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Stabilized Space–Time Fluid–Structure Interaction Techniques with the
Continuum Element

by

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Abstract

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We present the methods used to add continuum element functionality to the structure side of our Fluid–Structure Interaction (FSI) solver. The FSI solver, already capable of handling the interaction between membrane structure elements and fluid elements, can now accurately simulate fully 3D structure models as well. A few simple test calculations are presented in order to verify the proper implementation of these changes. Then we aim to establish the effectiveness of these methods by modeling blood flow through a cerebral sacular aneurysm. These computations are performed with three different structural models: linearly-elastic, hyperelastic (Mooney–Rivlin), and Neo-Hookean. Furthermore, each structure model is tested with two different pressure profiles and two different aneurysm thicknesses. Finally, we suggest a procedural change for further investigation: instead of assuming image-based geometry corresponds to zero blood pressure, use that image-based geometry to estimate the zero-pressure arterial geometry.
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Chapter 1

Introduction

Computer simulations of Fluid–Structure Interactions (FSI) are often used to broaden our understanding of problems that are difficult to test experimentally. However, the scope of applications for FSI simulations is limited by the capabilities of the available solvers. For example, solvers that are designed to handle membrane structures will be able to model flags and parachutes, but they will be insufficient in modeling thicker structures. These structures are better represented by continuum elements.

This thesis describes the methods used to add continuum element functionality to the structure side of our own FSI solver. A few brief examples of its operation are provided. Finally, we examine some practical applications in the field of cardiovascular fluid mechanics.

1.1 Motivation

The Team for Advanced Flow Simulation and Modeling (T*AFSM) has conducted research on the topic of Fluid–Structure Interactions (FSI) for over 15 years. The real
world application driving much of this research has been parachute simulation. We have relied on the thin membrane assumption to effectively model these parachutes. However, as we expand our focus to encompass other FSI applications, we need to model structures that are too thick for this assumption. As such, we have augmented our FSI capabilities with continuum element functionality for the structure side of our solvers.

We can test two different types of continuum elements: linearly-elastic and hyperelastic. The linearly-elastic model can be used to simulate problems such as Blade–Vortex Interactions (BVI) and turbine FSI. The hyperelastic model can be used to simulate the deformation of rubber and rubber-like material. We use Mooney–Rivlin continuum elements to model hyperelastic structures.

FSI simulation was recently applied to modeling arterial flow in the human body (1; 2; 3; 4; 5; 6; 7). These computations were performed using continuum elements. We use continuum elements as well, but we also use a stabilized space–time formulation and its newly-developed special versions. We test the artery as both a linearly-elastic structure and a hyperelastic structure.

Some research has already been conducted using linearly-elastic continuum elements with block-iterative coupling in the FSI computations (1; 2; 4; 5; 7). Our solver uses quasi-direct coupling. Details are provided in Chapter 4.
1.2 Overview

Chapter 2 provides the governing equations used by our FSI solver. Since the modeled artery has a relatively large diameter, we treat blood flow as an incompressible Newtonian fluid. As such, we use the Navier–Stokes equations for the fluid solver. The structure solver, in addition to using the standard linearly-elastic model, also draws upon the Mooney–Rivlin model for hyperelastic elements.

Chapter 3 details the finite element formulations for fluid flow and structural deformation. Fluid flow is modeled using the Deforming-Spatial-Domain/Stabilized Space–Time (DSD/SST) formulation. The structural deformation is modeled using a finite element formulation based on the principle of virtual work. We use the Stabilized Space–Time Fluid–Structure Interaction (SSTFSI) method to combine these formulations.

Chapter 4 describes how we solve the fully-discretized equations. It also explains the differences between three types of coupling: block-iterative, quasi-direct, and direct.

Chapter 5 provides examples of the solver's operation as a validation of the new continuum element functionality.

Chapter 6 establishes the effectiveness of these methods in modeling blood flow through a cerebral sacular aneurysm. It shows the results of several different simulations conducted with a variety of test conditions and material properties.
Chapter 2

Governing Equations

2.1 Fluid mechanics

Let \( \Omega_t \subset \mathbb{R}^{n+1} \) be the spatial domain with boundary \( \Gamma_t \) at time \( t \in (0, T) \). The subscript \( t \) indicates the time-dependence of the domain. The Navier–Stokes equations of incompressible flows are written on \( \Omega_t \) and \( \forall t \in (0, T) \) as

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{f} \right) - \nabla \cdot \mathbf{\sigma} = 0 ,
\]

\[
\nabla \cdot \mathbf{u} = 0 ,
\]

where \( \rho, \mathbf{u} \) and \( \mathbf{f} \) are the density, velocity and external force, respectively. The stress tensor \( \mathbf{\sigma} \) is defined as

\[
\mathbf{\sigma}(\rho, \mathbf{u}) = -p \mathbf{I} + 2\mu \mathbf{\varepsilon}(\mathbf{u}) ,
\]
with

\[ \varepsilon(u) = \frac{1}{2} (\nabla u + (\nabla u)^T) . \]  

(2.4)

Here \( p \) is the pressure, \( \mathbf{I} \) is the identity tensor, \( \mu = \rho \nu \) is the viscosity, \( \nu \) is the kinematic viscosity and \( \varepsilon(u) \) is the strain-rate tensor. The essential and natural boundary conditions for Eq. (2.1) are represented as

\[ u = g \text{ on } (\Gamma_t)_g, \]  

(2.5)

\[ n \cdot \sigma = h \text{ on } (\Gamma_t)_h, \]  

(2.6)

where \( (\Gamma_t)_g \) and \( (\Gamma_t)_h \) are complementary subsets of the boundary \( \Gamma_t \), \( n \) is the unit normal vector and \( g \) and \( h \) are given functions. A divergence-free velocity field \( u_0(x) \) is specified as the initial condition.

### 2.2 Structural mechanics

Let \( \Omega^s_t \subset \mathbb{R}^{n_{xd}} \) be the spatial domain with boundary \( \Gamma^s_t \), where \( n_{xd} = 3 \). The superscript "s" indicates the structure. The parts of \( \Gamma^s_t \) corresponding to the essential and natural boundary conditions are represented by \( (\Gamma^s_t)_g \) and \( (\Gamma^s_t)_h \). The equations of motion are written as

\[ \rho^s \left( \frac{d^2 y}{dt^2} + \eta \frac{dy}{dt} - f^s \right) - \nabla \cdot \sigma^s = 0, \]  

(2.7)
where $\rho^s$, $y^s$, $f^s$ and $\sigma^s$ are the material density, structural displacement, external force and Cauchy stress tensor, respectively. Here $\eta$ is an artificial damping coefficient, which is nonzero only in computations where time accuracy is not required, such as in determining the deformed shape of the structure for specified fluid mechanics forces acting on it. Such computations typically precede any fluid mechanics or fluid-structure interaction computations, and the artificial damping facilitates reaching that initial shape in a robust way. The stresses are expressed in terms of the second Piola-Kirchoff stress tensor $S$, which is related to the Cauchy stress tensor through a kinematic transformation.

2.2.1 Linearly-elastic continuum elements

For the continuum element made of linearly-elastic material, the expression for $S$ is given as

$$S_{ij} = (\lambda^s G^{ij} C_{kl} + \mu^s (G^{il} G^{jk} + G^{ik} G^{jl})) E_{kl}, \quad (2.8)$$

where $\lambda^s$ and $\mu^s$ are the Lamé constants, $G^{ij}$ are the contravariant components of the metric tensor in the original configuration and $E_{kl}$ are the components of the Green-Lagrange strain tensor.
2.2.2 Hyperelastic (Mooney–Rivlin) continuum elements

For the continuum element made of hyperelastic (Mooney–Rivlin) material, the expression for \( \mathbf{S} \) is given as

\[
S^{ij} = 2 \left( C_1 + C_2 G^{kl} g_{kl} \right) G^{ij} - 2 C_2 G^{ik} g_{kl} G^{lj} + \left( K_{\text{PEN}} \ln \left( \sqrt{T_3} \right) - 2 (C_1 + 2C_2) \right) g^{ij},
\]

(2.9)

where \( C_1 \) and \( C_2 \) are the Mooney–Rivlin material constants and \( g_{kl} \) and \( g^{ij} \) are the covariant and contravariant components of the metric tensor in the deformed configuration. The incompressibility constraint is enforced with the penalty term \( K_{\text{PEN}} \ln \left( \sqrt{T_3} \right) \) (see (8)). Here \( I_3 \) is the third invariant of the right Cauchy–Green deformation tensor, and \( K_{\text{PEN}} \) is a penalty parameter determined based on the expression given in (9) for the bulk modulus:

\[
K_{\text{PEN}} = \frac{2(C_1 + C_2)}{(1 - 2\nu_{\text{PEN}})},
\]

(2.10)

where \( \nu_{\text{PEN}} \) (with a value close to 0.50) is the “penalty” Poisson’s ratio we use in the expression in place of the actual Poisson’s ratio.
Chapter 3

