Title:
LDPC Code-Design for the Relay Channel

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Abstract:
We propose LDPC code designs for the half-duplex relay channel. Our designs are derived from the information theoretic random coding scheme for decode-and-forward relaying. An important advantage of our scheme is that it is built entirely of single-user codes. The optimization of relay LDPC code profiles presents unique challenges, which are met by modifying the density evolution algorithm by introducing additional constraints for relaying. To speed up optimization, we use a Gaussian approximation of density evolution that converts the infinite dimensional code profile optimization into a simple linear programming problem. The thresholds of the discovered relay code profiles are a fraction of a dB from the achievable lower bound for decode-and-forward relaying.

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***Abstract*— We propose LDPC code designs for the half-duplex relay channel. Our designs are derived from the information theoretic random coding scheme for decode-and-forward relaying. An important advantage of our scheme is that it is built entirely of single-user codes. The optimization of relay LDPC code profiles presents unique challenges, which are met by modifying the density evolution algorithm by introducing additional constraints for relaying. To speed up optimization, we use a Gaussian approximation of density evolution that converts the infinite dimensional code profile optimization into a simple linear programming problem. The thresholds of the discovered relay code profiles are a fraction of a dB from the achievable lower bound for decode-and-forward relaying.**

I. Introduction

Recent years have seen an explosion of interest in the field of relay coding. In this paper, we present LDPC code designs for the decode-and-forward half-duplex relay protocol. Our approach is inspired by the information theoretic random coding scheme. To translate the random coding scheme into practice, we employ LDPC codes, which are known for their capacity-approaching performance. The result is a coding scheme with noise thresholds a fraction of a dB of the theoretical achievable limit. The steps leading from information theory to practical codes are extremely general, and can be used for a variety of channels in network information theory.

The following are the main contributions of this paper. First, we present a coding theoretic interpretation of the information theoretic random coding scheme for decode-and-forward relaying. The coding scheme involves unique challenges in constituent code design. Although the half-duplex relay channel is a combination of a broadcast and a multiple-access channel, we show that a coding scheme consisting exclusively of single-user codes can achieve the decode-and-forward bound. Next, we judiciously make simplifications to reduce complexity of encoding and decoding the constituent codes without significantly compromising performance. We choose LDPC constituent codes because they can be optimized for a variety of channels using the density evolution algorithm, and optimized codes have thresholds within a fraction of a dB from the theoretical limit. Although the constituent codes in the relay coding scheme are all single-user codes, some of these codes must be jointly optimized. Our third contribution is the derivation of relationships between the profiles of constituent codes. These relationships act as additional constraints in the profile optimization problem that uses density evolution. In the optimization of constituent LDPC codes, several useful simplifications that can be made in implementing density evolution for single-user links (such as assuming concentrated check node distributions) no longer remain valid. The resulting increase in the complexity of density evolution poses a significant challenge. To reduce the complexity of density evolution, our final contribution is to modify the Gaussian approximation of density evolution [1] to reduce the infinite dimensional problem of tracking densities to a one-dimensional problem that is readily addressed with linear programming tools.

The information theoretic relay channel was first studied by van der Meulen [2]. Shortly afterwards, several fundamental capacity results on relaying were published in [3]. After the initial interest, however, the idea of relaying received little attention for nearly two decades. Recent years have seen a renewed interest in relaying in the context of wireless networks. In [4], [5], the feasibility of user cooperation in a wireless network was demonstrated by an information theoretic exposition of the gains, and a practical CDMA implementation. New information theoretic results were found in [6], [7], [8], [9]. The study of relay protocols and their outage analysis in fading environments was done in [10], [11]. It is worth noting that several of the above research contributions have been based on the premise of half-duplex relaying [8], [9], [12], [11], [10].

With significant advances in technology over the past two decades, the promise of relaying is very real. A large body of research is currently geared towards developing practical user-cooperation schemes to harvest the gains predicted by information theory. Solutions in this direction include [5], [13], [12], [14], [15], [16], [17], [18], [19], [20].

The remainder of this paper is organized is follows.

II. System description

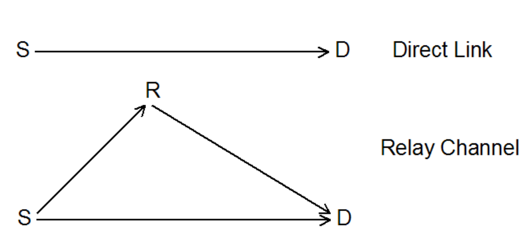

Fig. 1. Direct and relay channels.

In the relay channel (see Fig. 1), the source (S) sends data to the destination (D), and in doing so it is aided by the relay (R). In the particular case of a half-duplex relay channel, the relay

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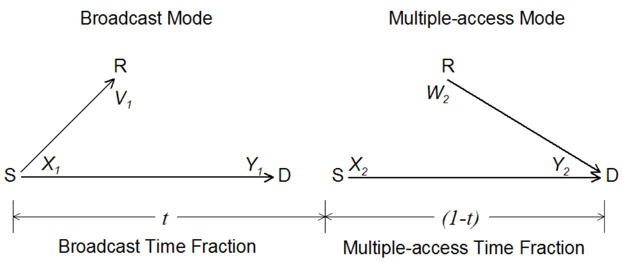

Fig. 2. Half-duplex relay modes.

cannot transmit and receive simultaneously in the same band.[1] We concentrate on time-division half-duplex relaying, where communication takes place over two time slots of (normalized) durations t and $\bar{t} = (1 - t)$. In the first time slot, S transmits information, which is received by both R and D. We call this the broadcast (BC) mode of communication. In the second time slot, both S and R transmit information to D. We refer to this as the multiple-access (MAC) mode. These two modes are depicted in Fig. 2. In the rest of this paper, whenever we mention the relay channel, we will be implicitly referring to the half-duplex relay channel described above.

