THE EFFECT OF INTERSYMBOL INTERFERENCE ON THE PERFORMANCE OF A DIGITAL FM SYSTEM

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ABSTRACT

The error performance of a digital FM system is studied in the presence of additive Gaussian noise. The digital system considered is a conventional one employing a voltage-controlled oscillator as the modulator and a limiter-discriminator followed by a low-pass filter as the demodulator.

The notion of "clicks" introduced by S.O. Rice, as used by Mazo and Salz, is adopted. The approach taken here, however, is aimed at covering the more general (band-limited) cases where intersymbol interference effects cannot be neglected. Special consideration has been given to binary FSK systems. For such systems, the probability of error is given in a closed form equation.

1. INTRODUCTION

The FM receivers have been the subject of many investigations for many years. At first such studies were primarily concerned with the handling of analog signals for best signal-to-noise (S/N) performance. The criterion of (S/N) transfer is not satisfactory, however, in digital data transmission. A better measure of performance in such channels is the probability of making wrong decisions at the receiver end of the system (called the probability of errors, or P_e). This probability cannot be directly related to their (S/N) ratio in nonlinear systems.

To handle this problem, the notion of "clicks" was first introduced by

S.O. Rice [2]. It was then observed that when the noise input to the FM receiver exceeds some value, the (S/N) value measured at the output of the receiver is much poorer than what was predicted by linearized analysis. Rice related this phenomenon to the expected number of clicks per second at the output.

Bennet and Salz [3] analyzed binary FM systems taking into consideration the distortion effects due to band limiting, but neglected the influence of the post detection filter. As will be shown in what follows, this filter strongly influences the proper selection of signals for best performance.

More recently, Mazo and Salz [1] using the notion of "clicks" [2] developed a theory for predicting the performance of digital FM systems with the post-detection filter included. No consideration was given, however, to the distortions produced by band limiting the FM wave. A general procedure towards the prediction of the performance of a digital FM system is hereby presented with the post-detection filter and the distortions due to band limitation taken into consideration.

2. THE SYSTEM

The block diagram of the digital FM system we shall investigate is shown in Fig. 1 and is similar to the one analyzed by Mazo and Salz in [1]. In what follows, we reiterate and use some of the notations and results given there, for completeness.

The data source (block 1) is a digital source producing one digit every T seconds. Each such digit can assume one out of N discrete values with no dependence on the values of past or future digits.

The pulse shaper (block 2) starts producing a rectangular pulse of duration T the moment the data source generates one digit. The height of this pulse depends on the value of the digit generated.

If $g(t)$ is the function representing the rectangular signaling pulse of unit amplitude and a_n is the value of the digit generated by the data source* at the time $(n-1)T$ then the input to the FM modulator (block 3) at time t (the "information bearing" signal [1]) is given by

$$s(t) = \omega_d \sum_{n=0}^{\infty} a_n g(t-(n-1)T) \tag{1}$$

where ω_d is a constant with frequency dimensions (that relates amplitudes to frequency shifts).

The FM modulator shifts its carrier frequency ω_c under the control of $s(t)$, its output $S(t)$ being given by the following equation

$$S(t) = A \cos(\omega_c t + \int_0^t s(\tau)d\tau + \theta) \tag{2}$$

where A is the constant amplitude of the FM wave and θ is the initial phase of the modulator.

The bandpass filter #1 (block 4) is added, in practice, for eliminating unnecessary waste in transmission power, since the channel can only utilize a limited portion of the FM spectral band. The noise interfering with the channel is assumed to be additive and Gaussian.

The bandpass filter #2 (block 5) is necessary for band limiting the spectrum of the noise that reaches the receiver. The signal finally reaches the limiter-discriminator (blocks 6 and 7) and is filtered at baseband (block 8)

* Assuming the data source starts transmitting at time zero.

before it is synchronously sampled at $t = nT$ for detecting the digit a_n (block 9).

3. THE MATHEMATICAL MODEL

Let the (noiseless, but bandlimited) signal at the input to the envelope-limiter, $S_1(t)$, be the sum of two processes $X(t)$ and $Y(t)$ that will be determined later, i.e.,

$$S_1(t) = X(t) \cos \omega_c t - Y(t) \sin \omega_c t \tag{3}$$

Let the noise at the FM demodulator input be given by

$$n(t) = a(t) \cos \omega_c t - b(t) \sin \omega_c t \tag{4}$$

where $a(t)$ and $b(t)$ are both considered Gaussian processes with zero mean and σ_o^2 variance and independent from each other. This additive Gaussian noise is band limited by filter #2 (Fig. 1) which is assumed to be sharply band-pass with its bands centered at the $\pm \omega_c$ frequencies.

We define now two new processes $x(t)$ and $y(t)$ by

$$\begin{aligned} x(t) &= X(t) + a(t) \\ y(t) &= Y(t) + b(t) \end{aligned} \tag{5}$$

The input to the envelope limiter after the noise has been added is now

$$S_2(t) = (x(t),y(t)) \tag{7a}$$

which, using the following change in variables

$$\rho(t) = \sqrt{x^2(t) + y^2(t)} \tag{8a}$$

$$\varphi(t) = \tan^{-1}[y(t)/x(t)] = \tan^{-1}[r(t)] \tag{8b}$$

$$r(t) = [y(t)/x(t)] \tag{8c}$$

becomes:

$$S_2(t) = \rho(t) \cos [\omega_c t + \varphi(t)] \tag{7b}$$

The FM discriminator (demodulator) output then equals

$$\dot{\varphi}(t) = \frac{x(t)\cdot\dot{y}(t) - y(t)\cdot\dot{x}(t)}{x^2(t) + y^2(t)}$$

where a dotted symbol represents time derivative.

The low-pass post-detection filter is usually approximated in the literature [1] by an ideal integrator with unity impulse response for T seconds and zero afterwards. To avoid intersymbol interference, where the distortions in the FM wave due to band-limitations are ignored [1], the period of integration is chosen equal to the pulse duration.

We are using the same approach here by considering the source post-detection filter. The quadrature signal components, however, are assumed to be distorted by the band limitation effects, producing the distorted FM waveform $S_1(t)$ given in (3).

No strict restrictions are imposed here on the band-pass filter #1. When their effects on the FM waveform is investigated, the two band-pass filters #1 and #2 are combined into a single equivalent -- one.

The output of the post-detection filter at time nT can now be calculated from

$$q_n = \int_{r[(n-1)T]}^{r(nT)} \dot{\varphi}(t)dt = \int_{r[(n-1)T]}^{r(nT)} \frac{dr(t)}{1 + r^2(t)} \tag{10}$$

with $\varphi(t)$ and $r(t)$ as given in (8b) and (8c) respectively. The decision about the value of the information digit a_n will be based on this, q_n , output.

The value of φ in (10) is not evaluated using some fixed branch of $\tan^{-1} y/x$. The point representing the noisy processes through $x(t),y(t)$ wanders about the xy-plane encircling the origin and occasionally taking large excursions. The value of q_n depends, therefore, on the path chosen by $\rho(t)$ in the x,y-plane. These paths are random, hence q_n is random with statistics related to the statistics of $r(t)$. Clearly, then, the distribution of q_n does not depend only on the elementary statistics of $r(t)$, it also depends on the statistics of the singularities of $r(t)$. It has been shown that q_n may be expressed as follows [1]

$$q_n = \tan^{-4} r(nT) - \tan^{-1} r[(n-1)T] + N(T)\pi \tag{11}$$

Here $\tan^{-1} x$ has its value derived from the principal branch of the function and $n(T)$ is an integer, related to the number of times $x(t)$ vanishes in the interval T , its sign being related to that of $y(t)$ when $x(t)$ vanishes.