Finite Element Formulations

3.1 DSD/SST formulation of fluid mechanics

In the DSD/SST method (10; 11; 12; 13), the finite element formulation is written over a sequence of $N$ space–time slabs $Q_n$, where $Q_n$ is the slice of the space–time domain between the time levels $t_n$ and $t_{n+1}$. At each time step, the integrations are performed over $Q_n$. The space–time finite element interpolation functions are continuous within a space–time slab, but discontinuous from one space–time slab to another. The notation $(\cdot)_n^-$ and $(\cdot)_n^+$ will denote the function values at $t_n$ as approached from below and above. Each $Q_n$ is decomposed into elements $Q_n^e$, where $e = 1, 2, \ldots, (n_{el})_n$. The subscript $n$ used with $n_{el}$ is for the general case where the number of space–time elements may change from one space–time slab to another. The essential and natural boundary conditions are enforced over $(P_n)_k$ and $(P_n)_h$, the complementary subsets of the lateral boundary of the space–time slab. The finite element trial function spaces $(S^e_u)_n$ for velocity and $(S^e_p)_n$ for pressure, and the test function spaces $(\mathcal{V}^e_u)_n$ and $(\mathcal{V}^e_p)_n = (S^e_p)_n$ are defined by using, over $Q_n$, first-order polynomials in space and time.
The DSD/SST formulation (from (13)) is written as follows: given \((u^h)^-\), find \(u^h \in (S^h_u)_n\) and \(p^h \in (S^h_p)_n\) such that \(\forall w^h \in (Y^h_u)_n\) and \(\forall q^h \in (Y^h_p)_n\):

\[
\int_{Q_n} w^h \cdot \rho \left( \frac{\partial u^h}{\partial t} + u^h \cdot \nabla u^h - f^h \right) \, dQ + \int_{Q_n} \varepsilon(w^h) : \sigma(p^h, u^h) \, dQ \\
- \int_{(P_n)_h} w^h \cdot h^h \, dP + \int_{Q_n} q^h \nabla \cdot u^h \, dQ + \int_{\Omega_n} (w^h)^+ \cdot \rho \left( (u^h)^+ - (u^h)^- \right) \, d\Omega \\
+ \sum_{e=1}^{(n_d)_n} \int_{Q^e_n} \frac{1}{\rho} \left[ \tau_{SUPG} \rho \left( \frac{\partial w^h}{\partial t} + u^h \cdot \nabla w^h \right) + \tau_{PSPG} \nabla q^h \right] \cdot [L(p^h, u^h) - \rho f^h] \, dQ \\
+ \sum_{e=1}^{(n_d)_n} \int_{Q^e_n} \nu_{LSIC} \nabla \cdot w^h \rho \nabla \cdot u^h \, dQ = 0 ,
\] (3.1)

where

\[
L(q^h, w^h) = \rho \left( \frac{\partial w^h}{\partial t} + u^h \cdot \nabla w^h \right) - \nabla \cdot \sigma(q^h, w^h) .
\] (3.2)

This formulation is applied to all space–time slabs \(Q_0, Q_1, Q_2, \ldots, Q_{N-1}\), starting with \((u^h)_0 = u_0\). Here \(\tau_{SUPG}, \tau_{PSPG}\) and \(\nu_{LSIC}\) are the Streamline-Upwind/Petrov-Galerkin (SUPG), Pressure-Stabilizing/Petrov-Galerkin (PSPG) and Least-Squares on Incompressibility Contraint (LSIC) stabilization parameters. There are various ways of defining these stabilization parameters. Here we provide the definitions given
in (14):

\[
\tau_{\text{SUPG}} = \left( \frac{1}{\tau_{\text{SUGN1}}} + \frac{1}{\tau_{\text{SUGN2}}} + \frac{1}{\tau_{\text{SUGN3}}} \right)^{-\frac{1}{2}},
\]

(3.3)

\[
\tau_{\text{SUGN1}} = \left( \sum_{a=1}^{n_{en}} \left| (u^h - v^h) \cdot \nabla N_a \right| \right)^{-1},
\]

(3.4)

\[
\tau_{\text{SUGN2}} = \frac{\Delta t}{2},
\]

(3.5)

\[
\tau_{\text{SUGN3}} = \frac{h_{\text{RGN}}^2}{4 \nu},
\]

(3.6)

\[
h_{\text{RGN}} = 2 \left( \sum_{a=1}^{n_{en}} |r \cdot \nabla N_a| \right)^{-1},
\]

(3.7)

\[
r = \frac{\nabla |u^h|}{\| \nabla |u^h| \|},
\]

(3.8)

\[
\tau_{\text{PSPG}} = \tau_{\text{SUPG}},
\]

(3.9)

\[
\nu_{\text{LSC}} = \tau_{\text{SUPG}} \| u^h - v^h \|^2.
\]

(3.10)

where \( v^h \) is the mesh velocity, \( n_{en} \) is the number of (space–time) element nodes and \( N_a \) is the space–time shape function associated with the space–time node \( a \). For more ways of calculating \( \tau_{\text{SUPG}}, \tau_{\text{PSPG}} \) and \( \nu_{\text{LSC}} \), see (15; 13; 16; 17).

Several of the remarks made in (14) are relevant to this thesis and are reproduced here as Remarks 1 through 13.

**Remark 1** Strictly speaking, the DSD/SST formulation given by Eq. (3.1) was introduced in (13) and is slightly different from the formulation given in (10; 11). The two formulations are equivalent if the stabilization parameters \( \tau_{\text{SUPG}} \) and \( \tau_{\text{PSPG}} \) are defined to be identical, \( \nu_{\text{LSC}} = 0 \), and \( \nabla \cdot (\mu \varepsilon (w^h)) \) is zero (which will be the case for
linear elements) or neglected.

Remark 2 As an alternative to the way the SUPG test function is defined in Eq. (3.1), we propose the SUPG test function option of replacing the term \( \left( \frac{\partial w^h}{\partial t} + u^h \cdot \nabla w^h \right) \) with \( (u^h - v^h) \cdot \nabla w^h \). This replacement is equivalent to excluding the \( \left( \frac{\partial w^h}{\partial t} \right) \) part of \( \left( \frac{\partial w^h}{\partial t} \right) \). We call this option "WTSE", and the option where the term is active "WTSA".

Remark 3 With the function spaces defined in the paragraph preceding Eq. (3.1), for each space-time slab velocity and pressure assume double unknown values at each spatial node. One value corresponds to the lower end of the slab, and the other one upper end. The option of using double unknown values at a spatial node will be called "DV" for velocity and "DP" for pressure. In this case, we use two integration points over the time interval of the space-time slab, and this time-integration option will be called "TIP2". This version of the DSD/SST formulation, with the options set DV, DP and TIP2, will be called "DSD/SST-DP".

Remark 4 We propose here the option of using, for each space-time slab, a single unknown pressure value at each spatial node, and we will call this option "SP". With this, we propose another version of the DSD/SST formulation, where the options set is DV, SP and TIP2, and we will call this version "DSD/SST-SP". Because the number of unknown pressure values is halved, the computational cost is reduced substantially.

Remark 5 To reduce the computational cost further, we propose the option of using only one integration point over the time interval of the space-time slab, and we call
this time-integration option "TIP1". With this, we propose a third version of the DSD/SST formulation, where the options set is DV, SP and TIP1, and we will call this version "DSD/SST-TIP1".

**Remark 6** As a third way of reducing the computational cost, we propose the option of using, for each space–time slab, a single unknown velocity value at each spatial node, and we will call this option "SV". In the SV option, of the two parts of Eq. (3.1), the one generated by $(w^h)^+_n$ is removed, and we explicitly set $(u^h)^+_n = (u^h)^-_n$, which makes the velocity field continuous in time. Based on the SV option, we propose a fourth version of the DSD/SST formulation, where the options set is SV, SP and TIP1, and we will call this version "DSD/SST-SV". With this version of the DSD/SST formulation, we propose to use the SUPG test function option WTSE.

**Remark 7** In terms of computational cost the DSD/SST-SV formulation would be quite comparable to the ALE formulations. This makes the DSD/SST-SV formulation very competitive in computational efficiency.

### 3.2 Semi-discrete formulation of structural mechanics

With $y^h$ and $w^h$ coming from appropriately defined trial and test function spaces, respectively, the semi-discrete finite element formulation of the structural mechanics
equations (see (18; 19; 20)) is written as

$$\int_{\Omega^h} w^h \cdot \rho \frac{d^2 y^h}{dt^2} d\Omega^s + \int_{\Omega^h} w^h \cdot \eta \frac{dy^h}{dt} d\Omega^s - \int_{\Omega^h} \delta \mathbf{E}^h : \mathbf{S}^h d\Omega^s = \int_{\Omega^h} w^h \cdot \left( \mathbf{t}^h + \rho^h \mathbf{f}^h \right) d\Omega^s . \quad (3.11)$$

The fluid mechanics forces acting on the structure are represented by vector $\mathbf{t}^h$. This force term is geometrically nonlinear and thus increases the overall nonlinearity of the formulation. The left-hand-side terms of Eq. (3.11) are referred to in the original configuration and the right-hand-side terms in the deformed configuration at time $t$.

