ON BEAMFORMING WITH FINITE RATE FEEDBACK IN MULTIPLE ANTENNA SYSTEMS

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Abstract

In this paper, we study a multiple antenna system where the transmitter is equipped with quantized information about instantaneous channel realizations. Assuming that the transmitter uses the quantized information for beamforming, we derive a universal lower bound on the outage probability for any finite set of beamformers. The universal lower bound provides a concise characterization of the gain with each additional bit of feedback information regarding the channel. Using the bound, it is shown that finite information systems approach the perfect information case as $(t-1)2^{-\frac{B}{t-1}}$, where B is the number of feedback bits and t is the number of transmit antennas. The geometrical bounding technique, used in the proof of the lower bound, also leads to a design criterion for good beamformers, whose outage performance approaches the lower bound. The design criterion minimizes the maximum inner product between any two beamforming vectors in the beamformer codebook, and is equivalent to the problem of designing unitary space time codes under certain conditions. Finally, we show that good beamformers are good packings of 2-dimensional subspaces in a $2t$-dimensional real Grassmannian manifold with chordal distance as the metric.

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I. Introduction

Recently, multiple antenna systems have received a great deal of attention, mainly due to their potential to meet the growing demand of data rates for current and future wireless systems. Current cellular standards have already made provisions for two antennas on the handsets while allowing for more at the basestations. The efficient deployment of multiple antenna elements in practical systems requires algorithms which achieve the desired spectral efficiency at a low computational cost.

In this paper, we derive a universal bound on the performance of a beamforming system, in which the transmitter is provided with B bits of information regarding the instantaneous channel vector. By assuming perfect channel information at the receiver and noiseless, zero-delay feedback link, we clearly quantify the impact of quantization on the outage probability of resulting beamforming system. Interestingly, the bounding technique also results in a geometrically intuitive construction criterion to design finite rate beamformer codebooks. We show several constructions whose performance follow closely the universal lower bound, thereby showing tightness of the lower bound.

Our primary motivation for the proposed problem formulation with limited transmitter information stems from several inter-related reasons. First, the outage probability when both transmitter and receiver are equipped with perfect channel information is much lower than that with no information at the transmitter (see Figure 1). Further, the gain over no transmitter information is significant ($\log(t)$ dB, where t is the number of transmit antennas) even when the transmitter does not use power control over time (see Figure 1). Note that *if* neither transmitter nor receiver know the channel, then a non-coherent transmission is optimal. However, with receiver feedback regarding the channel, estimating and feeding back a few bits can appreciably outperform a non-coherent system [1], especially if the coherence time of the channel is sufficiently large.

Second, the proposed formulation directly captures the finite rate nature of the feedback link along with the available feedback channel capacity. In the process, it also acts as a model for several practical communication systems, which do have feedback mechanisms to assist the transmitter [2]. Third, for large number of antennas, systems which use channel knowledge at the transmitter are typically lower in decoder complexity than those which do not. For example, in our case, beamforming concatenated with a scalar codebook (rank one signaling) requires lower decoding complexity than a typical space-time code (full rank codebook).

A similar quantized feedback model was first analyzed in [3] to maximize average mutual information and average received SNR, for a multiple transmit and single receive antenna system. The use of Lloyd algorithm for quantization was suggested as a possible solution for the design of beamformers with quantized channel information at the transmitter. While the approach of [3] gives useful results in the asymptotics of the number of transmit antennas, it does not provide insight into the structure and performance of good beamformer codebooks with finite number of antennas or feedback bits. Design of quantized power control for multiple antennas was considered in [1], with a full-rank code (like space-time code). Analytical and numerical tools were used to design the quantizers to minimize outage and it was shown that substantial gains in outage can be obtained by employing a few bits of feedback for power control. The problem of designing beamformers to minimize the pair-wise error probability of codewords with certain structure imposed on the feedback channel was also addressed in [4, 5].

In this paper, we consider a multiple transmit antenna system along with a single receive antenna. The channel is assumed to be quasi-static in time and hence outage probability can be used as the relevant performance metric [6]. We assume that the channel realization is known perfectly at the receiver. Even though this is not possible in practice, such an assumption allows us to focus on the effects of finite rate feedback on the system performance. We assume the existence of a feedback channel from the receiver to the transmitter as shown in Figure 2. The capacity of the feedback channel is finite and limited to B bits/frame. Further, the feedback channel is assumed to be free from errors. This feedback model is especially appropriate for a frequency division duplex (FDD) system where the channels for the uplink and the downlink are different, so that training in the reverse direction is not possible. With the feedback channel model under consideration, the transmitter knowledge comprises of the channel statistics along with limited information (B bits) about the channel realization made available by the feedback channel.

The design of the feedback channel is tightly coupled with the transmission strategy. In this paper, the transmission strategy is fixed to be beamforming. Note that beamforming is optimal for a multiple antenna system with a single receive antenna when the channel is known perfectly at the transmitter[1]. Further, beamforming with quantized feedback is a practical transmission scheme and is supported in the third generation cellular standards employing two transmit antennas along with two and four bits of feedback [2]. We do not incorporate temporal power control in our analysis.

[1] The conditions for the optimality of beamforming with partial channel information at the transmitter is investigated in [3, 7, 8] with mutual information as the performance metric.

The current analysis would be applicable for a system with short term power constraints where the power saved in one frame cannot be used in another.

The contributions of this paper are as follows.

1. *Performance bounds on beamforming*: We derive a universal lower bound on the outage probability of an arbitrary beamformer with finite feedback bits, hence finite number of beamforming vectors. The derived lower bound is valid for all SNR and any number of transmit antennas. The bound is found to be fairly tight in all regimes of SNR, transmit antennas and feedback bits, and is shown to be asymptotically tight in the number of beamforming vectors (or equivalently feedback bits). We show that the outage probability of beamforming with quantized feedback approaches the outage probability of beamforming with perfect channel feedback exponentially as $(t-1)2^{-\frac{B}{t-1}}$, where t is the number of transmit antennas and B is the number of feedback bits per frame. Our analysis also allows us to define a distortion measure for finite rate feedback, based on the resulting outage probability, which behaves similar to the rate distortion measure in the theory of source coding.

2. *Construction of beamformers*: We derive the structure of the quantizer which minimizes the outage probability with finite rate feedback. We show that near-optimal beamformer codebooks can be constructed by minimizing the maximum inner product between any two beamforming vectors in the codebook. This design criterion is similar to the design criterion of signal constellations for Gaussian channels, which maximizes the minimum Euclidean distance between any two signal points in the constellation.

   The proposed beamformer design criterion turns out to be equivalent to the design criterion for unitary space-time constellation. The equivalence between the two criteria holds with the following substitutions: (a) coherence time for unitary space time constellation is set equal to the number of transmit antennas for the beamformer design problem and (b) number of transmit antennas for unitary space time constellation is set equal to one. Alternately, the beamformer design problem can be interpreted as the problem of packing 2-dimensional subspaces in $2t$-dimensional real Grassmannian manifold with chordal distance as the metric, where t is the number of transmit antennas.

The rest of the paper is organized as follows. The system model is described in Section II. In Section III, the beamforming methodology is described and a universal lower bound on outage probability of beamforming with finite feedback is derived. The construction of good beamformer codebooks is discussed in Section IV while Section V concludes the paper.

II. System Model

Consider a wireless communication system with t transmit antennas and a single receive antenna. We assume a narrowband channel so that the channel is frequency non-selective. Let $\mathbf{h} = [h_1, h_2, ..., h_t]$ be the $1 \times t$ channel vector from the transmitter to the receiver. We assume that the entries in $\mathbf{h}$ are *i.i.d* and for each i, h_i is a circularly symmetric complex Gaussian random variable with zero mean and unit variance per complex dimension. Let X be the $1 \times t$ vector transmitted along the t transmit antennas at a given instant, where $X = [x_1, x_2, ...x_t]$, and x_i represents the transmission from the ith antenna. The average power of transmission is limited to P so that $\mathbb{E}[XX^\dagger] \leq P$, where $\mathbb{E}[.]$ represents the expectation operation while the superscript † represents conjugate transposition of a matrix. If y denotes the corresponding signal received at the receiver when X is transmitted, then we have

$$y = \mathbf{h}X^\dagger + \eta, \tag{1}$$

where η is the additive noise which is assumed to be circularly symmetric complex Gaussian random variable with zero mean and unit variance.

With the above model for the channel statistics, the distribution of $\mathbf{h}$ conditioned on norm, $\|\mathbf{h}\|^2 = \gamma$, is given by

$$p_{\mathbf{h}}(\mathbf{h} \mid \|\mathbf{h}\|^2 = \gamma) = \frac{e^{-\frac{\gamma^2}{2}}}{\sqrt{\det(2\pi I_t)}} \tag{2}$$

where $\det$ is the matrix determinant and I_t is the $t \times t$ identity matrix. Hence, conditioned on γ, $\mathbf{h}$ is uniformly distributed on the surface of a t-dimensional complex hypersphere of radius $\sqrt{\gamma}$ centered at origin.

The channel statistics are assumed to be quasi-static in time i.e., the channel realization is independent across different frames of transmission while the channel does not vary within a given frame. The channel is assumed to be known perfectly at the receiver while the transmitter has B bits of information about the channel provided by the feedback channel (see Figure 2). The transmission strategy is beamforming without power control. The transmitted vector X can be written as $X = C_i x$, where C_i is the beamforming vector while x is the information bearing scalar. The transmitter chooses the appropriate beamforming vector C_i from the set of beamforming vectors $\{C_1, C_2, ..., C_{2^B}\}$ based on the B feedback bits.

III. Performance Limits on Beamforming with Quantized Feedback

In this section, we introduce outage probability as our performance metric and explain the methodology of beamforming with quantized channel information at the transmitter. We then derive a lower bound on outage probability of beamforming with finite rate feedback and demonstrate the tightness of the bound in several scenarios. The bound is also useful in characterizing the improvement of outage with increasing feedback bits.

A. Performance metric: Outage probability

In this work, the channel is assumed to be quasi-static and the ergodic capacity of such a channel is zero [9], since for any desired transmission rate R, there is a finite probability of experiencing a channel realization that cannot support the rate R. The concept of outage probability was introduced for delay limited applications by Ozarow et al.[6] where ergodic capacity is not applicable. Outage is defined as the event that the channel realization cannot support the desired rate of transmission, the outage probability measures the frequency of such an event. Further, outage probability gives a lower bound for frame error rates in practical coded systems and hence yields useful bounds on system performance. We use outage probability as the performance metric for the design and analysis of the beamforming schemes in this paper.

B. Outage probability with beamforming

The channel realization is known perfectly at the receiver. The receiver quantizes the channel vector and transmits the information about the quantized channel vector to the transmitter over the feedback channel. Since the rate of feedback channel is limited to B bits/frame, the set of quantized vectors can comprise of at most $N = 2^B$ vectors. Further, we can assume that the set of quantized vectors at the receiver is the same as the set of beamforming vectors at the transmitter.

Let $\mathcal{C} = \{C_1, C_2, ...C_N\}$ be an arbitrary beamformer codebook, where $C_i \in \mathbb{C}^t$ along with $\|C_i\|_2 = 1$ for each $1 \leq i \leq N$. Note that $\mathcal{C}$ is also the set of quantizers used at the receiver for quantizing the channel vector. The beamforming vector C_i will be chosen for transmission when the feedback information corresponds to i. For each $1 \leq i \leq N$, let H_i be the set of all channel realizations which will be quantized to C_i and hence the beamforming vector C_i will be chosen for transmission whenever $\mathbf{h} \in H_i$. We will address the problem of quantizing the channel vector $\mathbf{h}$ to one of the vectors in the codebook $\mathcal{C}$ in the next section. The received signal when $\mathbf{h} \in H_i$ is given by (using $X = C_i x$ for beamforming in (1))

$$y = \langle \mathbf{h}, C_i \rangle x^* + \eta, \tag{3}$$

where $\langle ., . \rangle : \mathbb{C}^t \times \mathbb{C}^t \to \mathbb{C}$ and is given by $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}\mathbf{v}^\dagger$. Since this is a Gaussian channel for a given beamforming vector C_i and channel realization $\mathbf{h}$, the mutual information conditioned on $\mathbf{h}$ and C_i is maximized by choosing Gaussian distribution for x. Hence,

$$\max_{P(X|\mathbf{h},C_i)} I(X; y \mid \mathbf{h}, C_i) \;=\; \log_2(1 + P|\langle \mathbf{h}, C_i \rangle|^2). \tag{4}$$

Also, note that the event of $\mathbf{h} \in H_i$ is the same as the event of choosing the beamforming vector C_i for transmission. Hence, the outage probability conditioned on choosing C_i for transmission is given by

$$\begin{aligned} P_{out}(R, P \mid C_i) &= \text{Prob}\left[I(X; y \mid \mathbf{h}) < R \mid C_i \right] \\ &= \text{Prob}\left[\log_2(1 + P|\langle \mathbf{h}, C_i \rangle|^2) < R \mid C_i \right] \\ &= \text{Prob}\left[|\langle h, C_i \rangle|^2 < \frac{2^R - 1}{P} \mid \mathbf{h} \in H_i \right]. \end{aligned} \tag{5}$$

Thus, the outage probability while using the beamformer codebook $\mathcal{C}$ is given by the total probability rule

$$P_{out}(R, P) \;=\; \sum_{i=1}^{N} P(C_i) P_{out}(R, P|C_i), \tag{6}$$

where $P(C_i)$ is the probability of choosing C_i for transmission and $P(C_i) = P(\mathbf{h} \in H_i)$.

