AN ALGORITHM TO FACTOR POLYNOMIAL MATRICES

by

A. KONTOS

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ABSTRACT

A new algorithm for factoring polynomial matrices, based on the concept of elementary factors, is presented in this report. This algorithm does not use numerically unsatisfactory Euclidean type operations.

*Department of Electrical Engineering, Rice University, Houston, Tx. 77001
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1. Introduction

In this report a new way of factoring polynomial matrices is presented. By factoring a polynomial matrix $A(\lambda)$, we mean to find two polynomial matrices $B(\lambda)$ and $C(\lambda)$ such that $A(\lambda) = B(\lambda)C(\lambda)$ and $B(\lambda)$ and $C(\lambda)$ have their zeroes in two given disjoint subsets of the complex plane. It is well known that such a factorization is always possible and it can be done using Smith decomposition of the polynomial matrix.

To obtain the Smith decomposition we need to use Gaussian elimination without pivoting on the coefficients of the entries of the polynomial matrix [8], and such a procedure is numerically unsatisfactory [13] [14].

Dennis _et al_. [1] show how to factor a polynomial matrix into a product of polynomial matrices of the form $\lambda I-S$. They require, though, linearly independent latent vectors (to be defined in Section 2) for all zeroes of $\lambda I-S$. Moreover, if the number of zeroes to be removed from an mxn polynomial matrix $A(\lambda)$ is not $n, 2n, \ldots$ the method in [1] does not apply.

We will present an algorithm for factoring polynomial matrices that does not use Smith forms, and it can be considered as a generalization of the algorithm in [1]. We will only consider matrices with more rows than columns, or square matrices. In the other case, i.e., if we have matrices with more columns than rows, all the results can be applied to their transpose.

2. Elementary Factors

Let $A(\lambda)$ be an $m \times n$ polynomial matrix of full column rank. We can write

$$A(\lambda) = A_o + \lambda A_1 + \ldots + \lambda^{\ell} A_{\ell} \quad (2.1)$$

where

$$A_i \in \mathcal{R}^{m \times n}, \quad i = 0,1,\ldots,\ell.$$

A zero ρ of $A(\lambda)$ is a value of λ that reduces the column rank of $A(\lambda)$, i.e., if ρ is a zero of $A(\lambda)$, then there exists a non-zero constant vector b, such that

$$A(\rho)b = 0 \quad (2.2)$$

Definition. [1] The vector b as defined in (2.2) is called a right latent vector (or simply latent vector) of A corresponding to the zero ρ.

Let $\rho_1,\ldots\rho_k$ be zeroes of $A(\lambda)$ and $b_1,\ldots,b_k$ corresponding latent vectors. If $b_1,\ldots,b_k$ are linearly independent then the matrix

$$B = [b_1,\ldots,b_k] \quad (2.3)$$

has full column rank. By the Gramm-Schmidt theorem [3] we know we can find an orthonormal basis for the column space of B, i.e., we can find a matrix $U \in \mathcal{R}^{n \times k}$ and a non-singular matrix $T \in \mathcal{R}^{k \times k}$ such that

$$B = UT \quad (2.4a)$$

and

$$U^T U = I_k \quad (2.4b)$$

Numerically U and T can be obtained using singular value decomposition [6].

Define also

$$R = \begin{bmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_k \end{bmatrix} \quad (2.5)$$

and

$$M = TRT^{-1} \tag{2.6}$$

The following lemma is now in order.

Lemma 2.1. Let U and M be as defined in (2.3) - (2.6). Define $A(X) = A_0+A_1X+\ldots+A_\ell X^\ell$ where X is a square matrix. Then

$$A(UMU^T)U = 0 \tag{2.7}$$

Proof. First, notice that

$$(UMU^T)^2 = UMU^TUMU^T$$
$$= UM^2U^T$$

and in general

$$(UMU^T)^j = UM^jU^T \quad \text{for} \quad j = 1,2,\ldots \quad .$$

We now have

$$A(UMU^T)U = [A_0 + A_1UMU^T+\ldots+ A_\ell UM^\ell U^T]U$$
$$= A_0U + A_1UM +\ldots+ A_\ell UM^\ell$$
$$= [A_0B + A_1BR +\ldots+ A_\ell BR^\ell]T^{-1}$$
$$= [A(\rho_1)b_1, A(\rho_2)b_2,\ldots, A(\rho_k)b_k]T^{-1}$$
$$= 0 \quad .$$