The distribution of $n(T)$ has been thoroughly studied by S.O. Rice [2] and related by him to the FM "clicks." The nature of those "clicks" has been given an interesting interpretation by Mazo and Salz in [1].

We now attempt an analysis of the exact probability distribution of q_n from which the probability of an erroneous detection at the output of the system will be derived. Following Mazo and Salz [1], we first assume that the random variables involved in (11) above are all independent from each other. For a flat Gaussian noise, the assumption that $\tan^{-1} r(nT)$ and $\tan^{-1} r[(n-1)T]$ are independent provides a good approximation of $T \geq 1/W$ with W the noise bandwidth.

The independence assumption on $n(T)$ and between $W(T)$ and $\tan^{-1}(nT)$, $\tan^{-1}[(n-1)T]$ is expressing the intuitive feeling that the "clicks" occur rarely and are of short duration. In general, this is the case if the signal-to-noise ratio is large, which will be one of the main assumptions in our analysis also.

We search now for the distributions of the three independent variables in (11).

4. THE INITIAL DISTRIBUTIONS

To find the distribution of $\tan^{-1} r(nT)$ and $\tan^{-1} r(n-1)T$ we first compute $\tan^{-1} r(T)$ for arbitrary t with

$$r(t) = \frac{y(t)}{x(t)}$$

and $y(t)$, $x(t)$ independent Gaussian processes with means $Y(t)$, $X(t)$ respectively and some variance σ_o^2.

The distorted signal considered is as given in (3) and we assume it has passed an (equivalent) bandpass filter that combines the effects of those given in blocks 4 and 5 of Fig. 1. Also (as already mentioned), only the second of the above filters (#2, block 5) is assumed to be sharply band limiting the signal. As shown in the following section, when considering intersymbol interference, $Y(t)$ and $X(t)$ are, in general, functions of <u>all</u> the information digits generated by the data source up to time t. Hence, for a given sequence of information digits that need be transmitted, the $y(t)$ and $x(t)$ processes are Gaussian with $Y(t)$ and $X(t)$ as their means respectively.

The joint density $f_{x(t),y(t)}(u,v)$ of the two processes $y(t)$ and $x(t)$, under the conditions mentioned above, is given by the following equation:

$$f_{x(t),y(t)}(u,v) = f_{x(t)}(u)\cdot f_{y(t)}(v) =$$

$$= \frac{1}{2\pi\sigma_o^2} \exp\left\{-\frac{1}{2\sigma_o^2}\left[(u-X(t))^2 + (v-Y(t))^2\right]\right\} \quad (12)$$

By using variables $\rho(t),\varphi(t)$ as defined in (8a) and (8b), we reach a different expression for the same joint density, as follows:

Let $\quad x(t) = \rho(t) \cos \varphi(t) \quad (13)$

$\quad y(t) = \rho(t) \sin \varphi(t)$

Then

$$f_{x(t),y(t)}(u,v)dudv = f_{x(t),y(t)}[\rho \cos \varphi(t), \rho \sin \varphi(t)]\rho d\rho d\varphi$$

$$\triangleq f_{\rho(t),\varphi(t)}(\rho,\varphi)d\rho d\varphi \quad (15)$$

Directly from relationships (15) and (12) one gets:

$$f_{\rho(t),\varphi(t)}(\rho,\varphi) = \rho\cdot f_{x(t),y(t)}(\rho \cos \varphi, \rho \sin \varphi)$$

$$= \frac{\rho}{2\pi\sigma_o^2} \exp\left\{-\frac{1}{2\sigma_o^2}\left[\rho^2 - 2\rho(X(t) \cos \varphi + Y(t) \sin \varphi) + (X^2(t) + Y^2(t))\right]\right\} \quad (16)$$

The variable $\rho(t)$ takes values in the range $[0,\infty]$ and the variable $\varphi(t)$ in the range $[-\pi,\pi]$. Integrating expression (16) with respect to ρ in its whole range, we get the marginal density of the variable $\varphi(t)$ as follows:

$$f_{\varphi(t)}(\varphi) = \int_0^\infty f_{\rho(t),\varphi(t)}(\rho,\varphi)d\rho =$$

$$= \frac{1}{2\pi} \exp\left\{- \frac{1}{2\sigma_o^2} [X^2(t) + Y^2(t)]\right\} + \frac{X(t) \cos \varphi + Y(t) \sin \varphi}{\sqrt{2\pi}\, \sigma_o} *$$

$$* \exp\left\{- \frac{1}{2\sigma_o^2} [X(t) \sin \varphi - Y(t) \cos \varphi]^2\right\} * \left[1 - \Phi\left(\frac{-[X(t) \cos \varphi + Y(t) \sin \varphi)}{\sigma_o}\right)\right] \tag{17}$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(- \frac{u^2}{2}\right) du$$

If we want the angle $\varphi(t)$ to vary over the range $[- \frac{\pi}{2}, \frac{\pi}{2}]$, we will have a new density $f^1_{\varphi(t)}(\varphi)$ given by the density $f_{\varphi(t)}(\varphi)$ in (17) through the following transformation

$$f^1_{\varphi(t)}(\varphi) = f_{\varphi(t)}(\varphi) + f_{\varphi(t)}(\varphi + \pi) \tag{18}$$

From (17) and (18) we conclude that

$$f^1_{\varphi(t)}(\varphi) = \frac{1}{\pi} \exp\left\{- \frac{1}{2\sigma_o^2} [X^2(t) + Y^2(t)]\right\}$$

$$+ \frac{X(t) \cos \varphi + Y(t) \sin \varphi}{\sqrt{2\pi}\, \sigma_o} \cdot \exp\left\{- \frac{1}{2\sigma_o^2} [X(t) \sin \varphi - Y(t) \cos \varphi]^2\right\}$$

$$* \operatorname{erf}\left(\frac{X(t) \cos \varphi + Y(t) \sin \varphi}{\sqrt{2}\, \sigma_o}\right) \tag{19}$$

Expression (19) can be simplified if the following notations are introduced:

$$R(t) = \frac{X^2(t) + Y^2(t)}{2\sigma_o^2} \tag{20}$$

$$A(t) = \tan^{-1}\left(\frac{Y(t)}{X(t)}\right) \tag{21}$$

where $A(t)$ has the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Equation (19) can then be written as follows:

$$f^1_{\varphi(t)}(\varphi) = \frac{1}{\pi}\exp\{-R(t)\} + \sqrt{\frac{R(t)}{\pi}}\cos(\varphi - A(t)) *$$

$$* \exp\{-R(t)\sin^2(\varphi - A(t))\} \cdot \mathrm{erf}\,(\sqrt{R(t)} \cdot (\varphi - A(t))) \tag{22}$$

Expression (22) can also be written in the following way:

$$f^1_{\varphi(t)}[\varphi + A(t)] = \frac{1}{\pi}\exp\{-R(t)\} + \sqrt{\frac{R(t)}{\pi}}\cos\varphi$$

$$* \exp\{-R(t)\sin^2\varphi\} \cdot [\mathrm{erf}(\sqrt{R(t)}\cos\varphi)] \tag{23}$$

This equation (except for a few changes in notation) is looking almost the same as equation (15) in reference [1]. The additional angle $A(t)$ introduced here is the only significant change from the above and it includes the true value of the information digits appearing in the history of the system.

