RICE UNIVERSITY

Horizontal Thermal Contractional Strain of Oceanic Lithosphere: The Ultimate Limit to the Rigid Plate Hypothesis

by

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ABSTRACT

Horizontal Thermal Contractional Strain of Oceanic Lithosphere: The Ultimate Limit to the Rigid Plate Hypothesis

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Depth-averaged horizontal strain rates in oceanic lithosphere due to thermal contraction are determined. Calculated strain rates range from $\approx 10^{-2}$ Myr$^{-1}$ (near the mid-ocean ridge) to $\approx 10^{-5}$ Myr$^{-1}$ (for the oldest oceanic lithosphere). The average thermal contractional strain rate in oceanic lithosphere is $\approx 10^{-4}$ Myr$^{-1}$. Newly created lithosphere is displaced toward old ocean basins at a rate that is 1.35% of the half-rate of seafloor spreading, giving displacement rates of 0.1 to 1.1 km Myr$^{-1}$. The bias in plate displacement rates estimated from marine magnetic anomalies, expressed as a percentage of the full spreading rate, is 0.60% or 0.85% depending on the age of the magnetic isochron used to estimate current plate velocity. The displacement rate due to thermal contraction parallel to a mid-ocean ridge could be as large as $\approx 10$ mm/yr. Strain rates due to thermo-elastic stresses are an order of magnitude smaller than the strain rate calculated when these stresses are neglected.
Acknowledgements

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Chapter 1

INTRODUCTION

The central assumption of plate tectonics is that the plates are rigid [Wilson 1965; Gordon 1998; Gordon & Royer, 2006]. In contrast to this assumption, a great success of the mobile view of the solid Earth applied to the marine realm has been the explanation of the broad features of submarine topography in terms of the cooling and subsidence of the lithosphere with age [McKenzie 1967; Turcotte & Oxburgh 1967; Parsons & Sclater 1977; Stein & Stein 1992; McKenzie et al. 2005]. Quantitative models of lithospheric subsidence explicitly assume that the lithosphere contracts with age. Coefficients of thermal expansion for olivine differ little in crystallographic orientation [Bouhifd et al., 1996], which indicates that contraction should be dominantly isotropic. Thus, these two widely accepted pillars of global tectonics are mutually incompatible. It follows that oceanic plates cannot be rigid.

Surprisingly, we are aware of no prior efforts to quantify the size of the predictable horizontal thermal contraction of the lithosphere. Here we use widely accepted models for subsidence of the oceanic lithosphere to estimate the horizontal thermal contraction of the lithosphere. We find that contraction rates are large enough to require significant modifications to the rigid-plate hypothesis.

Thermal contraction of oceanic lithosphere and plate non-rigidity

In standard models for cooling of the lithosphere, the base of the lithosphere is defined by an isotherm of about 1300 °C. The seismogenic portion of the lithosphere is bounded at depth approximately by the 600 °C isotherm [McKenzie et al. 2005], which
indicates an approximate average temperature of the upper lithosphere of about 300 °C. Thus the overall cooling of mature seismogenic lithosphere is about 10³ °C. As discussed further below, current estimates of the linear coefficient of thermal contraction averaged over these temperature and pressure ranges are $1.0 \times 10^{-5}$ (°C)$^{-1}$ to $1.1 \times 10^{-5}$ (°C)$^{-1}$. Thus mature seismogenic lithosphere has contracted by about 1% in each direction. Here our focus is not on the total contraction, but on the rate of contraction and its possible effects on estimates of geologically instantaneous plate motions, such as the NUVEL-1A set of relative angular velocities [DeMets et al. 1994] and in their comparison with relative velocities and relative angular velocities estimated from space geodetic data [e.g., Heflin et al. 1992; Argus & Gordon 1991; Sella et al. 2002].

METHODS AND ASSUMPTIONS

To simplify the analysis, we apply half-space cooling models [Turcotte & Oxburgh, 1967], which indicate that the temperature $T$ in the cooling half-space is given by

$$ T = T_m \text{erf} \left( \frac{z}{2\sqrt{\kappa t}} \right) $$

(1) (Appendix A)

where $z$ is the depth, $t$ is time, $\kappa$ is thermal diffusivity, and $T_m$ is temperature at the base of the lithosphere. Such models are known to predict seafloor depths greater than observed for lithosphere ages exceeding 80 Ma. This too large of prediction is associated with greater cooling in the mantle at depths approaching 100 km than is predicted by plate models of cooling, which fix the temperature near this depth (e.g., Parson & Sclater [1977]). Thus, given our focus on cooling of the crust and uppermost mantle at depths much shallower than 100 km, the use of a simple half-space cooling model is valid. To
further simplify the analysis, we also neglect the effects of thermo-elastic stresses. Prior work indicates that these stresses are compressional for lithosphere more than a few million years old [Sandwell 1986, Haxby & Parmentier, 1988]. Thus, the main effect of thermo-elastic stresses would be to give moderately higher rates of contraction than we estimate for all but very young lithosphere.

Relevant thermal parameters, including the coefficient of thermal contraction and thermal diffusivity, are known to vary with temperature and pressure. To account for these variations in an analytical model, we use values that correspond to the values that are the approximate average over the range of temperatures and pressures that we investigate (Appendix B).

Approach

Rates of cooling, and hence contraction, vary with depth in the lithosphere. We assume that the potentially observable horizontal contraction of oceanic lithosphere will be an average of the rate of contraction over a range of depths. This range of depths begins with the surface of the lithosphere down to a depth that we will refer to as the base of the competent lithosphere. Below this base, we assume that stresses are relaxed by creeping flow, which accommodates the contraction of the overlying competent layer.

One long-held view is that the depth of the base of the competent lithosphere is thermally controlled, i.e., it should follow an isotherm [Turcotte & Oxburg, 1967]. We explore this family of models by considering the competent lithosphere to be bounded by an isotherm of three alternative temperatures, 600°C, 700°C, and 800°C.
The choice of the 600°C isotherm as the lowest temperature follows from the work of McKenzie et al. [2005], who showed that the base of the seismogenic lithosphere approximately follows this temperature. Thus, it gives a minimum thickness for a temperature-controlled competent layer. The limiting depth of earthquakes is generally thought to be the boundary between unstable sliding and stable sliding. How far beneath this transition lies the brittle-plastic transition (i.e., the transition from semi-brittle deformation to deformation by creeping flow [Kohlstedt et al. 1995]) is poorly known, but in some prior work has been assumed to occur at a depth about 20% greater than the base of the seismogenic zone (e.g., Martinod & Molnar [1995]). We consider this possibility by also considering models with limiting depths for the competent layer at temperatures of 700°C and 800°C, which respectively lie at depths 20% and 41% greater than the depth of the 600°C isotherm, with 700°C being our preferred value. Let \( T_1 \) be the temperature at the base of the competent layer. Reorganizing equation (1), \( Z_c \), the thickness of the competent lithosphere is given by

\[
Z_c = 2\sqrt{\kappa t \text{ erf}^{-1}
\begin{bmatrix}
\frac{T_1}{T_m}
\end{bmatrix}}
\]

where \( t \) is the age of the lithosphere, \( T_m \) is the temperature at the base of the lithosphere, and \( \kappa \) is the diffusivity. The expression shows that the thickness of the lithosphere increases proportionally to \( \sqrt{t} \).

Let \( A = \text{erf}^{-1}\left[\frac{T_1}{T_m}\right] \). It follows (Appendix A) that the depth-averaged rate of cooling for a lithosphere bounded by a constant isotherm is given by
\[
\frac{\partial T}{\partial t} = \frac{T_m}{2At\sqrt{\pi}} \left[ \exp\left(\frac{x^2}{4At}\right) - 1 \right]
\]  

(3)

A second, more recent, view on what controls the thickness of the lithosphere is the proposal that the lithosphere extends to the depth in the mantle to which significant de-watering due to partial melting has occurred [Hirth & Kohlstedt, 1996; Karato, 2003]. Thus we also consider two models with constant thickness of the competent layer with particular values for thickness of 20 km and 40 km.

**Parameters and Quantitative Methods**

Prior work has relied on a wide range of potential temperatures, \( T_p \), to fit the observed data for seafloor bathymetry and heat loss. Following Parsons & Sclater [1977], McKenzie et al. [2005] use a value of \( T_p \) of 1315°C at the base of the lithosphere. They argue that the implied value of 1408°C for \( T_p \) used by Stein & Stein [1992] would produce too much melt, which would in turn produce a crust of 16 km thickness, much thicker than the observed thickness of 7 km. In contrast a value of \( T_p \) of 1315°C produces a crustal thickness of 7 km, equal to that observed. For this reason we do not consider values of \( T_p \) as large as that inferred by Stein & Stein [1992] and instead consider only two values, 1300°C and 1350°C, which bracket the value used by McKenzie et al. [2005]. We set \( T_m \), the initial temperature of the half-space cooling model, to one of these two values in each of our models. As will be shown below, the lower of these two temperatures produces slightly lower predicted rates of contraction and leads to more conservative estimates of contraction rates. It is also the closer of the two to the value preferred by McKenzie et al. [2005] and thus is our preferred value.
Values of thermal diffusivity used by the scientific community in recent years lie between $0.8 \times 10^{-6}$ and $1.0 \times 10^{-6}$ mm s$^{-1}$ [Wessel, 1992; Turcotte, 1974, Lee et al., 2005]. We use these as end-member values for the models with constant thickness of the competent lithosphere. For a competent layer defined by an isotherm, the depth-averaged rate of contraction is independent of the thermal diffusivity (Appendix A).

