SYNTHESIS OF LINEAR MULTIVARIABLE REGULATORS

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ABSTRACT

This paper generalizes the results obtained by the authors in [1] and presents new results on the construction of proper controllers for linear multivariable regulators. These results appear to lead to efficient computational methods that do not involve Smith forms nor the order relations used in [1] to construct proper controllers. The methods are based on the recently developed theory of Λ-generalized polynomials [2]. We discuss several new computational algorithms and how they can be applied to the regulator problem.

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1. Introduction

In a recent paper [1], the authors solved a general linear regulator problem. The constructive proofs of the main results of [1] relied on Smith forms of polynomial matrices and certain special types of order relations to prove the existence of proper controllers that solve the regulator problem. That approach does not appear to lead to efficient computational methods to construct controllers. In this paper we present different proofs to a more general problem using some of Pernebo's recently developed theory of Λ-generalized polynomials [2]. Our new approach does not rely on special forms and techniques and demonstrates the general types of computational problems that must be solved in order to compute controllers in an efficient way. In addition we discuss several new algorithms and show how they are applied to the solution of the regulator problem.

2. The Model

We will restrict our attention to systems described as follows:

$$\begin{bmatrix} y(s) \\ z(s) \end{bmatrix} = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} \begin{bmatrix} u(s) \\ e(s) \end{bmatrix} \tag{1}$$

where y(s) is the Laplace transform of the measured variable, z(s) the transform of the regulated variable and u(s) and e(s) represent transforms of the control input and the exogenous input respectively. The G_{ij} terms represent proper rational functions*. In general y(s), z(s), u(s), and e(s) are vectors and the $C_{ij}(s)$ are transfer function matrices. An alternate way of describing our systems will be

* A proper rational function has no poles at infinity.

$$\begin{aligned} T^*(s)\ \xi &= U_1^*(s)\ u + U_2^*(s)\ e \\ y &= V_1^*(s)\ \xi \\ z &= V_2^*(s)\ \xi \end{aligned} \tag{2}$$

where u, e, y, and z are the same as before and ξ is some internal variable. We assume $T^*(s)$ is square and nonsingular and that the greatest common left divisor (GCLD) of $T^*(s)$ and $U_1^*(s)$ and the greatest common right divisor (GCRD) of $T^*(s)$ and $V_1^*(s)$ have no zeros in the closed right half plane.

Following [2], we use the transformation

$$\lambda = \frac{1}{s-a} \tag{3}$$

so that (2) becomes

$$\begin{aligned} T(\lambda)\ \xi &= U_1(\lambda)\ u + U_2(\lambda)\ e \\ y &= V_1(\lambda)\ \xi \\ z &= V_2(\lambda)\ \xi \end{aligned} \tag{4}$$

and finally, we will write

$$\begin{aligned} y &= -H\ u + G_1\ e \\ z &= P\ u + G_2\ e \end{aligned} \tag{5}$$

where

$$\begin{aligned} -H &= V_1\ T^{-1}\ U_1 \\ P &= V_2\ T^{-1}\ U_1 \\ G_1 &= V_1\ T^{-1}\ U_2 \\ G_2 &= V_2\ T^{-1}\ U_2 \end{aligned}$$

and the indeterminate λ has been dropped for simplicity. This model is more general than that used in [1] and is essentially the same as that of [2] and [3].

Under the transformation (3), a "bad" region of the complex s-plane is transformed into a "bad" region Λ of the complex λ-plane when a is a real number belonging to the "good" region in the s-plane. This transformation is used because of the simplification it furnishes in dealing with problems of constructing proper controllers. In terms of our previous assumptions, the closed right half s-plane including the infinity point maps into a circle passing through the origin of the λ-plane and centered on the positive real axis (when $a < 0$). The interior of this circle including the boundary we will call Λ and if we exclude the origin we will call this region Λ_o.

Our assumption on greatest common left and right divisors of (T, U_1) and (V_1, T) means that all the zeros of $|T(\lambda)|$* in Λ_o appear as poles of $V_1 T^{-1} U_1$. Our assumption of properness in (1) means that in (5) each of the transfer functions is analytic at $\lambda = 0$.

3. Internal Stability

In (5), let e = 0 and consider

$$y = -Hu \ .$$

Let the controller be described by

$$u = Cy$$

and define the minimal representations

$$H = A^{-1}B$$

$$C = P_C Q_C^{-1} \ .$$

Our system will be called internally stable if the zeros of the characteristic polynomial

* $|T|$ means determinant of the square matrix T.

$$|AQ_C + BP_C|$$

are outside Λ_o. This corresponds to the system having no roots of its characteristic polynomial in the closed right half s-plane. If the system is internally stable we will also say that it is Λ_o-stable.

