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RELATED TO SPLINE FUNCTIONS

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Abstract: The problem of determining a minimum energy control for a dynamically interconnected set of  $p$  single-input single-output finite-dimensional linear time-varying dynamical systems, for which the outputs are constrained to assume prescribed values at different points in time is considered. It is shown that the solution (optimal controller) is obtained by performing a linear operation on an appropriate vector-valued generalized interpolating spline.

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## 1. Introduction

This note presents a result developed in [1]. It shows how the solution to a class of minimum energy control problems with output constraints is related to interpolation by generalized splines.

For complete generality, we consider a large-scale system consisting of  $p$  linear time-varying  $m^{\text{th}}$  order differential dynamical systems interconnected by a linear time-varying dynamic compensator (see Fig. 1). For such a composite system, we show that the optimal solution (trajectory) in the space of outputs appears as a vector-valued generalized  $L$  spline, where  $L$  is a matrix of high order differential operators, satisfying the output constraints; and we calculate the corresponding minimum energy control function by applying the operator  $L$  to such a spline. One of the consequences of this study of ours is the generalization of splines arising from a single high order differential operator  $\mathcal{L}$  to splines generated by a  $p \times n$  matrix  $L$  of  $m^{\text{th}}$  order differential operators, as we have indicated above.

While minimum energy control problems have been discussed at length in the control theory literature (see, for example, [2] and [3]), the main point of the present note is simply to establish the connection between these control theory results and those in spline approximation (see, for example, [4] and [5]) which seemingly were developed in a separate context. It is believed that such a connection will be helpful toward providing insights and stimulating research in both areas, as has been the case for example with the connection, developed by Kimeldorf and Wahba [6] - [8] and Weinert and Kailath [9] [10], between Bayesian estimation of stochastic processes and interpolation and smoothing by splines.

We do not discuss the generalization of our results to the  $L_g$  spline case since it follows from the considerations available in the

literature [4] [11] at the expense of complicating the notation.

Since [1] appeared, Weinert and Kailath have made interesting contributions [9], [12] to the problem discussed here in that they consider the case of an Lg-spline generated by a single high order differential operator and use a reproducing kernel Hilbert space approach to compute the minimum energy control.

## 2. Problem Statement

The output vector  $y(t) = \text{col}(y_1(t), \dots, y_n(t))$  of the composite continuous-time, continuous-state, linear, time-varying, finite-dimensional system under consideration is assumed to satisfy a vector differential equation of the form

$$\hat{L} y(t) = u(t), \quad 0 \leq t \leq a < \infty, \quad (1)$$

where  $u(t) = \text{col}(u_1(t), \dots, u_p(t)) \in \mathbb{R}^p$ ,  $p \leq n$ , and  $\hat{L}$  is a  $p \times n$  matrix with elements  $\hat{L}_{ij}$ ,  $i = 1, \dots, p$ ,  $j = 1, \dots, n$ , of the form

$$\hat{L}_{ij} y_j(t) = \sum_{k=0}^m \hat{l}_{ijk}(t) D^k y_j(t), \quad D = d/dt, \quad (2)$$

where  $m$  is a positive integer and  $\hat{l}_{ijk}$  are real-valued functions on  $[0, a]$ , the conditions on which are to be stated in connection with (6).

We suppose that we are given the performance index

$$\hat{I}(u) = \int_0^a u^T(t) R(t) u(t) dt, \quad (3)$$

where the superscript  $T$  denotes the transpose,

$$R(t) = M^T(t) M(t), \quad (4)$$

$M(t)$  is a  $p \times p$  matrix of full rank over  $t \in [0, a]$ , with elements in  $C^m[0, a]$ ,

and such that

$$L \equiv M(t) \hat{L} \quad (5)$$

has the following properties: in the  $ij^{\text{th}}$  element  $L_{ij}$  of  $L$ , which is of the form

$$L_{ij} = \sum_{s=1}^p M_{is}(t) \hat{L}_{sj}$$

$$\begin{aligned}
&= \sum_{s=1}^p \sum_{k=1}^m M_{is}(t) \hat{\ell}_{sjk}(t) D^k \\
&\equiv \sum_{k=0}^m \ell_{ijk}(t) D^k,
\end{aligned} \tag{6}$$

$\ell_{ijk}(\cdot)$  are real-valued functions belonging to  $C^k[0,a]$  and  $\ell_{ijm}(\cdot)$  is uniformly bounded away from zero on  $[0,a]$ .

