THE LIMITING DENSITY OF A NONLINEAR SYSTEM

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ABSTRACT

The RC filter-hard limiter-RC filter nonlinear system shown in Fig. 1 is the subject of this paper. Because of computational difficulties implicated in the analysis of the above system, only its response to the zero mean Gaussian system input has been analytically investigated [2,3,5]. An approximate output density has also been found for nonzero mean Gaussian, while verified to be "close" to the real one for finite means [4]. In the present paper, a close form of the system output density is obtained when the input mean tends to infinity. For that, ε-upcrossing methods were used.

1. INTRODUCTION

The system in Fig. 1 is considered where [2]

$n(t)$: white, Gaussian, zero mean noise

$\frac{1}{a}, \frac{1}{b}$: time constants of the two RC filters

$x(t)$: Gaussian, mean $(-m)$, markov process with autocorrelation

$$R_x(J) = e^{-a|J|}$$

$z(t)$: $\operatorname{sgn}(x(t))$

$y(t)$: the output process

Hence, the hypothesis here is that the system input $n(t) - m$ is Gaussian and white with mean $(-m)$.

The steady-state output $y(0)$ of the system which will be the subject of our investigation is given by the following equation [2]:

$$y(0) = \int_{-\infty}^{0} be^{bs} z(s)ds = \int_{0}^{\infty} be^{-bs} z(-s)ds \qquad (1)$$

From (1) it is obvious that $-1 \leq y(0) \leq 1$. A first look at expression (1) makes one think that, since the integral in it changes sign each time the process $x(s)$ does, applications of zero-crossing methods might lead to some kind of expression for the density of $y(0)$.

If one defines the new process $w(t)$ as follows:

$$w(s) = x(s) + m \qquad (2)$$

then a crossing of the zero level by the process $x(s)$ is equivalent to the crossing of the level (+m) by the zero-mean process $w(s)$. The process $w(s)$ has, of course, the same autocorrelation with the process $x(s)$. That is:

$$R_w(J) = R_x(J) = e^{-a|J|} \qquad (3)$$

The equation giving the sign process $z(s)$ as a function of $w(s)$, is the following:

$$z(s) = \begin{cases} 1 \ , & w(s) \geq m \\ -1, & w(s) < m \end{cases} \qquad (4)$$

The definition of the process $w(s)$ in (2) places the problem of defining the density of $y(0)$ into the category of crossings of the (+m) level by a zero-mean Gaussian process. James Pickands III [1] developed formulas and density-expressions for such crossings when the level (+m) is infinitely increasing. Specifically, he developed expressions for ε-upcrossings of a high level, as they will be defined in the process of the introduction, which happened to be independent of the chosen ε. Hence, he was able to conclude that his results were independent of ε and so one could talk about simply upcrossings of a certain high level $(+m)$ by a zero-mean Gaussian process. We will proceed by giving a few definitions and formulas from Pickands' paper which we will use to determine the density of the $y(0)$ in (1).

Definition 1

ε-upcrossing at time t of a level $(+m)$ by a process $w(s)$ is called the event that $w(t) = +m$ and $w(J) < +m$, for every J such that $t - \varepsilon \leq J < t$. The number ε is an arbitrary number and it can be as small as desired.

Definition 2

$N(\varepsilon,m,t)$ is defined as the number of ε-upcrossings of the level $+m$ by the process $w(s)$ and in a time length t.

If the process $w(s)$ is zero mean, Gaussian and with autocorrelation $R_w(J)$ such that

$$R_w(J) = 1 - aJ + 0(J) \tag{5}$$

for J tending to zero from positive values (the autocorrelation $R_w(J) = e^{-a|J|}$ satisfies property (5)), then the number $N(\varepsilon,m,t)$ is given by the following formula:

$$\lim_{m\to\infty} \frac{E\{N(\varepsilon,m,t)\}}{\frac{m}{\sqrt{2\pi}} \cdot e^{-(m^2)/2} \cdot t} = a \tag{6}$$

Expression (6) was proven to be true by Pickands [1]. Pickands also found the following expression for the distribution of $N(\varepsilon,m,t)$:

$$\lim_{m\to\infty} P\{N(\varepsilon,m,\tfrac{\lambda}{\mu}) = k\} = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \tag{7}$$

where

$$\mu = \mu(\varepsilon,m) = E\{N(\varepsilon,m,t)\}/t \tag{8}$$

Hence, μ is actually the expected number of ε-upcrossings of the level m, per unit time, which in the limit of $m \to \infty$ is given by the following equation (substitutions of (8) into (6)):

$$\lim_{m\to\infty} \frac{\mu(\varepsilon,m)}{\frac{m}{\sqrt{2\pi}}\, e^{-(m^2)/2}} = a \tag{9}$$

Furthermore, one can write (7) in the following different way:

$$\lim_{m\to\infty} P\{N(\varepsilon,m,t) = k\} = e^{-t\mu}\, \frac{(t\mu)^k}{k!} \tag{10}$$

which shows that the number $N(\varepsilon,m,t)$, for level m increasing, tends to be Poisson distributed with parameter $t\mu$.

Also, from expression (9) one concludes that in the limit $(m \to \infty)$, the expected number of ε-upcrossings per unit time decreases with increasing m as fast as $m/(\sqrt{2m})\ e^{-(m^2)/2}$. If one calls $d = d(\varepsilon,m)$ the distance between two consecutive ε-upcrossings of the level m , then:

$$\lim_{m\to\infty} P\{d(\varepsilon,m) > T\} = P\{0 \ \ \varepsilon\text{-upcrossings in a time interval of length } T\}$$
$$= e^{-\mu T} \tag{11}$$

In the development of (11), formula (10) was used. From (11) we have, of course, that

$$\lim_{m\to\infty} P\{d \leq T\} = 1 - e^{\mu T} \tag{12}$$

$$\lim_{\substack{m\to\infty \\ dT\to 0}} P\{T < d \leq T + dT\} = \mu e^{-\mu T} dT \tag{13}$$

With formula (13) which gives the limiting density of the distance between two consecutive ϵ-upcrossings, we have given to the reader all the formulas that we will use from Pickands' paper and we have completed the introduction. In the following sections we will give expressions of the system output $y(0)$ using reasonable approximations and we will finally reach a close limiting expression $(m \to \infty)$ of the output density based on some intuitive assumptions.

2. AN APPROXIMATE LIMITING EXPRESSION OF $y(0)$

It is obvious from expression (1) that the value of $y(0)$ depends on the instants that the process $x(s)$ changes sign. That is equivalent to saying that the value of $y(0)$ depends on the instants that the process $w(s)$, as defined in (2), crosses the level $+m$. One can express $y(0)$ as a function of these last crossing instants. Such an expression will be different depending on whether or not the process $w(s)$ finds itself at a level below or above $+m$ at the observation time $t = 0$. The two figures 2 and 3 are drawn with the process $w(t)$ actually moving in time from right $(t = -\infty)$ to left $(t = 0)$, until it reaches its final value $w(0)$. Figure 2 indicates the changes of the process when it ends at a level lower than $+m$, while figure 3 indicates the similar changes when $w(0) > +m$. The following variables were introduced in figures 2 and 3:

t_i : instant at which the i^{th} closest to $t = 0$ upcrossing happens

θ_i : time difference between the i^{th} closest to $t = 0$ upcrossing and the downcrossing which immediately follows

$t_{i,i+1}$: instant at which the downcrossing between the i^{th} closest to $t = 0$ upcrossing and the $(i+1)^{th}$ one

