A FOURIER-PRONY TAUBERIAN APPROACH TO THE ANALYSIS OF A MIXTURE OF DELAYED SIGNALS

by

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Let x and y be signals (i.e. real-valued functions of time) of finite duration and energy. In the present paper, we develop a frequency domain Prony approach for interpolating, or, in general, approximating y(t) by $\sum_{i=1}^{M} a_i x(t - \tau_i)$; where $\underline{a} = (a_1, \ldots, a_M)$ and $\underline{\tau} = (\tau_1, \ldots, \tau_M)$ are real

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1. INTRODUCTION

In a number of natural and man-made systems, the output signal y consists of a linear combination of differently delayed versions of the input signal x. More generally, some of these delayed versions of x may be subjected to linear operations before they are superimposed with the remaining ones. Finally, in some applications, the observed signal y may be a superposition of M different signals, each differently delayed. In the present paper, we are concerned with the problem of finding, in such situations, the time delays and other system parameters from the input-output data or only from the output data.

Manifestations of the above-mentioned behavior occur in radar systems and in seismological applications. The set of delays and system parameters referred to above can then be used as feature vectors in appropriate system identification algorithms, as will be discussed in a separate paper [1].

Typically, if the input (incident) signal x and the output (reflected) signal y are continuous real-valued functions of time t having finite duration and energy, then, in the simplest case mentioned above, they are related by

$$y(t) = \sum_{i=1}^{M} a_i \, x(t - \tau_i), \tag{1}$$

where, in a typical scattering problem, the real numbers a_i and τ_i are respectively the reflection coefficient and delay associated with the i^{th} reflection.

The first problem that we consider is:

Problem 1. Given x, M and a finite set of samples of y at equally spaced time instants, namely $y(t_0 + n\Delta t)$, n=0, ... , N-1, find $\underline{\tau} = (\tau_1, \ldots, \tau_M)$ and $\underline{a} = (a_1, \ldots, a_M)$ such that (1) is satisfied.

In section 2 of this paper, we convert the above time domain problem into a frequency domain problem. Then, since the time delays in x appear

$$\underline{a} = (a_{11}, \ldots, a_{1M_1}, a_{21}, \ldots, a_{2M_2}, \ldots, a_{M_0K_{M_0}}),$$

and

$$\underline{\tau} = (\tau_1, \ldots, \tau_{M_0}).$$

In section 3, we present some numerical simulations of the above technique with very satisfactory results.

In section 4 we introduce as Problem 3 a noisy version of Problem 1. Specifically, we consider the case in which the data is corrupted by additive noise, and present a development, based on a paper by Evans and Fischl [7], which permits us to obtain the desired solution iteratively. These results are illustrated by a numerical simulation described in section 5.

In seismic exploration by reflection methods, the received signal is a superposition of signals x_i resulting from the interaction of the input signal x with dynamical systems representing various lithological configurations in its path. A similar situation occurs in some radar applications in which the received signal y is the superposition of several signals x_i reflected from a large object, which differ from one another not only in delay but also in shape. The differences in signal shape result from the fact that the target, at different points where the reflections occur, has different responses. All these cases can be treated assuming that x_i can be represented in the parametric form

$$x_i(t) = \sum_{j=1}^{L_i} b_{ij}\, \varphi(t-\eta_{ij}), \quad i=1, 2, \ldots, M, \qquad (6)$$

where φ is an appropriate known "potential function," L_i, $i = 1, \ldots, M$, are positive integers, and b_{ij} and $\eta_{ij} = 1, \ldots, L_i$, are suitable real constants. A similar representation can be used, when appropriate, for the target impulse responses h. The question of analyzing such a mixture of signals by the Prony approach mentioned above is discussed in section 6.

If (6) is an L^2 approximation of x_i, then it can be viewed as a Tauberian approximation in the sense of Wiener [8]. In section 7, we establish the connection between the Prony approach described here and a Tauberian the theorem of Wiener [8].

Concluding remarks are in section 8.

2. THE FREQUENCY DOMAIN PRONY TECHNIQUE FOR THE NOISE - FREE CASE

To solve Problem 1, we take the Fourier transform of (1) and then rewrite the resultant expressions in the form

$$\Psi(\underline{a}, \underline{\tau}, \omega) = H(\omega) \tag{7}$$

where

$$\Psi(\underline{a}, \underline{\tau}, \omega) = \sum_{i=1}^{M} a_i \, \ell^{-j\tau_i \omega} , \tag{8}$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} \tag{9}$$

and X and Y are Fourier transforms of x and y.

We then select a set of equally spaced points on the frequency axis, say $\omega_k = \omega_o + k\Delta\omega$, $k = 0, \ldots, K-1$, where $K \geq 2M$ and $N \geq K$. Note that, under appropriate conditions, $X(\omega_k)$ and $Y(\omega_k)$ may be quite accurately approximated by their discrete Fourier transforms (DFT's) at ω_k calculated from the data $x(t_o + n\Delta t)$, and $y(t_0 + n\Delta t)$, $n = 0, 1, \ldots, N-1$.

Problem 1 is then reduced to:

<u>Problem 1a</u>. Under the conditions given, fint $\underline{\tau}$ and $\underline{a}$ such that

$$\Psi(\underline{a}, \underline{\tau}, \omega_k) = H(\omega_k), \; k = 0, 1, \ldots, K-1, \tag{10}$$

where $H(\omega_k)$ are calculated from the data as indicated above.

