densities at drift resonant frequencies. Later work by Elkington et, al
[1999; 2003] discovered that particles drifting in asymmetric background magnetic fields can experience radial diffusion in new resonance modes, and given an estimation of the corresponding diffusion coefficients.

In this work, I have derived analytical radial diffusion coefficients. In addition to reproducing the early results of Fälthammar [1965], in this work the effects of asymmetric background fields are included. The derivations also improve upon the estimations by [Elkington, et al., 1999; , 2003]. The newly-derived analytic diffusion coefficients are compared with numerical calculations (test particle calculations in model fields) later in the chapter.

In this section I will first derive the electric diffusion coefficients, and the magnetic diffusion coefficients will be presented in the following section.

2.4.1 Field model

The electric field variations perturb the drift motion of the radiation belt electrons and may cause radial diffusion. The electric fields may include potential electric field and induced electric field contributions. We can assume the perturbation fields to be random functions of time, with time-independent statistical properties. In other words, they can be considered as realizations of stochastic processes that are individually and jointly stationary and ergodic. These are the same assumptions as made in Fälthammar [1965]. We will assume the particles are confined to the equatorial plane and we will use standard polar coordinates \((r, \phi)\) in that plane. We decompose the perturbation electric field into radial and azimuthal components:
\[ E(r, \phi, t) = E_\phi \hat{\phi} + E_r \hat{r}. \]  
(2-8)

Both components can be further expanded into Fourier series with azimuthal mode number \( m \):

\[ E_\phi = \sum_mE_{\phi m}(r, t) \cos(m\phi + \gamma_{\phi m}), \text{ and} \]  
(2-9)

\[ E_r = \sum_mE_{rm}(r, t) \cos(m\phi + \gamma_{rm}). \]  
(2-10)

The Fourier amplitudes \( E_{\phi m} \) and \( E_{rm} \) are random functions of time, and \( \gamma_{\phi m} \), \( \gamma_{rm} \) are phase constants.

We use the following day-night asymmetric equatorial magnetic field model as the unperturbed magnetic field and we assume the direction of the magnetic field is perpendicular to the equatorial plane.

\[ B(r, \phi) = \frac{B_E R_E^3}{r^3} + \Delta B \cos \phi. \]  
(2-11)

\[ = B_0(r) + \Delta B \cos \phi \]

In equation (2-11) the first term represents the Earth’s internal dipole magnetic field, where \( B_E \) is the magnetic field strength at Earth surface (0.32 gauss). The second term represents the day-night asymmetry of the compressed magnetic field. Here \( \phi = 0 \) at local noon. The value of \( \Delta B \) can be estimated from measurements of magnetic field values, or from a magnetic field model. In this analytical derivation I assume that \( \Delta B \ll B_0 \) (i.e., a “weak asymmetry” assumption).
2.4.2 Particle motion

The guiding center drift velocity of equatorial mirroring particles is:

\[ v_d = \frac{E \times B}{B^2} + \frac{M}{q\gamma B^2} B \times \nabla B. \]  (2-12)

Here \( M = \frac{p_z^2}{2m_B} \) is the first adiabatic invariant. The drift velocity consists of ‘E cross B’ drift and ‘Grad B’ drift.

For high-energy radiation belt particles, the E cross B drift velocity is small compared to the gradient drift. Neglecting Shabansky orbits [Shabansky, 1971], the grad B drift will keep a particle of constant M on the contour of constant B. Following Elkington [2000] we assume the constant-B value on the drift trajectory is given by \( B_d(L) = \frac{B_E}{L^3} \).

Then, using \( \Delta B \ll B_0 \), the drift path can be expressed as a function of its \( L \) value [Elkington, 2000] and azimuthal angle:

\[ r(\phi) = L R_E \left( 1 - \frac{\Delta B}{B_E} L^3 \cos \phi \right)^{-1/3}. \]  (2-13)

2.4.3 Cross L shell motion

The wave perturbations can cause the particle drift velocity to have a component perpendicular to the unperturbed drift orbit. This velocity component causes the particle to deviate from the original \( L \)-shell. We can calculate this deviation from a dot product of the guiding center drift velocity and the normal vector of the unperturbed drift orbit:
\[ \frac{d\rho}{dt} = v_d \cdot \hat{n}_B. \]  \hspace{1cm} (2-14)

Here \( \hat{n}_B \) is the unit vector normal to the unperturbed drift orbit (which is the same as a constant-\( B \) contour), and \( \rho \) is the distance of deviation in the direction perpendicular to the \( L \)-shells (not to be confused with gyro radius). Figure 2-3 illustrates the particle’s drift orbit in an asymmetric magnetic field.

![Diagram of asymmetric drift orbit]

**Figure 2-3** Asymmetric drift orbit

The dashed circles are of constant radial distances; the solid curve is the particle drift orbit.

From the gradient of the magnetic field strength

\[ \nabla B = \frac{\partial B}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial B}{\partial \phi} \hat{\phi} = -\frac{3B_0}{r} \hat{r} - \frac{\Delta B \sin \phi}{r} \hat{\phi}, \]  \hspace{1cm} (2-15)
one can calculate the normal direction of the $L$-shells (contours of constant $B$ value) as:

$$\hat{n}_B = -\frac{\nabla B}{|\nabla B|} = \frac{3B_0}{\sqrt{9B_0^2 + \Delta B^2 \sin^2 \phi}} \hat{r} + \frac{\Delta B \sin \phi}{\sqrt{9B_0^2 + \Delta B^2 \sin^2 \phi}} \hat{\phi}$$

(2-16)

Next, consider the effect of the electric field perturbation on the particle’s unperturbed drift motion. The drift velocity due to electric field is $v_E = \frac{\vec{E} \times \vec{B}}{B^2}$, and the rate of deviation from the unperturbed orbit is:

$$\frac{d\rho}{dt} = v_E \cdot \hat{n}_B = \frac{\vec{E} \times \vec{B}}{B^2} \cdot \hat{n}_B.$$  

(2-17)

Substituting Equation (2-16) into Equation (2-17), and keeping the first order terms of $\frac{\Delta B}{B_0}$, we find

$$\frac{d\rho}{dt} = \frac{1}{B} E_\phi - \frac{\Delta B \sin \phi}{3B^2} E_r$$

(2-18)

The first term does not depend on the asymmetry factor $\Delta B / B$. For a symmetric background field, only the first term remains. From now on, we will name this term the ‘symmetric’ term. It agrees with equation 2.2-(2) in [Fälthammar, 1965] which is derived using simple dipole field model. We name the second term which contains $\Delta B / B_0$ as the ‘asymmetric’ term.
We want to calculate \( \frac{dL}{dt} \) from \( \frac{d\rho}{dt} \). The following drawing (Figure 2-4) illustrates the relation between \( dL \) and \( d\rho \). Recall that \( dL \) is the change in \( L \) value, \( d\rho \) is the change in distance.

![Figure 2-4 Relation between \( dL \) and \( d\rho \)]

Solid lines are the original \( L \)-shell and the new \( L \)-shell; the dotted line is a circle centered at the earth. The solid arrow is in the direction of \( d\rho \), the dotted arrow is the \( dr \) direction.

First we calculate \( \frac{\Delta r}{\Delta \rho} \). Geometrically, \( \Delta r \) and \( \Delta \rho \) are the distances between two close \( L \) shells measured in two different ways: \( \Delta r \) is measured in the radial direction \( \hat{r} \) (with constant \( \phi \)), while \( \Delta \rho \) is measured in the normal direction \( \hat{n}_B \) (the shortest
geometric distance between two $L$-shells). Define $\theta$ as the angle between $\hat{n}_B$ and $\hat{r}$, as illustrated in Figure 2-4. From equation (2-16) and $\hat{n}_B \cdot \hat{r} = \cos \theta$,

$$
\cos(\theta) = \frac{3B_0}{\sqrt{9B_0^2 + \Delta B^2 \sin^2 \phi}}.
$$

(2-19)

Then

$$
\frac{\Delta r}{\Delta \rho} = \frac{1}{\cos(\theta)} \approx 1 + \frac{1}{6} \left( \frac{\Delta B}{B_0} \right)^2 \sin^2 \phi.
$$

(2-20)

The difference between $\Delta r$ and $\Delta \rho$ is larger when $\phi$ is close to 90 and 270 degrees, and this difference is of second order in $\frac{\Delta B}{B_0}$.

So, to first order accuracy in $\frac{\Delta B}{B_0}$, $\frac{\Delta r}{\Delta \rho} = 1$.

Next, we calculate $\frac{dr}{dL} \big|_{\phi}$. Using the relation between $r$ and $L$ as in Equation (2-13), to first order we have

$$
\frac{dr}{dL} = R_E \left( 1 + \frac{4}{3} \frac{\Delta B}{B_E} L^3 \cos \phi \right),
$$

(2-21)

the difference between $r$ and $L$ is of first order, largest at noon and midnight.

Thus, by using the chain rule and keeping terms of first order in $\Delta B / B_0$, we have:
\[
\frac{dL}{dt} = \frac{d\rho}{dt} \frac{\Delta r}{\Delta \rho} \frac{dL}{dr} = \frac{d\rho}{dt} \frac{1}{R_E} \left( 1 - \frac{4}{3} \frac{\Delta B}{B_E} L^3 \cos \phi \right).
\]  
(2-22)

Applying equations (2-18) and (2-20), and keeping first order terms:

\[
\frac{dL}{dt} = \frac{1}{B_E R_E} L^3 \cdot E_\phi - \frac{1}{3} \frac{\Delta B \sin \phi}{B_E^2 R_E} L^6 \cdot E_r - \frac{4}{3} \frac{\Delta B \cos \phi}{B_E^2 R_E} L^6 \cdot E_\phi.
\]  
(2-23)

This is the rate of \( L \)-shell deviation caused by electric field perturbations. The first term is the symmetric term and the second and third terms are asymmetric terms.

We allow \( E_r \sim E_\phi \) for ULF waves. Then the third term is a factor of 4 larger than the second term. This factor becomes a factor of 16 in the final diffusion coefficient and so for simplicity we ignore the second term, leaving

\[
\frac{dL}{dt} = \frac{1}{B_E R_E} L^3 \cdot E_\phi - \frac{4}{3} \frac{\Delta B \cos \phi}{B_E^2 R_E} L^6 \cdot E_\phi.
\]  
(2-24)

The effect of the second term on the diffusion coefficient is easily added later if needed (for example, if \( E_r > 4E_\phi \)).

Using the expression for electric field in equation (2-9), equation (2-24) becomes:

\[
\frac{dL}{dt} = \frac{L^3}{B_E R_E} \cdot \sum_m E_{\phi m}(r,t) \cdot \cos (m\phi + \gamma_m)

- \frac{4}{3} \frac{\Delta B \cos \phi}{B_E^2 R_E} L^6 \cdot \sum_m E_{\phi m}(r,t) \cdot \cos (m\phi + \gamma_m).
\]  
(2-25)
2.4.4 Effect of asymmetric angular velocity

In equation (2-25), $\phi$ is the angular location of the particle, $\phi - \phi_0 = \int \Omega \, dt$. Here $\Omega$ is the particle angular velocity and $\phi_0$ is the initial azimuthal angle of the particle. The grad B drift velocity for a particle is $v_d = \frac{M}{q\gamma B^2} B \times \nabla B$. Then the angular velocity is

$$\Omega(\phi) = \frac{v_d \cdot \hat{\phi}}{r(\phi)} = \frac{1}{r(\phi) q\gamma} \frac{M B \times \nabla B \cdot \hat{\phi}}{B^2}, \quad (2-26)$$

where $r(\phi)$ is the particle’s radial position given in equation (2-13).

From equation (2-11)

$$B(r,\phi) = (B_0 + \Delta B \cos(\phi)) \hat{z},$$

so

$$\nabla B = \frac{\partial B}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial B}{\partial \phi} \hat{\phi} = -\frac{1}{r} \left( 3B_0 \hat{r} + \Delta B \sin \phi \hat{\phi} \right)$$

Substituting this into equation (2-26), the angular velocity becomes

$$\Omega(\phi) = \frac{1}{r(\phi)} \left[ \frac{M}{q\gamma} \frac{(B_0 + \Delta B \cos \phi) \hat{z} \times \left( -\frac{1}{r} \right) \left( 3B_0 \hat{r} + \Delta B \sin \phi \hat{\phi} \right)}{(B_0 + \Delta B \cos \phi)^2} \right] \cdot \hat{\phi}$$

$$= -\frac{1}{r(\phi)^2 q\gamma (B_0 + \Delta B \cos \phi)} \frac{3B_0}{(B_0 + \Delta B \cos \phi)}$$

$$\approx -\frac{1}{r(\phi)^2 q\gamma} \frac{M}{3} \left( 1 - \frac{\Delta B}{B_0} \cos \phi \right)$$

Using $r(\phi) = LR_E \left( 1 - \frac{\Delta B}{B_0} \cos \phi \right)^{-1/3}$ as in equation (2-13),
\[ \Omega (\phi) \approx -\frac{1}{L^2 R_E^2 q \gamma} \frac{M}{3} \left( 1 - \frac{5}{3} \frac{\Delta B}{B_0} \cos \phi \right), \text{ or} \]

\[ \Omega (\phi) \approx \Omega_0 \left( 1 - \frac{5}{3} \frac{\Delta B}{B_0} \cos \phi \right). \quad (2-28) \]

Here \( \Omega_0 = \frac{3}{L^2 R_E^2 q \gamma} \frac{M}{3} \) is the angular velocity of the particle in a symmetric dipole magnetic field, which is constant.

Integrating to obtain the angular location as a function of time:

\[ \phi (t) \approx \Omega_0 t - \frac{5}{3} \frac{\Delta B}{B_0} \sin (\Omega_0 t + \phi_0) + \phi_0 \quad (2-29) \]

The first term is the same as in a symmetric dipole magnetic field, and the second term is a first order correction.

Then

\[
\cos (m \phi + \gamma_m) = \cos \left( (m \Omega_0 t + m \phi_0 + \gamma_m) - m \frac{5}{3} \frac{\Delta B}{B_0} \sin (\Omega_0 t + \phi_0) \right) \\
\approx \cos (m \Omega_0 t + m \phi_0 + \gamma_m) + m \frac{5}{3} \frac{\Delta B}{B_0} \sin (m \Omega_0 t + m \phi_0 + \gamma_m) \sin (\Omega_0 t + \phi_0) \quad , \quad (2-30)
\]

and

\[
\sin (m \phi + \gamma_m) \\
\approx \sin (m \Omega_0 t + m \phi_0 + \gamma_m) - m \frac{5}{3} \frac{\Delta B}{B_0} \cos (m \Omega_0 t + m \phi_0 + \gamma_m) \sin (\Omega_0 t + \phi_0) \quad . \quad (2-31)
\]

Submitting equation (2-30) into equation (2-25), the first term in equation (2-25) becomes:
\[ \frac{L^3}{B_E R_E} \cdot \sum E_{m,\phi} \cos (m\phi) = \frac{L^3}{B_E R_E} \cdot \sum E_{m,\phi} \cos (m\Omega_0 t + m\phi_0 + \gamma_m) \]
\[ + \frac{5}{3} \frac{\Delta B}{B_E^2 R_E} L^6 \cdot \sum mE_{m,\phi} \sin (m\Omega_0 t + m\phi_0 + \gamma_m) \sin (\Omega_0 t + \phi_0) \]  

(2-32)

The second term in equation (2-25) is already a first order term, so we can simply use \( \phi = \Omega_0 t + \phi_0 \) for that term.

Then equation (2-25) becomes

\[
\frac{dL}{dt} = \frac{L^3}{B_E R_E} \cdot \sum E_{\phi m}(r, t) \cdot \cos (m\Omega_0 t + m\phi_0 + \gamma_m) 
\]
\[ + \frac{5}{3} \frac{\Delta B}{B_E^2 R_E} L^6 \cdot \sin (\Omega_0 t + \phi_0) \sum mE_{\phi m}(r, t) \sin (m\Omega_0 t + m\phi_0 + \gamma_m). \]
\[ - \frac{4}{3} \frac{\Delta B}{B_E^2 R_E} L^6 \cdot \cos (\Omega_0 t + \phi_0) \sum mE_{\phi m}(r, t) \cdot \cos (m\Omega_0 t + m\phi_0 + \gamma_m) \]  

(2-33)

The second term is due to the asymmetry of the drift angular velocity (time asymmetry), and the third term is due to the asymmetry of the geometry of the drift orbit (geometry asymmetry).

Using

\[
\sin (m\Omega_0 t + m\phi_0 + \gamma_m) \cdot \sin (\Omega_0 t + \phi_0) 
\]
\[ = -\frac{1}{2} \cdot (\cos ((m+1)(\Omega_0 t + \phi_0) + \gamma_m) - \cos ((m-1)(\Omega_0 t + \phi_0)) + \gamma_m) \]

\[
\cos (m\Omega_0 t + m\phi_0 + \gamma_m) \cdot \cos (\Omega_0 t + \phi_0) 
\]
\[ = \frac{1}{2} \cdot (\cos ((m+1)(\Omega_0 t + \phi_0) + \gamma_m) + \cos ((m-1)(\Omega_0 t + \phi_0)) + \gamma_m) \]

equation (2-33) becomes
\[
\frac{dL}{dt} = \frac{L^3}{B_E R_E} \cdot \sum_m E_{\phi m}(r, t) \cdot \cos(m(\Omega_0 t + \phi_0) + \gamma_m) \\
-\left(\frac{\Delta B}{B_E}\right) \frac{L^5}{B_E R_E} \cdot \sum_m \left(\frac{5}{6} m + \frac{2}{3}\right) E_{\phi m}(r, t) \cos((m + 1)(\Omega_0 t + \phi_0) + \gamma_m). \\
+\left(\frac{\Delta B}{B_E}\right) \frac{L^6}{B_E R_E} \cdot \sum_m \left(\frac{5}{6} m - \frac{2}{3}\right) E_{\phi m}(r, t) \cos((m - 1)(\Omega_0 t + \phi_0) + \gamma_m). 
\] (2-34)

2.4.5 Radial diffusion coefficients

Both the symmetric term and asymmetric term depend on the perturbation electric field \(E_\phi(t)\), so they are correlated and should be treated together. Define

\[
u_1 = \frac{L^3}{B_E R_E} \cdot \sum_m E_{\phi m}(r, t) \cdot \cos(m(\Omega_0 t + \phi_0) + \gamma_m),
\] (2-35)

and

\[
u_2 = -\left(\frac{\Delta B}{B_E}\right) \frac{L^5}{B_E R_E} \cdot \sum_m \left(\frac{5}{6} m + \frac{2}{3}\right) E_{\phi m}(r, t) \cos((m + 1)(\Omega_0 t + \phi_0) + \gamma_m) \\
+\left(\frac{\Delta B}{B_E}\right) \frac{L^6}{B_E R_E} \cdot \sum_m \left(\frac{5}{6} m - \frac{2}{3}\right) E_{\phi m}(r, t) \cos((m - 1)(\Omega_0 t + \phi_0) + \gamma_m),
\] (2-36)

Then

\[
\frac{dL}{dt} = \nu_1 + \nu_2.
\] (2-37)

Since \(D_{LL}\) at \(L = L_0\) is defined as \(D_{LL} = \frac{\langle (L - L_0)^2 \rangle}{2t}\), where \(\langle \ldots \rangle\) denotes an ensemble average, we wish to calculate \(\langle (L - L_0)^2 \rangle\). We calculate \(\langle (L - L_0)^2 \rangle\) from
a time integral of \( \frac{d}{dt} \langle (L - L_0)^2 \rangle \), which can be obtained using equation (2-37) as follows.