$\lambda_{i,i+1}$: $t_{i+1} - t_{i,i+1}$

J_i : $t_{i+1} - t_i$

ρ_i : $t_{i+1,i+2} - t_{i,i+1}$

Applying the above table of variables to formula (1), one can easily break the integral in it into the sum of integrals as follows: If $w(0) < +m$ and, using Fig. 2, one gets

$$\underset{w(0)<+m}{y(0)} = -\int_{o}^{t_1} be^{-bs}ds + \sum_{i=1}^{\infty} [\int_{t_i}^{t_{i,i+1}} be^{-bs}ds - \int_{t_{i,i+1}}^{t_{i+1}} be^{-bs}ds]$$

$$= e^{-bt_1} - 1 + \sum_{i=1}^{\infty} [e^{-bt_i} + e^{-bt_{i+1}} - 2e^{-bt_{i,i+1}}]$$

$$= -1 + 2\sum_{i=1}^{\infty} e^{-bt_i} - 2\sum_{i=1}^{\infty} e^{-bt_{i,i+1}} =$$

$$= -1 + 2e^{-bt_1}\left(1 + \sum_{i=1}^{\infty} e^{-b\sum_{j=1}^{i} J_j}\right) - 2e^{-bt_{12}}\left(1 + \sum_{i=1}^{\infty} e^{-b\sum_{j=1}^{i} \rho_j}\right) \qquad (14)$$

Using Fig. 3 for the case that $w(0) \geq +m$ and applying the same table to expression (1), one gets

$$\underset{w(0) \geq +m}{y(0)} = \int_o^{t_{12}} be^{-bs}ds + \sum_{i=1}^{\infty} \left[-\int_{t_{i,i+1}}^{t_i} be^{-bs}ds + \int_{t_i}^{t_{i+1,i+2}} be^{-bs}ds\right]$$

$$= 1 - e^{-bt_{12}} + \sum_{i=1}^{\infty} \left[-e^{-bt_{i,i+1}} + 2e^{-bt_i} - e^{-bt_{i+1,i+2}}\right]$$

$$= 1 - 2\sum_{i=1}^{\infty} e^{-bt_{i,i+1}} + 2\sum_{i=1}^{\infty} e^{-bt_i}$$

$$= 1 - 2e^{-bt_{12}}\left(1 + \sum_{i=1}^{\infty} e^{-b\sum_{j=1}^{i} \rho_j}\right) + 2e^{-bt_1}\left(1 + \sum_{i=1}^{\infty} e^{-b\sum_{j=1}^{i} J_j}\right) \qquad (15)$$

Our objective will now be to simplify expressions (14) and (15) using reasonable approximations. The variables that appear in both formulas are t_1, t_{12}, ρ_i and J_i, with the last two forming infinite exponential sums. From expressions (11) and (9) one can easily see that, for the level m tending to plus infinity, the probability of J_i being larger than a number m^n, for any arbitrary integer n, is tending to one. This combined with the fact that the J_i's are independent because of the Markov character of $w(t)$, give the following equality with probability are:

$$\lim_{m\to\infty} \sum_{i=1}^{\infty} e^{-b\sum_{j=1}^{i} J_j} \sim e^{-bJ_1} \qquad (16)$$

Similarly, one expects that in the limit $(m \to \infty)$, since the upcrossings happen very rarely, so do downcrossings. Furthermore, because of the Markov character of $w(t)$, the ρ_i's are also independent. Hence, with probability one we can write:

$$\lim_{m\to\infty} \sum_{i=1}^{\infty} e^{-b\sum_{j=1}^{i}\rho_j} \sim e^{-b\rho_1} \qquad (17)$$

After these two approximations, one gets, with probability one, the following simplified formulas of $y(0)$:

$$\lim_{\substack{m\to\infty \\ w(0)<m}} y(0) \sim -1 + 2e^{-bt_1}(1 + e^{-bJ_1}) - 2e^{-bt_{12}}(1 + e^{-b\rho_1}) \qquad (18)$$

$$\lim_{\substack{m\to\infty \\ w(0)\geq m}} y(0) \sim 1 - 2e^{-bt_{12}}(1 + e^{-b\rho_1}) + 2e^{-bt_1}(1 + e^{-bJ_1}) \qquad (19)$$

If we express $y(0)$ through the variables θ_i and λ_{12} we will get the following two new expressions

$$\lim_{\substack{m\to\infty \\ w(0)<m}} y(0) \sim -1 + 2e^{-bt_1}[1 - e^{-b\theta_1}(1 + e^{-b\lambda_{12}}[e^{-b\theta_2} - 1])] \qquad (20)$$

$$\lim_{\substack{m\to\infty \\ w(0)\geq m}} y(0) \sim 1 - 2e^{-bt_{12}}[1 + e^{-b\rho_1}(1 - e^{b\theta_1}[1 + e^{-bJ_1}])] \qquad (21)$$

3. PARTIAL DISTRIBUTIONS

In the limiting case of the level $+m$ increasing towards infinity, one expects the distances θ_i to tend to zero, the way the distance J_i tends to infinity. In other words, intuitively we expect the density of θ_i to tend in the limit to a delta function. In this case, the distribution of the variable $\lambda_{i,i+1}$ will be the same with the distribution of J_i in the limit. This is because

$$\Pr\{J_i \leq \chi\} = \Pr\{\lambda_{i,i+1} + \theta_i \leq \chi\} = \int_o^\infty \Pr\{\lambda_{i,i+1} \leq \chi - u,\ \theta_i = u\} du$$

Because of the Markov character of $w(t)$, $\lambda_{i,i+1}$ and θ_i are independent; hence

$$\Pr\{J_i \leq \chi\} = \int_o^\infty \Pr\{\lambda_{i,i+1} \leq \chi - u\}\cdot f_{\theta_i}(u) du \qquad (22)$$

where $f_{\theta_i}(u)$ is the density of the variable θ_i.

From (22) one directly gets that if $f_{\theta_i}(u)$ tends toward a delta function for $m \to \infty$, and since $\Pr\{J_i \leq \chi\}$ practically zero for χ very large, then:

$$\lim_{m\to\infty} \Pr\{J_i \leq \chi\} \sim \Pr\{\lambda_{i,i+1} \leq \chi\} \qquad (23)$$

Equation (23) is the formal consequence of the intuitive expressions that $f_{\theta_i}(u)$ is in the limit a delta function. Another consequence of the claim that $f_{\theta_i}(u)$ tends to a delta function is that in the limit $(m \to \infty)$, J_i and θ_i tend to become two independent random variables. Indeed, then:

$$\begin{aligned}\lim_{m\to\infty} \Pr\{J_i \leq \chi\} &= \lim_{m\to\infty} \int_o^\infty \Pr\{J_i \leq \chi,\ \theta_i = u\} du \\ &= \lim_{m\to\infty} \int_o^\infty \Pr\{J_i \leq \chi/\theta_i = u\} f_{\theta_i}(u) du \\ &= \Pr\{J_i \leq \chi/\theta_i = u_o\} \end{aligned} \qquad (24)$$

where u_o is the value close to zero that θ_i can take.