From this formulation at each time step we obtain a nonlinear system of equations. In solving that nonlinear system with an iterative method, we use an incremental form (see (18; 19; 20; 21)), which is expressed as

$$\left[ \frac{M}{\beta \Delta t^2} + \frac{(1 - \alpha) \gamma C}{\beta \Delta t} + (1 - \alpha) K \right] \Delta \mathbf{d}^i = \mathbf{R}^i . \quad (3.12)$$

Here $M$ is the mass matrix, $C$ is the artificial-damping matrix, $K$ is the consistent tangent matrix associated with the internal elastic forces, $\mathbf{R}^i$ is the residual vector at the $i^{th}$ iteration and $\Delta \mathbf{d}^i$ is the $i^{th}$ increment in the nodal displacements vector $\mathbf{d}$. The artificial-damping matrix $C$ is used, as mentioned in Section 2.2, only in computations where time-accuracy is not required, and for spatially-constant $\eta$ it can be written as $C = \eta M$. All of the terms known from the previous iteration are lumped into the residual vector $\mathbf{R}^i$. The parameters $\alpha, \beta, \gamma$ are part of the Hilber–Hughes–Taylor (22)
scheme, which is the time-integration technique used here.

### 3.3 Stabilized Space–Time Fluid–Structure Interaction (SSTFSI) method

This section describes the SSTFSI method based on the finite element formulations given by Eqs. (3.1) and (3.11), with a slight change of notation and with a clarification of how the fluid–structure interface conditions are handled. In this notation subscripts 1 and 2 refer to fluid and structure, respectively. Furthermore, while subscript $I$ refers to the fluid–structure interface, subscript $E$ refers to "elsewhere" in the fluid and structure domains or boundaries. Then the equations representing the SSTFSI method are written as follows:

$$
\int_{Q_1} \mathbf{w}_1^h \cdot \rho \left( \frac{\partial \mathbf{u}_1^h}{\partial t} + \mathbf{u}_1^h \cdot \nabla \mathbf{u}_1^h - \mathbf{f}_1^h \right) \, dQ + \int_{Q_n} \mathbf{e}(\mathbf{w}_{1E}^h) : \sigma(\mathbf{p}_E^h, \mathbf{u}_E^h) \, dQ - \int_{(P_2)_n} \mathbf{w}_2^h \cdot \mathbf{h}_2^h \, d\mathbf{P} + \int_{Q_n} q_{1E}^h \nabla \cdot \mathbf{u}_E^h \, dQ + \int_{\Omega_n} (\mathbf{w}_{1E}^h)_{\n}^+ \cdot \rho \left( (\mathbf{u}_E^h)_{\n}^+ - (\mathbf{u}_E^h)_{\n}^- \right) \, d\Omega
$$

$$
+ \sum_{e=1}^{(n_{el})_n} \int_{Q_e^h} \frac{1}{\rho} \left[ \tau_{\text{SUPG}} \left( \frac{\partial \mathbf{w}_{1E}^h}{\partial t} + \mathbf{u}_E^h \cdot \nabla \mathbf{w}_{1E}^h \right) + \tau_{\text{PSPG}} \nabla q_{1E}^h \right] \cdot \left[ L(p_E^h, \mathbf{u}_E^h) - \rho \mathbf{f}_E^h \right] \, dQ + \sum_{e=1}^{(n_{el})_n} \int_{Q_e^h} \nu_{\text{LSIC}} \nabla \cdot \mathbf{w}_{1E}^h \rho \nabla \cdot \mathbf{u}_E^h \, dQ = 0 ,
$$

(3.13)

$$
\int_{Q_n} q_{1E}^h \nabla \cdot \mathbf{u}_E^h \, dQ + \sum_{e=1}^{(n_{el})_n} \int_{Q_e^h} \frac{1}{\rho} \left[ \tau_{\text{PSPG}} \nabla q_{1E}^h \right] \cdot \left[ L(p_E^h, \mathbf{u}_E^h) - \rho \mathbf{f}_E^h \right] \, dQ = 0 ,
$$

(3.14)
\[ \int_{(\Gamma_{21})_{\text{REF}}} (w_{11}^h)^- \cdot ((u_{11}^h)^- - u_{21}^h) \, d\Gamma = 0, \quad (3.15) \]

\[ \int_{(P_1)^h \cap (P_n)^h} (w_{11}^h)^- \cdot h_{11}^h \, dP = - \int_{(P_n)^h} (w_{11}^h)^- \cdot \rho_2 \, dP + \int_{Q_n} 2\mu \varepsilon((w_{11}^h)^- \cdot \varepsilon(u)) \, dQ + \int_{Q_n} (w_{11}^h)^- \cdot \nabla \cdot (2\mu \varepsilon(u)) \, dQ, \quad (3.16) \]

\[ \int_{(\Omega_{21})_{\text{REF}}} w_{21}^h \cdot (h_{21}^h + (h_{11}^h)_A + (h_{11}^h)_B) \, d\Omega = 0, \quad (3.17) \]

\[ \int_{(\Omega_2)^0} w_2^h - \frac{d^2 y^h}{dt^2} \, d\Omega + \int_{(\Omega_2)^0} w_2^h \cdot \eta \frac{dy^h}{dt} \, d\Omega + \int_{(\Omega_2)^0} \delta E^h : S^h \, d\Omega \\
= \int_{\Omega_2} w_2^h \cdot \rho_2 \xi_2^h \, d\Omega + \int_{\Omega_{2E}} w_{2E}^h \cdot h_{2E}^h \, d\Omega + \int_{\Omega_{2I}} w_{2I}^h \cdot h_{2I}^h \, d\Omega. \quad (3.18) \]

Here \((\Gamma_{21})_{\text{REF}}\) and \((\Omega_{21})_{\text{REF}}\) represent some reference configurations of \(\Gamma_{21}\) and \(\Omega_{21}\), respectively. In reconciling the slightly modified notation used here with the notation we used in Eqs. (3.1) and (3.11), we note that \(\rho_2 = \rho\), \(\xi_2^h = \xi\), \((\Omega_2)^0 = \Omega_0\), \(\Omega_2 = \Omega_t\), and \(\Omega_{2I}\) and \(\Omega_{2E}\) indicate the partitions of \(\Omega_2\) corresponding to the interface and “elsewhere”. We also note that \(h_{2I}^h = t^h\), and \((h_{11}^h)_A\) and \((h_{11}^h)_B\) represent the values
of $h^h_i$ associated with the fluid surfaces above and below the membrane structure.

The symbol $h^h_{2E}$ denotes the prescribed external forces acting on the structure in $\Omega_{2E}$, which is separate from $f^h_2$. In this formulation, $(u^h_{1i})_{n+1}$, $h^h_{1i}$ and $h^h_{2i}$ (the fluid velocity, fluid stress and structural stress at the interface) are treated as separate unknowns, and Eqs. (3.15), (3.16) and (3.17) can be seen as corresponding to these three unknowns, respectively. The structural displacement rate at the interface, $u^h_{2i}$, is derived from $y^h$.

The formulation above is based on allowing for cases when the fluid and structure meshes at the interface are not identical. If they are identical, the same formulation can still be used. If the structure is represented by a 3D continuum model instead of a membrane model, the formulation above would still be applicable if the domain integrations over $\Omega_{2E}$ and $\Omega_{2i}$ in the last two terms of Eq. (3.18) are converted to boundary integrations over $\Gamma_{2E}$ and $\Gamma_{2i}$. In such cases, $h^h_{2E}$ would represent the prescribed forces acting “elsewhere” on the surface of the structure.

We note that, for constant viscosity, the term $\nabla \cdot (2\mu e(u))$ in Eq. (3.16) vanishes for tetrahedral elements and in most cases can be neglected for hexahedral elements. The same statement can be made also in the context of that term being a part of the expression $L(p^h, u^h)$ appearing in Eqs. (3.13) and (3.14).

**Remark 8** The versions of the SSTFSI method corresponding to the DSD/SST-DP, DSD/SST-SP, DSD/SST-TIP1 and DSD/SST-SV formulations (see Remarks 3–6) will be called “SSTFSI-DP”, “SSTFSI-SP”, “SSTFSI-TIP1” and “SSTFSI-SV”, re-
spectively.

**Remark 9** In terms of computational cost the SSTFSI-SV formulation would be quite comparable to the ALE FSI formulations. This makes the SSTFSI-SV formulation very competitive in computational efficiency.
Chapter 4

Solution of the Fully-Discretized Coupled Equations

Full discretization of the FSI formulation described in Subsection 3.3 leads to coupled, nonlinear equation systems that must be solved at every time step. In a conceptual form that is partitioned with respect to the models represented, such nonlinear equation systems can be written as:

\[
N_1 \left( d_1, d_2, d_3 \right) = F_1 ,
\]

\[
N_2 \left( d_1, d_2, d_3 \right) = F_2 ,
\]

\[
N_3 \left( d_1, d_2, d_3 \right) = F_3 ,
\]

where \( d_1, d_2 \) and \( d_3 \) are the vectors of nodal unknowns corresponding to generic unknown functions \( u_1, u_2 \) and \( u_3 \), respectively. In the context of an FSI problem, the generic functions \( u_1, u_2 \) and \( u_3 \) represent the fluid, structure and mesh unknowns, respectively. For the space–time formulation of the fluid mechanics problem, \( d_1 \) represents unknowns associated with the finite element formulation written for the
space–time slab between the time levels $n$ to $n+1$ (see (10; 11; 12; 13)). Solving these equations with the Newton–Raphson method would necessitate solving the following linear equation system at every Newton–Raphson step:

\begin{align}
A_{11}x_1 + A_{12}x_2 + A_{13}x_3 &= b_1, \\
A_{21}x_1 + A_{22}x_2 + A_{23}x_3 &= b_2, \\
A_{31}x_1 + A_{32}x_2 + A_{33}x_3 &= b_3,
\end{align}

(4.4) (4.5) (4.6)

where $b_1$, $b_2$ and $b_3$ are the residuals of the nonlinear equations, $x_1$, $x_2$ and $x_3$ are the correction increments for $d_1$, $d_2$ and $d_3$ and $A_{\beta\gamma} = \partial N_{\beta}/\partial d_\gamma$.

