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Coding-Spreading Tradeoff for Lattice Codes

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Abstract

A fixed bandwidth expansion can be achieved either by coding or spreading, while each have different effect on the resultant signal space. Coding increases both Shannon and Fourier bandwidth whereas spreading only increases the Fourier bandwidth. In this document we are looking for the optimum combination of coding and spreading, in a Code Division Multiple Access (CDMA) system, that minimizes the average frame error rate under fading channel with multiple antennas at transmitter and receiver. Using the theory of lattice code, we show that in a system with K users, the optimum spreading factor N equals K. Simulation results support the analysis. In simulations we used Minimum Mean Square Error (MMSE), and Matched Filter (MF) as multi-user detector. We also assumed that receiver knows the Channel State Information (CSI). In case of multiple antennas Alamouti scheme [Ala98] at transmitter and Maximum Ratio Combining (MRC) at the receiver are applied.

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Chapter 1

Introduction

As Shannon showed in his fundamental paper [Sha48], bits are the building blocks of information. In a digital communication system there exists an information source which generates information bit stream. This could be a binary representation of a quantized analog source. Information source generates the bit stream with a rate, which is a parameter of the source. Each bit generated by source is represented by a signal. The bandwidth of this signal is what we will refer to as required bandwidth. The focus of this thesis is on the signals and signal space prior to modulation. Note that one might refer to available bandwidth in the channel as required bandwidth. In this case for consistency in analysis one should consider the bandwidth efficiency factor provided by modulation in order to find the required and available bandwidth at the encoder.

In a multi-user wireless communication system, in addition to noise there exist multi-path fading and multi-user interference that degrade users' signals. In order to combat with these effects each user has to introduce redundancy into the system. This causes increase in rate of transmission without increase in the rate of information, and hence transmitted signal requires more bandwidth. This is what is referred to as bandwidth expansion.

A fixed bandwidth expansion can be achieved either by coding or by spreading. Although spreading can be viewed as a repetition code, as we will see in Chapter 2, they are different in nature. Channel codes are usually designed to increase the resistance of transmitted signal against thermal and background noises, and other unstructured interferences. Whereas spreading introduces diversity into the system that helps to reduce the structured interference, and allows algorithms to exploit multi-path fading for improving the performance.

Coding is a linear operation in finite fields, however a nonlinear operation in the real space. Using stronger codes reduces the probability of error, however stronger codes have greater decoding complexity at the receiver. In contrary, spreading is a linear operation in the space of reals with minimal complexity at the receiver.

In a multi-user system we need a scheme to enable us to distinguish among users at the receiver. This user separation can be done in time by allocating a time slot to each user such that at each interval one and only one user transmits its own data at all the available bandwidth. This system is known as Time Division Multiple Access (TDMA). Another way is to separate users in frequency. In this method, available bandwidth is divided to frequency bins, and these bins are allocated to users in a way that each bin is used by one and only one user at all times. This system is known as Frequency Division Multiple Access (FDMA). In a Code Division Multiple Access (CDMA) system, all users may have access to channel at all times and all frequencies. In this case separation is performed by either designing a multi-user code or by assigning a unique spreading pattern to each user. This unique spreading pattern allows us at the receiver to separate users.

Throughout this work, we deal with CDMA systems. Hence we have two choices for user separation: Design of a multi-user code or spreading. A multi-user code requires a high dimensional code book. This is equivalent to having a more complex decoder at the receiver. Whereas spreading enables us to separate users by a linear operation, and hence it has a lower complexity in comparison with a multi-user code.

In this thesis we are searching for the best combination of coding rate and spreading factor that minimizes the average frame error rate in a multi-user Direct Sequence Code Division Multiple Access (DS-CDMA) system over a frequency non selective fading channel. We modeled the uplink (user to base station) in which the users are assumed to be synchronous and have the same average power at the receiver. We perform the simulations for both single user Matched Filter (MF) detection and Minimum Mean Square Error (MMSE) detection.

The main contribution is a new approach to the problem that establishes proper links among the tools that have already been developed in the field of information theory. We use the theory of lattice codes and the performance analysis of such codes to understand and quantify coding gain. We consider spreading as a linear mapping that partitions the whole space into subspaces and allocates each of these subspaces to a user. In this case coding rate is equivalent to the dimension of each subspace, and spreading factor is equivalent to the number of partitions in the available space.

We show that for single user MF detection, more spreading is desirable. Performance of an MMSE system is more dependent on Signal to Noise Ratio (SNR). At low SNRs an MMSE system has a better performance when it uses all the bandwidth expansion via spreading. At higher SNRs such a system tends to have a better performance when it uses a combination of coding and spreading. In this case we showed that the best case is when the spreading factor is equal to the number of users.

The remaining parts of the thesis is organized as follows. In Chapter 2 we will explain the nature of bandwidth expansion through coding and spreading. In Chapter 3 we will review the works that have been done in literature. Chapter 4 contains our attempts for an analytical solution. Simulation results and our analysis are main theme of Chapter 5. Finally, summary and future directions of this problem is discussed in Chapter 6.

Chapter 2

Bandwidth Expansion

In this chapter we will introduce bandwidth expansion, dimension of signal space in which transmitted signal lies, and the true effect of coding and spreading on the dimension of this space. Also we articulate the tradeoff between coding and spreading from a multi-user code design point of view.

2.1 Bandwidth Expansion

Throughout this work we are using the following definitions introduced in [Mas94, VM00]. Let $S(T,W)$ be the space of all signals with time support of $[0,T]$ and band-limited to a baseband bandwidth of W. A discussion on time limited and band limited signals can be found in [Sle76]. Dimension of such a signal space $S(T,W)$, is given by [Sle76, WJ65]:

$$D_F \approx 2WT. \tag{2.1}$$

This is known as the *Fourier dimension* of the signal space. Hence every signal $x(t)$ in this space can be represented as a vector $x = (x_1, x_2, ..., x_{D_F})$ in a D_F-dimensional space with

$$x(t) = \sum_{i=1}^{D_F} x_i \phi_i, \tag{2.2}$$

where ϕ_i's are basis for the signal space $S(T,W)$.

Assume that the set of symbols $\mathcal{S}$ is finite, and each symbol $s \in \mathcal{S}$ is mapped to a signal $s(t)$, and that $s(t) \in S(T,W)$. Let $X(T,W)$ be the set of all signals $s(t)$ defined as above. Hence $X(T,W) \subseteq S(T,W)$ by construction. Under the above conditions, the Shannon dimension of $X(T,W)$ is defined to be the dimension of $Span(X(T,W))$, and is denoted by D_S, where

$$Span(X(T,W)) = \sum_{i=1}^{M} \alpha_i s_i(t) \quad \text{for all} \quad s_i(t) \in X(T,W), \quad \alpha_i \in \mathbb{R} \tag{2.3}$$

and $M = \| \mathcal{S} \|$ is the number of symbols in $\mathcal{S}$. The ratio of $D_S/2T$ is known as the *Shannon bandwidth* of the signal set $X(T,W)$ and is denoted by B [Sha49]. Since $X(T,W) \subseteq S(T,W)$ then $D_S \leq D_F$, which implies $B \leq W$.

Using the notion of Shannon bandwidth, Massey [Mas94] defined a spread spectrum system as one for which $B \ll W$, and the spreading factor is given by:

$$N = \frac{W}{B}. \tag{2.4}$$

By simple manipulation one can see that:

$$\begin{aligned} N &= \frac{2TW}{2TB} \\ &= \frac{D_F}{D_S}. \end{aligned} \tag{2.5}$$

Assume that each bit is represented by $bp(t)$, where $b \in \{-1, 1\}$ and $p(t) \in S(T,W)$ is the bit waveform. Suppose we want to add redundancy to k bits of information such that the resultant message consists of n bits and still be transmitted in kT seconds, then we need n signals each with duration kT/n to represent each coded bit. Therefore the rate has been increased and consequently the bandwidth has been expanded to $W_0' = nW/k$.