Integrating equation (2-37) once gives:

\[
L - L_0 = \int_{\xi=0}^{t} (u_1 + u_2) \, d\xi .
\]  
\[ (2-38) \]

We are free to interchange the ensemble average and the time derivative, so

\[
\frac{d}{dt} \langle (L - L_0)^2 \rangle = \left\langle \frac{d}{dt} (L - L_0)^2 \right\rangle = 2 \left\langle (L - L_0) \frac{dL}{dt} \right\rangle 
\]  
\[ (2-39) \]

\[
= 2 \left\langle (u_1(t) + u_2(t)) \int_{\xi=0}^{t} (u_1(\xi) + u_2(\xi)) \cdot d\xi \right\rangle 
\]

\[
= 2 \sum_{i,j=1,2} \int_{\xi=0}^{t} \left\langle u_i(t) u_j(\xi) \right\rangle \, d\xi .
\]  
\[ (2-40) \]

Integrating equation (2-40) gives

\[
\langle (L - L_0)^2 \rangle = \int_{\tau=0}^{t} \frac{d\langle (L - L_0)^2 \rangle}{dt} \, d\tau
\]

\[
= 2 \sum_{i,j=1,2} \int_{\tau=0}^{t} \int_{\xi=0}^{\tau} \left\langle u_i(\tau) u_j(\xi) \right\rangle \, d\xi \, d\tau
\]  
\[ (2-41) \]

Then the radial diffusion coefficient is:
\[
D_{LL} = \frac{1}{t} \int_{t=0}^{t} \int_{\xi=0}^{\tau} \langle u_1(\tau) u_1(\xi) \rangle \, d\xi \, d\tau \\
+ \frac{1}{t} \int_{t=0}^{t} \int_{\xi=0}^{\tau} \langle u_2(\tau) u_2(\xi) \rangle \, d\xi \, d\tau \\
+ \frac{1}{t} \int_{t=0}^{t} \int_{\xi=0}^{\tau} \langle u_2(\tau) u_1(\xi) \rangle \, d\xi \, d\tau \\
+ \frac{1}{t} \int_{t=0}^{t} \int_{\xi=0}^{\tau} \langle u_2(\tau) u_2(\xi) \rangle \, d\xi \, d\tau
\] (2-42)

This integration leads to a number of terms, and is quite complicated. Next, I will show the details of the integration of the first term.

(1) The \( u_1 u_2 \) term

Letting \( \xi = \tau - \eta \), we have \( d\xi = -d\eta \) and

\[
\int_{\xi=0}^{\tau} \langle u_1(\tau) u_1(\xi) \rangle \, d\xi = \int_{\eta=0}^{\tau} \langle u_1(\tau) u_1(\tau - \eta) \rangle \, d\eta.
\] (2-43)

So the contribution of the first term to \( D_{LL} \) is

\[
D_{LL}^{(1)} = \frac{1}{t} \int_{\tau=0}^{t} \int_{\eta=0}^{\tau} \langle u_1(\tau) u_1(\tau - \eta) \rangle \, d\eta \, d\tau
\] (2-44)

This term is the integration of the ‘self-correlation function’ of the symmetric term \( u_1 \).

It is similar to the result in [Falhammar, 1965], since in that work only a symmetric magnetic model is used.

Using the expression of \( u_1 \) shown in equation (2-35), the integrand is
\[
\langle u_1(\tau) u_1(\tau - \eta) \rangle = \frac{L^6}{B_E^2 R_E^2} \left( \sum_m E_{\phi m}(\tau) \cdot \cos(m\Omega_0 \tau + \varphi_m) \right) \cdot \left( \sum_n E_{\phi n}(\tau - \eta) \cdot \cos(n\Omega_0 (\tau - \eta) + \varphi_n) \right)
\]

where \( \varphi_m = m\phi_0 + \gamma_m \), \( \varphi_n = n\phi_0 + \gamma_n \).

Substituting this into equation (2-54), and using the identity
\[
\cos(A - B) = \cos A \cos B + \sin A \sin B,
\]
the integrand becomes
\[
\langle u_1(\tau) u_1(\tau - \eta) \rangle = \frac{L^6}{B_E^2 R_E^2} \left( \sum_m E_{\phi m}(\tau) \cdot \cos(m\Omega_0 \tau + \varphi_m) \right) \cdot \left( \sum_n E_{\phi n}(\tau - \eta) \cdot \left[ \cos(n\Omega_0 \tau + \varphi_n) \cos(n\Omega_0 \eta) + \sin(n\Omega_0 \tau + \varphi_n) \sin(n\Omega_0 \eta) \right] \right).
\]

Splitting the products into terms with \( m = n \) and \( m \neq n \) gives

\[
\langle u_1(\tau) u_1(\tau - \eta) \rangle = \frac{L^6}{B_E^2 R_E^2} \left\{ \sum_n \langle E_n(\tau) \cdot E_n(\tau - \eta) \rangle \cdot \cos^2(n\Omega_0 \tau + \varphi_n) \cdot \cos(n\Omega_0 \eta) \right. \\
+ \sum_n \langle E_n(\tau) \cdot E_n(\tau - \eta) \rangle \cdot \cos(n\Omega_0 \tau + \varphi_n) \cdot \sin(n\Omega_0 \tau + \varphi_n) \cdot \sin(n\Omega_0 \eta) \\
+ \sum_{m\neq n} \sum_n \langle E_n(\tau) \cdot E_m(\tau - \eta) \rangle \cdot \cos(n\Omega_0 \tau + \varphi_n) \cdot \cos(m\Omega_0 \tau + \varphi_m) \cdot \cos(n\Omega \eta) \\
+ \sum_{m\neq n} \sum_n \langle E_n(\tau) \cdot E_m(\tau - \eta) \rangle \cdot \cos(n\Omega_0 \tau + \varphi_n) \cdot \sin(m\Omega_0 \tau + \varphi_m) \cdot \sin(n\Omega \eta) \right\}.
\]

The ensemble average \( \langle E_n(t) \cdot E_m(t - \tau) \rangle \) is independent of \( t \) because of the stationary assumption, thus \( \langle E_n(t) \cdot E_m(t - \tau) \rangle = \langle E_n(t + \tau) \rangle \). The electric field \( E \) consists of an averaged part and a time-dependent part:
\[
E(t) = \langle E \rangle + \tilde{E}(t),
\]
then
\[
\langle E_s(t) \cdot E_m(t - \tau) \rangle = \langle E_m(t) \cdot E_s(t + \tau) \rangle \\
= \langle E_n \rangle \cdot \langle E_n \rangle + \langle \tilde{E}_n(t) \cdot \tilde{E}_n(t + \tau) \rangle
\] (2-46)

Next, substitute equation (2-46) into (2-45). The first term of the integration is proportional to

\[
\int_0^\infty \sum_n \langle E_n(\tau) \cdot E_n(\tau + \eta) \rangle \cdot \cos^2(n\Omega_0 \tau + \varphi_n) \cdot \cos(n\Omega_0 \eta) \cdot d\eta \\
= \sum_n \cos^2(n\Omega_0 \tau + \varphi_n) \cdot \int_0^\infty \left( \langle E_n \rangle \cdot \langle E_n \rangle + \langle \tilde{E}_n(\tau) \cdot \tilde{E}_n(\tau + \eta) \rangle \right) \cdot \cos(n\Omega_0 \eta) \cdot d\eta \\
= \sum_n \cos^2(n\Omega_0 \tau + \varphi_n) \cdot \left( \langle E_n \rangle^2 \cdot \frac{\sin(n\Omega_0 \tau)}{n\Omega_0} + \int \langle \tilde{E}_n(\tau) \cdot \tilde{E}_n(\tau + \eta) \rangle \cdot \cos(n\Omega_0 \eta) \cdot d\eta \right)
\]

Define

\[
I_{nn1} = \int \langle \tilde{E}_n(\tau) \cdot \tilde{E}_n(\tau + \eta) \rangle \cdot \cos(n\Omega_0 \eta) \cdot d\eta. 
\] (2-47)

By the stationary property \( \langle \tilde{E}_n(\tau) \cdot \tilde{E}_n(\tau + \eta) \rangle = \langle \tilde{E}_n(0) \cdot \tilde{E}_n(\eta) \rangle \). This autocorrelation function is assumed to be peaked at \( \eta = 0 \) with characteristic width \( \tau_{ac} \), the autocorrelation time. Then when \( \tau > \tau_{ac} \), \( I_{nn1} \) is constant with respect to \( \tau \).

With the definition (2-47) the first term of the integration is proportional to

\[
\sum_n \cos^2(n\Omega_0 \tau + \varphi_n) \cdot \left[ \langle E_n \rangle^2 \cdot \frac{\sin(n\Omega_0 \tau)}{n\Omega_0} + I_{nn1} \right].
\] (2-48)

The rest of the terms of the integration \( \int \langle u_i(\tau) u_i(\tau - \eta) \rangle \cdot d\eta \) can be found similarly.

The result is:
\[ \int_0^L \langle u_i(\tau) u_i(\tau - \eta) \rangle \cdot d\eta = \frac{L^5}{B_E^2 R_E^2} \{ \]

\[ \sum_n \cos^2(n\Omega_0^2 + \varphi_n) \cdot \langle \{ E_n^2 \} \cdot \frac{\sin(n\Omega_0^2 + \varphi_n)}{n\Omega_0^2} + I_{m1} \rangle \]

\[ + \sum_n \cos(n\Omega_0^2 + \varphi_n) \cdot \sin(n\Omega_0^2 + \varphi_n) \cdot \langle \{ E_n^2 \} \cdot \frac{1 - \cos(n\Omega_0^2 + \varphi_n)}{n\Omega_0^2} + I_{m2} \rangle \]  \hspace{1cm} (2-49)

\[ + \sum_{m, n, m} \cos(n\Omega_0^2 + \varphi_n) \cdot \cos(m\Omega_0^2 + \varphi_m) \cdot \langle \{ E_n \} \langle E_m \} \cdot \frac{\sin(m\Omega_0^2 + \varphi_m)}{m\Omega_0^2} + I_{m1} \rangle \]

\[ + \sum_{m, n, m} \cos(n\Omega_0^2 + \varphi_n) \cdot \sin(m\Omega_0^2 + \varphi_m) \cdot \langle \{ E_n \} \langle E_m \} \cdot \frac{1 - \cos(m\Omega_0^2 + \varphi_m)}{m\Omega_0^2} + I_{m2} \rangle \}

Here \( I_{m2} = \int_0^\tau \langle \bar{E}_n(\tau) \cdot \bar{E}_n(\tau + \eta) \rangle \cdot \sin(n\Omega_0^2\eta) \cdot d\eta \),

\( I_{m1} = \int_0^\tau \langle \bar{E}_n(\tau) \cdot \bar{E}_n(\tau + \eta) \rangle \cdot \cos(n\Omega_0^2\eta) \cdot d\eta \), and

\( I_{m2} = \int_0^\tau \langle \bar{E}_n(\tau) \cdot \bar{E}_n(\tau + \eta) \rangle \cdot \sin(n\Omega_0^2\eta) \cdot d\eta \) are constant with respect to \( \tau \) (when \( \tau \gg \tau_{ac} \)).

Compared to the expression for the radial diffusion coefficient defined in equation

\[ (2-44) \ D_{LL} = \frac{1}{t} \int_{\tau=0}^\tau \int_{\eta=0}^\tau \langle u_1(\tau) u_1(\tau - \eta) \rangle \ d\eta \ d\tau \], equation (2-49) needs to be integrated again:
\[ \int_{\tau=0}^{\tau} \int_{\eta=0}^{\eta} (u_i(\tau)u_i(\tau-\eta)) \, d\tau \, d\eta = \frac{L^6}{B E^2 R E^2} \{ \]
\[ \sum_n \int_0^\tau \cos^2(n\Omega_0 \tau + \varphi_n) \cdot \left( \langle E_n \rangle \right)^2 \cdot \frac{\sin(n\Omega_0 \tau)}{n\Omega_0} + I_{nn1} \} \, d\tau 
+ \sum_n \int_0^\tau \cos(n\Omega_0 \tau + \varphi_n) \cdot \sin(n\Omega_0 \tau + \varphi_n) \cdot \left( \langle E_n \rangle \right)^2 \cdot \frac{1 - \cos(n\Omega_0 \tau)}{n\Omega_0} + I_{nn2} \} \, d\tau \]

(2-50)
\[ + \sum_m \sum_n \int_0^\tau \cos(m\Omega_0 \tau + \varphi_m) \cdot \cos(m\Omega_0 \tau + \varphi_m) \cdot \left( \langle E_m \rangle \langle E_n \rangle \right) \cdot \frac{\sin(m\Omega_0 \tau)}{m\Omega_0} + I_{mn1} \} \, d\tau 
+ \sum_m \sum_{n=m} \int_0^\tau \cos(m\Omega_0 \tau + \varphi_m) \cdot \sin(m\Omega_0 \tau + \varphi_m) \cdot \left( \langle E_m \rangle \langle E_n \rangle \right) \cdot \frac{1 - \cos(m\Omega_0 \tau)}{m\Omega_0} + I_{mn2} \} \, d\tau \]

For \( t \gg 1/\Omega_0 \), \( \int_0^\tau \cos^2(n\Omega_0 \tau) \, d\tau \propto \frac{t}{2} \), but all the other integrals are \( O(1) \) with respect to \( t \):

\[ \int_0^\tau \cos^2(m\tau) \cdot \sin(\tau) \, d\tau = O(1), \]
\[ \int_0^\tau \cos^2(m\tau) \, d\tau = \frac{1}{2} t, \]
\[ \int_0^\tau \cos(m\tau) \cdot \cos(n\tau) \cdot \sin(m\tau) \, d\tau = O(1), \int_0^\tau \cos(n\tau) \cdot \cos(m\tau) \cdot \sin(n\tau) \, d\tau = O(1), \]

(2-51)
\[ \int_0^\tau \cos(m\tau) \cdot \cos(n\tau) \, d\tau = O(1), \]
\[ \int_0^\tau \cos(m\tau) \cdot \sin(n\tau) \, d\tau = O(1), \int_0^\tau \cos(n\tau) \cdot \sin(n\tau) \, d\tau = O(1) \]

(where \( m \neq n \)).

The integration becomes
\[
\int_{\tau=0}^{t} \int_{\eta=0}^{\eta} \left\langle u_i(\tau) u_i(\tau-\eta) \right\rangle \, d\eta \, d\tau = \frac{L^6}{B_E^2 R_E^2} \sum_n \left\{ \frac{\langle E_n \rangle^2}{\Omega^2} O(1) + \frac{1}{2} I_{en1} O(1) + \frac{I_{en2}}{\Omega} O(1) \right\}
\]

+ \sum_m \sum_{n \neq m} \left\{ \frac{\langle E_m \rangle \langle E_n \rangle}{\Omega^2} O(1) + \frac{I_{en1}}{\Omega} O(1) + \frac{I_{en2}}{\Omega} O(1) \right\}

(2-52)

Now it can be argued that the $\frac{1}{2} I_{en1}$ term dominates, as follows.

(a) The time scale $t$ is much longer than a drift period ($t \gg 1/\Omega_0$), which implies term (2) is much larger than terms (3), (5) and (6).

(b) The radial excursion in one drift period is approximately $\frac{\langle E_n \rangle}{\Omega_0 B_0}$, where $B_0 \approx \frac{B_E}{L^3}$ is the local magnetic field strength. This is assumed to be small compared to the initial radial position $r_0$, and we may define

\[
\frac{\langle E_n \rangle}{\Omega_0 B_0} = \varepsilon \cdot r_0,
\]

(2-53)

with $\varepsilon \ll 1$.

However, the averaged squared accumulated displacement should be much greater than $(\varepsilon \cdot r_0)^2$, so from equation (2-41) we have

\[
\left\langle (r-r_0)^2 \right\rangle \sim 2 \int_{\tau=0}^{t} \int_{\eta=0}^{\eta} \left\langle u_i(\tau) u_i(\tau-\eta) \right\rangle \, d\eta \, d\tau \gg (\varepsilon \cdot r_0)^2.
\]

Using equation (2-53),
\[ \int_{\tau=0}^{\tau} \int_{\eta=0}^{\tau} \langle u_i(\tau)u_i(\tau-\eta) \rangle \, d\eta \, d\tau \approx \frac{1}{B_0^2 R_E^2} \frac{\langle E^2_\eta \rangle}{\Omega_0^2} . \]

However, according to equation (2-52),

\[ \int_{\tau=0}^{\tau} \int_{\eta=0}^{\tau} \langle u_i(\tau)u_i(\tau-\eta) \rangle \, d\eta \, d\tau \approx \frac{1}{B_0^2 R_E^2} \frac{\langle E^2_\eta \rangle}{\Omega_0^2} + I_{\text{nl}} \cdot t . \]

So \( \langle E^2_\eta \rangle / \Omega_0^2 \) must be much smaller than \( I_{\text{nl}} \cdot t \). Thus terms (1) and (4) must be much smaller than term (2) in equation (2-52).

Because term (2) dominates, equation (2-52) becomes

\[ \int_{\tau=0}^{\tau} \int_{\eta=0}^{\tau} \langle u_i(\tau)u_i(\tau-\eta) \rangle \, d\eta \, d\tau = \frac{L}{2 B_0^2 R_E^2} \sum_n \int \langle \tilde{E}_n(\tau) \cdot \tilde{E}_n(\tau+\eta) \rangle \cdot \cos(n\Omega_0 \eta) \, d\eta . \]

Then from equation (2-44) we have

\[ D_{\text{LL}} = \frac{L}{2 B_0^2 R_E^2} \sum_m \int \langle \tilde{E}_m(\tau) \cdot \tilde{E}_m(\tau+\eta) \rangle \cdot \cos(m\Omega_0 \eta) \, d\eta . \] (2-54)

The Wiener-Khinchin theorem [Press, et al., 1992] states that the power spectral density of a stationary random process is the Fourier transform of the corresponding autocorrelation function. In our case

\[ 4 \int R(t) \cos(\omega t) \, dt = P(\omega) , \] (2-55)

were \( R(t) \) is an autocorrelation function \( R(t) = \langle f(t_0) \cdot f(t_0 + t) \rangle \), and \( P(\omega) \) is the power spectral density of \( f(t) \) at frequency \( \omega \) (i.e., \( P(\omega) = 2 |F(\omega)|^2 \), where \( F(\omega) \) is the Fourier transform of \( f(t) \)).
Using equation (2-55) and equation (2-54), the radial diffusion coefficient can be written as a function of the power spectral density of electric field perturbations at the resonant frequency \( m\Omega_0 \):

\[
D_{LL,E,Sym} = \frac{L^6}{8B_E^2 R_E^2} \sum_m P_m (m\Omega_0).
\] (2-56)

This electric symmetric diffusion coefficient agrees with [Fälthammar, 1965; , 1968]. The diffusion coefficient has \( L^6 \) dependence and is proportional to the power spectral density at frequency \( m\Omega_0 \). This indicates that this diffusion mode is due to the resonance with perturbations at frequencies equal to multiples of the particle drift frequency. In the next subsection we will show a quantitative comparison with the diffusion rate measured by numerical test particle simulation in model fields.

2) The \( u_1 u_2 \) term

Now we consider the \( u_1 u_2 \) "cross term" in equation (2-40). Note that, compared to the symmetric term \( u_1 \), the asymmetric term \( u_2 \) contains an extra \( \cos (\Omega dt) \) and \( \sin (\Omega dt) \) factor, which makes \( \langle u_1 u_2 \rangle \) quite different from \( \langle u_1 u_1 \rangle \). For example, the first term in \( u_2 \) from equation (2-36), gives a contribution to \( \langle u_1 u_2 \rangle \) proportional to

\[
\sum_m \sum_n (E_m (t - \tau) \cdot E_n (t)) \cdot \cos (n\Omega_0 t) \cdot \left\{ \cos ((m + 1)\Omega_0 t) \cos ((m + 1)\Omega_0 \tau) + \sin ((m + 1)\Omega_0 t) \sin ((m + 1)\Omega_0 \tau) + \cos ((m - 1)\Omega_0 t) \cos ((m - 1)\Omega_0 \tau) + \sin ((m - 1)\Omega_0 t) \sin ((m - 1)\Omega_0 \tau) \right\}
\]

As in equation (2-51), the integration of \( \langle u_1 u_2 \rangle \) leads to two dominant terms (i.e., terms proportional to \( t \))
\[
\sum_{m} \sum_{n} \cos(n \Omega_0 t) \cos((m + 1) \Omega_0 t) \cdot \int \langle \vec{E}_m(t - \tau) \cdot \vec{E}_n(\tau) \rangle \cdot \cos((m + 1) \Omega_0 \tau) d\tau ,
\]

(2-57)

and

\[
\sum_{m} \sum_{n} \cos(n \Omega_0 t) \cos((m - 1) \Omega_0 t) \int \langle \vec{E}_m(t - \tau) \cdot \vec{E}_n(\tau) \rangle \cdot \cos((m - 1) \Omega_0 \tau) d\tau .
\]

(2-58)

As in equation (2-41), we need to integrate once again to get \( D_{LL} \). The integration of \( \cos(n \Omega_0 t) \cos((m + 1) \Omega_0 t) \) is non-zero only when \( n = m + 1 \), and the integration of \( \cos(n \Omega_0 t) \cos((m - 1) \Omega_0 t) \) is non-zero only when \( n = m - 1 \). This indicates that we can have non-zero \( D_{LL} \) only if:

\[ n = m + 1 \text{ or } n = m - 1 \quad (a) \]

Assuming the amplitudes \( E_m(t) \) of different azimuthal \( m \) modes are uncorrelated in time, then the integrations \( \int \langle E_m(t - \tau) \cdot E_n(\tau) \rangle \cdot \cos((m \pm 1) \Omega_d \tau) d\tau \) are non-zero only when:

\[ n = m \quad (b) \]

Similar comments apply for the second term of \( u_z \).