Corollary 2.2. Let $\rho, \bar{\rho}$ be a pair of complex conjugate zeroes of $A(\lambda)$ and $b,\bar{b}$ the corresponding latent vectors. Then, if

$$x_1 = b + \bar{b} \quad \text{and} \quad x_2 = j(b - \bar{b})$$

are linearly independent, define U, T and M as follows:

$$[x_1\ x_2] = UT \quad \text{as in (2.4)}$$

and

$$M = T \begin{bmatrix} p & -q \\ q & p \end{bmatrix} T^{-1}$$

where

$$\rho = p + jq.$$

Then

$$A(UMU^T)U = 0$$

and U and M are real matrices

Definition. Given $U \in \mathbb{R}^{nxk}$ $(k \leq n)$ such that $U^TU = I_k$ and $M \in \mathbb{R}^{kxk}$, the elementary factor $H(\lambda)$, associated with U and M is defined by

$$H(\lambda) = \lambda UU^T + I_n - U(I_k + M)U^T \quad . \tag{2.8}$$

Lemma 2.3. Let $C = UU^T$ and $D = I_n - U(I_k + M)U^T$. Then, the following is true.

(1) D is non-singular if and only if M is non-singular, in which case

$$D^{-1} = I_n - U(I_k + M^{-1})U^T \tag{2.9}$$

and

$$(-D^{-1}C)^j = (UMU^T)^{\ell-j}\ UM^{-\ell}U^T, \quad j = 1,2,\ldots,\ell \tag{2.10}$$

(2) $$\left|H(\lambda)\right| = \left|\lambda I_k - M\right| \tag{2.11}$$

(3) U is a basis for the right null space of H(M).

Proof.

(1) Since $U^TU = I_k$ there exists a matrix $U_1 \in \mathbb{R}^{nx(n-k)}$ such that the matrix

$$\tilde{U} = [U\ U_1]$$

is unitary, i.e., $\tilde{U}^T\tilde{U} = \tilde{U}\tilde{U}^T = I_n$.

Obviously

$$D = \tilde{U}\begin{bmatrix} -M & 0 \\ 0 & I_{n-k} \end{bmatrix}\tilde{U}^T$$

which directly implies (2.9).

To prove (2.10), first notice that

$$\begin{aligned} -D^{-1}C &= (-I_n + U(I_k + M^{-1})U^T)UU^T \\ &= -UU^T + UU^T + UM^{-1}U^T \\ &= UM^{-1}U^T \quad . \end{aligned}$$

Now we have

$$\begin{aligned} (-D^{-1}C)^j &= UM^{-j}U^T \\ &= (UMU^T)^{\ell-j}UM^{-\ell}U^T, \quad j=1,2,\dots,\ell \end{aligned}$$

(2)

$$H(\lambda) = \tilde{U}\begin{bmatrix} \lambda I_k - M & 0 \\ 0 & I_{n-k} \end{bmatrix}\tilde{U}^T$$

and

$$|H(\lambda)| = |\lambda I_k - M| \quad .$$

(3) We have

$$H(M) = I - UU^T \quad .$$

Obviously

$$H(M)U = 0 .$$

Now let

$$H(M)x = 0 ,$$

then

$$x - U(U^T x) = 0$$

or

$$x = Ug$$

for some vector g, i.e., any vector satisfying $H(M)x = 0$ can be written as a linear combination of the columns of U. Moreover, the columns of U are linearly independent, i.e., U is a basis for the right null space of U.

Q.E.D.

We now present the main result of this section.

Theorem 2.4. Let $A(\lambda) = A_o + A_1\lambda + \ldots + A_\ell\lambda^\ell$ be an mxn polynomial matrix and U,M be defined in (2.3) - (2.6). If M is non-singular, then there exists a polynomial matrix $Q(\lambda)$ such that

$$A(\lambda) = Q(\lambda)H(\lambda) \quad (2.12)$$

where

$$H(\lambda) = \lambda UU^T + I_n - U(I_k + M)U^T = \lambda C + D$$

C and D are defined in Lemma 2.3.