Suppose finally that we are given a set of values

$$Y = \{y^i \in \mathbb{R}^n : i = 0, 1, \dots, N\}, \tag{7}$$

as well as a mesh

$$\Delta = t_0 < t_1 < \dots < t_N, \tag{8}$$

where  $t_{-1} \equiv 0 < t_0$  and  $t_N < a \equiv t_{N+1}$ .

The problem under consideration may be stated as follows:

Problem 1: Find  $u^*$  which minimizes the performance index  $\hat{I}(u)$ , defined by (3), over all  $u \in \mathcal{L}_p^2(0,a)^\#$ , under the differential dynamical constraint (1) and the output constraints

$$y(t_i) = y^i, \quad i = 0, 1, \dots, N. \tag{9}$$

### 3. Physical Interpretation

Before we go any further, it is worth seeking the meaning of the equations (1) in terms of an actual system. For this purpose, consider the system of  $p$  single-input single-output systems, interconnected as in Fig.1, described by the set of equations

$$\begin{aligned}
P_i z_i(t) \equiv \sum_{j=0}^{r_i} a_{ij}(t) D^j z_i(t) &= u_i(t) + w_i(t), \\
i &= 1, \dots, p,
\end{aligned} \tag{10}$$

where  $r_i$  are nonnegative integers,  $a_{ij}$  are appropriate real-valued functions on  $[0,a]$ ,  $u_i$  are the controls whose energy (i.e., square of the  $\mathcal{L}^2(0,a)$ -norm) is to be minimized, and  $w_i$  are inputs from an existing compensator.

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$^\# \mathcal{L}_p^2(0,a)$  is the linear space of  $p$ -tuples of elements of  $\mathcal{L}^2(0,a)$ .

With the notation

$$P = \text{diag} (P_1, \dots, P_p) \quad (11)$$

$$z = \text{col} (z_1, \dots, z_p) \quad (12)$$

$$w = \text{col} (w_1, \dots, w_p), \quad (13)$$

(10) may be rewritten as

$$P z = u + w. \quad (14)$$

Similarly, the dynamic compensator equations may be written as

$$Qy(t) = B(t) z(t) \quad (15)$$

where  $B(t)$  is a time-dependent  $n \times p$  real matrix of rank  $p$ ,

$$Q = \text{diag} (Q_1, \dots, Q_n) \quad (16)$$

$$Q_j y_j(t) = \sum_{q=0}^{q_j} g_{jq}(t) D^q y_j(t), \quad j = 1, \dots, n, \quad (17)$$

$q_j$  are nonnegative integers, and  $g_{jq}(\cdot)$  are appropriate real-valued functions on  $[0, a]$ .

The output  $y$  of the compensator is connected to the input of the systems being controlled by the interconnection  $p \times n$  matrix  $C(t)$ , that is,

$$w(t) = C(t) y(t) \quad (18)$$

Now (14), (15), and (18), together, are equivalent to (1) if we set

$$\hat{L} = P (B^T(t) B(t))^{-1} B^T(t) Q - C(t). \quad (19)$$

#### 4. An Auxiliary Problem and Its Solution

Returning to our original question, it turns out that it pays to formulate the problem, which we are considering, in the  $y$ -space (output space) rather than in the  $u$ -space (input space), for then the results and insights from the spline theory can be brought to bear upon the problem solution. This is exactly what we now proceed to do.