Thus the integration of equations (2-57) and (2-58) must be zero since the two conditions (a) and (b) cannot be satisfied at the same time. So this \( u_1u_2 \) ‘cross term’ makes no contribution to the radial diffusion coefficient. Similarly, the \( u_2u_1 \) term (the 3rd term in equation (2-40)) also makes no contribution.

3) The \( u_2u_2 \) term
The last term \( \int_0^t \int_0^\xi \left( u_2(\xi) \cdot u_2(\xi - \tau) \right) \cdot d\tau \cdot d\xi \) is the integration of the self correlation function of the asymmetric term \( u_2(t) \) given in equation (2-36):

\[
\begin{align*}
  u_2 &= -\left( \frac{\Delta B}{B_E} \right) \frac{L^6}{B_E R_E} \cdot \sum_m \left( \frac{5}{6} m + \frac{2}{3} \right) E_{\phi m}(r, t) \cos\left( (m + 1)(\Omega_d t + \phi_0) + \gamma_m \right) \\
  &+ \left( \frac{\Delta B}{B_E} \right) \frac{L^6}{B_E R_E} \cdot \sum_m \left( \frac{5}{6} m - \frac{2}{3} \right) E_{\phi m}(r, t) \cos\left( (m - 1)(\Omega_d t + \phi_0) + \gamma_m \right).
\end{align*}
\]

Comparing the expression of \( u_2(t) \) to the \( u_1(t) \) term in equation (2-35), they are very similar except for the factor in front, and \( \cos((m + 1)\Omega_d t) \), \( \cos((m - 1)\Omega_d t) \) dependence instead of \( \cos(m\Omega_d t) \). Skipping the details, using a similar treatment to section the \( u_1u_1 \) term, the radial diffusion coefficient due to the \( u_2u_2 \) term is:

\[
D_{LL, Asym} = \frac{1}{8} \frac{(\Delta B)^2}{B_E^4 R_E^4} L^{12} \sum_m \left( \frac{5}{6} m + \frac{2}{3} \right)^2 \left( P_m((m + 1)\Omega_d) \right) \\
+ \frac{1}{8} \frac{(\Delta B)^2}{B_E R_E^4} L^{12} \sum_m \left( \frac{5}{6} m - \frac{2}{3} \right)^2 \left( P_m((m - 1)\Omega_d) \right).
\]

\[\text{(2-59)}\]

This asymmetric radial diffusion coefficient is proportional to the power spectral density of \( \tilde{E}_m \) at frequencies \( (m + 1)\Omega_d \) and \( (m - 1)\Omega_d \). So that for each azimuthal wave mode \( m \) there are two resonance conditions: \( \omega = (m + 1)\Omega_d \) and \( \omega = (m - 1)\Omega_d \). This is consistent with the result of the analysis in [Elkington, 2000], but we obtain different factors multiplying the power spectral densities. The magnitude of this radial diffusion coefficient is also proportional to the square of the asymmetry factor \( \frac{\Delta B}{B_E} \), and has \( L^{12} \) dependence, which is steeper than the \( L^6 \) dependence of the symmetric electric diffusion coefficient.
As a brief summary, in this subsection I obtained radial diffusion coefficients due to perturbations of the electric field. The symmetric diffusion coefficient is (equation (2-56)):

\[
D_{LL}^{E, Sym} = \frac{1}{8B_E^2 R_E^2 L^6} \sum_m P_m \left(m \Omega_d \right),
\]

and the asymmetric diffusion coefficient is (equation (2-59))

\[
D_{LL}^{E, Asym} = \frac{1}{8B_E^4 R_E^4 L^{12}} \sum_m \left(\frac{5}{6} m + \frac{2}{3}\right)^2 \left(P_m \left((m + 1) \Omega_0\right)\right)
\]

\[
+ \frac{1}{8B_E^4 R_E^4 L^{12}} \sum_m \left(\frac{5}{6} m - \frac{2}{3}\right)^2 \left(P_m \left((m - 1) \Omega_0\right)\right)
\]

The symmetric diffusion coefficient is consistent with earlier works using a simple symmetric dipole magnetic field, and represents particle acceleration in resonance with electric waves close to frequencies \(m \cdot \Omega_0\). Using a more general asymmetric magnetic field, I found new resonance modes at wave frequencies of \((m \pm 1) \cdot \Omega_0\) resulting from the asymmetry of the drift orbit. The symmetric and asymmetric diffusion coefficients are both proportional to the electric wave power spectral density at resonant frequencies, but with different constant factors and different \(L\) dependence \((L^{12}\) compared to the \(L^6\)). The quantitative comparison with numerical test particle calculations will be shown in the next subsection.

The total diffusion coefficient should be the sum of both symmetric and asymmetric diffusion coefficients, for all \(m\) modes. Note that, as mentioned immediately after equation (2-23), the effect of the \(E_r\) component can be added by replacing \(G_m\) by the power spectral density of \(E_r\) and dividing by 16 in equation (2-59).
2.4.6 Numerical test particle diffusion

To verify the analytical radial diffusion coefficients derived in the previous subsections, we employ numerical test particle simulations using model electric and magnetic field to obtain numerical diffusion coefficients, and compare them with the analytical diffusion coefficients.

The numerical test particle calculation tracks the particles’ equatorial guiding center motion under model magnetic and electric fields. The details of the test particle simulation method can be found in [Elkington, 2000; Elkington, et al., 2003].

For the comparison with analytical electric diffusion coefficients of equation (2-56) and equation (2-59), we use the following model fields: The background magnetic field is calculated using a simple static asymmetric field model

\[ B(r) = B_E \frac{R_E^3}{r^3} + \Delta B \cos(\phi), \]

where \( B_E = 30500 \text{nT} \) is the earth surface magnetic field strength, and \( \Delta B \) is the asymmetric term. We set \( \Delta B = 0 \) for the comparison of the symmetric diffusion coefficients, and set \( \Delta B = 30 \text{nT} \) for the comparison of the asymmetric diffusion coefficients.

The perturbation electric field is the summation of series of cosine waves:

\[ E_\phi(\phi, t) = \sum_{m} \sum_{k=k_1}^{k_2} \delta E_{k,m} \cdot \cos(k \cdot \Delta \omega + \gamma_k) \cdot \cos(m \cdot \phi + \gamma_m), \quad (2-60) \]

where \( k \) is the frequency number, \( \Delta \omega \) is the width of each frequency bin, \( \gamma_k \) is the phase angle of the \( k \)th frequency bin, \( m \) is the azimuthal mode number, \( \gamma_m \) is the phase angle of
mth azimuthal mode, and $\delta E_{k,m}$ is the electric perturbation at frequency $k$, and azimuthal mode $m$.

In this simulation, both phase angle $\gamma_k$ and $\gamma_m$ are random. The size of $\Delta \omega$ is small, the inner summation $\sum_{k=k_1}^{k_2} \delta E_{k,m} \cdot \cos(k \cdot \Delta \omega + \gamma_k)$ approximates a frequency continuum from $\omega_1 = k_1 \Delta \omega$ to $\omega_2 = k_2 \Delta \omega$. We choose a range of $k_1, k_2$ to ensure the frequency range $(k_1 \cdot \Delta \omega - k_2 \cdot \Delta \omega)$ covers only the resonance frequency of interest, but away from other resonance modes. The outer summation combines azimuthal modes.

The size of the frequency bin for this numerical simulation is $\Delta f = 0.05$ mHz, with $\Delta \omega = 2\pi \Delta f$. The amplitude of the electric field perturbation is $\delta E = \frac{0.1}{\sqrt{2}}$ mV/m, which is constant over all frequencies. Then the power spectral density is:

$$P = \frac{\delta E^2}{2\Delta f} = 5 \times 10^{-5} V^2/m^2Hz.$$ 

The test particles are initialized with a fixed first adiabatic invariant $M$, and placed at initial radial location $L_0$. In each run we put in 360 particles. Their initial angular position are uniformly distributed with one degree spacing. The length of each run is 2 hours. The runs are repeated 100 times, then the results are averaged to calculate the radial diffusion coefficient using $D_{LL} = \frac{\langle dL^2 \rangle}{2\tau}$, where $\langle dL^2 \rangle$ is the mean square of particle’s $L$ value deviation from $L_0$ after time $\tau$ ($\tau = 2hr$). We repeat this experiment with different initial values of $L_0$ to further explore the correctness of the $L$ dependence of the radial diffusion coefficient.
First, we use this method to estimate the value of the symmetric mode radial diffusion coefficient. Setting $m = 1$ for the fundamental azimuthal mode, and setting the frequency range to cover the resonance condition $\omega = \Omega_d$, the value of $D_{LL}$ calculated by the numerical test particle simulation at different $L$-shells are plotted in the following figure.

![Electric diffusion, m=1 symmetric](image)

Figure 2-5 Numerical test particle calculation of symmetric electric diffusion.

Diamonds are the numerical $D_{LL}$s, and they are fitted into the solid straight line. The dotted line is the analytical $D_{LL}$; it is barely visible because it is very close to the solid line.

The $D_{LL}$ obtained from the numerical test particle simulation agrees very well with the analytical result. The $L$ dependence of the numerical result is $5.98 \pm 0.08$, which is
in great agreement with the analytical result of $L^6$ dependence. The magnitude of the diffusion coefficient also agrees very well, the difference is within 2%.

Next, we calculate the asymmetric mode radial diffusion coefficients. The frequency range is changed to cover $\omega = 2 \cdot \Omega_d$ which is the $m + 1$ asymmetric mode resonance frequency at $m = 1$. The background magnetic field includes a non-zero asymmetric term, and $\Delta B = 30\text{nT}$. The result of numerical calculation is shown in Figure 2-6.

![Electric diffusion, $(m+1)=2$ asymmetric](image)

**Figure 2-6** Numerical test particle calculation of asymmetric electric diffusion.

Diamonds are the numerical $D_{LL}$ s, and fit into the solid straight line. The dotted line is the analytical $D_{LL}$. 
The radial diffusion coefficients \( D_{LL} \) obtained from numerical test particle simulation agree very well with the analytical asymmetric diffusion coefficients. The \( L \) dependence of numerical result is 11.84 ± 0.04, it agrees with the analytical result of \( L^2 \). The magnitude of the diffusion coefficient also agrees very well, the difference is within 20%.

2.5 Magnetic diffusion coefficient with asymmetric term

2.5.1 Particle motion under magnetic perturbation

Now we consider the diffusion caused by magnetic perturbations. Considering only the magnetic guiding center drift term, the rate of deviation from the unperturbed orbit is:

\[
\frac{d\rho}{dt} = \nu_d \cdot \hat{n}_B = \frac{M}{q\gamma B^2} (\mathbf{B} \times \nabla B) \cdot \hat{n}_B.
\]  

(2-61)

Remember that the normal vector \( \hat{n}_B \) is defined using the unperturbed magnetic field, but \( \mathbf{B} \) is the total magnetic field that includes the time dependent perturbed field. This leads to a non-zero \( \frac{d\rho}{dt} \). From now on, I write the total magnetic field which includes the perturbation as \( B_{\text{total}} \), then

\[
\frac{d\rho}{dt} = \frac{M}{q\gamma B_{\text{total}}^2} (\mathbf{B}_{\text{total}} \times \nabla B_{\text{total}}) \cdot \hat{n}_B.
\]

The unperturbed field is the asymmetric magnetic field model presented in subsection 2.4.1: \( B = B_0 - \Delta B \cos \phi \). The normal vector \( \hat{n}_B \) of the constant field contour is calculated based on this unperturbed magnetic field \( B \).
Adding a time dependent magnetic field perturbation \( b(r, \phi, t) \) (in the z direction), the total magnetic field is

\[
\mathbf{B}_{\text{total}}(t) = \mathbf{B} + \mathbf{b}(t)
\]

\[
= (B_0 + \Delta B \cos \phi + b(t)) \hat{z}
\]

Then the rate of deviation from the drift shell is:

\[
\frac{d\rho}{dt} = \mathbf{v}_d \cdot \hat{n}_B = \frac{M}{q \gamma (B + b)^2} (\mathbf{B} + \mathbf{b}) \times \nabla (B + b) \cdot \hat{n}_B
\]

\[
= \frac{M}{q \gamma (B + b)^2} (\mathbf{B} + \mathbf{b}) \times \nabla B \cdot \hat{n}_B + \frac{M}{q \gamma (B + b)^2} (\mathbf{B} + \mathbf{b}) \times \nabla b \cdot \hat{n}_B.
\]

In which \((\mathbf{B} + \mathbf{b}) \times \nabla B \cdot \hat{n}_B = 0\) since \(\nabla B\) and \(\hat{n}_B\) are the same direction. So the first term is zero and only the second term remains:

\[
\frac{d\rho}{dt} = \frac{M}{q \gamma (B + b)^2} \hat{z} \times \nabla b \cdot \hat{n}_B.
\] (2-62)

The magnetic perturbation field \( b \) consists of an axisymmetric part, and a local time dependent part (Akasofu and Chapman 1964). In the equatorial plane the disturbance model is decomposed as:

\[
b = S(t) + \sum_{m=1}^{\infty} A_m(r, \phi, t) \cdot \cos m\phi.
\] (2-63)

Then

\[
\nabla b = \sum_m \frac{\partial A_m}{\partial r} \cdot \cos m\phi \cdot \hat{r} - \frac{1}{r} \sum_m m \cdot A_m \cdot \sin m\phi \cdot \hat{\phi}.
\] (2-64)

Using equation (2-63), equation (2-64), and the expression for the normal vector as in equation (2-16) in equation (2-62), and keeping first order terms, the rate of deviation from the drift shell is:
\[
\frac{d\rho}{dt} \approx \frac{M}{q\gamma Br} \sum_m mA_m \cdot \sin m\phi \\
- \frac{M}{q\gamma Br} \frac{\Delta B}{B} \sum_m mA_m \cdot \sin m\phi \cdot \cos \phi \\
+ \frac{1}{3} \frac{M}{q\gamma B} \frac{\Delta B}{B} \sum_m \frac{\partial A_m}{\partial r} \cdot \cos m\phi \cdot \sin \phi
\]

Using equation (2-13) \( r(\phi) = LR_E \left( 1 - \frac{\Delta B}{B} \frac{L^3 \cos \phi}{L^3} \right)^{-1/3} \), \( \frac{d\rho}{dt} \) becomes

\[
\frac{d\rho}{dt} \approx \frac{M}{q\gamma Br_0} \sum_m mA_m \cdot \sin m\phi \\
- \frac{4}{3} \frac{M}{q\gamma Br_0} \frac{\Delta B}{B} \sum_m mA_m \cdot \sin m\phi \cdot \cos \phi \\
+ \frac{1}{3} \frac{M}{q\gamma B} \frac{\Delta B}{B} \sum_m \frac{\partial A_m}{\partial r} \cdot \cos m\phi \cdot \sin \phi
\] (2-65)

The next step is to calculate \( \frac{dL}{dt} \) from \( \frac{d\rho}{dt} \). The unperturbed background magnetic field is the same as in section 2.4.5. Using the same \( \frac{dL}{dt} \) and \( \frac{d\rho}{dt} \) relation as in equation (2-20), (2-21), gives

\[
\frac{dL}{dt} = \frac{M}{q\gamma B^n E^2} L^2 \sum_m mA_m(t) \cdot \sin m\phi \\
- \frac{8}{3} \frac{M}{q\gamma B^n E^2} \frac{\Delta B}{B} L^5 \sum_m mA_m(t) \cdot \sin m\phi \cdot \cos \phi \\
+ \frac{1}{3} \frac{M}{q\gamma B^n E} \frac{\Delta B}{B} L^5 \sum_m \frac{\partial A_m(t)}{\partial r} \cdot \cos m\phi \cdot \sin \phi
\] (2-66)

Using the expression of angular position \( \phi \) in equation (2-29) and equation (2-31), the first term in \( \frac{dL}{dt} \) can be expanded into a symmetric term and a first order asymmetric term. \( \frac{dL}{dt} \) becomes
\[
\frac{dL}{dt} = \frac{M}{q\gamma B_E R_E^2} L^2 \sum_m mA_m(t) \cdot \sin m\Omega_0 t \\
- \frac{5}{3} \frac{M}{q\gamma B_E R_E^2} \frac{\Delta B}{B_E} L^5 \sum_m m^2 A_m(t) \cdot \cos m\Omega_0 t \cdot \sin \Omega_0 t \\
- \frac{8}{3} \frac{M}{q\gamma B_E R_E^2} \frac{\Delta B}{B_E} L^5 \sum_m mA_m(t) \cdot \sin m\Omega_0 t \cdot \cos \Omega_0 t \\
+ \frac{1}{3} \frac{M}{q\gamma B_E R_E^2} \frac{\Delta B}{B_E} L^6 \sum_m \frac{\partial A_m(t)}{\partial r} \cdot \cos m\Omega_0 t \cdot \sin \Omega_0 t
\]

This can be further written as:

\[
\frac{dL}{dt} = \frac{M}{q\gamma B_E R_E^2} L^2 \sum_m mA_m(t) \cdot \sin m\Omega_0 t \\
- \frac{M}{q\gamma B_E R_E^2} \frac{\Delta B}{B_E} L^5 \sum_m \left( \frac{5}{6} m^2 + \frac{4}{3} m \right) A_m(t) \cdot \sin ((m + 1)\Omega_0 t) \\
- \frac{M}{q\gamma B_E R_E^2} \frac{\Delta B}{B_E} L^5 \sum_m \left( \frac{4}{3} m - \frac{5}{6} m^2 \right) A_m(t) \cdot \sin ((m - 1)\Omega_0 t) \\
+ \frac{1}{3} \frac{M}{q\gamma B_E R_E} \frac{\Delta B}{B_E} L^6 \sum_m \frac{\partial A_m(t)}{\partial r} \cdot \cos m\Omega_0 t \cdot \sin \Omega_0 t
\]  \hspace{1cm} (2.67)

The first term is the symmetric term, it is similar to the symmetric term of the electric diffusion derivation. The next two terms are the asymmetric terms, also similar to the terms in the electric diffusion derivation. The last term can be neglected because its contribution to the diffusion coefficient will be only approximately \(1/25\) that of the second term. In the next subsection I give the corresponding diffusion coefficients.

### 2.5.2 Diffusion coefficient

Comparing Equation (2.66) to Equation (2.23), the form of the magnetic \(L\) shell deviation equation is similar to that of the electric deviation equation, except they have different \(L\) dependence and azimuthal mode number \(m\) dependence. Recall the
diffusion coefficient is calculated from the double integration of \( \frac{dL}{dt} \bigg|_\xi \) and \( \frac{dL}{dt} \bigg|_{\xi-\tau} \).

Very similar to the \( \langle u_i u_i \rangle \) term in the electric diffusion, the self-product of the symmetric term (the first term) in Equation (2-66) will lead to the symmetric magnetic diffusion coefficient. The resulting diffusion coefficient is

\[
D_{LL^{M,Sym}} = \frac{M^2}{8q^2 \gamma^2 B_E^2 R_E^4 L^4} \sum_m m^2 P_m (m\Omega_0). \tag{2-68}
\]

Here \( P_m (m\Omega_d) \) is the power spectral density of the magnetic field perturbation (i.e., the compressional magnetic component in the equatorial plane).

The symmetric magnetic diffusion coefficient is proportional to the power spectral density of the perturbation magnetic field at resonance frequencies \( \omega = m\Omega_d \). The \( L \) dependence is not a simple power law because \( \gamma \) is also a function of \( L \). In the ultra-relativistic limit, \( \gamma \sim L^{-3/2} \), so the diffusion coefficient has roughly \( L^7 \) dependence, and it is also proportional to the \( L \) dependence in power spectral density.

The cross terms from the symmetric and asymmetric terms again make no contribution to the net diffusion.

The product of the asymmetric terms leads to asymmetric radial diffusion coefficients.