In considering the results of our analysis it turns out, however, that this small formal change has important practical implications. Its influence on the results is expressed by the shift it is calling for in the p.d.f. for any given angle φ. The magnitude of this shift is representative of the level of intersymbol interference that distorted the modulating signal.

Returning now to the value of q_n (equation (11)), we would like the difference $\tan^{-1}r(nT) - \tan^{-1}r[(n-1)T]$ to vary over the principal range $(-\pi, +\pi)$. Each of the variables $[\tan^{-1}r(nT) - A(nT)]$ and $[\tan^{-1}r((n-1)T) - A(n-1)T]$ will therefore have, in our case, to vary over the range $[-\pi/2, \pi/2]$ and their density will be given by $f^1_{\varphi(nT)}$ and $f^1_{\varphi((n-1)T)}$ respectively.

Assuming the two variables $\tan^{-1} r(nT)$ and $\tan^{-1} r((n-1)T)$ are independent, the probability, P_Φ, of their difference being larger than some angle Φ is given by the expression

$$P_\Phi \triangleq P_r(\tan^{-1} r(nT) - \tan^{-1} r((n-1)T) > \Phi) \qquad \text{for } \Phi - \lambda_n > 0$$

which can be written as follows:

$$P_\Phi = P_r([\tan^{-1} r(nT) - A(nT)] - [\tan^{-1} r((n-1)T) - A((n-1)T)] >$$

$$> \Phi - [A(nT) - A((n-1)T)]) \tag{24a}$$

which also leads to the following expression

$$P_\Phi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2} - \Phi + [A(nT) - A((n-1)T)]} d\varphi_2 \times$$

$$\times \int_{\varphi_2 + \Phi - [A(nT) - A((n-1)T)]}^{\frac{\pi}{2}} d\varphi_1 f^1_{\varphi(nT)}(\varphi_1 + A(nT)) \cdot f^1_{\varphi(n-1)T}(\varphi_2 + A(n-1)T)) \tag{24b}$$

Note that the values of $A(nT)$ and $A((n-1)T)$ could be considered constant for a given (distorting) "history" and also that in (24) we assumed $|\Phi - [A(nT) - A(n-1)T)]| \leq \pi$. The calculation of (24) in closed form will be the subject of the next section.

The final item that we will discuss in the present section is the density of $N(T)$ in expression (11). That density, as already mentioned, is connected to the concept of "clicks." We now follow again the results obtained by Mazo and Salz based on this notion of "clicks."

Assuming that the process $R(t)$ in (20) varies slowly with time (which it does in most cases, as will be shown in the following section), we are concerned with the type of modulation in which the instantaneous frequency deviates by ω_d from the carrier for a time T. The number of clicks N_+,(N_-) that tend to increase (decrease) the value of q_n in equation (11) by 2π, are then "asymptotically" [1] equal to:

$$\begin{aligned} N_+ &\sim 0 \\ N_- &\sim f_d \exp(-R(t)) \end{aligned} \quad \text{for} \quad \frac{\omega_d}{2\pi} = f_d > 0 \text{ and } R(t) \text{ large} \tag{25}$$

We will see later that a large value for $R(t)$ means a large signal-to-noise ratio at the input of the envelope limiter. It can also be shown [1] that by using (25) the following simple Poisson distribution for the number of clicks $N(T)$ can be obtained

$$P(N(T)) = \frac{[\exp\{-(N_-)T\}] \times [(N_-)T]^{N(T)}}{[N(T)]!} \tag{26}$$

Finally, the probability of getting K or more clicks can be considered to be approximately equal to the probability of getting exactly K clicks, when large signal-to-noise ratios, i.e., large $R(t)$, are involved. The above assumptions now permit us to proceed to the following analysis.

5. THE PROBABILITY OF A DETECTION ERROR

If, as we already assumed, the process $R(t)$ in (20) changes slowly in time, then we can also assume with very good approximation that

$$R(nT) = R((n-1)T) = R_n \tag{27}$$

If this is the case, then, from expressions (22) one can write:

$$f_n^1(\varphi) = f^1_{\varphi(nT)}(\varphi + A(nT)) = f^1_{\varphi((n-1)T)}(\varphi + A((n-1)T) =$$

$$= \frac{1}{\pi} \exp\{-R_n\} + \sqrt{\frac{R_n}{\pi}} \cos\varphi * \exp(-R_n \sin^2\varphi) * \text{erf}(\sqrt{R_n}\cdot\cos\varphi) \tag{28}$$

Let

$$A[nT] - A[(n-1)T] \triangleq \lambda_n \tag{29a}$$

and

$$\varphi_n = \varphi_1 - \varphi_2 \triangleq \tan^{-1} r(nT) - \tan^{-1} r[(n-1)T] \tag{29b}$$

Substituting (28) and (29) into equation (24b), one gets

$$P_\Phi = \int_{-\pi/2}^{\pi/2 - \Phi + \lambda_n} d\varphi_2 \int_{\varphi_2 + \Phi - \lambda_n}^{\pi/2} f_n^1(\varphi_1)\cdot f_n^1(\varphi_2)\, d\varphi_1 \tag{30}$$

Assuming R_n introduced in (27) to be very large (theoretically infinity) the probability function given by equation (30) could be expressed in three different ways. The expression to be used will depend on the range of values permitted for $\Phi - \lambda_n$. In the most general case we should allow this sum to vary in the range $(-\pi,\pi)$.

For symmetry reasons, clearly

$$P_r\{[\tan^{-1} r(nT) - \tan^{-1} r(n-1)T] < \Phi_o\} = 1 - P_{-\Phi_o+\lambda_n}$$

if $\Phi_o - \lambda_n < 0$. Hence by covering the probability of errors only for those cases where $\Phi - \lambda_n > 0$, the negative values of the sum are also covered.

Using again results given by Mazo and Salz in Ref. [1], one has:

i) If $0 < \Phi - \lambda_n < \pi/2$, then

$$P_\Phi \sim \frac{1}{\sqrt{8\pi}} \frac{\cot[(\Phi-\lambda_n)/2]}{\sqrt{\cos(\Phi-\lambda_n)}} \cdot \frac{\exp[-2R_n \sin^2(\Phi-\lambda_n)/2]}{\sqrt{R_n}} \tag{31}$$

ii) If $\Phi - \lambda_n = \pi/2$, then

$$P_\Phi \sim \frac{1}{4} \exp(-R_n) \tag{32}$$

iii) If $\Phi - \lambda_n > \pi/2$, then

$$P_\Phi \sim \frac{\exp[-R_n[1 + \cos^2(\Phi - \lambda_n)]]}{2\pi\sqrt{\pi}\, R_n \sqrt{R_n}\, \sin(\Phi - \lambda_n)\cdot\cos^2(\Phi - \lambda_n)} . \tag{33}$$

Recalling now that

$$\lambda_n = \tan^{-1}\frac{Y(nT)}{X(nT)} - \tan^{-1}\frac{Y(n-1)T}{X(n-1)T}$$

it is important to observe that λ_n represents a measure of the true value of the digit about to be detected (a_n) . Its exact value is therefore important and will be stated and studied in a following section.

We will now relate the expressions giving the probability P_Φ to the probability of false detection.