For simplicity, we will use the notation

$$H(\omega_o + k\Delta\omega) = H_k. \tag{11}$$

We assume that the reader is familiar with Prony's method [2], [3]. In what follows, we outline the application of its basic steps to our case, and then describe a variant of the standard procedure which, for computational purposes, is especially suited to our problem.

The first step in the Prony method is to note that satisfaction of (10) (together with (8) and (7)) implies that there holds the following difference equation in the complex coefficients $c_o, \ldots, c_M$, with $c_o = 1$ and $c_1, \ldots, c_m$ to be determined:

$$\sum_{\ell=o}^{M} H_{k-\ell}\, c_\ell = 0,\ k = M,\ M+1, \ldots, K+1, \tag{12}$$

whose characteristic equation is

$$\sum_{\ell=o}^{M} c_\ell\, z^\ell = 0 \tag{13}$$

with roots

$$z_i = e^{-j\tau_i \Delta\omega},\ i = 1, 2, \ldots, M. \tag{14}$$

Letting

$$\underline{H} = \begin{bmatrix} H_{M-1} & \cdots & H_o \\ \vdots & & \vdots \\ H_{K-2} & \cdots & H_{K-M-1} \end{bmatrix} \tag{15a}$$

$$c = \begin{bmatrix} c_1 \\ \vdots \\ c_M \end{bmatrix}, \quad h = \begin{bmatrix} H_M \\ \vdots \\ H_{K-1} \end{bmatrix}, \tag{15b,c}$$

the set of equations (12) can be rewritten as

$$\underline{H}\, c = -h. \tag{16}$$

If $K = 2M$ and $\underline{H}$ is nonsingular, (16) has a unique solution given by

$$c = -\underline{H}^{-1}\, h. \tag{17}$$

If $K > 2M$ and the rank of $\underline{H}$ is equal to M, then the least squares solution of (16) is

$$c = -(\underline{H}^T \underline{H})^{-1} \underline{H}^T h. \tag{18}$$

So, in the Prony approach, after first calculating c from (17) or (18), one substitutes the components of c into (13) and solves this equation for z obtaining the roots $z_i = e^{-j\tau_i \Delta\omega}$ and hence the delays τ_i. Finally, substituting these τ_i into (10) and solving the resultant linear equations in a_i, one obtains these contants.

The above is the conventional Prony procedure. However, for the sake of simplicity and computational efficiency, we solve Problem 1a in the real domain by means of the following variant of that procedure.

Let, in equation (11),

$$H_{Rk} = \text{Re}\,\{H_k\}\ ,\ H_{Ik} = \text{Im}\,\{H_k\}. \tag{19}$$

Then, since a_i, i = 1, . . . , M, are assumed to be real,

$$H_{Rk} = \sum_{i=1}^{M} a_i \cos \tau_i \omega_k$$

$$= \sum_{i=1}^{2M} \tilde{a}_i . e^{\tilde{\tau}_i \omega_k} \quad , \tag{20}$$

where

$$\tilde{a}_i = \tfrac{1}{2} a_i \tag{21a}$$

$$\tilde{\tau}_i = - \tau_{M+i} = j\tau_i,\ i = 1, ..., M. \tag{21b}$$

It follows from (20) that H_{Rk} satisfies a difference quation (with coefficients $\tilde{c}_M = 1$ and $\tilde{c}_\ell$, $\ell = 0, ... , 2M$, $\ell \neq M$, to be determined) of the form

$$\sum_{\ell=o}^{2M} H_{R,k-\ell}\, \tilde{c}_\ell = 0,\ k = 2M, ... , K - 1\ , \tag{22}$$

with the characteristic equation

$$\sum_{\ell=o}^{2M} \tilde{c}_\ell \, z^\ell = 0, \tag{23}$$

in which M roots are as in (14) and the remaining M roots are obtained from the fact that if z_i is a root then z_i^{-1} is also a root. This fact also implies that*

$$\tilde{c}_\ell = \tilde{c}_{2M-\ell}, \; \ell = 0, \ldots, M-1 \tag{24}$$

A similar treatment applies to H_{Ik}.

The above leads to the set of equations

$$\sum_{\ell=0}^{M-1} (H_{R,k+\ell} + H_{R,k+2M-\ell}) \, \tilde{c}_\ell = -H_{R,k+M}, \quad k=0, 1, \ldots, K-2M-1 \tag{25a}$$

$$\sum_{\ell=o}^{M-1} (H_{I,k+\ell} + H_{I,k+2M-\ell}) \, \tilde{c}_\ell = -H_{I,k+M} \quad k=0, 1, \ldots, K-2M-1 \tag{25b}$$

---

*Since both z_i and z_i^{-1} are roots of (23) and $z_i \neq 0$, we have

$$\sum_{\ell=o}^{2M} \tilde{c}_\ell \, z_i^\ell = 0 \tag{A}$$

and $\sum_{\ell=o}^{2M} \tilde{c}_\ell \, z_i^{-\ell} = 0$, which implies

$$\sum_{\ell=o}^{2M} \tilde{c}_{2M-\ell} \, z_i^\ell = 0 \tag{B}$$

Comparing (A) and (B), one gets (24).

where the $\tilde{c}_\ell$ associated with $\{H_{Rk}\}$ and $\{H_{Ik}\}$ are the same since the sets of characteristic roots in the two cases are the same.