The integration is similar to the previous electric asymmetric diffusion coefficient calculation. The final result of magnetic asymmetric diffusion coefficient is:
\[ D_{LL}^{Asym} = \frac{M^2}{q^2 \gamma^2 B_E^2 R_E^4} \left( \frac{\Delta B}{B_E} \right)^2 L^{10} \sum_m \left( \frac{5}{6} m^2 + \frac{8}{6} m \right)^2 P_m ((m + 1) \Omega_0) \\
+ \frac{M^2}{q^2 \gamma^2 B_E^2 R_E^4} \left( \frac{\Delta B}{B_E} \right)^2 L^{10} \sum_m \left( \frac{5}{6} m^2 - \frac{8}{6} m \right)^2 P_m ((m - 1) \Omega_0) \]

(2-69)

The magnetic asymmetric diffusion contains two resonance modes: \( \omega = (m + 1) \Omega_0 \) and \( \omega = (m - 1) \Omega_0 \). In the non-relativistic limit it has \( L^{10} \) dependence, and in the ultra-relativistic limit it has \( L^{13} \) dependence.

In the next subsection the values and \( L \) dependences of these magnetic radial diffusion coefficients will be verified by comparing to numerical test particle calculations.

2.5.3 Numerical test particle diffusion

For the verification of the analytical magnetic radial diffusion coefficients, we performed numerical test particle simulation in model fields. The numerical method for this test particle calculation is the same as described in 2.4.6. The difference is that the magnetic field model includes both unperturbed background field and the perturbation magnetic field,

\[ B(r, t) = B_E \frac{R_E^3}{r^3} + \Delta B(r) \cos(\phi) + b(r, \phi, t). \]

The electric field is zero for this simulation.

The time dependent perturbation term \( b(r, \phi, t) \) is modeled as the sum of discrete frequency components:
\[ b(\phi, r, t) = \sum_{m} \sum_{k=k_1}^{k_2} \delta A_{k,m}(r) \cdot \cos(k \cdot \Delta \omega + \gamma_k) \cdot \cos(m \cdot \phi + \gamma_m), \] (2-70)

here \( k \) is an integer, \( \Delta \omega \) is the size of each frequency bin, \( \gamma_k \) is the phase angle of the \( k \)th frequency bin, \( m \) is the azimuthal mode number, \( \gamma_m \) is the phase angle of \( m \)-th azimuthal mode, and \( \delta A_{k,m} \) is the magnitude of the magnetic perturbation at frequency bin \( k \), and azimuthal mode \( m \). In this simulation \( k \cdot \Delta \omega \) (\( k \) ranges from \( k_1 \) to \( k_2 \)) covers the range of frequency of interest. The term \( \delta A_{k,m} \) is constant, and the power spectral density of the magnetic perturbation is \( P_m = \pi \frac{\delta A_{k,m}^2}{\Delta \omega} \).

The result of numerical tests for symmetric mode magnetic resonance of \( m = 1 \) mode is shown in the following figure.
Figure 2-7: Numerical test particle estimation of magnetic symmetric diffusion coefficient. Diamonds are the numerical $D_{LL}$, the solid line is a fit to the numerical $D_{LL}$, and the dots are the analytic formula.

The analytical diffusion coefficient (Equation (2-68)) and numerical results are in very good agreement, in both magnitude and L dependence. The difference in magnitude is about 2%, and the difference in the $L$ exponent ($n$ in $L'$) is about 9%.

The numerical diffusion coefficient calculation for the asymmetric magnetic resonance $m+1$, for $m = 1$ is shown in the following figure.
**Figure 2-8** Numerical test particle estimation of magnetic symmetric diffusion coefficient. Diamonds are the numerical $D_{LL}$, the solid line is a fit to the numerical $D_{LL}$, and the dots are the analytic formula.

The analytical asymmetric magnetic diffusion coefficient (Equation (2-69)) agrees with the numerical estimation. The L dependences agreed very well (~13), and the magnitude of the diffusion coefficients also agrees, the difference is within 25%.
2.6 Summary

In this chapter I derived analytical electric and magnetic wave driven radial diffusion coefficients of general form, where the background magnetic field can be asymmetric.

For particles drifting in an asymmetric background magnetic field, the perturbation electric field and the perturbation magnetic field can cause the particles to deviate from the original L-shell. The radial diffusion coefficient for electric perturbations is the sum of a symmetric resonance term and an asymmetric resonance term. The symmetric diffusion coefficient exists when the background magnetic field is either symmetric or asymmetric, it doesn’t depend on the asymmetry factor $\frac{\Delta B}{B_E}$. It has $L^6$ dependence and is proportional to the electric wave power spectral density at resonant frequencies $\omega = m\Omega_d$. The asymmetric diffusion coefficient exists only when the background field is asymmetric, and it is proportional to $\left(\frac{\Delta B}{B_E}\right)^2$. The resonant frequencies are $\omega = (m \pm 1)\Omega_d$, and the diffusion coefficient has $L^{12}$ dependence.

The form of the radial diffusion coefficients for magnetic perturbations is similar to that of the electric diffusion coefficients, including the dependency on $\frac{\Delta B}{B_E}$, and resonant frequencies. However, the symmetric magnetic diffusion coefficient has $\frac{L^4}{\gamma^2}$ dependence, and the asymmetric diffusion coefficient has $\frac{L^{10}}{\gamma^2}$ dependence.
The newly derived diffusion coefficients are compared with numerical calculations of particle motion in model fields and good agreement is obtained.

In real life the $L$ dependency of these diffusion coefficients are more complicated, because it also depends on the $L$ dependence in the wave power spectral density, and in the asymmetric factor $\frac{\Delta B}{B_E}$. The relative strength of the symmetric diffusion coefficients compared to the asymmetric diffusion coefficients depends on the asymmetry factor $\frac{\Delta B}{B_E}$, and the power spectral density at resonant frequencies. More realistic values of these diffusion coefficients will be presented in Chapter 6.
Chapter 3

Numerical Radial Diffusion Model

The radial diffusion equation (2-7) was introduced in the previous Chapter. This equation can be solved numerically to calculate the phase space density values of the radiation belt electrons. The stationary solution gives the steady-state radiation belt phase space density radial profile, and is relevant to the quiet-time particle distribution. Time-dependent radial diffusion calculations are able to describe dynamic particle transport due to the following processes: diffusion, internal sources, external sources, and losses.

In this chapter I will first describe the numerical radial diffusion code, and then show the calculation results of several idealized test cases. The input parameters for the code include boundary values, radial diffusion coefficients, and source and loss terms. How these values affect the radial diffusion transport will be analyzed. The test cases are: 1) Steady-state solutions; 2) Three time-dependent solutions: with an internal source, with an external source, and with loss.

3.1 Numerical method for the radial diffusion calculation

3.1.1 Numerical method

The radial diffusion equation is a second-order time-dependent partial differential equation (PDE) of the form
\[
\frac{\partial f}{\partial t} = L^2 \frac{\partial}{\partial L} \left( \frac{D_{LL}}{L^2} \frac{\partial f}{\partial L} \right) + S - \frac{f}{\tau}.
\]

The diffusion coefficient \( D_{LL} \), the source term \( S \), and the loss term \( \frac{f}{\tau} \) are all general functions of \( L \) and \( t \). The boundary conditions may also be time dependent.

This diffusion equation can be expanded into:

\[
\frac{\partial f}{\partial t} = D_{LL} \frac{\partial^2 f}{\partial L^2} + \left( \frac{\partial D_{LL}}{\partial L} - 2 \frac{D_{LL}}{L} \right) \frac{\partial f}{\partial L} + S - \frac{f}{\tau}.
\]

(3-1)

The first term at the right hand side, containing \( \frac{\partial^2 f}{\partial L^2} \), is a classic diffusion term (similar to second-derivative term in the \( I-D \) heat diffusion equation). The second term, containing \( \frac{\partial f}{\partial L} \) is an advection term. The radial diffusion process is a combination of diffusion and advection.

In general, the inner and outer boundary values are functions of \( t \). The location of the outer boundary is also variable in order to track the changes of the particle outer trapping boundary. In our simulations the inner boundary is at a fixed location (around \( L \sim 2 \)) and of low constant value to resemble the inner zone or the slot region of the inner radiation belt, depending on the simulation.

A central differencing method with second order accuracy is used for the spatial discretization. The computation domain and discretization is illustrated in Figure 3-1.
Define $f_j^i$ as the value of the phase-space density (PSD) at the $i$-th $L$ grid point, and at time step $j$. The first and second derivative operator can be written as:

$$\frac{\partial f}{\partial L} = \frac{f_j^{i+1} - f_j^{i-1}}{2dl},$$

and

$$\frac{\partial^2 f}{\partial L^2} = \frac{f_j^{i+1} + f_j^{i-1} - 2f_j^i}{dl^2},$$

where $dl$ is the grid spacing in $L$.

The Crank-Nicholson method is used for solving the PDE. It is a second-order trapezoidal implicit finite-difference method. The grid is uniform in both $L$ and $t$. The major calculation in the Crank-Nicholson method involves the numerical solution of a triangular matrix system, for which efficient numerical methods exist [Press, et al., 1992].
3.1.2 Verification

For verification purposes, we compare the code’s numerical results with the analytical solution to a test problem:

\[ \frac{\partial f}{\partial t} = D_{LL} \cdot \frac{\partial^2 f}{\partial L^2}, \]

where the diffusion coefficient \( D_{LL} \) is constant, inner and outer boundary values are zero, and the initial phase space density is a Gaussian function

\[ f(L, t = 0) = f_0(L) = Ce^{-\frac{(L-L_0)^2}{W}}. \]

Applying a Fourier transform yields the analytical solution

\[ f(L, t) = \frac{1}{\sqrt{4\pi D_{LL}t}} \int f_0(t) e^{-\frac{(L-L_1)^2}{\pi D_{LL}t}} dl. \]

This integral can be easily accomplished numerically.

According to this formula we will expect the solution to have a peak that decreases with time, and the width of the peak increases with time. Figure 3-2 (a) is a plot of the analytical solution, and Figure 3-2 (a) shows the result of the numerical radial diffusion code. The differences between them are smaller than \( 10^{-3} \) everywhere.

And the first two time dependent test cases in section 3.3 agrees with the results in [Selesnick and Blake, 2000].
Figure 3-2  (a) Analytical solution (top), and (b) Numerical diffusion result (bottom).

3.2 Steady state solution

In this section we will present the results of steady state solutions using the radial diffusion code. The steady-state solution of the radial diffusion equation is an approximate representation of the radiation belt after a prolonged period of magnetically
quiet conditions. Generally, the phase space density of relativistic electrons remains a near-constant low value at low $L$, and at higher $L$ the phase space density has much higher values.

We set up the parameters for the numerical calculation as follows: the inner boundary location is $L = 2$ with zero inner boundary value; the outer boundary is located at $L = 9$, with a positive high value; there are no loss or source terms. Data-based radial transport studies often assumed the radial diffusion coefficient can be approximated in a power-law form [Brautigam and Albert, 2000; Schulz and Lanzerotti, 1974; Selesnick, et al., 1997]

$$D_{LL} = D_0 L^n,$$

in which $n$ ranges from about 6 to 14. In this calculation I use $n = 6$ and $n = 10$. The simple case $n = 0$ is also shown for comparison. The result of numerical calculations of steady-state solutions are shown in Figure 3-3.
The common feature of the solutions in the three cases ($n = 0, 6, 10$) is that the phase space density is monotonically increasing with increasing $L$ in all cases. However, for higher $n$, the slope is steeper at lower $L$, and flatter at higher $L$. For example, when $n = 10$, from $L = 4$ to 9 the phase space density is high and almost flat, while from $L = 2$ to 4 the phase space density increases very fast with $L$.

Note that the steady state solution may not agree with averaged data-based models such as the AE-8 model [Vette, 1991]. This is because these models represent an averaged estimation of the radiation belt, which includes active times in which the fluctuations are high at high $L$, so the averaged phase space density at high $L$ is expected to be lower (for example, as shown in Figure 4-6, for the AE8 model).
3.3 Time dependent solution

During magnetically active times the outer boundary value and $D_{LL}$ can be highly dynamic, and sources and losses may appear in the radiation belt. The radial diffusion process transport particles in the radiation belt and varies the phase space density profile.

3.3.1 Diffusion effect and Advection effect

Equation (2-1) indicates that the radial diffusion equation contains two types of terms: an advection term $\left( \frac{\partial D_{LL}}{\partial L} - 2 \frac{D_{LL}}{L} \right) \frac{\partial}{\partial L}$, and a diffusion term $D_{LL} \frac{\partial^2}{\partial L^2}$.

The radial diffusion coefficient $D_{LL}$ is always positive, so the diffusion term $D_{LL} \frac{\partial^2 f}{\partial L^2}$ lowers the height of a peak, and increases the bottom of a valley. It also smoothes/spreads out small scale features (peaks or valleys). These effects are illustrated in Figure 3-4. When $D_{LL}$ increases with $L$, the diffusion rate is generally faster at higher $L$ than at lower $L$.

![Figure 3-4 The effect of the diffusion term](image)
The advection term acts to translate a peak or valley to higher or lower \( L \) shells, where the direction of this translation is determined by the sign of
\[
\left( \frac{\partial D_{LL}}{\partial L} - 2 \cdot \frac{D_{LL}}{L} \right).
\]
As mentioned above, data-based studies which assume \( D_{LL} = D_0 L^n \) yield values of \( n \) in the range 6 to 14. When \( n > 2 \), \( \left( \frac{\partial D_{LL}}{\partial L} - 2 \cdot \frac{D_{LL}}{L} \right) \) is positive, so in most conditions we expect the advection operator to move the phase space density profile inward, and the advection rate to be higher at higher \( L \). Figure 3-5 illustrates the effect of the advection term.

![Figure 3-5 The effect of the advection term](image)

Note that in Figure 3-5 the advection effect increases the phase space density at lower \( L \) (by moving a peak inward), even though there is no source active at that region. Thus, a satellite at low \( L \) may see a steady increase of the phase space density due to such an advection process, and simply measuring the phase space density temporal variation at a fixed \( L \) shell location can not determine whether an increase is due to an internal source or due to advection (or diffusion).
In order to specify the possible location of a source, both spatial and temporal coverage are needed. This will be further analyzed in the following tests. The first two test cases employ sources at different spatial regions (the setup is similar to that of [Selesnick and Blake, 2000]), the third test case simulates the effect of loss processes with the presence of an external source. From the results we can study the necessary requirements for identifying internal sources and external sources.

3.3.2 Test cases

Test case (1). Radial diffusion with a local source

This test case simulates a local source of high phase space density during the initial period of the simulation. The numerical experiment is set up as follows: The inner boundary location is \( L = 2 \), the outer boundary location is 9. Both inner and outer boundary values are kept at zero. The initial phase space density is zero at all \( L \) values. The artificial diffusion coefficient is \( D_{LL} = 10^{-8} L^{10} \). The source \( S \) is localized between \( L = 4.5 \) and \( L = 5 \), and it is turned on at \( t = 0 \) and then turned off after \( t = 5 \).

The result of this calculation is plotted in Figure 3-6. For the subplot to the left, the \( Y \) axis is \( L \)-shell value, the \( X \) axis is time, and the phase space density is color-coded. The subplot to the right is a 3D line view of the same result.
Figure 3-6 Time dependent test case 1 – Internal source

From $t = 0$ to 5, the radial diffusion process spreads the local source in both inward and outward directions. However, the inward diffusion and advection is relatively slow due to the $L^{10}$ dependence of the radial diffusion coefficient $D_{LL}$. After the source is turned off at $t = 5$, particles continue to diffuse rapidly outward due to the low (zero) value at the outer boundary, and the phase space density decreases in the outer region because there is no internal source to keep populating this region. Due to the inward advection, the location of the maximum phase space density continues to move inward and raise the phase space density in the region below the initial location of the peak. The peak value continues to decrease due to diffusion. After $t = 5$ the main shape of the phase space density ($L$ profiles) changes very slowly.

Test case (2). Diffusion from an source at the outer boundary

This test case is intended to simulate an external source acting for a finite period of time. The set up is the same as test case (1) except that the internal source $S$ is zero and
the boundary value is a high positive value during times $0 < t < 5$. The result of this radial diffusion calculation is shown in Figure 3-7.

![Image of Figure 3-7](image)

**Figure 3-7** Time dependent test case 2 – External source

During early times $t = 0 \sim 5$ the high diffusion rate and high advection rate in the outer region brings in phase space density from the external source to the middle $L$ region (from $L = 8$ to $L \sim 4$) very effectively. After the source is turned off after $t = 5$, the outer boundary value becomes zero, so the outer boundary becomes a 'sink', and the outer radiation belt phase space density is removed rapidly through the outer boundary. The phase space density decreases very fast in the region $L > 5$, and a peak is formed near $L = 4.5$ at $t \sim 10$. After $t = 10$, the evolution of the phase space density in Figure 3-7 is very similar to that of Figure 3-6: a peak at $L \sim 4.5$, and the phase space density over time decreases slowly at all $L$-shells. In order to distinguish internal source and external source, one needs to follow the time evolution of profiles when the source was activated (it is not enough to look at a profile at one time or to look at one $L$ for many times).

**Test case (3). Adding loss**
In this test case a loss term is added to the radiation belt fed by an external source. I use an empirical diffusion coefficient and loss rate given by Selesnick, et al. [1997]:

\[ D_{LL} = 2.1 \times 10^{-3} (L / 4)^{11.7} \]

\[ \tau^{-1} = 5 \times 10^{-2} (L / 4)^{7.6} \]

These are obtained by fitting observational data during magnetic quiet times. The initial state (phase space density profile at \( t = 0 \)) is calculated from the AE8MIN model (an electron flux model created from observations during solar minimum. I leave the details of the particle flux to phase space density conversion to Chapter 4 – page 80). The inner boundary value is zero, and the outer boundary keeps a high value to populate the belt. The result of this diffusion calculation is shown in Figure 3-8.

![Figure 3-8](image)

**Figure 3-8** Time dependent test case 3 – Diffusion with loss

In the outer regions, the diffusion rate is greater than the loss. Thus, fast diffusion can continue to bring in the high phase space density from the external source and overcome the loss. In the middle region (\( L \sim 5 \)) the loss term dominates, creating a local minimum quickly after the beginning of the simulation. In the low \( L \) region (\( L < 4.5 \)), the loss term also dominates, and diffusion works in the same direction as the loss process to decrease the phase space density. However both diffusion rate and loss
rate are very slow compared to the mid $L$ region, so the phase space density variation is very small in the low $L$ region. A peak at $L \sim 4$ is created quickly and decays very slowly. The peak is created due to the slower decrease of phase space density at $L < 4$ compared to nearby regions. The height of the peak decreases with time.

By comparing these three time dependent test cases we can conclude that, in order to distinguish the cause of phase space density peak from internal source, external source or loss: 1) Observations should be made during the time that the source is active, and 2) The measurements must cover a large enough range of $L$ shells that contains the phase space density peak.

3.4 Summary and discussion

In this chapter I first presented a numerical code for solving radial diffusion equations. The code is able to calculate steady-state phase space density profiles, and is also able to track the time dependent evolution of the phase space density. This radial diffusion solver allows general $L$- and time-dependent radial diffusion coefficients, losses, sources, and inner, outer boundary conditions. The accuracy of the code was verified by comparing to an analytical solution.

Next, the radial diffusion code was used to solve several test cases. The first test case was the steady state solution with several different analytical diffusion coefficients in power law form. The result shows that different $L$ power law dependence in the diffusion coefficient can greatly change the phase space density profile. With high power law dependence, the profile increases very steeply at low $L$, and at high $L$ the phase
space density is much higher and flatter. For all cases, the phase space density value is increasing with increasing $L$.

Then I presented three test cases of time-dependent diffusion. In the first test case, the outer boundary is zero, and an internal source acts for a short time. In the second test case, the outer boundary is also zero, but an external source acts for a short time. Both cases result in similar phase space density radial profiles after a long period of time: namely, a peak at $L \sim 4$. To distinguish an internal source from an external source one must observe the phase space density when the source is active, and time-dependent multipoint measurements are necessary. The third time-dependent test case had a steady loss term and a steady external source. The phase space density changes at locations higher than the peak are very similar to the external source test case (before the external source was turned off). This indicates that measurement at one location is not enough to distinguish a loss process from an external source case. An internal peak is also created in this test case although there is no internal source, and the external source is steady. Again, in order to determine whether the change in phase space density is due to the internal source, external source or loss, one must make time-dependent observations during the period that the source is active, and the observation must be made over a range of $L$ shells that covers the region of the phase space density peak.