This redundancy could be imposed through coding or spreading. Figure 2.1 further explains the bandwidth expansion through coding and spreading. In Figure2.1.a, k bits of information are being transmitted in T seconds, hence the original bandwidth occupied by the signal is

$$W = \frac{1}{2T_b}\ , \tag{2.6}$$

where T_b is the bit period and we have $T = kT_b$. By applying an (n, k) coding scheme, in Figure 2.1.b, n coded bits are being transmitted in the same time interval, hence the bandwidth of the coded signal is:

$$\begin{aligned} W'_0 &= \frac{1}{2T_c} \\ &= \frac{n}{k}\frac{1}{2T_b} \\ &= \frac{1}{R}W, \end{aligned} \tag{2.7}$$

where $nT_c = kT_b = T$, and $R = k/n$ is the coding rate. In Figure 2.1.c each coded bit is being spread by a spreading sequence of length N. At the end of this process the bandwidth of the coded and spread signal can be written as:

$$\begin{aligned} W' &= \frac{1}{2T_s} \\ &= \frac{N}{R}W, \end{aligned} \tag{2.8}$$

where $NT_s = T_c$, and $\eta = \frac{N}{R}$ is known as the total bandwidth expansion due to coding and spreading. However there is a major difference between these two forms of expansion as Massey pointed out in his paper [Mas94].

Spreading is a linear transformation which maps from the set of information signals to the set of coded signals. Since it is linear, the dimension of the image of the transformation is the same as the dimension of the domain, i.e., the input and output of spreader have the same Shannon bandwidth. Therefore one can say that spreading does not affect the Shannon bandwidth, however it increases the bandwidth of the signal, which is referred to as *Fourier bandwidth.* On the other hand since coding is a nonlinear operation in real space, a nontrivial coding is capable of increasing the dimension of code book $\mathcal{C}$ with respect to the input symbol set $\mathcal{S}$. Hence the major contribution of coding is expansion of the Shannon bandwidth.

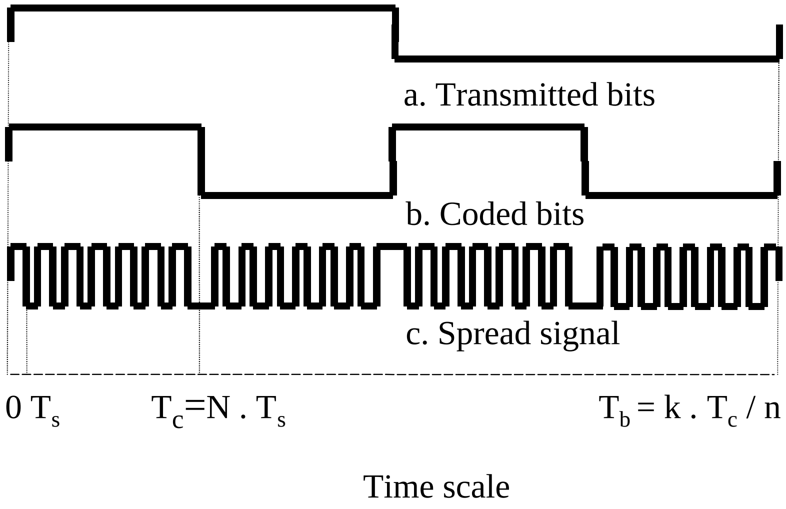

Figure 2.1 Bandwidth expansion due to coding and spreading.

2.2 Coding Point of View

The idea of a multi-user code started from designing a multiple access communication environment. Such system aside from conventional building blocks of a communication link shown in Figure 2.2, requires another feature; user separation. This requires that at the transmitter each user identifies its own data in a way that at the receiver it can be uniquely separated from other users' transmitted data. This separation can be obtained by designing a multi-user code. Assume that there are K users in the system. Also assume that the desired code book $\mathcal{C}$ has dimension D_0. Hence

$$D_0 = dim(\mathcal{C}),$$

Let C_i, $i = 1...K$ be the code book of the i^{th} user. Suppose that all C_i's have the same dimension D_1. There are two interesting cases. The first case is when $D_0 = D_1$. In this case we have a pure multi-user code. Since all C_i's, $i = 1...K$, are sharing the same space, there should be either a jointly encoder, or each user's encoder should have

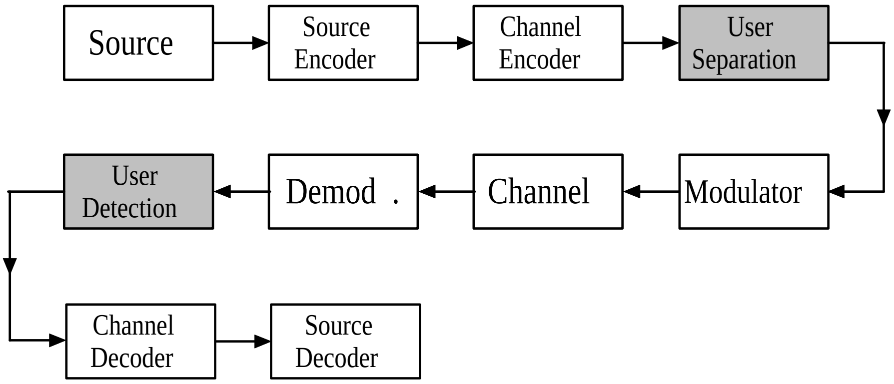

Figure 2.2 A communication link with additional blocks for supporting multi-user.

the information about all other users' code books in order to prevent any undesired overlap that may cause ambiguity in the system.

Second case is when $D_0 > D_1$. In this case the space generated by span of each C_i, $i = 1...K$, is a subspace of the span of $\mathcal{C}$. The advantage of this scheme is that all the users can use the same coding scheme to generate a D_1 dimension code book. Then by allocating a unique unitary linear transformation, each user is able to transfer its code book into $\mathcal{C}$. Hence the design of a multi-user code is now being divided into two sequential processes, i.e., design of single user code followed by user separation.

Define N to be

$$N = \lfloor \frac{D_0}{D_1} \rfloor, \tag{2.10}$$

where $\lfloor x \rfloor$ is the greatest integer smaller than x. This means that we can divide the D_0-dimension space into N subspaces of dimension D_1. Hence there are at most N linear transformation available to map the code book of each user into a subset of the desired multi-user code book. Therefore, N must be greater or equal to K, otherwise two or more users will be mapped into the same subset which leads us to the first case in a lower dimension. Having $N \geq K$, guarantees that we can find K

linear transformations that map each user's code book into a linearly independent subspace of our multi-user signal space. If such a linear transformation provides zero coding gain and preserves the energy, then it is referred to as spreading [Vee99], and N is known as spreading factor. Current spreading schemes in CDMA systems including Direct Sequence (DS) and Frequency Hopping (FH) systems, satisfy the above definition.

Now we can look at the problem of tradeoff between coding and spreading as a tradeoff between Shannon and Fourier dimensions. Shannon dimension is the dimension of the code book of each user. Fourier dimension is the dimension of the underlying signal space. In Chapter 5 we will analyze such a tradeoff when user's code books are n-dimensional lattices.

Chapter 3

System Performance Analysis

There have been two major attempts to find the optimum coding rate or equivalently spreading factor. In this chapter we will discuss these two approaches.

