On Diversity and Multiplexing Gain of Multiple Antenna Systems with Transmitter Channel Information

Ahmad Khoshnevis and Ashutosh Sabharwal
Department of Electrical and Computer Engineering
Rice University
Houston, Texas 77005
Email: {farbod,ashu}@rice.edu

Abstract

We quantify the multiplexing-diversity tradeoff of a multiple-input multiple-output (MIMO) system, when the channel state information (CSI) is known perfectly at the receiver and partially at the transmitter. The partial knowledge of CSI at the transmitter consists of the quantized value of one of the eigenvalues and perfect knowledge of eigenvectors of the channel matrix. The key result is that while multiplexing gain cannot be increased beyond minimum number of transmit and receive antennas, diversity order for each multiplexing gain can be substantially increased by using only a few bits of feedback at the transmitter. For example, with 1 bit of feedback in a 2×3 system, for multiplexing gains of 0, 1, and 2, diversity gains of 42, 6, and 2 can be achieved, respectively. Thus, while the tradeoff between diversity advantage and multiplexing gain is still present, its behavior is significantly changed by channel knowledge at the transmitter. The major reason for this different tradeoff can be attributed to addition of long-term power control, which allows the transmitter to switch between modes for reducing outage and increasing throughput based on signal to noise ratio along different eigenvalues.

1 Introduction

Introduction of diversity-multiplexing tradeoff in [1] brought the two branches of space-time codes, namely codes for increasing multiplexing gain, such as VBLAST, and codes for increasing diversity, like Alamouti, under a common framework. Moreover, it gave a more abstract understanding of multiple-input multiple-output (MIMO) channels in terms of degrees of freedom in a signal space [2]. In this paper we address the diversity-multiplexing tradeoff when partial channel state information (in the form of finite number of bits) is available at the transmitter.

Feedback is commonly used in many practical systems to provide limited channel information at the transmitter, to enable power control and beamforming combined with channel-adaptive coding. Feedback is known to significantly change system behavior. For example, a multiple antenna system with perfect CSIT does not have a finite diversity order [2] and the diversity orders of finite-rate feedback systems are much higher than systems with no CSIT [3]. Thus, with partial CSIT, we do expect that a diversity-multiplexing tradeoff will still be present but its specific behavior will be different from systems with no CSIT.

In this paper, we derive the diversity-multiplexing tradeoff in a system with partial channel knowledge at the transmitter (CSIT) and complete knowledge at the receiver. The derivation is based on a constructive method, which builds on two key steps. The first is a quantized power control method using only finite number of feedback bits to carry information about only one of the eigenvalues of the channel matrix, and the second is a hybrid coding scheme, which switches between two codebooks based on the channel state.

In the first step, we find the diversity order of a multiple antenna system when *only* one of the ordered eigenvalues of the channel matrix is being quantized and fed back to the transmitter. We use the same technique as in [3] to evaluate the diversity order. It is interesting to note that the diversity order of a MIMO system with finite rate feedback is finite and grows unboundedly with increase in the number of bits in feedback. For example, a system with 2 transmit and 4 receive antennas and 1 bit of feedback has diversity order of 72 (see Equation (28)). By adding one bit of feedback to the set up of the previous example the diversity order increases to 4680!

In the next step, we propose a coding scheme in which we use one bit of feedback to choose between two codebooks. One codebook has a fixed rate for a given average available power, but its rate grows with SNR. Another codebook has a fixed rate for all SNRs. We show that by use of such a hybrid codebook we can achieve an arbitrary small probability of error from the SNR-dependent codebook. Hence the total error is dominated by the error in the fixed rate codebook, which can be characterized by the outage probability. Finally, by combining this coding scheme and the power allocation scheme developed in the first step, we characterize a lower bound on the diversity-multiplexing tradeoff of a MIMO system with finite rate feedback.

The rest of the paper is organized as follows. In Section 2, we outline our system model which includes the channel model, our performance metrics and the feedback model. Section 3 contains the main result on diversity-multiplexing tradeoff and the proof. We conclude in Section 4.