In fact, substituting (1) in (3) and using the notation (5), (3) may be rewritten in the form

$$I(y) = \int_0^a (L y(t))^T (L y(t)) dt. \quad (20)$$

Also, let  $H^k$  denote the linear space of real-valued functions  $f$  on  $[0, a]$  such that  $D^{k-1}f$  is absolutely continuous and  $D^k f \in \mathcal{L}^2(0, a)$ ; and denote by  $H_n^m$  the linear space of  $n$ -tuples of functions in  $H^m$ . Then Problem 1 may be reformulated in the output space as follows:

Problem 2. Find  $y^*$  which minimizes  $I(y)$  defined by (20) over all  $y \in H_n^m$  under the output constraints

$$y(t_i) = y^i, \quad i = 0, 1, \dots, N. \quad (21)$$

The solution to the above problem is embodied by the following theorem, which represents a generalization to splines generated by matrix differential operators of a wellknown result [5, Chap. 6] for splines pertaining to scalar differential operators.

Let us define the formal adjoint of  $L$  as the  $n \times p$  matrix  $L^*$  whose  $ij^{\text{th}}$  element,  $i = 1, \dots, n$ ,  $j = 1, \dots, p$ , is

$$L^*_{ij} = (L_{ji})^*, \quad (22)$$

where  $(L_{ji})^*$  is the formal adjoint of the element  $L_{ji}$  of  $L$ , i.e.

$$(L_{ji})^*(\cdot) = \sum_{k=0}^m (-1)^k D^k (\ell_{jik}(t) (\cdot)). \quad (23)$$

Theorem 1. The solution to Problem 2 always exists. It is unique if and only if

$$N(L) \cap \Omega = \Phi, \quad (24)$$

where  $N(L)$  is the null space of  $L$ ,  $\Omega$  is the set

$$\Omega = \{ f \in H_n^m : f(t_i) = \underline{0}, \quad t_i \in \Delta \}, \quad (25)$$

and  $\Phi$  is the null subspace. Then the solution  $S(L, \Delta, Y; \cdot)$  to Problem 2 is the unique function  $y^*$  satisfying the following set of equations:

$$(a) L^*L y^*(t) = 0, \quad t_{i-1} \leq t < t_i, \quad i = 1, \dots, N; \quad (26a)$$

$$(b) y^*(t_i) = y^i, \quad i = 0, 1, \dots, N; \quad (26b)$$

$$(c) y_j^* \in C^{2m-2}, \quad j = 1, \dots, n; \quad (26c)$$

$$\text{and } (d) L y^*(t) = 0, \quad 0 \leq t < t_0 \text{ and } t_N \leq t \leq a. \quad (26d)$$

(Such a function  $y^*$  is called a generalized L spline interpolating  $Y$  over  $\Delta$ ).

Sketch of the Proof. The proof of the above theorem is based on the same arguments as the ones for the scalar case [5, Chap. 6], the extensions needed to validate the results being merely of a formal nature. For this reason, we omit the details and only sketch the steps involved in the generalization. First, one writes the vector version of the scalar Lagrange identity proved in [13, p.86]. Then one uses this vector Lagrange identity to prove a vector version of the "first integral relation" [5, p. 195] which then is used to establish a vector version of the "minimum norm property" [5, Chap. 6]. By this property,  $y^*$  satisfying (26a) through (26d) minimizes (20) over  $H_n^m$ , under (21). The proof that the uniqueness of  $y^*$  is equivalent to the satisfaction of (24) is identical to that for the scalar case [9][11].

##### 5. Return to Problem 1

From the preceding it is clear that now only a small additional step is needed to bridge the gap between the solution to Problem 2, expressed by (26a) through (26d), and that to Problem 1. The required step consists of (1) and (5) which, together, send  $y^*$  to  $u^*$ .