In matrix form equations (25a) and (25b) may be expressed as:

$$\underline{H}_R \tilde{c} = - h_R \tag{26a}$$

$$\underline{H}_I \tilde{c} = - h_I \tag{26b}$$

where

$$\underline{H}_R = \begin{bmatrix} H_{R,0} + H_{R,2M} & H_{R,1} + H_{R,2M-1} & \cdots & H_{R,M-1} + H_{R,M+1} \\ H_{R,1} + H_{R,2M+1} & H_{R,2} + H_{R,2M} & \cdots & H_{R,M} + H_{R,M+2} \\ \vdots & & & \vdots \\ H_{R,K-2M-1} + H_{R,K-1} & & \cdots & H_{R,K-M-2} + H_{R,K-M} \end{bmatrix}, \tag{27a}$$

$$\tilde{c} = \begin{bmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \vdots \\ \tilde{c}_{M-1} \end{bmatrix}, \qquad h_R = \begin{bmatrix} H_{R,M} \\ H_{R,M-1} \\ \vdots \\ H_{R,K-M-1} \end{bmatrix}, \tag{27b,c}$$

and $\underline{H}_I$ and h_I are the same as in (27a) and (27c) with the subscript R replaced by I.

Remarks:

(1) If one were to determine the delays solely by DFT techniques, i.e. by calculation of DFT's X(k) and Y(k) of x and y and inverse DFT of X(k)/Y(k), then it would not be possible to take care of the case in which the delays were not integer multiples of the sampling interval. This disadvantage is not present in the method proposed in this paper.

(2) The present approach has another advantage over direct transform techniques. The points $\omega_o, \omega_1, \ldots, \omega_{K-1}$ in our case can be selected where $H(\omega)$ (= $Y(\omega)/X(\omega)$) is well behaved (away from the zeros of $X(\omega)$) and well determinable.

(3) From the above we conclude also that the present approach has a "filtering property", since it is based on a set of spectral samples at equally spaced points on a given frequency interval. In particular, if these points lie in a frequency interval $[0, \omega_{K-1}]$, then our procedure involves low-pass filtering.

This concludes our remarks.

In the general case of Problem 2, taking the Fourier transform of (5), we are led to the approximating function

$$\psi(\underline{a}, \underline{\tau}, \omega) = \sum_{i=1}^{M_o} \sum_{\ell=1}^{M_i} a_{i,\ell} \, (j\omega)^{\ell-1} \, e^{-j\tau_i\omega} \tag{28}$$

and hence to:

Problem 2a. Same as Problem 1a with ψ as in (28) rather than as in (8).

Application of the Prony method, as done with Problem 1a, leads to the difference equation (12) (see [4]), where now the degree of the equation is

$$M = \sum_{i=1}^{M_0} M_i, \tag{29}$$

and the characteristic equation has the form (13), having however M_1 characteristic roots z_1, M_2 roots with value z_2, ... , and M_{M_o} roots with value z_{M_o}. Thus with this clarification, we use (12) and (13) in the same way as before to obtain the solution to Problem 2a.

3. NUMERICAL EXAMPLES FOR THE NOISE-FREE CASE

Several computer simulations were carried out to test the method described in the preceding section and we have found very close agreement between the calculated and the exact parameter values. The following four examples illustrate those results. In each case, x is defined by

$$x(t) = \sin(2\pi t), \quad 0 \le t \le 1 \qquad\qquad (30)$$
$$= 0, \qquad \text{elsewhere.}$$

Example 1:

The data consisted of the samples of x and y in the form x(n/16) and y(n/16) for n=0, 1, ... ,127. We assumed also that the output signal y was related to x by

$$y(t) = x(t-1) - \alpha\, x(t-3) \qquad\qquad (31)$$

where α was assigned, in different cases, the values of 1, .9, .8, and .7.

Table 1 shows the results for the two delays τ_1 and τ_2 calculated by our method for the above four cases (the exact values being respectively τ_1 = 1 and τ_2 = 3).

Table 1

α	τ_1	τ_2
1	.9998668	3.000132
.9	.9998838	3.000141
.8	.9999226	3.000118
.7	.9999658	3.000068

Example 2:

The data samples in our second example were x(n/8) and y(n/8) for n= 0, 1, ... , 63, and we generated y from x by

$$y(t) = x(t-1) - \alpha\, x(t - 2.5 + \frac{1}{16})$$
$$= x(t-1) - \alpha\, x(t - 2.4375) \qquad (32)$$

Note that the second delay shifts x from a given sampling instant to the midpoint between two other consecutive sampling instants.

The calculated values for the delays $\tau_1=1$ and $\tau_2=2.4375$ as $\alpha= 1$, .9, .8, .7, and .6 are shown in Table 2.

Table 2

α	τ_1	τ_2
1	1.000565	2.437036
.9	1.000271	2.437042
.8	1.000029	2.437049
.7	.9998399	2.437057
.6	.9997038	2.437064

Example 3:

Our third example is the same as Example 2 except that the second delay sends x from a sampling instant to very close to (but not coincident with) another sampling instant. Thus equation (32) is replaced by (with $\alpha=1$)

$$y(t) = x(t-1) - x(t-2.5+\frac{\frac{1}{8}}{16})$$
$$= x(t-1) - x(t-2.4921875). \qquad (33)$$

Our technique yielded the values $\tau_1=.9994552$ and $\tau_2=2.492167$ (the exact values being $\tau_1=1$ and $\tau_2=2.4921875$).