The detection of the digit a_n is clearly based on the output q_n of the sampler at the time $t = nT$. Had the clicks not existed, q_n would have had the distribution of φ_n and its range would have been $[-\pi + \lambda_n$ to $\pi + \lambda_n]$. The presence of the clicks causes q_n to extend theoretically from $-\infty$ to ∞. Lobes of decreasing area are added to the main lobe in the density graph as shown in Fig. 2. For large signal-to-noise ratios though, the only side lobe that has an area comparable to the main one is the first side lobe. Its area (see expressions (25) and (26)) equals:

$$A_L = f_d T \cdot \exp(-R_n) \tag{34}$$

Comparing the error contribution of the clicks (given by (34)) to the error expressions caused by the variables $\tan^{-1} r(nT)$ and $\tan^{-1} r((n-1)T)$ (given by (31), (32), (33)), one observes that the clicks contribute considerably only in the case where

$$\Phi - \lambda_n = \pi/2$$

Suppose now that each information digit a_i can take on M values (M levels). The range $[-\pi,\pi]$ is, in this case, divided into M equal parts in such a way that the first value V_1 is given by:

$$\omega_d V_1 = -\pi + \pi/M$$

The last value V_M is given by

$$\omega_d V_M = \pi - \pi/M$$

and the distance between consecutive values is equal to $2\pi/M$.

The probability of making an error when the clicks are ignored corresponds now to

$$V_k = -\pi + \pi/M + (k-1)(2\pi/M) \qquad K < M$$

$$= -\pi + (2k-1)(\pi/M) \tag{35a}$$

Letting now

$$V_k + \pi/M = V_k^1 \;\; ; \; V_k - \pi/M = V_k^2 \tag{35b}$$

This probability of making an error will be

$$P_e = \int_{\lambda_n : V_k^1 - \lambda_n > 0} [1 - P_{V_k^1}(\lambda_n) d\lambda_n] + \int_{\lambda_n : -V_k^1 + \lambda_n > 0} P_{-V_k^1 + 2\lambda_n}(\lambda_n) d\lambda_n$$

$$+ \int_{\lambda_n : V_k^2 - \lambda_n < 0} P_{V_k^2}(\lambda_n) d\lambda_n + \int_{\lambda_n : V_k^2 + \lambda_n < 0} [1 - P_{-V_k^2 + 2\lambda_n}(\lambda_n)] d\lambda_n \quad (36)$$

Obviously the above model with bit values distributed at equal distances is just one of the many possibilities.

Consider now the binary case (binary FSK). The two possible values of the information digits are usually chosen here in such a way that they are symmetric about zero but not necessarily equal to $\pm \pi/2$ when multiplied by ω_d. Indeed we may choose the following two arbitrary values

$$\varphi_o = \omega_d(+1)$$

$$-\varphi_o = \omega_d(-1)$$

In this case we can base our decision about a_n on the sign of q_n. That is

we decide $a_n = 1$, if $q_n \geq 0$

we decide $a_n = -1$, if $q_n < 0$

Hence, if the clicks are ignored, the probability of error in such a binary system is given by the following expression:

$$P_e = P_r\{q_n \geq 0/a_n = -1\} + P_r\{q_n < 0/a_n = 1\}$$

$$= \int_{-\lambda_n > 0} (\lambda_n/a_n = -1) d\lambda_n + \int_{-\lambda_n > 0} [1 - P_o(\lambda_n/a_n = +1)] d\lambda_n$$

$$+ + \int_{-\lambda_n < 0} [1 - P_{+2\lambda_n}(\lambda_n/a_n = -1)] d\lambda_n + \int_{-\lambda_n < 0} P_{+2\lambda_n}(\lambda_n/a_n = +1) d\lambda_n$$

(37)

If the result in (37) is comparable to the formula (34), the error from the clicks is added. Otherwise it is neglected.

In the following section we will see how λ_n changes with the actual sequence of information digits transmitted by the data source and how the distortions caused by band limitation are included in λ_n and in R_n.

If no distortions are assumed (i.e., if the signal at the input of the envelope limiter is taken to be the same as the FM modulated signal), then the processes X(t) and Y(t) defined in (4) are now given by the following expressions

$$X(t) = A \cos (\theta + \int_0^t s(\tau)d\tau) \tag{38a}$$

$$Y(t) = A \sin (\theta + \int_0^t s(\tau)d\tau) \tag{38b}$$

In this case we also have:

$$R(t) = \frac{X^2(t) + Y^2(t)}{2\sigma_o^2} = \frac{A^2}{2\sigma_o} = R_n = R \tag{39a}$$

$$A(t) = \tan^{-1} \frac{\sin(\theta + \int_o^t s(\tau)d\tau)}{\cos(\theta + \int_o^t s(\tau)d\tau)} = \theta + \int_o^t s(\tau)d\tau \tag{39b}$$

and from (39) we immediately derive

$$\lambda_n = \int_o^{nT} s(\tau)d\tau - \int_o^{(n-1)T} s(\tau)d\tau = \pm\omega_d T a_n \tag{40}$$

It is obvious from (40) and (39a) that R and λ_n are independent of the past information digits in this case. This could have been expected since no distortions in the FM modulated signal was assumed.

For a binary FSK system, the probability of error first expressed by (37) for the general case is given now by the following expression:

$$P_e = \begin{cases} \dfrac{1}{\sqrt{2\pi}} \dfrac{\cot[(\omega_d T)/2]}{\sqrt{\cos(\omega_d T)}} \cdot \dfrac{\exp\{-2R\sin^2(\omega_d T)/2\}}{\sqrt{R}} & , \text{ if } 0 < \omega_d T < \pi/2 \\ \frac{1}{2}\exp\{-R\} + \frac{1}{2}\exp\{-R\} & , \text{ if } \omega_d T = \pi/2 \\ \dfrac{\exp\{-R[1 + \cos^2\omega_d T]\}}{\pi\sqrt{\pi}\, R\sqrt{R}\, \sin(\omega_d T)\cos^2(\omega_d T)} & , \text{ if } \frac{\pi}{2} < \omega_d T \end{cases} \qquad (41)$$

From (41) we notice that the probability of falsely detecting the a_n digit is given by three different expressions. Which of those expressions is to be used will depend on where in the interval $[0,\pi]$ the angle $\omega_d T$ is chosen (i.e., depends on the original signal design). When the two signals are orthogonal ($\omega_d T = \pi/2$), the error from the clicks equals the one produced when the two signals are not distinguishable. This error should be added to the general expression. When the two signals are not orthogonal, however, the errors due to clicks are negligible.

6. THE DISTORTED SIGNAL COORDINATES

In the previous section, a general expression of the probability of error of the FM system in Fig. 1 was given. The signal at the input of the envelope limiter was arbitrary. Its coordinates $X(t)$ and $Y(t)$ include any possible distortion caused by filtering or other disturbances.

In this section the coordinates $X(t)$ and $Y(t)$ will be expressed explicitly when the distortion of the FM signal is caused by the combined band limitations from filters #1 and #2. The impulse response, $h(t)$, of the combined filter will be given by a sum of exponential responses, i.e.,

$$h(t) = \left[\sum_{k=1}^{Q} c_k e^{-b_k t}\right] \cos \omega_c t \tag{42}$$

Expression (42) is a typical representation of a rational band-pass filter with poles located at the points b_k of the complex plane and bands centralized at the frequencies $\pm \omega_c$.

When the FM modulated signal $S(t)$ in (2) passes through the filter described by (42) it takes on the following form:

$$\begin{aligned} S_o(t) &= \int_o^t h(t-\tau)S(\tau)d\tau \\ &= \sum_{k=1}^{Q} c_k S_k(t) \end{aligned} \tag{43}$$

where

$$S_k(t) = \cos \omega_c t \int_o^t e^{-b_k(t-\tau)} S(\tau)d\tau \tag{44}$$

The function $S(\tau)$ in (44) is the FM modulated signal given by (2) and after substitution one can find (appendix) the following expression of $S_k(t)$ at the sampling instants.

$$S_k(nT) = X_k(nT) \cos \omega_c nT - Y_k(nT) \sin \omega_c nT \tag{45}$$

where $X_k(nT)$ and $Y_k(nT)$ are given by the expressions (A7),(A8) of the appendix and are independent of the carrier frequency ω_c.