Formally, we summarize the above considerations as

Proposition 1. The solution to Problem 1 is determined by the set of equations (26a) through (26d), satisfied by  $y^*$ , and the equation

$$u^* = M^{-1} L y^*. \quad (27)$$

It is immediately clear that if we eliminate this last equation by substituting it in (26a) through (26d), the above proposition is equivalent to



Proposition 2. The solution  $u^*$  to Problem 1 is determined by the set of conditions:

$$(a) L^* M u^*(t) = 0, \quad t_{i-1} \leq t < t_i, \quad i = 1, \dots, N; \quad (28a)$$

$$(b) Mu^* \in \{ Ly : y \in H_n^m, y(t_i) = y^i, i = 0, \dots, N \}; \quad (28b)$$

$$(c) u_j^* \in C^{m-2}, \quad j = 1, \dots, p; \quad (28c)$$

$$(d) u^* = 0, \quad 0 \leq t < t_0 \text{ and } t_N \leq t \leq a. \quad (28d)$$

In reference [1], the result of the present paper was formulated in the form of Proposition 1. Recently, Weinert and Kailath, in their work [12] with Lg splines pertaining to scalar operators, essentially presented the above result in the form of Proposition 2, indicating some of the advantages of looking at the solution this way. Functions  $u^*$  satisfying a system such as (28) belong to the general class of "LM splines" introduced and discussed by one of us elsewhere [14].<sup>#</sup>

## 6. Examples

To illustrate the results presented above, we give two simple examples for which  $p = 1$ , and  $n = 1$  and  $2$  respectively.

### Example 1

Consider a system described by the scalar differential equation

$$\dot{y}(t) = u(t), \quad \dot{\phantom{y}} = d/dt, \quad 0 \leq t \leq a, \quad (29)$$

subject to the constraints (9).

Then,

$$y^*(t) = \frac{1}{t_i - t_{i-1}} [y(t_i)(t - t_{i-1}) + y(t_{i-1})(t_i - t)],$$

$$t_{i-1} \leq t < t_i, \quad i = 1, \dots, N, \quad (30a)$$

$$y^*(t) = y^0, \quad 0 \leq t < t_0, \quad (30b)$$

$$y^*(t) = y^N, \quad t_N \leq t \leq a. \quad (30c)$$

and the optimal control is given by

<sup>#</sup>Note that the symbol M in the denomination of an LM spline stands for a differential operator and not the matrix M appearing in the present paper.

$$u^*(t) = \frac{1}{t_i - t_{i-1}} (y(t_i) - y(t_{i-1})), t_{i-1} \leq t < t_i, i = 1, \dots, t_N, \quad (31a)$$

$$u^*(t) = 0, 0 \leq t < t_0, t_N \leq t \leq a. \quad (31b)$$

### Example 2

For the system

$$\ddot{y}(t) + 5\dot{y}(t) + 4y(t) = u(t), 0 \leq t < 1, \quad (32)$$

subject to the constraints

$$y(ih) = \frac{1}{10} \sin(i10h), h = 0.1, i = 1, \dots, 10, \quad (33)$$

we have

$$y^*(t) = \sum_{k=1}^4 A(i,k) e^{\alpha_k t}, (i-1)h \leq t < ih, i = 1, \dots, 10, \quad (34)$$

where

$$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \{-1, -4, 1, 4\}, \quad (35)$$

and  $A(i,k)$  are determined by the set of conditions (26).

Hence, the resultant optimal controller assumes the form

$$u_k(t) = \sum_{k=3}^4 \alpha_k A(i,k) e^{\alpha_k t}, (i-1)h \leq t < ih, i = 1, \dots, 10,$$

where  $\{A(i,k) : i = 1, \dots, 10, k = 3, 4\}$  are given in Table 1.

TABLE 1

	i	1	2	3	4	5	6	7	8	9	10
$A(i,3)$	0	-.117	.080	.028	.036	.028	.033	.006	.087	0	
$A(i,4)$	0	.022	-.005	.000	.000	.000	.000	.000	.000	-.001	0

## 7. Conclusion

A meaningful relationship has been established between a class of minimum energy control problems and spline interpolation. Even though the results presented here are for finite-time systems, they easily generalize to infinite-time systems.