In Chapter 6, I will apply this radial diffusion model to simulate the radiation belt dynamics during the September 24-26, 1998 magnetic storm. In order to simulate this real event, the inputs should be more realistic than the ones used in the idealized test cases in this chapter. Furthermore, observations such as particle flux and magnetic field are made in coordinates such as spherical coordinates or GSM coordinates, but the radial
diffusion equation uses the Roederer L-value coordinate. The coordinate conversion and
the conversion of differential flux to phase space density will be presented in the next
chapter.
Chapter 4

Roederer $L$ shell and Phase Space Density Evaluation

4.1 $L$-shell calculation

The Roederer $L$-shell is defined in Equation (2-6) in Chapter 2: $L = \frac{2\pi k_0}{R_E \Phi}$, where $\Phi = \oint_C \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot dS$ is the magnetic flux enclosed by a drift orbit. The numerical calculation of $L$ value involves the integration of magnetic flux.

The integrating contour $C$ is the particle guiding center drift orbit $CI$ shown in the schematic plot Figure 4-1. The surface integration of magnetic field can be done on surface $S1$, which is enclosed by $CI$. However such an integration requires the knowledge of the magnetic field over a large region from the inside the earth to the particle’s drift orbit. This is problematic since the magnetic field values inside the earth are not known. There is a more convenient and accurate method to calculate this flux integration which uses the conservation of magnetic flux (or, equivalently $\nabla \cdot B = 0$).

As illustrated in Figure 4-1, magnetic field lines starting from a point on the drift orbit $CI$ will intersect with the Earth’s surface. The intersection points from a new contour $C2$ on the Earth’s surface. Due to the conservation of magnetic flux, the magnetic flux enclosed by $C2$ is equal to the negative of the flux enclosed by $C1$, which is $\Phi$. At the Earth’s surface we can use the IGRF (The International Geomagnetic Reference Field) or dipole magnetic field as a good approximation.
The magnetic flux calculation can be summarized as the following three steps: (i) Trace the particle drift orbit (contour $C1$). (ii) Trace a set of field lines starting from the points on this drift orbit until the field line intersects the Earth's surface. This is the contour $C1$. (iii) Integrate the magnetic flux over surface $S2$ using the IGRF or dipole magnetic field model. Next I will describe the details of each calculation step:

For $K = 0$ particles the drift shell can be calculated by tracing the guiding center motion of a particle with 90 degree pitch angle. I use a fourth-order Runge-Kutta method with adaptive stepsize control for numerically integrating the $\nabla B$ drift equation

$$V_G = \frac{m v^2}{2 q B^3} (B \times \nabla B).$$

Note that the integration is done in a “snapshot” of the magnetic field at a given instant.

The magnetic field line tracing in step ii) is an ODE integration of the field line direction vector. I also used a fourth-order Runge-Kutta method with adaptive stepsize control for this integration.
The final step is a straightforward surface integration. Using a dipole magnetic field model to approximate the surface field, then

$$\Phi = -\frac{k_0}{R_E} \int_0^{2\pi} \cos^2(\lambda_e(\phi)) \, d\phi,$$

where $\lambda_e(\phi)$ is the dipole latitude, which is a function of longitude of the Earth surface intersection point at azimuthal position $\phi$.

For motion in a pure dipole magnetic field, $L$ is equal to $R/R_E$, the radius of the circular drift orbit in units of Earth radius. For the magnetic field of the Earth, the $L$ value is not equal to $R/R_E$, and may vary with time depending on geomagnetic conditions. In the inner magnetosphere, the major cause of the deviation from the dipole field is the southward magnetic field generated by the ring current, which can be associated with the $Dst$ index. During the main phase of a magnetic storm, the $Dst$ value drops to a lower value because the enhanced ring current weakens the northward magnetic field. Consequently, at a fixed radial location, $\Phi$ increases and the $L$ value becomes smaller.

An example of the variations of $L$ values during the pre-storm and main phase of the September 24-26, 1998 storm are shown in Figure 4-2 and Figure 4-3. The $L$ values are calculated using several magnetic field models, including the Lyon-Fedder-Mobarry (LFM) global MHD model [Lyon, et al., 2004; Lyon, et al., 1998; Wiltberger, et al., 2000; Wiltberger, 1998] and the 1996 and 2001 Tsyganenko models (the Tsyganenko models are empirical, data based models) [Tsyganenko, 2002a; Tsyganenko, 2002b; Tsyganenko and Stern, 1996], and plotted versus radial distance at midnight (in units of $R_E$). The curves of $L$ values are calculated by starting from a point at midnight close to the earth (3
for the MHD $L$ curve, 2 $R_E$ for Tsyganenko $L$ curves), and increasing the radial
distance of the starting point by 0.2 $R_E$ until the particle drift orbit is no longer closed.
The last closed drift shell is an approximation of the particle trapping boundary, beyond
which the particle is no longer trapped. For a dipole magnetic field $L = R / R_E$, but in a
magnetospheric magnetic field with a ring current we expect $L < R / R_E$.

Figure 4-2 Roederer $L$ value at 2200UT on Sep24, 1998, Pre-storm
Figure 4-3 Roederer $L$ value at 0300UT on Sep25, 1998, Main phase

Figure 4-2 shows that the $L$ values calculated using the Tsyganenko models are consistent with the presence of a ring current, since the Tsyganenko 1996 and the Tsyganenko 2001 models have $L$ values smaller than $R/R_E$. However, the MHD model does not. MHD models are known to have weak ring currents, and having $L$ values larger than $R/R_E$ is probably due to the effect of magnetopause or tail currents. In Figure 4-3, both Tsyganenko models and the MHD model show decreasing $L$ values during the main phase. This is due to the enhancement of ring current during the storm time, so that $L$ values are lower than the pre-storm values. Figure 4-3 also shows the lowering of the last closed drift shell (in all models), this is due to the dayside compression.

The $R$ to $L$ mapping is usually relatively small ($<20\%$), but it can cause substantial variations of the particle flux. When all the three adiabatic invariants $M, K, L$ are conserved, the phase space density (in the coordinates of these invariants) is also
conserved, however the value of particle flux as a function of energy and radial distance may vary. One example is the so called ‘$Dst$ effect’ [Kim and Chan, 1997]: Fluxes observed at constant energy will track changes in the magnetic field strength because electrons move to a different radial distance to stay on the same $L$-shell. A major part of the high energy electron flux decrease during the main phase of a magnetic storm and part of the flux level increases in the recovery phase can be explained by this $Dst$ effect. Details of flux and phase space density conversion will be described in the next section.

4.2 Flux and phase space density conversion

Spacecraft particle detectors measure differential or integral particle flux, rather than the phase space density. The differential directional flux $j(E, \vec{\theta})$ is defined as the number of particles passing through an area $dA$ in unit time, at energy $E$ within an energy rang $dE$, at specified direction $\vec{\theta}$ within a solid angle $d\Omega$. In the magnetosphere, the most convenient reference direction is the magnetic field vector, and pitch angle $\alpha$ is commonly used as the direction indicator. Then particle flux can be written as:

$$j = j(E, \alpha).$$

The omni-directional flux is defined as $j(E) = \int_0^\pi j(E, \alpha) 2\pi \sin \alpha \cdot d\alpha$, which is the total flux regardless of the particle’s direction. Some particle detectors on spacecraft measure energy integral flux: $j(E > E_0) = \int_{E_0}^\infty dE \int_0^\pi j(E, \alpha) 2\pi \sin \alpha \cdot d\alpha$, which is the total omni-directional flux of particles above a certain energy threshold $E_0$. 
The phase space density $f$ is related to the particle differential flux $j$ by the equation

$$f (q, p) = \frac{j(E, \theta)}{p^2}, \text{ or}$$

$$f (M, K, L) = \frac{j(E, \theta)}{p^2}.$$  \hspace{1cm} (4-1)

Here the energy $E$ and first adiabatic invariant $M$ are related by

$$M = \frac{p_\perp^2}{2mB},$$ \hspace{1cm} (4-2)

where $B$ is the local magnetic field strength, and

$$E = \sqrt{m^2c^4 + p_\perp^2c^2} - mc^2.$$ \hspace{1cm} (4-3)

We assume equatorially-mirroring particles, so $p = p_\perp, K=0$, and $\theta$ is the direction perpendicular to the field line direction. $L$ can be calculated using the method described in the previous section. $B(r)$ can be obtained using a magnetic field model or by measurement. The coordinate conversion is as follows

- If given particle energy $E$ and location $r$, then $M$ can be obtained using equations (4-3) and (4-2), and $L$ corresponding to $r$ can be calculated using the method described in section 4.1.

- If given $M$ and $L$, then particle location $r$ can be obtained by looking up in a pre-calculated $L$ to $r$ mapping table, then obtaining the magnetic field strength $B$ at location $r$ using a field model or measurement, finally obtain $E$ using equations (4-2) and (4-3).
The following plots illustrate the relation between phase space density $f$ and differential flux $j$. The flux distribution is set to be a constant high value over a square domain in the energy-$R$ plane, and zero value everywhere else. This distribution is converted to phase space density $f$ in the $M$-$L$ plane. Note that a simple dipole magnetic field is used in this example ($L = R / R_E$).

![Graph](image)

**Figure 4-4** Conversion of flux $j(E,L)$ to phase space density $f(M,L)$

There are two main effects in the conversion from $j$ to $f$: the mapping of the ‘energy-$R$’ domain into the ‘$M - L$’ domain, and the scaling factor $p_{\perp}^2$. The ‘mapping’ effect will stretch the higher part of the domain further to the right, because at higher $L$ the same energy $E$ maps to larger $M$ due to the relation $M = \frac{p_{\perp}^2}{2mB}$, where $B$ is smaller at larger $L$. The ‘scaling’ effect decreases the phase space density of flux $j(E)$ at higher energy $E$. This is due to the relation $f = j / p^2$ where $p$ is proportional to $E$. 
Figure 4-5 shows the conversion of a square distribution in phase space density to flux. Again, there are two effects: mapping of the energy-R and M-L domains, and the $p^2$ scaling factor.

![Phase space density](image1)

$f$ (phase space density)

![Flux](image2)

$j$ (flux)

**Figure 4-5** Conversion of phase space density to flux

Figure 4-6 shows the phase space density obtained from a more realistic flux distribution: the AE8MAX radiation belt electron flux model. The AE8 models are statistical models based on the omni-directional differential (in energy) flux maps obtained by averaging satellite particle flux measurements. This phase space density profile will be used as the initial condition for the simulation of September 24-26, 1998 magnetic storm which will be presented in Chapter 6. Here, a dipole magnetic field is used; for storm-time simulation a more realistic magnetic field model will be used. The two-zone structure (inner zone and outer zone separated by a slot region between 2 $R_E$ and 3 $R_E$) is apparent in the flux ($j$) plot, but it is barely identifiable in the PSD ($f$) plot. This is because the inner zone at low $R$ maps to such a small region of small $M$ and $L$ in
the $M - L$ plane. Also, note that the location of the peak of the phase space density at $L \sim 6$ differs from the peaks in flux which occur at $R \sim 1.5$ Re and $R \sim 4$ Re.

![Figure 4-6 AE8MAX flux and phase-space density](image)

(a) Log10 of AE8MAX model electron flux ($cm^{-2}s^{-1}MeV^{-1}$). (b) Log10 of AE8MAX phase space density ($cm MeV/c^{-3}$), using dipole field ($L = R / R_E$)

The units for phase-space density are ($cm MeV/c)^{-3}$ in this plot and in all later plots.

### 4.3 Summary

In this chapter I presented the algorithm and numerical method for Roederer $L$-shell calculations, and the method for converting flux to phase space density (and vice versa).

The Roederer $L$ value is proportional to the inverse of the magnetic flux enclosed by a particle drift orbit. I calculate the magnetic flux by integrating particle guiding center motion in the magnetic field model, tracing magnetic field lines, and integrating the
surface magnetic flux. As a test case, by varying the geo-magnetic conditions we can see that the $L$-values drop to smaller values during storm times. This agrees with theoretical estimates and the results of previous work on the ‘$Dst$ effect’ [Kim and Chan, 1997]. We note that the MHD magnetic field shows evidence of only a very weak ring current.

Phase space density as a function of three adiabatic invariants can be calculated from particle differential flux. I explained the conversion method, and used two examples, one simple trivial test case and the other using an AE8 radiation belt flux model, to show the main features of this conversion.

In the next chapter, using this phase space density calculation method, we will introduce a new technique for estimating the phase space density gradient at the geosynchronize region, utilizing data from flux detectors on the two GOES geosynchronous satellites.
Chapter 5

Calculation of Phase Space Density Radial Gradient Using GOES Satellite Measurements

5.1 Introduction

Identifying source and loss processes is one of the key issues for the understanding and modeling of radiation belt dynamics. As discussed in Chapter 3, observational studies of phase space densities as a function of adiabatic invariants are especially important for identifying source and loss processes. Previous work [Green and Kivelson, 2004; Hilmer, et al., 2000; Selesnick and Blake, 2000] has either used measurements from a single satellite orbiting across different \( L \) shells, or two satellites located at two distant \( L \) shells. These studies are inconclusive because of the lack of simultaneous data coverage, and also because of the difficulty of reliably calculating phase space density from measurements made at widely-separated \( L \) shell locations.

In this section we will introduce a technique of measuring radiation belt phase space density gradient, which uses the simultaneous measurements from two GOES satellites located at different local times. Briefly, from integrated flux measurements on the GOES satellites we calculate the phase space densities at two very closely-spaced \( L \) shells, then we further examine the phase space density radial gradient at geostationary locations. This work was led by Dr. Terry Onsager at the Space Environment Center, NOAA, Boulder, Colorado. Dr. Anthony Chan and I participated in this work while on a sabbatical visit in 2002. This work has been published in Journal of Geophysical
Research [Onsager, et al., 2004]. My main contributions to this work were in the
calculation of the Roederer L-shell values and participation in discussions of the methods
and results.

The GOES-8 and GOES-9 satellites are both located on the geographic equatorial
plane, at a constant distance of 6.6 Earth radii from the earth center. However, they are
located at different longitudes: the GOES-8 satellite is located at 75° west longitude, and
GOES-9 is located 135° west longitude, which is 4 hours local time separation. The tilt
direction of the Earth’s dipole magnetic field is 71° west longitude, which is quite close
to the GOES-8 longitude. Thus the two satellites are located at different magnetic
latitudes: GOES-8 is at geomagnetic latitude of about 11°, and GOES-9 is at about 4°.
Figure 5-1 and Figure 5-2 are schematic illustrations of the locations of GOES-8 and
GOES-9 spacecraft. These figures show that the two spacecrafts are located at magnetic
field lines with different L shell values, and thus they are measuring particles belonging
to different L shells.
Figure 5-1 GOES-8 and GOES-9 satellite locations and $L$-shells.

In Figure 5-1, the solid line is the geographic equator at a distance of 6.6 Earth radii; the GOES satellites are located on this line. The solid dots are the locations of GOES-8 and GOES-9 satellites. Arrowed curves are the magnetic field lines passing through the satellite locations. Open dots are the intersection points of the spacecraft field line and the geomagnetic equator, and dashed lines are the corresponding $L$ shells.
Figure 5-2 GOES-8 and GOES-9 locations and field lines.

In Figure 5-2, on the left the GOES-8 and GOES-9 locations are sketched in the same plane, to show how the associated magnetic fields map to different equatorial locations. The figure on the right shows the local-time configuration of the spacecraft [Onsager, et al., 2004].

The Earth's external magnetic field is asymmetric, mainly due to the magnetosphere being compressed on the dayside by the solar wind, and also due to the magnetic field produced by the partial ring current. Since this asymmetry is basically fixed in the sun-earth coordinate system, geostationary satellites that co-rotate with the earth will experience a diurnal variation of $L$-shell value. Thus, each single satellite can also measure a small range of $L$-shells as they orbit around the earth during a day. This type of single-satellite measurement can be helpful for verifying the phase space density gradient calculated using the two satellites separately. The $L$-shell results to be presented in the next section will show that the diurnal variation is large enough to decrease the $L$-
shell value at GOES-8 to be smaller than that of GOES-9 at certain times. The times when GOES-8 and GOES-9 are at the same $L$-shell are important for the phase-space density calculation, which will be described in Section 5.3.

5.2 $L$-shell value of GOES satellites

I calculate $L$-shell values of the GOES satellites by the following steps: **Step 1.** Trace a magnetic field line from the satellite and find the location of minimum $|\mathbf{B}|$. The locus of minimum $|\mathbf{B}|$ points corresponds to the drift orbit of 90 degree pitch angle particles. I used a fourth-order Runge-Kutta method to trace the field lines in north and south directions, until they hit the Earth surface or until a minimum $|\mathbf{B}|$ is found. An example of this calculation is shown in Figure 5-3, when satellite GOES-8 crossed the GSM noon meridian on Feb 3, 1996. In this figure and throughout this chapter, the Tsyganenko 2001 (T01) magnetic field model [Tsyganenko, 2002a; 2002b] is used. **Step 2.** Starting from the minimum $|\mathbf{B}|$ point, trace the electron’s guiding center drift trajectory, then calculate the enclosed magnetic flux and the corresponding $L$-shell value. For this step I used the same $L$ shell calculation method described in Section 4.1.
Figure 5-3 Field line tracing for geomagnetic equator.

GOES-8 crossing the GSM noon meridian on Feb 3, 1996. The solid dot is the satellite location, the open dot is the magnetic equator (minimum B) point. The dashed lines are the dipole axis and dipole equator plane.

The input parameters of the T01 model include $Dst$, solar wind pressure, solar wind magnetic field $By$, $Bz$ components, and solar wind velocity $Vx$, $Vy$, $Vz$. These inputs and the GOES satellite measurements are available in the GSM (Geocentric Solar Magnetospheric) coordinate system. The GSM system has its $X$-axis directed from the Earth to the Sun. The $Y$-axis is defined to be perpendicular to the Earth's magnetic dipole so that the $X-Z$ plane contains the dipole axis. The positive $Z$-axis is chosen to be in the same sense as the northern magnetic pole. All solar wind measurements have been propagated (using the average solar wind $Vx$ and average position of the solar wind...
monitor) to the Earth's location, and interpolated to the GOES time grid. $Dst$ has likewise been interpolated to the GOES time grid.

Figure 5-4 shows the resulting $L$ shell values of GOES satellites from Feb 1 1996 to Feb 12 1996, calculated using the Tsyganenko 2001 magnetic field model.

**Figure 5-4** $L$ value of GOES-8 and GOES-9 calculated using the T01 magnetic field model over 11 days, from Feb 1 to 12, 1996. The blue curve is the GOES-9 $L$ shell, the brown curve is the $L$ shell of GOES-8.

Note the following features of Figure 5-4:

1. The $L$ values of each satellite show clear diurnal variation: the $L$ value is larger when the satellite is in the night side and lower in the dayside. This is caused by the day-night asymmetry of Earth's magnetic field. Earth's magnetosphere is compressed by the
solar wind from the dayside, so the geostationary satellites located at constant radial distance from the Earth will have larger $L$ value at the night side than the dayside.

(2) The peaks and troughs $L$-shell curves of the GOES-8 and GOES-9 satellites are approximately shifted in time by 4 hours. This is consistent with the fact that the two GOES satellites are separated by roughly 4 hours in local time (section 5.1).

(3) The highest $L$ value of GOES-8 is higher than the highest $L$ values of GOES-9, and the lowest $L$-shell value of GOES-8 is also higher than that of GOES-9. Shifting the GOES-9 $L$-shell curve back 4 hours, one can see that the GOES-8 $L$ value is generally higher than that of GOES-9. This is expected due to the different geomagnetic latitude of the two satellites, as reasoned in section 5.1.

(4) The two curves cross twice a day. This happens because the $L$-shells of the two satellites are close enough (the geomagnetic latitudes of the two satellites are close enough), and the local time shift and diurnal variation are large enough. These conditions may not be satisfied for other satellites. Having $L$-shell crossing for GOES 8 and GOES 9 satellites is a lucky ‘coincidence’. This feature is very important for improving the accuracy of the phase space density gradient calculation, as described in the next section.