C. Quantization metric

From the previous section, we observe that the outage probability depends on the inner product of the channel vector $\mathbf{h}$ and the beamforming vector C_i. It turns out, as we show in the following lemma, the quantizer which minimizes the outage probability is related to the inner product and is the one which chooses the quantization vector with the smallest inner product norm. It should also be pointed out that the beamformer codebook $\mathcal{C}$ is fixed for the analysis in this section while we address the problem of assigning the channel realization $\mathbf{h}$ to the vectors in $\mathcal{C}$. We address the problem of designing good beamformer codebooks in Section IV.

Lemma 1 (Optimal quantizer) *For any beamformer codebook* $\mathcal{C} = \{C_1, C_2, .., C_N\}$ *, the outage probability is minimized by choosing for each channel realization* $\mathbf{h}$*, the vector* $C_i \in \mathcal{C}$ *which maximizes* $|\langle \mathbf{h}, C_i \rangle|$.

Proof : Let Q be any quantizer such that $Q : \mathbb{C}^t \to \mathcal{C}$. Further, let Q^* be a quantizer such that $Q^*(\mathbf{h}) = \arg\max_{C_i \in \mathcal{C}} |\langle \mathbf{h}, C_i \rangle|$. By definition, we have

$$|\langle \mathbf{h}, Q^*(\mathbf{h}) \rangle| \geq |\langle \mathbf{h}, Q(\mathbf{h}) \rangle|, \ \ \forall \, \mathbf{h} \in \mathbb{C}^t \tag{7}$$

Let $P_{out}(R, P)$ be the outage probability resulting from using the beamformer codebook $\mathcal{C}$ along with the quantizer Q, while $P^*_{out}(R, P)$ is the resulting outage probability of using $\mathcal{C}$ along with Q^*. It suffices to show that $P^*_{out}(R, P) \leq P_{out}(R, P)$. Define the indicator function $I_Q : \mathbb{C}^t \to \{0, 1\}$ such that

$$I_Q(\mathbf{h}) \ = \ \begin{cases} 1 & \text{if } \log_2(1 + P|\langle \mathbf{h}, Q(\mathbf{h}) \rangle|^2) < R, \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Equivalently, we have

$$I_Q(\mathbf{h}) \ = \ \begin{cases} 1 & \text{if } |\langle \mathbf{h}, Q(\mathbf{h}) \rangle|^2 < \frac{2^R - 1}{P}, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Similarly, define I_{Q^*} using the quantizer Q^* as

$$I_{Q^*}(\mathbf{h}) \ = \ \begin{cases} 1 & \text{if } |\langle \mathbf{h}, Q^*(\mathbf{h}) \rangle|^2 < \frac{2^R - 1}{P}, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Then, from (7), we have

$$I_{Q^*}(\mathbf{h}) \ \leq \ I_Q(\mathbf{h}), \ \forall \, \mathbf{h} \in \mathbb{C}^t. \tag{11}$$

Finally, the outage probability of the codebook $\mathcal{C}$ with the quantizer Q is given by

$$\begin{aligned} P_{out}(R,P) &= \mathbb{E}_{\mathbf{h}}[I_Q(\mathbf{h})] \\ &\geq \mathbb{E}_{\mathbf{h}}[I_{Q^*}(\mathbf{h})], \text{ using (11)} \\ &= P^*_{out}(R,P), \end{aligned} \tag{12}$$

where $\mathbb{E}_{\mathbf{h}}[.]$ stands for expectation operation with respect to the distribution of $\mathbf{h}$. ■

Hence, for each realization $\mathbf{h}$, choosing $C_i \in \mathcal{C}$ such that $|\langle \mathbf{h}, C_i \rangle|$ is maximized, minimizes the outage probability of the beamformer $\mathcal{C}$. In the sequel, the quantizer used will be Q^* unless otherwise mentioned. With this quantizer, the Voronoi region corresponding to a beamforming vector C_i and channel norm $\sqrt{\gamma}$ is given by

$$V_i(\gamma) = \{\mathbf{h} : \|\mathbf{h}\|^2 = \gamma, |\langle \mathbf{h}, C_i \rangle| \geq |\langle \mathbf{h}, C_j \rangle|, j \neq i\}. \tag{13}$$

Since the beamforming vectors have unit norm, the quantization metric is independent of the norm of $\mathbf{h}$ and depends only on the "direction" of $\mathbf{h}$. Hence, if a particular $\mathbf{h}$ belongs to $V_i(\gamma)$, then for any $\alpha \in \mathbb{C}$, $\mathbf{h}' = \alpha \mathbf{h}$ belongs to $V_i(|\alpha|^2\gamma)$. Let $A(V_i(\gamma))$ denote the surface area (sometimes referred to as $(n-1)$ content [10]) of $V_i(\gamma)$.

We will now characterize the subsets of $V_i(\gamma)$ which can support a given rate of transmission (*no-outage regions*) and those that cannot support the given rate (*outage regions*). We will show that the no-outage regions in particular have a nice structure which permits the computation of their measure, thus enabling the evaluation of the complement of the required outage probability.

D. *No-outage regions*

Consider the beamformer codebook $\mathcal{C}$ comprising of the beamforming vectors $C_1, C_2, ..C_N$. Let $V_1(\gamma), V_2(\gamma), ..V_N(\gamma)$ be the corresponding Voronoi regions on the surface of the hypersphere $\|\mathbf{h}\|^2 = \gamma$, as described in (13). Our beamforming methodology is such that C_i is chosen for transmission if and only if $\mathbf{h} \in V_i(\gamma)$. By Lemma 1, this minimizes outage probability for beamformer $\mathcal{C}$. We will first characterize the event where the transmission along C_i is not in error in order to evaluate the outage probability of the codebook $\mathcal{C}$. Let $\Theta_i(\gamma)$ be the subset of $V_i(\gamma)$ such that for all $\mathbf{h} \in \Theta_i(\gamma)$, the desired transmission rate R can be supported (without outage). We will refer to the regions, $\Theta_i(\gamma)$ for different i and γ, as *no-outage regions* in the sequel. Recall from (5) that outage does not occur when C_i is chosen for transmission if and only if $|\langle \mathbf{h}, C_i \rangle|^2 \geq \gamma_0$, where

$\gamma_0 = \frac{2^R - 1}{P}$. Hence, the no-outage region corresponding to $V_i(\gamma)$ is given by

$$\Theta_i(\gamma) = \{\mathbf{h} : \mathbf{h} \in V_i(\gamma), |\langle \mathbf{h}, C_i \rangle|^2 \geq \gamma_0\}. \tag{14}$$

Finally, the probability of successful transmission, denoted by P^c_{out}, when C_i is chosen for transmission is given by

$$\begin{aligned} P^c_{out}(R, P \mid \mathbf{h} \in V_i(\gamma)) &= \text{Prob}[\mathbf{h} \in \Theta_i(\gamma) \mid \mathbf{h} \in V_i(\gamma)] \\ &= \text{Prob}[|\langle \mathbf{h}, C_i \rangle|^2 \geq \gamma_0 \mid \mathbf{h} \in V_i(\gamma)] \end{aligned} \tag{15}$$

For small values of γ i.e., $\gamma < \gamma_0$, all the channel realizations $\mathbf{h} \in V_i(\gamma)$ will be in outage for all i. This is true irrespective of the number of beamforming vectors, i.e., there is a sphere of radius $\sqrt{\gamma_0}$ centered at origin, which is always in outage. However, as we will show later, the volume of this sphere goes to zero as the number of dimensions (or equivalently the number of transmit antennas) goes to infinity[2]. As a result of the preceding observation, we have $\Theta_i(\gamma) = \emptyset$ for $\gamma < \gamma_0$. As γ increases beyond γ_0, non-empty no-outage regions start forming around each of the beamforming vectors. We will use $A(\Theta_i(\gamma))$ to denote the surface area of $\Theta_i(\gamma)$.

E. *Outage probability conditioned on* $\|\mathbf{h}\|^2 = \gamma$

The probability of successful transmission with the beamformer $\mathcal{C}$ transmitting at rate R, conditioned on $\|\mathbf{h}\|^2 = \gamma$ can be calculated as

$$P^c_{out}(R, P \mid \|\mathbf{h}\|^2 = \gamma) = \sum_{i=1}^{N} P(C_i \mid \|\mathbf{h}\|^2 = \gamma)\, P^c_{out}(R, P \mid C_i, \|\mathbf{h}\|^2 = \gamma), \tag{16}$$

where $P(C_i \mid \|\mathbf{h}\|^2 = \gamma)$ is the probability of choosing the beamforming vector C_i for a channel realization with the squared norm given by γ, while $P^c_{out}(R, P \mid C_i, \|\mathbf{h}\|^2 = \gamma)$ is the probability that the resulting transmission along the vector C_i will not result in outage. Since $\mathbf{h}$ is distributed uniformly for a given norm from (2), $P(C_i \mid \|\mathbf{h}\|^2 = \gamma)$ is given by the fraction of the total surface area of the t-dimensional hypersphere $\|\mathbf{h}\|^2 = \gamma$ occupied by $V_i(\gamma)$, i.e.,

$$P(C_i \mid \|\mathbf{h}\|^2 = \gamma) = \frac{A(V_i(\gamma))}{A(\gamma)} \tag{17}$$

[2] One of the outcomes of this observation is that for any finite transmission rate R, the probability of outage can be driven to zero by providing enough transmit antennas and feedback bits.

where $A(\gamma)$ denotes the surface area of a t-dimensional complex hypersphere of radius $\sqrt{\gamma}$. The surface area of a complex hypersphere is given by the following lemma.

Lemma 2 *The surface area of a t-dimensional complex hypersphere of radius $\sqrt{\gamma}$ is given by*

$$A(\gamma) \;=\; \frac{2\pi^t(\sqrt{\gamma})^{(2t-1)}}{(t-1)!} \tag{18}$$

Proof : See Appendix A.. ■

The uniform distribution of $\mathbf{h}$ for a given norm also simplifies the calculation of $P^c_{out}(R,P \mid C_i, \|\mathbf{h}\|^2=\gamma)$. Noting that (a) conditioned on $\|\mathbf{h}\|^2=\gamma$, the event of choosing C_i for transmission is the same as the event $\mathbf{h}\in V_i(\gamma)$ and (b) when C_i is chosen, outage does not occur for $\mathbf{h}\in\Theta_i(\gamma)$, we have

$$\begin{aligned}
P^c_{out}(R,P \mid C_i, \|\mathbf{h}\|^2=\gamma) &= \text{Prob}\left[\mathbf{h}\in\Theta_i(\gamma) \mid \mathbf{h}\in V_i(\gamma), \|\mathbf{h}\|^2=\gamma\right] \\
&= \frac{\text{Prob}\left[\mathbf{h}\in\Theta_i(\gamma),\ \mathbf{h}\in V_i(\gamma) \mid \|\mathbf{h}\|^2=\gamma\right]}{\text{Prob}\left[\mathbf{h}\in V_i(\gamma) \mid \|\mathbf{h}\|^2=\gamma\right]} \\
&= \frac{\text{Prob}\left[\mathbf{h}\in\Theta_i(\gamma) \mid \|\mathbf{h}\|^2=\gamma\right]}{\text{Prob}\left[\mathbf{h}\in V_i(\gamma) \mid \|\mathbf{h}\|^2=\gamma\right]},\ \text{since } \Theta_i(\gamma)\subseteq V_i(\gamma) \\
&= \frac{A(\Theta_i(\gamma))}{A(V_i(\gamma))}.
\end{aligned} \tag{19}$$

From (19), we see that the distribution of $\mathbf{h}$ remains uniform conditioned on $\mathbf{h}\in V_i(\gamma)$.