The equivalent coordinates $X(nT)$, $Y(nT)$ of the signal $S_o(t)$ are then given by the following expressions:

$$X(nT) = \sum_{k=1}^{Q} c_k X_k(nT) \tag{46a}$$

$$Y(nT) = \sum_{k=1}^{Q} c_k Y_k(nT) \tag{46b}$$

It is shown in the appendix that when the fact that the carrier frequency ω_c is much larger than the transmission frequency ω_d is taken into consideration, the following final forms for the coordinates $X(t)$ and $Y(t)$ are obtained

$$X(nT) = \sigma_o \sqrt{2R_{n-1}} \cos(\theta - \gamma_{n-1}) \tag{47a}$$

$$Y(nT) = \sigma_o \sqrt{2R_{n-1}} \sin(\theta - \gamma_{n-1}) \tag{47b}$$

where σ_o is the variance of the $a(t)$, $b(t)$ noise coordinates (as expressed in the introduction) and

$$R_{n-1} = \frac{A^2}{4} \frac{\{1\}^2 + \{2\}^2}{2\sigma_o^2} \tag{48}$$

$$\gamma_{n-1} = \tan^{-1} \frac{\{1\}}{\{2\}} \tag{49}$$

with

$$\{1\} = \sum_{\ell=1}^{n-1} (a_\ell - a_{\ell+1}) \cos(\omega_d T \sum_{i=0}^{\ell} a_i) * \sum_{k=1}^{Q} c_k \frac{\omega_d}{b_k^2 + \omega_d^2} e^{-b_k(n-\ell)T}$$

$$- \sin(\omega_d T \sum_{i=0}^{n} a_i) \sum_{k=1}^{Q} c_k b_k \left[\frac{1}{b_k^2 + \omega_d^2} + \frac{1}{b_k^2 + 4\omega_c^2}\right]$$

$$+ \cos(\omega_d T \sum_{i=0}^{n} a_i) \sum_{k=1}^{Q} c_k \left[\frac{\omega_d a_n}{b_k^2 + \omega_d^2} + \frac{2\omega_c}{b_k^2 + 4\omega_c^2}\right] \tag{50}$$

$$\{2\} = \sum_{\ell=1}^{n-1} (a_\ell - a_{\ell+1}) \sin(\omega_d T \sum_{i=0}^{\ell} a_i) \sum_{k=1}^{Q} c_k \frac{\omega_d}{b_k^2 + \omega_d^2} *$$

$$* e^{-b_k(n-\ell)T} + \cos(\omega_d T \sum_{i=0}^{n} a_i) * \sum_{k=1}^{Q} c_k b_k \left[\frac{1}{b_k^2 + \omega_d^2} + \frac{1}{b_k^2 + 4\omega_c^2}\right]$$

$$+ \sin(\omega_d T \sum_{i=0}^{n} a_i) \sum_{k=1}^{Q} c_k \left[\frac{\omega_d a_n}{b_k^2 + \omega_d^2} + \frac{2\omega_c}{b_k^2 + 4\omega_c^2}\right] \tag{51}$$

These last expressions clearly show how fast information digits influence the coordinates of the signal $X(nT)$, $Y(nT)$ and therefore the probability of a false detection. If the filter characteristics are such that R_{n-1} changes slowly with time, then it will be approximately equal to R_{n-2} and the error expressions in section 5 apply.

Also, the expression for λ_n, as given in section 5, becomes equal to $Y_{n-2} - Y_{n-1}$ and is therefore obviously independent of the initial phase θ. That was expected because of the differentiating effect of the discriminator (demodulator) in the system.

7. THE ERROR EXPRESSIONS FOR FAST CHANGING R(t)

The expressions (31-33) in section 5 are true only when the filter characteristics of the system are such that $R(t)$ is slowly varying. If the case is not such, though, the probability of error is a function of the values R_{n-1}, R_{n-2} of the process $R(t)$ at the edges of the sampling interval $[(n-1)T,nT]$. Indeed, application of the same methods that for large signal-to-noise ratio gave expressions (31-33), now gives the following modified results:

i) If $0 < \Phi - \lambda_n < \pi/2$

$$P_\Phi \sim \frac{1}{\sqrt{8\pi}} \exp\left\{-\frac{R_{n-1} + R_{n-2}}{2} + \sqrt{R_{n-1}^2 + R_{n-2}^2 + 2R_{n-1}R_{n-2}\cos(2\lambda_n + 2\Phi)}\right\}$$

$$* \frac{1}{\sqrt[4]{R_{n-1}^2 + R_{n-2}^2 + 2R_{n-1}R_{n-2}\cos(-2\lambda_n + 2\Phi)}}$$

$$* \frac{R_{n-1} + R_{n-2} + \sqrt{R_{n-1}^2 + R_{n-2}^2 + 2R_{n-1}R_{n-2}\cos(-2\lambda_n + 2\Phi)}}{R_{n-1} + R_{n-2} - \sqrt{R_{n-1}^2 + R_{n-2}^2 + 2R_{n-1}R_{n-2}\cos(-2\lambda_n + 2\Phi)}} \tag{52}$$

ii) If $\Phi - \lambda_n = \pi/2$

$$P_\Phi \sim \sqrt{R_{n-2}/R_{n-1}} \exp\left(-\frac{R_{n-1}}{4}\right) \tag{53}$$

iii) If $\Phi - \lambda_n > \pi/2$

$$P_\Phi \sim \frac{\exp(-[R_{n-1} + R_{n-2}])}{4\pi\sqrt{\pi}\,|\sin(\Phi - \lambda_n)|\cos^2(\Phi - \lambda_n)} *$$

$$* \left[\frac{\exp\{R_{n-1}\sin^2(\Phi - \lambda_n)\}}{R_{n-1}\sqrt{R_{n-1}}} + \frac{\exp\{R_{n-2}\sin^2(\Phi - \lambda_n)\}}{R_{n-2}\sqrt{R_{n-2}}}\right] \tag{54}$$

The error from the "clicks" is influenced, too, by the fast variation of the process $R(t)$. Indeed, this is still comparable to the probability of mixing the signals only in the case that $\Phi - \lambda_n = \pi/2$, but is now given by the following expression.

$$A_L = \frac{\omega_d T}{2\pi} \exp\left(-\frac{R_{n-1} + R_{n-2}}{2}\right) \tag{55}$$

8. CONCLUDING COMMENTS

From the discussion in the sections 5, 6, and 7, one easily concludes that the performance of the FM system studied in this paper is described by a complicated function of the filter parameters as well as the signaling frequency used. The design of the (rational) combined band-pass filter that will be part of the most efficient system needs a numerical search. This search will have to make use of the probability and signal expressions given in sections 5, 6, and 7. Expressions (50) and (51) in section 6 show the influence of the history of the system on the correct detection of the present information digit a_n. That is, the probability of correctly detecting the message sent in the time interval $[(n-1)T, nT]$ is strongly influenced by all the messages that passed through the system from the time that it started working. From the factors $e^{-b_k(n-\ell)T}$ present in our expressions, it is obvious that the influence of the past messages, a_ℓ, decreases fast with the increase in their time distance $(n-\ell)T$ from the presently detected message a_n. Hence, when calculating expressions (50) and (51) one can neglect the terms with high $b_k(n-\ell)T$ exponents. The results obtained this way are highly simplified in some cases, therefore easy to compute but less accurate. For example, if the filter characteristics are such that:

(a) $e^{-b_k \cdot mT} \ll 1$, $\forall K$ and $m \geq 1$

(b) $|a_n| = 1$ (binary case with $a_i = \pm 1$)

and

(c) $$\left[\sum_{k=1}^{Q} c_k b_k \left[\frac{1}{b_k^2 + \omega_d^2} + \frac{1}{b_k^2 + 4\omega_c^2}\right]\right]^2 + \left[\sum_{k=1}^{Q} c_k \frac{\omega_d}{b_k^2 + \omega_d^2}\right]^2 + \left[\sum_{k=1}^{Q} c_k \frac{2\omega_c}{b_k^2 + 4\omega_c^2}\right]^2 \gg 2\left[\sum_{k=1}^{Q} c_k \frac{\omega_d}{b_k^2 + \omega_d^2}\right]\left[\sum_{k=1}^{Q} c_k \frac{2\omega_c}{b_k^2 + 4\omega_c^2}\right]$$

it can be easily found from (48-51) that

$$R = R_{n-1} = R_{n-2} \sim \frac{A^2}{8\sigma_o^2}\left\{\left[\sum_{k=1}^{Q} c_k b_k \left(\frac{1}{b_k^2 + \omega_d^2} + \frac{1}{b_k^2 + 4\omega_c^2}\right)\right]^2 + \left[\sum_{k=1}^{Q} c_k \frac{\omega_d}{b_k^2 + \omega_d^2}\right]^2 + \left[\sum_{k=1}^{Q} c_k \frac{2\omega_c}{b_k^2 + 4\omega_c^2}\right]^2\right. \tag{56}$$

$$\lambda_n = -\omega_d T a_n + \tan^{-1} \frac{\sum_{k=1}^{Q} c_k \left[\frac{\omega_d a_n}{b_k^2 + \omega_d^2} + \frac{2\omega_c}{b_k^2 + 4\omega_c^2}\right]}{\sum_{k=1}^{Q} c_k b_k \left[\frac{1}{b_k^2 + \omega_d^2} + \frac{1}{b_k^2 + 4\omega_c^2}\right]}$$

$$- \tan^{-1} \frac{\sum_{k=1}^{Q} c_k \left[\frac{\omega_d a_{n-1}}{b_k^2 + \omega_d^2} + \frac{2\omega_c}{b_k^2 + 4\omega_c^2}\right]}{\sum_{k=1}^{Q} c_k b_k \left[\frac{1}{b_k^2 + \omega_d^2} + \frac{1}{b_k^2 + 4\omega_c^2}\right]} \tag{57}$$

From (56) and (57) one then observes that in the highly simplified case where the combined band-pass filter displays a fairly wide bandwidth, the parameters R, λ_n characterizing the efficiency of the system will depend only on the most recent (past) information digit a_{n-1}. The probability of a false detection for the digit a_n is then averaged only over a_{n-1}.

As an additional remark we would like to quote Mazo and Salz [1] on the importance of the post-detection filter as a tool to reduce the dependence of the systems performance on ω_d. This fact should allow the system designer more freedom in his choice of information signals for optimizing performance when the post-detection filter is included.

We feel that the main contribution of this paper is in the accurate theoretical expression it provides for relating the error probability achievable in an FM system to the band-limitation (i.e., intersymbol interference) produced by incorporating a post-detection filter in it. The complexity of those relationships obviously depends on the filter characteristics and signal design and may be traded off for less accurate predictions for system performance by using customary simplifying assumptions.

Another rather interesting result emerging from our analysis is in the additional insight obtained from looking at expressions (22), (23). The basic structure of these p.d.f. for the angles involved in the detection process follows exactly the one that has been known to cover the case in which intersymbol interference is neglected. They do translate, however, the intersymbol interference disturbances into shifts of those p.d.f.s, the amount of those shifts corresponding to the magnitude of the disturbances.

ACKNOWLEDGMENTS

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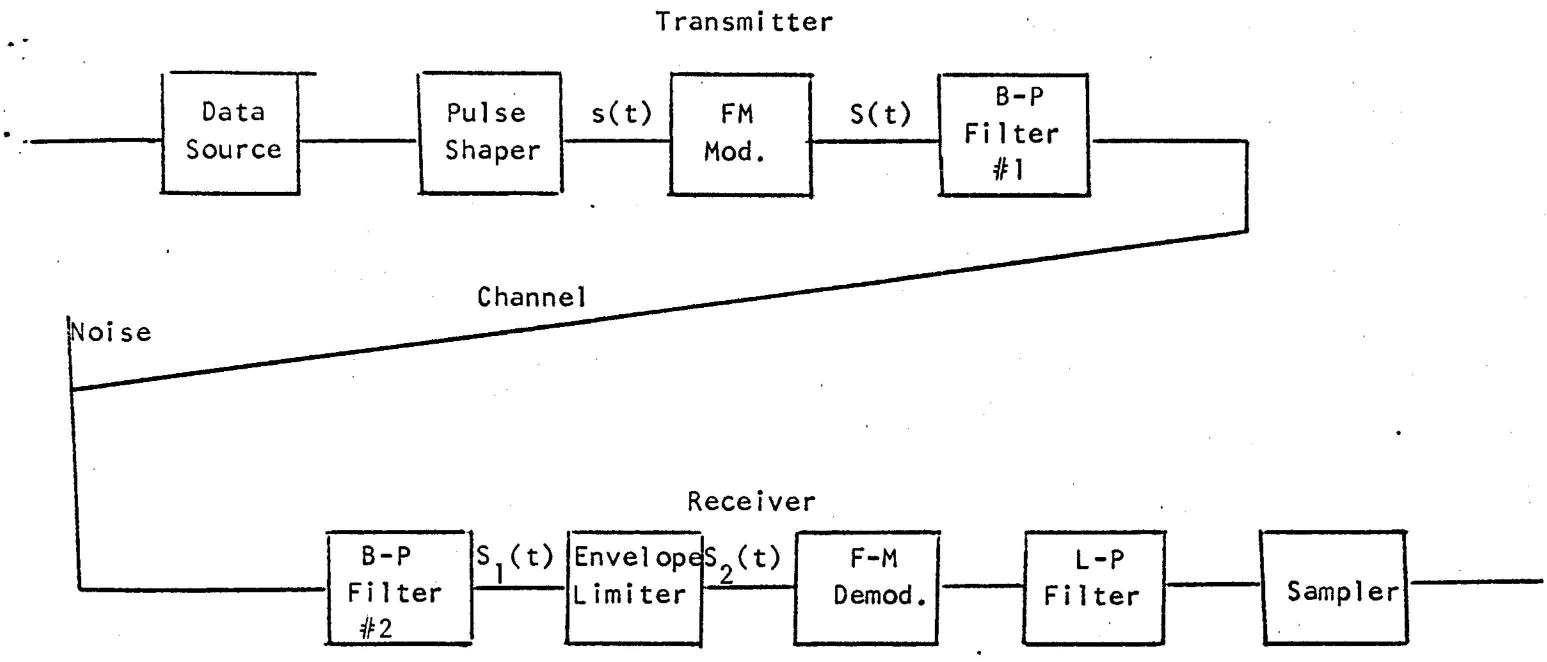
Transmitter
Data Source
Pulse Shaper
s(t)
FM Mod.
S(t)
B-P Filter #1
Channel
Noise
Receiver
B-P Filter #2
$S_1(t)$
Envelope Limiter
$S_2(t)$
F-M Demod.
L-P Filter
Sampler