Example 4:

Our fourth example represented a case in which the delayed versions of x overlapped. In this case, we assumed

$$y(t) = 0.8\ x(t-1) - 0.33\ x(t-1.5+\frac{1}{2}\cdot\frac{1}{16})$$

$$= 0.8\ x(t-1) - 0.33\ x(t-1.46875)\ . \qquad (34)$$

with the data $x(n/16)$ and $y(n/16)$, $n=0, 1, \ldots, 127$.

In this example we only used the frequency domain samples $\frac{Y(k)}{X(k)}$, $k=1, 2, \ldots, 15$, since there are singularity points at $k=0$ and $k=16$, which are shown as Table 3. The results for the exact and calculated values of the two coefficients τ_i, $i=1,2$ and two delays a_i, $i=1,2$ are shown in Table 3 also which consists of a computer printout.

Table 3

MAGNITUDE OF X(K)

0	0.14857E-12 ← Singularity Points	20	0.10796E 01
1	0.38213E 01	21	0.74441E 00
2	0.14401E 02	22	0.35764E 00
3	0.29317E 02	23	0.87061E-01
4	0.45224E 02	24	0.13217E-10
5	0.58686E 02	25	0.62243E-01
6	0.66992E 02	26	0.18240E 00
7	0.68731E 02	27	0.26961E 00
8	0.64000E 02	28	0.27572E 00
9	0.54210E 02	29	0.20697E 00
10	0.41592E 02	30	0.10736E 00
11	0.28565E 02	31	0.28023E-01
12	0.17149E 02	32	0.28505E-11
13	0.85841E 01	33	0.22656E-01
14	0.32170E 01	34	0.70119E-01
15	0.64090E 00	35	0.10905E 00
16	0.80688E-11 ← Singularity Points	36	0.11696E 00
17	0.33686E 00	37	0.91806E-01
18	0.87376E 00	38	0.49669E-01
19	0.11606E 01	39	0.13492E-01

OBTAINED SOLUTION OF DELAYS: 0.1000909E 01 0.1472269E 01
(EXACT SOLUTION: 1.000000; 1.468750)

OBTAINED SOLUTION OF COEF: 0.7987887E 00 -0.3310153E 00
(EXACT SOLUTION: 0.8; -0.33)

4. AN ITERATIVE PRONY ALGORITHM FOR THE NOISY CASE

If the data is corrupted by additive noise, the values of $\tilde{H}_k$ obtained from it are correspondingly noisy. This leads to an amendment of equations (25a,b) to

$$\sum_{\ell=0}^{M-1} (\tilde{H}_{R,k+\ell} + \tilde{H}_{R,k-\ell+2M})\tilde{c}_\ell = -\tilde{H}_{R,k+M} + e_{R,k+M}, \quad k=0, 1, \ldots, K-2M-1 \tag{35a}$$

$$\sum_{\ell=0}^{M-1} (\tilde{H}_{I,k+\ell} + \tilde{H}_{I,k-\ell+2M})\tilde{c}_\ell = -\tilde{H}_{I,k+M} + e_{I,k+M} \quad k=0, 1, \ldots, K-2M-1, \tag{35b}$$

where $e_{R,k+M}$ and $e_{I,k+M}$ denote the error in $\tilde{H}_{R,k+M}$ and $\tilde{H}_{I,k+M}$. In matrix form, (35a,b) are written as

$$\underline{\tilde{H}}_R \tilde{c} = -\tilde{h}_R + e_R \tag{36a}$$

$$\underline{\tilde{H}}_I \tilde{c} = -\tilde{h}_I + e_I \tag{36b}$$

where

$$\underline{\tilde{H}}_R = \begin{bmatrix} (\tilde{H}_{R,2M} + \tilde{H}_{R,0}) & (\tilde{H}_{R,2M-1} + \tilde{H}_{R,1}) & \cdots & (\tilde{H}_{R,M+1} + \tilde{H}_{R,M-1}) \\ (\tilde{H}_{R,2M+1} + \tilde{H}_{R,1}) & (\tilde{H}_{R,2M} + \tilde{H}_{R,2}) & \cdots & (\tilde{H}_{R,M+2} + \tilde{H}_{R,M}) \\ \vdots & & & \vdots \\ (\tilde{H}_{R,K-1} + \tilde{H}_{R,K-2M-1}) & & \cdots & (\tilde{H}_{R,K-M} + \tilde{H}_{R,K-M-2}) \end{bmatrix} \tag{37a}$$

$$\tilde{h}_R = \text{col}\,(\tilde{H}_{R,M}, \cdots, \tilde{H}_{R,K-M-1}) \tag{37b}$$

$$e_R = \text{col}\,(e_{R,M}, \ldots, e_{R,K-M-1}) \tag{37c}$$

$\underline{\tilde{H}}_I$, $\tilde{h}_I$, and e_I are defined in the same way as $\underline{\tilde{H}}_R$, $\tilde{h}_R$, and e_R in (37a-c) with the subscript R replaced by I.