Substituting (17) and (19) in (16), we have

$$\begin{aligned}
P^c_{out}(R,P \mid \|\mathbf{h}\|^2=\gamma) &= \sum_{i=1}^{N} \frac{A(V_i(\gamma))}{A(\gamma)}\frac{A(\Theta_i(\gamma))}{A(V_i(\gamma))} \\
&= \frac{\sum_{i=1}^{N} A(\Theta_i(\gamma))}{A(\gamma)}.
\end{aligned} \tag{20}$$

F. Spherical caps on hyperspheres

As equation (20) suggests, we need to evaluate the area of the no-outage regions $\Theta_i(\gamma)$ in order to compute the probability of outage for a particular beamformer $\mathcal{C}$. In general, it is not easy to compute the area of the no-outage regions. For this reason, we define *spherical caps* on the surface of the hypersphere, which closely approximate the no-outage regions. These spherical caps possess enough structure to allow area computation, which in turn yields a good approximation for the area of the no-outage regions.

For each $1 \le i \le N$ and $\gamma > 0$, consider the regions $S_i(\gamma)$ given by

$$S_i(\gamma) = \{\mathbf{h} : \|\mathbf{h}\|^2 = \gamma, |\langle \mathbf{h}, C_i \rangle|^2 > \gamma_0\}. \tag{21}$$

$S_i(\gamma)$ is the portion of the hypersphere $\|\mathbf{h}\|^2 = \gamma$ cut off by the subspace $|\langle \mathbf{h}, C_i \rangle|^2 \ge \gamma_0$. For the sake of illustration, if we let $\mathbf{h} \in \mathbb{R}^3$, then $S_i(\gamma)$ will be a spherical cap on the surface of the sphere as shown in Figure 3. Following this intuition, we will refer to $S_i(\gamma)$ as a spherical cap in the general t-dimensional complex case. Note that $S_i(\gamma)$ is congruent to $S_j(\gamma)$ for all i and j, i.e., if C_i is rotated to coincide with C_j, then $S_i(\gamma)$ will coincide with $S_j(\gamma)$.

From the definitions given in (14) and (21), it is clear that $\Theta_i(\gamma) \subseteq S_i(\gamma)$. On the other hand, it can be shown that the union of the no-outage regions $\Theta_i(\gamma)$ actually equals the union of the spherical caps $S_i(\gamma)$. This is an interesting property since the no-outage regions are always non-overlapping while the spherical caps will overlap for large γ. Hence, the no-outage regions partition the region formed by the union of the spherical caps into non-overlapping regions leading to the following lemma.

Lemma 3 *For all* $\gamma > 0$, $\bigcup_{i=1}^N \Theta_i(\gamma) = \bigcup_{i=1}^N S_i(\gamma)$.

Proof : For $\gamma < \gamma_0$, both $\Theta_i(\gamma)$ and $S_i(\gamma)$ are empty sets for all i and hence $\bigcup_{i=1}^N \Theta_i(\gamma) = \bigcup_{i=1}^N S_i(\gamma)$ holds trivially.

Suppose $\gamma \ge \gamma_0$. Then, from the definition of $\Theta_i(\gamma)$ and $S_i(\gamma)$, we know that $\Theta_i(\gamma) \subseteq S_i(\gamma)$ for each i so that $\bigcup_{i=1}^N \Theta_i(\gamma) \subseteq \bigcup_{i=1}^N S_i(\gamma)$. Now consider $\mathbf{h} \in \bigcup_{i=1}^N S_i(\gamma)$. Then, $\mathbf{h} \in S_j(\gamma)$ for some j. From the definition of $S_j(\gamma)$, it follows that $|\langle \mathbf{h}, C_j \rangle|^2 \ge \gamma_0$. Hence, $\max_{1 \le j \le N} |\langle \mathbf{h}, C_j \rangle|^2 \ge \gamma_0$ so that $\mathbf{h} \in \Theta_i(\gamma)$ for some i.

Therefore, $\mathbf{h} \in \bigcup_{i=1}^N \Theta_i(\gamma)$ and $\bigcup_{i=1}^N S_i(\gamma) \subseteq \bigcup_{i=1}^N \Theta_i(\gamma)$. Hence, the result follows. ∎

We can rewrite (20) in the light of the above lemma. Since $\Theta_i(\gamma)$ are non-overlapping for all γ, we have

$$\begin{aligned} \sum_{i=1}^N A(\Theta_i(\gamma)) &= A\left(\bigcup_{i=1}^N \Theta_i(\gamma)\right) \\ &= A\left(\bigcup_{i=1}^N S_i(\gamma)\right), \text{ using Lemma 3.} \end{aligned} \tag{22}$$

Hence, equation (20), i.e., the probability of successful transmission, for a given channel norm, can also written as

$$
\begin{aligned}
P_{out}^{c}(R, P \mid \|\mathbf{h}\|^2 = \gamma) &= \frac{\sum_{i=1}^{N} A(\Theta_i(\gamma))}{A(\gamma)} \\
&= \frac{A\left(\bigcup_{i=1}^{N} S_i(\gamma)\right)}{A(\gamma)}. \qquad (23)
\end{aligned}
$$

We will now relate $A\left(\bigcup_{i=1}^{N} S_i(\gamma)\right)$ to the area of individual spherical caps, $A(S_i(\gamma))$, for different values of γ. We noted earlier that $S_i(\gamma)$ are all congruent so that $A(S_i(\gamma))$ is the same for all i and hence it suffices to compute the area of $S_1(\gamma)$. To simplify the computation of $A(S_1(\gamma))$, we can assume, without loss of generality, that C_1 coincides with one of the co-ordinate axes, i.e., $C_1 = [1, 0, 0, .., 0]$. If C_1 is different from this, then by multiplying the vectors in $\mathcal{C}$ with a suitable unitary matrix, C_1 can be forced to coincide with one of the co-ordinate axes. The performance of the beamformer codebook $\mathcal{C}$ is invariant of such rotations since the distribution of $\mathbf{h}$ is symmetric about the origin.

With our choice for C_1, we have $S_1(\gamma) = \{\mathbf{h} : \|\mathbf{h}\|^2 = \gamma, |h_1|^2 \geq \gamma_0\}$ and its area $A(S_1(\gamma))$ is given by the following lemma.

Lemma 4 *The surface area of the spherical cap formed by the intersection of the subspace* $|h_1|^2 \geq \gamma_0$ *and the complex hypersphere* $\|\mathbf{h}\|^2 = \gamma$ *is given by*

$$
A(S_1(\gamma)) = \frac{2\pi^t \sqrt{\gamma}(\gamma - \gamma_0)^{t-1}}{(t-1)!}, \ \gamma \geq \gamma_0 \qquad (24)
$$

Proof : See Appendix B. ■

For all those γ for which no two $S_i(\gamma)$ overlap, Lemma 4 characterizes the exact area of the region corresponding to successful transmission in (23). If $S_i(\gamma)$ overlaps $S_j(\gamma)$ for some i and j, then we can arrive at an upper bound on the area of the region corresponding to successful transmission via Lemma 4. We define *overlap radius* of a beamformer $\mathcal{C}$ as the channel norm at which this transition takes place.

Definition 1 : *For any beamformer codebook* $\mathcal{C}$*, the overlap radius is defined as the largest channel norm squared for which no two spherical caps overlap. It is denoted by* $\gamma^*(\mathcal{C})$.

If $\Gamma = \{\gamma : \gamma > 0, S_i(\gamma) \bigcap S_j(\gamma) = \emptyset, \forall i, j \ \ i \neq j\}$, then $\gamma^*(\mathcal{C}) = \sup \Gamma$. Clearly, $\gamma^*(\mathcal{C}) \geq \gamma_0$. In the sequel, we will use the short notation γ^* to denote $\gamma^*(\mathcal{C})$ whenever there is no ambiguity about the beamformer codebook under discussion. The overlap radius γ^* is a key parameter which will be used in the derivation of the lower bound on outage probability as well as in the design of beamformer codebooks.

Above observations can be used to bound the numerator in (23). In particular, when $\gamma < \gamma_0$, the spherical caps are all empty sets so that

$$A\left(\bigcup_{i=1}^{N} S_i(\gamma)\right) = 0,\ \gamma < \gamma_0. \tag{25}$$

When $\gamma_0 \leq \gamma < \gamma^*$, the spherical caps form disjoint sets. In this case, we have

$$\begin{aligned} A\left(\bigcup_{i=1}^{N} S_i(\gamma)\right) &= \sum_{i=1}^{N} A(S_i(\gamma)) \\ &= N\left(\frac{2\pi^t\sqrt{\gamma}(\gamma-\gamma_0)^{t-1}}{(t-1)!}\right),\ \gamma_0 \leq \gamma < \gamma^*. \end{aligned} \tag{26}$$

Finally, when $\gamma \geq \gamma^*$, the spherical caps are no longer disjoint and hence we have

$$\begin{aligned} A\left(\bigcup_{i=1}^{N} S_i(\gamma)\right) &\leq \sum_{i=1}^{N} A(S_i(\gamma)) \\ &= N\left(\frac{2\pi^t\sqrt{\gamma}(\gamma-\gamma_0)^{t-1}}{(t-1)!}\right),\ \gamma \geq \gamma^*. \end{aligned} \tag{27}$$

Substituting (18) and the above set of relations in (23), we get

$$\begin{aligned} P_{out}^c(R, P \mid \|\mathbf{h}\|^2 = \gamma) &= 0,\ \gamma < \gamma_0 \\ &= N\left(1 - \frac{\gamma_0}{\gamma}\right)^{(t-1)},\ \gamma_0 < \gamma < \gamma^* \\ &\leq N\left(1 - \frac{\gamma_0}{\gamma}\right)^{(t-1)},\ \gamma^* < \gamma \end{aligned} \tag{28}$$

Let $\gamma_1 > \gamma_0$ be such that $N\left(1 - \frac{\gamma_0}{\gamma_1}\right)^{(t-1)} = 1$, so that

$$\gamma_1 = \frac{\gamma_0}{\left(1 - (\frac{1}{N})^{(\frac{1}{t-1})}\right)}. \tag{29}$$

Considering that we will eventually derive a lower bound on outage probability of the beamformer

$\mathcal{C}$, we can rewrite the set of inequalities in (28) as

$$\begin{aligned} P_{out}^{c}(R, P \mid \|\mathbf{h}\|^2 = \gamma) &= 0,\ \gamma < \gamma_0 \\ &\leq N\left(1 - \frac{\gamma_0}{\gamma}\right)^{(t-1)},\ \gamma_0 < \gamma < \gamma_1 \\ &\leq 1,\ \gamma_1 < \gamma. \end{aligned} \tag{30}$$

Note that the set of relations given in (28) are still useful and we will revisit them while discussing the design criterion for beamformers.

G. *Lower bound on outage probability with feedback*

In this section, we will derive a lower bound on the outage probability of beamformer codebooks with finite number of beamforming vectors. It should be noted that the lower bound derived holds for an arbitrary beamformer codebook $\mathcal{C}$. Thus, the quality of any beamformer codebook $\mathcal{C}$ can be determined by measuring its performance against the lower bound.

We will make use of the distribution of γ along with conditional distribution of successful transmission given by (30) to arrive at the lower bound on the probability of outage for the beamformer codebook $\mathcal{C}$. Note that $\gamma = \sum_{i=1}^{t} |h_i|^2$, where each h_i is distributed as complex proper Gaussian with zero mean and unit variance. Therefore, γ is given by the sum of the squares of $2t$ Gaussian random variables with zero mean and variance equal to $1/2$. Hence, γ has a central chi-square distribution with $2t$ degrees of freedom. The probability density function (pdf) of γ is given by [11]

$$p_\gamma(\gamma) = \frac{\gamma^{t-1} e^{-\gamma}}{(t-1)!}. \tag{31}$$

The cumulative distribution function (cdf) of γ is given by [11]

$$\begin{aligned} F_\gamma(\gamma) &= \int_0^\gamma p_\gamma(u)\, du \\ &= 1 - e^{-\gamma} \sum_{k=0}^{t-1} \frac{\gamma^k}{k!}. \end{aligned} \tag{32}$$

We can now integrate the conditional distribution of successful transmission given in (30) using the distribution of γ given in (31) to arrive at the complement of the probability of outage of the beamformer. The first main result of this paper is given in the following theorem.