3.1 Throughput Analysis

In his series of works [VA96, Vee97, Vee99, MV00, VM00], Veeravalli analyzed the coding spreading tradeoff on total *throughput* in a CDMA system. In the first two papers [VA96, Vee97] authors modeled the multi-user CDMA system as a Binary Symmetric Channel (BSC). In such a system the total throughput η for a system with K users is given by

$$\eta_K = \sum_{k=1}^{K} (1 - H(P_{e,k})), \tag{3.1}$$

where

$$H(p) = -p\ln(p) - (1-p)\ln(1-p), \tag{3.2}$$

is the entropy of a Bernoulli random variable with probability of success equal to p. The probability of error in 3.1 was computed from the results in [Ver98] for both MMSE and MF detectors. Figure 3.1 shows the simulation results in [VA96, Vee97]. The optimum spreading factor N, for MMSE detection is slightly smaller than the number of users K. Based on the simulation results and numerical analysis the authors concluded that the coding is desirable for both synchronous and asynchronous single user detection MF systems. This result is different from ours presented in Chapter 5. However, it is also mentioned that in a single user detection MF system

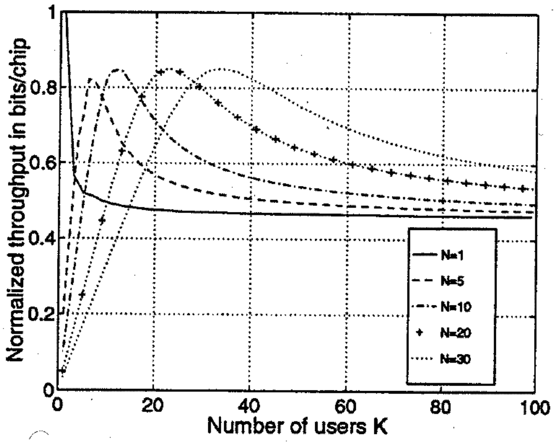

Figure 3.1 Normalized throughput for MMSE detector, and synchronous users.

the highest throughput is achieved for $N = K$, which is the same as our results for high Signal to Noise Ratios (SNR).

Later in [MV00] the tradeoff between spreading and coding has been studied in order to maximize the spectral efficiency, in evaluating which the effect of multi-cell has also been taken into account. Table 3.1 shows the peak spectral efficiencies achieved by MF and the MMSE detectors. The approach in [MV00] is slightly different from [VA96, Vee97, VM00]. In this work the analysis is trying to find the optimum number of users for a given pair of coding rate and spreading factor.

	MF	MMSE
Single Cell Hard Decision	0.2	1
Single Cell Soft Decision	0.48	1
Multi-cell Hard Decision	0.14	0.26
Multi-cell Soft Decision	0.3	0.41

Table 3.1 Peak spectral efficiencies in bits/chip.

This approach has defined the coding-spreading tradeoff as an optimization problem on throughput and bandwidth efficiency. Therefore this approach to the problem for a fading channel still is not applicable.

3.2 Capacity Analysis

The main theme of [EBT99] is to find the optimum coding-spreading tradeoff in a single-cell CDMA system with block-fading, block-synchronous transmission, and power control. In this approach an optimization problem is defined as follows; minimize the outage probability subject to a fixed given bandwidth expansion. The paper studies the behavior of the instantaneous capacity of the system vs. number of users in the system. The outline of the solution is trivial and appealing, but since it is based on a closed form formula for channel capacity it is not applicable for a multi-user fading system, since there is no known closed form analytical formula for channel capacity, outage probability, or code word probability of error for a multi-user fading CDMA system.

The outline of the proposed solution is as follows. Let H be a random variable corresponding to channel state. Then the instantaneous conditional mutual information between transmitted and received signals can be written as

$$I(X;Y|H) = \sum_{x \in X} \sum_{y \in Y} p(x,y|H) \log_2\left(\frac{p(x,y|H)}{p(x|H)p(y|H)}\right). \tag{3.3}$$

Note that if H is a random variable then so is $I(X,Y|H)$. Hence usually in analysis either conditional average mutual information obtained by averaging $I(X;Y|H)$ with respect to H, or probability of outage is being used. Probability of outage is defined as

$$P_{out}(R) = P(I(X;Y|H) < R), \tag{3.4}$$

where R is the code rate in bit/symbol. It is shown in [EBT99] that the information theoretic outage probability defined by Equation 3.4 is equal to the codeword probability of error averaged over all possible channel realizations and random code book. Now let $\eta = N/R$ be the bandwidth expansion, where N is the spreading factor. Also let K denote the number of users in the system. Then the capacity of the system for a desired outage probability P_{out} is defined by

$$C_{sys} = \frac{1}{\eta} \max\{K : P(I(X;Y|H)/N < 1/\eta) < P_{out}\}. \tag{3.5}$$

The tradeoff between coding and spreading from this point of view is elaborated in [EBT99] by implementing the solution to a CDMA system on an AWGN channel. It is assumed that the signals of all users are received with the same amplitude A. Also α is defined to be K/N. In this way the capacity of the system can be written as

$$C_{sys} = \begin{cases} \log_2(1+\beta)\max\left\{\left(\frac{1}{\beta} - \frac{N_0}{A}\right), 0\right\}, & \text{MF} \\ \log_2(1+\beta)(1+\beta)\max\left\{\left(\frac{1}{\beta} - \frac{N_0}{A}\right), 0\right\}, & \text{MMSE} \end{cases} \tag{3.6}$$

where N_0 is the variance of additive Gaussian noise, and β is the desired Signal-to-Interference plus Noise Ratio *(SINR)* at the receiver. In this case in order to have zero outage probability we must have $R = \log_2(1+\beta)$. Replacing in 3.6

$$\alpha = \begin{cases} \max\left\{\left(\frac{1}{\beta} - \frac{N_0}{A}\right), 0\right\}, & \text{MF} \\ (1+\beta)\max\left\{\left(\frac{1}{\beta} - \frac{N_0}{A}\right), 0\right\}. & \text{MMSE} \end{cases} \tag{3.7}$$

However, as mentioned before this analysis is for a CDMA system with AWGN channel and fading is not taken into consideration. Another disadvantage of information theoretic analysis is that it is based on asymptotic argument.

In this approach the coding-spreading tradeoff is defined as an optimization problem on system capacity and outage probability. Therefore this approach is also not applicable to the fading channel.

In next few chapters we try to introduce another approach to the problem, that on one hand enables us to explain the effect of coding and spreading on frame error rate, and on the other hand outlines another view that may lead us to a solution to the optimum coding-spreading pair.

Chapter 4

Analysis on Frame Error Rate

Most of the channel codes, which are used in practice, are lattice codes. Lattice codes include block codes such as cyclic codes, Reed-Solomon codes, Goppa codes, convolutional codes, and all other codes with a group structure. In this chapter we will introduce the required definition on lattice codes, and then we will discus the two approaches that we take in analyzing the behavior of frame error rate with respect to the coding rate.

4.1 Lattice Code

The theory of error correcting code in an Additive White Gaussian Channel (AWGN) is closely entangled with three classical problems of sphere packing, covering, and kissing numbers [CS99].

4.1.1 Sphere Packing

Unsolved even today, the sphere packing problem is to find out how densely a large number of identical spheres can be packed together in a region. The problem is that spheres do not fit together and there is always some wasted space in between them. No matter how the spheres are arranged around each other, about twenty five percent of the space will not be used [CS99].

4.1.2 Kissing Numbers

The problem of kissing numbers asks how many balls can be arranged so that they all just touch other balls of the same size. Applications of kissing numbers are in sampling and quantization [CS99].

4.1.3 Covering

The problem of covering is the dual for sphere packing problem. It looks for the most economical way of covering n-dimensional Euclidean space with equal overlapping spheres.