2 System Model

We consider a multiple antenna system with M transmit and N receive antennas, where for a given average available power, P_{av}, the rate of transmission is fixed. The complex Gaussian channel coefficients h_{ij}'s, from the jth transmit antenna to the ith receive antenna, are assumed to be independent. Moreover $H = [h_{ij}]$ is assumed to be perfectly known at the receiver and to remain constant for the duration of transmission of a codeword. In other words, coherence time of the channel is assumed to be much larger than the time required for transmission of a codeword. Then, the received signal in a matrix notation can be represented as

$$Y = HX + W, \tag{1}$$

where $H \in \mathcal{C}^{N\times M}$, X is an $M \times 1$ input signal, and W is an $N \times 1$ noise vector with i.i.d. circular complex Gaussian entries.

Using the standard procedure of singular value decomposition [4], H can be represented as $H = UDV$, where $U \in \mathcal{C}^{N\times m}$ and $V \in \mathcal{C}^{m\times M}$ are unitary matrices, with $m = \min(M, N)$. The $m \times m$ diagonal matrix D has $\lambda_i^{1/2}$'s on its main diagonal, where λ_i's are eigenvalues of the positive semi-definite Hermitian Wishart matrix, Z. If $N < M$, $Z = HH^\dagger$, otherwise, $Z = H^\dagger H$. [1]

In a system where the transmitter has perfect knowledge of the channel matrix, it utilizes the matrices Λ and V to adapt the transmited signal to the channel conditions. In this work, we divide the total power eqaully on all transmit antennas, hence the transmitter does not require knowledge of V matrix, and we consider a system in which quantized knowledge of the $DD^\dagger$ matrix (with finite bits) is provided at the transmitter. With these assumptions, the $M \times N$ system modeled in Equation (1) can be reduced to a system with m parallel channels, represented in matrix format by

$$\tilde{Y} = D\tilde{X} + \tilde{W}. \tag{2}$$

[1] $\dagger$ is complex conjugate transpose operator.

2.1 Performance Metric

We use the definition provided in [1] for multiplexing and diversity of a multiple antenna system. For completeness we restate these definitions here. The multiplexing gain of a system is denoted by r and is defined by

$$r = \lim_{P_{av} \to \infty} \frac{R(P_{av})}{\log P_{av}}, \tag{3}$$

where P_{av} is the average available power and $R(P_{av})$ is an achievable rate for a given power at the transmitter. In order to define the diversity order of a system, first we need to define the outage probability, which in turn relies on the conditional instantaneous mutual information given by

$$I(X;Y|D) = \log\det\left(I + QDD^{\dagger}\right). \tag{4}$$

Throughout the paper we are assuming a full rank Gaussian codebook being used with independent and identically distributed (iid) symbols. So (4) can be rewritten as

$$I(X;Y|D) = \sum_{i=1}^{m} \log(1 + P(D)\lambda_i), \tag{5}$$

where $P(D)$ is the power allocation strategy with $mE[P(D)] \leq P_{av}$. If the instantaneous mutual information is smaller than the transmission rate, R, then reliable communication is not possible, and an outage event is occurred. Therefore, outage probability can be expressed as

$$\Pi(R, P_{av}) = \Pr(I(X;Y|D) < R). \tag{6}$$

The diversity order of a system as defined in [1] is denoted by d and evaluated as:

$$d = \lim_{P_{av} \to \infty} -\frac{\log(\Pi(R, P_{av}))}{\log(P_{av})}. \tag{7}$$

2.2 Feedback Design

In a system with finite rate feedback, the problem of choosing the best quantity to be fed back to the transmitter that optimizes the system performance (e.g. minimizing outage, frame error rate, or bit error rate, or maximizing average throughput, etc.) is largely open. Here, we simplify the design of feedback by assuming that the transmitter has perfect knowledge of beamforming matrix, V. Note that this assumption has no effect on the diversity order of the system [5]. We consider the case when the receiver quantizes *only* one of the eigenvalues of the channel matrix, λ_i's, $1 \leq i \leq m$, to be fed back to the transmitter. The effect of choice of the eigenvalue (e.g. smallest or largest, etc.) on outage probability is discussed in Section 3.