Proof. Let

$$Q(\lambda) = Q_o + \lambda Q_1 + \ldots + \lambda^\ell Q_\ell$$

where the Q's are defined by

$$
\begin{aligned}
Q_o D &= A_o \\
Q_o C + Q_1 D &= A_1 \\
&\vdots \\
Q_{\ell-1} C + Q_\ell D &= A_\ell \quad .
\end{aligned}
\tag{2.13}
$$

Since M is invertible, by Lemma 2.3, D is invertible and (2.13) has a unique solution for $Q_o, Q_1, \ldots, Q_\ell$. Now

$$
\begin{aligned}
Q(\lambda)H(\lambda) &= (Q_o + \lambda Q_1 + \ldots + \lambda^\ell Q_\ell)(C\lambda + D) \\
&= Q_o D + (Q_o C + Q_1 D)\lambda + \ldots \\
&\quad + (Q_{\ell-1} C + Q_\ell D)\lambda^\ell + Q_\ell C\lambda^{\ell+1} \\
&= A(\lambda) + Q_\ell C\lambda^{\ell+1}
\end{aligned}
$$

We need to show that $Q_\ell C = 0$. From (2.13) we have

$$
Q_j = \left[A_o(-D^{-1}C)^j + A_1(-D^{-1}C)^{j-1} + \ldots + A_j \right] D^{-1} \quad , \qquad j = 0, 1, \ldots, \ell,
$$

and

$$
\begin{aligned}
Q_\ell C &= -\left[A_o(-D^{-1}C)^{\ell+1} + \ldots + A_\ell(-D^{-1}C) \right] \\
&= -\left[A_o + A_1 UMU^T + \ldots + A_\ell (UMU^T)^\ell \right] UM^{-\ell-1}U^T \\
&= -A(UMU^T) \;\; UM^{-\ell-1}U^T \\
&= 0
\end{aligned}
$$

where (2.7) and (2.10) were used.

Q.E.D.

The matrix M is non-singular if and only if its eigenvalues are non-zero. If $A(\lambda)$ has zeroes at $\lambda = 0$ which are to be factored out, then Theorem 2.4 cannot be used since D is not invertible. In such a case $Q(\lambda)$ is constructed as follows. Assume

$$A(0)b = 0$$

for some $b \neq 0$, and define

$$H(\lambda) = \lambda bb^T + I_n - bb^T .$$

Then

$$H^{-1}(\lambda) = \frac{1}{\lambda} bb^T + I_n - bb^T$$

and

$$A(\lambda)H^{-1}(\lambda) = (A_o + \lambda A_1 + \ldots + \lambda^{\ell} A_{\ell}) \times \left(\frac{1}{\lambda} bb^T + I - bb^T\right)$$

$$= Q_o + \lambda Q_1 + \ldots + \lambda^{\ell} Q_{\ell}$$

where

$$Q_i = A_i(I - bb^T) + A_{i+1}bb^T, \quad i=0,1,\ldots,\ell-1$$

$$Q_{\ell} = A_{\ell}(I - bb^T)$$

is the desired factorization.

Remark: With Q and H as previously defined it is easy to prove [4] that

$$Z_A = Z_Q \cup Z_H$$

where Z_A denotes the collection of all the zeroes of A. We will now illustrate the method with the following examples.

Example. The polynomial matrix

$$A(\lambda) = \begin{bmatrix} 0 & 12 \\ -2 & 14 \end{bmatrix} + \begin{bmatrix} -1 & -6 \\ 2 & 9 \end{bmatrix} \lambda + I_2 \lambda^2$$

has zeroes 1, 2, 3 and 4 with corresponding latent vectors $(1,0)^T$, $(0,1)^T$ $(1,1)^T$, $(1,1)^T$. [1]

It is obvious that the zeroes 3 and 4 cannot be removed simultaneously since their corresponding latent vectors are linearly dependent.

It is easy to check that

$$A(\lambda) = Q(\lambda) H_2(\lambda) H_1(\lambda)$$

where

$$Q(\lambda) = \frac{1}{6} \begin{bmatrix} -33 & 39 \\ -45 & 51 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 15 & -15 \\ 33 & -33 \end{bmatrix} \lambda + \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \lambda^2$$

and

$$H_i(\lambda) = (\lambda - \rho_i) b_i b_i^T + I - b_i b_i^T$$

$$\rho_1 = 4, \quad \rho_2 = 3$$
$$b_1 = b_2 = \frac{1}{\sqrt{2}} (1,1)^T$$

The method proposed in [1] cannot be applied if only one zero, for example $\lambda = 4$, is to be factored out, or if both zeroes 3 and 4 are to be factored out.