Although one GOES satellite on its own orbit can also experience an $L$ shell variation from $L \sim 6$ to $L \sim 7$ twice a day, one single satellite is not able to measure gradient in the time scale shorter than 12 hours, and more importantly, it is highly affected by other factors such as azimuthal changes of phase space density, and magnetosphere variations one time scales of less than half a day. Only the simultaneous $L$ shell differences between the two satellites can allow us to calculate the simultaneous phase space density gradient.
5.3 Conversion of GOES satellite flux measurements to phase space density

Next we calculate the value of phase space density from the GOES electron flux measurements. The GOES spacecrafts are three-axis stabilized, and the onboard electron sensors have a large field of view pointed westward [Onsager, et al., 1996]. The GOES particle sensors measure directional, integral electron flux at energies higher than 2 MeV: $j(E > 2 \text{ MeV}, \vec{r})$, where $\vec{r}$ is the location of satellite. We can assume that the GOES detectors’ wide-aperture measurements are representative of the 90° pitch angle flux at local latitude. Figure 5-5 shows GOES-8 and GOES-9 electron flux measurements in the beginning of February, 1996.
Figure 5-5  Electron flux measured by GOES 8 and GOES 9

(a) Electron flux $> 2$ MeV at GOES-8 (solid) and GOES 9 (dashed). (b-f) Solar wind conditions. (g) $Dst$ [Onsager, et al., 2004]

The electron flux in subplot (a) shows diurnal variations in the measurements by both GOES-8 and GOES-9 satellites. There is a 4 hour time shift between the two flux curves.
The integral flux measured by GOES-9 is generally higher than that of GOES-8. But this does not necessarily imply that the electron phase space density at GOES-9 is also higher. The phase space density of the two satellites must be compared at a constant first adiabatic invariant $M = \frac{p^2}{2mB}$. For example, when GOES-9 is located at a lower $L$-shell than GOES-8 (as analyzed in the previous subsection), the magnetic field strength $B$ is stronger at GOES-9 than GOES-8, thus the corresponding energy for the same adiabatic invariant is higher for GOES-9. Since the electron energy spectrum has roughly an exponential distribution with lower flux at higher energy, this effect will bring down the phase space density of GOES-9, and it could be lower than that of GOES-8. Also, since the two satellites are located at different magnetic latitudes, they are measuring particles with different pitch angles. Due to these complications, we cannot determine the sign of the phase space density gradient by simply comparing the flux measurements, more accurate analysis is needed.

The phase space density gradient $\frac{\partial f}{\partial L}$ needs to be evaluated at fixed $M$ and $K$, and at two locations with different but very close $L$-shell values. Two main assumptions are made to allow the calculation of phase space densities from the satellite flux measurements $j(E > 2 \text{ MeV}, \vec{r})$. The first assumption is that the electron phase space density has an exponential distribution in energy:

$$f(E) = f_0 e^{-\frac{E}{E_0}}$$  \hspace{1cm} (5-1)
The phase space density is related to the differential flux as \( f = \frac{j(E)}{p^2} \), and

\[ p^2c^2 = E^2 + 2mc^2E, \]

where \( E \) is the particle’s kinetic energy measured by spacecraft sensors. Remember that integrated flux is \( J(>E) = \int_E^\infty j(E')dE' \). Then the integrated flux can be expressed as:

\[
J(>E) = \int_E^\infty \frac{f(E')}{c^2}(E'^2 + 2mc^2E')dE'
\]  \hspace{1cm} (5-2)

Using the assumption of equation (5-1) and integrating, we obtain the phase space density at energy \( E \) as a function of the integrated flux:

\[
f(E) = \frac{c^2}{(E_0^2 + EE_0)2mc^2 + 2E_0^3 + 2EE_0^2 + E^2E_0} J(>E).
\]  \hspace{1cm} (5-3)

With magnetic field information (which can be obtained either from the satellite magnetometer or a model field), we can convert the first adiabatic invariant \( M \) to \( E \), and obtain \( f(M) \) at a given \( M \). This is the phase space density at the satellite location, and the second adiabatic invariant \( K \), which is related to the latitude, is still a variable. We need further assumptions to determine the phase space density at a fixed \( K \).

The second main assumption is that the pitch angle distribution at the geomagnetic equator has the following form:

\[
f_{eq}(E, \alpha_{eq}) = f_{eq}(E) \sin^m \alpha_{eq}.
\]  \hspace{1cm} (5-4)

Here \( \alpha_{eq} \) is the pitch angle of a particle at the equator, \( f_{eq}(E) \) is the omni-directional phase space density at the equator, and \( f_{eq}(E, \alpha_{eq}) \) is the equatorial phase space density.
at pitch angle $\alpha_{eq}$. Assuming constant first invariant, the pitch angle is a function of magnetic field strength:

$$\sin \alpha_{eq} = \left( \frac{B_{eq}}{B_\lambda} \right)^{\frac{1}{2}} \sin \alpha_\lambda = \left( \frac{B_{eq}}{B_\lambda} \right)^{\frac{1}{2}},$$

where $B_\lambda$ is the magnetic field strength at latitude $\lambda$. At the satellite location, the pitch angle is approximately $\alpha_\lambda = 90^\circ$. Substituting into equation (5-4), we can relate the phase space density at the equator to the phase space density at any latitude $\lambda$ as follows:

$$f_\lambda (E, 90^\circ) = f_{eq} (E) \left( \frac{B_{eq}}{B_\lambda} \right)^{\frac{m}{2}}.$$  

(5-5)

Using this in the left hand side of equation (5-3), we can relate the equatorial phase space density to the measured integral flux as:

$$f_{eq} (E, 90^\circ) = \frac{c^2}{(E_0^2 + E E_0) 2mc^2 + 2E_0^2 + 2EE_0^2 + E^2E_0} \left( \frac{B_{eq}}{B_\lambda} \right)^{\frac{m}{2}} J (> E).$$  

(5-6)

This equation is used to calculate the equatorial phase space density at the $L$-shell of each satellite. We define $f_8$ and $f_9$, the equatorial phase space density values at $L$-shells $L_8$, $L_9$, where $L_8$, $L_9$ are the $L$-shell value of the GOES-8 and GOES-9 satellites, respectively. Then the normalized phase space density gradient is simply:

$$\frac{1}{f} \frac{\Delta f}{\Delta L} \approx \frac{1}{f_9} \frac{f_0 - f_8}{L_9 - L_8}.$$  

(5-7)

### 5.4 Results
In this analysis we chose the value of the first adiabatic invariant to be $M = 6000$ MeV/G, which corresponds to 2 MeV electrons at geostationary orbit, and we set the second adiabatic invariant to be $K = 0$ for equatorial mirroring particles. For this first study, the time period we chose is a magnetically quiet period from February 1 to February 8, 1996, because the currently available magnetic field models are more reliable at quiet times, and the electron energy spectrum model and the pitch angle distribution model are also more accurate.

For the calculation of phase space densities we need two parameters: $E_0$ in the energy distribution equation (5-1), and $m$ in the pitch angle distribution equation (5-4). We can determine the pitch angle index $m$ by comparing the phase space density of two satellites at the time when they were at equal $L$ values. As shown in section 5.2, the $L$-shell curves of GOES-8 and GOES-9 cross twice a day. At the moment that the two satellites were at the same $L$ value, the equatorial phase space density should also be equal. Using equation (5-6) we have:

$$
\frac{f_8(E)}{f_0(E)} = \frac{J_8(>2)}{J_0(>2)} \left( \frac{B_{8\lambda}}{B_{0\lambda}} \right)^m = 1.
$$

From this equation we can estimate the pitch angle index $m$.

The phase space density gradient must be evaluated at a fixed $M$. Equation (5-6) indicates that the value of $E_0$ does not change the sign of $(f_0 - f_8)$, thus it does not change the sign of phase space density gradient $\frac{1}{f_0} \frac{f_8 - f_0}{L_0 - L_8}$, although it may change the magnitude of the gradient. Based on the values of $E_0$ measured by Cayton, et al. [1989] and McAdams, et al. [2001], we choose $E_0 = 250$ keV.
Figure 5-6 shows the results of this phase space density gradient analysis for the 3-day period 3-6 February 1996. In order to make the phase space density equal at L-shell crossing points, the value for pitch angle index was found to be in the range from 0.25 to 3. We assumed the pitch angle index to have an intermediate value of 2.

The normalized phase space gradient $\frac{1}{f} \frac{\Delta f}{\Delta L}$ is shown in Figure 5-6d. At the L shell crossing points, the calculated phase space gradient value goes to large positive or negative values. This is due to the divisor $\Delta L$ approaching zero at these points. Except at these special points, the gradient is positive over the 3 day period - the phase space density is higher at higher L-shell.
Figure 5-6 Phase space gradient

(a) Phase space density at GOES-8 (solid) and GOES-9 (dashed). (b) Difference in phase space density. (c) Difference in L shell. (d) Normalized phase space density gradient.

[Onsager, et al., 2004]

The phase space densities of each satellite are shown in Figure 5-6a. They both show diurnal variations with higher values of phase space density at the night side than dayside. Because on each satellite's orbit, the $L$ value is higher at the night side than dayside, this supports the result that the phase space density gradient is positive. This result also agrees with previous studies of phase space density gradient during geomagnetic quiet
times [Green and Kivelson, 2004; Hilmer, et al., 2000]. This positive phase space density gradient supports the existence of inward radial diffusion source located at high $L$ regions.

The main advantage of this method is that the radial gradient of the phase space density is measured simultaneously and we are able to provide continuous results over a long period of time. In the future we can improve the accuracy by using a local-time dependent pitch angle index. Also, with more reliable magnetic field models we can also apply this method to magnetic active times.

5.5 Conclusion

In this chapter we introduced a new technique for estimating the phase space density gradient at geo-synchronous orbit utilizing the flux detectors on two GOES satellites. Although GOES 8 and GOES 9 are both at distance of 6.6 Re on the geographic equator, due to their difference in longitude they are located at different geomagnetic latitudes, and have different $L$ values. They both experience diurnal $L$-shell variations, but their $L$-shell curves are shifted by 4 hours, and they cross twice a day. By calculating the ratio of phase space densities at these two different $L$-shells for the same first invariant, mapped to the minimum-B location, we can obtain the equatorial phase space density gradient.

We established formulae for estimating phase space density from the measured omni-directional integrated flux. Using the fact that the phase space density should be the same at the two crossing points, we fixed the pitch angle distribution index $m$. 
Then using this formula, we calculated phase space densities at the L shells of the two satellites for a period of three days during geomagnetic quiet time. The result shows persistent positive gradient in this period (the phase space density at higher L shell is higher than lower L shell). This result supports inward radial diffusion from a source at the high L regions.

Chapter 6

Radial Diffusion Calculation of September, 98 Storm

6.1 Introduction

As mentioned in Chapter 1, in an MHD-particle simulation of the September 1998 magnetic storm the evolution of the radiation belt electron radial flux profile appears to be diffusive, and diffusion caused by ULF waves has been invoked as the probable mechanism. In order to separate adiabatic and non-adiabatic effects, and to investigate the radial diffusion mechanism during this storm, in this Chapter I solve a radial diffusion equation with ULF-wave diffusion coefficients and a time-dependent outer-boundary condition, and compare the results with the phase space density of the MHD-particle simulation. The diffusion coefficients include contributions from both symmetric resonance modes and asymmetric resonance modes derived in Chapter 2. ULF-wave power spectral densities used in calculating the radial diffusion coefficients are obtained from a Fourier analysis of the electric and magnetic fields of the MHD simulation. The asymmetric diffusion coefficients are proportional to the magnetic field asymmetry,
which is also calculated from the MHD field. The diffusion calculation simulates a 42-hour period during the September 24-26, 1998 magnetic storm, starting just before the storm sudden commencement and ending in the late recovery phase. The differential flux calculated in the MHD-particle simulation is converted to phase space density. Phase-space densities in both simulations (diffusion and MHD-particle) are functions of Roederer $L$-value for fixed first and second adiabatic invariants. The Roederer $L$-value is calculated using drift shell tracing in the MHD magnetic field, and particles have zero second invariant.

A number of approaches of radiation belt modeling have been proposed, including data-based empirical models, test particle simulation models, and radial diffusion models. Previous radial diffusion studies have introduced various diffusion coefficients, applicable in certain regions, time scales, and geomagnetic conditions. Some work assumes time-independent radial diffusion coefficients and dipole magnetic field

In this chapter I present radial diffusion calculations of the September 24-26 1998 magnetic storm. This study uses the ULF wave driven diffusion coefficients derived in Chapter 2, the numerical radial diffusion method presented in Chapter 3, and the \( L \) shell and phase space density calculation method presented in Chapter 4. The new diffusion simulation in this chapter covers all phases of a magnetic storm, and uses a wave-driven diffusion coefficient calculated based on the MHD wave power spectrum. The diffusion code also employs dynamic outer boundary locations and outer boundary values.

The results of this radial diffusion calculation are compared with the results of the MHD-particle simulation of the same storm. The particle simulation directly tracks the motion of individual test particles in the electric and magnetic fields provided by the global MHD simulation.

The remainder of this chapter is organized as follows: First we show the method and results of the MHD-particle simulation of the September 24-26 1998 magnetic storm. Then in Section 3 we discuss the details of the radial diffusion calculation, including the electric and magnetic ULF power, the radial diffusion coefficients and the boundary conditions. The results of the radial diffusion calculation and the comparison with the MHD-particle simulation are given in Section 4.

6.2 MHD-test particle simulation
6.2.1 Introduction

The MHD/particle technique used to compare with the radial diffusion calculations in this study is discussed in detail in [Elkington, et al., 2004], relevant points are briefly reviewed in this section.

The test particle simulation tracks the physical motion of a large number of test particles in the global MHD electric and magnetic fields. Three main approximations can be made to make the test particle simulation much more efficient than directly solving the 6-D Vlasov equation' but still accurate enough for the radiation belt electrons:

(1) The high energy electrons only contribute a very small portion of the total ring current energy density, so the feedback electric and magnetic field generated by the simulated particles can be ignored. One can use only the external model field to push the particles. The test particle simulation by [Li, et al., 1993] used a simple analytical field model, and the simulation by [Elkington, et al., 2002] employed the electric and magnetic field produced by global MHD simulation.

(2) The first adiabatic invariant is generally conserved for radiation belt electrons. The guiding center approximation is applicable and can greatly reduce the numerical computation time by a factor of roughly $\rho/L_N$ (the ratio of the electron gyro radius to the magnetic field scale length).

(3) The equatorially mirroring particles are of the most interest, since the majority of the radiation belt population has pitch angles near $90^\circ$. Tracking only equatorially mirroring particles can reduce the 3-D guiding center equation into 2-D.
The MHD-particle simulations used for the comparison with my radial diffusion calculation is discussed in [Elkington, 2000; 2002]. This model numerically tracks the equatorial guiding center motion of about $10^6$ test particles in the electric and magnetic field produced by the Lyon-Fedder-Mobarry (LFM) global MHD simulation. The LFM global MHD code [Lyon, et al., 2004; Lyon, et al., 1998; Wiltberger, et al., 2000; Wiltberger, 1998] consists of two coupled simulations: a Solar wind-Magnetosphere simulation, and a Magnetosphere-Ionosphere simulation. The LFM MHD simulation is applicable to magnetic storms when it is driven by realistic solar wind inputs [Wiltberger, et al., 2000]. Each test particle in the simulation (representing an ensemble of particles with similar phase space coordinates) was given an initial weight dictated by the NASA AE-8 trapped electron flux model [Vette, 1991], and the time evolving state of the radiation belts was calculated by evolving the flux associated with each test particle in a manner satisfying Liouville’s theorem along the individual test particle trajectories [Elkington, et al., 2004]. Particles failing to maintain the first adiabatic invariant (according to the Chirikov criterion [Chirikov, 1987]) and particles drifting through the magnetopause and beyond the boundaries of the simulation were removed from the simulation. The MHD-test particle simulation gives the differential flux of equatorial particles $j(E,r,t)$ at any given equatorial location $r$ and energy $E$.

6.2.2 MHD-particle simulation of September 24-26, 1998 storm

The LFM global MHD code was used to model the time-evolving geomagnetic environment during the September 24-26, 1998 magnetic storm. This event was
characterized by high solar wind speeds and an order-of-magnitude increase in outer-zone energetic electron fluxes. Solar wind conditions from the upstream monitor WIND were used to provide time-dependent boundary conditions for the LFM model. The computation domain of the LFM MHD simulation is three dimensional, extending from 30 $R_E$ upstream to 300 $R_E$ tailward, $\pm 100 R_E$ at the flanks, with about the inner boundary located at around 2.3 $R_E$. The MHD-particle simulation provides electron differential flux $j$ of 90° pitch angle on the equatorial plane, covering radial distances from 2.3 $R_E$ to 9 $R_E$, with energy from 0 to 10 MeV.

The storm sudden commencement on September 24 led to a limited injection and energization of electrons, but the time scale of bulk acceleration in the simulations suggests another acceleration mechanism was active. Examples of equatorially-mirroring electron fluxes calculated by an MHD-particle simulation of this storm are shown in Figure 6-1, for the periods prior to storm onset (top left); following sudden commencement (top right); and during the initial period of the recovery phase (bottom). Contours of constant first adiabatic invariant are indicated in grey. The simulation results show electrons have been transported inward. These particles gain energy particularly in the main phase and recovery phase.
Figure 6-1  Electron flux profiles at three different times of MHD-particle simulation of the September 24, 1998 geomagnetic storm. Flux is plotted in color vs energy and radial distance at local midnight. Grey contours indicate lines of constant first adiabatic invariant, $M$, calculated in a dipole magnetic field.

In Figure 6-2 and Figure 6-3 we compare the results of the MHD/particle simulations with geosynchronous observations of the particles and fields during the time period modeled. Figure 6-2 shows the observed (red) and simulated (blue) magnetic fields observed at GOES-8 from the beginning of the simulation on September 24th through mid-day during the recovery phase on September 26th. While the large scale fluctuations observed during the main phase are not reproduced in exact detail by the simulation, the onset of the sudden commencement, the increased dynamic activity during the main phase, and the large-scale magnetic trends during the recovery phase are all captured by the MHD simulation.

A comparison of the simulated electron fluxes to those observed at geosynchronous orbit is shown in Figure 6-3. Here integral electron fluxes $>2$ MeV are plotted for a 30 hour period at the location of GOES-8. The character of the fluxes seen in the simulation at this spacecraft location match those observed at the spacecraft both qualitatively and,
to a large degree, quantitatively. Substantial variations over three orders of magnitude are seen in this period, and are well reproduced in the simulations. While a good deal of the variation can be attributed to variations in the local magnetic field and the motion of the spacecraft across $L$ shells as it moves in a circular orbit within the asymmetric dipole of the Earth, the agreement between the measured and simulated fields and fluxes suggest that the MHD/particle simulations do a reasonably good job of quantitatively tracking the evolution of the fields and particles over an extended period of time.

![Graph showing simulated and observed magnetic fields](image)

**Figure 6-2** Simulated and observed magnetic fields at the location of GOES-8 during the September 24-26, 1998 geomagnetic storm.
Figure 6-3 Simulated (top) and observed (bottom) electron fluxes at the location of GOES-8, for a 30 hour period spanning storm main phase and recovery phase.

6.2.3 Phase space density from the MHD-particle simulation

We note that the nature of the guiding-center simulations conducted for this study precludes inclusion of any non-adiabatic effects that may have been present in the real magnetosphere (e.g. local heating and loss due to wave-particle interactions). However, the fact that these results are strictly limited to processes related to the radial transport of particles makes them especially useful as a means of testing radial transport models. In particular, recent work has shown that large-amplitude Pc5 ULF waves (approximately in the 1.5 to 10 mHz frequency range) are correlated with electron flux increases, and there is evidence that Pc5 oscillations play a fundamental role in storm-time particle dynamics over periods of hours and longer by enhancing transport through radial diffusion [Baker, et al., 1998; Elkington, et al., 1999; , 2003; Mathie and Mann, 2000; Rostoker, et al., 1998]. A comparison between the MHD/particle simulation presented above and a
purely diffusion formulation of radiation belt dynamics can therefore provide insight into the extent to which radial transport may be treated as a diffusive process, and the accuracy of current models of radial diffusion in the outer zone.

Using the $L$ shell and phase space density calculation method described in Chapter 4, the equatorial differential electron flux of this MHD-particle simulation is converted into phase space density as a function of the adiabatic invariants. The phase space density $f(M, L, t)$ is a function of Roederer $L$, at $K = 0$ (which corresponds to equatorially mirroring particles), and at fixed first invariant $M = 1870$ MeV/G which corresponds to 1 MeV electrons at 6.6 $R_E$.