4.1.4 Definitions

Let us elaborate the problem in a more profound way [VTZ99]. Let $\{\mathbb{R}^{\mathbf{n}}, d(x, y)\}$ be a metric space and Λ be an infinite set in $\mathbb{R}^{\mathbf{n}}$. Then Λ is said to be a sphere packing if $d(\Lambda) > 0$, where

$$d(\Lambda) = \inf_{x,y \in \Lambda, x \neq y} d(x, y).$$

If Λ is group under addition in $\mathbb{R}^{\mathbf{n}}$, then Λ is said to be a lattice packing, or simply a lattice. From now on we assume that Λ is a lattice. Define ρ to be equal to $d(\Lambda)$, then ρ is known as the radius of sphere packing.

Let R be defined by

$$R = \sup_{x \in \mathbb{R}^{\mathrm{n}}} \inf_{\lambda \in \Lambda} (d(x, \lambda)).$$

Then spheres centered around each point of Λ with radius R will cover $\mathbb{R}^{\mathbf{n}}$, and R is referred to as covering radius.

The *Voronoi cell* Π around each point $\lambda \in \Lambda$, is a convex poly-tope which consists of points of $\mathbb{R}^{\mathbf{n}}$ that are at least as close to λ as to any other point $\gamma \in \Lambda$. In other words

$$V(\lambda_i) = \{x \in \mathbb{R}^{\mathrm{n}} : d(x, \lambda_i) \leq d(x, \lambda_j) \text{ for all } j \neq i\}. \tag{4.3}$$

The vertices of a Voronoi cell are points of $\mathbb{R}^{\mathbf{n}}$ whose distances from Λ are local maxima, known as *holes* in $\mathbb{R}^{\mathbf{n}}$. If there is a point in $\mathbb{R}^{\mathbf{n}}$ whose distance from Λ is absolute maximum then it is called a *deep hole* and its distance from Λ is the covering radius R.

4.1.5 Channel Coding

The fundamental link between channel coding and lattice theory goes back to sampling theorem and Shannon's theorem. Sampling theorem states that if $f(t)$ is a signal containing no components of frequency greater than W cycles per second, then $f(t)$ is completely specified by its samples taken every $1/(2W)$ seconds [Sha49, WJ65]. In this case if the signal $f(t)$ lasts for T seconds [Sle76], then there are $\mathbf{n}= 2WT$ samples needed to uniquely represent $f(t)$. These samples are the coordinates of a point in $\mathbb{R}^{\mathbf{n}}$. So the sampling theorem makes it possible to represent $f(t)$ by a single point in $\mathbb{R}^{\mathbf{n}}$.

By above argument, a transmitted signal $C(t)$ can be represented by a point $C = (c_1, ..., c_n)$ in $\mathbb{R}^{\mathbf{n}}$. We can model the signal at the receiver by the vector equation of $y = C + n$. The decoder finds the closest code point to the received vector and reconstructs the signal from it. If noise is small or the code points are placed far enough, then it is more likely to reconstruct the original signal from the noisy received signal.

Above argument can be stated more precisely through the expression of probability of error P_e. Suppose that the code book consists of M code words $C_1, ..., C_M$ in $\mathbb{R}^{\mathbf{n}}$, and let $V(C_k)$ be the Voronoi cell for C_k. Knowing that C_k is transmitted, the decoder makes a correct decision if and only if the received point lies inside $V(C_k)$. Hence probability of correct decision can be written as

$$P_c(C_k) = \int_{V(C_k)} f(x)dx$$

where $f(x)$ is the distribution of noise. If all of the Voronoi cells are congruent to some poly-tope Π, then the average probability of error is

$$\begin{aligned} P_e &= 1 - P_c \\ &= 1 - \frac{1}{M}\sum_{k=1}^{M} P_c(C_k) \\ &= 1 - \int_{\Pi} f(x)dx. \end{aligned} \tag{4.5}$$

Now the problem of channel coding can be stated as follows. Given the dimension n, the number of code words M, and power constraint $\|C_k\| \leq nP, k = 1, ..., M$, find a code book $\mathcal{C} \subset \mathbb{R}^{\mathbf{n}}$ satisfying the power constraint such that the probability of error P_e given by 4.5 is minimized. The lattice version of this problem is to find $\mathcal{C} \subset \mathbb{R}^{\mathbf{n}}$ such that $\mathcal{C}$ be a lattice, which satisfies the power constraint and minimizes the probability of error given by 4.5.

Let Λ be a lattice with Voronoi cell Π of volume 1. Then the difference among the three main lattice problems is as follows. For sphere packing problem we maximize the in-radius of Π, for the covering problem we minimize the circumradius of Π, and for the channel coding problem we minimize P_e. It is also worth to mention that for the quantization problem we minimize the second moment $G(\Pi)$ given by

$$G = G(\Pi) = G(V(\lambda_i)) = \frac{1}{n} Vol(V(\lambda_i))^{-1-\frac{2}{n}} \int_{V(\lambda_i)} \|x - \lambda_i\|^2 dx. \tag{4.6}$$

The quantity $\gamma_S(\Pi) = 1/12G(\Pi)$ is known as the *shaping gain*. Figure 4.1 shows the probability of error vs. SNR for different known lattice codes. As one can see, the

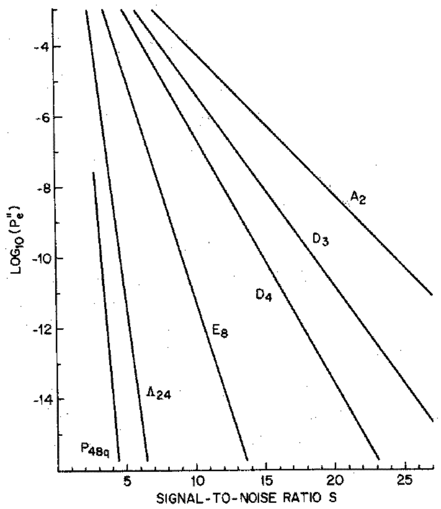

Figure 4.1 P_e vs. SNR for different lattice codes.

probability of error and the slope of the curve decreases as the dimension of the code increases.

We will use the above definitions and relations in next two sections to explain the lower bound and error exponent for lattice codes.

4.2 Lower Bound on Probability of Error

Seeking the solution for an optimum coding rate subject to a fixed bandwidth expansion, and in following the path of solving an optimization problem on the frame error rate, we decided to analyze the best known lower bound on the frame error rate on lattice codes [VTZ99].

In the case of an AWGN channel the noise is Gaussian, and hence the Equation 4.5 can be written as:

$$\begin{aligned} P_e &= 1 - \int_{\Pi} f(x)dx \\ &= 1 - \frac{1}{(\sqrt{2\pi\sigma})^n}. \end{aligned} \tag{4.7}$$

However it is difficult and sometimes impossible to compute the integral of a Gaussian distribution over a Voronoi cell.