In the design of the quantizer we follow the model in [3], where a sub-optimum channel quantizer has been developed which allocates equal total power at each quantization bin. Authors in [3] have shown that the performance of the developed quantizer (with respect to the outage probability) closely follows that of the optimum quantizer.

Figure 1 depicts an example of the quantizer with 5 quantization bins. The channel quantizer can be represented by the set of thresholds γ_i's, $1 \leq i \leq L-1$, where L is the number of quantization bins. We allocate a constant power level, p_i, $0 \leq i \leq L-1$, to each quantization interval, $[0, \gamma_1), \ldots, [\gamma_i, \gamma_{i+1}), \ldots, [\gamma_{L-1}, \infty)$, such that for $i \geq 1$ a zero outage communication is guaranteed. In this fashion, the fixed power allocated to the first bin, p_0, is not necessarily sufficient to guarantee zero outage transmission for all channel states, $0 \leq \gamma < \gamma_1$. Hence, there exists a channel state,

$\gamma_0 \in [0, \gamma_1)$ and $\gamma_0 = (2^R - 1)/P_0$, such that for the channel states smaller than which the system faces outage in transmission of information. As a result, Equation (6) can be rewritten as

$$\Pi(R, P_{av}) = \Pr(\gamma < \gamma_0). \tag{8}$$

We will characterize the outage probability of Equation (8) with respect to average available power and feedback rate in Section 3. Throughout the paper the channel parameter that is measured is denoted by λ_i and quantization parameters are denoted by γ_j. Hence, while $f(\lambda_i)$ is a function of channel parameter λ_i, $f(\gamma_j)$ is the same function evaluated at one of the quantization thresholds.

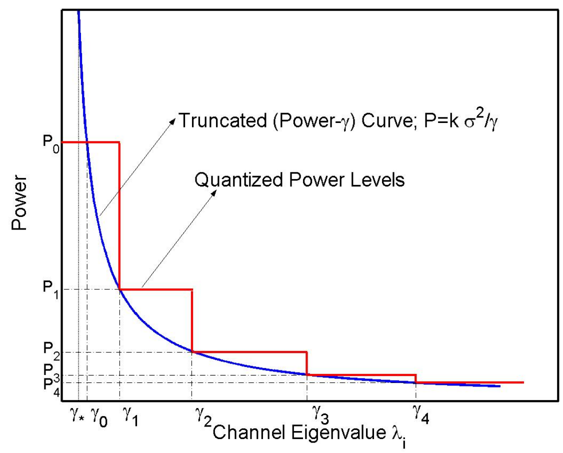

Figure 1: An example of a 5-bin channel quantizer with channel inversion power allocation.

3 Main Result

In this section we state and prove the main result of this paper which quantifies the tradeoff between diversity and multiplexing in a multiple antenna system.

Theorem 3.1 (multiplexing-diversity tradeoff) *For a system with M transmit and N receive antennas with B bits of feedback ($L = 2^B$ level quantizer, $B \geq 1$), where the feedback only quantizes the ith maximum eigenvalue of the channel matrix, the maximum achievable multiplexing gain, r, and diversity order, d, are given by:*

$$\begin{aligned} r &= i, \text{ for } i = 1, 2, \ldots, m \\ d &= (n-r+1)(m-r+1)[1 + (n-r+1)(m-r+1) + \cdots + ((n-r+1)(m-r+1))^{2^B-2}] \qquad (9) \\ d &= nm(1 + nm + \cdots + (nm)^{2^B} - 1) \text{ for } r = 0, \qquad (10) \end{aligned}$$

where $m = \min(M, N)$ and $n = \max(M, N)$.

Since each of the parallel channels is similar to a single antenna system, a maximum multiplexing gain of one can be achieved on each of the m parallel channels. In order to achieve multiplexing factor of $r < m$ it suffices to use r out of m channels for multiplexing gain. If we guarantee the multiplexing gain of one on the channel corresponding to the rth largest eigenvalue of the channel matrix, then we also get the multiplexing gain of one on $r - 1$ channels corresponding to the larger eigenvalues. Hence the total multiplexing gain of r is achieved. The diversity order of such a system also can be bounded by the outage performance of the channel corresponding to the rth eigenvalue.