In the results that follow, depth-averaging of cooling is determined analytically except for the constant-thickness models. We ignore all the cooling of oceanic lithosphere with ages between 0-Myr and 0.1-Myr old to avoid the breakdown of the validity of the conductive cooling model as the axis of the mid-ocean ridge is approached. A coefficient of thermal expansion $\alpha_i = 10^{-5} \text{C}^{-1}$ has been used for all the calculations.
Figure 1.1: Isotropic contraction of a block of material cooled from above. Arrow shows the direction in which heat is getting lost. $\dot{e}_{xx}$, $\dot{e}_{yy}$ and $\dot{e}_{zz}$ are strain rates in three perpendicular directions. In our analysis we assume that these strain rates are equal and contractional.
RESULTS

Rate of cooling as a function of depth and age

The rate of cooling varies considerably with depth and age (Figure 1.2). The locus of the greatest rate of cooling is near the surface for very young oceanic lithosphere, but is at greater depths in older lithosphere. At any given age the rate of cooling increases with depth until it reaches a maximum (black dashed line in the lower panel of Figure 1.2), below which it decreases again.
**Figure 1.2:** Upper panel: Rate of cooling and linear rate of thermal contractional strain versus age and depth of oceanic lithosphere. Lower panel: Contours of cooling rate (°C Myr\(^{-1}\)) as a function of depth and age. This is essentially a contoured birds-eye view of the upper panel. Dashed black curve: locus of fastest cooling, which corresponds to an isotherm of 888°C. Contours shrink to the zero depth at the youngest part of the lithosphere. Thick red, blue, and black curves respectively show the 600°C, 700°C, and 800°C isotherms.
**Depth-averaged rate of cooling:**

Depth-averaged rates of cooling vary from maximum values approaching or exceeding $10^3$ °C Myr$^{-1}$ for very young lithosphere to low values of a few degrees per million years for lithosphere near 50 Ma in age (Figure 1.3). For the models that assume that the competent lithosphere is bounded by an isotherm, the curve for the 800°C isotherm always lies above that for the 700°C isotherm, which in turn always lies above that for the 600°C isotherm. Thus, the higher the temperature assumed for the base of the competent lithosphere, the greater the depth-averaged rate of cooling.

For the models that assume a constant-thickness (of 20 km or 40 km) for the competent lithosphere, the two curves cross at an age of about 6 Ma. Depth-averaged cooling is greater for the 20-km model for ages less than 6 Ma and greater for the 40-km model for ages exceeding 6 Ma (Figure 1.3). Unsurprisingly, the curves for the constant isotherm models are similar to that of the 20-km model for younger ages and are similar to the 40-km model for older ages (Figure 1.3).
Figure 1.3: Depth-averaged rate of cooling versus age of the oceanic lithosphere. Red, blue, and black curves correspond to models for which the base of the competent lithosphere is defined by the 600°C, 700°C, and 800°C isotherms respectively. Green and yellow curves correspond to models for which the base of the competent lithosphere is defined by a constant thickness of 20 and 40 km respectively.
**Depth- and age-averaged rate of cooling**

The depth-averaged rates of cooling were further averaged with respect to age from the youngest lithosphere that we considered (0.1 Ma) to increasingly older ages (Figure 1.4). The curves of the depth- and age-averaged rate of cooling have maximum rates of cooling of about $10^3 \, ^\circ$C Myr$^{-1}$ for the youngest ages and minimum rates of cooling of 20 to 30 $^\circ$C Myr$^{-1}$ for the average out to 50 Myr-old lithosphere, except for the model with 40-km thick competent lithosphere, which lies consistently beneath the other curves. The depth- and age-averaged rate of cooling for our preferred model (i.e., the 700$^\circ$C isotherm model) is about $10^2 \, ^\circ$C Myr$^{-1}$ for 7.25-Myr old lithosphere, which is similar to the depth-averaged rate over a constant thickness of 20 km (Figure 1.4, Table 1).
Figure 1.4: Depth- and age-averaged rate of cooling versus age of the oceanic lithosphere. Red, blue, and black curves correspond to models for which the base of the competent lithosphere is defined respectively by the 600°C, 700°C, and 800°C isotherms. Green and yellow curves correspond to models for which the base of the competent lithosphere is defined by a constant thickness respectively of 20 km and 40 km.
Thermal contractional strain rates

Using a value for $a_1$ of $10^{-5} \, (\text{oC})^{-1}$, the curves of rates of cooling can be simply transformed to one-dimensional rates of contraction. Thus, a cooling rate of $10^3 \, \text{oC} \, \text{Myr}^{-1}$ corresponds to a contraction rate of $10^{-2} \, \text{Myr}^{-1}$ (1% Myr$^{-1}$), a cooling rate of $10^2 \, \text{oC} \, \text{Myr}^{-1}$ corresponds to a contraction rate of $10^{-3} \, \text{Myr}^{-1}$ (0.1% Myr$^{-1}$), a cooling rate of $10^0 \, \text{oC} \, \text{Myr}^{-1}$ corresponds to a contraction rate of $10^{-4} \, \text{Myr}^{-1}$ (0.01% Myr$^{-1}$), etc. For our preferred model, contractual strain rate exceeds $10^{-2} \, \text{Myr}^{-1}$ for very young lithosphere, is $10^{-4} \, \text{Myr}^{-1}$ for $\approx 15$-Myr-old lithosphere, and is $\approx 3 \times 10^{-5}$ for 50-Myr-old lithosphere (Figure 1.3).

For our preferred model, the age- and depth-averaged thermal contractual strain rate exceeds $10^{-2} \, \text{Myr}^{-1}$ for very young lithosphere, is $10^{-3} \, \text{Myr}^{-1}$ for 7.25 Myr-old lithosphere, and is $\approx 2.15 \times 10^{-4}$ for 50 Myr-old lithosphere (Figure 1.4).

Our results give depth-averaged or age- and depth-averaged cooling rates and contraction rates as a function of age. These rates can be used to transform standard global catalogs of the age of the seafloor [Müller et al. 1993] into maps of depth-averaged or age- and depth-averaged cooling rate or contraction rate [Figure 1.5]. The maps predict that significant portions of Earth’s oceanic lithosphere are thermally contracting at high rates. For example, the dark blue region in the lower world map shows a large region—in particular that flanking the East Pacific Rise—that is contracting, on average, at $10^{-3} \, \text{Myr}^{-1}$. An even larger region, (the dark blue and light blue regions combined) is contracting, on average, at $\approx 3 \times 10^{-4} \, \text{Myr}^{-1}$. This dark blue region corresponds to 7.4% of total area of ocean floor; while light blue and dark blue combined correspond to 34% of the total area. The combined area marked with green, light blue, and dark blue on the
same map is contracting at an average strain rate of $10^{-4}$ Myr$^{-1}$ and constitutes 88% of the ocean floor.

**Thermal contractional displacement rates**

Displacement rate across a contracting region can be estimated by integrating predicted strain rate over a given distance. The global average for depth-averaged contractional strain rate of oceanic lithosphere is $\approx 10^{-4}$ Myr$^{-1}$ (Figure 1.5). If the average oceanic thermal contraction rate is used along a hypothetical circumferential great circle path (i.e. $\approx 40,000$ km long), then a contractional displacement rate of 4 km Myr$^{-1}$ would result from this global oceanic average. Local or regional paths of integration typically give lower displacement rates, but some can be higher.

For each plate pair separating along a mid-ocean ridge, the displacement rates due to horizontal thermal contraction can be decomposed into ridge-parallel and ridge-perpendicular components. The assumption that spreading rate is constant over time for any plate pair permits a simple, universal, quantitative assessment of the ridge-perpendicular component of displacement rate due to horizontal thermal contraction. The ridge-perpendicular component of displacement rate of 0.1 Myr-old lithosphere relative to lithosphere of any older age can be expressed as a percentage of the spreading half rate, which increases monotonically with age up to 1.35% for 80-Myr-old lithosphere (Figure 1.6). For the ultra-fast segments of the East Pacific Rise (spreading half rate of 80 km Myr$^{-1}$), this gives a displacement rate of only 1.1 km Myr$^{-1}$, which is small, but potentially detectable with space geodetic data (assuming the existence of enough islands
in optimum locations). At the other extreme, for ultra-slow spreading (half rate of 5 km Myr\(^{-1}\)), the displacement rate is minuscule, 0.1 km Myr\(^{-1}\), and presumably unmeasurable.

Because of ridge-perpendicular horizontal thermal contraction, plate velocities estimated from marine magnetic anomalies across mid-ocean ridges are biased estimators of the relative velocities of plate interiors. The bias is not the full 1.35%, however, because significant thermal contraction occurs in the across-ridge interval spanned by the magnetic anomalies used to estimate the spreading rate. If this interval spans 3.2 Myrs, as is the case for the NUVEL-1A set of relative angular velocities, then the bias is the displacement rate of 3.2 Myr-old lithosphere relative to 80-Myr-old lithosphere, which is 0.60% (1.35% for 80-Myr-old lithosphere minus 0.75% for 3.2 Myr-old lithosphere; Figure 1.6). Thus we predict that the velocities of stable plate interiors are on average about 0.6% slower than those predicted by NUVEL-1A or any other estimate constructed from spreading rates averaged over the past 3.2 Myr. For fast-moving plate pairs, this is potentially detectable from space geodetic data. For example, the effect on Pacific-Nazca spreading rate is 1.0 km Myr\(^{-1}\) (= 0.6% x 1.60 km Myr\(^{-1}\)).