Continuing on the same intuitive assumption on θ_i , we find the density of ρ_i . ρ_i is the sum of the two independent variables $\lambda_{i,i+1}$ and θ_{i+1}. Hence

$$\lim_{m\to\infty} \Pr\{\rho_i \le \chi\} = \lim_{m\to\infty} \int_0^\infty \Pr\{\lambda_{i,i+1} \le \chi - u\} \cdot f_{\theta_{i+1}}(u)du$$

$$= \Pr\{\lambda_{i,i+1} \le \chi\} = \Pr\{J_i \le \chi\} \tag{25}$$

Hence, from (25) one concludes that in the limit ($m \to \infty$) the three variables J_i, $\lambda_{i,i+1}$ and ρ_i have the same distribution given by expressions (11), (12) and (13). Furthermore, for similar reasons expressed for the variable J_i , the two variables ρ_i and θ_{i+1} are in the limit independent. From figure #2, one can see that since $w(t)$ is a Markov process, the variable t_1 in formula (20) is independent of the variables θ_1, λ_{12} and θ_2 . Similarly, from Figure 3 one can conclude that the variable t_{12} is independent of the variables ρ_1, θ_1 and J_1 that appear in (21). The only thing that remains now is to express the distributions of the above mentioned variables t_1 and t_{12} .

Using figure #4 first, where the trajectory of the process $w(t)$ in figure #3 was extended to the left until the downcrossing instant, one has

$$\lim_{m\to\infty} \Pr\{x \le t_{12} \le x + dx\} = \lim_{m\to\infty} \int_0^\infty \Pr\left\{\begin{matrix} x \le t_{12} \le x + dx, \\ \theta = y \end{matrix}\right\} dy$$

$$= \lim_{m\to\infty} \int_x^\infty \Pr\{x \le t_{12} \le x + dx/\theta = y\} f_\theta(y)dy \tag{26}$$

The choice of the origin 0 inside the interval of length θ in figure #4 is unbiased. Hence one has

$$\Pr\{x \leq t_{12} \leq x + dx/\theta = y\} = (1/y)dx \tag{27}$$

Application of (27) in (26) will give the density of the variable t_{12} as an integral:

$$\lim_{m\to\infty} f_{t_{12}}(x) = \int_x^\infty (1/y) f_\theta(y)dy \tag{28}$$

From figure #5 we similarly express the density of the variable t_1 after we have used the equal choice equation

$$\Pr\{x \leq t_1 \leq x + dx/\lambda = y\} = (1/y)dx \tag{29}$$

Indeed, we have from application of (11) and (25):

$$\mathrm{Pf}\{x \leq t_1 \leq x + dx\} = \int_x^\infty \Pr\{x \leq t_1 \leq x + dx/\lambda = y\} \cdot \Pr\{\lambda = y\}dy$$

$$= \int_x^\infty \mu e^{-\mu y}(1/y)dy = \mu e^{-\mu x} \int_o^\infty \frac{e^{-\mu y}}{y+x} dy \tag{30}$$

Expression (30) can be simplified if one remembers formula (9). Indeed for any $y < m^n$, where n is any arbitrary integer, the product $\mu \cdot y$ is in the limit zero. Hence, for $m \to \infty$, the exponential $e^{-\mu y}$ is equal to one then. Because of that one can write:

$$\lim_{m\to\infty} \int_o^\infty \frac{e^{-\mu y}}{y+x} dy = \lim_{m\to\infty} \int_o^{m^n} \frac{e^{-\mu y}}{y+x} dy + \lim_{m\to\infty} \int_{m^n}^\infty \frac{e^{-\mu y}}{y+x} dy +$$

$$= \lim_{m\to\infty} \int_o^{m^n} \frac{1}{y+x} dy + \lim_{m\to\infty} \int_{m^n}^\infty \frac{e^{-\mu y}}{y+x} dy \tag{31}$$

For fixed x and $y > m^n$, the sum $y+x$ is approximately equal to y. Hence, expression (31) finally becomes:

$$\lim_{m\to\infty} \int_0^\infty \frac{e^{-\mu y}}{y+x}\, dy = \lim_{m\to\infty} \int_0^{m^n} \frac{1}{y+x}\, dy + \lim_{m\to\infty} \int_{m^n}^\infty \frac{e^{-\mu y}}{y}\, dy =$$

$$= \lim_{m\to\infty} \left[\ell n\left(\frac{m^n}{x} + 1\right) + \int_{m^n}^\infty \frac{e^{-\mu y}}{y}\, dy \right] =$$

$$= \lim_{m\to\infty} \left[n\ell n(m) - \ell n x + \int_{m^n}^\infty \frac{e^{-\mu y}}{y}\, dy \right] =$$

$$= \text{(for fixed } x\text{)} \lim_{m\to\infty} \left[n\ell n(m) + \int_{m^n}^\infty \frac{e^{-\mu y}}{y}\, dy \right]$$

$$= \text{independent of } x \,.$$

Applying this last result to (30) we finally get

$$\lim_{m\to\infty} f_{t_1}(x) = c(\mu) \cdot \mu e^{-\mu x} \tag{32}$$

where $f_{t_1}(x)$ is the density of the variable t_1 and $c(\mu)$ is a function of μ .

Up to this point the densities of all the variables that appear in expressions (20) and (21) have been covered. About the variables θ_1 and θ_2 though, only the claim that in the limit their density function should be a delta function has been expressed. Here we will conjecture this density to be a Rayleigh one. This conjecture will be again based on our intuition. So, we will write

$$\lim_{m\to\infty} f_{\theta_1}(x) = \lim_{m\to\infty} f_{\theta_2}(x) = \frac{x}{\sigma^2} \exp\left(-\frac{x^2}{2\sigma^2}\right) \tag{34}$$

The parameter σ^2 will have to be very small so that in the limit the two densities $f_{\theta_1}(x)$ and $f_{\theta_2}(x)$ approach a delta function and it will be left arbitrary at the moment.

Application of expression (34) to (28) will give the following expression for the limiting density of the variable t_{12}

$$\lim_{m\to\infty} f_{t_{12}}(x) = \frac{\sqrt{2\pi}}{\sigma}[1 - \Phi(x/\sigma)] \tag{35}$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-(u^2/2)} du \tag{36}$$

With (36) we conclude this section. Our objective in the following section will be to find the distribution of the system output $y(0)$ from expressions (20) and (21) as well as (25), (32), (34), (35) and (11).

4. THE DISTRIBUTION OF $y(0)$

The distribution $F_{y(0)}(y)$ of the system output $y(0)$ can be broken into two parts. That is:

$$\lim_{m\to\infty} F_{y(0)}(x) = \lim_{m\to\infty} \Pr\{y(0) \leq y,\ w(0) < m\} + \lim_{m\to\infty} \Pr\{y(0) \leq y,\ w(0) \geq m\} \tag{37}$$

The two terms in (37) correspond to the two expressions (20) and (21) of the introduction. Hence,

$$\lim_{m\to\infty} \Pr\{y(0) \leq y,\ w(0) < m\} = \Pr\left\{-1 + 2e^{-bt_1}[1 - e^{-b\theta_1}(1 + e^{-b\lambda_{12}}[e^{-b\theta_2} - 1])] \leq y\right\} \tag{38}$$

and

$$\lim_{m\to\infty} \Pr\{y(0) \leq y,\ w(0) \geq m\} = \Pr\left\{1 - 2e^{-bt_{12}}[1 + e^{-b\rho_1}(1 - e^{b\theta_1}[1 + e^{-b\tau_1}])] \leq y\right\} \tag{39}$$

According to the previous section, all the variables involved in expressions (38) and (39) are in the limit independent from each other. Their densities are all given in this same section with only the density $f_{\theta_i}(x)$ conjectured. Hence, both distributions in (38) and (39) can be found. Indeed, from appendices 1 and 2 (expressions (14) and (10) correspondingly) one gets:

$$\lim_{m\to\infty} f_{y(0)}(y) = \lim_{m\to\infty} f_{y(0)}(y,\ w(0) < m) + \lim_{m\to\infty} f_{y(0)}(y,\ w(0) \geq m)$$

$$= K_2(\mu,b)(1+y)^{\mu/b} + [K_3(\mu,b) + K_4(\mu,b)] \qquad (40)$$

The functions $K_2(\mu,b)$, $K_3(\mu,b)$ and $K_4(\mu,b)$ are either constants with respect to y or very slowly changing with it when compared with $(1 + y)^{\mu/b}$.