The following example is taken from [11], where the greatest common right divisor of the rows of $A(\lambda)$ is determined. The same example is given here to show that the greatest common divisors can be found using the results presented in this report.

Example: Consider the polynomial matrix

$$A(\lambda) = \begin{bmatrix} -1 & -1 \\ 0 & 4 \\ -1 & -3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -4 & 2 \\ 3 & 0 \\ -3 & 0 \end{bmatrix} \lambda + \begin{bmatrix} 0 & 0 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \lambda^2$$

$A(\lambda)$ will be factored in two different ways, i.e.,

$$A(\lambda) = Q_1(\lambda)R_1(\lambda) = Q_2(\lambda)R_2(\lambda)$$

where

$$Z_{Q_1} = Z_A - \{\pm 1\}, \quad Z_{R_1} = \{\pm 1\}$$

$$Z_{Q_2} = \emptyset, \quad Z_{R_2} = Z_A .$$

For this example,

$$Q_1(\lambda) = \begin{bmatrix} 1 & 1 \\ -4 & 0 \\ 3 & 1 \\ -2 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ -2 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R_1(\lambda) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$Q_2(\lambda) = \begin{bmatrix} 1 & 1 \\ -3 & -1 \\ 2.5 & 1.5 \\ -1.5 & -0.5 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & -1 \\ 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \lambda$$

$$R_2(\lambda) = \begin{bmatrix} 0.5 & -1.5 \\ -1.5 & 0.5 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \lambda + \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \lambda^2$$

Notice that $R_2(\lambda)$ is the greatest common right divisor of the rows of $A(\lambda)$. This is essentially the same answer as in [11] since this result here is obtained by multiplying the answer in [11] from the left by the unimodular matrix

$$\frac{1}{2} \begin{bmatrix} 1-\lambda & -3-\lambda \\ -3+\lambda & 1+\lambda \end{bmatrix}$$

In general, given any mxn $(m > n)$ polynomial matrix $A(\lambda)$, of full column rank, and a factorization $A(\lambda) = Q(\lambda)R(\lambda)$ with $Z_A = Z_R$ and $Z_Q = \emptyset$, the matrix $R(\lambda)$ is the greatest common right divisor of the rows of $A(\lambda)$.

Comments.

(1) The factor $\lambda I - S$ considered in [1] is an elementary factor with $k = n$. To see that, let $k = n$ in

$$H(\lambda) = \lambda UU^T + I_n - U(I_k + M)U^T$$

Since $U^T U = I_n$ and $UMU^T = B.\text{diag}(\rho_i).B^{-1}$ (see 2.3 and 2.4) it follows that

$$H(\lambda) = \lambda I - S$$

with

$$S = B.\text{diag}(\rho_i).B^{-1}$$

exactly as defined in [1].

(2) Using corollary 2.2, we can use only real number arithmetic to get the desired factorization, provided that x_1 and x_2 (defined in corollary 2.2) are linearly independent. If this is not the case, complex number arithmetic should be used.

(3) The elementary factor $H(\lambda)$ is uniquely defined by U and M. So we need only store U and M, which reduces storage requirements.

(4) The zeroes and the latent vectors need to be found, as it is obvious from the previous discussion. The next section discusses this problem.

3. Zeroes and Latent Vectors of a Polynomial Matrix

Before we proceed we need some definitions and a few results from the theory of pencils.

Definition. By the term pencil we mean an mxn polynomial matrix whose elements are of the form $\lambda p-q$, i.e., any pencil can be written as

$$\lambda P - Q,$$

where both P and Q are mxn constant matrices.

Definition. Two pencils λP_1-Q_1 and λP_2-Q_2 are said to be strictly equivalent if and only if there exist non-singular matrices M and N of order mxm and nxn, respectively, such that

$$M(\lambda P_1-Q_1)N = \lambda P_2-Q_2 .$$

Theorem 3.1. Any pencil $\lambda P-Q$ is strictly equivalent to a pencil λP_c-Q_c of the form

$$\lambda P_c-Q_c = \text{diag}\left[R_{\epsilon_1},\ldots,R_{\epsilon_p},L_{\eta_1},\ldots,L_{\eta_q},\lambda N-I,\lambda I-J\right] \tag{3.1}$$

where

(i) R_ϵ is a $\epsilon\times(\epsilon+1)$ bidiagonal pencil of the form

$$R_\epsilon \triangleq \begin{bmatrix} \lambda & -1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & \cdot \\ & & & \lambda & -1 \end{bmatrix} \tag{3.2}$$