An alternative magnetic anomaly used in some analyses of current spreading rate is the outside edge of the Central Anomaly, corresponding to the Brunhes-Matuyama reversal boundary (0.78 Ma). In this case, the bias is the displacement rate of 0.78-Myr-old lithosphere relative to 80-Myr-old lithosphere, which is 0.85% (1.35% for 80-Myr-old lithosphere minus 0.50% for 0.78-Myr-old lithosphere; Figure 1.6). Thus we predict that the velocities of stable plate interiors are on average about 0.85% slower than predicted by any estimates from spreading rates averaged over the past 0.78 Myr.
Displacement rates parallel to mid-ocean ridges depend strongly on the age of the lithosphere and are proportional to the distance along which the strain rate is integrated. Displacement rates are greatest where the lithosphere is the youngest. Because of the fast-spreading along the East Pacific Rise and the Pacific-Antarctic Rise, the Pacific basin is the location with the planet’s largest concentration of young, rapidly contracting oceanic lithosphere (Figure 5). Consider the isochron (5.3 Ma) in the Pacific plate corresponding to a thermal contractual strain rate of $3 \times 10^{-4}$ Myr$^{-1}$ ("$10^{-3.5}$", the border between light blue and green in the upper panel of Figure 5). This isochron spans about 10,000 km in latitude. The implied north-south contractual displacement rate thus is 3 km Myr$^{-1}$. Pacific plate seafloor to the west of this isochron contracts at rates between about $1 \times 10^{-5}$ Myr$^{-1}$ and $3 \times 10^{-4}$ Myr$^{-1}$. Over north-south distances of 10,000 km (say from 60°S to 30°N), these contractual strain rates correspond to displacement rates of 0.1 to 3 km Myr$^{-1}$. In contrast, Pacific plate lithosphere east of this isochron contracts at rates between $3 \times 10^{-4}$ Myr$^{-1}$ and $\approx 3 \times 10^{-2}$ Myr$^{-1}$. Over north-south distances of 10,000 km, these correspond to displacement rates of 3 km Myr$^{-1}$ to $\approx 300$ km Myr$^{-1}$!

The greatest of these hypothetical displacement rates would occur along very narrow strips of very young seafloor. These highest rates of contraction may be realized, at least in part, by widening of active transform faults [Turcotte, 1974]. To find displacement rates that may lead to measurable shrinking of the plate parallel to the isochrones over longer distances we employ some ridge-perpendicular spatial averaging. The lower panel of Figure 5 shows thermal contractual strain rates spatially averaged from the youngest (i.e., 0.1-Myr old) lithosphere out to various lithospheric ages. Consider the isochron (32.4 Ma) in the Pacific plate delineated by the border between light blue and green
regions in the lower panel of Figure 5. The average contractional rate for lithosphere between this isochron and the 0.1 Ma isochron is $3 \times 10^{-4}$ Myr$^{-1}$ ("10$^{-3.5}$", this region encompasses both the light blue and the dark blue areas of the Pacific plate in the lower panel of Figure 5). For a north-south distance of 10,000 km, the average implied north-south contractional displacement rate is 3 km Myr$^{-1}$, a value significant for many determinations of plate velocity. Now consider the isochron (7.25 Ma) in the Pacific plate delineated by the border between dark blue and light blue regions in the lower panel of Figure 5. The average contractional rate for lithosphere between this isochron and the 0.1 Ma isochron is $10^{-3}$ Myr$^{-1}$ (this region encompasses only the dark blue area of the Pacific plate in the lower panel of Figure 5). For a north-south distance of 10,000 km, the average implied north-south contractional displacement rate is 10 km Myr$^{-1}$, which is a large displacement rate relative to the presumed uncertainties in estimates of geologically instantaneous plate motions (e.g., DeMets et al. [1990, 1994]) or relative to plate motions estimated from space geodetic data.
Figure 1.5: Upper map shows depth-averaged thermal contractional strain rates predicted for oceanic lithosphere. Red shows those locations for which we do not have age data or shows continents. Lower map shows depth- and age-averaged thermal contractional strain rate predicted for oceanic lithosphere. The average is taken from very young (0.1 Myr old) lithosphere to the age of the lithosphere at the given location.
Figure 1.6: Cumulative displacement rate as percentage of plate velocity. The blue line shows the age (0.78 Ma) for the old edge of the central anomaly. The red line shows the age (3.2 Ma) for the middle of marine magnetic anomaly 2A, which was used in estimating the NUVEL-1A global set of relative plate angular velocities. Displacement rate parallel to the plate motion due to thermal contraction for any other velocity can be found by linearly interpolating the curve. Plate interiors thus move ≈1% slower (i.e., 0.1 mm yr\(^{-1}\) to 0.8 mm yr\(^{-1}\) than estimated from spreading half rates determined from marine magnetic anomalies.
APPLICATION:

The dark blue region in Fig 1.5, corresponds to average contractional strain rate of $10^{-3}$ Myr$^{-1}$ intersects a substantial portion of southern Baja California, a critical region for estimating the geologically instantaneous velocity of the Pacific plate relative to the North American plate [DeMets et al. 1987]. The (dark blue) region of high thermal contractional strain rate continues far to the south along the East Pacific Rise and the Pacific-Antarctic Rise. The north-south distance between Baja California and the Pacific-Antarctic Rise is about 10,000 km. Thus, southern Baja California is expected to be displaced southward relative to Pacific lithosphere near the Pacific-Antarctic Rise at a rate of about 10 km Myr$^{-1}$, which should have an easily observed effect on estimates of the velocity of the Pacific plate relative to North America. Recent analysis of global plate motion circuits indicates that the global Pacific-Antarctic-Nubian-North American plate motion circuit is inconsistent with the directly estimated Pacific-North America velocity and the difference is described by a nearly north-south velocity with a length of $15 \pm 3$ km Myr$^{-1}$ (95% confidence limits) [Gordon et al. 2006]. It thus appears that predictable horizontal thermal contractional strain is the main explanation of the misfit in the global plate motion circuit documented by Gordon et al. [2006].
TABLE 1(a):

Depth-average:

<table>
<thead>
<tr>
<th>Log of strain rate</th>
<th>Depth variables</th>
<th>$T_n = 1300$</th>
<th>$T_n = 1350$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>$\kappa = 1.0$</td>
<td>$\kappa = 1.0$</td>
</tr>
<tr>
<td>-3</td>
<td>$600^\circ C$</td>
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TABLE 1(b):

**Depth-and age-averaged:**

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DISCUSSION

Depth variation of thermal parameters

Thermal parameters vary with temperature and pressure. One way to account for these variations is to incorporate the temperature and pressure dependence in numerical estimates of the temperature profile of oceanic lithosphere (e.g., McKenzie et al. [2005]). Another way to account for these variations, which was used herein, is to use the depth-averaged values of the thermal parameters for the competent layer of the lithosphere. Our calculations show that such depth-averaged values can differ by as much as 10% from the values applicable to the shallowest part of the lithosphere (Appendix B). That the depth-averaged parameters differ by no more than 10% from the near-surface values suggests, but by no means proves, that a more complex numerical model with temperature- and pressure-dependent thermal parameters would not substantially change the results obtained herein.

Effect of elasticity of lithosphere

In a companion study (Chapter 2), we assess how much the results change if the effects of thermo-elasticity are considered. We show that these effects are an order of magnitude smaller than the contractional strain rates we estimate here, and can reasonably be neglected.
Detection of plate non-rigidity due to horizontal thermal contraction

Herein we argue for high rates of horizontal contractional strain in young oceanic lithosphere and, therefore, that the plates are not strictly rigid. If the non-rigidity is as large as we estimate, and is potentially contributing about 10 mm yr$^{-1}$ displacement rate of non-closure to the global plate motion circuit [Gordon et al., 2006], why hasn’t it been detected before through plate reconstructions? Stein & Gordon [1984] and Gordon et al. [1987] devised statistical tests for plate circuit closure, which have been applied in many later works (e.g. DeMets et al. [1990]) and failed to unambiguously identify any non-rigidity that may be due to thermal contraction.

In most plate geometries, the change in plate size that we hypothesize cannot be detected from plate motion data. Figure 1.7 shows a cartoon of a stepped ridge-transform plate boundary between Plate A and Plate B. On the left ("before") is a cartoon showing the size and shapes of portions of these plates if there were no horizontal contraction due to cooling. It illustrates the typical idealized concept of plate rigidity. On the right ("after") is a cartoon with the same plate portions after undergoing an isotropic homogeneous horizontal contraction due to cooling. They can be fit together just as precisely as the plates hypothetically assumed to have undergone no contraction. The amount of contraction for real-world plates will depend on the age of the lithosphere in the left panel, but we simplify to illustrate the point more clearly. An age-dependent contraction would be more complex but lead to the same conclusion. This follows because the homologous elements of the two plates are of the same age and shrink by the same amount.
Figure 1.8 shows that the argument applies equally well to three plates meeting at a ridge-ridge-ridge triple junction and having mutual ridge-transform plate boundaries. On the left ("before"), the three plates (or more precisely portions of plates) are assumed to have been perfectly rigid with no contraction. Thus they can be fit precisely together. On the right ("after") are the same three plates after undergoing isotropic homogeneous horizontal contraction due to cooling. They can be fit together just as precisely as the plates hypothetically assumed to have undergone no contraction. As for the two-plate example (Figure 1.7), the amount of contraction for real world plates will depend on the age of lithosphere in the left panel, but lead to the same conclusion.