**Remark 10** In FSI computations with a fluctuating traction boundary condition at the outflow, we propose to calculate the initial guess for $p_{n+1}$ with the expression $(p_{n+1})^0 = p_n + (\Delta p_{\text{OUT}})_n$ to improve the convergence of the nonlinear iterations. In this expression, $(\Delta p_{\text{OUT}})_n$ is a measure of the change in the outflow traction from time level $n$ to $n+1$.

### 4.1 Block-iterative coupling

In the block-iterative coupling (23; 24; 16; 25; 26; 27; 28; 29), the fluid, structure and mesh systems are treated as separate blocks, and the nonlinear iterations are carried out one block at a time. In solving a block of equations for the associated block of unknowns, we use the most current values of the other blocks of unknowns.
Assuming a cyclic order of $1 \rightarrow 2 \rightarrow 3$, in an iteration step taking us from iterative solution $i$ to $i + 1$, the following three blocks of equations are solved:

$$
\frac{\partial N_1}{\partial d_1}(d_1^i, d_2^i, d_3^i) (\Delta d_1^i) = F_1 - N_1(d_1^i, d_2^i, d_3^i), \quad (4.7)
$$

$$
\frac{\partial N_2}{\partial d_2}(d_1^{i+1}, d_2^i, d_3^i) (\Delta d_2^i) = F_2 - N_2(d_1^{i+1}, d_2^i, d_3^i), \quad (4.8)
$$

$$
\frac{\partial N_3}{\partial d_3}(d_1^{i+1}, d_2^{i+1}, d_3^i) (\Delta d_3^i) = F_3 - N_3(d_1^{i+1}, d_2^{i+1}, d_3^i). \quad (4.9)
$$

Each of the three blocks of linear equations systems given by Eqs. (4.7)–(4.9) is also solved iteratively, using the GMRES search technique (30). In the block-iterative implementation of the T×AFSM, at each time step the cycle starts with block number 2 (i.e., $2 \rightarrow 3 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \ldots$).

In FSI computations where the structure is light, structural response becomes very sensitive to small changes in the fluid mechanics forces. In such cases, when the coupling between the three blocks of equations given by Eqs. (4.4)–(4.6) is handled with a block-iterative coupling technique rather than a direct coupling technique, convergence becomes difficult to achieve. In Subsections 4.2 and 4.3 we describe "more direct" techniques for handling the coupling. A shortcut approach was proposed in (23; 24; 16) (and was also described in (25; 26; 27; 28; 29)) for improving the convergence of the block-iterative coupling technique. In this approach, to re-
duce "over-correcting" (i.e., "over-incrementing") the structural displacements during the block iterations, the mass matrix contribution to $A_{22}$ is increased. This is achieved without altering $b_1$, $b_2$ or $b_3$ (i.e., without altering $F_1 - N_1 (d_1, d_2, d_3)$, $F_2 - N_2 (d_1, d_2, d_3)$ or $F_3 - N_3 (d_1, d_2, d_3)$), and therefore when the block iterations converge, they converge to the solution of the problem with the correct structural mass.

4.2 Quasi-direct coupling

In the quasi-direct coupling (26; 27; 28), the fluid+structure and mesh systems are treated as two separate blocks, and the nonlinear iterations are carried out one block at a time. In solving a block of equations for the associated block of unknowns, we use the most recent values of the other block of unknowns. In an iteration step taking us from iterative solution $i$ to $i + 1$, the following two blocks of equations are solved:

$$
\frac{\partial N_1}{\partial d_1} \Bigg|_{(d_1', d_2', d_3')}(\Delta d_1') + \frac{\partial N_1}{\partial d_2} \Bigg|_{(d_1', d_2', d_3')}(\Delta d_2') = F_1 - N_1 (d_1', d_2', d_3'), \quad (4.10)
$$

$$
\frac{\partial N_2}{\partial d_1} \Bigg|_{(d_1', d_2', d_3')}(\Delta d_1') + \frac{\partial N_2}{\partial d_2} \Bigg|_{(d_1', d_2', d_3')}(\Delta d_2') = F_2 - N_2 (d_1', d_2', d_3'), \quad (4.11)
$$

$$
\frac{\partial N_3}{\partial d_3} \Bigg|_{(d_1^{i+1}, d_2^{i+1}, d_3')}(\Delta d_3') = F_3 - N_3 (d_1^{i+1}, d_2^{i+1}, d_3'). \quad (4.12)
$$
Each of the two blocks of linear equations systems given by Eqs. (4.10), (4.11) and Eq. (4.12) is also solved iteratively using the GMRES search technique.

**Remark 11** In the iterative solution of the combined fluid+structure (i.e. 1+2) block with the GMRES search technique and a diagonal preconditioner, depending on the nature of the problem, one of these two parts might pose a greater convergence challenge than the other one. The convergence challenges might be created by an incompressibility constraint, having thin or shallow computational domains, or some other factors. The scaling provided by diagonal preconditioning is unlikely to remedy such disparities in the convergence challenges offered by the two parts, which are typically exhibited as disparities in the residual-decay rates for the two parts rather than disparities in the residual magnitudes. In some cases, the scaling provided by diagonal preconditioning might not even be able to properly account for the disparities in the residual magnitudes corresponding to the fluid and structure parts. Here we propose “Selective Scaling” to place, in GMRES iterations, greater emphasis on the part posing greater convergence challenge. With this additional scaling (beyond diagonal preconditioning), in constructing the Krylov vectors of the GMRES search technique, the relative weights given to the residual vectors associated with the fluid and structure parts are determined based on the relative convergence challenges posed by those two parts. We propose to determine those relative weights on a case-by-case basis as well as on a more automated basis, where the weights increase with decreasing residual-decay rates.
4.3 Direct coupling

In the direct coupling (26; 27; 28), the fluid+structure+mesh system is treated as a single block, and the linear equation system given by Eqs. (4.4)–(4.6) is solved iteratively:

\[ P_{11} z_1 + P_{12} z_2 + P_{13} z_3 = b_1 - (A_{11} x_1 + A_{12} x_2 + A_{13} x_3), \quad (4.13) \]
\[ P_{21} z_1 + P_{22} z_2 + P_{23} z_3 = b_2 - (A_{21} x_1 + A_{22} x_2 + A_{23} x_3), \quad (4.14) \]
\[ P_{31} z_1 + P_{32} z_2 + P_{33} z_3 = b_3 - (A_{31} x_1 + A_{32} x_2 + A_{33} x_3), \quad (4.15) \]

where each \( P_{\beta\gamma} \) represents a block of the preconditioning matrix \( P \). The most computing-intensive part here is the evaluation of the matrix–vector products of the form \( A_{\beta\gamma} x_\gamma \) (for \( \beta, \gamma = 1, 2, \ldots, N \) and no sum). In the FSI computations carried out by the T*AFSM, those evaluations are performed with the element-vector-based (EVB) computation techniques (see (31; 32; 33; 16; 29)), which do not require computation of any matrices, not even at the element level. EVB computations can be carried out numerically (NEVB) or analytically (AEVB).

4.3.1 NEVB computations

In the NEVB computation technique, which is also called the matrix-free computation technique (see (31; 32)), a matrix-vector product of the form \( Ax \), which is the
directional derivative of \( \mathbf{N} \) in \( \mathbf{x} \) direction, is evaluated by the expression:

\[
A \mathbf{x} = \frac{1}{\epsilon} \sum_{e=1}^{n_e} \left[ \frac{N^e(d + \epsilon \mathbf{x}) - N^e(d)}{\epsilon} \right],
\]

where \( N^e \) is the element-level vector representing the contribution of element \( e \) to \( \mathbf{N} \), and \( \epsilon \) is a small parameter used in the numerical calculation of the limit representing the directional derivative. This concept was extended in (33; 16; 29) to FSI computations, where we evaluate matrix-vector products of the form \( A_{\beta \gamma} \mathbf{x}_\gamma \):

\[
A_{\beta \gamma} \mathbf{x}_\gamma = \frac{1}{\epsilon} \sum_{e=1}^{n_e} \left[ \frac{N^e_{\beta}(\ldots, d_\gamma + \epsilon_{\beta \gamma} \mathbf{x}_\gamma, \ldots) - N^e_{\beta}(\ldots, d_\gamma, \ldots)}{\epsilon_{\beta \gamma}} \right],
\]

where \( N^e_{\beta} \) is the element-level vector representing the contribution of element \( e \) to \( \mathbf{N}_\beta \), and \( \epsilon_{\beta \gamma} \) is the limit-evaluation parameter selected for the unknown set \( \gamma \) in the equation set \( \beta \). If we decide to use a single limit-evaluation parameter \( \epsilon_\beta \) for all the unknown sets in the equation set \( \beta \), then the computations can be carried out as

\[
\sum_{\gamma=1}^{\gamma=N} A_{\beta \gamma} \mathbf{x}_\gamma = \frac{1}{\epsilon} \sum_{e=1}^{n_e} \left[ \frac{N^e_{\beta}(d + \epsilon_\beta \mathbf{x}) - N^e_{\beta}(d)}{\epsilon_\beta} \right].
\]

**Remark 12** Using a single limit-evaluation parameter for all unknown sets in the equation set \( \beta \) would be more computationally economical. However, using a different limit-evaluation parameter for each unknown set gives us the option of taking into account the dependence of \( \mathbf{N}_\beta \) on each unknown \( \mathbf{d}_\gamma \), separately, including how \( \frac{\partial \mathbf{N}_\beta}{\partial \mathbf{d}_\gamma} \) varies with \( \mathbf{d}_\gamma \). This is an important consideration because of the multi-physics and
multi-scale nature of FSI computations.