Theorem 1 (Outage lower bound) *Consider a wireless system with t transmit antennas along with a single receive antenna transmitting at rate R bits/sec/Hz and SNR P over a quasi-static Rayleigh fading channel. The outage probability of such a system employing a beamformer codebook with N beamforming vectors (corresponding to $\log_2(N)$ bits of feedback per frame) is bounded below as*

$$P_{out}(R,P) \;\; \geq \;\; 1 - Ne^{-\gamma_0} + e^{-\gamma_1} \sum_{k=0}^{t-1} \left(\frac{N(\gamma_1 - \gamma_0)^k - \gamma_1^k}{k!} \right) \tag{33}$$

where

$$\gamma_0 = \frac{2^R - 1}{P} \textit{ and } \gamma_1 = \frac{\gamma_0}{\left(1 - (\frac{1}{N})^{(\frac{1}{t-1})}\right)} \tag{34}$$

Proof : Let $\mathcal{C}$ be an arbitrary beamformer codebook with N beamforming vectors. Let $P^c_{out}(R,P)$ be the probability of no-outage when $\mathcal{C}$ is used for beamforming. We have

$$P^c_{out}(R,P) \;\; = \;\; \int_{\gamma=0}^{\infty} P^c_{out}(R,P \mid \|\mathbf{h}\|^2 = \gamma) p_\gamma(\gamma)\, d\gamma$$

Using the upper bound on $P^c_{out}(R,P \mid \|\mathbf{h}\|^2 = \gamma)$ from (30) and the distribution of γ in (31), we get

$$\begin{aligned} P^c_{out}(R,P) \;\; &= \;\; \int_{\gamma=0}^{\infty} P^c_{out}(R,P \mid \|\mathbf{h}\|^2 = \gamma) p_\gamma(\gamma)\, d\gamma \\ &\leq \;\; \int_{\gamma=\gamma_0}^{\gamma_1} N \left(1 - \frac{\gamma_0}{\gamma}\right)^{(t-1)} p_\gamma(\gamma)\, d\gamma + \int_{\gamma=\gamma_1}^{\infty} 1\, p_\gamma(\gamma)\, d\gamma \end{aligned} \tag{35}$$

The second term in (35) above can be simplified as

$$\int_{\gamma=\gamma_1}^{\infty} 1\, p_\gamma(\gamma)\, d\gamma \;\; = \;\; 1 - F_\gamma(\gamma_1), \tag{36}$$

where $F_\gamma(\gamma)$ is given by (32). The first term in (35) can be simplified as follows. We have

$$\begin{aligned} \int_{\gamma=\gamma_0}^{\gamma_1} N \left(1 - \frac{\gamma_0}{\gamma}\right)^{(t-1)} p_\gamma(\gamma)\, d\gamma \;\; &= \;\; \int_{\gamma=\gamma_0}^{\gamma_1} \frac{N}{(t-1)!} \left(1 - \frac{\gamma_0}{\gamma}\right)^{(t-1)} \gamma^{t-1} e^{-\gamma}\, d\gamma \\ &= \;\; \frac{N}{(t-1)!} \int_{\gamma=\gamma_0}^{\gamma_1} (\gamma - \gamma_0)^{t-1} e^{-\gamma}\, d\gamma \\ &= \;\; \frac{N}{(t-1)!} \int_{u=0}^{\gamma_1-\gamma_0} u^{t-1}\, e^{-u}\, e^{-\gamma_0}\, du \qquad (37) \\ &= \;\; Ne^{-\gamma_0} F_\gamma(\gamma_1 - \gamma_0), \qquad (38) \end{aligned}$$

where we have used the substitution $u = \gamma - \gamma_0$ in (37). Using (36) and (38) in (35) and using $P_{out}(R,P) = 1 - P_{out}^c(R,P)$, we get

$$\begin{aligned} P_{out}(R,P) &\geq F_\gamma(\gamma_1) - N\, e^{-\gamma_0}\, F_\gamma(\gamma_1 - \gamma_0) \\ &= \left(1 - e^{-\gamma_1} \sum_{k=0}^{t-1} \frac{\gamma_1^k}{k!}\right) - N\, e^{-\gamma_0} \left(1 - e^{-(\gamma_1-\gamma_0)} \sum_{k=0}^{t-1} \frac{(\gamma_1 - \gamma_0)^k}{k!}\right) \\ &= 1 - N\, e^{-\gamma_0} + e^{-\gamma_1} \sum_{k=0}^{t-1} \left(\frac{N(\gamma_1 - \gamma_0)^k - \gamma_1^k}{k!}\right) \end{aligned}$$

■

The lower bound given above is valid at all SNR and for an arbitrary beamformer comprising of N beamforming vectors. We have observed that the bound is fairly tight for various cases of t and $N \geq t$ which were investigated (see Section IV(B.)). Note that even though the lower bound is valid when $N < t$ (i.e., fewer beamforming vectors than the transmit antennas), it is very loose in that case. In particular, with the beamforming methodology used in this paper, the rate of decay of outage probability with SNR for beamforming will be $\min(t, N)$ while the bound decays at the rate of t as we show in the next section. In the next section, we also show that the bound is asymptotically tight in the number of beamforming vectors. The bound will be used to explicitly characterize the behaviour of outage probability of good beamformers as a function of the number of transmit antennas and the number of beamforming vectors. Finally, the procedure used in arriving at the bound will also prove useful in characterizing near-optimal beamformer codebooks and will be used to derive a design criterion for good beamformer codebooks.

We will now evaluate the outage probability of beamforming when perfect channel information is available at the transmitter. Note that this can be treated as the case when N is made very large. The case of perfect channel information at the transmitter serves as a useful reference in evaluating the performance of beamformer codebooks with finite number of beamforming vectors.

1. *Perfect feedback bound*

Suppose that the channel is known perfectly at the transmitter and the receiver. It was shown in [12] that the optimal scheme for minimizing outage consists of a Gaussian code followed by a beamformer. The optimal beamformer is given by a unit vector in the direction of $\mathbf{h}$, where $\mathbf{h}$ is the channel realization. The received signal in this case is given by

$$y = \|\mathbf{h}\| x + \eta. \tag{39}$$

The mutual information with Gaussian codes is given by

$$I(x;y \mid \mathbf{h}) = \log_2(1 + P\|\mathbf{h}\|^2) \tag{40}$$

and the outage probability for a rate R is given by

$$\begin{aligned} P_{out}(R,P) &= \text{Prob}[\log_2(1 + P\|\mathbf{h}\|^2) < R] \\ &= \text{Prob}[\gamma < \frac{2^R - 1}{P}] \\ &= 1 - e^{-\gamma_0} \sum_{k=0}^{t-1} \frac{\gamma_0^k}{k!} \end{aligned} \tag{41}$$

where we have used the central chi-squared distribution of γ given in (31) and also substituted γ_0 for $\frac{2^R-1}{P}$.

The bound given in (33) can be used to understand the effect of increasing the capacity of feedback channel. Figure 4 compares the performance predicted by the lower bound in (33) with the outage probability of beamforming with perfect feedback given in (41), for the case of 4 transmit antennas and a single receive antenna at a fixed SNR of 10dB. It can be seen that the performance gap between 2 bit quantized feedback and perfect feedback is significant. But the gap is considerably reduced by about 8 bits of feedback. The bound suggests that about 8 bits of feedback is sufficient to perform very close to the case of perfect channel information with 4 transmit antennas and further increase in the number of feedback bits will result in very little improvement in the outage performance.

We will now illustrate the behaviour of the bound as a function of the number of beamforming vectors N and SNR.

2. *Asymptotic tightness in N*

In this section, we will investigate the effect of increasing N on the lower bound on outage probability given in (33). Note that perfect channel information is the limiting case of quantized channel information as N grows without bound. In particular, we will show that the lower bound in (33) converges to (41) for large N.

Proposition 1 (Asymptotic tightness in N) *The lower bound on the outage probability of beamforming with finite rate feedback given by Theorem 1 is asymptotically tight in the number of beamforming vectors N.*

Proof : Let $K_N = \left(1 - (\frac{1}{N})^{(\frac{1}{t-1})}\right)$. Then, $\gamma_1 - \gamma_0 = \frac{\gamma_0(1-K_N)}{K_N}$, so that the lower bound on outage in (33) can be written as

$$P_{out}(R,P) \;\; \geq \;\; 1 - Ne^{-\gamma_0} + e^{-\frac{\gamma_0}{K_N}} \sum_{k=0}^{t-1} \frac{\gamma_0^k}{k!} \left(\frac{N(1-K_N)^k - 1}{K_N^k} \right) \tag{42}$$

Using the series expansion of e^x, the right hand side of (42) can be written as

$$1 - Ne^{-\gamma_0} + Ne^{-\frac{\gamma_0}{K_N}} \left(e^{\frac{\gamma_0(1-K_N)}{K_N}} - \sum_{k=t}^{\infty} \frac{\gamma_0^k}{k!} \left(\frac{(1-K_N)^k}{K_N^k} \right) \right) - e^{-\frac{\gamma_0}{K_N}} \sum_{k=0}^{t-1} \frac{\gamma_0^k}{k! K_N^k}. \tag{43}$$

The second and third terms in (43) can be simplified as

$$-e^{-\frac{\gamma_0}{K_N}} \sum_{k=t}^{\infty} \frac{\gamma_0^k}{k!} \left(\frac{N(1-K_N)^k}{K_N^k} \right). \tag{44}$$

Now consider the term $\left(\frac{N(1-K_N)^k}{K_N^k}\right)$ in the above expression. Substituting for K_N, we have

$$\begin{aligned} \frac{N(1-K_N)^k}{K_N^k} &= \frac{N}{\left(N^{\frac{1}{t-1}} - 1\right)^k} \\ &\approx \frac{N}{N^{\frac{k}{t-1}}} \text{ for large } N \\ &\to 0 \text{ for } k \geq t \end{aligned} \tag{45}$$

Hence, the expression in (44) goes to 0 for large N. Finally, noting that $K_N \to 1$, for large N, we see that (43) can be simplified to give the lower bound asymptotically as

$$P_{out}(R,P) \;\; \geq \;\; 1 - e^{-\gamma_0} \sum_{k=0}^{t-1} \frac{\gamma_0^k}{k!}, \tag{46}$$

which is the same as the outage probability of beamforming with perfect feedback given in (41). ■

Therefore, the lower bound given by Theorem 1 is asymptotically tight as the number of feedback bits gets large. This also suggests that, for sufficient number of feedback bits, the lower bound can be used as a good approximation to the outage probability of "good" beamformers. We will discuss the construction of good beamformers whose outage performance will be close to that predicted by Theorem 1 in Section IV.

3. *High SNR approximation*

In this section, we will investigate the behaviour of the outage probability of good beamformers with increasing SNR. From the above discussion and Theorem 1, the outage probability of good beamformers, for sufficiently large N, can be approximated by

$$\begin{aligned} P_{out}(R,P) &\approx 1 - Ne^{-\gamma_0} + e^{-\gamma_1}\sum_{k=0}^{t-1}\left(\frac{N(\gamma_1-\gamma_0)^k - \gamma_1^k}{k!}\right) \\ &= 1 - Ne^{-\gamma_0} + e^{-\gamma_1}\left(N\left(e^{\gamma_1-\gamma_0} - \sum_{k=t}^{\infty}\frac{(\gamma_1-\gamma_0)^k}{k!}\right) - \left(e^{\gamma_1} - \sum_{k=t}^{\infty}\frac{\gamma_1}{k!}\right)\right) \quad (47) \\ &= \sum_{k=t}^{\infty}\left(\frac{\gamma_1^k - N(\gamma_1-\gamma_0)^k}{k!}\right), \quad (48) \end{aligned}$$

where we have used the series expansion of e^x twice in (47). Note that $\gamma_0 = \frac{2^R-1}{P}$ so that as P increases, γ_0 goes to zero. Substituting for γ_1 in terms of γ_0 in (48) and ignoring the higher powers of γ_0, we get,

$$\begin{aligned} P_{out}(R,P) &\approx \frac{\gamma_0^t}{t!}\frac{1 - N(\frac{1}{N})^{\frac{t}{t-1}}}{\left(1 - (\frac{1}{N})^{\frac{1}{t-1}}\right)^t} \\ &= \frac{\gamma_0^t}{t!}\frac{1}{\left(1 - (\frac{1}{N})^{\frac{1}{t-1}}\right)^{t-1}}. \quad (49) \end{aligned}$$

We can do a similar high SNR analysis of outage probability of beamforming with perfect feedback. In particular, from (41), we get

$$\begin{aligned} P_{out}(R,P) &= 1 - e^{-\gamma_0}\sum_{k=0}^{t-1}\frac{\gamma_0^k}{k!} \\ &= 1 - e^{-\gamma_0}\left(e^{\gamma_0} - \sum_{k=t}^{\infty}\frac{\gamma_0^k}{k!}\right) \\ &= e^{-\gamma_0}\sum_{k=t}^{\infty}\frac{\gamma_0^k}{k!} \\ &\approx \frac{\gamma_0^t}{t!} \text{ for large } P, \quad (50) \end{aligned}$$

where we have neglected the higher order terms in (50).