Proof: [3.1] The proof consists of three steps. In the first step we assume that the rate is fixed for all P_{av} and the receiver measures the ith largest eigenvalue, quantizes its value, and feeds it back to the transmitter. We find the diversity order of such a system as a function of average available power and feedback rate by characterizing the asymptotic behavior of the distribution of the ith largest eigenvalue of the channel matrix for large values of average available power.

In the second step, we propose a hybrid coding scheme which exploits one bit of feedback. This bit of feedback enables us to choose between two codebooks, one with a rate which varies with average available power, and another codebook with a fixed rate. In doing so, we can separate the error event (and outage event) from the codebook that achieves multiplexing.

Finally in the third step, we combine the power allocation scheme of step one, and the hybrid coding scheme introduced in step two, to evaluate the simultaneously achievable multiplexing gain and diversity order in a system with channel feedback.

Step1, Full diversity order: In order to quantify the diversity order, we consider the ordered statistics of the eigenvalues of the Wishart matrix, Z, described in Section 2. The distribution of the ordered eigenvalues of Wishart matrix, $\lambda_m > \lambda_{m-1} > \cdots > \lambda_1$, is given in [6]

$$f(\lambda_m, \lambda_{m-1}, \ldots, \lambda_1) = e^{-\sum_{i=1}^{m} \lambda_i} \prod_{i=1}^{m} \lambda_i^{n-m} \prod_{i<j} (\lambda_i - \lambda_j)^2. \tag{11}$$

Here, we evaluate the outage probability when the channel quantizer chooses a fixed index i, $1 \leq i \leq m$, and only feeds back information regarding λ_i to the transmitter. In this case, the design of the quantizer depends on average available power, P_{av}, and the distribution of λ_i, $f(\lambda_i)$, which can be calculated by integration of Equation (11) over all λ_j's, $1 \leq j \leq m$, $j \neq i$. Define $F(\lambda_i)$ as

$$F(\lambda_i) = \int f(\lambda_i) d\lambda_i. \tag{12}$$

Then, the outage probability (8) can be expressed as

$$\begin{aligned} \Pi(R, P_{av}) &= \int_0^{\gamma_0} f(\lambda_i) d\lambda_i \\ &= F(\gamma_0) - F(0). \end{aligned} \tag{13}$$

Note that as P_{av} goes to infinity, γ_0 approaches zero. Hence, the diversity order depends on behavior of $F(\cdot)$ around the origin, which is characterized in Lemma 3.2.

Lemma 3.2 *Let λ_i be the ith smallest eigenvalue and $F(\lambda_i)$ be defined as in Equation (12), then the Taylor expansion of $F(\lambda_i)$ around the origin is given by*

$$F(\lambda_i) = F(0) + \beta \lambda_i^{i(n-m+i)}, \tag{14}$$

where β is a constant.

Proof: [3.2] Let the joint distribution of the λ_i's, $1 \leq i \leq m$, be given by Equation (11). Then the marginal distribution of λ_i's, can be written as

$$f(\lambda_i) = \int_0^{\infty} \int_0^{\lambda_m} \cdots \int_0^{\lambda_{i+2}} \int_0^{\lambda_i} \cdots \int_0^{\lambda_2} f(\lambda_m, \lambda_{m-1}, \ldots, \lambda_1) d\lambda_m d\lambda_{m-1} \ldots d\lambda_{i+1} d\lambda_{i-1} \ldots d\lambda_1. \tag{15}$$