Define Q_K as follows

$$AQ_C + BP_C = Q_K \ . \tag{6}$$

Since A and B are relatively left prime (rℓp), there exist Y_1 and X_1 such that [4]

$$U = \begin{bmatrix} A & B \\ -Y_1 & X_1 \end{bmatrix} \tag{7}$$

is unimodular. If P_K is defined by

$$U \begin{bmatrix} Q_C \\ P_C \end{bmatrix} = \begin{bmatrix} Q_K \\ P_K \end{bmatrix} \tag{8}$$

then

$$\begin{bmatrix} Q_C \\ P_C \end{bmatrix} = U^{-1} \begin{bmatrix} Q_K \\ P_K \end{bmatrix} \tag{9}$$

characterizes a stabilizing controller when $|Q_K|$ has no roots in Λ_o. Define X, Y, A_1, B_1 by

$$U^{-1} = \begin{bmatrix} X & -B_1 \\ Y & A_1 \end{bmatrix} . \tag{10}$$

Then

$$C = P_C Q_C^{-1} = (Y+A_1K)(X-B_1K)^{-1} \tag{11}$$

where

$$K = P_K Q_K^{-1} \tag{12}$$

and this corresponds to the YJB[5] characterization of stabilizing controllers in terms of a Λ_o-generalized polynomial (i.e., a rational function with no poles in Λ_o [2]).

In this paper we are primarily interested in controllers C that are analytic at $\lambda = 0$. Under our previous assumption of properness in (1), our closed-loop transfer functions should also be proper. In order to insure this and to avoid situations where Λ_o-generalized polynomials K can produce unrealizable controllers C and to avoid algebraic loops we will restrict our attention to Λ-generalized polynomials K. In particular we will deal only with Λ-stable systems which means that $|Q_K(0)| \neq 0$. In [2] Pernebo assumes that $G_{11}(s)$ in (1) is strictly proper which leads to the conclusion that Λ_o-stable systems are also Λ-stable. This is no longer true in general when $G_{11}(s)$ is only proper and our restriction to Λ-stable systems will avoid functions with poles at $\lambda = 0$ and not result in any serious loss of generality. Our approach to controller synthesis will be to construct a controller C satisfying (11) after having found a Λ-generalized polynomial K with the necessary properties. This is the approach used in [1] and is useful since nonlinear functions of C become linear functions of K. This is demonstrated in the next section where we formulate the regulator problem.

4. The Regulator Problem

Let the exogenous inputs be described by

$$e = R^{-1}M\,w$$

where R and M are rℓp and w is a constant vector. In terms of the function K, the transfer function from w to z is

$$Z = Z_1 R^{-1} M \tag{13}$$

where

$$Z_1 = PA_1 \, KAG_1 + PYAG_1 + G_2 \tag{14}$$

and the object of regulation is to find a stable function K which results in a stable function Z. More precisely, define a region Ω of the λ-plane such that $\Lambda \subset \Omega$ and the complement of which contains the desired poles of Z. For example if Λ represents the image of the extended half-plane $\{s|\text{real } s \geq 0\}$ under the mapping (3), then Ω might represent the image of the extended half plane $\{s|\text{real } s \geq -1\}$ under the same mapping. In the following if K is a Λ-generalized polynomial and Z is a Ω-generalized polynomial we will use the abbreviated terminology Λ-stable K and Ω-stable Z. In this way we formulate the regulator problem as follows:

Regulator Problem (RP)

Given the system described by (5). Under what conditions does there exist a Λ-stable K such that Z in (13) is Ω-stable and

$$|X - B_1 K| \not\equiv 0$$

where X and B_1 are defined in (10).

The problem that was solved in [1] involved first determining conditions for the existence of a Λ_o-stable K such that Z was Λ_o-stable. It was then shown that a K could be constructed that resulted in a C which was analytic at $\lambda = 0$. Here we will use the same approach to find conditions for the existence of a Λ-stable K and a Λ-stable Z and then show how these results can be used to determine the existence of a Λ-stable K and a Ω-stable Z. Our problem formulation differs from Pernebo's [2] in that he requires both Z and Z_1 in

(13) to be Ω-stable. It turns out that when K is Λ-stable, Z_1 is Λ-stable and is almost always Ω-stable when Z is Ω-stable. Therefore the requirement that Z_1 be Ω-stable is not imposed here.

5. Solution of RP

We first solve the following problem:

Regulator Problem 1 (RP1)

Under what conditions does there exist a Λ-*stable* K *such that* Z *in* (13) *is* Λ-*stable.*

Define the following quantities

$$\begin{aligned}
PA_1 &= B_2A_2^{-1} \\
D_1 &= G_1R^{-1}M \\
D_2 &= G_2R^{-1}M \\
\{AD_1\}_\Lambda &= P_{d_1}Q_{d_1}^{-1} = \tilde{Q}_{d_1}^{-1}\tilde{P}_{d_1} \\
\{PYAD_1+D_2\}_\Lambda &= P_{d_2}Q_{d_2}^{-1} \\
\pi_1P_{d_1}+\pi_2Q_{d_1} &= I
\end{aligned}$$

where $\{\ \}_\Lambda$ represents that part of a partial fraction expansion having poles in Λ and the fractional representations are minimal. The solution of RP1 is:

Theorem 1. RP1 *is solvable if and only if there exist polynomial matrices* N, V, *and* W *such that*

$$Q_{d_2}^{-1}Q_{d_1} = N \tag{15}$$

and

$$B_2V + W\tilde{Q}_{d_1} = P_{d_2}N\pi_1 \ . \tag{16}$$

When a solution exists, there exist Λ-generalized polynomials V_o and W_o such that

$$B_2V_o + W_o \tilde{Q}_{d_1} = 0 \tag{17}$$

in which case K *is given by*

$$K = A_2(V_o - V) \ . \tag{18}$$

Recall our previous assumptions that in the system representation (4) the GCLD of T and U_1 has no zeros in Λ_o(A-1), the GCRD of V_1 and T has no zeros in Λ_o(A-2) and each of the transfer functions in (5) is analytic at $\lambda = 0$ (A-3).

Lemma 1. Under the assumptions (A-2) *and* (A-3)

$$\{PA_1\}_\Lambda = 0 \ .$$

Proof: From (5) and (10) it follows that

$$-H = V_1T^{-1}U_1 = -B_1A_1^{-1} \ .$$

Let

$$V_1 = \tilde{V}_1 G$$
$$T = \tilde{T} G$$

where G is the GCRD of V_1 and T and has no zeros in Λ_o by (A-2). Then

$$\tilde{V}_1\tilde{T}^{-1}U_1A_1 = -B_1 \quad \text{(a polynomial)}$$

implies that

$$\tilde{T}^{-1}U_1A_1 = L_1 \quad \text{(a polynomial)}$$

and so

$$PA_1 = V_2T^{-1}U_1A_1 = V_2T^{-1}\tilde{T}L_1$$
$$= V_2G^{-1}L_1$$

which has no poles in Λ_o. By (A-3), P is analytic at $\lambda = 0$ and so

$$\{PA_1\}_\Lambda = 0 . \qquad \text{Q.E.D.}$$

Lemma 2. Under the assumptions (A-1) and (A-3)

$$\{AG_1\}_\Lambda = 0 .$$

Proof: The proof is the same as the proof of Lemma 1 and will not be repeated.

Lemma 3. Under the assumptions (A-1), (A-2) and (A-3)

$$\{PYAG_1+G_2\}_\Lambda = 0 .$$

Proof: First note that

$$H(YB-I) = -XB$$

is a polynomial matrix. Then

$$H(YB-I) = V_1(I+T^{-1}U_1YAV_1)T^{-1}U_1 .$$

Let S_1 be the GCLD of T and U_1 so that

$$T = S_1\tilde{T} , \quad U_1 = S_1\tilde{U}_1 .$$

Then it follows that

$$V_1(I+T^{-1}U_1YAV_1)\tilde{T}^{-1} = M_1 \quad \text{(a polynomial)} .$$

Let S_2 be the GCRD of V_1 and T

$$V_1 = \hat{V}_1S_2 , \quad T = \hat{T}S_2 .$$

Then

$$\hat{T}^{-1}(T+U_1YAV_1)\tilde{T}^{-1} = M_2 \quad \text{(a polynomial)}$$

and it follows that

$$PYAG_1 + G_2 = V_2S_2^{-1}M_2S_1^{-1}U_2 .$$

By (A-1) and (A-2) this is a Λ_o-generalized polynomial. By (A-3), P, G_1, and G_2 are analytic at $\lambda = 0$ so

$$\{PYAG_1+G_2\}_\Lambda = 0 \ . \qquad \text{Q.E.D.}$$

Proof of Theorem 1:

Conditions (15) and (16) were proved in Theorem 1 [1]. The only new results are (17) and (18) and are established as follows:

Define

$$PA_1K = A_3^{-1}B_3$$

K is Λ-stable by assumption and PA_1 is Λ-stable by Lemma 1. Therefore if Z is Λ-stable

$$B_3 P_{d_1} Q_{d_1}^{-1} + A_3 P_{d_2} Q_{d_2}^{-1} = M$$

where M is a polynomial and

$$B_3 P_{d_1} + A_3 P_{d_2} N = MQ_{d_1}$$

which easily leads to

$$B_3 = -A_3 P_{d_2} N\pi_1 + L\tilde{Q}_{d_1}$$

for some polynomial matrix L. Therefore

$$A_3^{-1}B_3 = PA_1K = -P_{d_2} N\pi_1 + A_3^{-1}L\tilde{Q}_{d_1}$$

and from (16)

$$B_2V + W\tilde{Q}_{d_1} = -PA_1K + A_3^{-1}L\tilde{Q}_{d_1}$$

and

$$B_2(V+A_2^{-1}K) + (W-A_3^{-1}L)\tilde{Q}_{d_1} = 0 \ .$$

Clearly

$$V_o = V + A_2^{-1}K$$

$$W_o = W - A_3^{-1}L$$

are Λ-generalized polynomials and

$$K = A_2(V_o - V) \quad . \qquad \text{Q.E.D.}$$

Theorem 1 gives a general expression for the matrix K. In order to obtain a practical method to construct K, we will examine the solution of (17).