(ii) L_η is a $(\eta+1)\times\eta$ bidiagonal pencil of the form

$$L_\eta = \begin{bmatrix} \lambda & & & \\ -1 & \cdot & & \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & & \lambda \\ & & & -1 \end{bmatrix} \tag{3.3}$$

(iii) N has all its eigenvalues at zero and is in Jordan canonical form, and

(iv) J is in Jordan canonical form. We say that a matrix is in Jordan canonical form if it is in block diagonal form, each diagonal block being an elementary Jordan block. These blocks are of the form

$$J_k(\alpha) = \begin{bmatrix} \alpha & & & 0 \\ 1 & \cdot & & \\ & \cdot & \cdot & \\ & & \cdot & \cdot \\ & & 1 & \alpha \end{bmatrix}, \quad J_k(\alpha) \in R^{k \times k} .$$

The above theorem was developedbby Kronecker and Weierstrass and a proof can be found in [2].

Definition.

(i) The indices $\epsilon_1, \ldots, \epsilon_p$ are called the minimal column indices of the pencil $\lambda P-Q$.

(ii) The indices $\eta_1, \ldots, \eta_q$ are called the minimal row indices of the pencil $\lambda P-Q$.

Let J and N be defined as in Theorem 3.1, and of the form

$$J = \operatorname{diag}\left[J_{k_1}(\alpha), \ldots, J_{k_r}(\alpha)\right] \tag{3.4}$$

and

$$N = \operatorname{diag}\left[J_{\ell_1}(0), \ldots, J_{\ell_s}(0)\right] \tag{3.5}$$

Definition. The polynomials $(\lambda - \alpha_i)^{k_i}$, $i=1,\ldots,r$ and λ^{ℓ_j}, $j=1,\ldots,s$ are called the finite elementary divisors and infinite elementary divisors of the pencil, respectively.

Theorem 3.2. [13] Given P and Q, mxn constant matrices, there exist unitary matrices U and V $(UU^T = U^TU = I_m$ and $VV^T = V^TV = I_n)$ such that

$$U(\lambda P-Q)V = \begin{bmatrix} \lambda P_\eta - Q_\eta & 0 & 0 & 0 \\ X & \lambda P_f - Q_f & 0 & 0 \\ X & X & \lambda P_\infty - Q_\infty & 0 \\ X & X & X & \lambda P_\epsilon - Q_\epsilon \end{bmatrix} \quad (3.6)$$

where

X's are some appropriately defined pencils, and

$\lambda P_f - Q_f$ is a square regular pencil (i.e., P_f is non-singular) containing the finite elementary divisors, and

$\lambda P_\infty - Q_\infty$, $\lambda P_\eta - Q_\eta$ and $\lambda P_\epsilon - Q_\epsilon$ contain the infinite elementary divisors, minimal column and row indices respectively.

We now return to our discussion of the zeroes of a polynomial matrix $A(\lambda)$.

Zeroes of a Polynomial Matrix

Let

$$A(\lambda) = A_o + \lambda A_1 + \ldots + \lambda^\ell A_\ell$$

and define

$$P = \begin{bmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & A_\ell \end{bmatrix} \quad (3.7a)$$

and

$$Q = \begin{bmatrix} 0 & I & & \\ & & \ddots & \\ & & & I \\ -A_0 & \cdot & \cdots & -A_{\ell-1} \end{bmatrix} \tag{3.7b}$$

Lemma 3.3. *If* $A(\lambda)$ *has full column rank then* $\lambda P-Q$, *with* P,Q *as defined in* (3.7) *has no part corresponding to minimal row indices, i.e., there exist* U *and* V *as in Theorem 3.2 such that*

$$U(\lambda P-Q)V = \begin{bmatrix} \lambda P_\eta - Q_\eta & 0 & 0 \\ X & \lambda P_f - Q_f & 0 \\ X & X & \lambda P_\infty - Q_\infty \end{bmatrix} \overset{\Delta}{=} \lambda\tilde{P}-\tilde{Q} \tag{3.8}$$

Proof. Let, for the contrary,

$$U(\lambda P-Q)V = \begin{bmatrix} \lambda\tilde{P}-\tilde{Q} & 0 \\ \times & \lambda P_\epsilon - Q_\epsilon \end{bmatrix}$$

as in theorem 3.2.