It takes an unusual plate geometry for the thermal contraction to cause misfits that are observable with plate motion data. Figure 1.9 is a cartoon illustrating one such geometry. In this case Plate A mainly consists of young lithosphere recently created at a mid-ocean ridge shared with Plate B. In contrast, the main part of Plate C is composed of old lithosphere. The young lithosphere cools faster than the old lithosphere and thus Plate A shrinks faster than Plate C. If the plates were rigid and did not shrink, then the plate boundary of "Before", which is a paleo-plate boundary in the "After" panel, should also fit together. If the young lithosphere contracts more than the old lithosphere, as expected, then Plate A and Plate C cannot be fit together millions of years later, as shown in the "After" panel.

**Thermal contraction and the rigid plate hypothesis**

The rigid-plate hypothesis has survived many rigorous tests over time. For example, determinations of global sets of relative plate angular velocities constitute a rigorous test
of the rigid-plate hypothesis (e.g., DeMets et al. [1990]). Some prior non-closures of plate circuits have been interpreted as deformation across broad, diffuse plate boundaries between essentially rigid component plates [Wiens et al. 1985; Royer & Gordon 1997; Gordon 1998]. Precisely how rigid are plate interiors is poorly known. A lower bound of about $10^6$ Myr$^{-1}$ can be established from the rate of moment release of intraplate earthquakes [Gordon 1998]. This rate may be unrepresentative of typical lithosphere, however, if the moment release is concentrated in a few narrow belts. On the other hand, the upper bound on strain rate of stable plate interiors is more than two orders of magnitude greater than this. Argus & Gordon [1996] use geodetic results from very long baseline interferometry to place an upper bound on the laterally averaged strain rate of a large portion of the North American plate of about $10^{-17}$ s$^{-1}$ ($3 \times 10^{-4}$ Myr$^{-1}$). The actual areally averaged deformation rate could be substantially less than this upper bound.

Thus, the thermal contractional strain rate, as inferred from our models, may now be better known than the background intraplate strain rate associated with earthquakes and faulting. Relevant strain rates compared in Figure 1.10. What is new is the arrow showing the range of horizontal contractional strain rates due to cooling of oceanic lithosphere, as well as two key values along it (shown by a black-filled square and a black-filled circle). In our model, the lowest depth-averaged rate of contraction in oceanic lithosphere is about $10^{-3}$ Myr$^{-1}$ ($10^{-5}$ Myr$^{-1}$ corresponds to the 167 Ma isochron in our model). Thus the lowest depth-averaged contractional strain rate anywhere in oceanic lithosphere is an order of magnitude greater than the strain rate inferred from seismic moment release. If we instead compare lateral averages, the difference becomes wider. The average strain rate for oceanic lithosphere having an age of 118 Ma or less,
which constitutes 88% of all oceanic lithosphere, is $10^4 \text{ Myr}^{-1}$. Thus, the average thermal contractional strain rate of oceanic lithosphere is roughly two orders of magnitude greater than the strain rate released through intraplate oceanic earthquakes. These calculations give new lower bounds, substantially greater than found before, on the non-rigidity of stable oceanic plate interiors [Gordon, 1998]. Horizontal thermal contraction appears to be the ultimate limit to the rigid plate hypothesis.
Figure 1.7: Cartoon illustrating why the horizontal components of isotropic thermal contraction are generally expected to be undetectable from reconstructions of two plates meeting at a ridge and transform fault boundary. On the left is shown the usual model of a rigid plate that has not deformed or changed size. On the right are the same plates reduced in size by a homogeneous isotropic contraction. The contracted pair of plates can be fit together as well as the rigid pair of plates. Of course, plates do not contract homogeneously as the amount of vertically averaged isotropic contraction is a function of age, but one would obtain the same result because homologous isochronal features would shrink by the same amount in each plate.
Figure 1.8: Cartoon illustrating why the horizontal components of isotropic thermal contraction are generally expected to be undetectable from reconstructions of three plates meeting at a ridge-ridge-ridge triple junction. On the left is shown the standard model of three rigid plates that have not deformed or changed size. On the right are the same plates reduced in size by a homogeneous isotropic contraction. The three contracted plates can be fit together as well as the three rigid plates. Thus almost all two-plate and three-plate reconstructions would not detect any isotropic contraction of the plates.
Figure 1.9: A cartoon illustrating special conditions in which the horizontal components of isothermal thermal contraction are detectable from plate reconstructions. The top panel shows three plates at 10 Ma. The bottom panel illustrates an unsuccessful attempt to fit together the 10-Myr isochrons of Plate A and Plate C today. Because it was composed of younger lithosphere than Plate C, Plate A has contracted more than Plate C and a satisfactory fit cannot be achieved.
Figure 1.10: Comparison range of thermal constructional strain rate to the ranges of regionally averaged strain rates associated with large-scale tectonic processes (modified from Gordon [2000]). Square: Age- and depth-averaged strain rate for oceanic lithosphere up to 119 Myrs in age. Circle: Age- and depth-averaged strain rate for oceanic lithosphere up to 7 Myrs in age. The low end of the range of thermal contraction strain rates is the depth-averaged rate for the oldest oceanic lithosphere on Earth and essentially defines a new lower-bound on non-rigidity of the stable interior of oceanic plates.
CONCLUSION

Because of the predictable effects of thermal contraction, we conclude that oceanic lithosphere is not rigid. The ocean-wide-average, depth-averaged strain rate due to thermal contraction of oceanic lithosphere is roughly two orders of magnitude greater than the average strain rate inferred from earthquake moment release in intraplate oceanic lithosphere. Thermal contractional strain rates are greatest for young lithosphere. Misfits due to horizontal thermal contraction have not been observed when analyzing the fit of single pairs of plates or three plates meeting at a triple junction because of the symmetry of thermal contraction. Thermal contraction likely leads to displacement rates large enough to be measured in the analysis of plate motion circuits with less symmetrical arrangements of plates and isochrons. For example, the misfit to Pacific-North America plate velocities documented by Gordon et al. [2006] is likely due to ongoing thermal contraction of Neogene Pacific plate lithosphere created along the East Pacific Rise and Pacific-Antarctic Rise.
Chapter 2

INTRODUCTION

In chapter 1, we quantified the size of the predictable horizontal thermal contraction of the lithosphere using widely accepted models for thermal contraction and subsidence of oceanic lithosphere. We found that horizontal contractional strain rates are large enough to require significant modifications to the rigid-plate hypothesis, with predicted displacement rates being as large as about 10 km Myr$^{-1}$. In Chapter 1 we ignored the effect of thermo-elastic stresses in modifying the predicted rates of thermal contraction strains and displacements. Here we explicitly model these effects and find that they are at least an order of magnitude smaller than the effects predicted in Chapter 1. The incorporation of thermo-elastic stresses thus require only minor changes to our prior results and no modification of our prior conclusions.

Published models of thermo-elastic stresses for finite plates based on analytical solutions have been unable to satisfy all the natural boundary conditions (i.e., no applied tractions) along all the edges of a finite plate. What such models do instead is to satisfy the no traction condition along two opposing edges, assumed to correspond to transform faults or fracture zones, bounding a strip of oceanic lithosphere. Such models accept non-zero tractions along the other two edges, one of which corresponds to a mid-ocean ridge [Sandwell, 1986]. For this chapter we construct both analytic and finite-element models for thermo-elastic stresses of a finite plate of oceanic lithosphere. With the analytic models, we obtain results like those obtained before. With the finite-element model we are able for the first time—as far as we know—to satisfy the traction-free boundary condition along all four edges of the plate. The resulting field of thermo-elastic
stresses differs significantly from prior models in the vicinity of the mid-ocean ridge and has different implications for the state of stress of oceanic lithosphere than found before.

**Thermo-elastic stresses in the oceanic lithosphere**

Thermo-elastic stresses in oceanic lithosphere occur because adjacent regions of oceanic lithosphere of different age cool at different rates. Different rates of cooling also occur as a function of depth, but here we consider only the vertically averaged cooling rates of the competent upper lithosphere. It is the horizontal variations of these vertical averages that are the source of the stress we model.

The thermal stress predicted by a model depends on the assumed boundary conditions and on the values assumed for the thermal parameters. If oceanic lithosphere is not allowed to contract then large tensile stresses will be generated throughout the lithosphere. These stresses have a magnitude of $-\alpha E \Delta T$, where $\alpha$ is the coefficient of linear thermal expansion, $E$ is the Young’s modulus of elasticity, and $\Delta T$ is the change with time in temperature. The negative sign shows that positive tensile stresses occur if cooling takes place. On the other hand, if lithosphere has free lateral boundaries, then both compressive and tensile stresses will be generated [Sandwell, 1986; Parmentier & Haxby, 1986; Haxby & Parmentier, 1988]. Sandwell [1986] uses an analytical approach to calculate the accumulation of thermal stress in the oceanic lithosphere, but this model fails to satisfy all the boundary conditions. In contrast, Haxby & Parmentier [1988] calculate the accumulation of thermal stress in the oceanic lithosphere and the effect of variation in plate thickness, but they do not assume a finite plate. Here we use a finite-
element approach to solve for thermal stresses in a constant-thickness lithosphere of finite lateral extent.