For consistency, we use the magnetic field provided by the LFM MHD code for this storm, which is the same magnetic field used in the test particle simulation. In the L shell calculation we define the last closed drift shell as $L_{\text{max}}$. The phase space density was calculated at L shells ranging from 2.3 to $L_{\text{max}}$.

Phase space density $f_{\text{MHD}}(L, t)$ calculated from the flux result of the MHD-particle simulation is shown in Figure 6-4. The storm time solar wind data and Dst index are shown in Figure 6-5.
Figure 6-4  Phase-space density of the MHD-particle simulation

From 22UT September 24 to 12UT September 26.
Figure 6-5 Solar wind conditions observed at the spacecraft WIND in the period between 12 UT, September 24, 1998 and 12 UT September 26, 1998. The geomagnetic response as characterized by the hourly $Dst$ index is plotted in panel (f).

During the storm sudden commencement and main phase (23UT September 24, 1998 to 2UT September 25, 1998) the magnetopause has been compressed from larger than 9
$R_E$ into approximately 5 $R_E$, and the trapped electrons at high $L$ shells were lost during this period. In the early recovery phase (2UT to 14UT, September 25) the location of the trapping boundary $L_{\text{max}}$ gradually relaxed outward, but was still quite dynamic. Particles appear diffuse to both inward and outward from the center of the phase space density peak at $L = 4.5$. In the late recovery phase the location of $L_{\text{max}}$ was comparatively steady and the value of phase space density was lower at higher $L$ shells and the location of the peak continued to move inward spread from the peak at $L \sim 4.2$ to higher and lower $L$ shells. However, during this late recovery period the diffusion and advection rates were apparently rather slow.

6.3 Radial diffusion calculation

The radial diffusion calculation is based on the numerical radial diffusion code described in Chapter 3, and the $L$ shell and phase space density calculation method has been described in Chapter 4. In order to examine the relative importance of different acceleration processes during the storm we have examined different diffusion coefficients, including our newly derived analytical diffusion coefficient as a function of electric and magnetic ULF wave power spectrum densities (derived in Chapter 2).

6.3.1 Initial condition and computation domain

The initial condition of this radial diffusion calculation is the phase space density $f_0(L)$ converted from the AE8MAX electron flux model. The September 98 MHD test particle simulation uses the same electron flux model as initial condition: the initial test
particle distribution is allocated to represent the flux profile of AE8MAX. The computation domain is uniform in $L$, ranging from 2.3 to 9, and uniform in time from 0 to 60 hours. The radial diffusion coefficient $D_{rL}$, the outer boundary location $L_{\text{max}}(t)$, and the value of the outer boundary phase-space density $f(L_{\text{max}}, t)$ are updated at each time step.

6.3.2 Boundary condition

During magnetic storms, the outer trapping boundary of radiation belt electrons is dynamic. Trapped particles can drift across the magnetopause and be lost, and magnetosheath particles may enter and become trapped. The value of the phase-space density at the outer trapping boundary is determined by both the processes of particle loss through the boundary and by the source population outside of the trapping boundary. In this radial diffusion calculation we determine the location of the outer boundary by tracing test particle guiding center motion in the MHD magnetic field and find the last closed drift shell $L_{\text{max}}$. Figure 6-6 shows the temporal variations of $L_{\text{max}}$ in the September 1998 storm. $L_{\text{max}}$ varies from about 4.5 to 10 during the storm. At 2UT and 6UT September 25 $L_{\text{max}}$ went down to 4.5 and 5.5, corresponding to the peaks in the solar wind dynamic pressure (see Figure 6-5) during the same time. In the early recovery phase (2UT to 14UT, September 25) the location of the trapping boundary $L_{\text{max}}$ gradually relaxed outward, but was still quite dynamic. In late recovery phase, the location of $L_{\text{max}}$ was comparatively steady. The $L_{\text{max}}$ calculated using our drift-shell tracing is slightly inside the particle boundary of the MHD-particle simulation.
Figure 6-6  $L_{\text{max}}$ vs. time from 22 UT, September 24 to 16 UT, September 26

Calculated by tracing particle guiding center motion and finding the last closed drift shell

Time-dependent values of phase space density at the location of the last closed drift shell ($L_{\text{max}}$) are used as the outer boundary condition $f_{BC}$. Initially I used a constant outer-boundary value, but the result was in poor agreement with MHD-particle result $f_{MHD}$. Figure 6-7 shows the values of the phase-space density at the outer boundary location $L_{\text{max}}$. 
Figure 6-7 Outer boundary value of phase-space density calculated from MHD-particle result at $L_{\text{max}}$. The time axis is the same as Figure 6-4

The outer boundary value $f_{BC}$ has a fairly constant high value before the storm onset, then after the sudden commencement and during the beginning of main phase, the trapping boundary location moved quickly inward to $\sim 4.5$ $R_E$ and the value of $f_{BC}$ decreases. During the main phase and the beginning of the recovery phase both the location of the outer boundary and the value of $f_{BC}$ are very dynamic. After 18UT Sep 25, the trapping boundary stays at larger $L$ and $f_{BC}$ decreases to more steady low values.

6.3.3 Electric and magnetic ULF wave power

The analytical radial diffusion coefficients (derived in Chapter 2) are functions of power spectral density of ULF electric and magnetic perturbation waves. We measure
the power spectral density from the MHD magnetic field model during the September 1998 storm. Following the method of [Holzworth and Mozer, 1979], an FFT series expansion is taken in azimuthal angle to obtain the power spectral density in each of the $m = 1, 2, 3, 4$ modes. Then, for each mode, an FFT transform is taken in the time domain to calculate the power spectral density $P(\omega)$.

The result of the MHD simulation is interpolated to uniform time steps of 15 seconds, and the time window of the FFT calculation is 2 hours. The resulting frequency range of this spectrum analysis is 0.27 mHz to 10 mHz, which covers the electron resonance frequencies in the ULF Pc5 band.

Theoretically the power spectral density of stationary random signals can by obtained by using the Welch method: split a long time series of signal into small segments, and calculate the power spectra of each segment (which is noisy), then ensemble average all the power spectra of each segment to get an averaged smooth power spectrum. For the storm time power spectral density calculation, especially during a single storm, there is not enough data for such an averaging method, and the power spectral density calculated using one single event is quite noisy. So we apply smooth Gaussian filters to reduce the random noise and to approximate the averaged power spectral density.

The storm-time power spectral densities for the resonance frequencies are shown in Figure 6-8 for electric fields and Figure 6-9 for magnetic fields. The vertical axis is $R$ and the horizontal axis is time. Each subplot shows the power spectral density at a certain resonance condition (multiples of the particle’s drift frequency) for electrons with first adiabatic invariant $M = 1870$ MeV/G. The resonance frequencies are:

$$\omega = (m - 1) \omega_d, \quad \omega = m \omega_d, \quad \omega = (m + 1) \omega_d$$

for $m = 1, 2, 3, 4$. (Of course, for
the $m = 1$ mode there are only $m$ and $m + 1$ modes, but no $(m - 1)$ mode.) The plots are arranged as follows: the first row is $m = 1$, the second row is $m = 2$, etc. The left column shows $m - 1$ resonances of the asymmetric diffusion coefficient, the center column shows $m$ resonances of the symmetric diffusion coefficient, and the right column shows $m + 1$ resonances of the asymmetric diffusion coefficient.

From these plots we can see the following features: (1) in all cases there is more power in the main phase than in other times, (2) there is more power at higher L-shells, (3) there are more power in low-$m$ modes, especially in the lowest $m = 1$ and $m = 2$ modes. Note that the $m + 1$ resonance with $m = 1$, the $m$ resonance with $m = 2$, and the $m - 1$ resonance with $m = 3$ all correspond to the same frequency (diagonal subplots) and note that, there is more power in the low-$m$ modes at these common frequencies.

The electric radial diffusion coefficients derived in Chapter 2 are proportional to the electric power spectral density at these resonant frequencies. We use these ULF wave power obtained from the global MHD simulation to calculate the diffusion coefficients.
Figure 6-8 Electric ULF power at resonance frequencies of an $M=1870$ MeV/G electron ($V^2/\text{m}^2/\text{Hz}$). The first zero on the time axis is 0UT of September 25.
The magnetic field power spectral density is plotted in Figure 6-9. The subplots are in the same format, and are arranged in the same way as the electric power spectral density plots (Figure 6-8). Compared to the electric power spectral density, the magnetic power spectra density shares all the common features listed above. However, the magnetic power spectral density has steeper $L$ dependence. Also, the magnetic wave power spectral density decreases faster in higher $m$ modes than the electric power spectral density. Note that the magnetic radial diffusion coefficients have an extra $m^2$ factor, which will elevate the effect of high-$m$ magnetic power.
Figure 6-9  Magnetic ULF power spectral densities
at resonance frequencies of an M=1870 electron $nT^2$/Hz. (The first zero on the time axis is OUT of September 25.)
As discussed in Chapter 2, the final form of the diffusion coefficient is the sum of diffusion coefficients at all azimuthal modes $m = 1, 2, 3, 4, \ldots$. Since most of the power is in the lowest $m$ modes, in this study we only utilize $m = 1, 2, 3, 4$ modes. Contributions from higher $m$ modes do not alter the results significantly. For $m = 1$ mode there are only two resonance frequencies $\omega = m\omega_d$ and $\omega = (m + 1)\omega_d$, and for $m \geq 2$ there are three resonances $\omega = (m - 1)\omega_d$, $\omega = m\omega_d$ and $\omega = (m + 1)\omega_d$ for each $m$ mode.

6.3.4 Asymmetry factor

In addition to the ULF wave power spectrum, the asymmetric electric and magnetic diffusion coefficients (as in section 2.4.5 and 2.5.2) also depend on the asymmetry factor $\frac{\Delta B}{B_0}$, which is a measure of the day-night asymmetry of the magnetic field. The compression coefficient $\Delta B$ is determined from the time-varying magnetic field of the MHD simulation using noon and midnight values of the equatorial magnetic field strength [Elkington, et al., 2003]:

$$\Delta B = \frac{(B_{\text{noon}} - B_{\text{midnight}})}{2}$$

The variation of the asymmetric coefficient $\frac{\Delta B}{B_0}$ during this storm is shown in the following plot.
The magnetic asymmetry is enhanced through the main phase of the storm from 0UT to 12UT, September 25, and it begins to diminish in the early recovery phase. In the following recovery phase, the field asymmetry in middle and high L regions increases, and the asymmetry is higher at larger L. This indicates the asymmetric diffusion coefficient can be more important in the recovery phase and at high L regions.
The analysis of the asymmetric diffusion coefficients assumes that $\Delta B$ is smaller than local magnetic field ($B_0$). The values shown in Figure 6-10 indicate this assumption is valid.

6.3.5 Storm time radial diffusion coefficient

As mentioned above, the total radial diffusion coefficient is the sum of the symmetric electric diffusion coefficient (2-56), asymmetric electric diffusion coefficient (2-59), symmetric magnetic diffusion coefficient (2-68), and asymmetric magnetic diffusion coefficient (2-69).

Figure 6-11 shows the total ULF driven diffusion coefficient at three selected times: 22UT September 24 (pre-storm), 6UT September 25 (main phase), and 11UT on September 26 (late recovery phase). In the main phase, the diffusion coefficient is about 1 to 2 orders of magnitude higher than the recovery phase, because both ULF power density and magnetic field asymmetry are both higher.

For comparison, we also plot a simple static power-law diffusion coefficient of the form $D_{LL} = D_0 L^n$, where $D_0 = 6.25 \times 10^{-8}$ and $n = 8.5$. The value of $D_0$ and $n$ are chosen that this power law diffusion coefficient represents an approximate average value and $L$ dependence of the time varying ULF wave driven diffusion coefficient. (Note that an empirical study of the quiet-time radial diffusion coefficient [Selesnick, et al., 1997] shows $n = 11.7$ in $L$ dependence.)
Figure 6-11 Radial diffusion coefficient at three selected times during the storm.

Figure 6-12 shows the separate contributions of the symmetric and asymmetric terms to the total diffusion coefficient, at 6UT September 25 (main phase). The asymmetric diffusion coefficient increases more strongly with $L$ than the symmetric contribution, and when $L$ is greater than about 5 it dominates. This is also true in the early recovery phase, but at other times (before the storm and in the late recovery phase) the asymmetric diffusion coefficient is generally smaller than the symmetric coefficient.
Figure 6-12 Comparison of symmetric (dashed line) and asymmetric (dotted line) contributions to the total diffusion coefficient.

It has been shown in Chapter 2 that, for an $L$-independent ULF power spectrum, the asymmetric electric diffusion coefficient has roughly an $L^{12}$ power law dependence, and the asymmetric magnetic diffusion coefficient has roughly an $L^{2.5}$ power law dependence. While the symmetric electric diffusion coefficient has roughly a $L^6$ power law dependence, symmetric magnetic diffusion coefficient has roughly $L^7$ dependence. The ULF power spectral density obtained from the MHD simulation result (Figure 6-8) is higher at higher $L$, so the asymmetric and the symmetric diffusion coefficients (dotted lines) in this figure show higher power law dependency. The static power law of Figure 6-11 is also plotted in Figure 6-12, for reference.
6.4 Radial diffusion calculation result and comparison

The radial diffusion calculation start time is 23UT of September 24, right before the storm sudden commencement, and the ending time is 17UT of September 26, late in the recovery phase. The total simulated time duration is approximately 42 hours.

The results of the storm-time radial diffusion calculations are shown in Figure 6-14a through Figure 6-18a. All the plots are for fixed value of first adiabatic invariant $M = 1870 \text{ MeV} / \text{G}$. The upper panel of each plot is the radial diffusion result (phase space density $f$). To better compare with the MHD-test particle result, the lower panels (Figure 6-14b to Figure 6-18b) show the LOG10 difference between the diffusion result and the MHD-particle result: $\log_{10} \left( \frac{f}{f_{\text{MHD}}} \right)$, where $f_{\text{MHD}}$ is the phase space density of the MHD-particle simulation shown in Figure 6-4. These difference plots show clearly when and where the radial diffusion result is over estimating or under estimating the phase space density.

We also define a global RMS deviation $\sigma$ as an indicator of overall difference between the two simulations,

$$
\sigma = \sqrt{\frac{\sum_{L, t} \left( \log \left( \frac{f(L, t)}{f_{\text{MHD}}(L, t)} \right) \right)^2}{n}}
$$

(6-8)

Where the summation is over all $L$ and time, and $n$ is the total number of values of $f$ (or $f_{\text{MHD}}$).
We have conducted this storm time radial diffusion calculation with several different choices of diffusion coefficients. Figure 6-14 shows radial diffusion calculation results using the 'total' ULF wave driven radial diffusion coefficient, with both electric, magnetic, symmetric and asymmetric terms. Figure 6-15 and Figure 6-16 shows the results using only symmetric or only asymmetric terms, respectively. Figures in the following section shows results using several other diffusion coefficients.

### 6.4.1 Radial diffusion using ULF wave diffusion coefficients

Comparing Figure 6-14a to Figure 6-4 one can see that the radial diffusion calculation using ULF-wave driven diffusion coefficients is able to reproduce the main features of the electron transport shown in the MHD-particle simulation. The value of the radial diffusion coefficient is plotted in Figure 6-13. During main phase and early recovery phase (from 0UT to 3UT of September 25), the trapping boundary was deeply compressed into \( L \sim 5 \) and phase-space density at higher \( L \) is depleted. During this period, high ULF wave power and high radial distortion of the magnetosphere lead to enhanced radial diffusion, and both the MHD-particle simulation and the diffusion calculation show the peak of phase space density moves inward rapidly from \( L > 6 \) to \( L \sim 4.5 \). From 4UT to 15UT, during the recovery phase, the trapping boundary location is still quite dynamic and gradually expands outward. The ULF power spectral density during this time period is still considerable high and leads to effective radial diffusion; the peak in phase-space density moves in from \( L \sim 4.5 \) to \( L \sim 4.2 \), and also spreads out into both lower \( L \) and higher \( L \). In the late recovery phase (15 UT of September 25 to 17 UT of September 26), the outer boundary location is more steady, the outer boundary
value remains low, the ULF wave power also drops to low values, the radial diffusion rate is very slow, and the peak moves only slightly inward. Generally radial diffusion gives a phase space density close to that of MHD-particle simulation in all phases of the storm. The MHD-particle simulation results shows small-scale fluctuations (due to the noise associated with counting statistics), while the radial diffusion result is smoother, as would be expected.

The difference plot in Figure 6-14b can better display regions where the two simulations agree or disagree. Positive values mean the phase space density of the radial diffusion calculation is higher than that of the MHD-particle result, and vice versa. Overall, the two simulations agree to within a factor of approximately 3, which is small compared to the several-order-of-magnitude range of $f$.

Figure 6-13 ULF wave radial diffusion coefficient
Figure 6-14 (a) Radial diffusion using both symmetric and asymmetric terms. (b) Log difference with MHD-particle simulation result.
Figure 6-15 (a) Radial diffusion using only symmetric. (b) difference plot.
Figure 6-16 (a) Radial diffusion using only asymmetric terms. (b) difference plot.
Figure 6-15 shows the result of the radial diffusion calculation using only the symmetric diffusion coefficients. The differences between $f$ and $f_{MHD}$ is slightly larger than that of the result using the complete symmetric and asymmetric diffusion coefficients (shown in Figure 6-14), at later times and higher $L$ region. This is because the symmetric diffusion is not high enough at higher $L$ to diffuse phase space density outward through the outer boundary.

Figure 6-16 shows the results obtained using only the asymmetric diffusion coefficients. We can clearly see that the asymmetric diffusion coefficient is inefficient at lower $L$ because the radial diffusion rate at lower $L$ is too slow. The phase space density $f_{MHD}$ is not very well reproduced.

Overall then, the time-dependent ULF wave radial diffusion with both symmetric and asymmetric terms in the radial diffusion coefficients shows good agreement with the MHD-particle simulation.

### 6.4.2 Other radial diffusion coefficients

**Quiet-time power law diffusion coefficient**

For comparison, I also performed radial diffusion calculations for this storm using several previously introduced radial diffusion coefficients. The work by [Selesnick, et al., 1997] determined a magnetic quiet-time radial diffusion coefficient by measuring averaged behavior of electrons as they gradually diffused toward a relaxed configuration over months. The resulting radial diffusion coefficient is independent of time, with power law $L$ dependence: $D_{LL}^{Selesnick} = D_0 \cdot (L/4)^{11.7 \pm 1.3}$, where $D_0 = 2.1 \pm 0.2$ days$^{-1}$. It will be
interesting to see whether this power law form is applicable to storm time. Since the constant factor $D_0$ is obtained for quiet time, in storm time calculations the Selesnick et al. diffusion coefficient is thought to increase by a factor of about 10. An initial run showed that a factor of 10 is too low, and that a factor of 20 was better. The result of the diffusion calculation using $20 \cdot D_{Selesnick}^{L_{\perp}}$, and the difference from the MHD-particle simulation are shown in Figure 6-17. The global deviation $\delta = 0.268$ is larger than the diffusion result using ULF wave driven diffusion coefficients. Also, the plots show that in the recovery phase the phase space density is under-estimated in both high $L$ and low $L$ regions. This indicates that the $L^{1.7}$ power law dependence in the radial diffusion coefficient is too steep. The diffusion rate is not fast enough in the low $L$ region to move the phase space density peak down to this region; at the same time, at high $L$ the diffusion rate is too high, too many particles are diffused through the outer boundary.
Figure 6-17  Radial diffusion using diffusion coefficient of the form of [Selesnick 1997]
**Optimized power law diffusion coefficient**

From Figure 6-17, a power law radial diffusion coefficient $D_{LL} = D_0 (L/4)^n$ with a lower value of $n$ is expected to give better radial diffusion result. A new power law radial diffusion coefficient optimized for this 1998 storm is found by choosing the value of $D_0$ and $n$ so that $D_0 (L/4)^n$ represents an approximate of the time varying ULF wave driven diffusion coefficient. The values are: $D_0 = 0.197$ day$^{-1}$ and $n = 8.5$ (plotted as the solid line in Figure 6-11). Figure 6-18 shows the radial diffusion using this diffusion coefficient. There is good overall agreement with the MHD-particle result, the global deviation is lowered to $\delta = 0.223$. However, in the late recovery phase the phase space density is underestimated at high $L$, and overestimated at low $L$. Further changing the power law dependence $n$ does not help in this situation: if $n$ is increased, the high $L$ region will be more under-estimated; if $n$ is decreased, the lower $L$ region will be more over-estimated. The main reason of the deviation is because a time-dependent diffusion coefficient is needed. The value of this time-independent diffusion coefficient is high enough to capture the radial transport in the main phase and early recovery phase fairly well, but in the late recovery phase, when the particle transport rate has decreased with the decreasing geomagnetic activity level, the time-independent coefficient still has a relatively high value.
Figure 6-18 (a) Radial diffusion using static $L^{3.5}$ power law diffusion coefficient. (b) difference plot.
**KP-Dependent radial diffusion coefficient**

I made further comparisons by performing this radial diffusion calculation using the Brautigam and Albert [Brautigam and Albert, 2000] diffusion coefficients. Brautigam and Albert [2000] developed a time-dependent radial diffusion coefficient parameterized by $Kp$. The $Kp$ index is obtained from a number of magnetometer stations at mid-latitudes by taking the logarithm of perturbations over a 3-hour period and placing it on a scale from 0 to 9 [Bartels, et al., 1939]. Direct measurements of magnetospheric electric and magnetic fields have shown a correlation between $Kp$ and wave power. Brautigam and Albert [2000] parameterized the electric and magnetic radial diffusion coefficients based on the expressions of $D_{\parallel L}^E$ and $D_{\parallel L}^M$ derived by [Fälthammar, 1965], which are proportional to $L^6$ and $L^{10}$ respectively. The proportionality factors in each diffusion coefficient are parameterized by $Kp$ as follows:

$$D_{\parallel L}^E = \frac{1}{4} \left( \frac{cE_{rms}}{B_0} \right)^2 \left( \frac{T}{1+(\omega_d T/2)^2} \right) L^6,$$

and

$$D_{\parallel L}^M = 10^{(0.506 Kp-9.325)} L^{10}, \quad Kp = 1 \text{ to } 6.$$  

Here $\omega_d$ is the electron drift frequency $\omega_d = \left( \frac{3Mc}{eL^2 R_E^2} \left( 1 + \frac{2MB}{E_0} \right)^{-1/2} \right)$, and $E_{rms}$ is the root mean square of the electric field amplitude. For their study of the October 9, 1990 magnetic storm, $E_{rms}$ was assumed to be a linear function of $Kp$:

$$E_{rms} = 0.26(Kp-1) + 0.1 \text{ mV/m}, \quad Kp = 1 \text{ to } 6.$$  

$T$ is the exponential decay time (0.75 hour), $B_0 = 0.31$ gauss, and $E_0$ is the electron rest energy. The total radial diffusion coefficient is the sum of the electrostatic ($D_{\parallel L}^E$) and
electromagnetic ($D_{ll}^M$) contributions, from now on I refer to this diffusion coefficient as $D_{ll}^{BT}$.