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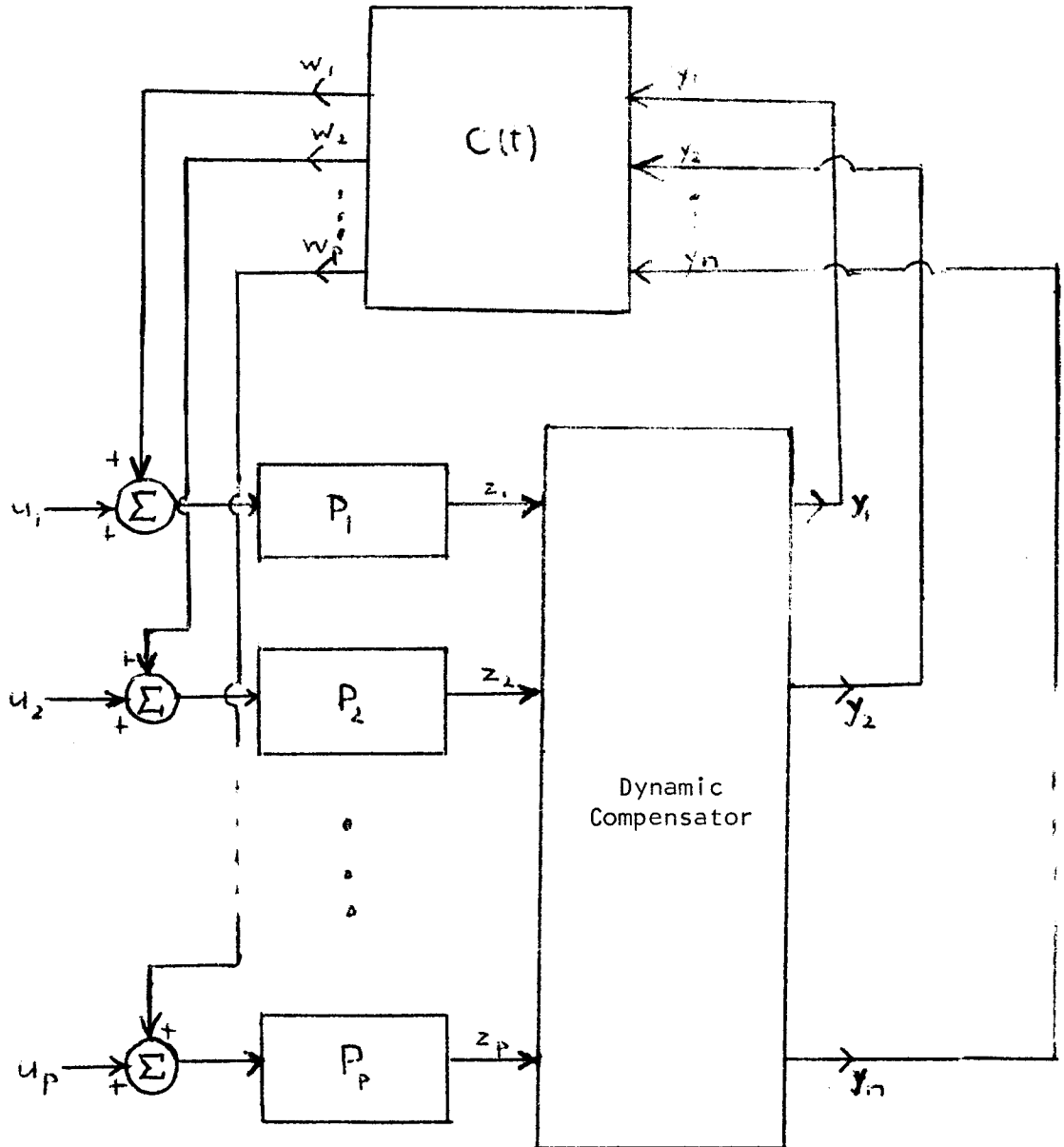


Fig. 1: Interconnected Dynamical Systems.