The high SNR approximation of outage probability of beamforming with finite feedback given by (49) can be reinterpreted in the light of (50). In particular, (49) consists of two terms, where the first term is the same as the outage probability of beamforming with perfect channel information as given by (50). The second term in (49) accounts for the finite nature of the feedback and captures the loss incurred by finite rate feedback. The loss in outage performance with finite rate feedback compared to perfect feedback is characterized by

$$L(t,N) = \left(1 - \left(\frac{1}{N}\right)^{\frac{1}{t-1}}\right)^{-(t-1)} \tag{51}$$

Note that $L(t,N) \geq 1$ and decays to 1 with increasing N for a fixed t. In the next section, we define a distortion measure based on the above high SNR analysis of outage probability which has a close resemblance to the rate distortion function in the theory of source coding.

4. *Distortion measure for quantization*

Let $P_{out}^{\infty}(R,P)$ denote the outage probability of beamforming with perfect channel information at the transmitter, for large P. Also, let $P_{out}^{B}(R,P)$ denote the outage probability of a good beamformer with B bits of feedback, for large P. We define a measure of distortion, $F(B)$, as follows.

Definition 2 : *The distortion measure $F(B)$ is defined as the relative loss in outage performance of the best beamformer with B bits of feedback when compared to a beamformer designed with perfect channel information at the transmitter, for large SNR.*

Using the notation developed earlier, we have

$$F(B) = \frac{P_{out}^{B}(R,P) - P_{out}^{\infty}(R,P)}{P_{out}^{\infty}(R,P)}. \tag{52}$$

Substituting (49) and (50) in (52), we get

$$\begin{aligned} F(B) &= \frac{1}{\left(1 - (\frac{1}{N})^{\frac{1}{t-1}}\right)^{t-1}} - 1 \\ &\approx \left(1 + (t-1)(\frac{1}{N})^{\frac{1}{t-1}}\right) - 1, \text{ for sufficiently large } N \\ &= (t-1)(\frac{1}{N})^{\frac{1}{t-1}} \\ &= (t-1)2^{-(\frac{B}{t-1})}, \text{ using } N = 2^B \end{aligned} \tag{53}$$

Hence, the performance loss of quantized beamforming compared to perfect channel feedback decays exponentially with the number of feedback bits B for sufficiently large B and SNR. Figure 5 plots $\log(F(B))$ as a function of the number of feedback bits B for different number of transmit antennas. The slope of the $F(B)$ curve is inversely proportional to the number of transmit antennas. This means that the marginal performance gain obtained by the addition of a feedback bit reduces as we increase the number of transmit antennas. This is not surprising since the number of channel coefficients to be quantized increases with the transmit antennas. It is interesting to note that with the mean squared error as distortion metric D, between $\mathbf{h}$ and C_i, the classical rate distortion theory predicts that $D(B) = 2^{-\frac{B}{t}}$. Hence, our distortion measure based on the outage probability behaves similar to the distortion metric based on the mean squared error in B, though not in t.

Narula et al. [3] have arrived at a similar result with received SNR as the performance metric. We will recast their result in a form suitable for easy comparison with (53). If we define $\Gamma(B)$ as the fraction loss in SNR with quantized beamforming compared to a beamformer designed with perfect channel information, we have from [3], after suitable manipulations,

$$\Gamma(B) \approx \frac{t-1}{t} 2^{-(\frac{B}{t})}. \tag{54}$$

Hence, the SNR metric also displays an exponential decay with respect to the number of feedback bits similar to the outage metric given by (53), though the behaviour is different in t.

We should also point out the different conditions under which (53) and (54) are attained. The distortion rate function $F(B)$ defined in this paper is valid for high SNR and large number of beamforming vectors. We do not need the number of transmit antennas to be large. On the other hand, the distortion rate function in (54) is derived using the mean squared distortion rate function $D(B)$ of the classical rate distortion theory. The bounds of the rate distortion theory can be approached by coding over large blocks of source outputs (channel is the source to be encoded in this problem) with independent realizations. Since, quasi-static fading assumptions for the channel do not allow for independent realizations over time, Narula et al. [3] resort to independent realizations over space (obtained via large number of transmit antennas). Hence, (53) requires high SNR while (54) requires large number of transmit antennas.

IV. Construction of beamformers

In the previous section, we presented a lower bound on the outage probability of beamformer codebooks with finite number of vectors. In this section, we will discuss the construction of beamformer codebooks approaching the bound in their outage performance. We will see that the performance of the beamformer codebooks thus constructed comes very close to the lower bound presented in the last section. The design criterion for beamformer codebooks is also related to the problem of designing unitary space time codes for multiple antennas as well as the problem of subspace packing in real Grassmannian spaces.

A. Design criterion

We can draw inferences about the structure of good beamformer codebooks from the proof methodology of the lower bound discussed in the last section. Recall from Section III(F.), the probability of successful transmission with a beamformer codebook $\mathcal{C}$, conditioned on the channel norm is given as

$$\begin{aligned} P_{out}^{c}(R, P \mid \|\mathbf{h}\|^2 = \gamma) &= 0, \ \gamma < \gamma_0 \\ &= N\left(1 - \frac{\gamma_0}{\gamma}\right)^{(t-1)}, \ \gamma_0 < \gamma < \gamma^* \\ &\leq N\left(1 - \frac{\gamma_0}{\gamma}\right)^{(t-1)}, \ \gamma^* < \gamma, \end{aligned} \tag{55}$$

where γ^* was defined as the overlap radius of the beamformer codebook $\mathcal{C}$. For $\gamma < \gamma^*$, the spherical caps do not overlap and hence the total area corresponding to successful transmissions is given by the sum of the areas of individual spherical caps. On the other hand, for $\gamma \geq \gamma^*$, some of the spherical caps overlap and as a result, the area corresponding to the successful transmissions is less than that of the sum of the individual spherical caps. This leads to the inequality for $\gamma \geq \gamma^*$ in (55) above. It is clear that the performance of a beamformer can be improved by increasing its overlap radius so that the spherical caps do not intersect upto a larger radius. The lower bound in Theorem 1 can be achieved with equality if the beamformer is such that the spherical caps do not overlap till the entire surface area is covered for a certain γ. For such a beamformer, $\gamma^* = \gamma_1$, where γ_1 was defined such that

$$\left(1 - \frac{\gamma_0}{\gamma_1}\right)^{(t-1)} = 1. \tag{56}$$

Hence, the design criterion for a good beamformer will be to maximize the overlap radius γ^*. The design of good beamformers is reduced to the following optimization problem

$$\max_{\mathcal{C} \in \mathbb{C}^t} \gamma^*(\mathcal{C}). \tag{57}$$

We will now use the definitions of the spherical caps and γ^* in particular to calculate the overlap radius for a given beamformer codebook. We will show that the overlap radius is inversely proportional to the angle between the two closest vectors in the codebook, where the angles are defined via the inner product norm. Hence, the design criterion for good beamformer codebook seeks to maximize the angular distance between any two vectors in the codebook. In this respect, we will see that (57) is similar to the design criterion for signal constellations for Gaussian channels. In both the cases, the goal is to maximize the minimum distance between any two members in the set. While Euclidean distance is the relevant metric for signal constellation design, inner product norm serves the purpose of metric in the case of beamformer codebook design. We show the calculation of γ^* for a beamformer codebook in the following lemma.

Lemma 5 (Overlap radius) *For any beamformer codebook $\mathcal{C}$, we have*

$$\gamma^*(\mathcal{C}) = \left(\frac{2\gamma_0}{1 + \max_{i,j:i \neq j} |\langle C_i, C_j \rangle|} \right) \tag{58}$$

Proof : Let $\mathcal{C} = \{C_1, C_2, ..C_N\}$ be any beamformer comprising of N beamforming vectors, with $C_i \in \mathbb{C}^t$ for each i. Following the notation developed earlier, for a given channel norm $\sqrt{\gamma}$, let $V_i(\gamma)$ and $\Theta_i(\gamma)$ be the Voronoi region and no-outage region corresponding to C_i, for each i. Further, let $S_i(\gamma)$ be the spherical cap around C_i as described in Section III(E.). Let $\sqrt{\gamma_{ij}}$ be the smallest channel norm for which $S_i(\gamma)$ and $S_j(\gamma)$ intersect. Then, it is clear that γ^* can be calculated as $\gamma^* = \min_{1 \leq i < j \leq N} \gamma_{ij}$. Hence, we will first evaluate γ_{ij} for a given i and j and then calculate γ^*.

Consider the smallest channel norm $\sqrt{\gamma_{12}}$ at which $S_1(\gamma)$ intersects $S_2(\gamma)$. By the definition of $S_i(\gamma)$, if $\mathbf{h}$ is a vector which lies on the boundary of $S_i(\gamma)$, then $\mathbf{h}$ satisfies $|\langle \mathbf{h}, C_i \rangle|^2 = \gamma_0$. Let $\mathbf{h}$ be a vector which lies on the boundary of $S_1(\gamma)$ and $S_2(\gamma)$. Also, let $||\mathbf{h}||^2 = \gamma$. Then, γ_{12} is the smallest value of γ which satisfies the following conditions.

$$\begin{aligned} \mathbf{h} &= \sqrt{\gamma}(\alpha C_1 + \beta C_2 + \delta C) && (59) \\ \|\mathbf{h}\|_2^2 &= \gamma && (60) \\ |\langle h, C_1\rangle|^2 &= \gamma_0 && (61) \\ |\langle h, C_2\rangle|^2 &= \gamma_0 && (62) \end{aligned}$$

where $C \in \mathbb{C}^t$ is a unit vector orthogonal to both C_1 and C_2. Also, α, β and γ are complex scalars.

On equating the left hand side of (61) and (62), we get

$$\begin{aligned} |\alpha|^2 + |\beta|^2|\langle C_1, C_2\rangle|^2 &= |\beta|^2 + |\alpha|^2|\langle C_1, C_2\rangle|^2 \\ \Rightarrow |\alpha| &= |\beta|. && (63) \end{aligned}$$

(where we have ignored the less interesting case of $|\langle C_1, C_2\rangle| = 1$ which happens when C_1 is a scalar multiple of C_2. In this case, we can drop one of the two vectors from $\mathcal{C}$ without affecting the outage performance of $\mathcal{C}$.)

Without loss of generality, let α be real and $\beta = \alpha e^{j\theta}$. Substituting this in (61), we get

$$\gamma_0 = \gamma\alpha^2 \left(1 + |\langle C_1, C_2\rangle|^2 + 2\text{Re}\left(e^{-j\theta}\langle C_1, C_2\rangle\right)\right). \quad (64)$$

Similarly, substituting for α and β in (59), we get

$$2\alpha^2 \left(1 + \text{Re}\left(e^{-j\theta}\langle C_1, C_2\rangle\right)\right) + |\delta|^2 = 1 \quad (65)$$

Now, substituting the value of α from (65) in (64) and simplifying, we get

$$\gamma = \frac{2\gamma_0 \left(1 + \text{Re}\left(e^{-j\theta}\langle C_1, C_2\rangle\right)\right)}{(1 - |\delta|^2)\left(1 + |\langle C_1, C_2\rangle|^2 + 2\text{Re}\left(e^{-j\theta}\langle C_1, C_2\rangle\right)\right)} \quad (66)$$

Finally, note that γ_{12} is the minimum value of γ which satisfies this equation. To evaluate γ_{12}, we minimize γ with respect to θ and δ. Clearly, γ is minimized when $|\delta| = 0$ (i.e., the spherical caps $S_1(\gamma)$ and $S_1(\gamma)$ first intersect in the subspace formed by C_1 and C_2). Let γ assume its minimum value at $\theta = \theta_{12}$. Differentiating γ with respect to θ and setting it to zero at $\theta = \theta_{12}$, we get

$$\begin{aligned} \text{Re}\left(-je^{-j\theta_{12}}\langle C_1, C_2\rangle\right) &= 0 \\ \Rightarrow \phi - \theta_{12} &= k\pi && (67) \end{aligned}$$

where $\phi = \text{phase}\left(\langle C_1, C_2\rangle\right)$ and k is any integer.