4.3.2 AEVB computations

The AEVB computation technique (see (33; 16; 29)) can be used to evaluate the matrix-vector products of the form \( A_{\beta\gamma} x_{\gamma} \) if deriving expressions for such matrix-vector products is not too difficult and we prefer not to use limit-evaluation parameters and numerical evaluation of directional derivatives.

Let us suppose that the nonlinear vector function \( N_{\beta} \) corresponds to a finite element integral form \( B_{\beta}(W_{\beta}, u_1, \ldots, u_N) \). Here \( W_{\beta} \) represents the vector of nodal values associated with the weighting function \( w_{\beta} \), which generates the nonlinear equation block \( \beta \). Let us also suppose that we are able to, without major difficulty, derive the expressions for the first-order terms in the expansion of \( B_{\beta}(W_{\beta}, u_1, \ldots, u_N) \) in \( u_\gamma \). Those first-order terms in \( \Delta u_\gamma \) will be represented by the finite element integral form \( G_{\beta\gamma}(W_{\beta}, u_1, \ldots, u_N, \Delta u_\gamma) \). For example, \( G_{11}(W_1, u_1, \ldots, u_N, \Delta u_1) \) will represent the first-order terms obtained by expanding the finite element formulation of the fluid mechanics equations (i.e., momentum equation and incompressibility constraint) in fluid mechanics unknowns (i.e., fluid velocity and pressure). We note that the integral form \( G_{\beta\gamma} \) will generate \( \frac{\partial N_{\beta}}{\partial d_{\gamma}} \). Consequently, as it was pointed out in (33; 16; 29), the product \( A_{\beta\gamma} x_{\gamma} \) can be evaluated as follows:

\[
A_{\beta\gamma} x_{\gamma} = \frac{\partial N_{\beta}}{\partial d_{\gamma}} x_{\gamma} = \sum_{\varepsilon=1}^{n_{EL}} G_{\beta\gamma}(W_{\beta}, u_1, \ldots, u_N, v_{\gamma}) ,
\]

(4.19)

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where $v_\gamma$ is a function interpolated from $x_\gamma$ in the same way $u_\gamma$ is interpolated from $d_\gamma$.

In the mixed AEVB/NEVB computation technique (33; 16; 29), either technique can be used to evaluate $A_{\beta,\gamma}$ for each combination of $\beta$ and $\gamma$, depending on the nature of the particular evaluation. In the direct coupling approach (26; 27; 28; 29), the matrix-vector product $A_{13}x_3$ is computed with the NEVB technique:

$$A_{13}x_3 = \frac{1}{n_{\epsilon}} \left[ \frac{N^e_1(d_1, d_2, d_3 + \epsilon_{13}x_3) - N^e_1(d_1, d_2, d_3)}{\epsilon_{13}} \right].$$

(4.20)

Remark 13: Remark 11, with the wording expanded to the combined fluid+structure+mesh (i.e., 1+2+3) system, becomes applicable to the direct coupling technique.
Chapter 5

Validation

This chapter covers some of the simple tests conducted to verify proper handling of continuum elements in T*AFSM solvers.

5.1 Elongation of a bar

We ran two tests with a cylindrical bar made up of linearly-elastic continuum elements. The bar is 1 m long and has a diameter of 0.1 m.

5.1.1 Axial load

The Young’s Modulus for the bar is 1,000 Pa. One end of the bar is held fixed while the other end is acted upon by an evenly distributed load of 100 Pa.

We can calculate a simple solution to this problem if we model the bar as a single line element (34). Let $P$ equal the force exerted on the bar. Let $u$ be the elongation of the bar as a function of $x$, the distance from the fixed end. Let $u_2$ be the elongation at the free end of the bar. It can be shown that $u$ is linear in this case. Given this, Eqs. (5.1) through (5.7) show a derivation of the analytical solution.
\[ u = \frac{x}{L} u_2, \]  
(5.1)

\[ \varepsilon = \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2, \]  
(5.2)

\[ \varepsilon = \left( \frac{u_2}{L} \right) + \frac{1}{2} \left( \frac{u_2}{L} \right)^2, \]  
(5.3)

\[ \pi = \frac{1}{2} \int_0^L AE \varepsilon^2 \, dx - P u_2, \]  
(5.4)

\[ \pi = \frac{AEL}{2} \left[ \left( \frac{u_2}{L} \right)^2 + \left( \frac{u_2}{L} \right)^3 + \frac{1}{4} \left( \frac{u_2}{L} \right)^4 \right] - P u_2, \]  
(5.5)

\[ \frac{\partial \pi}{\partial u_2} = 0, \]  
(5.6)

\[ 2 \left( \frac{u_2}{L} \right) + 3 \left( \frac{u_2}{L} \right)^2 + \left( \frac{u_2}{L} \right)^3 - \frac{2P}{EA} = 0. \]  
(5.7)

Using a numerical solver like the Newton–Raphson method, we find that the elongation of this model is 0.088 m.

In order to test continuum element functionality in our solver, we generated a mesh to represent this problem with 8,129 four-node tetrahedral elements. This mesh has 60 nodes along the axis of elongation. Using the same test conditions specified above, our structure solver predicts an elongation of 0.085 m. This is within 5\% of the analytical solution.

### 5.1.2 Gravity load

In this test, we set the Young’s Modulus for the bar to 2,000 Pa and the density to 1,000 kg/m³. For simplicity, we use a gravitational constant of 1 m/s².
We can find an approximate answer to this case by using the axial-load analysis (34). We use the expression given by Eq. (5.7), but we replace the force $P$ with half of the weight, $W = AL\rho g$:

$$2 \left( \frac{u_2}{L} \right) + 3 \left( \frac{u_2}{L} \right)^2 + \left( \frac{u_2}{L} \right)^3 - \frac{2}{EA} \left( \frac{W}{2} \right) = 0.$$  \hspace{1cm} (5.8)

The elongation of this model is 0.19 m. Using the same 8,129 element mesh to model this problem, we find a steady-state elongation of 0.18 m. This is within 5\% of the simple analysis solution.

### 5.2 Inflation of a slice of a cylindrical tube

This test is performed as a check on our hyperelastic model, which uses Mooney–Rivlin elements. Taking advantage of symmetry, we inflate a quarter arc segment of a slice of a cylindrical tube. We use the same geometry and material properties used in Section 6.4. The inner and outer radii of the arc are 1.5 mm and 1.8 mm, respectively. The width of this segment is 0.1 mm. The mesh consists of 72 nodes and 150 elements. The nodes are constrained in the axial direction.

For a given change in the inner radius, we can calculate the change in the outer radius if we assume the structure is incompressible. Let $R_i$ and $R_o$ stand for the initial inner and outer radii, and $r_i$ and $r_o$ for the final inner and outer radii.
\[ r_o^2 - r_i^2 = R_o^2 - R_i^2 \] (5.9)

We can also calculate the pressure required to inflate the artery to a particular size given the material constants \( C_1 \) and \( C_2 \) and the initial inner and outer radii of the cylinder (35). We use \( 1.1 \times 10^5 \, \text{N/m}^2 \) and \( 4.0 \times 10^4 \, \text{N/m}^2 \) for the Mooney–Rivlin material constants \( C_1 \) and \( C_2 \), respectively.

\[ P = 2 (C_1 + C_2) \left( \ln \left( \frac{r_i R_o}{R_i r_o} \right) - \frac{1}{2} \left( \frac{R_i^2}{r_i^2} - \frac{R_o^2}{r_o^2} \right) \right) \] (5.10)

Given Eqs. (5.9) and (5.10), we calculate the final inner and outer radii for a range of pressure values. Since we are working with arterial fluid mechanics, we are particularly interested in checking the performance of this model in the range of normal human blood pressure. Tables 5.1 and 5.2 compare analytical results from the above equations to computational results from our solver.
<table>
<thead>
<tr>
<th>Pressure</th>
<th>Analytical $r_i$</th>
<th>Computed $r_i$</th>
<th>Error for $r_i$</th>
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<tbody>
<tr>
<td>50 mm Hg</td>
<td>1.561 mm</td>
<td>1.559 mm</td>
<td>0.1%</td>
</tr>
<tr>
<td>80 mm Hg</td>
<td>1.603 mm</td>
<td>1.599 mm</td>
<td>0.2%</td>
</tr>
<tr>
<td>120 mm Hg</td>
<td>1.666 mm</td>
<td>1.661 mm</td>
<td>0.3%</td>
</tr>
<tr>
<td>160 mm Hg</td>
<td>1.740 mm</td>
<td>1.734 mm</td>
<td>0.4%</td>
</tr>
<tr>
<td>200 mm Hg</td>
<td>1.830 mm</td>
<td>1.822 mm</td>
<td>0.4%</td>
</tr>
</tbody>
</table>