METHODS AND ASSUMPTIONS:

For consistency with Chapter 1, we use a half-space cooling model. For such a model the temperature distribution in the oceanic lithosphere is given by

\[ T = T_m \text{erf}\left(\frac{z}{2\sqrt{\kappa t}}\right) \]

where \( T_m \) is the deep mantle temperature, \( z \) is depth, \( \kappa \) is thermal diffusivity, and \( t \) is the age of the lithosphere. The thickness of the competent layer is taken as that part of the lithosphere below which deformation is dominated by creeping flow. It is defined by the 700 °C isotherm [Kumar & Gordon, 2006]. For simplicity we integrate the cooling of the lithosphere above this isotherm. Thus the temperature field that we use is a function only of \( x \).

The total change in the temperature since lithosphere creation at any time and at any one location can be defined as

\[ \Delta T = \int_0^t (T - T_i) \, dz = 2T_m(1 - e^{-\kappa x}) \sqrt{\frac{\kappa x}{\pi v}} \]  \quad (1) 

where \( H = 2\sqrt{\kappa t \text{erf}^{-1}\left(\frac{T_i}{T_m}\right)} \) (Appendix C), \( T_i \) is the isotherm that defines the depth limit of the competent boundary of the layer, and \( A = \text{erf}^{-1}\left(\frac{T_i}{T_m}\right) \). Variables \( x \), \( y \), and \( z \) respectively are the directions (i) horizontal and perpendicular to the ridge, (ii) horizontal and parallel to the ridge, and (iii) vertical.
In our models, oceanic lithosphere has width $W$ and length $L$. We follow Sandwell [1986] in taking $W$ to be the distance between successive fracture zones. Sandwell [1986] found that the average value of $W/v$, where $v$ is the half rate of seafloor spreading, is 6.28 Myr$^{-1}$. Sandwell [1986] distinguishes between two end-member of the ridge-transform geometry: stepped and crenelate. The stepped pattern is characterized by a consistent sense of offset of a sequence of mid-ocean-ridge segments. In contrast a crenelate pattern is characterized by an alternating sense of offset of segments of a mid-ocean ridge at successive transform faults. In a stepped pattern of ridge-transform plate geometry, only one side of the strip has an active transform boundary, which can be considered traction free. It is observed that the stepped pattern is dominant in nature. To account for this, Sandwell [1986] uses a value of $W/v = 10$ Myr$^{-1}$ in his modeling. The idea is to use a value larger than the observed value of 6.28 Myr$^{-1}$ because his models (as well as ours) more closely resemble the geometry of the stepped pattern than the crenelate pattern. The observed value of $W/v$, however, is based on observations mainly from the crenelate geometry. Following Sandwell [1986], we also assume a value for $W/v$ of 10 Myr$^{-1}$ in our models. To simplify, we distribute the total cooling of the variable-thickness competent layer over a hypothetical lithosphere of constant thickness.

We also note that depth-integrated stress (N/m) does not depend on the value of thickness (Appendix C), but depth-distributed stress depends on the thickness of the plate. A model with a constant thickness of 20 km more closely resembles the properties of the young competent oceanic lithosphere than does a model with a constant thickness of 40 km [Chapter 1]. Thus our calculations for stress and strain are based on distributing the total cooling throughout a 20 km thick layer. We also assume a plane-stress
approximation, i.e., \( \sigma_{lz} = 0 \), for calculation of thermal stresses. Values of thermal parameters are kept constant and are the same as those used in the preferred model of Chapter 1: Young’s modulus, \( E \), is assumed to be \( 6.5 \times 10^{10} \) Pa and Poisson’s ratio, \( \nu \), is assumed to be 0.25.

If we define \( N_T = -\alpha E \Delta T \), then the differential equation that defines plane stress condition is written as

\[
\nabla^4 \phi = -\nabla^2 N_T \\
\text{(2) (Appendix C)}
\]

where \( \phi \) is the Airy stress function [Timoshenko & Goodier 1970; Boley & Weiner, 1960]. We also assume that all the boundaries are traction free, which results in the following boundary condition:

\[
\begin{align*}
N_{yy} = \frac{\partial^2 \phi}{\partial x^2} &= 0 \text{ along } y = \pm W/2 \\
N_{xx} = \frac{\partial^2 \phi}{\partial y^2} &= 0 \text{ along } x = 0 \text{ and } x = L \\
N_{xy} = \frac{\partial^2 \phi}{\partial x \partial y} &= 0 \text{ along } y = \pm W/2, \ x = 0 \text{ and } x = L
\end{align*}
\]

where, \( N_y = \int^L_0 \sigma_y \, dz \), \( W \) is the width, and \( L \) is the length of the plate.

We estimate the thermo-elastic stresses using two complementary methods, an analytical solution and a finite-element model. The analytical solution is similar to the approach taken by Sandwell [1986] and we obtain similar (though not identical) results. Our method of solution is explained in detail in Appendix C. As was true for Sandwell [1986], we were unable to find an analytical solution that satisfied all the boundary conditions outlined above. We forced the boundary conditions along the transform-faults
of the rectangular plate (i.e. \( \frac{\partial^2 \phi}{\partial x^2} = 0 \), along \( y = \pm W / 2 \) and \( \frac{\partial^2 \phi}{\partial x \partial y} = 0 \), along \( y = \pm W / 2 \)) to be satisfied. We could not find a solution that satisfied the boundary conditions along \( x = 0 \) and along \( x = L \) as only two parameters can be fixed from the general solution.

In contrast, the solution found with a finite element model satisfied all the boundary conditions. The strain predicted by the finite element model is converted to strain rate using the relationship \( \dot{\epsilon}_y = v \frac{\partial \epsilon_y}{\partial x} \) in the steady state, where \( v \) is the half-spreading rate of the plate and \( \frac{\partial \epsilon_y}{\partial x} \) is the derivative of strain \( \epsilon_y \) in the x-direction.

RESULTS:

Analytical results

In agreement with models of Turcotte [1974], Sandwell [1986], and Haxby & Parmentier [1988], ridge-parallel tensile thermo-elastic stresses are predicted for young lithosphere (Figure 2.1). The cross over from tensile to compressive thermo-elastic stresses takes place near the 5 Ma isochron (Figure 2.1). As discussed above, although boundary conditions are satisfied along the transform-faults, they are not satisfied along the faces parallel to the ridge. Although the net shear stress integrated along the width of the plate does vanishes at \( x=0 \) and \( x=L \), it does not vanish identically. The contour plot for \( \sigma_{xy} \) indicates that shear stress magnitudes vary along both horizontal directions. Along the x-axis the magnitude of shear stress is highest close to the ridge and falls sharply for the older portion of the oceanic lithosphere. This is due to decrease in the gradient of the total cooling of the oceanic lithosphere along the x-direction (Figure 2.1).
Along the y-axis, shear stress is highest close to the transform or fracture zone \( y = \pm W/2 \) and decays towards the center of the plate. Axial stresses \( \sigma_y \) and \( \sigma_x \) are generated due to gradient in the shear stress along x and y directions. \( \sigma_y \) vanishes along \( y = \pm W/2 \) and increases to maximum values at \( y = 0 \).

That the failure to satisfy the natural boundary conditions at \( x = 0 \) is a serious problem can be seen by examining Figure 2.1c. The very high levels of shear stress as \( x = 0 \) is approached implies that the implicitly assumed shear tractions along the ridge boundary are very high. These shear tractions cause the plate to be in ridge-parallel tension very near the ridge, and suggest that the ridge-parallel tensional stresses near the ridge are an artifact of the implicit boundary conditions and not necessarily a robust result. Thus, the state of thermo-elastic stress near the ridge may have been misinterpreted in past work.
Figure 2.1: Thermo-elastic stresses from the analytic model with $\frac{W}{v}=10$. Color contours are value of stresses in MPa. Boundary conditions are satisfied for $y = \pm \frac{W}{2}$ but not for $x = 0$ and $x = L$.
(a) $\sigma_{yy}$, (b) $\sigma_{xx}$, and (c) $\sigma_{yx}$. 
Results from a finite-element model

Figure 2.2 shows two-dimensional distributions of plane stresses as calculated using a finite-element model with the natural boundary conditions (i.e., no tractions along any edge). The spatial gradient for $\sigma_{yy}$ is very large near the ridge and a detailed distribution of it is shown in Figure 2.3. In contrast with the analytical results and prior work [Turcotte, 1974, Sandwell, 1986; Haxby & Parmentier, 1988], the finite-element model indicates not tensile but compressive thermo-elastic ridge-parallel stresses in very young oceanic lithosphere. Aside from the compressive $\sigma_{yy}$ at very young ages, the spatial distribution of stress is similar to what was indicated by the analytical results. Vertically averaged stresses are tensile for lithosphere older than 1 Myr but younger than 5 Myr in age.