Note that the integration over λ_j's for $j > i$ results in a constant and is not a function of λ_i. On the other hand, evaluation of $f(\lambda_i)$ involves $(i-1)$ integrations over λ_j's, $1 \leq j < i$, each contributing a factor of λ_i^{n-m+1} to the marginal distribution [2]. Also for every j and k smaller than i, since λ_i is assumed to be close to the origin, so are λ_j and λ_k ($\lambda_{j,k} < \lambda_i$). Hence, we can approximate $(\lambda_k - \lambda_j)^2$ for $j < k < i$ with λ_k^2, which results in a factor of $\lambda_i^{2(1+2+\cdots+(i-1))}$. Therefore, $f(\lambda_i)$ is of form

$$f(\lambda_i) = \lambda_i^{(n-m)+(i-1)(n-m+1)+2(1+2+\cdots+(i-1))} q(\lambda_i) e^{-\lambda_i}, \tag{16}$$

where $q(\lambda_i)$ is a function of λ_i containing polynomials and exponential functions of λ_i such that $q(0) \neq 0$. Let k be the exponent of λ_i in Equation (16). Since $\frac{dF(\lambda_i)}{d\lambda_i} = f(\lambda_i)$, the first k derivatives of $F(\lambda_i)$ evaluated at 0 are equal to 0. Thus, the Taylor expansion of $F(\lambda_i)$ around the origin can be written as

$$F(\lambda_i) = F(0) + \beta \lambda_i^{k+1}, \tag{17}$$

where $k + 1 = i(n - m + i)$. ■

Corollary 3.3 *For the ith largest eigenvalue we have*

$$F(\lambda_i) = F(0) + \beta \lambda_i^{(n-i+1)(m-i+1)}. \tag{18}$$

Proof: [3.3] The proof is immediate by change of variable i with $m - i + 1$ in Equation (14). ■

Corollary 3.4 *The probability of outage in a system with finite rate feedback in which λ_i, the ith smallest eigenvalue, is quantized and fed back is proportional to*

$$\Pi(R, P_{av}) \propto \gamma_0^{i(n-m+i)}. \tag{19}$$

Proof: [3.4] The proof is immediate by replacing Equation (14) in Equation (13). ■

Corollary 3.4 describes the asymptotic (as $P_{av} \to \infty$, γ_0 goes to zero) behavior of outage probability as a function of γ_0 and the choice of eigenvalue λ_i. Now, in order to fully characterize the probability of outage we need to describe γ_0 as a function of average available power, number of quantization bins, and choice of λ_i.

Lemma 3.5 *For a quantizer with L bins, designed for the ith smallest eigenvalue, λ_i, and average power constraint, P_{av},*

$$\gamma_0 \approx \frac{c_0}{P_{av}^{1+i(n-m+i)+\cdots+(i(n-m+i))^{L-1}}}. \tag{20}$$

Proof: [3.5] Proof by recursion. In proving Lemma 3.5 we start by solving the expression of the total power at the last quantization bin for γ_{L-1}. Replacing the resultant value for γ_{L-1} in the expression of the total power at the one to the last quantization bin, results in an approximation for γ_{L-2}. Repeating the process recursively results in an approximate solution for γ_0.

For simplicity let $k = i(n-m+1)$. Then, for the last bin of a sub-optimum quantizer as described in Section 2.2, we have

$$\frac{c}{\gamma_{L-1}}(F(\infty) - F(\gamma_{L-1})) = \frac{P_{av}}{L}, \tag{21}$$

where c is a constant depending on the transmission rate, R, and $F(\infty) = 0$ due to the exponential factor of $e^{-\gamma}$ in $F(\cdot)$ (Note that $F(\infty) - F(0) = 1$ and hence, $F(0) = -1$). Since as P_{av} goes to infinity, γ_{L-1} approaches zero, $F(\gamma_{L-1})$ in Equation (21) can be replaced with the right hand side of Equation (14). By doing so we have

$$\frac{c}{\gamma_{L-1}}(1 - \beta\gamma_{L-1}^k) = \frac{P_{av}}{L}. \tag{22}$$

[2] $\int_0^x y^\alpha e^{-y} dy \approx y^{\alpha+1}$, for small x.