Since $A(\lambda)$ has full column rank, there exists some ρ_o such that $A(\rho_o)$ has full column rank. But for $\lambda = \rho_o$ there exists a non-zero vector $\tilde{d}$ such that

$$(\rho_o P_\epsilon - Q_\epsilon)\tilde{d} = 0$$

and, if

$$d = V^{-1}\begin{bmatrix} 0 \\ \tilde{d} \end{bmatrix}$$

we have

$$(\rho_o P-Q)d = 0$$

or

$$\begin{bmatrix} \rho_o & -I & & & \\ & & & & \\ & & \rho_o & -I & \\ A_o & \cdot & \cdot & \cdot\ A_{\ell-2} & \rho_o A_\ell + A_{\ell-1} \end{bmatrix} \begin{bmatrix} d_o \\ d_1 \\ \\ d_{\ell-1} \end{bmatrix} = 0$$

or

$$A(\rho_o)d_o = 0 \quad \text{and} \quad d_o \neq 0; \text{ contradiction.}$$

Q.E.D.

Theorem 3.4. Let P_f, Q_f be as defined in (3.8).

Any zero of $A(\lambda)$ is a zero of the determinant $|\lambda P_f - Q_f|$ and conversely, any zero of $|\lambda P_f - Q_f|$ is a zero of $A(\lambda)$.

Proof. Let ρ be a zero of $A(\lambda)$, i.e., there exists a non-zero vector b_o such that

$$A(\rho)b_o = 0$$

or

$$(A_o + \rho A_1 + \ldots + \rho^\ell A_\ell)b_o = 0$$

Let

$$b_1 = \rho b_o$$

$$b_2 = \rho b_1$$

$$\cdot$$

$$\cdot$$

$$b_{\ell-1} = \rho b_{\ell-2}$$

We, therefore, have shown that the zeroes of a polynomial matrix $A(\lambda)$ can be found as the zeroes of a square pencil.

_i.e._, Given $A(\lambda) = A_o + \lambda A_1 + \ldots + \lambda^{\ell} A_{\ell}$, we construct P and Q as in (3.7) and we obtain P_f and Q_f as in Theorem (3.2). An algorithm for computing P_f and Q_f is given in [13] and an APL implementation of this algorithm can be found in [5].

The QZ algorithm applied to the pencil $\lambda P_f - Q_f$ gives the zeroes of $\lambda P_f - Q_f$ and, thus, the zeroes of $A(\lambda)$. The QZ algorithm is described in [9]* and essentially is a procedure to find unitary matrices M and N such that both MP_fN and MQ_fN are in upper triangular form.

_i.e._,

$$M(\lambda P - Q)N = \lambda \begin{bmatrix} p_1 & \times & \times \\ & \ddots & \times \\ 0 & & p_t \end{bmatrix} - \begin{bmatrix} q_1 & \times & \times \\ & \ddots & \times \\ 0 & & q_t \end{bmatrix}$$

Then, the zeroes of $\lambda P_f - Q_f$ are the numbers

$$\frac{q_i}{p_i}, \quad i = 1, \ldots, t \quad .$$

After the zeroes of $A(\lambda)$ are found the latent vectors must also be determined.

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*An APL implementation can be found in [6]

Determination of the latent vectors

Let $\ddot{\rho}$ be a zero of $A(\lambda)$, then b is a latent vector corresponding to $\ddot{\rho}$, if

$$A(\ddot{\rho})b = 0$$

So we need to find the right null space of $A(\rho)$.

This can be done using singular value decomposition (SVD). [12] [6]

Any constant mxn matrix, A, can be written as

$$A = U \begin{bmatrix} \Sigma & 0 \\ & \\ 0 & 0 \end{bmatrix} V^T \qquad (3.11)$$

where U and V are unitary matrices and $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots \sigma_r)$ and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0$. If we partition U and V accordingly as

$$U = [U_1 \; U_2] \quad \text{and } V = [V_1 \; V_2]$$

then

$$A = U_1 \Sigma V_1^T$$

and

$$AV_2 = 0.$$

U_1 is an orthonormal basis for the space spanned by the columns of A, and V_2 is an orthonormal basis for the right null space of A.

4. Conclusions

A new method to factor polynomial matrices is presented in this report, which avoids numerically unsatisfactory Euclidean type operations. This algorithm appears useful in the computation of structure matrices [10] and greatest common divisors.

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