Evaluating the second derivative and setting it greater than zero at θ_{12} (for a minima at θ_{12}), we get

$$\text{Re}\left(e^{-j\theta_{12}}\langle C_1, C_2\rangle\right) \; > \; 0 \tag{68}$$

From (67) and (68), we see that γ is minimized at $\theta_{12} = \phi + 2k\pi$, for any integer k. Set $k = 0$ to give $\theta_{12} = \phi$. Substituting this value of θ_{12} in (66), we get

$$\gamma_{12} \;=\; \frac{2\gamma_0}{1 + |\langle C_1, C_2\rangle|}. \tag{69}$$

Finally, $\gamma^*\left(\mathcal{C}\right)$ is given by

$$\begin{aligned} \gamma^*\left(\mathcal{C}\right) &= \min_{i,j:i\neq j} \gamma_{ij} \\ &= \left(\frac{2\gamma_0}{1 + \max_{i,j:i\neq j} |\langle C_i, C_j\rangle|}\right). \end{aligned} \tag{70}$$

■

Hence, the overlap radius is a function of the two closest vectors in the beamformer. The design criterion for beamformers $\max_{\{\mathcal{C}\}} \gamma^*(\mathcal{C})$, given in (57), is now equivalently given by the following lemma.

Lemma 6 (Design criterion) *The design criterion for a good beamformer codebook $\mathcal{C}$ comprising of N beamforming vectors for t transmit antennas is given by*

$$\min_{\mathcal{C} \in \underbrace{\mathbb{C}^t \times .. \times \mathbb{C}^t}_{N}} \; \max_{\substack{i,j: \\ 1 \leq i \neq j \leq N}} \; |\langle C_i, C_j\rangle|. \tag{71}$$

The norm of the inner product $|\langle C_i, C_j\rangle|$ is inversely proportional to the angular "distance" between C_i and C_j. In particular, $|\langle C_i, C_j\rangle|$ attains a maximum value of 1 when C_i is aligned with C_j (corresponding to minimum angular distance), while the minimum value of 0 is attained when C_i is orthogonal to C_j (corresponding maximum angular distance). Hence, the design criterion given in (71) seeks to maximize the angular distance between the two closest vectors in the codebook. In this respect, the design criterion is similar to the criterion of maximizing the minimum Euclidean distance used for signal constellation design for Gaussian channels. However, the Euclidean distance metric is markedly different from the norm of the inner product distance.

It is worthwhile to note that for the special case where the number of beamforming vectors equals the number of transmit antennas, the design criterion results in selection diversity.

Corollary 1 *When $N = t$, the design criterion in (71) leads to selection diversity with $\gamma^* = 2\gamma_0$.*

Proof : For any beamformer codebook $\mathcal{C}$, $\max_{i,j:i\neq j} |\langle C_i, C_j\rangle| \geq 0$, where the equality occurs if and only if C_i is orthogonal to C_j for all $i \neq j$. When $N = t$, we can achieve the equality by choosing $\mathcal{C}$ as an orthonormal basis of $\mathbb{C}^t$. For example, $\mathcal{C}$ could be standard basis of $\mathbb{C}^t$ given by $C_i = [0\ 0\ ..1..\ 0]$, where the 1 occurs for C_i in the ith position. This is nothing but choosing the best antenna for transmission, which is commonly referred to as *selection diversity*. ■

Hence, the design rule of equation (71) is consistent with our knowledge for $\log_2(t)$ bits of feedback information. This design rule can also be used to verify the asymptotic tightness of the lower bound, given in Theorem 1, in the number of beamforming vectors. We have already mentioned that the performance given by the lower bound can be achieved if $\gamma^* = \gamma_1$. Using (56) and (58), the lower bound will be achieved if

$$\begin{aligned} \frac{\gamma_0}{\left(1 - (\frac{1}{N})^{(\frac{1}{t-1})}\right)} &= \frac{2\gamma_0}{1 + \max_{i,j:i\neq j} |\langle C_i, C_j\rangle|} \\ \Rightarrow \max_{i,j:i\neq j} |\langle C_i, C_j\rangle| &= 1 - 2\left(\frac{1}{N}\right)^{\frac{1}{t-1}} \end{aligned} \tag{72}$$

As N grows, the right hand side in the above equation goes to 1. Hence, the equality holds if $\max_{i,j:i\neq j} |\langle C_i, C_j\rangle| \rightarrow 1$ which will be true for any beamformer codebook. Hence, lower bound is achievable by any non-degenerate beamformer codebook (i.e., $C_i \neq C_j$, $i \neq j$) as the number of beamforming vectors grows large.

B. Simulation results

We present the performance results of beamformers constructed using the design criterion given in (71). We use computer aided search process to design the beamformers. In particular, we use the `fminimax` function in the optimization toolbox of matlab, to arrive at the beamforming vectors. We provide simulation results for the outage probability performance of beamforming vectors in a quasi-static channel. For the sake of comparison, we also give the lower bound on outage probability from (33). The performance of constructed beamformers for four transmit antennas with 2, 3 and 4 bits of feedback is presented in Figure 6 for the case of $R = 2$ bits/sec/Hz. Note that

2 bits of feedback with 4 transmit antennas results in the selection diversity scheme as discussed above. From Figure 6 we see that the outage probability is very close to the corresponding lower bound predicted by (33), thus reflecting the tightness of the bound. The tightness of the bound improves as the number of feedback bits B is increased. Further, the outage probability gets closer to the case of perfect channel information as B increases. We also observe that a few bits of feedback can achieve most of the gains promised by beamforming with perfect channel information.

C. Similarity with unitary space time constellations

Unitary space time constellations for multiple antenna systems were introduced by Marzetta and Hochwald in [13]. In particular, it was shown that unitary space time constellations achieve the capacity of multiple antenna systems when the channel information is not available at either the transmitter or the receiver. The design criterion for unitary space time constellations to minimize the pair-wise error probability was given in [14]. A unitary space time constellation consists of signals $\Phi_1, \Phi_2, ..\Phi_N$, where $\Phi_i \in \mathbb{C}^{T \times M}$, where T is the block length of the code (less than or equal to the coherence time of the channel), M is the number of transmit antennas and N is the number of codewords in the constellation. Also, the unitarity condition implies that $\Phi_i^\dagger \Phi_i = I$ for each i. It was shown that the design criterion for unitary space time constellations, which minimize pair-wise error probability, is to minimize δ, where δ is defined as

$$\delta \;\; = \;\; \max_{1 \leq l < l' \leq N} \|\Phi_l^\dagger \Phi_{l'}\|, \tag{73}$$

where the norm used above is a scaled Frobenius norm of a matrix, the scaling factor being given by M in this case.

Recall that, for the finite feedback problem considered in this paper, if the beamformer codebook $\mathcal{C}$ comprises of $\{C_1, C_2, .., C_N\}$, then the design criterion for good beamformers is

$$\min_{\mathcal{C} \in \mathbb{C}^t} \max_{1 \leq l < l' \leq N} |\langle C_l, C_{l'} \rangle|, \tag{74}$$

where $\langle C_l, C_{l'} \rangle = C_l C_{l'}^\dagger$. The criterion (73) is equivalent to the criterion for beamformers given in (74), if we set $T = t$ and $M = 1$ in (73). We see an equivalence between coherence time, in the unitary constellation design problem, and the number of transmit antennas, in the beamforming design problem. Hence, the design of beamformers is equivalent to the problem of the design of unitary space time constellations under these conditions.

Hochwald et al. [15] addressed the problem of imposing structure on the unitary space time constellations for easier encoding/decoding as well as to reduce the dimensionality of the search space during the design process. The equivalence of good beamformers with good unitary space time constellations established in this paper allows the use of the systematic unitary space time constellations designed in [15] as beamformer codebooks. We reproduce the construction of systematic unitary space time codes from [15] to explain the equivalence with an example.

Consider the set of linear block codes defined by the $K \times T$ generator matrix U, whose elements are in R_q, ring of integers modulo-q. The code $\mathcal{C}$ represented by U comprises of codewords C_l given by $C_l = l.U$, where l is a $1 \times K$ vector with elements taken from R_q. Thus, the size of the codebook is given by $|\mathcal{C}| = q^K$. The codewords are mapped into signals by mapping the integers in the codeword into components of a complex signal via the transformation

$$\phi(j) = \frac{1}{\sqrt{T}} e^{i\frac{2\pi}{q}j}, j = 0, 1, ..q-1. \tag{75}$$

Finally, the unitary space time constellation is given by

$$\begin{aligned} \Phi_l &= \phi(C_l) \\ &= \frac{1}{\sqrt{T}} \begin{bmatrix} e^{i\frac{2\pi}{q}[C_l]_1} \\ e^{i\frac{2\pi}{q}[C_l]_2} \\ . \\ . \\ e^{i\frac{2\pi}{q}[C_l]_T} \end{bmatrix}, 1 \leq l \leq N \end{aligned} \tag{76}$$

Suppose, we wish to design a beamformer comprising of $N = 16$ vectors for $t = 8$ transmit antennas and a single receive antenna. Then the equivalent problem of designing unitary space time constellation reduces to designing a codebook of size $N = 16$ with a single transmit antenna (i.e., $M = 1$) and coherence time given by $T = 8$. One of the codes designed for this problem in [15] is characterized by $U = [1\ 0\ 3\ 14\ 15\ 11\ 10\ 8]$, $K = 1$ and $q = 16$.

The performance of this beamformer is given in Figure 7. The rate of transmission is $R =$ 2bits/sec/Hz. The corresponding performance predicted by the lower bound on outage is also given in the figure. It can be seen that the performance of the constructed beamformer comes very close (less than 0.05dB away) to the corresponding lower bound on outage probability. The performance of (full rank) space times codes, which do not require any feedback information, is also given in the plots for reference. The performance of the space time code was obtained analytically assuming an optimal code for the case of 8 transmit antennas [9]. Note that there is a gain of about 4dB with 4 bits of feedback compared to the space time codes. This example shows that the systematic unitary space time constellations serve as very good beamformers. Some more constructions of the beamformers for $t = 8$ transmit antennas can be obtained from [15] for the cases of $N = 64, 133, 256, 529, 1296, 2209$ vectors.

D. Packings in Grassmannian spaces

We conclude the discussion on the construction of beamformer codebooks by showing similarities to the problem of packings in Grassmannian spaces, which was discussed in detail by Conway et al. in [16]. The similarity could potentially lead to new construction methods for beamformer codebooks. Consider the vector $C_i \in \mathbb{C}^t$. We can write $C_i = C_{iR} + \jmath C_{iI}$, where C_{iR} and C_{iI} are the real and the imaginary parts of C_i respectively. Now consider the two dimensional subspace, F_i, of $\mathbb{R}^{2t}$ formed by the span of the columns of the matrix M_i given by

$$M_i = \begin{bmatrix} C_{iR} & C_{iI} \\ C_{iI} & -C_{iR} \end{bmatrix}. \tag{77}$$

A measure of distance, called the *chordal distance*, between the subspaces F_i and F_j is given by,

$$d_c(F_i, F_j) = \sqrt{\sin^2(\theta_1) + \sin^2(\theta_2)}, \tag{78}$$

where θ_1 and θ_2 are the principal angles between F_i and F_j [17]. It can be shown that $|\langle C_i, C_j \rangle|^2 = 1 - \frac{d_c^2(F_i,F_j)}{2}$ [see Appendix B], so that the design criterion in (71) can be restated as

$$\max_{\{F\}} \min_{1 \le i < j \le N} d_c^2(F_i, F_j), \tag{79}$$

where F is a set of cardinality N, consisting of 2 dimensional subspaces of $\mathbb{R}^{2t}$ formed by matrices of the nature given by (77). This is the packing problem discussed in [16].

V. Discussion and Conclusions

In this paper, we have provided a geometrical framework for the design and analysis of beamformers for multiple antenna systems. The framework was used to derive fundamental bounds on performance as well as a design criterion for good beamformer codebooks.