Since γ_{L-1} is approaching zero, the term γ_{L-1}^k in Equation (22) is negligible compared to 1. Applying this approximation in Equation (22) results in the following approximate solution for γ_{L-1}

$$\gamma_{L-1} \approx \frac{c_{L-1}}{P_{av}}. \tag{23}$$

The total power at the $(L-1)$st bin is given by

$$\frac{c}{\gamma_{L-2}}(F(\gamma_{L-1}) - F(\gamma_{L-2})) = \frac{P_{av}}{L}. \tag{24}$$

Applying the large P_{av} approximation to Equation (24) we get

$$\frac{c}{\gamma_{L-2}}(\beta\gamma_{L-1}^k - \beta\gamma_{L-2}^k) = \frac{P_{av}}{L}. \tag{25}$$

By neglecting γ_{L-2}^k compared to γ_{L-1}^k in Equation (25) and replacing γ_{L-1} with the expression in (23), we get

$$\gamma_{L-2} \approx \frac{c_{L-2}}{P_{av}^{1+k}}. \tag{26}$$

Repeating this procedure L times, the expression in Equation (20) is achieved. ■

Theorem 3.6 *Consider a multiple antenna system with M transmit and N receive antennas, with finite rate feedback which carries quantized information of ith ($1 \leq i \leq m$) smallest eigenvalue of the Wishart matrix of the channel. The diversity order of such a system is given by*

$$d = i(n-m+i)[1 + i(n-m+i) + \cdots + (i(n-m+i))^{L-1}]. \tag{27}$$

Proof: [3.6] The result in Equation (27) is obtained by replacing (20) in (19). ■
Rewriting Equation (27) in terms of the ith largest eigenvalue, we obtain

$$d = (m-i+1)(n-i+1)[1 + (m-i+1)(n-i+1) + \cdots + ((m-i+1)(n-i+1))^{L-1}]. \tag{28}$$

We use the ith smallest eigenvalue for the simpler evaluation of the marginal distribution. In the rest of the paper we will use only the distribution and diversity order based on the ith largest eigenvalue. An interesting conclusion drawn from Equation (28) is that the diversity order is the largest when $i = 1$, or equivalently when the power allocation is based on the realization of the largest eigenvalue. This means that the allocated power is sufficient enough only for the successful transmission on the channel with the largest eigenvalue and as a result all other channels would experience outage.

Step 2, Feedback-based coding scheme: For a given P_{av}, the rate of transmission is fixed. However, in order to achieve a non-zero multiplexing gain, rate of transmission must increase in a logarithmic fashion with P_{av}. On the other hand, since a block fading channel model is considered, the coding scheme is different from what is used for ergodic channels. In [1], a codebook with fixed rate and constant power is used to achieve the multiplexing gain when CSI is only available at the receiver. When CSI is also available at the transmitter the optimum power allocation and coding scheme are found in [7]. We use the codebook introduced in [1] with one modification. In our modification to the codebook introduced in [1], we introduce a threshold γ_{th} on the channel state such that a) γ_{th} vanishes as P_{av} increases, and b) for channel states $\gamma > \gamma_{th}$ we use the codebook in [1] with a constant power αP_{av}, where α is a fixed constant (for all P_{av}'s) between zero and one, we call this codebook *variable rate code* and c) for channel states $\gamma < \gamma_{th}$ we use a codebook with a fixed transmission rate, R', for all P_{av}'s, and we call this codebook *fixed rate code.*

Note that the coding scheme in [1] with a block length greater than $m+n-1$ achieves the diversity order and multiplexing gain. This is due to the fact that for block lengths greater than $m+n-1$ the probability of error for the zero outage region is smaller than the probability of outage. Therefore, outage is the dominant contributor to the error event. Thus probability of error has a similar asymptotic behavior as of probability of outage. With quantized eigenvalues, the probability of outage depends on the number of bits in feedback. Therefore in order to make the outage a dominant factor, the block length of the code also depends on the number of bits in feedback. So one can select an appropriate code length (for a given feedback rate) such that the error event at the zero outage region can be neglected in comparison with the outage event.