Let $V_o = P_v Q_v^{-1}$ and $W_o = Q_w^{-1} P_w$.

Proposition 1

Given polynomial matrices B_2 and $\tilde{Q}_{d_1}$ where $|\tilde{Q}_{d_1}|$ has zeros only in Λ. There exist polynomial matrices P_v, Q_v, Q_w, and P_w with $|Q_v|$ and $|Q_w|$ having no zeros in Λ such that

$$B_2 P_v Q_v^{-1} + Q_w^{-1} P_w \tilde{Q}_{d_1} = 0 \tag{19}$$

if and only if there exist polynomial matrices Q_k, Q_d and B_k such that

$$\tilde{Q}_{d_1} Q_v = Q_k Q_d \tag{20}$$

$$B_2 P_v + B_k Q_d = 0 \tag{21}$$

with

$$|Q_k| = |Q_v| \quad \text{and} \quad |Q_d| = |\tilde{Q}_{d_1}| \tag{22}$$

Proof: Assume (19) is true. Since $|Q_v|$ and $|\tilde{Q}_{d_1}|$ have no zeros in common it is clear that there exist Q_k and Q_d such that (20) and (22) hold. Equation (19) can then be written as

$$B_2 P_v Q_d^{-1} = -Q_w^{-1} P_w Q_k \ .$$

The left hand side has poles only in Λ, the right hand side has no poles in Λ and therefore

$$B_2 P_v = -B_k Q_d$$

for some polynomial matrix B_k.

On the other hand assume (20), (21), and (22) are true and write (21) as

$$B_2 P_v Q_v^{-1} + B_k Q_d Q_v^{-1} = 0$$

or

$$B_2 P_v Q_v^{-1} + B_k Q_k^{-1} \tilde{Q}_{d_1} = 0$$

Q_w and P_w are then defined by

$$Q_w^{-1} P_w = B_k Q_k^{-1} \; . \qquad \text{Q.E.D.}$$

Remark: Equations (20), (21), and (22) are always solvable since $|Q_v|$ and $|\tilde{Q}_{d_1}|$ have no zeros in common. In particular

$$P_v = N_o Q_d \quad \text{and} \quad B_k = -B_2 N_o$$

is a useful solution to (21) where N_o is any polynomial matrix.

The next result shows how K is constructed so that a controller C, analytic at $\lambda = 0$, is obtained.

Theorem 2. Assume RP1 is solvable and AD_1 is analytic at $\lambda = 0$. K can be constructed so that $|X - B_1 K| \neq 0$ at $\lambda = 0$ in the following manner.

i) Choose $X(\lambda)$ in (10) so that $|X(0)| \neq 0$. (This can always be done.)

ii) Choose Q_v so that $|Q_v|$ has no zeros in Λ.

iii) Compute N_o according to

$$N_o = V(0)Q_v(0)Q_d^{-1}(0)$$

where $V(\lambda)$ is a solution of (16) and $Q_d(\lambda)$ is a solution of (20) and (22).

iv) Compute

$$K(\lambda) = A_2(\lambda)[N_o Q_d(\lambda)Q_v^{-1}(\lambda)-V(\lambda)] .$$

v) The controller C is constructed as follows:

$$C = (Y+A_1K)(X-B_1K)^{-1} .$$

<u>Proof</u>: Since AD_1 is analytic at $\lambda = 0$, $\tilde{Q}_{d_1}(0)$ and thus $Q_d(0)$ are nonsingular. Let $K = P_K Q_K^{-1}$ be a minimal representation. From (iv) $|Q_K(0)| \neq 0$ and $P_K(0) = 0$ so using (10)

$$Q_C(0) = [X(0)-B_1(0)]\begin{bmatrix} Q_K(0) \\ 0 \end{bmatrix} = X(0)Q_K(0)$$

and thus is nonsingular. Q.E.D.

The situation can now be summarized as follows. Under the assumptions (A-1), (A-2), (A-3), and AD_1 analytic at $\lambda = 0$, conditions (15) and (16) represent the necessary and sufficient conditions for the existence of a solution to RP1 subject to the constraint $|X-B_1K| \neq 0$. A controller which is analytic at $\lambda = 0$ and thus proper as a function of s can be constructed according to Theorem 2. This procedure is a considerable improvement over that presented in [1].

Notice that our method of solution to RP1 can be generalized in the following sense. Given P, R, and S with P Λ-stable. Under what conditions does there exist a Λ-stable Q such that

$$PQR + S$$

is Λ-stable? In this sense Λ can be any symmetric region in the λ-

plane. We will use this observation in our solution of the main problem of interest which we now approach as follows:

Regulator Problem 2 (RP2)

Under what conditions does there exist a Λ-stable K such that Z in (13) is Ω-stable.