Our framework of using finite number of bits to quantize the channel vector naturally lends itself to rate-distortion like analysis. The key difference from conventional rate-distortion theory is that we are not interested in describing the channel vector accurately, but only in a description which leads to improved transmission efficiency in the forward channel. Thus, a quantizer which minimizes an L_p norm (like in most practical quantizer designs) to describe the channel is *not* well-suited to minimize outage performance of a beamformer based system (see Lemma 1). But, even though our metric of optimization is different from widespread quantization metrics, some of our results have a rate-distortion like flavour. We highlight two of our results which show similarities to conventional source coding (a) the exponential decay of the new distortion function (Equation 53), much like in the source coding of Gaussian sources [18] and (b) the equivalence of our quantization problem (or the beamformer design problem) with a known channel coding problem.

The high SNR approximation of the outage probability (used to derive the fractional loss from perfect information system) has behaviour similar to the rate-distortion function of the Gaussian sources. To achieve the rate-distortion bound requires vector quantization using increasingly large blocks of input. But the fractional loss in outage result is asymptotic in SNR and not the length of observation. In fact, the length of observation is always one for beamformer design, since the channel information regarding the current block is useful for transmission in the current block only and has to be thus received causally. The fact that we cannot impose asymptotics in block length to quantize the channel realizations is the major reason for the difficulty in obtaining exact performance characterizations. In the light of the above comments, it is satisfying to see that the current problem lends a nice geometrical structure and admits a relatively tight finite block analysis.

It is well known that the coding for Gaussian channels is dual to the problem of quantizing a Gaussian source with mean-squared distortion metric. In the same spirit, we show that the problem of source quantization (beamformer design with loss in outage probability as distortion metric) is closely related to non-coherent code design problem [14] for the same vector channel. This strong equivalence, though intuitively appealing once known, came as a surprise to us. In the quantization problem for beamforming, the transmitter is only interested in the direction of the channel vector and not in its channel norm. Thus, all the relevant information regarding the source lies in its directional information. As a dual, in non-coherent communications, unitary space-time codes transmit along different directions (along different subspaces) and place no information in the magnitude of the signal.

Many interesting problems in beamforming remain to be explored. We have addressed the problem of beamforming with a single receive antenna only. With multiple receive antennas, we will need to consider the multiple directions for transmission (given by the eigenvectors) along with the spatial power allocation among these directions. It can be shown that beamforming along multiple directions will result in additional gains over space time codes only when it is coupled with spatial power allocation. Hence, multiple receive antennas present a different set of problems which do not arise with a single receive antenna.

In this paper, we have addressed the design of feedback for beamforming only. Power control mechanism can also result in huge improvements in outage performance [1]. Joint design of power control and beamforming is an interesting problem which needs further investigation. It would also be interesting to see the conditions under which temporal power control and beamforming can be decoupled with quantized channel information at the transmitter. Note that the optimal scheme with perfect channel information at the transmitter does in fact decouple beamforming and power control at the transmitter [12]. Finally, we addressed the case of quantized feedback which is applicable to an FDD system. The framework presented in this paper is general enough to extend to the case of noisy channel estimates also which is applicable to a TDD system.

A. Proof of Lemmas

A. Proof of Lemma 2

We will use the methodology given in [10] for the calculation of the surface area of a t-dimensional real hypersphere to calculate the surface area of t-dimensional complex hypersphere. We will first calculate the volume of the complex sphere $||\mathbf{h}|| \leq r$, where $\mathbf{h} \in \mathbb{C}^t$. Let $\mathbf{h} = [h_1, h_2, ..h_t]$, where $h_j \in \mathbb{C}$ for $j = 1, ..t$. Further, for each j, let $h_j = x_{jr} + \jmath x_{ji}$, i.e., the real and the imaginary parts of h_j are given by x_{jr} and x_{ji} respectively. Now consider the transformation of variables given by

$$\begin{aligned} x_{jr} &= r_j \cos\theta_j \\ x_{ji} &= r_j \sin\theta_j, \end{aligned} \tag{80}$$

so that $h_j = r_j e^{\jmath\theta_j}$ for each j.

Proposition 2 *The Jacobian for the transformation in (80) is* $r_1 r_2 .. r_t$.

Proof : The Jacobian of the transformation given in (80) is given by $\det(\text{diag}(M_1, M_2, ..M_t))$ where

$$M_j = \begin{bmatrix} \cos(\theta_j) & \sin(\theta_j) \\ -r_j \sin(\theta_j) & r_j \cos(\theta_j) \end{bmatrix}. \tag{81}$$

The determinant of this block diagonal matrix can be shown equal to $\prod_{j=1}^{t} \det(M_j)$. On evaluation, we find that $\det(M_j)$ is given by r_j and hence, the proposition follows. ■

Hence, the volume elements are related by

$$dx_{1r}dx_{1i}dx_{2r}dx_{2i}...dx_{tr}dx_{ti} = r_1 r_2 ... r_n dr_1 dr_2 ... dr_t d\theta_1 d\theta_2 ... d\theta_t \tag{82}$$

The volume of the hypersphere $\|\mathbf{h}\| \leq r$ is given by

$$\begin{aligned} V_t(r) &= \int\int ... \int_{\sum_{j=1}^{t} x_{jr}^2 + x_{ji}^2 \leq r^2} dx_{1r}dx_{1i}dx_{2r}dx_{2i}...dx_{tr}dx_{ti} \\ &= \int\int ... \int_{\sum_{j=1}^{t} r_j^2 \leq r^2, 0 \leq \theta_i \leq 2\pi} r_1 r_2 ... r_t dr_1 dr_2 ... dr_t d\theta_1 d\theta_2 ... d\theta_t \\ &= (2\pi)^t \int\int ... \int_{\sum_{j=1}^{t} r_j^2 \leq r^2} r_1 r_2 ... r_t dr_1 dr_2 ... dr_t, \end{aligned} \tag{83}$$

where we have integrated over the t-variables $\theta_1, ..\theta_t$ in (83).

Now consider another transformation given by

$$\begin{aligned} r_1 &= u c_1 c_2 ... c_{t-3} c_{t-2} c_{t-1} \\ r_2 &= u c_1 c_2 ... c_{t-3} c_{t-2} s_{t-1} \\ r_3 &= u c_1 c_2 ... c_{t-3} s_{t-2} \\ &. \\ &. \\ r_{t-1} &= u c_1 s_2 \\ r_t &= u s_1, \end{aligned} \tag{84}$$

where we have used the notation $c_i = \cos \phi_i$ and $s_i = \sin \phi_i$, for $i = 1, 2, ..t-1$. Note that ϕ_i goes only from 0 to $\pi/2$ since $r_i \geq 0$ for each i. Further, u goes from 0 to r. The Jacobian of this transformation is given by [10]

$$u^{t-1}c_1^{t-2}c_2^{t-3}...c_{t-2}\ dud\phi_1 d\phi_2...d\phi_{t-1}. \tag{85}$$

Substituting (84) and (85) in (83), we get after simplification,

$$\begin{aligned} V_t(r) &= (2\pi)^t \int_{u=0}^{r} u^{2t-1}du \int_{\phi_1=0}^{\pi/2}\int_{\phi_2=0}^{\pi/2}...\int_{\phi_{t-1}=0}^{\pi/2} \qquad (86)\\ &\quad (c_1^{2t-3}s_1)(c_2^{2t-5}s_2)...(c_{t-2}^{3}s_{t-2})(c_{t-1}s_{t-1})d\phi_1 d\phi_2...d\phi_{t-1}\\ &= (2\pi)^t\frac{r^{2t}}{2t}\frac{1}{2t-2}\frac{1}{2t-4}..\frac{1}{4}\frac{1}{2}\\ &= \frac{(2\pi)^t r^{2t}}{2^t t!} \qquad (87)\end{aligned}$$

Finally, the surface area is obtained by differentiating the volume of the sphere. Hence,

$$\begin{aligned} A_t(r) &= \frac{dV_t(r)}{dr}\\ &= \frac{2\pi^t r^{2t-1}}{(t-1)!} \qquad (88)\end{aligned}$$

B. Proof of Lemma 4

Let $A_t(r, r_0)$ denote the area of the spherical cap formed by the intersection of the subspace $|h_1| > r_0$ and the t-dimensional complex hypersphere given by $\|\mathbf{h}\| = r$. We will use the same technique as in the proof of Lemma 2, where we first calculate the volume of a solid object and treat the surface area as a differential element of the volume at the required radius. Now consider the region formed by the intersection of the subspace $|h_1| > r_0$ with the interior of the hypersphere given by $\|\mathbf{h}\| \leq r$. In terms of the transformation given in (80), this region is given by $\{\mathbf{h} \in \mathbb{C}^t : h_i = r_i e^{\jmath\theta_j}, \sum_{i=1}^{t} r_i^2 \leq r^2,\ r_1 \geq r_0\}$. The volume of this region, denoted by $V_t(r, r_0)$ is given by the integral in (83) where the limits of r_i are now different for each i. In particular, we have

$$V_t(r, r_0) = (2\pi)^t \int_{r_1=r_0}^{r}\left(\int ... \int_{\sum_{j=2}^{t} r_j^2 \leq r^2 - r_1^2} r_2...r_n\ dr_2...dr_t\right) r_1 dr_1. \tag{89}$$

Note that the multiple integral in brackets in the expression above is nothing but the volume of $t-1$ dimensional complex hypersphere of radius $\sqrt{r^2 - r_1^2}$ with a scaling factor of $(2\pi)^{t-1}$, which is given by (87). On substitution, we get,

$$\begin{aligned} V_t(r, r_0) &= \frac{(2\pi)^t}{2^{t-1}(t-1)!} \int_{r_1=r_0}^{r} r_1(r^2 - r_1^2)^{t-1}\, dr_1 \\ &= \frac{(2\pi)^t}{2^{t-1}(t-1)!} \int_{u=r_0^2}^{r^2} \frac{(r^2-u)^{t-1}}{2}\, du, \text{ where } u = r_1^2 \\ &= \frac{(2\pi)^t}{2^t t!}(r^2 - r_0^2)^t. \end{aligned} \tag{90}$$

Finally, the surface area of the spherical cap is obtained by differentiating $V_t(r, r_0)$ given by

$$\begin{aligned} A_t(r, r_0) &= \frac{dV_t(r, r_0)}{dr} \\ &= \frac{2\pi^t}{(t-1)!}\, r(r^2 - r_0^2)^{t-1}. \end{aligned} \tag{91}$$

B. Relationship between chordal distance and norm of inner product

In this section we will show that

$$|\langle C_i, C_j \rangle|^2 = 1 - \frac{d_c^2(F_i, F_j)}{2} \tag{92}$$

Following the notation introduced in Section IV(D.), we see that

$$b_{i1} = \begin{bmatrix} C_{iR} \\ C_{iI} \end{bmatrix} \text{ and } b_{i2} = \begin{bmatrix} C_{iI} \\ -C_{iR} \end{bmatrix} \tag{93}$$

form an orthonormal basis of the subspace F_i. Similarly, we can define b_{j1} and b_{j2} corresponding to F_j. Define the matrix A as

$$\begin{aligned} A &= [b_{i1} \;\; b_{i2}]^T [b_{j1} \;\; b_{j2}] \\ &= \begin{bmatrix} b_{i1}^T b_{j1} & b_{i1}^T b_{j2} \\ b_{i2}^T b_{j1} & b_{i2}^T b_{j2} \end{bmatrix}. \end{aligned} \tag{94}$$

Then, the principal angles θ_1 and θ_2 between F_i and F_j are such that $\cos(\theta_1) = \sigma_1$ and $\cos(\theta_2) = \sigma_2$, where σ_1 and σ_2 are the singular values of A [17].