So, in the limit, all three functions are considered nonfunctions of y. Calling

$$K_3(\mu,b) + K_4(\mu,b) = C(\mu,b) \qquad (41)$$

we can write (40) as follows:

$$\lim_{m\to\infty} f_{y(0)}(y) = K_2(\mu,b) \cdot (1 + y)^{\mu/b} + C(\mu,b) \qquad (42)$$

where y is taking values in the interval [-1,1]. Both $K_2(\mu,b)$ and $C(\mu,b)$ are complicated limiting functions of μ and b given in appendices 1 and 2. Their values can be easily calculated though indirectly. Indeed, we must obviously have the following equation satisfied by $f_{y(0)}(y)$:

$$\int_{-1}^{1} f_{y(0)}(y)dy = 1 \qquad (43)$$

Also, from expression (1) we easily get that

$$E\{y(0)\} = \int_{-\infty}^{0} be^{bs}\, E\{z(s)\}ds \tag{44}$$

where $z(s)$ is given by (4). Hence,

$$E\{z(s)\} = Pr\{w(s) \geq m\} - Pr\{w(s) < m\} \tag{45}$$

But the process $w(s)$ is Gaussian zero mean with autocorrelation $R_w(J) = a^{-a|J|}$ (hence variance one). So,

$$E\{z(s)\} = 1 - 2Pr\{w(s) < m\} = 1 - 2\Phi(m) \tag{46}$$

where

$$\Phi(m) = \int_{-\infty}^{m} \frac{e^{-\frac{1}{2}u^2}}{\sqrt{2\pi}}\, du \tag{47}$$

Application of (45) to (44) gives:

$$E\{y(0)\} = 1 - 2\Phi(m) = \int_{-1}^{1} y\, f_{y(0)}(y)dy \tag{48}$$

If one solves the system of equations (43) and (48), with $f_{y(0)}(y)$ given in the limit by expression (42), one gets

$$K_2(\mu,b) = -\frac{[2\Phi(m) - 1](1 + \frac{\mu}{b})(2 + \frac{\mu}{b})}{\frac{\mu}{b} \cdot 2^{1 + \mu/b}} \tag{49}$$

$$C(\mu,b) = \frac{1}{2} + \frac{[2\Phi(m)-1](2 + \frac{\mu}{b})}{\frac{2\mu}{b}} \tag{50}$$

and

$$\lim_{m\to\infty} f_{y(0)}(y) = \frac{2\Phi(m) - 1}{\frac{\mu}{b}}\left[1 - \frac{(1 + y)^{\mu/b}}{2^{\mu/b}}\right] \tag{51}$$

We will here remind the reader that expression (51) corresponds to the limiting density of the steady-state output of the nonlinear system in Fig. 1 when the input is Gaussian, white and of mean (-m). As it was expected, the density $f_{y(0)}(y)$ tends in the limit to a delta function at the point $y = -1$.

It is very important that we point out here that the conjecture on the density $f_\theta(x)$ of the distance between one upcrossing of the level m and the following downcrossing was only used as a mathematical tool towards the calculation of the density $f_{y(0)}(y)$. The parameters of the density $f_\theta(x)$ as well as its exact form do not appear in the expression (51). Hence, the density $f_{y(0)}(y)$ is in the limit independent of the actual density $f_\theta(x)$.

We will conclude this section by pointing out that because of symmetry, the steady-state density of the system output in Fig. 1, when the input is mean (+m), is given in the limit by the following expression:

$$\lim_{m\to\infty} f_{y(0)}(y) = \frac{2\Phi(m) - 1}{\frac{\mu}{b}} \left[1 - \frac{(1 - y)^{\mu/b}}{2^{\mu/b}}\right] \tag{52}$$

COMMENTS

In reference [4] an approximation of the steady-state output density of the system in Fig. 1 was found based on a Markov assumption. The result was numerically proved to be "close" to the real density for white, Gaussian arbitrary finite mean input $n(t)$. It was pointed out, though, that the "tail values" of the real and the approximate density did not agree. In this paper this "tail value" was calculated for the first time in the limiting case of infinitely large input signal.