**Table 5.1**: Cylindrical segment inflation results for final inner radius

<table>
<thead>
<tr>
<th>Pressure</th>
<th>Analytical $r_o$</th>
<th>Computed $r_o$</th>
<th>Error for $r_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50 mm Hg</td>
<td>1.852 mm</td>
<td>1.849 mm</td>
<td>0.2%</td>
</tr>
<tr>
<td>80 mm Hg</td>
<td>1.888 mm</td>
<td>1.884 mm</td>
<td>0.2%</td>
</tr>
<tr>
<td>120 mm Hg</td>
<td>1.943 mm</td>
<td>1.936 mm</td>
<td>0.4%</td>
</tr>
<tr>
<td>160 mm Hg</td>
<td>2.008 mm</td>
<td>1.999 mm</td>
<td>0.5%</td>
</tr>
<tr>
<td>200 mm Hg</td>
<td>2.088 mm</td>
<td>2.076 mm</td>
<td>0.6%</td>
</tr>
</tbody>
</table>

**Table 5.2**: Cylindrical segment inflation results for final outer radius
Chapter 6

Cerebral Sacular Aneurysm

This chapter covers several arterial blood flow simulations through a cerebral artery with a sacular aneurysm. These simulations are a continuation of (36) using linearly-elastic and hyperelastic continuum elements to model the artery walls. The model used in this chapter is a close approximation to the patient-specific image-based geometry used in (7). The geometry used in (7) was extracted from the computed tomography model of a middle cerebral artery segment from a 57 year-old male. The diameter and length of the artery are 3.0 mm and 15 mm, and the size of the aneurysm is 6 mm. The location of this artery segment is shown in Figure 6.1 and the problem geometry is shown in Figure 6.2.

For each structure model, two different pressure profiles and two different aneurysm thicknesses are tested. Blood pressure varies from 80 to 120 mm Hg for the normal blood pressure (NBP) case and from 100 to 170 mm Hg for the high blood pressure (HBP) case. The aneurysm thickness is set to the artery thickness, 0.3 mm, in the uniform wall thickness (UWT) case. In the variable wall thickness (VWT) case, the aneurysm thickness transitions from 0.3 mm to 0.25 mm. This is illustrated in
Figure 6.3.

After reviewing these test cases, we will introduce a few changes to improve the accuracy of the results. Up to this point, we have used the computed tomography model to generate a mesh of the artery at zero pressure. A more realistic approach would be to use the image-based geometry as the arterial geometry corresponding to time-averaged pressure values. To accomplish this, we will use a rudimentary technique (37) to estimate the zero-pressure arterial geometry. Further details are given in Section 6.6

![Diagram of MCA and ICA](image)

**Figure 6.1:** Location of middle cerebral artery (MCA) and internal carotid artery (ICA). Picture taken from (39). Circle indicates modeled segment.

### 6.1 General conditions for the test computations

T☆AFSM runs simulations in a parallel computing environment using PC clusters. Our meshes are generated on a single node of that cluster. For these arterial blood flow simulations, the fluid and structure meshes are compatible at the fluid–structure interface. The computations are performed without any remeshing. The quasi-direct
coupling technique (see Section 4.2) is used to solve the fully-discretized, coupled fluid and structural mechanics and mesh-moving equations. The GMRES search technique (30) is used with a diagonal preconditioner to solve the linear equation systems involved at every nonlinear iteration.

6.1.1 Simulation sequence

In most of the simulations carried out by the T★AFSM, the FSI computations are preceded by a set of pre-FSI computations that provide a good starting point for the FSI computations. These pre-FSI computations include the fluid-only and structure-only computations. Two options were proposed in (38) for the pre-FSI generation of
Figure 6.3: Wall-thickness distribution for the artery segment with variable wall thickness (VWT).

the fluid mesh. We can generate it by starting with a mesh corresponding to the initial shape of the structure and updating it as the structure-only computation proceeds, or generate it after the structure-only computation is completed. These simulation sequences are described below.

Fluid→Structure→FSI (F→S→FSI) sequence

**Step 1:** Generate the fluid and structure meshes based on the shape of the unstressed structure.

**Step 2:** Compute a developed flow field while holding the structure rigid.

- The outflow traction should be set to a value close to 80 mm Hg for NBP and 100 mm Hg for HBP.
• The inflow velocity should be set to a value corresponding to the outflow traction.

**Step 3:** Compute the structural deformation with the fluid stresses at the interface held steady at their values from Step 2, and simultaneously update the fluid mesh.

• Structural deformation can be determined with a steady-state computation or a time-dependent computation that eventually yields a steady-state solution.

• For the steady-state computation, $\Delta t \to \infty$ and $\alpha = 0$ in Eq. (21) in (14), the number of time steps is one, and the initial displacement, velocity and acceleration are set to zero.

• The mesh quality obtained with the time-dependent computation is better than the one obtained with the steady-state computation.

**Step 4:** Compute the FSI with the inflow and outflow conditions held steady at the values used in Step 2.

• Sometimes, to prevent a sudden increase in the structural acceleration at the start of this step, it may be necessary to begin with an increased structural mass that will later be decreased back to its actual value. An unrealistically large acceleration can initiate an instability that is subsequently magnified.

**Step 5:** Compute the FSI with the inflow and outflow conditions pulsating.
Structure→Fluid→FSI (S→F→FSI) sequence

Analysis of the results obtained with the F→S→FSI sequence reveals that the fluid pressure is spatially almost uniform, even when outflow traction is pulsating and pressure changes as a function of time. The S→F→FSI sequence was built based on these considerations.

**Step 1:** Generate the structure mesh based on the shape of the unstressed structure.

**Step 2:** Compute the structural deformation with a uniform fluid pressure held steady at a value close to 80 mm Hg for NBP and 100 mm Hg for HBP.

- The structural deformation can be computed, as in Step 3 of the F→S→FSI sequence, with a steady-state computation or a time-dependent computation that eventually yields a steady-state solution.

**Step 3:** Generate the fluid mesh based on the shape of the deformed structure.

**Step 4:** Compute a developed flow field while holding the structure from Step 3 rigid.

- The outflow traction should be set to a value close to 80 mm Hg for NBP and 100 mm Hg for HBP.

- The inflow velocity should be set to a value corresponding to the outflow traction.

**Step 5:** Continue from Step 4 of the F→S→FSI sequence.
6.2 Case specifics

6.2.1 Fluid and structure properties

Although blood is known to be non-Newtonian in general, we are assuming it to be Newtonian here. The explanation we give for that here has been reproduced from the explanation given in (39). With the arterial diameter being approximately 3.0 mm and the flow rate 2.0 ml/s, the average shear rate in the artery is approximately \( \dot{\gamma} = 4Q/(\pi R^3) \sim 755 \text{ s}^{-1} \), where \( Q \) and \( R \) are the flow rate and radius. As it was pointed out in (39), the viscosity of the blood can assumed to be constant if the shear rate is high enough \(( > 150 \text{ s}^{-1}) \) (40).

We set the density and kinematic viscosity of the fluid to 1,000 kg/m\(^3\) and 4.0\times10^{-6} \text{ m}^2/\text{s}. The material density of the arterial wall is known to be close to that of the blood and we therefore set it to 1,000 kg/m\(^3\). Arteries are surrounded by tissues, and we expect those tissues to have a damping effect on the structural dynamics of the arteries. Therefore we add mass-proportional damping, which also helps in removing the high-frequency modes of the structural deformation. The damping coefficient \( \eta \) is set to 1.5\times10^4 \text{ s}^{-1}.

6.2.2 Boundary conditions

We have a single inflow boundary where we specify the velocity profile as a function of time. The shape of this velocity profile is similar to the one obtained by using the Womersley solution of a pulsating flow (41). We determine the magnitude of the
velocity profile by setting the volumetric inflow rate to a close approximation of the one used in (7), which can be found in (4; 5), and then dividing by the cross sectional area at the inflow boundary. The resulting velocity profile is shown in Figure 6.4.

At the outflow boundaries, we specify traction boundary conditions based on a pressure profile. Our pressure profile is an approximate solution (4) of the Windkessel model (42). We set the parameters of the Windkessel model in such a way that the range for the pressure profile is approximately from 80 to 120 mm Hg for NBP and from 100 to 170 mm Hg for HBP. The pressure profiles are shown in Figure 6.5.

On the arterial walls, we specify no-slip boundary conditions for the flow. In the structural mechanics part, as boundary condition at the ends of the arteries, we set the displacement to zero at those faces.