The state of ridge-parallel normal stress is generally determined by the relative magnitudes of shear stress on ridge-parallel vertical planes bounding an element of lithosphere of a given age. That older lithosphere has cooled more than an element of lithosphere of a given age acts to shorten that element of lithosphere in the ridge-parallel direction. On the other hand, that younger lithosphere has cooled less than that same element of lithosphere acts to lengthen that element of lithosphere in the ridge-parallel direction. If the effect of the shear stresses on the young side exceeds those on the old side, the element of lithosphere will be in ridge-parallel tension, and vice versa.

This reasoning makes it clear why compressive stresses ($\sigma_{yy}$) are expected for the oceanic lithosphere very near the mid-ocean ridge (in lithosphere less than 1 Myr in age for the parameters that we assume). The absence of shear tractions along the ridge
boundary leaves only the effect of the shear stress (on ridge-parallel vertical planes) from older lithosphere, which acts to shorten the youngest elements of lithosphere in the ridge-parallel direction, and thus puts them in a state of ridge-parallel compression.

Between lithosphere of \( \approx 1 \text{-Myr} \) in age and \( \approx 5 \text{-Myr} \) in age, the strong gradient in cooling causes a strong gradient with age of shear stresses on ridge-parallel vertical planes. This strong gradient in turn causes ridge-parallel extension to exceed ridge-parallel contraction and hence the thermo-elastic state of stress is tensile in the ridge-parallel direction. Beyond \( \approx 5 \text{-Myr} \) in age, the gradient in \( x \) of contraction is much smaller, and the size of the ridge-parallel normal stresses are also much smaller and tend to be slightly compressive.

The resulting 2D stresses and strain rates at \( y=0 \) are plotted in Figures 2.4 and 2.5. The magnitude of compressive stresses could be as high as \( \approx 80 \text{ MPa} \) for the youngest part of the sea-floor. Tensile \( \sigma_y \) is predicted between ages of 1 Myr old to 5 Myr old with a magnitude that remains between 0 MPa and 20 MPa.

Strain rates (assuming elastic behavior everywhere) due to these stresses are shown in Figure 2.5. As stresses are not constant throughout the width of the plate (i.e., in the ridge-parallel direction), we show strain rate averaged along the width as a function of lithosphere age in Figure 2.6. The average value of strain rate is smaller than its value at \( y = 0 \). Comparison of this average value of strain rate is shown in Figure 2.6 along with one-tenth of the intrinsic strain rate due to depth-averaged cooling predicted by a half-space cooling model. Strain rates due to thermal stresses are an order of magnitude smaller than the intrinsic strain rates.
Figure 2.2: Thermo-elastic stresses from the finite-element model. Color contours are value of stresses in MPa.
(a) $\sigma_{yy}$, (b) $\sigma_{xx}$, and (c) $\sigma_{xy}$.
Figure 2.3: $\sigma_{yy}$ for very young oceanic lithosphere as determined from the finite-element model.
Figure 2.4: Variation of stresses at the center of the plate ($y = 0$) with distance away from the ridge for $W/v = 10$ Myr$^{-1}$ and $L = 3200$ km. Green curve: $\sigma_{xx}$; red curve: $\sigma_{yy}$. $\sigma_{yy}$ is compressive for the youngest part of the lithosphere, but soon it becomes tensile and again transitions between tensile to compressive at a distance of $\approx 400$ km away from the ridge.
Figure 2.5: $\dot{\varepsilon}_{yy}$ due to thermal stress plotted against age of the lithosphere. Dashed-dotted curve: $\dot{\varepsilon}_{yy}$ averaged over $-W/2 \leq y \geq W/2$. Solid curve: $\dot{\varepsilon}_{yy}$ at $y=0$. 
Figure 2.6: Blue curve: \( \dot{\varepsilon}_{yy} \) averaged along the width of the plate as determined from a finite element model. Red curve: one-tenth of the intrinsic strain rate as a function of age [Kumar & Gordon, 2006]. Comparison shows that strain rate parallel to the ridge due to thermo-elastic stresses is an order of magnitude smaller than that calculated using a half-space cooling when the stiffness of oceanic lithosphere is neglected.
DISCUSSION

All the results are based on a hypothetical plate model with $W/v = 10$ Myr$^{-1}$ and half-spreading rate of 80 mm yr$^{-1}$; thus the results can be applied approximately to the Pacific plate. The analytical solution indicates that the lithosphere undergoes tensile stress if it is younger than 5 Myr and undergoes compressive stress if older. In contrast, our finite element solution indicates compressive stresses for lithosphere younger than 1 Ma and tensile stresses for ages between 1 and 5 Ma. Compressive stresses are insignificant for lithosphere older than 5 Ma. Comparison of the stress distributions obtained from the analytical solution and the finite element solution show similar stress distribution except very near the mid-ocean ridge (lithosphere $<\approx 1$ Myr old) where the solution found from the finite element model indicates compressive stresses for the oceanic lithosphere. As the finite element solution satisfies all the boundary condition for the problem while the analytical solution does not, the finite element solution is used to calculate strain rates due to thermal cooling.

Strain rates thus calculated when compared against the strain rates due to intrinsic cooling show that prior estimates need small or no modification to account for thermal elastic stresses as its effect is an order of magnitude smaller than that of intrinsic cooling.

While calculating strain rates due to thermo-elastic stresses we assumed that lithosphere is a simple thin plate model of constant thickness. We know that this is not true and that the thickness of the competent layer increases with age if it is defined using an isotherm. This assumption underestimates the thermal stresses for the youngest part of the lithosphere and overestimates the thermal stresses for the oldest part of the lithosphere. Thus, if the model is made more realistic by incorporating age varying
thickness of the lithosphere, then the strain rates due to thermal-elastic stresses will be of
greater magnitude for younger ages.

An unexpected result of our analysis was that—unlike prior studies—we found from
our finite-element model that the lithosphere is predicted to be in ridge-parallel
compression very near the ridge. It now can be seen that the ridge-parallel tension found
in our analytical solution and a previously published analytical solution is an artifact of
the boundary conditions assumed in those models. Thus, the widespread belief that the
mid-ocean ridges should be in a state of ridge-parallel thermo-elastic tension may not be
on as firm a foundation as previously believed. Some workers have cited normal-faulting
earthquakes with T axes parallel to the Central Indian and Southeast Indian Ridges as
evidence of ridge-parallel tension [Bratt et al., 1985]. We believe that these events,
however, simply reflect deformation in diffuse oceanic plate boundaries bounding the
Capricorn component plate [Wiens et al., 1985; Royer & Gordon, 1996; Gordon, 1998;
Zatman et al., 2002, 2005]. Absent these two zones of deformation, ridge-parallel
extensional deformation is rare.

CONCLUSION:

The effect of thermo-elastic stresses on the strain field of vertically averaged oceanic
lithosphere is an order of magnitude smaller than the strain due to intrinsic horizontal
thermal contraction. Thus prior estimates of horizontal strain rates and horizontal strain
rates within oceanic plates due to thermal contraction require little modification. Tensile
ridge-parallel normal stress very near mid-ocean ridges indicated by analytical models
appears to be an artifact of inappropriate boundary conditions. Our finite-element model,
which satisfies the natural boundary condition of no tractions on any boundary, indicates ridge-parallel compression very near the ridge (in lithosphere \( \approx 1 \) Myr in age or less). This result may explain the near absence of observations of normal faulting with \( T \) axes parallel to the ridge that were predicted from prior thermo-elastic models.
REFERENCES:


Appendix A

Half space cooling model

The heat diffusion equation can be written as

\[ \rho C_p \frac{\partial T}{\partial t} = \dot{A} + K \frac{\partial^2 T}{\partial z^2} \]  

(A-1)

assuming heat transfer takes place in only one direction, 'z', where \( \rho \) is density, \( C_p \) is specific heat capacity, \( K \) is thermal conductivity, and \( \dot{A} \) is rate of heat generation per unit volume [Turcotte, 1982]. Heat generation in oceanic lithosphere due to radioactive elements can be ignored because oceanic lithosphere is highly depleted of radiogenic elements [Jochum et al., 1983]. Thus, one is left with the heat diffusion equation in one dimension. Because the above partial differential is second order in \( z \) and first order in \( t \), two boundary conditions in \( z \) and one in \( t \) are required.

In the half-space cooling model oceanic lithosphere is assumed to be formed from hot mantle material with constant temperature of \( T_m \), which extends to infinity in depth. As soon as hot mantle material comes in the contact with the cool water (\( 0 \text{ }^\circ \text{C} \)) at the top, it loses heat to the ocean water above. As time progresses, deeper and deeper portions of lithosphere are cooled. Figure A.1 below will give a better idea of the temperature distribution at any time \( t \) and boundary condition for solving the above mentioned heat diffusion equation would be as follows-

- \( T = T_m, \ t = 0, \ z = \text{depth} > 0 \)

- \( T = T_0, \ z = 0, \ t > 0 \)

- \( T \to T_m, \ z \to \infty, \ t > 0 \)
Using the above boundary conditions, temperature profile at any time \( t \) and depth \( z \) can be written as

\[
T(t, z) = T_m \operatorname{erf} \left( \frac{z}{\sqrt{4kt}} \right)
\]  

(A-2)

where \( T_m \) is the mantle temperature. Differentiating the above equation will give us the rate of change in temperature at any depth and time given by-

\[
\frac{\partial T}{\partial t} = -\frac{T_m t^{-3/2} z}{\sqrt{4k\pi}} \exp \left\{ \frac{-z^2}{4kt} \right\}
\]  

(A-3)

It should be noted that the negative sign in the above equation has been ignored as we consider change in temperature due to cooling as a positive quantity. The depth-averaged rate of cooling can be calculated for an isotherm as well as constant depth.