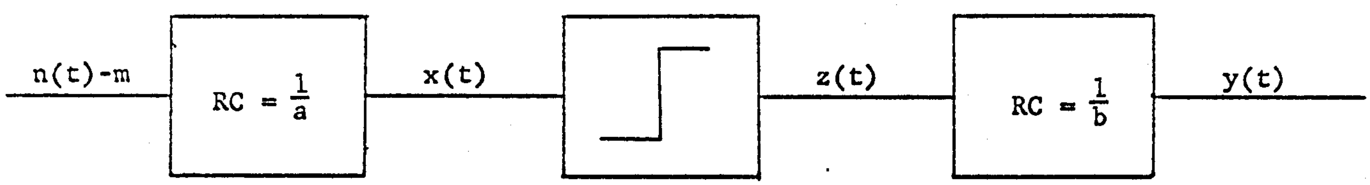

Fig. 1

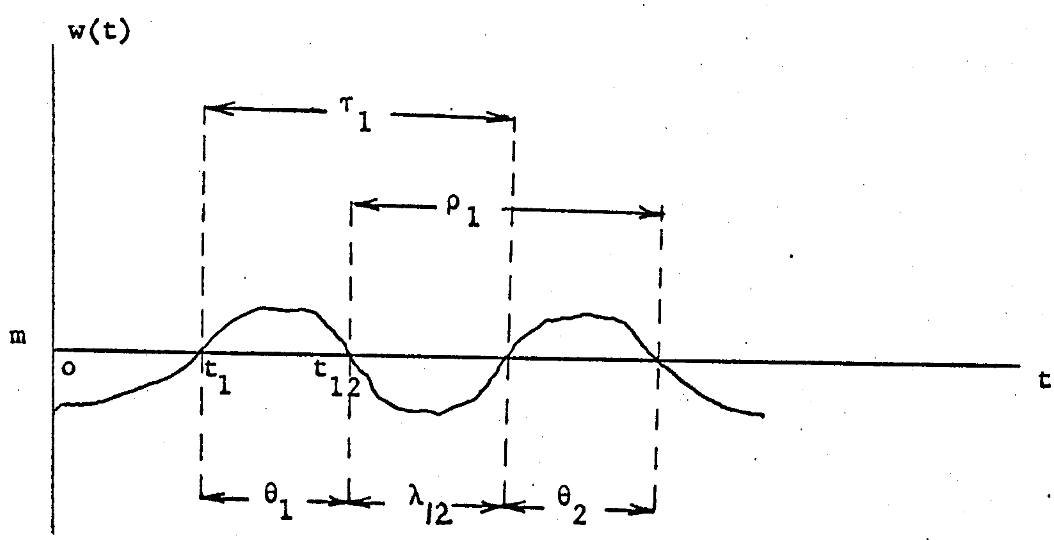

Fig. 2

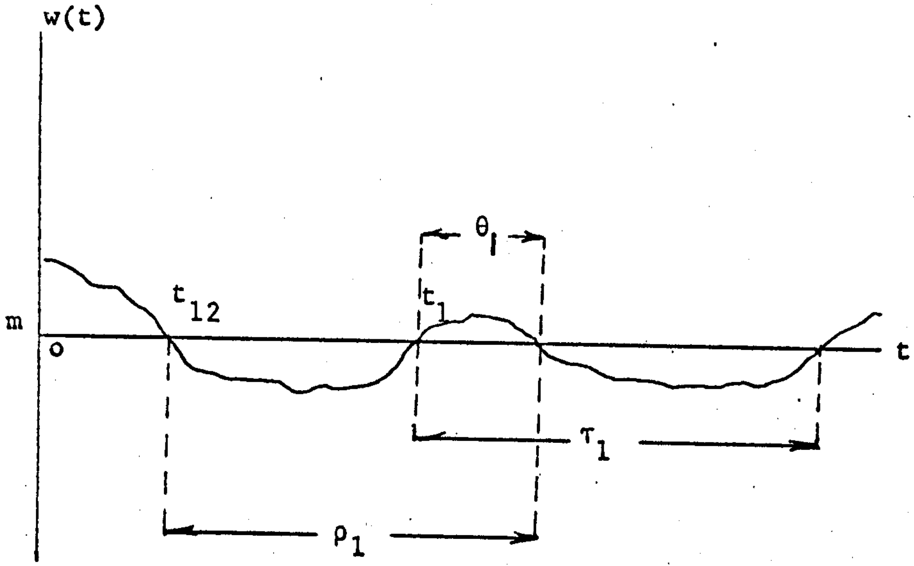

Fig. 3

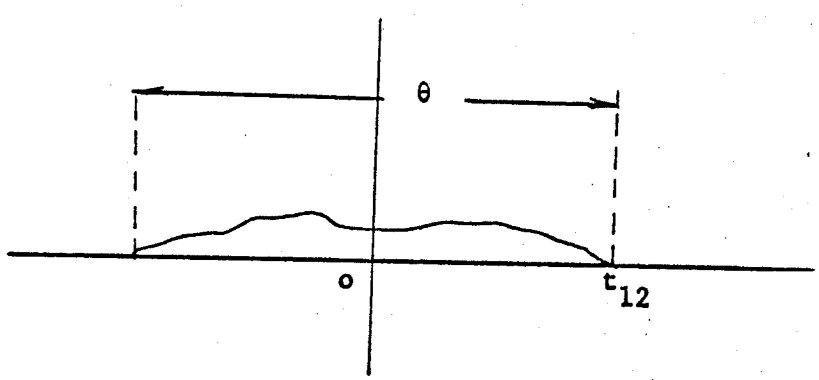

Fig. 4

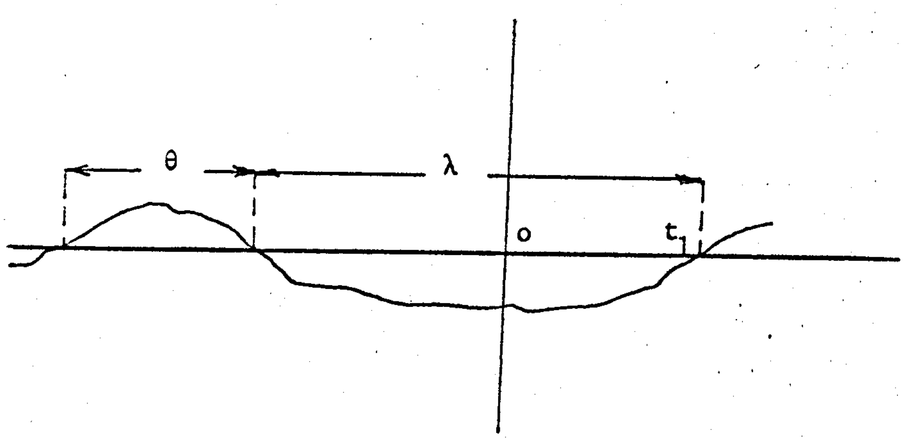

Fig. 5

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APPENDIX 1

In this appendix, the distribution $\lim_{m\to\infty} \Pr\{y(0) \le y,\ w(0) < m\}$ will be calculated. From expression (38) in section 4 we have:

$$\lim_{m\to\infty} \Pr\{y(0) \le y,\ w(0) < m\} =$$

$$= \Pr\left\{e^{-bt_1}\left[1 - e^{-b\theta_1}\left(1 + e^{-b\lambda_{12}}\left[e^{-b\theta_2} - 1\right]\right)\right] \le \frac{y+1}{2}\right\}$$

$$= \int_0^1 \Pr\left\{e^{-bt_1} = x\right\} \cdot \Pr\left\{e^{-b\theta_1}\left(1 + e^{-b\lambda_{12}}\left[e^{-b\theta_2} - 1\right]\right) \ge 1 - \frac{y+1}{2x}\right\} dx$$

$$= \int_0^1 f_{t_1}\left(-\frac{\ln x}{b}\right) dx \int_0^1 f_{\theta_1}\left(-\frac{\ln u}{b}\right) du \int_0^1 f_{\theta_2}\left(-\frac{\ln w}{b}\right) dw \cdot$$

$$\cdot \Pr\left\{e^{-b\lambda_{12}} \le \frac{1 - \frac{1}{u}\left[1 - \frac{y+1}{2x}\right]}{1 - w}\right\} \qquad (1)$$

In expression (1) the independence of the variables t_1, θ_1, θ_2, λ_{12}, as expressed in section 3 was used. The variable λ_{12} being always positive, the exponential $e^{-b\lambda_{12}}$ is always between zero and one. Hence,

$$\Pr\left\{e^{-b\lambda_{12}} \le \frac{1 - \frac{1}{u}\left[1 - \frac{y+1}{2x}\right]}{1 - w}\right\} =$$

$$= \begin{cases} 0 & , \quad 1 - \frac{1}{u}\left[1 - \frac{y+1}{2x}\right] < 0 \\[2ex] 1 - F_{\lambda_{12}}\left(-\dfrac{\ln\left(\dfrac{1 - \frac{1}{u}\left[1 - \frac{y+1}{2x}\right]}{1 - w}\right)}{b}\right) & , \quad 0 \le \dfrac{1 - \frac{1}{u}\left[1 - \frac{y+1}{2x}\right]}{1 - w} \le 1 \\[2ex] 1 & , \quad \dfrac{1 - \frac{1}{u}\left[1 - \frac{y+1}{2x}\right]}{1 - w} > 1 \end{cases}$$

therefore

$$\Pr\left\{e^{-b\lambda_{12}} \leq \frac{1 - \frac{1}{u}\left[1 - \frac{y+1}{2x}\right]}{1 - w}\right\}$$

$$= \begin{cases} 0 & , \; x > \dfrac{y+1}{2(1-u)} \\ 1 - F_{\lambda_{12}}\left(-\dfrac{\ell n\left(\dfrac{1 - \frac{1}{u}\left[1 - \frac{y+1}{2x}\right]}{1 - w}\right)}{b}\right) & , \; \dfrac{y+1}{2(1-uw)} \leq x \leq \dfrac{y+1}{2(1-u)} \\ 1 & , \; x < \dfrac{y+1}{2(1-uw)} \end{cases} \tag{2}$$

If one applies the results in (2) in expression (1), considering also the fact that $-1 \leq y(0) \leq 1$ in any case [2], one gets

$$\lim_{m \to \infty} \Pr\{y(0) \leq y, \, w(0) < m\}$$

$$= \int_0^1 \int_0^1 \int_{\frac{y+1}{2(1-uw)}}^{\min\left(1, \frac{y+1}{2(1-u)}\right)} dudwdx \; f_{t_1}\left(-\frac{\ell nx}{b}\right) f_{\theta_1}\left(-\frac{\ell nu}{b}\right) f_{\theta_2}\left(-\frac{\ell nw}{b}\right)$$

$$\cdot \left[1 - F_{\lambda_{12}}\left(-\frac{1}{b}\,\ell n\left[\frac{1 - \frac{1}{u}\, 1 - \frac{y+1}{2x}}{1 - w}\right]\right)\right]$$

$$+ \int_0^1 \int_0^1 \int_0^{\min\left(1, \frac{y+1}{2(1-uw)}\right)} dudwdx \; f_{t_1}\left(-\frac{\ell nx}{b}\right) f_{\theta_1}\left(-\frac{\ell nu}{b}\right) f_{\theta_2}\left(-\frac{\ell nw}{b}\right) \tag{3}$$