Let S_{Π} be a sphere of radius r, centered at the origin such that $V(S_{\Pi}) = V(\Pi)$, i.e., S_{Π} has the same volume as Π has. Using this *equivalent* sphere it is not difficult to see that

$$\int_{\Pi} f(x)dx \leq \int_{S_{\Pi}} f(x)dx, \tag{4.8}$$

and hence from Equation 4.5 one can conclude that

$$P_e \geq P_{e,\mathcal{S}}, \tag{4.9}$$

where $P_{e,\mathcal{S}}$ is the probability that the received signal falls outside the sphere S_{Π}, which is exactly what we were looking for. Instead of evaluating the left hand side in 4.8, we can simply compute the right hand side, which can be computed in closed form as in [VTZ99]

$$P_e \geq e^{-z}(1 + \frac{z}{(1)!} + \frac{z^2}{(2)!} + ... + \frac{z^{\frac{n}{2}-1}}{(\frac{n}{2}-1)!}) \quad n \text{ even} \tag{4.10}$$

$$P_e \geq \text{erfc}(\ z^{\frac{1}{2}})\ + e^{-z}(\frac{z^{\frac{1}{2}}}{(\frac{1}{2})!} + \frac{z^{\frac{3}{2}}}{(\frac{3}{2})!} + ... + \frac{z^{\frac{n}{2}-1}}{(\frac{n}{2}-1)!}) \quad n \text{ odd}, \tag{4.11}$$

where

$$z = \frac{6\Gamma(\frac{n}{2}+1)^{\frac{2}{n}}}{\pi}\gamma_S(\Pi)SNR_{norm} + o(1) \text{ and} \tag{4.12}$$

$$SNR_{norm} \stackrel{def}{=} \frac{P_{av}}{(2^{2R}-1)\sigma^2}, \tag{4.13}$$

where σ^2 is the noise variance per dimension and R is the coding rate.

We examined the behavior of Equation 4.10 in our simulation environment explained in Chapter 5. Figure 4.2 shows the variation of the lower bound given in 4.10 vs. SNR. The SNR_{norm} was calculated as the output SNR of the MMSE multi-user detector given in [Ver98]. However the bound was not tight enough so the variation

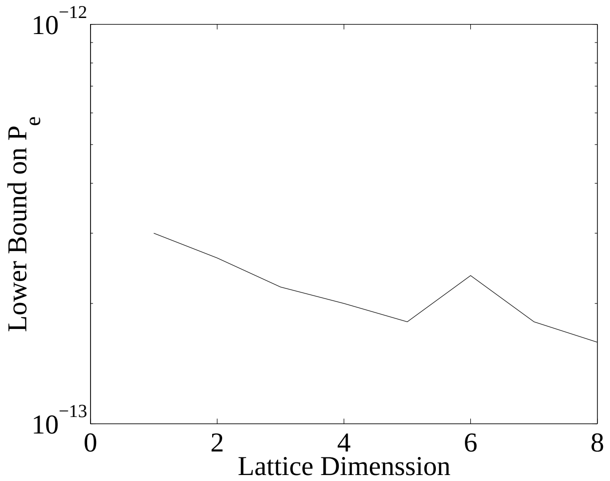

Figure 4.2 Lower bound for frame error rate vs. spreading factor for lattice codes.

of the code word probability of error with respect to the spreading gain can not be seen, and hence the analysis was inconclusive.

4.3 Error Exponent

In [AM99] P_e is upper bounded by a decaying exponential function. In another attempt to find an intuition to analytical solution for the optimum coding rate, we studied the behavior of the error exponent given by [AM99]. The error probability of error in a flat fading channel can be written as

$$P_e \leq (\frac{2e^{r\delta}}{\Xi})^2 \exp\{-E_r(q(x), \rho, r, R)\}, \tag{4.14}$$

where P_e is the average probability of error for decoding block codes of length N. Also $r \geq 0$, $0 \leq \rho \leq 1$, R is the transmission rate, $\delta \geq 0$, and Ξ is equal to the probability that the random variable Z, defined as $z = \sum_{n=1}^{N} |x_n|^2$, lies between its mean and δ less than the mean, and X_n's are the transmitted symbols independent and identically distributed given by distribution function $q(x)$. The exponent given in 4.14 is lower bounded by

$$E(\rho, N) \geq -\frac{1}{N} \ln E_h(1 + \frac{1}{1+\rho} \sum_{n=1}^{N} SNR_n |h_n|^2), \tag{4.15}$$

where h_n's are fading coefficients of the system, and SNR_n is the output SNR of each coded bit at the receiver. One can see that in 4.15 the expectation is taken with respect to the fading distribution.

We analyze the behavior of the error exponent given by 4.15 for different coding rates in order to find a pattern in harmony with simulation results for coding-spreading tradeoff. Figure 4.3 shows the variation of the error exponent with respect

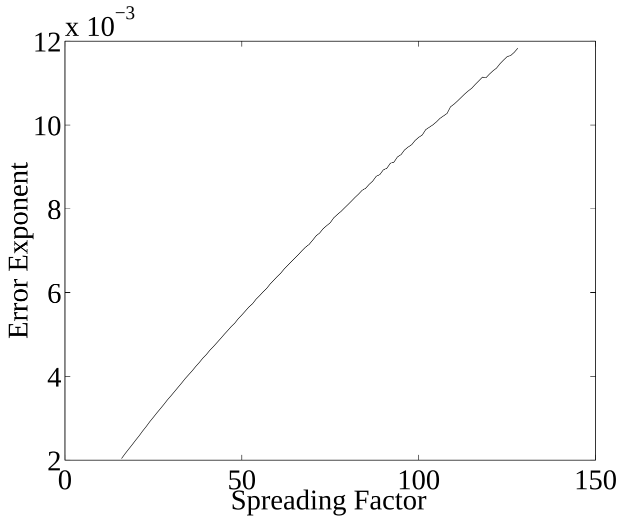

Figure 4.3 Error Exponent vs. Spreading factor.

to different spreading factors. The bound was not tight enough and hence the result was not conclusive.

Chapter 5

Signal Space Analysis

In this chapter we discuss the system model, different simulation scenarios and the analysis of each case.

5.1 System Model

We model the up-link channel (from user to base station) as follows: there are K users in the system, each with an independent *source*. The output of the source is modelled with independent identically distributed (iid) Bernoulli trials in which both events have equal probability. We also assume that all the sources generate the information bits with the same rate of 1 bit per T_b seconds.

The bit stream generated by each source is given to a $1/n$ convolutional encoder, however in analysis this could be any (n, k) block encoder, which generates a lattice code. The important point in comparing the performance of two coding rates is that they both must have the same number of input bits to the encoder. In this case we guarantee that the input signals to the encoders are in the same signal space $S(T, W)$, with $T = kT_b$ and k the information block length. Figure 5.1 shows the flow of information in the system. At this point a codeword is represented as an n-tuple living in an n-dimensional space.

In Figure 5.1 c_i's are $n \times 1$ vectors, which each indicates a point in $\mathbb{R}^{\mathbf{n}}$. This is the Shannon bandwidth of the signal. Considered to be a block code, the definition of codeword for a $1/n$ convolutional encoder and the dimension of the output of such

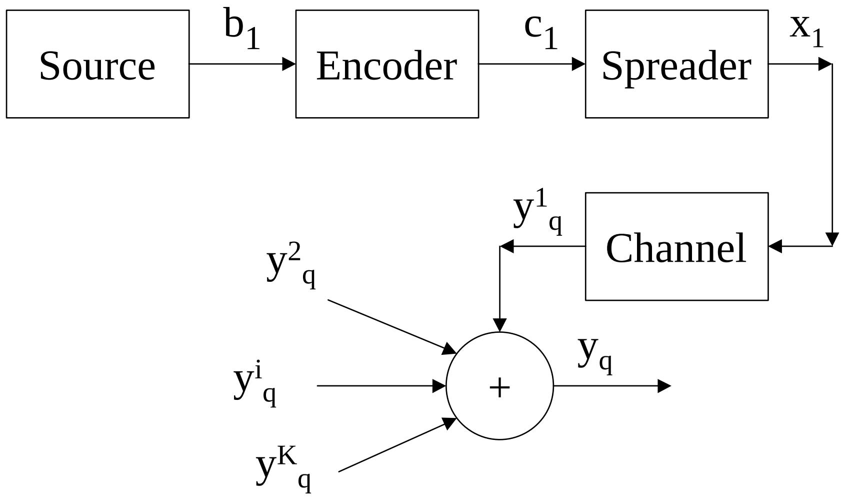

Figure 5.1 Block diagram of the system.