![Inflow Velocity Graph](image)

**Figure 6.4:** Pulsating inflow velocity.
Figure 6.5: Normal and high blood pressure (NBP and HBP) profiles.

6.2.3 Mesh properties and FSI techniques

In the uniform wall thickness (UWT) case, the mesh for the artery consists of 13,332 nodes and 52,944 four-node tetrahedral elements, with 4,444 nodes and 8,824 three-node triangular elements on the fluid-structure interface. In the variable wall thickness (VWT) case, we use the same fluid-structure interface, but we only have one layer of elements on top of it. This means the VWT artery mesh has 8,888 nodes and 26,472 elements. The fluid mechanics mesh contains 9,568 nodes and 43,960 four-node tetrahedral elements. The computations are carried out with the SSTFSI-SV technique (see Remarks 6 and 8) and the SUPG test function option WTSE (see Remark 2). The stabilization parameters used are those given by Eqs. (3.3) through (3.10). The time-step size is $3.333 \times 10^{-3}$ s. The number of nonlinear iterations per time step is 7, and the number of GMRES iterations per nonlinear iteration is 200.
We use the “Selective Scaling” technique (see Remark 11) to dynamically shift the emphasis between the fluid and structure parts. The scales used for the fluid and structure parts at each nonlinear iteration of a time step are given below:

Iteration 1: fluid scale = 1.00, structure scale = 0.00
Iteration 2: fluid scale = 0.00, structure scale = 1.00
Iteration 3: fluid scale = 1.00, structure scale = 0.00

Iterations 4–7: fluid scale = 1.00, structure scale = 0.01

The scales used in the first three nonlinear iterations reduce the fluid–structure coupling to block-iterative coupling (see Section 4.1). This allows the individual parts to converge significantly in the early stages of the iterations. In the subsequent nonlinear iterations, we change the scaling to maintain the quasi-direct coupling technique, but we place an emphasis on the fluid equations.

6.3 Computations with the linearly-elastic continuum element

For the arteries, we use a stiffness value of \(5.0 \times 10^5\) N/m\(^2\). This value is comparable to the stiffness values used in (1; 2; 4; 5; 7), where they were determined by comparison to experimental values (see (7)). The Poisson’s ratio is 0.45. The wall thickness for the artery is 0.3 mm at zero pressure. These computations follow the F→S→FSI sequence.

The computations with linearly-elastic continuum elements show reasonable de-
formations of the aneurysm and good mass balance. Figures 6.6, 6.8, 6.10, and 6.12 show the mass balance for the four test cases: NBP-UWT, NBP-VWT, HBP-UWT, HBP-VWT. A selection of snapshots are also included for each of these cases. The snapshots are taken at particular points in the heartbeat. See Figure 6.4.

While these computations went smoothly, we found that decreasing the initial aneurysm thickness below a certain amount leads to instability at higher blood pressures. This observation is consistent with what was pointed out to us recently by Hughes (43) — that this is not a suitable material model when the strains are not infinitesimal, especially if there are compressive strains. As such, we are abandoning the linearly-elastic model for arterial blood flow simulations.

![Graph](image)

**Figure 6.6:** NBP-UWT case computed with the continuum element made of linearly-elastic material. Verification of mass balance. Volumetric inflow rate, difference between the volumetric inflow and outflow rates and rate of change for the artery volume.
Figure 6.7: NBP-UWT case computed with the continuum element made of linearly-elastic material. Flow field at various instants. Velocity vectors colored by magnitude.
Figure 6.8: NBP-VWT case computed with the continuum element made of linearly-elastic material. Verification of mass balance. Volumetric inflow rate, difference between the volumetric inflow and outflow rates and rate of change for the artery volume.
Figure 6.9: NBP-VWT case computed with the continuum element made of linearly-elastic material. Flow field at various instants. Velocity vectors colored by magnitude.
Figure 6.10: HBP-UWT case computed with the continuum element made of linearly-elastic material. Verification of mass balance. Volumetric inflow rate, difference between the volumetric inflow and outflow rates and rate of change for the artery volume.
Figure 6.11: HBP-UWT case computed with the continuum element made of linearly-elastic material. Flow field at various instants. Velocity vectors colored by magnitude.
Figure 6.12: HBP-VWT case computed with the continuum element made of linearly-elastic material. Verification of mass balance. Volumetric inflow rate, difference between the volumetric inflow and outflow rates and rate of change for the artery volume.
Figure 6.13: HBP-VWT case computed with the continuum element made of linearly-elastic material. Flow field at various instants. Velocity vectors colored by magnitude.
6.4 Computations with the Mooney–Rivlin continuum element

These computations are set up just like the linearly-elastic simulations: the wall thickness for the artery is 0.3 mm at zero pressure and the computations followed the F→S→FSI sequence. The penalty Poisson ratio is set to 0.45.

In this section, we try two different sets of Mooney–Rivlin material constants. In our first attempt we use two equations to relate $C_1$ and $C_2$ to the modulus of elasticity used in the previous test cases. The relationship between the Mooney–Rivlin constants and the modulus of elasticity is taken from (9). We assume the ratio of $C_1$ to $C_2$ is the same as the ratio for rubber material.

\begin{align}
6(C_1 + C_2) &= E \\
\frac{C_1}{C_2} &= \frac{4}{3}
\end{align}

(6.1) \hspace{2cm} (6.2)

Using these relationships, we obtain Mooney–Rivlin material constants $C_1$ and $C_2$ of $4.8 \times 10^4$ N/m$^2$ and $3.6 \times 10^4$ N/m$^2$. Figure 6.14 shows the test run performed with these parameters has good mass balance. However, the corresponding frames of the simulation exhibit too much deformation in the arterial walls. As such, we will scrap these constants and approach the problem from a different perspective.

In (44), we propose a way to base our two-parameter Mooney–Rivlin model on a
five-parameter model given in (45). Specifically, we compare the strain energy from our model to the strain energy from the five-parameter model, shown in Eq. (6.3). The constants $a_{10}$ through $a_{30}$ are also taken from (45). In (44), to determine $C_1$ and $C_2$, we assume a given stretch value (associated with the inflation of a spherical structure) for the two tangential directions, where $\lambda_1 = \lambda_2 = \lambda$. We also assume that at this stretch level $C_1 (I_1 - 3) + C_2 (I_2 - 3)$ is equal to the strain energy given by the five-parameter model of (45):

$$W = a_{10} (I_1 - 3) + a_{01} (I_2 - 3) + a_{20} (I_1 - 3)^2 + a_{11} (I_1 - 3)(I_2 - 3) + a_{30} (I_1 - 3)^3.$$  \hspace{1cm} (6.3)

Note that $a_{30} = 0$ in (45). We also assume in (44) that:

$$(a_{10} (I_1 - 3) + a_{20} (I_1 - 3)^2 + a_{11} (0.5)(I_1 - 3)(I_2 - 3)) = C_1 (I_1 - 3)$$

$$(a_{01} (I_2 - 3) + a_{11} (0.5)(I_1 - 3)(I_2 - 3)) = C_2 (I_2 - 3).$$  \hspace{1cm} (6.4)

Eq. (6.4) can be rewritten as Eq. (6.5), which we use in (44) to estimate the Mooney–Rivlin parameters $C_1$ and $C_2$. Then, using these parameters, we run a structure deformation test to the peak pressure value of the HBP case. We calculate the numerical value of the stretch and compare it to our input parameters. If they
do not match, we choose a new stretch value and repeat the process.

\[
C_1 = \frac{(a_{10}(I_1 - 3) + a_{20}(I_1 - 3)^2 + a_{11}(0.5)(I_1 - 3)(I_2 - 3))}{(I_1 - 3)}
\]

\[
C_2 = \frac{(a_{01}(I_2 - 3) + a_{11}(0.5)(I_1 - 3)(I_2 - 3))}{(I_2 - 3)}
\]  \hspace{1cm} (6.5)

Using this process, we obtain Mooney–Rivlin material constants \( C_1 \) and \( C_2 \) of \( 1.1 \times 10^5 \) N/m² and \( 4.0 \times 10^4 \) N/m².