FIG. 1

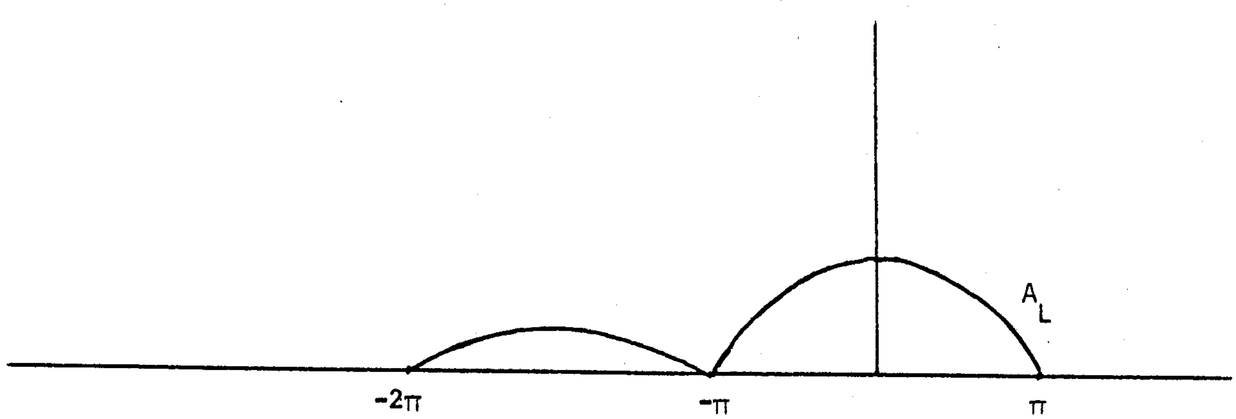
A_L
-2π
$-\pi$
π

FIG. 2

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APPENDIX

We consider a rational filter with an impulse response of the form

$$h(t) = \left(\sum_{k=1}^{a} c_k e^{-b_k t} \right) \cos \omega_c t \tag{A1}$$

to be presented on input of the form

$$S(\tau) = A \cos\left(\omega_c \tau + \theta + \omega_d T \sum_{i=0}^{\ell} a_i + \omega_d a_{\ell+1} (\tau - \ell T) \right) \tag{A2}$$

where $\ell T \leq \tau \leq (\ell+1)T$. The output will then be given by the following expression

$$S_o(t) = \sum_{k=1}^{Q} c_k S_k(t) \tag{A3}$$

where

$$S_k(t) = \int_o^t e^{-b_k(t-\tau)} \cos \omega_c (t-\tau) \cdot S(\tau) d\tau \tag{A4}$$

Substituting in (A4) $S(\tau)$ by its expression in (A2) one finds:

$$S_k(t) = \frac{A}{4} e^{-b_k t} \cdot e^{j\omega_c t} \left\{ e^{j\theta} \left[\sum_{\ell=0}^{n-2} \frac{\exp\left(j\omega_d T \sum_{i=0}^{\ell+1} a_i + b_k(\ell+1)T \right) - \exp\left(b_k \ell T + j\omega_d T \sum_{i=0}^{\ell} a_i \right)}{b_k + j\omega_d a_{\ell+1}} \right. \right.$$

$$+ \frac{\exp\left(j\omega_d T \sum_{i=0}^{n-1} a_i + j\omega_d T(t-[n-1]T) a_n + b_k t \right)}{b_k + j\omega_d a_n} - \frac{\exp\left(j\omega_d T \sum_{i=0}^{n-1} a_i + b_k [n-1]T \right)}{b_k + j\omega_d a_n}$$

$$\left. + \frac{\exp\left(b_k t + j\omega_d T \sum_{i=0}^{n-1} a_i + j\omega_d (t - [n-1]T) a_n \right)}{b_k + j2\omega_c + j\omega_d a_n} \right].$$

$$+ e^{-j\theta} \left[\sum_{\ell=0}^{n-2} \frac{\exp\left(b_k(\ell+1)T - j\omega_d T \sum_{i=0}^{\ell+1} a_i - j2\omega_c(\ell+1)T \right) - \exp\left(b_k \ell T - j2\omega_c \ell T - j\omega_d T \sum_{i=0}^{\ell} a_i \right)}{b_k - j2\omega_c - j\omega_d a_{\ell+1}} \right.$$

$$- \frac{\exp\left(b_k[n-1]T - j\omega_d T \sum_{i=0}^{n-1} a_i - j2\omega_c[n-1]T\right)}{b_k - j2\omega_c - j\omega_d a_n}\Bigg]\Bigg\} +$$

$$+ \frac{A}{4} e^{-b_k t} \cdot e^{-j\omega_c t} \Bigg\{ e^{j\theta} \Bigg[\sum_{\ell=0}^{n-2} \frac{\exp\left(b_k[\ell+1]T + j\omega_d T \sum_{i=0}^{\ell+1} a_i + j2\omega_c[\ell+1]T\right) - \exp\left(j\omega_d T \sum_{i=0}^{\ell} a_i + j2\omega_c \ell T + b_k \ell T\right)}{b_k + j2\omega + j\omega_d a_{\ell+1}}$$

$$- \frac{\exp\left(j\omega_d T \sum_{i=0}^{n-1} a_i + j2\omega_c[n-1]T + b_k[n-1]T\right)}{b_k + j2\omega_c + j\omega_d a_n}\Bigg]$$

$$+ e^{-j\theta} \Bigg[\sum_{\ell=0}^{n-2} \frac{\exp\left(b_k[\ell+1]T - j\omega_d T \sum_{i=0}^{\ell+1} a_i\right) - \exp\left(b_k \ell T - j\omega_d T - j\omega_d T \sum_{i=0}^{\ell} a_i\right)}{b_k - j\omega_d a_{\ell+1}}$$

$$+ \frac{\exp\left(b_k t - j\omega_d T \sum_{i=0}^{n-1} a_i - j\omega_d (t - [n-1]T) a_n\right)}{b_k - j2\omega_c - j\omega_d a_n}$$

$$+ \frac{\exp\left(b_k t - j\omega_d T \sum_{i=0}^{n-1} a_i - j\omega_d a_n (t - [n-1]T)\right) - \exp\left(b_k[n-1]T - j\omega_d T \sum_{i=0}^{n-1} a_i\right)}{b_k - j\omega_d a_n}\Bigg]\Bigg\} \quad \text{(A5)}$$

If one chooses $t = nT$ and ω_c equal to a multiple of $2\pi/T$, one gets the following simplified version of equation (A5):

$$S_k(nT) = \frac{A}{4} * \{1\} * \exp(j\omega_c nT) + \frac{A}{4} * \{2\} * \exp(-j\omega_c nT) \quad \text{(A6)}$$

where the brackets $\{1\}$ and $\{2\}$ correspond to the brackets in expression

(A5) and are not functions of the carrier frequency ω_c .