Noting that $b_{i1}^T b_{j1} = b_{i2}^T b_{j2}$ and $b_{i1}^T b_{j2} = -b_{i2}^T b_{j1}$, we have

$$\begin{aligned} \cos^2(\theta_1) + \cos^2(\theta_2) &= \sigma_1^2 + \sigma_2^2 \\ &= \text{trace}(A^T A) \\ &= 2((b_{i1}^T b_{j1})^2 + (b_{i1}^T b_{j2})^2) \\ &= 2|\langle C_i, C_j \rangle|^2 \end{aligned} \tag{95}$$

where the last equation follows from $\langle C_i, C_j \rangle = b_{i1}^T b_{j1} + \jmath b_{i1}^T b_{j2}$. Finally, we have

$$\begin{aligned} d_c^2(C_i, C_j) &= \sin^2(\theta_1) + \sin^2(\theta_2) \\ &= 2 - (\cos^2(\theta_1) + \cos^2(\theta_2)) \\ &= 2\left(1 - |\langle C_i, C_j \rangle|^2\right) \end{aligned} \tag{96}$$

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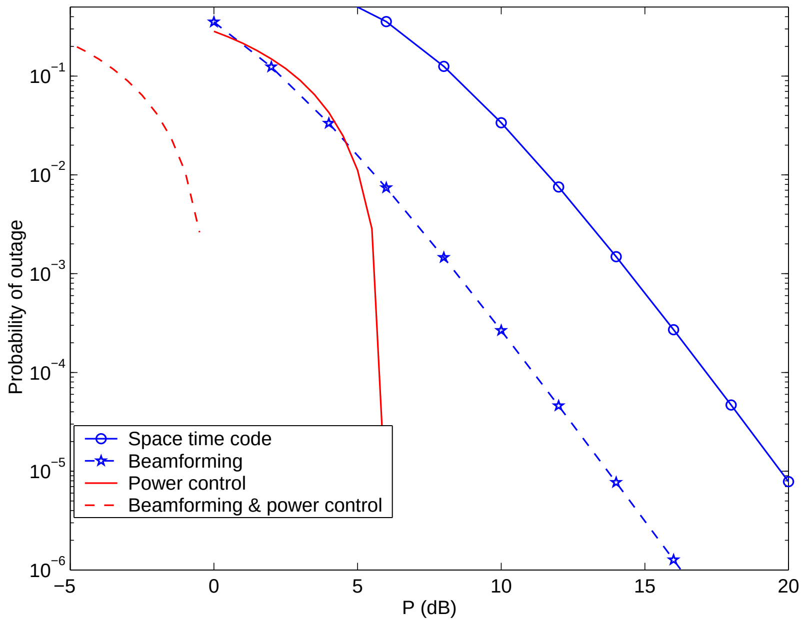

Figure 1: Performance gains of feedback based schemes compared to space time codes for four transmit antennas. The plot shows the predicted performance in theory when optimal codes are used with each of the transmission schemes. Space time code refers to full rank transmission with *i.i.d* Gaussian codes and beamforming refers to rank one transmission with Gaussian code. Power control refers to full rank space time code with temporal power control while the last case refers to unit rank transmission with temporal power control.

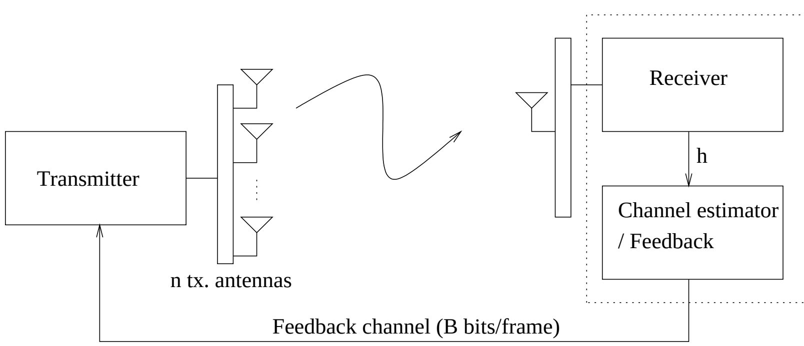

Figure 2: A feedback based wireless system with t transmit antennas and a single receive antenna. The feedback channel is error-free and has capacity B bits/frame

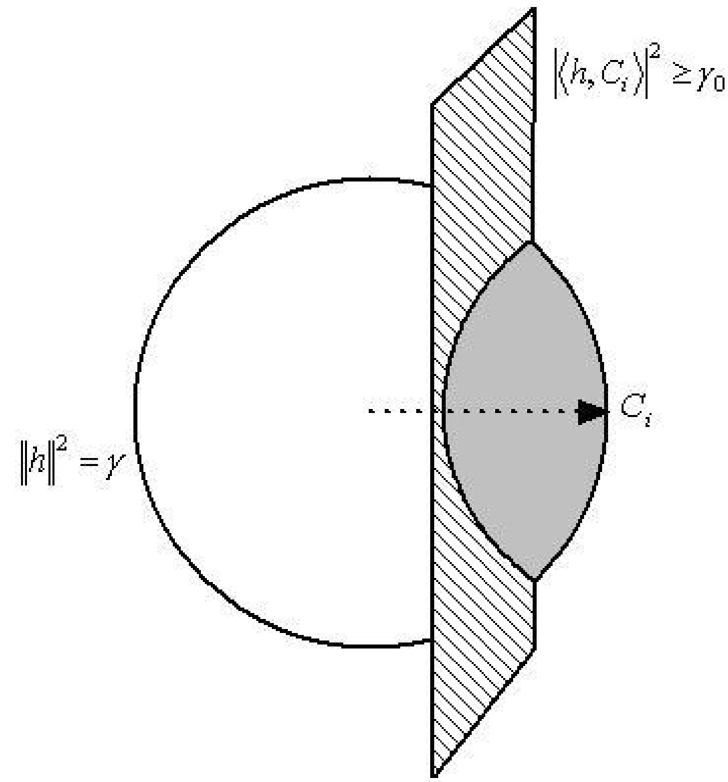

Figure 3: The nature of the region $S_i(\gamma)$ for the illustrative case of $\mathbb{R}^3$. Note that $S_i(\gamma)$ is a spherical cap denoted by the shaded region.

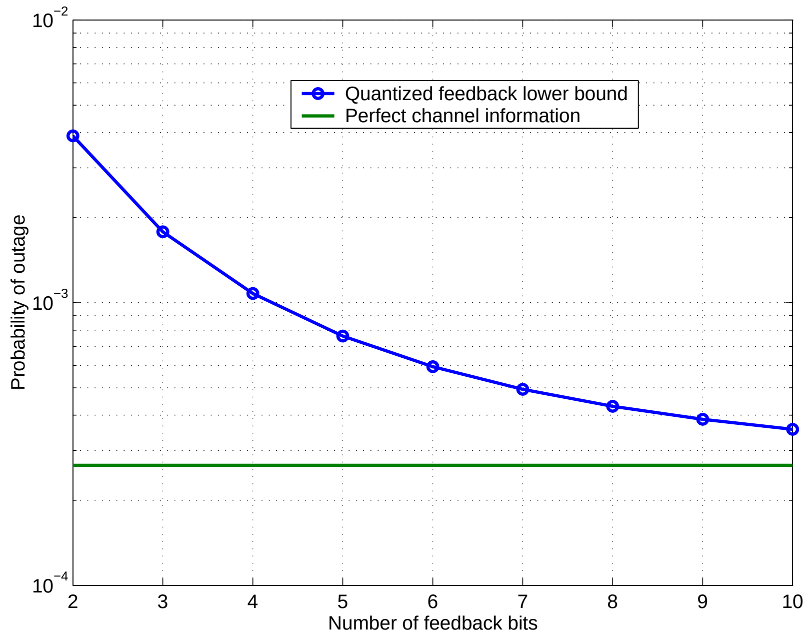

Figure 4: Improvement in outage probability with increasing feedback bits for four transmit antennas with a single receive antenna at SNR $=10$dB, as predicted by Theorem 1.

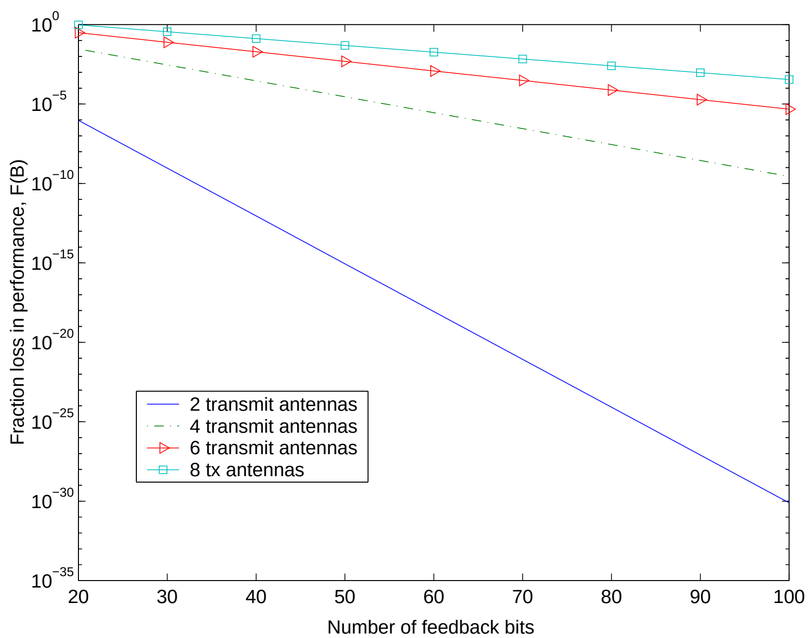

Figure 5: Improvement in outage with feedback bits for $2, 4, 6$ and 8 transmit antennas given by (53). The slope of the fraction loss plot is inversely proportional to the number of transmit antennas.

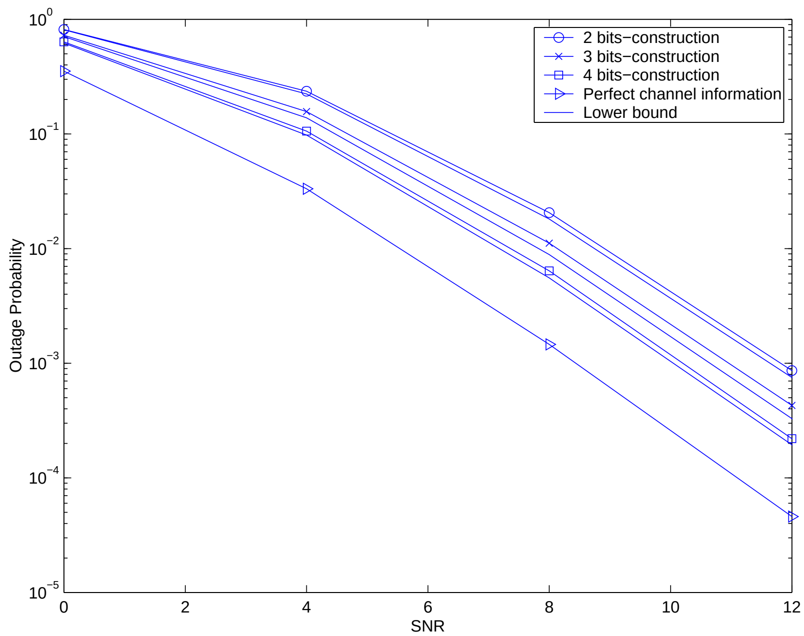

Figure 6: Simulated outage performance of the beamformers constructed for four transmit antennas for rate $R = 2$ bits/sec/Hz. For the sake of comparison, the lower bound on outage probability for each number of feedback bits given in Theorem 1 is also plotted.

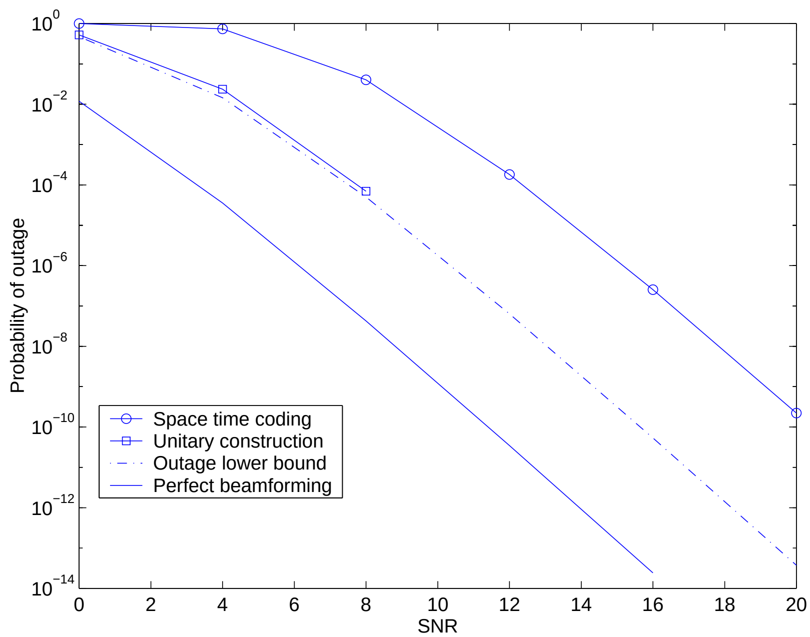

Figure 7: Performance of systematic unitary space time constellation as a beamformer. The beamformer consists of 16 vectors (4 bits of feedback) for 8 transmit antennas transmitting at 2 bits/sec/Hz. The lower bound on outage probability as well as the performances of beamformer with perfect channel information and (full rank) space time codes are given for comparison.