If we set

$$\frac{\partial}{\partial \tilde{c}} (e_R^T e_R + e_I^T e_I) = \underline{0} \tag{38}$$

with e_R and e_I defined by (36a) and (36b), and solve for $\tilde{c}$, we obtain

$$\hat{\tilde{c}} = -(\underline{\tilde{H}}_R{}^T \underline{\tilde{H}}_R + \underline{\tilde{H}}_I{}^T \underline{\tilde{H}}_I)^{-1} (\underline{\tilde{H}}_R{}^T \tilde{h}_R + \underline{\tilde{H}}_I{}^T \tilde{h}_I) \tag{39}$$

where we recall

$$\hat{\tilde{c}} = \text{col}\ (\hat{\tilde{c}}_o, \hat{\tilde{c}}_1, \ldots, \hat{\tilde{c}}_{M-1}) \tag{40}$$

$$\hat{\tilde{c}}_M = 1 \tag{41}$$

$$\hat{\tilde{c}}_{2M-\ell} = \hat{\tilde{c}}_\ell\ , \quad \ell = 0, 1, \ldots, M-1. \tag{42}$$

Once the constants $\hat{\tilde{c}}_\ell$ are obtained, they are substituted in (23) to yield the roots z_i and hence the delays, and from these delays the coefficients a_i are gotten in the same way as before.

The direct method discussed above can only give a suboptimal solution since it minimizes the error in the samples indexed from M to K-M-1. The following is a variant of the iterative refinement of Evans and Fischl [7] with the modifications introduced to account for the fact that our equations in $\tilde{H}_R$, $\tilde{H}_I$, and $\tilde{c}$ have the form (35a,b). This algorithm constitutes an improvement over the one embodied by (36a, b) because all the samples are taken into account in the calculations performed.

(35a) can be rewritten as

$$\begin{bmatrix} \tilde{H}_{R,2M} & \tilde{H}_{R,2M-1} & \cdots & \tilde{H}_{R,M+1} & \tilde{H}_{R,M} & \tilde{H}_{R,M-1} & \cdots & \tilde{H}_{R,0} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \tilde{H}_{R,K-1} & \tilde{H}_{R,K-2} & \cdots & \tilde{H}_{R,K-M} & \tilde{H}_{R,K-M-1} & \tilde{H}_{R,K-M-2} & \cdots & \tilde{H}_{R,K-2M-1} \end{bmatrix} \begin{bmatrix} \tilde{c}_0 \\ \tilde{c}_1 \\ \vdots \\ \tilde{c}_{M-1} \\ 1 \\ \tilde{c}_{M+1} \\ \vdots \\ \tilde{c}_1 \\ \tilde{c}_0 \end{bmatrix}$$

$$= \begin{bmatrix} e_{R,M} \\ \vdots \\ e_{R,K-M-1} \end{bmatrix} \triangleq e_R \ , \tag{43}$$

which by rearrangement can be expressed as

$$B^T \hat{h}_R = e_R \tag{44}$$

where

$$B = \begin{bmatrix} \tilde{c}_0 & & 0 \\ \tilde{c}_1 & \ddots & \vdots \\ \vdots & \ddots & 0 \\ & & \tilde{c}_0 \\ \tilde{c}_{M-1} & & \tilde{c}_1 \\ 1 & \ddots & \vdots \\ \tilde{c}_{M-1} & \ddots & \\ \vdots & \ddots & \tilde{c}_{M-1} \\ & & 1 \\ \tilde{c}_1 & & \tilde{c}_{M-1} \\ \tilde{c}_0 & \ddots & \vdots \\ 0 & \ddots & \\ \vdots & \ddots & \tilde{c}_1 \\ 0 & & \tilde{c}_0 \end{bmatrix} , \quad \hat{h}_R = \begin{bmatrix} H_{R,0} \\ H_{R,1} \\ \vdots \\ H_{R,K-1} \end{bmatrix} + \begin{bmatrix} d_{R,0} \\ d_{R,1} \\ \vdots \\ d_{R,K-1} \end{bmatrix}$$

where

$d_{R,i}$ = real part of error in $\tilde{H}_{R,i}$ due to noise, i.e.,

$$\tilde{H}_{R,i} = H_{R,i} + d_{R,i} \; . \tag{45}$$

Hence,

$$e_R = B^T d_R \; . \tag{46}$$

In a similar way, we can write for the errors in the imaginary part of the frequency domain samples

$$\tilde{H}_{I,i} = H_{I,i} + d_{I,i} \tag{47}$$

and

$$e_I = B^T d_I \; . \tag{48}$$

One wants to express d_R as a function of e_R. Since the dimension of d_R is greater than that of e_R, one can find the minimum norm solution of (45), obtaining

$$d_R = W \, e_R \; , \tag{49}$$

where

$$W = B(B^T B)^{-1} . \tag{50}$$

Similarly,

$$d_I = W \, e_I \tag{51}$$

then the minimization of $d_R^T d_R + d_I^T d_I$ with respect to $\tilde{c}$ is equivalent to

$$\min \; [(We_R)^T We_R + (We_I)^T We_I] \tag{52}$$

which, by differentiation with respect to $\tilde{c}$, leads to

$$(W\underline{\tilde{H}}_R)^T \left[W(\underline{\tilde{H}}_R \tilde{c} + \tilde{h}_R) \right] + (W\underline{\tilde{H}}_I)^T \left[W(\underline{\tilde{H}}_I \tilde{c} + \tilde{h}_I) \right] = 0 \tag{53}$$