If one applies the results in section 3 here, one gets:

$$\lim_{m\to\infty} \Pr\{y(0) \leq y,\ w(0) < m\}$$

$$= \int_0^1 \iint_{\frac{y+1}{2(1-uw)}}^{\min\left(1, \frac{y+1}{2(1-u)}\right)} du\ dw\ dx\ f_{\theta_1}\left(-\frac{\ln u}{b}\right) f_{\theta_2}\left(-\frac{\ln w}{b}\right) \cdot$$

$$\cdot\ c(\mu) \cdot \mu \cdot x^{\mu/b} \left[\frac{1 - \frac{1}{u}\left[1 - \frac{y+1}{2x}\right]}{1 - w}\right]^{\mu/b}$$

$$+ \int_0^1 \iint_0^{\min\left(1, \frac{y+1}{2(1-uw)}\right)} du\ dw\ dz\ f_{\theta_1}\left(-\frac{\ln u}{b}\right) f_{\theta_2}\left(-\frac{\ln w}{b}\right) \cdot$$

$$\cdot\ c(\mu) \cdot \mu \cdot x^{\mu/b} \qquad (4)$$

A few manipulations on (4) and integration with respect to x lead to the following expression:

$$\lim_{m\to\infty} \Pr\{y(0) \leq y,\ w(0) < m\}$$

$$= \frac{\mu c(\mu)}{1 + \frac{\mu}{b}} \cdot \int_0^1 \int \frac{du\ dw}{(1-w)^{\mu/b}} f_{\theta_1}\left(-\frac{\ln u}{b}\right) f_{\theta_2}\left(-\frac{\ln w}{b}\right) \cdot$$

$$\cdot\ \frac{u}{u-1} \left\{\left[\frac{y+1}{2u} - \frac{1-u}{u} \min\left(1, \frac{y+1}{2(1-u)}\right)\right]^{1+\frac{\mu}{b}} - \left[\frac{y+1}{2u} - \frac{(y+1)}{2(1-uw)} \frac{(1-u)}{u}\right]^{1+\frac{\mu}{b}}\right\}$$

$$+ \frac{\mu c(\mu)}{1 + \frac{\mu}{b}} \int_0^1 \int du\ dw\ f_{\theta_1}\left(-\frac{\ln u}{b}\right) f_{\theta_2}\left(-\frac{\ln w}{b}\right) \cdot \left[\min\left(1, \frac{y+1}{2(1-uw)}\right)\right]^{1+\frac{\mu}{b}}$$

$$(5)$$

From (5), one gets the following expression if one gets rid of the $\min(\cdot,\cdot)$ functions:

$$\lim_{m\to\infty} \Pr\{y(0) \le y,\ w(0) < m\}$$

$$= \frac{\mu c(\mu)}{(1+\frac{\mu}{b})2^{1+(\mu/b)}} (y+1)^{1+\frac{\mu}{b}} \left\{\int_0^1 dw(1-w) f_{\theta_2}\left(-\frac{\ell n w}{b}\right) \cdot \right.$$

$$\cdot \int_0^1 \frac{u}{(1-u)(1-uw)^{1+(\mu/b)}} f_{\theta_1}\left(-\frac{\ell n u}{b}\right) du$$

$$\left. + \int_0^1 \frac{dw}{w} f_{\theta_2}\left(-\frac{\ell n w}{b}\right) \int_0^{\frac{1-y}{2}} \frac{1}{(1-u)^{1+(\mu/b)}} f_{\theta_1}\left(-\frac{\ell n u}{b} + \frac{\ell n w}{b}\right) du\right\}$$

$$+ \frac{\mu c(\mu)}{1+\frac{\mu}{b}} \left\{\int_0^1 \frac{dw}{w} f_{\theta_2}\left(-\frac{\ell n w}{b}\right) \int_{\frac{1-y}{2}}^1 du\ f_{\theta_1}\left(-\frac{\ell n u}{b} + \frac{\ell n w}{b}\right)\right.$$

$$\left. - \left[\int_0^1 \frac{dw}{(1-w)^{\mu/b}} f_{\theta_2}\left(-\frac{\ell n w}{b}\right)\right] \int_{\frac{1-y}{2}}^1 du \frac{u}{1-u} f_{\theta_1}\left(-\frac{\ell n u}{b}\right)\left[\frac{y+1}{2u} + \frac{u-1}{u}\right]^{1+\frac{\mu}{b}}\right\} \tag{6}$$

Differentiating (6) with respect to y, one finds the following expression for the density $f_{y(0)}(y,\ w(0) < m)$:

$$\lim_{m\to\infty} f_{y(0)}(y,\ w(0) < m)$$

$$= \frac{\mu c(\mu)}{2^{1+(\mu/b)}} (1+y)^{\frac{\mu}{b}} \cdot \int_0^1 dw\ f_\theta\left(-\frac{\ell n w}{b}\right) \cdot$$

$$\cdot \left[\frac{1}{1-w} \int_0^w \frac{du}{(1-u)^{1+(\mu/b)}} f_\theta\left(-\frac{\ell n u}{b} + \frac{\ell n w}{b}\right)\right.$$

$$\left. + \frac{1}{w} \int_0^{\frac{1-y}{2}} \frac{du}{(1-u)^{1+(\mu/b)}} f_\theta\left(-\frac{\ell n u}{b} + \frac{\ell n w}{b}\right)\right] -$$

$$- \frac{\mu c(\mu)}{2}\left[\int_0^1 \frac{dw}{(1-w)^{\mu/b}} f_\theta\left(-\frac{\ell nw}{b}\right)\right] \cdot$$

$$\cdot \int_{\frac{1-y}{2}}^{1} \frac{du}{1-u} f_\theta\left(-\frac{\ell nu}{b}\right) \cdot \left[\frac{y+1}{2u} + \frac{u-1}{u}\right]^{\frac{\mu}{b}} \tag{7}$$

In (7) $f_{\theta_1}(\cdot) = f_{\theta_2}(\cdot) = f_\theta(\cdot)$ was put. We will remind the reader here that the density $f_\theta(-(\ell nw)/b)$ tends in the limit towards a delta function. Hence it is nonzero for w in the very close vicinity of the unity. For such w, the integral

$$\int_0^w \frac{du}{(1-u)^{1+(\mu/b)}} f_\theta\left(-\frac{\ell nu}{b} + \frac{\ell nw}{b}\right)$$

is larger than the integral

$$\int_0^{\frac{1-y}{2}} \frac{du}{(1-u)^{1+(\mu/b)}} f_\theta\left(-\frac{\ell nu}{b} + \frac{\ell nw}{b}\right)$$

for all y's except for $y = -1$ in which case the two integrals approach each other. So, for w's still very close to unity, the expression

$$\frac{1}{1-w}\int_0^w \frac{du}{(1-u)^{1+(\mu/b)}} f_\theta\left(-\frac{\ell nu}{b} + \frac{\ell nw}{b}\right)$$

being multiplied by $1/1-w$ which approaches infinity is always much larger than

$$\frac{1}{w}\int_0^{\frac{1-y}{2}} \frac{du}{(1-u)^{1+(\mu/b)}} f_\theta\left(-\frac{\ell nu}{b} + \frac{\ell nw}{b}\right).$$