Here we will write our exogenous signals

$$e = M_1 R_1^{-1} w$$

and so from (14)

$$PA_1 KAG_1 M_1 + (PYAG_1 + G_2) M_1 = ZR_1 \tag{23}$$

where

$$Z = Z_1 M_1 R_1^{-1} \ .$$

Let

$$(PYAG_1 + G_2) M_1 = M_o R_o^{-1} \ .$$

Then (23) can be written as

$$PA_1 KAG_1 M_1 R_o + M_o = ZR_1 R_o \ .$$

Let

$$PA_1 = B_2 A_2^{-1}$$

$$AG_1 = A_4^{-1} B_4$$

and factor B_2 and $B_4 M_1 R_o$ as follows [2]

$$B_2 = \hat{B}_2 \tilde{B}_2$$

$$B_4 M_1 R_o = \tilde{C}\hat{C}$$

where $\hat{B}_2$ is a polynomial matrix with linearly independent columns and having all its zeros in Λ. $\tilde{B}_2$ is right Λ-invertible [2]. $\tilde{C}$ is left

Λ-invertible and $\hat{C}$ has linearly independent rows and all its zeros in Λ.

Define

$$\tilde{X} = \tilde{B}_2 A_2^{-1} K A_4^{-1} \tilde{C}$$

which by Lemmas 1 and 2 is Λ-stable when K is Λ-stable.

The situation is now

$$\hat{B}_2 \tilde{X} \hat{C} + M_o = Z R_1 R_o \ . \tag{24}$$

Now observe that $\hat{B}_2 \tilde{X} \hat{C}$ is Ω-stable if Z is Ω-stable and this means that by Lemma 8.2 [2] and its dual $\tilde{X}$ is Ω-stable. Therefore we conclude that if there exists a Λ-stable K such that Z is Ω-stable, then there exists a Ω-stable $\tilde{X}$ satisfying (24). Rewrite (24) as follows

$$\hat{B}_2 \tilde{X} \hat{C} (R_1 R_o)^{-1} + M_o (R_1 R_o)^{-1} = Z$$

and we are back to RP1 with Λ replaced by Ω. Let

$$\{\hat{C}(R_1 R_o)^{-1}\}_\Omega = P_{d_1} Q_{d_1}^{-1} = \tilde{Q}_{d_1}^{-1} \tilde{P}_{d_1} \tag{25}$$

$$\{M_o (R_1 R_o)^{-1}\}_\Omega = P_{d_2} Q_{d_2}^{-1} \ . \tag{26}$$

We know from Theorem 1 the conditions for the existence of an Ω-stable $\tilde{X}$ such that Z is Ω-stable. If we can find such an $\tilde{X}$, we can construct K from

$$K = A_2 B_2^R \tilde{X} C^L A_4$$

where

$$\tilde{B}_2 B_2^R = I$$

$$C^L \tilde{C} = I$$

and since B_2^R and C^L can be made Λ-stable, K will be Λ-stable. Then clearly by reversing our steps we have created a Λ-stable K so that Z is Ω-stable and have solved RP2. We will state an intermediate result as follows:

Proposition 2

RP2 *is solvable if and only if there exist* Ω-*generalized polynomials* $\tilde{X}$ *and* Z *so that* (24) *is true*.

Remark: This result is stated only so that it can be compared with Theorem 8.4 in [2] which is Pernebo's solution of the regulator problem. It is important to note here that using the results of Theorem 1, Proposition 2 furnishes us with an alternate problem statement for RP2 which is, obviously, under what conditions do there exist Ω-generalized polynomials $\tilde{X}$ and Z so that (24) is true. The answer, of course, is:

Theorem 3: RP2 *is solvable if and only if there exist polynomial matrices* N, V, *and* W *such that*

$$Q_{d_2}^{-1} Q_{d_1} = N \tag{27}$$

$$\hat{B}_2 V + W\tilde{Q}_{d_1} = P_{d_2} N\pi_1 \tag{28}$$

When a solution exists, there exist Ω-*stable matrices* $\tilde{X}$, V_o, *and* W_o *such that*

$$\hat{B}_2 V_o + W_o \tilde{Q}_{d_1} = 0 \tag{29}$$

$$\tilde{X} = V_o - V \tag{30}$$

and K *is given by*

$$K = A_2 B_2^R \tilde{X} C^L A_4 \ . \tag{31}$$

Proof. Follows directly from Theorem 1 with the obvious substitutions. The construction procedure of Theorem 2 can be used to construct a K such that $|X-B_1K| \neq 0$ at $\lambda = 0$ and thus we have solved our main problem RP. The solution is:

Theorem 4: (Solution of RP) Assume (A-1), (A-2), (A-3) are satisfied and $\hat{C}(R_1R_o)^{-1}$ is analytic at $\lambda = 0$. There exists a Λ-stable K such that Z is Ω-stable and $|X-B_1K| \neq 0$ at $\lambda = 0$ if an only if conditions (27) and (28) of Theorem 3 hold. K can be constructed according to the procedure of Theorem 2 with step (iv) replaced by

iv)' $$K = A_2B_2^R(N_oQ_dQ_v^{-1}-V)C^LA_4 \quad .$$

A controller C is then constructed by

$$C = (Y+A_1K)(X-B_1K)^{-1}$$

Remark: The assumptions AD_1 analytic at $\lambda = 0$ and $\hat{C}(R_1R_o)^{-1}$ analytic at $\lambda = 0$ in Theorems 2 and 4 respectively are usually satisfied. Rarely, if ever, will the description of the exogenous signals $R^{-1}M = M_1R_1^{-1}$ contain poles at $\lambda = 0$. Since

$$AD_1 = AG_1R^{-1}M$$

it follows from Lemma 2 that AD_1 will be analytic at $\lambda = 0$ when $R^{-1}M$ is. Also recall that R_o arose as follows

$$(PYAG_1+G_2)M_1 = M_oR_o^{-1}$$

and from Lemma 3, $|R_o|$ has no zeros in Λ. Therefore $\hat{C}(R_1R_o)^{-1}$ will be analytic at $\lambda = 0$ when $M_1R_1^{-1}$ is which is almost always. Thus these assumptions represent no serious constraints.

6. Example*

Consider the following example [6],

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{s(s+1)} & 0 \\ \frac{1}{s(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{1}{s} & \frac{1}{s+1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{s+1} \\ 0 \\ 0 \end{pmatrix} d$$

where d represents a disturbance, u_1 and u_2 are control inputs and y_1, y_2, and y_3 are measured plant outputs. The objective is to design a controller so that in the presence of a constant disturbance d, y_1 tracks r_1 and y_2 tracks r_2 where r_1 is an arbitrary ramp and r_2 is an arbitrary step. We assume r_1 and r_2 are measured. Using the transformation

$$\lambda = \frac{1}{s+1}$$

and writing the system as in (5)

$$y = -Hu + G_1 e$$

$$z = Pu + G_2 e$$

we have

$$H(\lambda) = \begin{bmatrix} -\frac{\lambda^2}{1-\lambda} & 0 \\ -\frac{\lambda^2}{1-\lambda^2} & -\frac{\lambda^2}{1+\lambda} \\ -\frac{\lambda}{1-\lambda} & -\lambda \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad G_1(\lambda) = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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* This example was worked out in the M.S. Thesis of A. Kontos using algorithms described in section 7.

$$P(\lambda) = \begin{bmatrix} \frac{\lambda^2}{1-\lambda} & 0 \\ \frac{\lambda^2}{1-\lambda^2} & \frac{\lambda^2}{1+\lambda} \end{bmatrix} \quad G_2(\lambda) = \begin{bmatrix} \lambda & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} .$$

The exogenous signals are given by

$$e = E(\lambda)\ w$$

where

$$E(\lambda) = \begin{bmatrix} \frac{\lambda}{1-\lambda} & 0 & 0 & 0 \\ 0 & \left(\frac{\lambda}{1-\lambda}\right)^2 & \frac{\lambda}{1-\lambda} & 0 \\ 0 & 0 & 0 & \frac{\lambda}{1-\lambda} \end{bmatrix} .$$

For simplicity, we will use the formulation of Theorem 1 which means we will construct a Λ-stable K so that Z is Λ-stable.

First let $H = A^{-1}B$ where

$$A = \begin{bmatrix} 1 & 0 & -\lambda & 0 & 0 \\ 0 & \lambda+1 & -\lambda & 0 & 0 \\ 0 & 0 & \lambda-1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & \lambda^2 \\ 0 & 0 \\ \lambda & -\lambda^2+\lambda \\ 0 & 0 \\ 0 & 0 \end{bmatrix} .$$

Then find X_1, Y_1, A_1, B_1, X, and Y such that

$$\begin{bmatrix} A & B \\ -Y_1 & X_1 \end{bmatrix} \begin{bmatrix} X & -B_1 \\ Y & A_1 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} .$$

They are

$$X_1 = \begin{bmatrix} 1+\lambda & -\lambda^2 \\ -\lambda & 1-\lambda+\lambda^2 \end{bmatrix} \quad Y_1 = \begin{bmatrix} 0 & 0 & -\lambda & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1-\lambda & 0 \\ \lambda & 1+\lambda \end{bmatrix} \quad B_1 = \begin{bmatrix} \lambda^2 & 0 \\ \lambda^2 & \lambda^2 \\ \lambda+\lambda^2 & \lambda+\lambda^2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 0 & -(\lambda+\lambda^2) & 0 & 0 \\ 0 & 1-\lambda+\lambda^2 & -\lambda & 0 & 0 \\ 0 & \lambda^2 & -(1+\lambda) & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Y = \begin{bmatrix} 0 & 0 & \lambda & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \end{bmatrix} .$$