Let now:

$$X_k(nT) = \frac{A}{4}\,[\{1\} + \{2\}] \tag{A7}$$

$$Y_k(nT) = \frac{A}{4}\,[\{2\} - \{1\}] \tag{A8}$$

Substituting these notations into (A6) one gets:

$$S_k(nT) = X_k(nT)\cos\omega_c nT - Y_k(nT)\sin\omega_c nT \tag{A9}$$

If

$$S_o(t) = X(t)\cos\omega_c t - Y(t)\sin\omega_c t \tag{A10}$$

and because of (A3), the following expressions are obtained:

$$X(t) = \sum_{k=1}^{Q} c_k X_k(t) \tag{A11}$$

$$Y(t) = \sum_{k=1}^{Q} c_k Y_k(t) \tag{A12}$$

From (A5-A12) and some simple mathematical manipulations one gets:

$$\begin{aligned} X(nT) = {} & \frac{A}{2}\sum_{\ell=1}^{n-1}[\Gamma_{n-1,\ell}-\Gamma_{n,\ell+1}]\cos(\theta + \omega_d T\sum_{i=0}^{\ell} a_i) \\ & + \frac{A}{2}\sum_{\ell=1}^{n-1}[E_{n-1,\ell}-E_{n,\ell+1}]\sin(\theta + \omega_d T\sum_{i=0}^{\ell} a_i) \\ & + \frac{A}{2}\,[\Gamma_{n-1,n}\cos(\theta + \omega_d T\sum_{i=0}^{n} a_i) - \Gamma_{n,1}\cos\theta] \\ & + \frac{A}{2}\,[E_{n-1,n}\sin(\theta + \omega_d T\sum_{i=0}^{n} a_i) - E_{n,1}\sin\theta] \end{aligned} \tag{A13}$$

$$Y(nT) = \frac{A}{2} \sum_{\ell=1}^{n-1} [\Delta_{n-1,\ell} - \Delta_{n,\ell+1}] \sin(\theta + \omega_d T \sum_{i=0}^{\ell} a_i)$$

$$- \frac{A}{2} \sum_{\ell=1}^{n-1} [Z_{n-1,\ell} - Z_{n,\ell+1}] \cos(\theta + \omega_d T \sum_{i=0}^{\ell} a_i)$$

$$+ \frac{A}{2} [\Gamma_{n-1,n} \sin(\theta + \omega_d T \sum_{i=0}^{n} a_i) - \Delta_{n,1} \sin \theta]$$

$$- \frac{A}{2} [E_{n-1,n} \cos(\theta + \omega_d T \sum_{i=0}^{n} a_i) - Z_{n,1} \cos \theta] \qquad \text{(A14)}$$

where

$$\Gamma_{n-1,\ell+1} = \sum_{k=1}^{Q} c_k \exp(-b_k[n-\ell-1]T) \left[\frac{b_k}{b_k^2 + \omega_d^2} + \frac{b_k}{b_k^2 + (2\omega_c + \omega_d a_{\ell+1})^2} \right] \qquad \text{(A15)}$$

$$E_{n-1,\ell+1} = \sum_{k=1}^{Q} c_k \exp(-b_k[n-\ell-1]T) \left[\frac{\omega_d a_{\ell+1}}{b_k^2 + \omega_d^2} + \frac{2\omega_c + \omega_d a_{\ell+1}}{b_k^2 + (2\omega_c + \omega_d a_{\ell+1})^2} \right] \qquad \text{(A16)}$$

$$\Delta_{n-1,\ell+1} = \sum_{k=1}^{Q} c_k \exp(-b_k[n-\ell-1]T) \left[\frac{b_k}{b_k^2 + \omega_d^2} - \frac{b_k}{b_k^2 + (2\omega_c + \omega_d a_{\ell+1})^2} \right] \qquad \text{(A17)}$$

$$Z_{n-1,\ell+1} = \sum_{k=1}^{Q} c_k \exp(-b_k[n-\ell-1]T) \left[\frac{\omega_d a_{\ell+1}}{b_k^2 + \omega_d^2} - \frac{2\omega_c + \omega_d a_{\ell+1}}{b_k^2 + (2\omega_c + \omega_d a_{\ell+1})^2} \right] \qquad \text{(A18)}$$

The approximation $\omega_c \gg \omega_d$ is now used on (A13-A18) while taking into consideration the now simplified following relationships:

$$\Gamma_{n-1,\ell} - \Gamma_{n,\ell+1} = 0 \tag{A19}$$

$$\Delta_{n-1,\ell} - \Delta_{n,\ell+1} = 0 \tag{A20}$$

$$E_{n-1,\ell} - E_{n,\ell+1} = \sum_{k=1}^{Q} c_k \exp(-b_k[n-\ell]T) \frac{\omega_\ell(a_\ell - a_{\ell+1})}{b_k^2 + \omega_d^2} \tag{A21}$$

$$Z_{n-1,\ell} - Z_{n,\ell+1} = \sum_{k=1}^{Q} c_k \exp(-b_k[n-\ell]T) \frac{\omega_d(a_\ell - a_{\ell+1})}{b_k^2 + \omega_d^2} \tag{A22}$$

The modified equivalent expressions so obtained are:

$$X(nT) = \frac{A}{2} \sin\theta\ [1] + \frac{A}{2} \cos\theta\ [2] \tag{A23}$$

$$Y(nT) = \frac{A}{2} \cos\theta\ [1] + \frac{A}{2} \sin\theta\ [2] \tag{A24}$$

where

$$\begin{aligned}[1] = &\sum_{\ell=1}^{n-1} (a_\ell - a_{\ell+1}) \cos(\omega_d T \sum_{i=0}^{\ell} a_i) \sum_{k=1}^{Q} c_k \frac{\omega_d}{b_k^2 + \omega_d^2} \cdot \exp(-b_k[n-\ell]T) \\ &- \sin(\omega_d T \sum_{i=0}^{n} a_i) \sum_{k=1}^{Q} c_k b_k \left[\frac{1}{b_k^2 + \omega_d^2} + \frac{1}{b_k^2 + 4\omega_c^2}\right] \\ &+ \cos(\omega_d T \sum_{i=0}^{n} a_i) \sum_{k=1}^{Q} c_k \left[\frac{\omega_d a_n}{b_k^2 + \omega_d^2} + \frac{2\omega_c}{b_k^2 + 4\omega_c^2}\right]\end{aligned} \tag{A25}$$

$$\begin{aligned}[2] = &\sum_{\ell=1}^{n-1} (a_\ell - a_{\ell+1}) \sin(\omega_d T \sum_{i=0}^{\ell} a_i) \sum_{k=1}^{Q} c_k \frac{\omega_d}{b_k^2 + \omega_d^2} \exp(-b_k[n-\ell]T) \\ &+ \cos(\omega_d T \sum_{i=0}^{n} a_i) \sum_{k=1}^{Q} c_k b_k \left[\frac{1}{b_k^2 + \omega_d^2} + \frac{1}{b_k^2 + 4\omega_c^2}\right] \\ &+ \sin(\omega_d T \sum_{i=0}^{n} a_i) \sum_{k=1}^{Q} c_k \left[\frac{\omega_d a_n}{b_k^2 + \omega_d^2} + \frac{2\omega_c}{b_k^2 + 4\omega_c^2}\right]\end{aligned} \tag{A26}$$

By further manipulating expressions (A23-A26), one obtains:

$$X(nT) = \sigma_o\sqrt{2R_{n-1}}\cos(\theta - \gamma_{n-1}) \quad \text{(A27)}$$

$$Y(nT) = \sigma_o\sqrt{2R_{n-1}}\sin(\theta - \gamma_{n-1}) \quad \text{(A28)}$$

where

$$R_{n-1} = \frac{X^2(nT) + Y^2(nT)}{2\sigma_o^2} = \frac{A^2}{4}\frac{[1]^2 + [2]^2}{2\sigma_o^2} \quad \text{(A29)}$$

$$\gamma_{n-1} = \tan^{-1}\frac{[1]}{[2]} \quad \text{(A30)}$$

Expressions (A27-A30) are the necessary relationships for calculating the probability of errors.