Hence

$$\hat{\tilde{c}} = - V^{-1} g \tag{54}$$

where $\hat{\tilde{c}}$ denotes the solution of (53), and

$$V = (W\underline{\tilde{H}}_R)^T W\underline{\tilde{H}}_R + (W\underline{\tilde{H}}_I)^T W\underline{\tilde{H}}_I \tag{55}$$

$$g = (W\underline{\tilde{H}}_R)^T W\tilde{h}_R + (W\underline{\tilde{H}}_I)^T W\tilde{h}_I \tag{56}$$

Once the optimal $\hat{\tilde{c}}$ is obtained, the remaining steps are the same as before. In the above procedures, the most time consuming step is the calculation of $(B^T B)^{-1}$. Since $B^T B$ is Toeplitz, one can carry out its inversion recursively by the Levinson algorithm.

A summary of this procedure is given in the flow chart of Fig. 1.

Our experience in simulating this procedure on the computer indicates that it is sufficient to consider the real part of the frequency domain samples to obtain quite accurate answers.

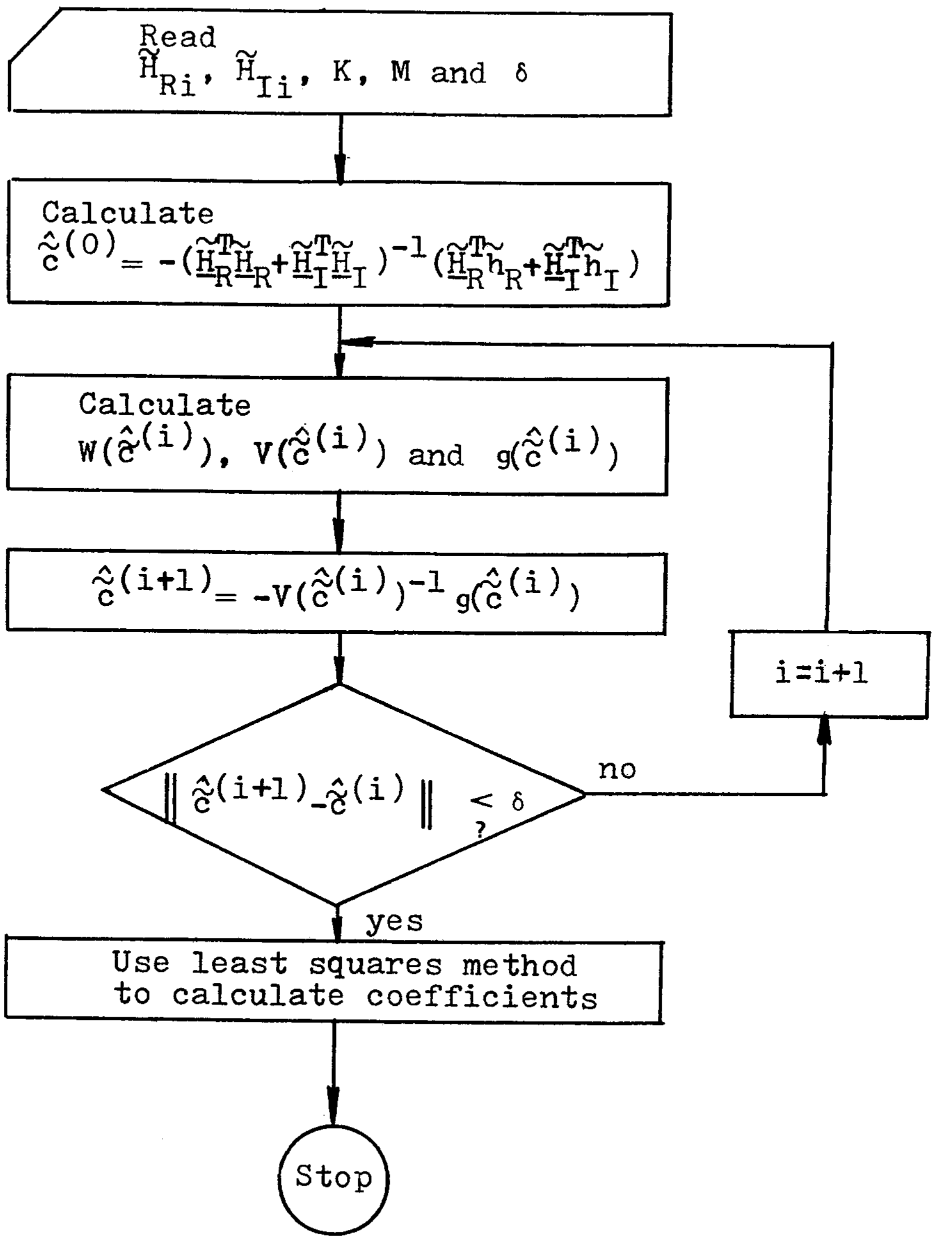

Fig. 1 An Iteration Refinement Algorithm of Frequency Domain Prony Method

5. A NUMERICAL SIMULATION OF THE NOISY CASE

To Illustrate the procedure that we just described we performed a numerical solution with x(t) as in (30) and y given by

$$y(t) = 0.8\,x(t-1) - 0.33\,x(t - 1.46875) + e(t) \qquad (57)$$

where e(t) was white noise with zero mean and variance 0.01.