Then

$$PA_1 = \lambda^2 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

so $B_2 = PA_1$ and $A_2 = I$

$$\{AD_1\}_\Lambda = P_{d_1} \, Q_{d_1}^{-1} = \tilde{Q}_{d_1}^{-1} \, \tilde{P}_{d_1}$$

$$P_{d_1} \, Q_{d_1}^{-1} = \begin{bmatrix} \lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}^{-1}$$

$$\tilde{Q}_{d_1}^{-1}\tilde{P}_{d_1} = \begin{bmatrix} 1-\lambda & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & (1-\lambda)^2 & 0 \\ 0 & 0 & 0 & 0 & 1-\lambda \end{bmatrix}^{-1} \begin{bmatrix} \lambda^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & \lambda-\lambda^2 & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$\pi_1 = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda-\lambda^2 & 0 \\ 0 & 0 & 0 & 3\lambda-2\lambda^2 & 0 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$

$$\{PYAD_1+D_2\}_\Lambda = P_{d_2} Q_{d_2}^{-1}$$

$$= \begin{bmatrix} \lambda^2 & -\lambda^2 & -\lambda-\lambda^2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1-\lambda & \lambda & \lambda & 0 \\ 0 & 1-\lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}^{-1} .$$

Now conditions (15) and (16) are checked

$$Q_{d_2}^{-1} Q_{d_1} = \begin{bmatrix} 1 & 0 & -\lambda & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Since this is a polynomial, (15) is satisfied. For (16), consider

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} V + \frac{1}{\lambda^2} W \tilde{Q}_{d_1} = \begin{bmatrix} \lambda & 0 & 0 & -3+2\lambda & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} .$$

Let

$$\overline{W} = \frac{1}{\lambda^2} W .$$

Then for any polynomial matrix $\overline{W}$, this equation is solvable for V. In fact different solutions will produce different controllers. With

$$V_1 = \begin{bmatrix} 1 & 0 & 0 & -3+2\lambda & 0 \\ -1 & 0 & 0 & 3-2\lambda & -1 \end{bmatrix}$$

and $V_o = 0$, $K = -A_2V_1$ from (18) which leads to

$$C_1^*(s) = \begin{bmatrix} s+2 & 0 \\ s & 2s \end{bmatrix}^{-1} \begin{bmatrix} -(s+1) & 0 & 0 & 3s+1 & 0 \\ s+1 & 2 & -2 & -(3s+1) & 2(s+1) \end{bmatrix}$$

a second-order controller. With

$$V_2 = \begin{bmatrix} 1 & 0 & 0 & -3+2\lambda & 0 \\ -1 & 0 & 1 & 3-2\lambda & -1 \end{bmatrix}$$

and $V_o = 0$, we find

$$C_2^*(s) = \begin{bmatrix} s+2 & 0 \\ 0 & s(s+2) \end{bmatrix}^{-1} \begin{bmatrix} -(s+1) & 0 & 0 & 3s+1 & 0 \\ (s+1)^2 & s+1 & s^2-1 & -(3s+1)(s+1) & (s+1)^2 \end{bmatrix}$$

a third-order controller.

An interesting situation arises with

$$V_3 = \begin{bmatrix} 1 & 0 & 0 & -3+2\lambda & 0 \\ -1 & 1 & 0 & 3-2\lambda & -1 \end{bmatrix}$$

and $V_o = 0$. We find

$$C_3^*(s) = \begin{bmatrix} s+2 & 0 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} -(s+1) & 0 & 0 & 3s+1 & 0 \\ s+1 & -(s+1) & 0 & -(3s+1) & (s+1) \end{bmatrix}$$

which is a second-order controller which does not use the measurement y_3. If we had formulated this problem in the time domain using only the measurements y_1, y_2, r_1, and r_2, we would have required a fourth-order observer using the method presented in [7] for the solution of RPIS, or if we used the extended state space methods of [9] we would again have required four additional integrators.

In each of the above cases, $Z^*(s)$ is the same and is given by

$$Z^*(s) = \begin{bmatrix} \frac{s+2}{(s+1)^3} & -\frac{s+3}{(s+1)^3} & -\frac{s(s+3)}{(s+1)^3} & 0 \\ 0 & 0 & 0 & -\frac{s+2}{(s+1)^2} \end{bmatrix}.$$

Using the controller $C_3^*(s)$ and setting the exogenous signals to zero, the loop-gain of the system is

$$-C_3^*(s)\ P^*(s) = \begin{bmatrix} \frac{1}{s(s+2)} & 0 \\ 0 & \frac{1}{s(s+2)} \end{bmatrix}$$

and is robust with respect to gain perturbations k_1 and k_2 introduced at the plant inputs u_1 and u_2. The system is not robust however with regard to asymptotic regulation if all plant and conptroller parameters are free to change. In order to achieve this type of robustness [8] the problem should be reformulated with the measured variables the same as the regulated variables. A higher order controller would then result which would be robust with respect to tracking but not necessarily robust with respect to stability margins.

Numerical methods to solve the regulator problem will be discussed in the next section.