Hence, for w's close to one, which is the only case counting for integrands of the form

$$\int_0^1 dw\ f_\theta\left(-\frac{\ell n w}{b}\right)\cdot\ g(w)$$

we have that:

$$\frac{1}{1-w}\int_0^w \frac{du}{(1-u)^{1+(\mu/b)}}\ f_\theta\left(-\frac{\ell n u}{b}+\frac{\ell n w}{b}\right) \gg$$

$$\gg \frac{1}{w}\int_0^{\frac{1-y}{2}} \frac{du}{(1-u)^{1+(\mu/b)}}\ f_\theta\left(-\frac{\ell n u}{b}+\frac{\ell n w}{b}\right)$$

and of course,

$$\lim_{m\to\infty}\int_0^1 dw\ f_\theta\left(-\frac{\ell n w}{b}\right)\left[\frac{1}{1-w}\int_0^w \frac{du}{(1-u)^{1+(\mu/b)}}\ f_\theta\left(-\frac{\ell n u}{b}+\frac{\ell n w}{b}\right)\right.$$

$$\left.+\frac{1}{w}\int_0^{\frac{1-y}{2}} \frac{du}{(1-u)^{1+(\mu/b)}}\ f_\theta\left(-\frac{\ell n u}{b}+\frac{\ell n w}{b}\right)\right] \approx$$

$$\approx \int_0^1 dw\ f_\theta\left(-\frac{\ell n w}{b}\right)\cdot\ \frac{1}{1-w}\int_0^w \frac{du}{(1-u)^{1+(\mu/b)}}\ f_\theta\left(-\frac{\ell n u}{b}+\frac{\ell n w}{b}\right)$$

$$= K_1(\mu,b) \tag{8}$$

Applying (8) to (7) one gets

$$\lim_{m\to\infty} f_{y(0)}(y,w(0)<m) = \frac{\mu c(\mu)}{2^{1+(\mu/b)}}\ K_1(\mu,b)\cdot(1+y)^{\frac{\mu}{b}}$$

$$-\ \frac{\mu c(\mu)}{2}\left[\int_0^1 \frac{dw}{(1-w)^{\mu/b}}\ f_\theta\left(-\frac{\ell n w}{b}\right)\right]\cdot$$

$$\int_{\frac{1-y}{2}}^1 \frac{du}{1-u}\ f_\theta\left(-\frac{\ell n u}{b}\right)\left[\frac{y+1}{2u}+\frac{u-1}{u}\right]^{\frac{\mu}{b}} \tag{9}$$

We will now concentrate on the integral

$$\varphi(y) = \int_{\frac{1-y}{2}}^{1} \frac{du}{1-u} f_\theta\left(-\frac{\ell nu}{b}\right)\left[\frac{y+1}{2u} + \frac{u-1}{u}\right]^{\frac{\mu}{b}} =$$

$$= \int_{\frac{1-y}{2}}^{1} \frac{du}{1-u} \left[1 - \frac{1-y}{2u}\right]^{\frac{\mu}{b}} f_\theta\left(-\frac{\ell nu}{b}\right) \qquad (10)$$

From (10) we can easily find

$$\varphi'(y) = \frac{\mu}{2b} \int_{\frac{1-y}{2}}^{1} \frac{du}{u(1-u)} \left[1 - \frac{1-y}{2u}\right]^{-1+(\mu/b)} f_\theta\left(-\frac{\ell nu}{b}\right) \qquad (11)$$

At the same time, if we call

$$g(y) = (1+y)^{\frac{\mu}{b}} \qquad (12)$$

we have

$$g'(y) = \frac{\mu}{b}(1+y)^{-1+(\mu/b)} \qquad (13)$$

Comparison of (11) and (13) shows that while both derivatives $\varphi'(y)$ and $g'(y)$ are positive for every y (increasing functions $\varphi(y)$ and $g(y)$), they are approaching each other towards zero for y's close to $+1$. For y's smaller than $-1+(\mu/b)$, $g'(y)$ increases to a value larger than one and approaching infinity as $y \to -1$. For similar y values, the derivative $\varphi'(y)$ remains finite approaching zero for $y \to -1$. Thus one can write from (9)

$$\lim_{m\to\infty} f_{y(0)}(y, w(0) < m) = K_2(\mu,b)\cdot(1+y)^{\frac{\mu}{b}} + K_3(\mu,b) \qquad (14)$$

where

$$K_2(\mu,b) = \frac{\mu c(\mu)}{2^{1+(\mu/b)}} K_1(\mu,b) \tag{15}$$

$$K_3(\mu,b) = -\frac{\mu c(\mu)}{2}\left[\int_0^1 \frac{dw}{(1-w)^{\mu/b}} f_\theta\left(-\frac{\ell n w}{b}\right)\right] \cdot$$

$$\cdot \int_{\frac{1-y}{2}}^{1} \frac{du}{1-u} f_\theta\left(-\frac{\ell n u}{b}\right)\left[\frac{y+1}{2u} + \frac{u-1}{u}\right]^{\frac{\mu}{b}} \tag{16}$$

APPENDIX 2

Using expression (21) in section 2, we will now express the probability:

$$\lim_{m\to\infty} \Pr\{y(0) \leq y,\ w(0) \geq m\}$$

Indeed, we have:

$$\lim_{m\to\infty} \Pr\{y(0) \leq y,\ w(0) \geq m\} =$$

$$= \Pr\left\{1 - 2e^{-bt_{12}}\left[1 + e^{-b\rho_1}\left(1 - e^{b\theta_1}\left[1 + e^{-b\tau_1}\right]\right)\right] \leq y\right\}$$

$$= \int_0^1 f_{t_{12}}\left(-\frac{\ell n x}{b}\right) \Pr\left\{e^{-b\rho_1}\left(1 - e^{b\theta_1}\left[1 + e^{-b\tau_1}\right]\right) \geq \frac{1-y}{2x} - 1\right\}dx$$

$$= \int_0^1 f_{t_{12}}\left(-\frac{\ell n x}{b}\right) dx \int_0^1 f_{\rho_1}\left(-\frac{\ell n n}{b}\right)du \Pr\left\{e^{b\theta_1}\left[1 + e^{-b\tau_1}\right] \leq \right.$$

$$\left. \leq 1 - \frac{1}{u}\left[\frac{1-y}{2x} - 1\right]\right\}$$

$$= \int_0^1 f_{t_{12}}\left(-\frac{\ell n x}{b}\right) dx \int_0^1 f_{\rho_1}\left(-\frac{\ell n n}{b}\right)du \int_1^\infty f_\theta\left(\frac{\ell n w}{b}\right) dw \cdot$$

$$\cdot \Pr\left\{e^{-b\tau_1} \leq \frac{1}{w}\left[1 - \frac{1}{u}\left[\frac{1-y}{2x} - 1\right]\right] - 1\right\}$$

$$= \int_0^1 f_{t_{12}}\left(-\frac{\ell n x}{b}\right) dx \int_0^1 f_{\rho_1}\left(-\frac{\ell n u}{b}\right)du \int_0^1 f_\theta\left(-\frac{\ell n w}{b}\right) \cdot \frac{dw}{w^2} \cdot$$

$$\cdot \Pr\left\{e^{-b\tau_1} \leq w\left[1 - \frac{1}{u}\left(\frac{1-y}{2x} - 1\right)\right] - 1\right\} \qquad (1)$$