The simulation procedures is outlined below:

(I) Find the DFT of $x(\frac{n}{16})$ and $y(\frac{n}{16})$ as X (k) and Y(k) for n and k = 0, 1, ..., 127;

(II) Consider $H_k = Y(k)/X(k)$ at k = 1, ..., 15 (omitting k = 0 and 16 to avoid singularities at those points);

(III) Pick $H_{R,k}$ = Real Part of H_k, k = 1, ..., 15 shown in the computer printout of TABLE 4 as "GIVEN SEQUENCE HR";

(IV) Use the iteration algorithm to calculate $\tilde{c}_i$, i = 0, 1, ($\tilde{c}_2$ is set equal to 1) and then to solve for τ_i, i=1,2 (the exact values of which are, according to (57), $\tau_1 = 1$, $\tau_2 = 1.46875$) and a_i, i = 1, 2 (the exact values of which are $a_1 = 0.8$ and $a_2 = -0.33$).

It is clear from the computer printout that $\tilde{c}_i$'s converge rapidly to their exact values, in fact to within the order of 10^{-9} in three iterations.

Table 4 also provides a comparison of the values for the delays obtained without iteration and with iteration. It is seen that iteration significantly improves parameter estimates driving these estimates to very near the actual parameter values.

TABLE 4

GIVEN SEQUENCE HR :

0.4235125	0.2377980	-0.2690070	-0.7668938	-0.8619426
-0.2738070	0.6362737	1.1200461	0.7403171	-0.1819472
-0.9201753	-0.9009999	-0.3175050	0.2798404	0.5547632

ITERATION RESULTS OF THE COEFF. OF THE CHAR. EQ. :

ITERATIONS	$\tilde{c}_0$	$\tilde{c}_1$
INITIATL	0.3098771442	-0.7087249238
1	0.3180122575	-0.7071953943
2	0.3179789242	-0.7072153650
3	0.3179791271	-0.7072152470
4	0.3179791259	-0.7072152477

COMPARISON OF THE VALUE OF DELAYS OBTAINED
WITHOUT ITERATION AND WITH ITERATION :

EXACT VALUE	1.0000000	1.4687500
VALUE OBTAINED WITHOUT ITERAT.	0.9879550	1.4343451
VALUE OBTAINED WITH ITERATION	0.9992820	1.4697246

6. EXTENSIONS TO THE MULTIPLE SIGNAL CASE AND TO DISTRIBUTED RESPONSE

All the preceding developments extend to the following two general cases

6.1 Superposition of Different signals.

One general case of interest is one in which the output y is a linear combination of a set of differently delayed and differently shaped signals x_i, $i = 1, \ldots, M$. i.e.,

$$y(t) = \sum_{i=1}^{M} a_i \, x_i \, (t-\tau_i) \,. \tag{58}$$

The fundamental assumption that we make is that each x_i can be represented as in (6). Then replacement of (6) into (58) leads to

$$y(t) = \sum_{i=1}^{M} \sum_{j=1}^{L_i} a_{ij} \, \varphi(t-\tau_{ij}), \tag{59}$$

where

$$a_{ij} = a_i \, b_{ij} \tag{60}$$

$$\tau_{ij} = \tau_i + \eta_{ij} \,. \tag{61}$$

Since now

$$H(\omega) = \frac{Y(\omega)}{\Phi(\omega)} = \sum_{i=1}^{M} \sum_{j=1}^{L_i} a_{ij} \, e^{-j\tau_{ij}\omega} \tag{62}$$

the Prony method of the preceding sections permits us to obtain the parameters a_{ij} and τ_{ij}, $i = 1, \ldots, M$; $j = 1, \ldots, L_i$.

Since b_{ij} and η_{ij} are assumed known, then using the fact that

$$\tau_{ij} - \tau_{kl} = \eta_{il} \,, \text{ if } j=k, \tag{63}$$

$$(a_{ij}/a_{kl}) = b_{il} \,, \text{ if } j=k, \tag{64}$$

and by matching the differences and ratios between the calculated parameters in the left sides of (63) and (64) to their knwon values expressed by the rights sides of (63) and (64), it is possible to group all τ_{ij} that belong to the same i togehter. The same applies to a_{ij}. Then it is a simple matter to determine a_i and τ_i for the i^{th} signal using (60) and (61).

6.2 Distributed Response

Another generalization of the preceding ideas is that the expression (1) may be viewed as the convolution of x with the impulse response

$$h_o(t) = \sum_{i=1}^{M} a_i\ \delta(t-\tau_i). \tag{65}$$

A natural generalization of (65) is one in which the delta functions in (65) are replaced by a window (potential) function φ (See Fig. 2), i.e.,