The rate of cooling averaged to a constant isotherm is given by

\[
\frac{\partial \bar{T}}{\partial t} = \frac{T_m}{2At\sqrt{\pi}} \left[ 1 - e^{-A^2} \right]
\]  

(A-4)

where \( A = \operatorname{erf}^{-1} \left[ \frac{T_i}{T_m} \right] \). Similarly, the depth-averaged rate of cooling over a constant thickness is given by

\[
\frac{1}{H} \frac{\partial T}{\partial t} \frac{\partial T}{\partial z} = \frac{1}{H} \int_0^H \frac{T_m t^{-3/2} z}{\sqrt{4k\pi}} \exp \left\{ \frac{-z^2}{4kt} \right\} \partial z = \frac{T_m}{H \sqrt{\pi}} \sqrt{\frac{\kappa}{4kt}} \left( 1 - e^{-\frac{H^2}{4kt}} \right).
\]  

(A-5)

where \( H \) is the constant thickness of the competent layer.

The depth- and age-averaged rate of cooling for an isotherm can be calculated by integrating \( \frac{\partial \bar{T}}{\partial t} \) over the time interval \( \Delta t = t_2 - t_1 \) and averaging it over the time period.
\[
\left[ \frac{1}{t_2-t_1} \right]^{t_2}_{t_1} \frac{\partial T}{\partial t} \, dt = \frac{T_m}{2(t_2-t_1)A\sqrt{\pi}} \left[ 1-e^{-\lambda t} \right] \ln \left( \frac{t_2}{t_1} \right) \tag{A-6}
\]

The lower limit of integration should not start from \( t_1 = 0 \) because the half-space cooling model is invalid at time \( t = 0 \) and a limiting value of the above equation exists when \( t_2 = t_1 \).

\[
Lt \left[ \frac{T_m}{2(t_2-t_1)A\sqrt{\pi}} \left[ 1-e^{-\lambda t} \right] \ln \left( \frac{t_2}{t_1} \right) \right] = \frac{T_m}{2(t_1)A\sqrt{\pi}} \left[ 1-e^{-\lambda t_1} \right] \tag{A-7}
\]

Similarly we find the depth- and age-averaged rate of cooling for a constant thickness model

\[
\frac{1}{t_2-t_1} \int^{t_2}_{t_1} \frac{T_m}{H \sqrt{\pi t}} \left( 1-e^{-\frac{H^2}{4\alpha t}} \right) \, dt = \frac{T_m}{H(t_2-t_1) \sqrt{\pi}} \left[ 2 \left( 1-e^{-\frac{H^2}{4\alpha t_2}} \right) \sqrt{t_2} - 2 \left( 1-e^{-\frac{H^2}{4\alpha t_1}} \right) \sqrt{t_1} \right. \\
\left. \sqrt{\frac{\pi}{\kappa}} \left( \text{erf} \left( \frac{H}{\sqrt{2\kappa t_2}} \right) - \text{erf} \left( \frac{H}{2\sqrt{\kappa t_1}} \right) \right) \frac{\pi}{\kappa} \right] \tag{A-8}
\]
Figure A.1: Figure shows different stages of half space cooling model. Hot mantle material comes in contact with ocean water and cools off. Figure on the extreme left shows the temperature distribution with depth just before the hot mantle material comes in contact with the surface water. Figure in the middle shows temperature profile as soon as oceanic lithosphere forming mantle matter starts to cool. In this case only the top layer takes the temperature of ocean water. Figure on the extreme right shows temperature profile at any time $t>0$. 
Figure B.1: Variation of volumetric coefficient of thermal expansion for forsterite with temperature [Bouhifd, 1996].
Appendix B

Temperature dependence of thermal parameters

The rate of cooling predicted by half-space cooling models depends mainly on thermal diffusivity while strain rates depend on coefficient of linear thermal expansion $\alpha_i$. We know that these thermal parameters are not constants and that they vary with temperature and pressure. Here we try to understand their behavior with the change in temperature and pressure from the surface to the base of the lithosphere where temperature can easily reach up to $1300^\circ C$.

Experimental studies done on olivine to understand the variation of thermal diffusivity with the temperature and pressures shows that its value decreases by $\approx 50\%$ for a $1000^\circ C$ change in the temperature. An exact empirical relationship for olivine is given by the following equation

$$\kappa = \kappa_{298} \left( \frac{298}{T} \right)^n \left( 1 + a \, p \right) \quad (\text{Xu et al., 2004}) \quad (B-1)$$

where $n = 0.681$, $a = 0.036 \text{ GPa}^{-1}$ and $\kappa_{298} = 1.1$ when extrapolated for surface pressure. Temperature $T$ in the above equation is in Kelvin. Using the above equation, the depth averaged value for thermal diffusivity can be calculated for the whole lithosphere where the temperature difference between the top and the base of the lithosphere is the same as the preferred model. The change in the pressure is taken as $2 \text{ GPa}$. Our calculation shows that the averaged value of thermal diffusivity would be as high as $1.07 \text{ mm}^2 \text{ s}^{-1}$. 
The coefficient of linear thermal expansion can be defined by \( \alpha = \left( \frac{1}{V} \right) \left( \frac{\partial V}{\partial T} \right) \rho \)

and it is the measure of change in the volume of a substance due to the change in temperature. Many researchers in the past have tried to estimate a relationship between \( \alpha \) and temperature based on experimental observation. Figure B.1 [Bouhifd et al., 1996] shows the observed value of the coefficient of volumetric thermal expansion of forsterite versus temperature. Each and every experimental study predicts at least a 50% change in \( \alpha \) with a change in temperature of 1500 K. The plotted experimental data has been shown to be easily fitted by the following polynomial

\[
\alpha_i(T) = a_0 + a_1 T + a_2 T^2
\]  

(B-2)

where, \( a_0, a_1, \) and \( a_2 \) are constants and \( T \) is temperature. Ignoring the third term in the above equation we get a linear relationship between \( \alpha \) and temperature. According to Bouhifd et al., [1996], a linear relationship between \( \alpha \) and temperature, \( T \), can be written as

\[
\alpha_i(T) = 2.832 \times 10^{-5} + 0.785 \times 10^{-8} T
\]  

(B-3)

The above relationship was established for a temperature range of 300 to 2150 K, but can be easily extrapolated for temperatures between 273 and 300 K. The average value of coefficient of linear expansion for a constant isotherm is found independent of age. When averaged over an isotherm of 900 C, the average value of linear expansion is \( 1.1 \times 10^{-5} \text{ C}^{-1} \), which is 10% more than the value used.
Appendix C

Analytical approach to calculate thermal stresses in the oceanic lithosphere

Sandwell [1986] proposed the solution for thermo-elastic stresses which shows the distribution of stresses in an elastic plate assuming no net traction at the boundaries. Here we re-derive his results using only odd functions for the analytic solution in the y-direction. A step by step procedure is shown below:

Strains are given by \( e_{xx} = \frac{\partial u}{\partial x}, e_{yy} = \frac{\partial v}{\partial y} \) and \( e_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \).

Differentiating \( e_{xx} \) twice with respect to \( y \), \( e_{yy} \) with \( x \), and \( e_{xy} \) once with \( x \) and \( y \) will lead to the strain compatibility relationship:

\[
\frac{\partial^2 e_{xx}}{\partial y^2} + \frac{\partial^2 e_{yy}}{\partial x^2} = 2 \frac{\partial^2 e_{xy}}{\partial x \partial y}
\]  
(C-1)

For a plane stress condition, the stress-strain relationships are as follows,

\[
e_{xx} = \frac{1}{E} \left[ \sigma_{xx} - \nu \sigma_{yy} \right] \quad (C-2)
\]

\[
e_{yy} = \frac{1}{E} \left[ \sigma_{yy} - \nu \sigma_{xx} \right] \quad (C-3)
\]

\[
e_{xy} = \frac{1}{2G} \sigma_{xy} = \frac{(1+\nu)}{E} \sigma_{xy} \quad (C-4)
\]

Plugging these into the strain compatibility condition will lead to a relationship between spatial derivatives of different stress components. This can be further simplified using equilibrium conditions to give us the relationship \( \left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right] (\sigma_x + \sigma_y) = 0 \). This
relationship represents an isothermal elasticity problem when no body forces are acting. Keeping strain compatibility and equilibrium conditions the same and changing the stress-strain relationship to account for thermal strain will establish a stress compatibility relationship

\[ \left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right] \left( \sigma_x + \sigma_y \right) = -\alpha \mathbf{E} \nabla^2 T \] (C-5)

Now it is important to find stress components that must satisfy the equilibrium conditions and compatibility relationship. We can easily satisfy the compatibility relationship and equilibrium if the stresses are defined using a stress function, \( \varphi \), which is also called the Airy stress function.