If we choose γ_{th} such that the power allocated to the variable rate code, αP_{av}, can guarantee a zero outage transmission of codeword of variable rate code, then the outage only happens in the transmission of codeword of fixed rate code. Therefore the total throughput can be written as

$$R(P_{av},\gamma_{th}) = R' \Pr(\gamma_{out} < \gamma < \gamma_{th}) + \log(1+\alpha P_{av}\gamma_{th}) \Pr(\gamma > \gamma_{th}). \tag{29}$$

Since γ_{th} goes to zero as P_{av} approaches infinity, the first term in the right hand side of (29) goes to zero and the total throughput is dominated by the rate achieved from the variable rate code. Also note that since we would like to achieve a non-zero multiplexing gain, $\alpha P_{av}\gamma_{th}$ must increase unboundedly as P_{av} approaches infinity. Although we do not aim in choosing a threshold γ_{th} that maximizes the throughput and we only care about the multiplexing gain, an approximation to the threshold that maximizes the total throughput given in (29) could be a good candidate for γ_{th}. Since $\Pr(\gamma > \gamma_{th})$ is monotone decreasing with γ_{th} and $\log(1+\alpha P_{av}\gamma_{th})$ is monotone increasing with γ_{th}, there is a $\gamma_{th} \in \mathcal{R}^+$ that maximizes (29). Consider that γ has the distribution of λ_i, for some i, $1 \le i \le m$, and asymptotic approximation given in Lemma 3.2, then by setting the first derivative of (29) with respect to γ_{th} to zero we have

$$\frac{\alpha P_{av}}{1+\alpha P_{av}\gamma_{th}} - \log(1+\alpha P_{av}\gamma_{th}) f(\gamma_{th}) = 0, \tag{30}$$

where $f(\cdot)$ is probability density function of λ_i. Replacing $f(\gamma_{th})$ by approximation of (16) around zero and by algebraic simplification of (30) we get

$$\gamma_{th} \approx \frac{1}{(\log(\alpha P_{av}))^{1/k}}, \tag{31}$$

where $k = i(n-m+i)$. Note that

$$r_i = \lim_{\alpha P_{av}\to\infty} \frac{\log\left(1+\frac{\alpha P_{av}}{(\log(\alpha P_{av}))^{1/k}}\right)}{\log(\alpha P_{av})} = 1, \tag{32}$$

as expected. The rate of the *variable rate code* is set to $\log(1+\alpha P_{av}\gamma_{th})$. Since for channel states greater than γ_{th} the channel can support higher rates, it is guaranteed that for this choice of threshold and power, the transmission of codewords from variable rate code is outage free.

In this step we showed that by use of a single bit of feedback, which corresponds to threshold γ_{th}, we can exploit the knowledge of the channel to switch between two codebooks. This knowledge of channel enables us to separate outage event from the codebook that achieves multiplexing gain.

Step 3, Combining Steps 1 and 2: In Step 1, we derived the diversity order of a system with fixed rate codebook and finite rate feedback. In Step 2, we showed how one bit of feedback can be used to separate outage prone codebook from multiplexing achieving codebook. In this step by applying the power allocation strategy developed in Step 1 to the codebook developed in Step 2 we quantify the diversity-multiplexing tradeoff.

For multiplexing gain of zero, $r = 0$, the answer is immediate by replacing $i = m$ in Equation (28). Let assume a non-zero multiplexing gain, $r = i, i \neq 0$, then by use of one bit of feedback and the coding scheme introduced in Step 2, we separate the multiplexing and diversity. Let the quantizer choose power allocation and coding scheme based on the realization of the ith largest eigenvalue of the channel matrix. Then, there are $i-1$ eigenvalues larger than the quantized eigenvalue. Since the power allocation and coding scheme achieve the multiplexing of order $r_i = 1$ (from (32)) for the channel corresponding to the ith largest eigenvalue of the channel matrix, the same multiplexing gain is achievable for the other $i-1$ channels corresponding to the $i-1$ larger eigenvalues. Hence the total multiplexing gain will be $r = \sum_{j=1}^{i} r_j = i$, for the total i parallel channels.