$$h(t) = \sum_{i=1}^{M} a_i\ \varphi(t-\tau_i) \tag{66}$$

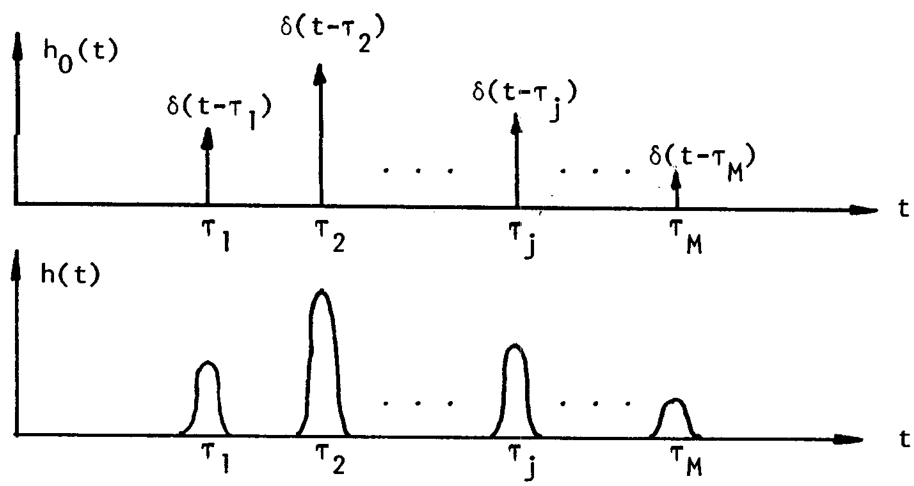

Fig. 2 Generalization of the impulse response $h_o(t)$ consisting of pure delays to a distributed delay impulse response h(t).

We assume that h, and hence a_i and τ_i, $i = 1, \ldots, M$, are not known a priori (even though φ is assumed known).

The output is then

$$y(t) = \sum_{i=1}^{M} a_i \, \Gamma(t-\tau_i), \tag{67}$$

where

$$\Gamma(t) = \int_0^t \varphi(t-t') \, x(t') \, dt'. \tag{68}$$

By taking the Fourier transform of (67) the constants a_i and τ_i, $i = 1, \ldots, M$, can be calculated by our Prony method, and thus h identified.

7. RELATIONSHIP TO WIENER'S TAUBERIAN THEOREMS

From the preceding developments, we deduce that (6) is a very useful signal representation for the types of problems which we are addressing. The following two Tauberian theorems assure us that such a representation is valid under very general conditions on the signals x_i (we will drop the subscript i for simplicity in notation). As before, X and Y will denote the Fourier Transforms of x and y.

Theorem 1 (Wiener [8]). If x belongs to $L_2(-\infty, \infty)$, and almost everywhere on $-\infty < t < \infty$,

$$\lim_{T\to\infty} \int_{-T}^{T} x(t)\, e^{-j\omega t} dt \neq 0, \tag{69}$$

then if $y \in L_2(-\infty, \infty)$ and $\varepsilon > 0$, there is an integer M, together with a set of real numbers τ_i, $i = 1, \ldots, M$ and a set of complex numbers a_i, $i = i, \ldots, M$ such that

$$\int_{-\infty}^{\infty} \left| y(t) - \sum_{i=1}^{M} a_i\, x(t - \tau_i) \right|^2 dt < \varepsilon. \tag{70}$$

Conversely, let $x \in L_2(-\infty, \infty)$, and let it be possible, whenever $y \in L_2(-\infty, \infty)$ and $\varepsilon > 0$, to find M, τ_i, and a_i as above for which (70) holds; then (69) is true almost everywhere.

Theorem 2 (Wiener [8]). If $X \in L_2(-\infty, \infty)$ and has a set of zeros of zero measure, then if Y belongs to $L_2(-\infty, \infty)$ and $\varepsilon > 0$, there is an integer M, together with a set of real numbers $\tau_1, \ldots, \tau_M$ and complex numbers $a_1, \ldots, a_M$ such that

$$\int_{-\infty}^{\infty} \left| Y(\omega) - X(\omega) \sum_{i=1}^{M} a_k e^{-j\tau_k\omega} \right|^2 d\omega < \varepsilon\ , \tag{71}$$

We now prove the following

Corrolary. If in Theorem 1, x and y are real, then the results of that theorem holds for $a_i, \ldots, a_M$ real.

Proof: Let $\tilde{a}_1, \ldots, \tilde{a}_M$ be the complex constants labeled as $a_1, \ldots, a_M$ in Theorem 1. Let a_i and b_i denote the real and imaginary parts of $\tilde{a}_i$. Then according to theorem 1,

$$\begin{aligned} \epsilon &> \int_{-\infty}^{\infty} \left|y(t) - \sum_{i=1}^{M} \tilde{a}_i \, x(t-\tau_i)\right|^2 dt \\ &= \int_{-\infty}^{\infty} \left\{\left| y(t) - \sum_{i=1}^{M} a_i x(t-\tau_i)\right|^2 + \left|\sum_{i=1}^{M} b_i \, x(t-\tau_i)\right|^2\right\} dt, \\ &\geq \int_{-\infty}^{\infty} \left| y(t) - \sum_{i=1}^{M} a_i x(t-\tau_i)\right|^2 dt , \end{aligned} \tag{72}$$

where in going from the first to the second lines, we have used the fact x and y are real. Q. E. D.

We thus see that the frequency domain Prony method constitutes a natural technique for determining the parameters appearing in Wiener's Tauberian approximation.

8. CONCLUSION

A frequency domain Prony approach has been presented for analyzing a superposition (mixture) of M differently delayed versions of one or more signals. We assumed that the data consisted of samples of the mixture and of the input. Both the noise-free and noisy cases were considered and advantages over direct frequency domain methods were pointed out. A connection between the present Prony approximation and certain Tauberian theorems of Wiener was established. Finally computer simulations showed a remarkable agreement between theory and experiment. The results are directly applicable to target identification by radar and to identification of geophysical structures by seismic reflection methods.

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