encoder is not clear, since such dimension depends on the constraint length of the encoder. At the spreader the codeword is multiplied by user's spreading waveform bit by bit and transmitted sequentially through the channel. The received signal of i^{th} user at the receiver can be written as

$$y_k[l] = h_k[l]c_k[l]S_k, \tag{5.1}$$

where $h_k[l]$ is the channel state during the transmission of l^{th} coordinate $c_k[l]$, of k^{th} user's codeword c_k, and S_k is k^{th} user's spreading waveform. In a DS-CDMA system spreading waveform is a signal consisting of a train of square pulses of length N, which in the vector format is an $N \times 1$ vector known as spreading sequence, where N is the spreading factor. Fading coefficient $h_k[l]$ is modeled as complex circular Gaussian random variable of form $x[l] + jy[l]$ where $x[l]$ and $y[l]$ are independent Gaussian random variables with zero mean and variance equal to $\frac{1}{2}$ so that the average channel power is normalized to one unit of power. For the case of multiple antennas, the received signal at the q^{th} receiver antenna can be written as

$$y_k^q[l] = \sum_{p=1}^{P} h_k^{pq}[l] c_k^p[l] S_k^p, \tag{5.2}$$

where there are P transmit antennas and each antenna could have its own code word and spreading waveform, and $h_k^{pq}[l]$ is the fading coefficient between the p^{th} transmit and q^{th} receive antenna during the transmission of l^{th} bit of k^{th} users codeword. For simplicity we assume that each user is assigned one unique spreading waveform, and hence in the case of multiple antennas each user applies the same spreading wave on each transmit antenna. The overall received signal of a transmitted codeword of length n bits at q^{th} receiver antenna can be written as

$$y_q = \sum_{l=1}^{n} \sum_{k=1}^{K} y_k^q[l] + w, \tag{5.3}$$

where w is the additive white Gaussian noise with zero mean and variance equal to N_0 per dimension. Note that $y_q \in S(T, \eta W)$, where $\eta = \frac{N}{R}$ is the bandwidth expansion factor as explained in Chapter 2. We assume that all users are synchronous, have the same transmit power, and have the same number of transmit antennas. Also in order to exploit the diversity attainable gain, a perfect interleaving is assumed so that each bit of a codeword would experience an independent fading state.

Each of the received signals at each of the receiver antennas then passes through a multi-user detector. We implemented MMSE detector since its performance is very close to maximum likelihood detector [Ver98]. Also we studied the simulation results of the hard decision on the output of single user MF detector. Since the channel state is assumed to be known at the receiver, in the case of multiple antennas at the receiver, *Maximum Ratio Combining* (MRC) [Ver98] is applied to improve the performance of the system.

5.2 AWGN Channel

In this section we study the behavior of our simulation environment in an AWGN channel. We considered single user and multi-user cases in our scenarios. Spreading can be seen as a repetition coding and despreading as the majority decoding. Hence in a single user AWGN system any code with a coding gain greater than coding gain of a repetition code is preferred over spreading. Also by spreading a signal over a larger frequency, the average power of signal at each frequency reduces. Therefore the signal to noise ratio decreases. Figure 5.2 depicts bit error rate vs. spreading factor

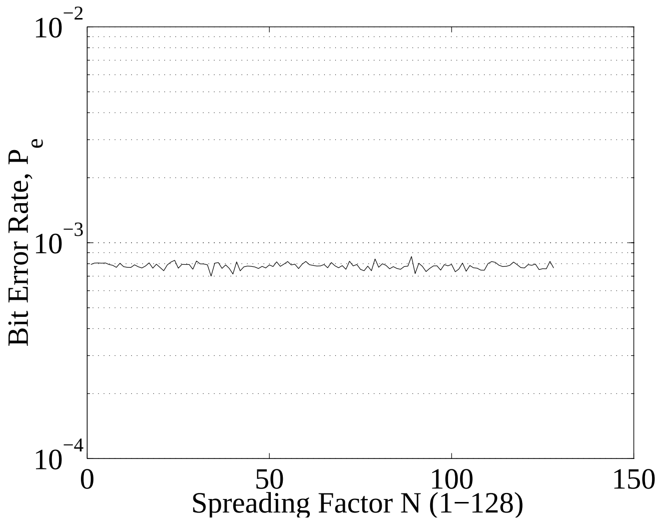

Figure 5.2 Bit error rate (P_e) vs. Spreading factor (N) for a single user AWGN channel.

in a single user AWGN channel. In this scenario user sends uncoded data, hence each point represents a different bandwidth expansion equal to the spreading factor.

In the case of multi-user case the result is different comparing to the single user case. In this case, spreading can cancel out the multi-user interference. Figure 5.3

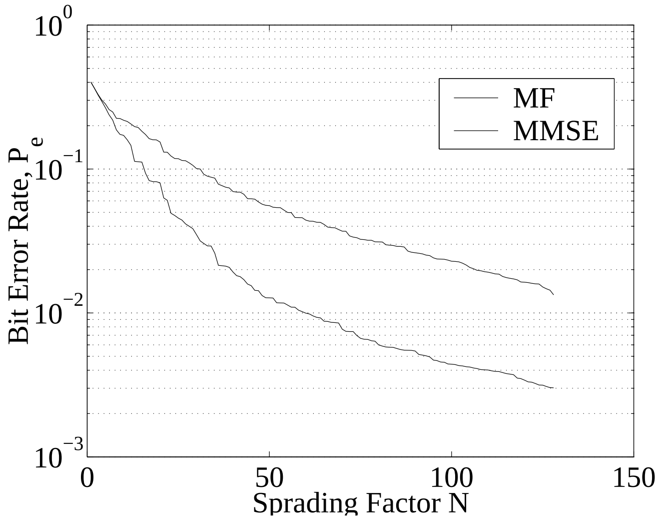

Figure 5.3 Bit error rate (P_e) vs. Spreading factor (N) for a multi user AWGN channel.

shows bit error rate P_e vs. spreading factor N for an AWGN channel with 16 users. It shows that as we increase the spreading factor, the bit error rate decreases.

Performance of a single user AWGN channel is shown in Figure 5.4. It shows that for a fixed bandwidth expansion, more spreading is preferred. We expect to see a linear relation between spreading factor, which is equivalent to coding rate, and $\log(P_e)$. This might be caused by coding scheme. Currently we are investigating better codes, in order to achieve the highest possible coding gain.

5.3 Multi-user Fading Channel

As discussed in Chapter 2, an (n, k) block code expands the bandwidth by a factor of $1/R = n/k$ and increases the Shannon dimension to n. Spreading in the DS-CDMA system plays the role of the linear mapping, and increases the Fourier bandwidth by a factor equal to the spreading factor N. Figure 5.5 shows the simulation results of average frame error rate P_e vs. spreading factor N, for single transmit and receive

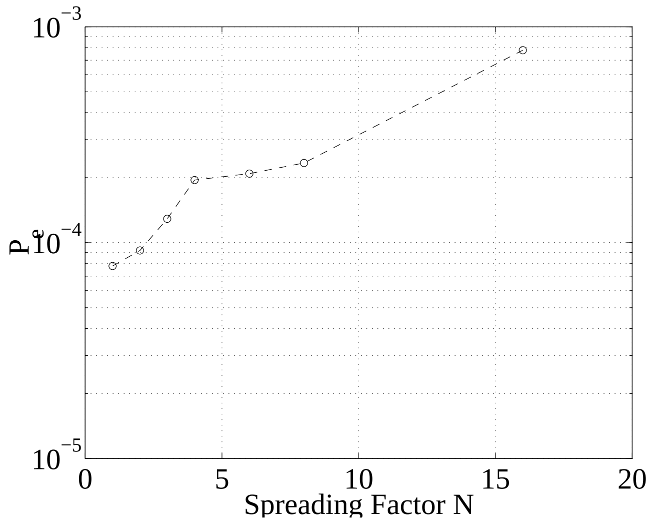

Figure 5.4 Bit error rate (P_e) vs. Spreading factor (N) for a single user AWGN channel, $\eta = 16$