Similarly, since the channels corresponding to all the $(i-1)$ largest eigenvalues are better than the channel of the ith largest eigenvalue, the outage is only related to the channel corresponding to the ith largest eigenvalue. Also, since we have used one bit of feedback for rate control, the total levels for power control is $L = 2^B - 1$. Then diversity order will be obtained by replacing the equivalent available quantization levels L and $i = r$ into (28). This completes the proof of Theorem 3.1 ■

Note that by use of one bit of feedback for choosing the codebook we reduce the boundaries of quantization from the right to $\gamma_L = \gamma_{th}$, instead of infinity. Hence the equations for calculating the threshold of quantizer need to be modified. It is not hard to show that such a modification does not affect the outage behavior of the power control scheme. In fact by replacing infinity with γ_{th} given by (31) in Equation (21) we get

$$\frac{c}{\gamma_{L-1}}\left(\frac{1}{(\log(P_{av}))^{1/k}} - \gamma_{L-1}^{k}\right) = (1-\alpha)\frac{P_{av}}{L}, \tag{33}$$

where $(1-\alpha)$ in (33) comes from the fact that we allocate α fraction of the P_{av} to the variable rate code. An approximate solution to (33) for large values of P_{av} is

$$\gamma_{L-1} \approx \frac{c}{P_{av}(\log(P_{av}))^{1/k}} \tag{34}$$

This new modification in thresholds does not affect the outage and diversity because

$$\lim_{P_{av}\to\infty} \frac{\log[\log(P_{av})]^{\eta}}{\log(P_{av})} = 0 \qquad \forall \eta,\ \ 0 < \eta < \infty. \tag{35}$$

Figure 2 shows multiplexing-diversity curve derived from the results of Theorem 3.1 along with multiplexing-diversity curve for a system with no CSI at the transmitter. It is assumed that $m = \min(M, N) = 2, n = \max(M, N) = 3$, and there are $B = 1$ bits of feedback (which corresponds to a quantizer with 2 bins). The three points on the (multiplexing, diversity) curve for the system with one bit of feedback are $(0, 42), (1, 6)$ and $(2, 2)$. Note that for a system with full multiplexing with finite rate feedback a nonzero diversity order is achievable.

4 Conclusions

In this work we derived the tradeoff between diversity order and multiplexing gain of a MIMO system with finite rate feedback. In order to find the tradeoff we showed two interesting intermediate properties of systems with finite rate feedback. First we showed that the diversity order of a system with finite rate feedback is finite and grows unboundedly as we increase number of feedback bits, such that as number of bits in feedback approaches infinity, the diversity order approaches infinity as well.

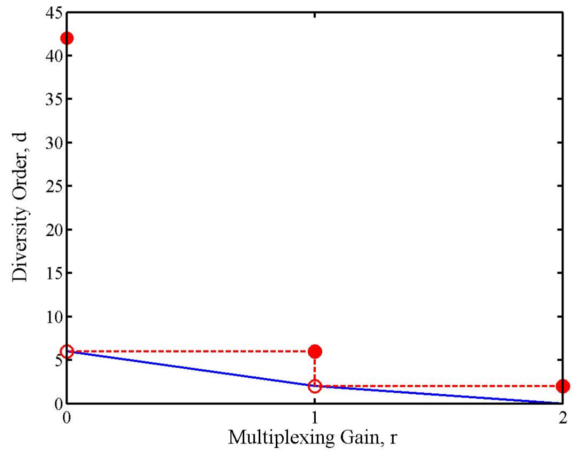

Figure 2: Multiplexing-diversity curve for systems with no and partial CSI at the transmitter with M=2 transmit and N=3 receive antennas, and B=1 bit of feedback. Solid curve is corresponding to the system with CSI only at the receiver. Dashed curve with circles corresponds to the system with perfect CSI at the receiver and partial knowledge at the transmitter. The filled circles and empty circles are corresponding to the closed and open end of the intervals respectively.

The other interesting intermediate step showed that by use of knowledge of channel we could separate the outage event from the codebook that achieves multiplexing gain. But since the multiplexing achieving code does not have an impressive performance with respect to outage, we use another codebook with proper power allocation, such that we gain the maximum possible diversity order.

Although the idea of diversity-multiplexing tradeoff introduced a measure for comparison of coding schemes that achieve different points on the diversity-multiplexing curve, the way to compare two different coding schemes that achieve the same diversity-multiplexing tradeoff still can be improved. We believe that one can define a better measure rather than error exponent to provide a way of comparing coding schemes with the same diversity and multiplexing gains.

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