7. Numerical Algorithms

Three recently developed algorithms [10], [11], [12] are quite useful in the numerical solution of the regulator problem. The basic problems to be solved are:

i) Computing coprime factorizations,

ii) Dual factorizations,

iii) Structural factorizations,

iv) Partial fraction expansions

v) Existence of solution to regulator problem.

Coprime Factorizations - A recently developed algorithm for extracting greatest common divisors by Silverman and Van Dooren [10] is quite useful here. An APL listing of this algorithm can be found in [13].

Dual Factorization - Given a left coprime factorization $A^{-1}B$. It is necessary to compute a dual factorization (i.e., $B_1A_1^{-1}$) as well as matrices X, Y, X_1, Y_1 such that

$$AX + BY = I$$

$$X_1A_1 + Y_1B_1 = I \; .$$

This can be accomplished using the algorithm described in [11] for computing a unimodular matrix with an APL listing given in [13].

Structural Factorization - The problem arises of factoring a polynomial matrix

$$B(\lambda) = \hat{B}(\lambda)\ \tilde{B}(\lambda)$$

where $\hat{B}(\lambda)$ has all its zeros in some region Λ of the complex plane. This can be done using programs described in [14], [15], and [12]. Basically the Q-Z algorithm is used to compute the zeros of a polynomial matrix and then the factorization algorithm described in [12] is used. APL listings are given in [13] and [15].

Partial Fraction Expansions - Let

$$A^{-1}B = A_1^{-1}B_1 + A_2^{-1}B_2$$

be a partial fraction expansion of a rational matrix where A_1 has its zeros in a region Λ and A_2 has its zeros in the complement $\overline{\Lambda}$. (A_1, B_1, A_2, B_2) are computed as follows.

Use the factorization algorithm [12] to determine A_1, $\tilde{A}_1$, A_2, $\tilde{A}_2$ from

$$A = \tilde{A}_1 A_1 = \tilde{A}_2 A_2$$

where A_1 and A_2 ($\tilde{A}_1$ and $\tilde{A}_2$) are coprime. The matrix

$$[\tilde{A}_1 \ \tilde{A}_2]$$

therefore has a right inverse A^R that can be computed using the algorithm for computing a unimodular matrix [11]. Then

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = A^R B \ .$$

An APL listing for this program is given in [13].

Existence of Solution to RP - The previous algorithms furnish the matrices used to check the existence of a solution to RP and to construct a controller when a solution exists. The situation is as follows. Given polynomial matrices Q_1, Q_2, A, B, and C, do there exist polynomial matrices X, Y, and Z such that

$$Q_1 = Q_2 Z$$

and

$$AX + YB = C \ .$$

The first condition is easily checked using the Silverman-Van Dooren algorithm [10]. Calculate the GCLD of Q_1 and Q_2, i.e.,

$$Q_1 = L\tilde{Q}_1 \quad \text{and} \quad Q_2 = L\tilde{Q}_2 \ .$$

Then the unimodular matrix algorithm [11] is used to check if $\tilde{Q}_2$ is unimodular. If not, RP is not solvable.

At the present time, we do not have a useful algorithm for the solution of the equation

$$AX + YB = C .$$

In the example given above, this equation was solved "by hand" and proved no real problem. When C = I, Wolovich has approached this problem in terms of "externally-skew-prime" polynomial matrices [16], but it is not clear to us that this theory will be helpful in constructing useful numerical algorithms even in the special cases studied in [16]. Some recent work on the characterization of solutions of this equation by Emre and Silverman [17] may offer a better approach.

8. Conclusions

In this paper, we have developed a new, more general solution to the regulator problem [1] which does not involve computing Smith forms of polynomial matrices. In comparing our solution with Pernebo's solution [2], the comparison is basically between Proposition 2 and Theorem 4. Pernebo's solution is given in terms of the existence of Ω-generalized polynomials while our conditions are in terms of the existence of polynomial matrices. Also, in a general sense, his solution is our problem formulation as pointed out by the Remark following Proposition 2.

We have solved an example and shown how the order of the controller depends on the selection of K, in particular on V and V_o. We have demonstrated a lower order controller using our frequency domain formulation than could be constructed using the presently available state space formulations. This is to be expected however and is a consequence of using a natural problem formulation and not having to introduce observers or state space extensions. These techniques, although useful for solving problems involving output feedback, never allowed any real progress to be made regarding understanding of control using output feedback. Perhaps this understanding is forthcoming now that a more proper problem formulation has been found.

We have discussed several new computational algorithms and shown how they can be applied to the solution of RP. A serious deficiency at the present time is no general computational approach to the solution of the of the equation $AX + YB = C$. When some progress has been made on this equation the synthesis of linear multivariable regulators in the frequency domain will perhaps become routine.

There are other problems of course which must be considered before really useful design procedures can be developed. An important practical problem is how to achieve robustness with respect to closed-loop stability. This problem involves determining what the major uncertainties will be in any particular system and then to design loop gains that will be robust with respect to these uncertainties.

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