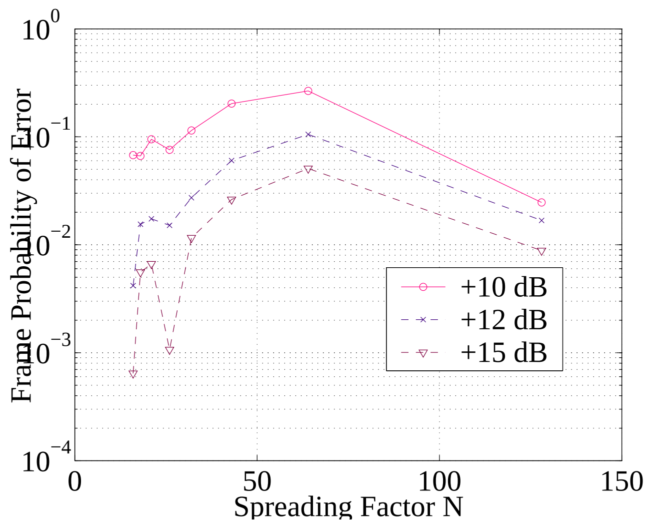

Figure 5.5 Frame Error Rate vs. Spreading Factor, for one transmit and one receive antenna, K=16, Convolutional Code $\nu = 3$, Random Spreading Code, $\eta = 128$, MMSE receiver.

antenna. Note that since the total bandwidth expansion $\eta = \frac{N}{R}$ is constant, changes in spreading factor implies changes in coding rate. In this case $\eta = 128$, hence the coding spreading pair of $(N = 128, R = 1)$ indicates transmission of un-coded data. Note that since there are $K = 16$ users in the system, based on discussion in Chapter 2 we have $N \geq K$. Hence the first point in the plot corresponds to the coding spreading pair of $(N = K = 16, R = 1/8)$. One can notice from Figure 5.5 that the behavior of the coding-spreading tradeoff varies as the SNR changes. This can be explained as follows: each user's code book $\mathcal{C}_k$ is an n-dimensional lattice around the origin. The smaller the SNR, the smaller the volume of these lattices. Consequently the lattice points, which correspond to the code words, are closer to each other and hence even a small additive noise may cause confusion in making a correct decision. On the other hand the points of different lattices are closer to each other now, and hence the amount of multi-user interference is larger in lower SNRs comparing to higher ones.

Lowering the code rate usually results in higher dimension code book, and as it can be seen in Figure 4.1, for a fixed SNR, the higher the dimension of the code book, the lower the code word probability of error. Hence considering a single user system, it is desired to have a code book with a dimension as high as possible. In other words the best choice for a single user is when its Shannon and Fourier bandwidth are equal. Under this condition, user is able to exploit the whole bandwidth on coding to minimize its code word error rate. However, in the multi-user case, the spreading provides a linear scheme for partitioning the available D-dimensional space, where $D = 2\eta WT$. Hence we are trying to divide the D-dimensional space into at least L subspaces with same dimensions, such that $L \geq K$, and each has the highest possible dimension. It is trivial that for the case of $L = K$ all the subspaces have the highest possible dimension. Since L is equal to the number of linearly independent linear mappings, hence one can conclude that in this case L=N=K is the best possible partitioning of the space, and each user will have the highest possible code rate of $R = K/\eta$. The next special case will elaborate the analysis.

Consider the Time Division Multiple Access (TDMA) as a special case of DS-CDMA, where the spreading sequence of each user consists of a train of pulses, with only one of them equal to 1 at the time slot dedicated to the user and equal to 0 elsewhere. In this case the spreading signals of all users are orthogonal to each other. As one can see, if the number of chips in the spreading sequence N, is greater than the number of users in the system K, then there are some time slots that the channel is idle. Each wasted time slot represents a subspace, which is not in use, and hence reduces the available dimension for active users. In this example there is no interference from other users and the only source of error is considered to be noise. In general in a CDMA system, orthogonality among the spreading sequences is not required. Linear independence is necessary condition for receiver to be able to separate the users' signals.

In case of multiple antennas at transmitter and/or receiver there are several parameters that may affect the performance of the system, but overall the same argument still holds. In the case of multiple antennas at the transmitter, since the total power is being divided among all transmit antennas, then the P_e vs. Spreading curve may lean towards spreading in higher SNRs. As long as all transmit antennas use the same spreading mapping, all of the transmit signals lie in the same subspace. Hence the factor that may help the performance of the system is diversity gain, and there is no coding gain to be exploited.

Figure 5.7 is the simulation result for two transmit and one receive antennas. In this case we applied Alamouti [Ala98] scheme. Comparing to Figure 5.5, this graph shows the same behavior but in a higher SNR as expected. Note that Alamouti scheme provides zero coding gain, and only improves the performance of the system

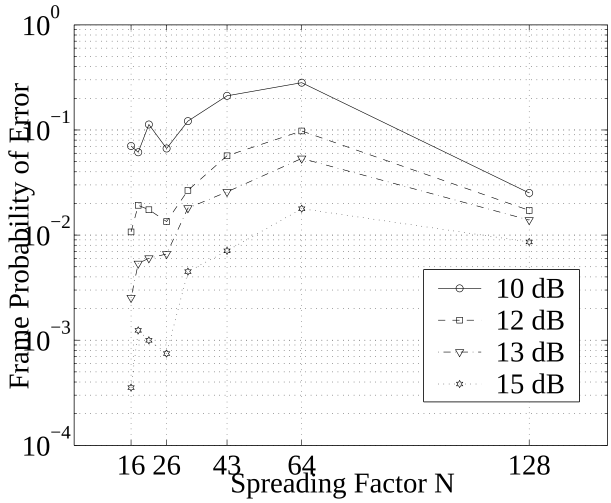

Figure 5.6 Frame Error Rate vs. Spreading Factor, for two transmit and one receive antennas, K=16, Convolutional Code $\nu = 3$, Random Spreading Code, $\eta = 128$, MMSE receiver.

through diversity gain. In the scenario of one transmit and two receive antennas, the diversity gain at the receiver improves the performance of the system comparing to the case of one transmit and one receive antenna at the same SNR.

As one can see from Figures 5.5, 5.6, and 5.7, at low SNRs spreading is preferred over coding, since spreading increases the space between users' subspaces. On the other hand, more spreading results in a lower dimension code book for each user. This means that there is more power per dimension, which increases the distance between code words and consequently reduces the code error rate.

Figures 5.8 and 5.9 depicts the frame error rate vs. spreading gain. As one can see even in high SNRs more spreading results in lower frame error rate. However by comparing the rate of change in frame error rate with respect to SNR in the very two end points of Figure 5.9, one can imagine that at very high SNRs more coding might be more desirable than spreading.