I used the $Kp$-dependent coefficients in a radial diffusion calculation of the September 1998 magnetic storm. The following plot shows the $Kp$ index during this storm.

![Figure 6-19 $Kp$ index during September 24-26 1998 magnetic storm](image)

The September 24-26, 1998 storm is stronger than the October 9, 1990 storm. In the 1998 storm the lowest value of the Dst index reached -200 nT, while the lowest Dst of the 1990 storm is -150nT. Also, the $Kp$ index of the 1998 storm reached 8+ while the highest $Kp$ of the 1990 storm is 6. In the $Kp$-dependent radial diffusion coefficients [Brautigam and Albert, 2000] the valid range of $Kp$ is from 1 to 6. In the calculation of $Kp$-dependent diffusion coefficients for this storm, I imposed the upper limit of $Kp$ to
be 6. Figure 6-20 shows the value of the $Kp$-dependent $D_{LL}^{BA}$. Compared to the value of the ULF wave radial diffusion coefficient (Figure 6-13), $D_{LL}^{BA}$ shows similar trends in time dependence during the storm: enhanced diffusion rate during the main phase and beginning of the recovery phase, and lower diffusion rate in the rest of the recovery phase. However the $D_{LL}^{BA}$ is higher than the ULF $D_{LL}$ from the beginning of the storm till late recovery phase, and it has lower time resolution.

![Figure 6-20 Values of $Kp$-dependent radial diffusion coefficient](image)

Figure 6-21 shows the result of a radial diffusion calculation using $D_{LL}^{BA}$. The deviation from the MHD-particle simulation result is $\sigma = 0.402$. 
Figure 6-21  Radial diffusion using $K_p$ diffusion coefficients
There is good agreement with the MHD-particle simulation near $L \sim 4$. This can be explained by the fact that the $D_{LL}^{M}$ was determined from data at $L = 4$ [Brautigan and Albert, 2000], and $D_{LL}^{H}$ dominates the diffusion coefficient (generally $D_{LL}^{M}$ is about 1 order higher than $D_{LL}^{E}$). However there are major differences from the MHD-particle simulation during the main phase and recovery phase. The radial diffusion calculation using $D_{LL}^{BA}$ over estimates phase space density at low $L$ ($L < 3.5$) and under-estimates the phase space density at mid- and high-$L$ ($L > 4.5$). The overall deviation is higher than for any of the other radial diffusion coefficients previously presented. During the recovery phase, the peak of phase space density has been advected to $L \sim 3.8$, and the lower edge of phase space density has been diffused to $L \sim 2.5$. Compared to the MHD-particle simulation result (Figure 6-4), in the recovery phase the peak is at $L \sim 4.2$, and the lower edge is at $L \sim 3$. One possible reason for this discrepancy is that $D_{LL}^{BA}$ is generally too high for this storm.

![Graph showing the comparison of $D_{LL}^{BA}$, ULF-$D_{LL}$, and static $D_{LL}$](image)

**Figure 6-22** $D_{LL}^{BA}$, ULF-$D_{LL}$ and static $D_{LL}$
In addition to Figure 6-20, Figure 6-22 shows the value of $D_{LL}^{BA}$, ULF-$D_{LL}$, and the static $n=8.5$ $D_{LL}$ at three selected times of this storm (2210UT Sept 24 pre storm, 0600UT Sept 25 main phase, and 1148UT Sept 26 late recovery phase). From pre storm till early recovery phase $D_{LL}^{BA}$ kept a very high value, higher than the value of ULF-$D_{LL}$ at the main phase (which is the maximum value of ULF-$D_{LL}$). During pre storm and early recovery phase $D_{LL}^{BA}$ is approximately 1 or 2 orders higher than the ULF-$D_{LL}$. Only in the late recovery phase has $D_{LL}^{BA}$ dropped to a value comparable to the recovery phase ULF-$D_{LL}$ value. The high value of $D_{LL}^{BA}$ diffuses more particles to lower $L$ shells which causes the over-estimation of $f$, and at the same time it diffuses more particles through the outer boundary, which under-estimates $f$ in higher $L$ regions.

The values of the RMS deviation $\sigma$ for each diffusion calculation are listed in Table 6-1. Table 6-1 supports the conclusion that the best agreement is obtained using ULF wave driven symmetric and asymmetric contributions.

<table>
<thead>
<tr>
<th>$D_{LL}$ type</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symmetric and Asymmetric (Figure 6-14)</td>
<td>0.213</td>
</tr>
<tr>
<td>Symmetric only (Figure 6-15)</td>
<td>0.222</td>
</tr>
<tr>
<td>Asymmetric only (Figure 6-16)</td>
<td>0.328</td>
</tr>
<tr>
<td>Power law of form [Selesnick 97] $L^{11.7}$ (Figure 6-17)</td>
<td>0.268</td>
</tr>
<tr>
<td>Static power law, $L^{8.5}$ (Figure 6-18)</td>
<td>0.223</td>
</tr>
<tr>
<td>$Kp$ dependent diffusion coefficient $D_{LL}^{BA}$ (Figure 6-21)</td>
<td>0.402</td>
</tr>
</tbody>
</table>

Table 6-1 RMS deviation between radial diffusion result and MHD-pile simulation result.
6.5 Summary

The test-particle simulation combined with a global MHD simulation of the September 24-26, 1998 magnetic storm yields electron flux profiles which appear to evolve diffusively on time scales of several hours. In order to investigate the diffusive nature of the particle motion, we performed radial diffusion calculations of phase space density evolution of this storm. The differential flux of the MHD-particle simulation result was converted into phase-space density at $M = 1870$ MeV/G, assuming equatorially-mirroring particles. Different radial diffusion coefficients have been investigated, and the location and value of the outer boundary are dynamic.

We used analytical radial diffusion coefficients which contain symmetric and asymmetric field contributions. The diffusion coefficients are proportional to the electric and magnetic ULF wave power spectral density at drift resonance frequencies. Apart from the $L$ dependence in the ULF wave power spectra, the asymmetric diffusion coefficient also depends on field asymmetry, which increases during periods of significant radial distortion. The ULF power spectral density is calculated using the electric and magnetic fields of the MHD simulations. The wave power is higher during the main phase and in the beginning of recovery phase, and increases with $L$, which is consistent with observations [Mathie and Mann, 2000; , 2001]. During the main phase and beginning of the recovery phase, the diffusion coefficient can be 1 to 2 orders of magnitude higher than that of the pre-storm and late recovery phase because of enhanced ULF wave power and greater asymmetry of the geomagnetic field.

In the radial diffusion calculation we saw strongly enhanced inward diffusion during the main phase and early recovery phase. The rate of inward diffusion is much slower in
the late recovery phase when the peak of phase space density moves slowly from $L \sim 4.5$ to $L \sim 4$. The differences between the radial diffusion calculation and the MHD-particle simulation are mostly small-scale at higher $L$ regions. Overall, the two simulations are in good agreement.

By comparing the symmetric and asymmetric contributions separately, we conclude that for this storm the symmetric terms dominate but the asymmetric terms make a significant contribution in higher $L$ regions. For the symmetric diffusion (Figure 6-15), there is a minor discrepancy with the MHD-particle result in the recovery phase at higher $L$ ($t > 15$, $L > 4$) where the outer boundary value has dropped to low values but the outward diffusion is not fast enough to deplete particles. For asymmetric-only diffusion, the diffusion is fast enough at higher $L$ to deplete particles in the recovery phase, but the main discrepancy with the MHD-particle result is at middle and low $L$, where inward diffusion is too weak. Adding asymmetric diffusion coefficients to symmetric diffusion coefficients gives a better result than the symmetric mode alone, although the improvement is small for this September 24-26, 1998 storm. Since the asymmetric diffusion coefficients are proportional to the square of field asymmetry term $\Delta B / B$, for storms with greater compressional asymmetry we expect the asymmetric modes be more important.

The radial diffusion coefficient of [Selesnick 97] has the form of

$$D_{\perp L} = D_0 \cdot \left( L / 4 \right)^{1.7 \pm 1.3},$$

which was determined by measuring the average behavior of electrons in quiet times. $D_0$ has been increased by a factor of 20 to compensate for the high level of geomagnetic activity during storm time. The radial diffusion result shows
this radial diffusion coefficient is not suitable for describing storm time particle transport, and indicates that the $L$ dependence ($n = 11.7 \pm 1.3$) is too steep for storm time.

A static power law diffusion coefficient ($D_L = D_0 (L/4)^n$), where $D_0$ and $n$ are chosen to give an average fit to the ULF diffusion coefficients can reproduce the overall phase space density of the MHD-particle result, but it is not able to capture the dynamics during the storm. For example, in the recovery phase the static diffusion is too fast, so it underestimates phase space density at higher $L$ and overestimates at low $L$.

The Brautigam and Albert [Brautigam and Albert, 2000] radial diffusion coefficient has similar time dependent dynamics compared to the ULF diffusion coefficients. However it lacks details in both time and $L$, and is generally too large for this storm, especially during pre storm and early recovery phase. The diffusion result shows only good agreement in $L \sim 4$ regions, but large differences in other areas. It underestimates at high $L$, and overestimates at low $L$.

Overall, the ULF-wave-driven radial diffusion coefficients did a good job of reproducing the relativistic electron transport of the MHD-particle simulation. We emphasize that this study is a model-model comparison (i.e., a comparison of the MHD-particle simulation with the radial diffusion simulation), primarily for the purpose of better quantifying ULF-wave radiation belt electron transport. In order to make these simulations more realistic and to enable more meaningful comparisons with spacecraft measurements, additional processes (such as convective transport, impulsive injections, and acceleration and loss by high-frequency wave-particle interactions, for example) should be added.
Chapter 7

Summary and Discussion

This dissertation describes efforts to model dynamics of relativistic electrons in the Earth’s radiation belts. The research has focused, in particular, on the radiation belt particle phase space density as a function of the adiabatic invariants, and the radial diffusion process as a mechanism for particle transport and acceleration.

Radial diffusion coefficients for radiation belt transport

Assuming the first and second adiabatic invariants are conserved, the full transport equation reduces to a phase-averaged radial diffusion equation. The radial diffusion coefficient controls the rate of diffusion and advection at different times and in different regions. Based on general electric field and magnetic field perturbations, I derived analytical wave-driven diffusion coefficients. Both electric and magnetic perturbations are considered, and the background magnetic field can be asymmetric. The newly derived diffusion coefficients are compared with numerical tests of particle motion in model fields and they are in good agreement both qualitatively and quantitatively. The symmetric diffusion coefficients are proportional to the field power spectral density at resonance frequencies $\omega = m\Omega_d$, with $L$-dependent factors of the form $L^6$ and $L^4/\gamma^2$, for electric and magnetic diffusion coefficients, respectively. (Note that, in the ultra relativistic limit $1/\gamma^2$ is proportional to $L^3$. ) Asymmetric modes have two resonance frequencies: $\omega = (m + 1)\Omega_0$ and $\omega = (m - 1)\Omega_0$, and a higher $L$ dependence: $L^{12}$
and $L^{10}/\gamma^2$ for electric and magnetic diffusion coefficients. The asymmetric diffusion coefficients are also proportional to the field asymmetry factor $(\Delta B / B_E)^2$.

The analytical radial diffusion coefficients derived in Chapter 2 are functions of fundamental physical quantities such as the power spectral density of electric and magnetic fields. However sometimes these in-situ quantities are not easy to obtain from observations; there are limited numbers of spacecraft measurements available and spacecraft may not have electric field instruments. Future work could relate these analytical diffusion coefficients with other observable quantities, especially ground based observations of ULF waves. It would be very useful to be able to relate ground-based ULF wave measurements to power spectral densities in space, for use in the radial diffusion coefficients.

**Numerical radial diffusion model**

A numerical radial diffusion code was developed which allows time-dependent and L-dependent radial diffusion coefficients, loss rates, sources, and inner and outer boundary conditions. This code calculates the time-evolution of phase space density profiles in the radiation belts. Different $L$ dependences of the diffusion coefficients greatly change the steady-state phase space density profile. I considered three test cases of time-dependent solutions: the first two test cases use the same setup as [Selesnick and Blake, 2000], for comparing effects of an internal source to an external source; the third test case includes loss. The results of the first two cases agrees with the results of [Selesnick and Blake, 2000]. The internal source and external source cases result in similar phase space density radial profiles after a long period of time. With the loss term,
the evolution of phase space density at locations higher than the peak is very similar to
the external source case. In order to determine whether a change in phase space density
is due to an internal source, external source or loss, one must make time-dependent
observations during the period that the source is active, and the observation must be made
over a range of $L$ shells that covers the region of the phase space density peak.

Local heating by breaking the first adiabatic invariant is also important, and the
pitch angle scattering (breaking the first and second adiabatic invariants) causes particle
loss into the loss cone. Cross terms in the diffusion tensor might be also important.
Future work in this area includes the development of 2D and 3D diffusion codes which
solve the Fokker-Planck transport equation with the two by two, or full three by three
diffusion tensor.

Roederer $L$ shell and Phase Space Density Evaluation

In Chapter 4 I described the method of the Roederer $L$-shell value calculation, and
the method of conversion between phase space density and particle differential flux.
These calculations are needed in order to use the Roederer $L$-shell as a phase space
coordinate and to compare phase space density and particle flux.

Calculation of Phase Space Density Radial Gradient Using GOES Satellite
Measurements

Based on the Roederer $L$-shell and phase space density calculation methods and
electron flux observations from GOES satellites, we have introduced a new technique for
estimating the phase space density gradient in the geosynchronous region. This is
described in Chapter 5. The $L$-shell calculation results show that although GOES 8 and
GOES 9 are both at a radial distance of 6.6 Re on the geographic equator, due to their different local time they are located at different geo-magnetic latitude and have different $L$ values. They both experience diurnal L shell variations but shifted by 4 hours, and their $L$-shells cross twice a day. By calculating the ratio of phase space densities, we can obtain the phase space density gradient.

We established a formula for estimating phase space density from the measured omni-directional integrated flux. Using the fact that the phase space density should be the same at the two crossing points, we calibrated the constant factors in the formula. Using this formula, we calculated phase space densities at the $L$-shells of the two satellites for a period of three days during geomagnetic quiet time. The result shows persistent positive gradient in this period (the phase space density at higher L shell is higher than lower L shell). This result supports inward radial diffusion from a source at the high L regions.

Future work could use a more accurate local-time dependent pitch angle model, and a more robust storm time magnetic field model to extend this phase space density gradient measurement technique to geomagnetic active times. Also, by combining the geosynchronous data with measurements by other satellites in different regions, we may be able to better locate sources and losses in the radiation belt.

**Radial Diffusion Calculation of September, 98 Storm**

In Chapter 6 I applied the radial diffusion model to calculate radiation belt dynamics using the September 24-26, 1998 magnetic storm, and the results are compared to an MHD-particle simulation. For the radial diffusion calculation I use the newly derived analytical diffusion coefficients including both asymmetric and symmetric terms.
and electric and magnetic perturbations. The diffusion coefficients are proportional to
the ULF power spectral density at resonance frequencies, and the values of the power
spectral density are calculated from the MHD fields. The powers are higher during the
main phase and in the beginning of the recovery phase and increase with $L$, which is
consistent with observations [Mathie and Mann, 2000; , 2001]. During the main phase
and the beginning of the recovery phase the diffusion coefficients can be 1 to 2 orders of
magnitude higher than that of the later recovery phase because of enhanced ULF wave
power and greater asymmetry of the magnetospheric field.

In the radial diffusion calculation using ULF diffusion coefficients we saw strongly
enhanced inward diffusion during the main phase. The rate of inward diffusion is much
slower in the late recovery phase, when the peak moved steadily from $L = 4.5$ to $L = 4$.
The differences between the results of radial diffusion calculation and the MHD-particle
simulation are mostly small-scale. The overall comparison shows good agreement. The
ULF-wave-driven diffusion coefficient is the sum of symmetric and asymmetric diffusion
coefficients. By comparing the diffusion calculation result using the two terms separately,
we conclude that symmetric terms play important roles in particle transport during the
storm, however the asymmetric terms are non-negligible and do improve the agreement
at higher $L$ shells during the storm recovery phase.

I have also studied radial diffusion using other radial diffusion coefficients. (1) The
static diffusion coefficient ($D_{LL} = D_0 L^n$, with constant values of $D_0$ and $n$ optimized
for this storm) can roughly reproduce the result of the MHD-particle simulation, but it is
not able to capture the variable diffusion rate during the storm. The main difference is
that in the recovery phase the diffusion rate is too fast. (2) The quiet time diffusion
coefficient of Selesnick, et al., [1997] has a power law dependence of $L^{1.7}$. Increasing the value of the constant factor $D_0$, this diffusion coefficient can not reproduce storm time particle evolution very well because the power law exponent is too high. (3) The time-dependent radial diffusion coefficient of Brautigam and Albert [2000] is semi-empirical, and it uses the $Kp$ index to estimate wave activity. The radial diffusion result using this $D_{RL}$ only gives good agreement at the region $L \sim 4$, but this radial diffusion coefficient is generally too high, and lacks of details of variations in time and $L$.

Overall, the ULF-wave-driven radial diffusion coefficients do a good job of reproducing the relativistic electron transport observed in the MHD-particle simulation. We emphasize that this storm time radial diffusion study is a model-model comparison (i.e., a comparison of the MHD-particle simulation with the radial diffusion simulation), primarily for the purpose of better quantifying ULF-wave-driven radiation belt electron transport.

Closing Comments

The radial diffusion process is very important for transporting and accelerating energetic electrons in the radiation belts, and the ULF wave resonant acceleration is an efficient driver. The work in this thesis has increased understanding of radial diffusion by generalizing derivations of radial diffusion coefficients, by helping to develop a data analysis method for calculating phase space density gradient at geosynchronous orbit, and by making the first detailed comparisons of radial diffusion simulations and MHD-
particle simulations. However, more work is needed and the study of radiation belt transport and acceleration continues to be an active and exciting research area.
References


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