\[ \sigma_{xx} = \frac{\partial^2 \varphi}{\partial y^2}; \sigma_{yy} = \frac{\partial^2 \varphi}{\partial x^2}; \sigma_{xy} = \frac{\partial^2 \varphi}{\partial x \partial y} \] (C-6)

When we substitute the Airy stress function for stresses in equation, we form the following differential equation

\[ \nabla^4 \varphi = -\alpha \mathbf{E} \nabla^2 T \] (C-7)

Here we are dealing with depth integrated temperature to a constant isotherm, and stresses used in equation 2 are also integrated to a constant depth. This is done to obtain an approximation for a plate of constant thickness, and as a result of which we are dealing with forces (N/m) in place of stresses. Thus an equivalent equation for equation C-3 in terms of forces would be

\[ \left[ \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} \right] \left( N_{xx} + N_{yy} \right) = -\nabla^2 N_T \] (C-8)

where, \( N_{ij} = \int_0^{\ell} \sigma_{ij} \, dz \)
and \( N_T = -\alpha E \int_0^{\frac{T_h}{T_c}} (T - T_h) \, dz \). One should note that the integral for \( N_T \) varies with the depth of elastic layer with age for a constant isotherm. However, we are distributing that change in the temperature over a plate of constant thickness for mathematical simplicity. The above equation is written in the form of depth-integrated stresses. The solution, consists of an Airy stress function composed of two functions \( F_p \) and \( F_h \), which are called the particular and the homogeneous solution. \( F_p \) can be found by integrating the right hand side of the equation four times with respect to \( x \). \( F_h \) would be the solution of the homogeneous biharmonic equation with the following boundary conditions applied:

\[
\nabla^4 F_h = 0 \quad \text{(C-9)}
\]

\[
\begin{align*}
\frac{\partial^2 F_h}{\partial x^2} & = -\frac{\partial^2 F_p}{\partial x^2} = -2\alpha ET_m \left(1-e^{-\frac{A^2}{2}}\right) \sqrt{\frac{\kappa x}{\pi V}}, \quad y = \pm \frac{W}{2} \\
\frac{\partial^2 F_h}{\partial y^2} & = 0, \quad y = \pm \frac{W}{2}, x = 0, L \\
\frac{\partial^2 F_h}{\partial y^2} & = 0 \quad x = 0, L
\end{align*} \quad \text{(C-10)}
\]

The above method of solving the initial differential equation

\[
\frac{\partial^2}{\partial y^2} \left( N_{xx} + N_{yy} \right) = -\nabla^2 N_T
\]

is analogous to the approach given by Timoshenko [1970] for solving thermo-elastic problems. According to him, the thermo-elastic problem can be reduced to a more simplified problem of isothermal elasticity. Suppose we have a plate of a certain length and width and that there is a temperature distribution along the length due to which thermal stresses will be generated in the plate. Timoshenko says we should apply a force of \(-\alpha ET(x)\) at each and every element of the
plate. $E$ is the Young's modulus of the elasticity and $T$ is temperature. The negative sign shows that if temperature is positive we will have to apply compressive stresses. If the boundaries are stress free, similar to our boundary condition, then one should apply tensile stresses of magnitude $-\alpha ET(x)$ at the boundary only and solve for $\nabla^4 \varphi = 0$. When the resultant stress distribution is superimposed on the initial compressive stresses in the plate, it will generate stress-free boundary condition.

Assuming we are able to reduce our problem to a homogeneous biharmonic differential equation, we have to look for the possible solutions. In the absence of any body forces, a low-order polynomial Airy stress function satisfies the biharmonic differential equation

$$\varphi = \sum_m \sum_n A_{mn} x^m y^n$$  \hspace{1cm} (C-11)

Equation 6 is a solution for the biharmonic differential equation if $m+n \leq 3$, but polynomials of higher order can be a solution too. Solutions for simple problems can be formulated using these low-order polynomials [Timoshenko, 1970, Soutas-Little, 1973, Selvadurai, 2000; Malvern, 1969]. A much more rigorous form of solution to the biharmonic equation can be found using Fourier series solution involving trigonometric functions. According to Soutas-Little [1973], it is advantageous to separate the problem into a combination of even and odd functions in the $x$ and $y$ directions. A function can be easily divided into even and odd functions if it is possible to find a Fourier series expansion of any function. If we are able to split the boundary conditions in even and odd functions we should be able to choose the Airy stress function accordingly.
According to Table [7-1] of Soutas-Little [1973], if the boundary condition is an odd function of x, then the Airy stress function, $\phi$, should be an odd function in x too. If you look at the boundary condition for the complementary solution at $y=W/2$, it is a function of the square root in x. We can always expand this as a half range series in sine or cosine functions. I chose Airy stress function to be an odd function (sine) in x because it satisfies the boundary condition for $N_{xx}$ automatically at x=0.

I assume that the stress function can be separated easily in x and y and can be written as

$$F_c(x, y) = \sum_{n=1}^{\infty} \phi_n(y) \sin(n\pi x / L)$$  \hspace{1cm} (C-12)

plugging equation C-12 into the homogeneous biharmonic equation gives a general solution in the form,

$$F_c(x, y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[\left(C_n + B_n y \frac{n\pi}{L}\right) \sinh\left(\frac{n\pi y}{L}\right) + \left(A_n + D_n y \frac{n\pi}{L}\right) \cosh\left(\frac{n\pi y}{L}\right)\right]$$  \hspace{1cm} (C-13)

If we look at the problem, then we find that ours is a case when stresses should be symmetric about $y = 0$. Thus, the above equation can be further reduced to an even function in y.

$$F_c(x, y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[\left(B_n y \frac{n\pi}{L}\right) \sinh\left(\frac{n\pi y}{L}\right) + A_n \cosh\left(\frac{n\pi y}{L}\right)\right]$$  \hspace{1cm} (C-14)

Stress components can be written as

$$\begin{align*}
N_{yy} &= \frac{\partial^2 \phi}{\partial x^2} = \sum_{n=1}^{\infty} -k_n^2 \sin(k_n x) \left[A_n \cosh(k_n y) + B_n k y \sinh(k_n y)\right] \\
N_{xx} &= \frac{\partial^2 \phi}{\partial y^2} = \sum_{n=1}^{\infty} k_n^2 \sin(k_n x) \left[A_n \cosh(k_n y) + B_n \left(k y \sinh(k_n y) + 2 \cosh(k_n y)\right)\right] \\
N_{xy} &= \frac{\partial^2 \phi}{\partial x \partial y} = \sum_{n=1}^{\infty} -k_n^2 \cos(k_n x) \left[A_n \sinh(k_n y) + B_n \left(k_n y \cosh(k_n y) + \sinh(k_n y)\right)\right]
\end{align*}$$  \hspace{1cm} (C-15)
where \( k_n = \frac{n\pi}{L} \)

Since, \( \frac{\partial^2 \varphi}{\partial x \partial y} = 0 \) at \( x = \pm W / 2 = \pm w \)

\[
A_n \sinh(k_n w) + B_n \left( k_n \cosh(k_n w) + \sinh(k_n w) \right) = 0 \quad \text{or,}
\]

\[
A_n = -B_n \left( k_n \cosh(k_n w) + \sinh(k_n w) \right) \left[ \sinh(k_n w) \right]^{-1} \tag{C-16}
\]

Also,

\[
\frac{\partial^2 \varphi}{\partial x^2} = -2\alpha ET_m (1 - e^{-A^2}) \sqrt{\frac{\kappa x}{\pi v}} \text{or} - \beta \sqrt{x}, \text{at} \ y = \pm w \tag{C-17}
\]

\[
\sum_{n=1}^{\infty} -k_n^2 \sin(k_n x) \left[ A_n \cosh(k_n w) + B_n k w \sinh(k_n w) \right] = -\beta \sqrt{x} \tag{C-18}
\]

To compare this we need to expand \( \sqrt{x} \) in Fourier series for \( x=0 \) to \( L \), where \( L \) is the length of the strip. Since the left hand side of the equation is an odd function of \( x \) and we can find a half range series of \( \sqrt{x} \) in cos or sin.

\[
b_n = \frac{2}{L} \int_0^L \sqrt{x} \sin(k_n x) dx \tag{C-19}
\]

Thus the Fourier series expansion of \( \sqrt{x} \) would be \( \sum b_n \sin(k_n x) \) for \( n=\pm 1, 2, 3... \infty \) which when substituted in the equation results in

\[
\sum_{n=1}^{\infty} -k_n^2 \sin(k_n x) \left[ A_n \cosh(k_n w) + B_n k w \sinh(k_n w) \right] = \sum_{n=1}^{\infty} -\beta b_n \sin(k_n x) = \sum_{n=1}^{\infty} b_n' \sin(k_n x)
\]

or

\[
-b_n' k_n^2 = A_n \cosh(k_n w) + B_n k w \sinh(k_n w) \tag{C-20}
\]
From equations 10 and 8 we can find the value for $A_n$ and $B_n$

\[
B_n = \frac{2b_n k_n^{-2} \sinh(k_n w)}{2k_n w + \sinh(2k_n w)}
\]  \hspace{1cm} (C-21)

\[
A_n = -2b_n k_n^{-2} \left[ \frac{k_n w \cosh(k_n w) + \sinh(k_n w)}{2k_n w + \sinh(2k_n w)} \right]
\]  \hspace{1cm} (C-22)