The computations with these Mooney–Rivlin continuum elements show more reasonable deformation of the aneurysm as well as good mass balance. Figures 6.16, 6.18, 6.20, and 6.22 show the mass balance for the four test cases: NBP-UWT, NBP-VWT, HBP-UWT, HBP-VWT. As before, the following pages also contain snapshots of the artery deformation and velocity profile.
Figure 6.14: Test run computed with the continuum element made of Mooney–Rivlin material ($C_1 = 4.8 \times 10^4$ N/m², $C_2 = 3.6 \times 10^4$ N/m²). Verification of mass balance. Volumetric inflow rate, difference between the volumetric inflow and outflow rates and rate of change for the artery volume.
Figure 6.15: Test run computed with the continuum element made of Mooney-Rivlin material ($C_1 = 4.8 \times 10^4 \text{ N/m}^2$, $C_2 = 3.6 \times 10^4 \text{ N/m}^2$). Flow field at various instants. Velocity vectors colored by magnitude.
Figure 6.16: NBP-UWT case computed with the continuum element made of Mooney-Rivlin material ($C_1 = 1.1 \times 10^5$ N/m$^2$, $C_2 = 4.0 \times 10^4$ N/m$^2$). Verification of mass balance. Volumetric inflow rate, difference between the volumetric inflow and outflow rates and rate of change for the artery volume.
Figure 6.17: NBP-UWT case computed with the continuum element made of Mooney-Rivlin material ($C_1 = 1.1 \times 10^8 \text{ N/m}^2$, $C_2 = 4.0 \times 10^4 \text{ N/m}^2$). Flow field at various instants. Velocity vectors colored by magnitude.
Figure 6.18: NBP-VWT case computed with the continuum element made of Mooney-Rivlin material ($C_1 = 1.1 \times 10^5$ N/m², $C_2 = 4.0 \times 10^4$ N/m²). Verification of mass balance. Volumetric inflow rate, difference between the volumetric inflow and outflow rates and rate of change for the artery volume.
Figure 6.19: NBP-VWT case computed with the continuum element made of Mooney–Rivlin material ($C_1 = 1.1 \times 10^5$ N/m$^2$, $C_2 = 4.0 \times 10^4$ N/m$^2$). Flow field at various instants. Velocity vectors colored by magnitude.
Figure 6.20: HBP-UWT case computed with the continuum element made of Mooney–Rivlin material ($C_1 = 1.1 \times 10^5$ N/m², $C_2 = 4.0 \times 10^4$ N/m²). Verification of mass balance. Volumetric inflow rate, difference between the volumetric inflow and outflow rates and rate of change for the artery volume.
Figure 6.21: HBP-UWT case computed with the continuum element made of Mooney-Rivlin material \(C_1 = 1.1 \times 10^5 \text{ N/m}^2, C_2 = 4.0 \times 10^4 \text{ N/m}^2\). Flow field at various instants. Velocity vectors colored by magnitude.
Figure 6.22: HBP-VWT case computed with the continuum element made of Mooney-Rivlin material ($C_1 = 1.1 \times 10^5$ N/m$^2$, $C_2 = 4.0 \times 10^4$ N/m$^2$). Verification of mass balance. Volumetric inflow rate, difference between the volumetric inflow and outflow rates and rate of change for the artery volume.
Figure 6.23: HBP-VWT case computed with the continuum element made of Mooney-Rivlin material \( (C_1 = 1.1 \times 10^5 \text{ N/m}^2, C_2 = 4.0 \times 10^4 \text{ N/m}^2) \). Flow field at various instants. Velocity vectors colored by magnitude.
6.5 Computations with the Neo-Hookean continuum element

We have also considered using a special case of the Mooney–Rivlin model: the Neo-Hookean model. This model uses the same equations, but specifies that $C_2$ must be set to zero. So, while all of our other parameters for these simulations are identical to the Mooney–Rivlin cases, the strain energy calculated in Eq. (6.3) is allocated entirely to $C_1$, while $C_2$ is set to zero:

$$C_1 = \frac{(a_{10}(I_1 - 3) + a_{20}(I_1 - 3)^2 + a_{11}(I_1 - 3)(I_2 - 3) + a_{01}(I_2 - 3))}{(I_1 - 3)} ,$$

$$C_2 = 0.0 . \quad (6.6)$$

Using the same input stretch as in Section 6.4, we find a value of $1.6 \times 10^5 \, \text{N/m}^2$ for $C_1$ and 0.0 for $C_2$. The computations with the Neo-Hookean continuum elements show reasonable deformation of the aneurysm as well as good mass balance. Figures 6.24, 6.26, 6.28, and 6.30 show the mass balance for the four test cases: NBP-UWT, NBP-VWT, HBP-UWT, HBP-VWT. As before, the following pages also contain snapshots of the artery deformation and velocity profile.
Figure 6.24: NBP-UWT case computed with the continuum element made of Neo-Hookean material. Verification of mass balance. Volumetric inflow rate, difference between the volumetric inflow and outflow rates and rate of change for the artery volume.
Figure 6.25: NBP-UWT case computed with the continuum element made of Neo-Hookean material. Flow field at various instants. Velocity vectors colored by magnitude.
Figure 6.26: NBP-VWT case computed with the continuum element made of Neo-Hookean material. Verification of mass balance. Volumetric inflow rate, difference between the volumetric inflow and outflow rates and rate of change for the artery volume.
Figure 6.27: NBP-VWT case computed with the continuum element made of Neo-Hookean material. Flow field at various instants. Velocity vectors colored by magnitude.
Figure 6.28: HBP-UWT case computed with the continuum element made of Neo-Hookean material. Verification of mass balance. Volumetric inflow rate, difference between the volumetric inflow and outflow rates and rate of change for the artery volume.
Figure 6.29: HBP-UWT case computed with the continuum element made of Neo-Hookean material. Flow field at various instants. Velocity vectors colored by magnitude.
Figure 6.30: HBP-VWT case computed with the continuum element made of Neo-Hookean material. Verification of mass balance. Volumetric inflow rate, difference between the volumetric inflow and outflow rates and rate of change for the artery volume.
Figure 6.31: HBP-VWT case computed with the continuum element made of Neo-Hookean material. Flow field at various instants. Velocity vectors colored by magnitude.
6.6 Computed with estimated zero-pressure arterial geometry

In all of the previous cases, we use the image-based geometry (with nominal diameter 3.0 mm) as the arterial geometry corresponding to zero blood pressure. But as we point out in (37), it is more realistic to use that image-based geometry as the arterial geometry corresponding to the time-averaged value of the blood pressure. To accomplish this, we must construct a zero-pressure arterial geometry that will inflate to the image-based data at the time-averaged value of the blood pressure. In (37), we use a rudimentary technique to estimate that zero-pressure arterial geometry. We use an estimated configuration for the zero-pressure geometry and try different values for the zero-pressure wall thickness until the arterial geometry obtained with the time-averaged pressure value has the same diameter and wall thickness as the approximate image-based geometry. We recognize that only two of the parameters that represent the image-based geometry are matched, but considering that a full matching would be quite challenging, this rudimentary technique is a good start in estimating the zero-pressure arterial geometry.

In order to accurately calculate the diameter of the artery at the time-averaged pressure value, we relax the boundary conditions for the artery. The nodes on the inflow and outflow boundary are allowed to move in the plane. As the blood pressure increases, the diameter of the artery at the boundary inflates with the rest of the artery. The computations in this section use the S→F→FSI simulation sequence
because the fluid information is more accurate if calculated after the initial inflation of the structure.

In (37), we also propose a change to the Mooney–Rivlin constants. The strain energy of the model used in Section 6.4 is significantly higher than other models at the strain rates we are observing, so we are changing the Mooney–Rivlin constants to more closely approximate the strain energy of other constitutive models. Figure 6.32 shows the strain energy of our model with the old Mooney–Rivlin constants. We compare this to the five-parameter model (45) and the Fung-type strain energy function model (46). Note that the strain energy of the two-parameter model at lower stretch values is significantly higher than either of the other models.

Figure 6.32: Strain energy comparison with the old Mooney–Rivlin constants ($C_1 = 1.1 \times 10^5$ N/m$^2$, $C_2 = 4.0 \times 10^4$ N/m$^2$).
Figure 6.33 shows the strain energy of our model with both Mooney–Rivlin constants $C_1$ and $C_2$ set to $5.0 \times 10^4$ N/m$^2$. Up to a stretch of 1.15, the strain energy of the two-parameter model is a reasonable approximation of the five-parameter model and Fung-type model. We also conduct a test inflation to the highest blood pressure value to ensure that the deformation of the artery under these conditions is reasonable. We use a penalty Poisson’s ratio of 0.45.

![Graph showing strain energy comparison](image)

**Figure 6.33:** Strain energy comparison with the new Mooney–Rivlin constants ($C_1 = C_2 = 5.0 \times 10^4$ N/m$^2$).

Figure 6.34 shows the mass balance of the NBP-UWT case from (37). To provide some comparison to the results from the Mooney–Rivlin NBP-UWT case from (38), we can measure the average aneurysm thickness. The case from (37) starts with an average aneurysm thickness of 0.301 mm and drops to an average aneurysm thickness...
of 0.283 mm in the third frame of Figure 6.35. The case from (38) starts with an average aneurysm thickness of 0.280 mm and drops to 0.267 mm in the third frame of Figure 6.17.

**Figure 6.34:** Computations using estimated zero-pressure arterial geometry. NBP-UWT case computed with the continuum element made of Mooney–Rivlin material \( (C_1 = C_2 = 5.0 \times 10^4 \text{ N/m}^2) \). Verification of mass balance. Volumetric inflow rate, difference between the volumetric inflow and outflow rates and rate of change for the artery volume.
Figure 6.35: Computations using estimated zero-pressure arterial geometry. NBP-UWT case computed with the continuum element made of Mooney–Rivlin material ($C_1 = C_2 = 5.0 \times 10^4$ N/m²). Flow field at various instants. Velocity vectors colored by magnitude.
Chapter 7

Conclusion

The simulation results presented in Chapters 5 and 6 have shown that the structure side of our FSI solver has been successfully upgraded with continuum element functionality. We can accurately model both linearly-elastic structures and hyperelastic structures.

Chapter 6 has shown that we can use continuum elements to model arterial walls in blood flow simulations. We have seen that linearly-elastic continuum elements do not perform well at these strain levels, but we have much more promising results from hyperelastic models. We successfully tested a number of Mooney–Rivlin models, including the special Neo-Hookean case.

We have begun to investigate methods of generating more realistic zero-pressure arterial geometry from patient-specific image-based geometry. This topic needs further investigation, as the initial thickness of the artery and aneurysm plays a role in the deformation of the aneurysm.
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