Here, we have as in appendix 1:

$$\Pr\left\{e^{-b\tau_1} \leq w\left[1 - \frac{1}{u}\left(\frac{1-y}{2x} - 1\right)\right] - 1\right\} =$$

$$= \begin{cases} 0 & , w\left[1 - \frac{1}{u}\left(\frac{1-y}{2x} - 1\right)\right] - 1 < 0 \\ 1 - F_{\tau_1}\left(-\frac{1}{b}\ln\left[w\left(1 - \frac{1}{u}\left[\frac{1-y}{2x} - 1\right]\right) - 1\right]\right) & , 0 \leq w\left[1 - \frac{1}{u}\left(\frac{1-y}{2x} - 1\right)\right] - 1 \leq 1 \\ 1 & , w\left[1 - \frac{1}{u}\left(\frac{1-y}{2x} - 1\right)\right] - 1 > 1 \end{cases} \quad (2)$$

Equations (2) are equivalent to:

$$\Pr\left\{e^{-b\tau_1} \leq w\left[1 - \frac{1}{u}\left(\frac{1-y}{2x} - 1\right)\right] - 1\right\} =$$

$$= \begin{cases} 0 & , u > \frac{w}{1-w}\left[1 + \frac{y-1}{2x}\right] \\ \left[w\left(1 - \frac{1}{u}\left[\frac{1-y}{2x} - 1\right]\right) - 1\right]^{\frac{\mu}{b}} & , x > \frac{1-y}{2} \text{ and } \frac{w}{2-w}\left[1 + \frac{y-1}{2x}\right] \leq u \leq \frac{w}{1-w}\left[1 + \frac{y-1}{2x}\right] \\ 1 & , x > \frac{1-y}{2} \text{ and } u < \frac{w}{2-w}\left[1 + \frac{y-1}{2x}\right] \end{cases} \quad (3)$$

Applying (3) to (1), one gets:

$$\lim_{m\to\infty} \Pr\{y(0) \leq y,\ w(0) > m\}$$

$$= \int_0^1 \frac{f_\theta\left(-\frac{\ln w}{b}\right)}{w^2} dw \int_{\frac{1-y}{2}}^1 f_{t_{12}}\left(-\frac{\ln x}{b}\right) dx \int_{\frac{w}{2-w}\left[1 + \frac{y-1}{2x}\right]}^{\min\left(1, \frac{w}{1-w}\left[1 + \frac{y-1}{2x}\right]\right)} \left[w\left(1 - \frac{1}{u}\left[\frac{1-y}{2x} - 1\right]\right) - 1\right]^{\frac{\mu}{b}} \cdot$$

$$\cdot f_{\rho_1}\left(-\frac{\ln u}{b}\right) du$$

$$+ \int_0^1 \frac{f_\theta\left(-\frac{\ln w}{b}\right)}{w^2} dw \int_{\frac{1-y}{2}}^1 f_{t_{12}}\left(-\frac{\ln x}{b}\right) dx \int_0^{\frac{w}{2-w}\left[1 + \frac{y-1}{2x}\right]} f_{\rho_1}\left(-\frac{\ln u}{b}\right) du \quad (4)$$

Application of density expressions from section 3 to (4) leads to the following expression:

$$\lim_{m\to\infty} \Pr\{y(0) \le y,\ w(0) \ge m\}$$

$$= \frac{-\mu}{1+(\mu/b)} \int_0^1 \frac{f_\theta\left(-\frac{\ell n w}{b}\right)}{w^2(1-w)}\, dw \int_{\frac{1-y}{2}}^1 f_{t_{12}}\left(-\frac{\ell n x}{b}\right) dx \cdot$$

$$\cdot \left[\left[w\left(1-\frac{1-y}{2x}\right) - (1-w)\min\left(1, \frac{w}{1-w}\left[1-\frac{1-y}{2x}\right]\right)\right]^{1+\frac{\mu}{b}} - \right.$$

$$\left. - \left[\frac{w}{2-w} - \frac{w}{2-w}\,\frac{1-y}{2x}\right]^{1+\frac{\mu}{b}}\right]$$

$$+ \frac{\mu}{1+(\mu/b)} \int_0^1 \frac{f_\theta\left(-\frac{\ell n w}{b}\right)}{w^2}\, dw \int_{\frac{1-y}{2}}^1 f_{t_{12}}\left(-\frac{\ell n x}{b}\right) dx \cdot \left[\frac{w}{2-w}\right]^{1+\frac{\mu}{b}} \left[1+\frac{y-1}{2x}\right]^{1+\frac{\mu}{b}} \qquad (5)$$

From (5) and by simple manipulation, one gets

$$\lim_{m\to\infty} \Pr\{y(0) \le y,\ w(0) \ge m\}$$

$$= \frac{\mu}{1+(\mu/b)} \left[\int_0^1 \frac{f_\theta\left(-\frac{\ell n w}{b}\right)}{w^{1-(\mu/b)}(1-w)(2-w)^{\mu/b}}\, dw\right] \cdot \int_{\frac{1-y}{2}}^1 \left[1-\frac{1-y}{2x}\right]^{1+\frac{\mu}{b}} f_{t_{12}}\left(-\frac{\ell n x}{b}\right) dx \qquad (6)$$

By differentiating (6), one gets the density in the limit:

$$\lim_{m\to\infty} f_{y(0)}(y, w(0) \ge m) =$$

$$= \frac{\mu}{2} \left[\int_0^1 \frac{f_\theta\left(-\frac{\ell n w}{b}\right)}{w^{1-(\mu/b)}(1-w)(2-w)^{\mu/b}}\, dw\right] \int_{\frac{1-y}{2}}^1 \left[1-\frac{1-y}{2x}\right]^{\frac{\mu}{b}} \frac{f_{t_{12}}\left(-\frac{\ell n x}{b}\right)}{x}\, dx \qquad (7)$$

The only y function appearing in (7) is the function:

$$h(y) = \int_{\frac{1-y}{2}}^{1} \left[1 - \frac{1-y}{2x}\right]^{\frac{\mu}{b}} \frac{f_{t_{12}}\left(-\frac{\ell nx}{b}\right)}{x} dx \tag{8}$$

with derivative

$$\lim_{m\to\infty} h'(y) = \int_{\frac{1-y}{2}}^{1} \frac{f_{t_{12}}\left(-\frac{\ell nx}{b}\right)}{x\left[x - \frac{1-y}{2}\right]} dx \tag{9}$$

The function $h'(y)$ is always finite, while the derivative $g'(y)$ of the function $g(y) = (1 + y)^{\mu/b}$ tends to infinity for y close to -1. For y's not close to -1, both $g(y)$ and $h(y)$ are practically not changing with y. Hence, in the presence of a term including the function $g(y) = (1 + y)^{\mu/b}$, the density $f_{y(0)}(y,w(0) \geq m)$ in (7) will be considered a constant with respect to y. Then the following notation will be used:

$$\lim_{m\to\infty} f_{y(0)}(y,w(0) \geq m) = K_4(\mu,b) \tag{10}$$

where

$$K_4(\mu,b) = \lim_{m\to\infty} \frac{\mu}{2} \left[\int_0^1 \frac{f_\theta\left(-\frac{\ell nw}{b}\right)}{w^{1-(\mu/b)}(1-w)(2-w)^{\mu/b}} dw\right] \cdot$$

$$\cdot \int_{\frac{1-y}{2}}^{1} \left[1 - \frac{1-y}{2x}\right]^{\mu/b} \frac{f_{t_{12}}\left(-\frac{\ell nx}{b}\right)}{x} dx \tag{11}$$