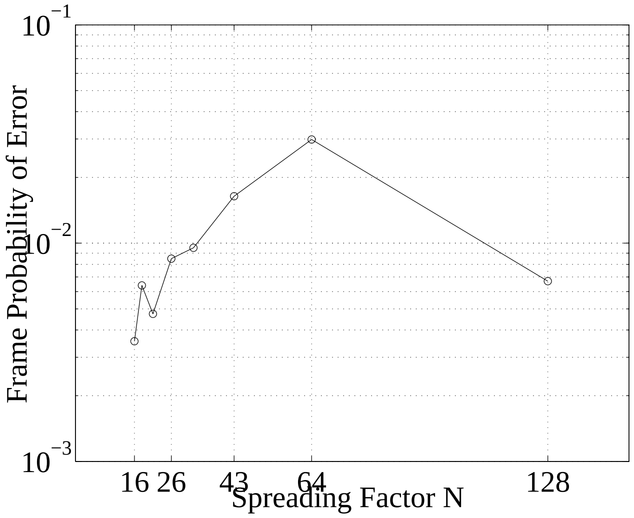

Figure 5.7 Frame Error Rate vs. Spreading Factor, for one transmit and two receive antennas, K=16, Convolutional Code $\nu = 3$, Random Spreading Code, $\eta = 128$, MMSE receiver.

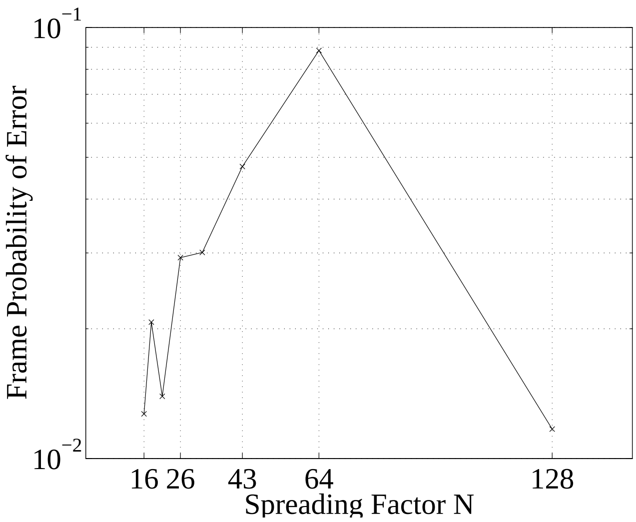

Figure 5.8 Frame Error Rate vs. Spreading Factor, for one transmit and two receive antennas, K=16, Convolutional Code $\nu = 3$, Random Spreading Code, $\eta = 128$, MF receiver, SNR=+7 dB.

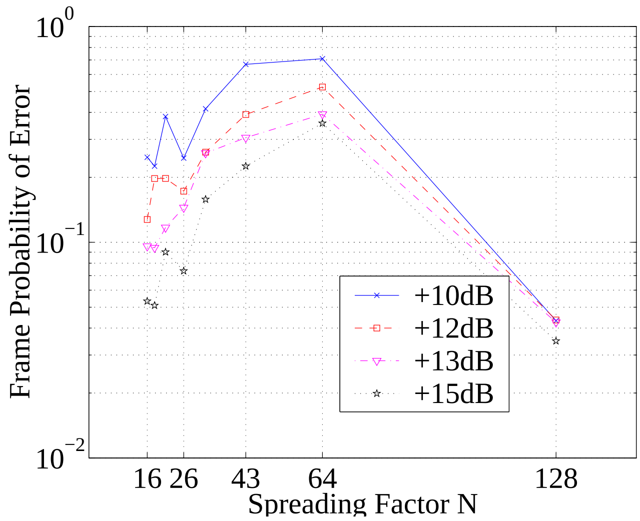

Figure 5.9 Frame Error Rate vs. Spreading Factor, for one transmit and two receive antennas, K=16, Convolutional Code $\nu = 3$, Random Spreading Code, MF, $\eta = 128$.

By observation, one can notice that in all above graphs there is a local minimum at $N = 26$, or $1/R = 5$. The explanation is as follows. Let P be the average power constraint. In this case the volume $V(\Pi)$ of the lattice code book can be written as

$$V(\Pi) = PV_n \tag{5.4}$$

where n is the dimension of the lattice, and V_n is volume of a unit sphere in $\mathbb{R}^n$, which is equal to [CS99]

$$V_n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} = \begin{cases} \frac{\pi^k}{k!}, & n = 2k \\ \frac{2^n \pi^k k!}{n!}. & n = 2k+1 \end{cases} \tag{5.5}$$

Figure 5.10 shows the volume of unit sphere in dimension 1-8. As one can see from the graph, the volume of unit sphere in $\mathbb{R}^5$ has the highest value. Hence the corresponding lattice code has the biggest volume, and consequently it increases the distance between code words. Therefore it is expected to have a better performance comparing to other lattices with dimensions relatively close to 5.

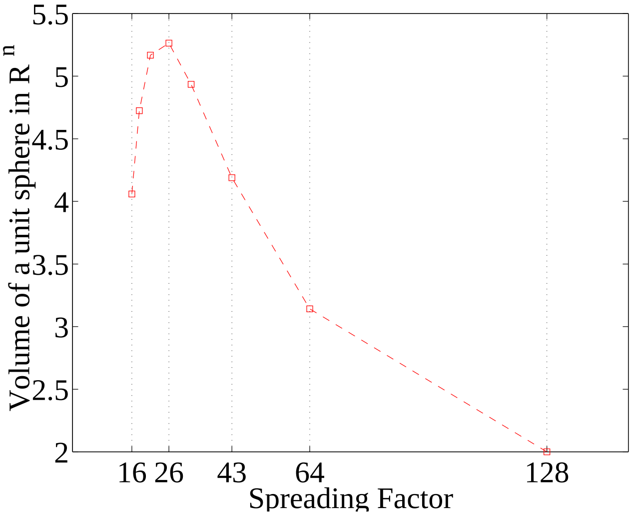

Figure 5.10 Volume of unit spheres in spaces of dimensions 1-8.

Chapter 6

Conclusion

In this thesis we have studied the tradeoff between coding and spreading in a spread spectrum system under the constraint of fixed bandwidth expansion. We showed and argued that the code word probability, in a CDMA system with flat fading channel and MMSE multi-user detection, is minimized when the spreading factor N, is equal to the number of users K, which is in agreement with results in literature. We also extended the analysis and simulations to multiple antennas scenarios. In this case as long as using multiple antennas only improves the performance through diversity gain, and does not provide any coding gain, the above argument still holds.

We also showed that in the same system, with MF multi-user detection, more spreading is preferred. However in very high SNRs (greater than 20 dB) the N=K is the optimum spreading factor, which minimizes the frame error rate.

In contrast with previous works, which were based on solving an optimization problem with constraint on the bandwidth expansion, in this work we looked at the coding-spreading tradeoff as a tradeoff between the Shannon and Fourier bandwidths. From this point of view, we look for the dimension of a subspace that optimizes the performance of the system. This dimension is set to be the Shannon bandwidth of the coded signal, i.e., the dimension of the code book of the signal.

As the future and related work to this research one can consider articulating and proving the argument as a theorem. Also since the argument here has been established for lattice codes under fading channel, one can do the same analysis for codes designed for fading channel. The extension to Massy's [Mas94] is still open, and for start, one can look into the Shannon bandwidth of a multi-user system. One of the exciting extensions to this work is the design of a concatenated code, in which the first layer is the regular channel encoder and second layer is a code designed for user separation. In this case we not only reduce the complexity of the decoder, but also increase the Shannon bandwidth in the process of user separation, and hence the coding gain increases through the separation process. Consequently one can imagine that such a system should outperform a spread spectrum system, with the cost of higher complexity; but not as complex as a joint multi-user code. However, still in such a system, each encoder should have some knowledge of system parameters such as number of users, and their separation coding scheme.

The current analysis can be also extended to the asynchronous case in which the users are not synchronous. Moreover one may consider the case in which at the receiver the users' signals have different average powers. The effect of feedback can also be analyzed in future work. How knowing the channel at the transmitter may change the analysis, or on the other hand, how imperfect knowledge of channel at the receiver may affect the coding-spreading tradeoff.

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