We use X, V, W and Y to denote the source transmitted signal, the relay received signal, the relay transmitted signal, and the destination received signal respectively (see Fig. 2). Subscripts 1 and 2 in conjunction with the channel inputs and outputs denote the BC and MAC modes respectively.

We normalize transmitted power with noise power, which is assumed to be the same at each receiver. An average global transmission power constraint is imposed on the nodes, denoted by the symbol Θ,

$$\Theta : tP_{S_{BC}} + \bar{t}(P_{S_{MAC}} + P_{R_{MAC}}) \leq P \tag{1}$$

where P represents the total system transmission power. Since noise power is normalized to unity, P is also the equivalent relay channel SNR in our plots. $P_{S_{BC}}$ is the source transmission power in the BC mode, $P_{S_{MAC}}$ is the source transmission power in the MAC mode, and $P_{R_{MAC}}$ is the relay transmission power in the MAC mode.

We compare relaying with direct communication. For fair comparison, we require that the sum of the source and relay transmission powers in the relay channel equals the source transmission power for the direct link. Therefore, for the direct link, the above power constraint (1) must hold when we set $t = 1$.

For the purpose of code design, we consider the case of time-invariant 1-D AWGN channels. An extension to circularly symmetric AWGN channels is straightforward. The distance between S and D is normalized to unity, and R is assumed to lie on the straight line joining S and D. The relay position, denoted by d, represents its distance from the source. The collinearity of S, R and D is not used in deriving any of our results, but it enables a simple characterization of the relay position. The SD channel gain is given by $\gamma_{SD} = 1$, the SR gain is $\gamma_{SR} = \frac{1}{d^\alpha}$, and the RD gain is $\gamma_{RD} = \frac{1}{(1-d)^\alpha}$, where α is the channel attenuation exponent. We will use $\alpha = 2$ in this paper.

Although we assume AWGN channels, our codes can be used on fading channels also, if global channel state information (CSI) is available at all communicating nodes. The availability of CSI can be ensured by a feedback mechanism.

[1]Most current radios are half-duplex. Full-duplex operation requires shielding and accurate interference cancellation between transmitted and received signals that differ in power typically by more than 100 dB, which is difficult to achieve in practice.

III. Structure of Relay Channel Codes

In this section, we describe the structure of the decode-and-forward coding scheme in detail and motivate the simplifications leading to the LDPC code design discussed in the next section.

A. *Achievable Rate and Code Structure*

For the general half-duplex relay channel, the decode-and-forward protocol achieves the following rate [8]

$$R_{DF} = \sup_{0\leq t\leq 1} \min\{tI(X_1;V_1) + \bar{t}I(X_2;Y_2|W_2), \\ tI(X_1;Y_1) + \bar{t}I(X_2,W_2;Y_2)\}, \tag{2}$$

The coding scheme which achieves the above rate sends a total of N symbols, such that a fraction t of them are sent first in the BC mode and the remaining $\bar{t}N$ symbols are sent in the MAC mode of the relay channel.[2] After all the N symbols have been transmitted, the destination proceeds to decode using the received signals from both modes.

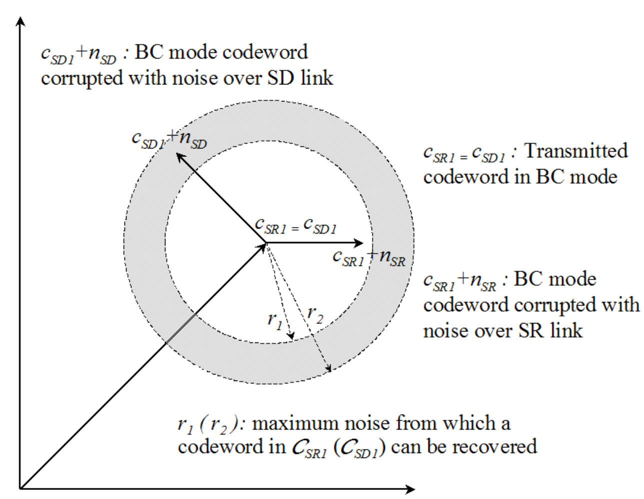

Fig. 3. Coding for decode-and-forward. S transmits codeword $c_{SR_1} = c_{SD_1}$ in the BC mode. The inner circle denotes noise from which a codeword in $\mathcal{C}_{SR_1}$ can be recovered. The received signal at R is inside this circle, therefore R can decode correctly, but the signal at D is not decodable. In the MAC mode, S and R send additional information to D about c_{SR_1}. Consequently, the same codeword c_{SD_1} now effectively belongs to a lower rate code $\mathcal{C}_{SD_1}$, and can be recovered from more noise (shown by the outer circle) at D.

[2]The codeword length, tN, may not be an integer. However, the loss of accuracy is negligible for large N, therefore we will treat it as an integer in the rest of this paper.

The information bits at the source are first divided into two independent parts (ω, ν). In the BC mode, the source encodes ω using the tN symbol-long codeword $c_{SR_1} \in \mathcal{C}_{SR_1}$ with a rate of $I(X_1;V_1)$ bits per channel use. The relay can decode c_{SR_1} reliably but the destination cannot, since the capacity of the source-destination (SD) link is less than that of the source-relay (SR) link. It can be seen from (2) that decode-and-forward relaying outperforms direct communication only if the SR link is better than the SD link ($I(X_1;V_1) > I(X_1;Y_1)$).

In the MAC mode, both relay and source transmit albeit different kinds of information. The destination already has $tNI(X_1;Y_1)$ bits of information, in the form of the noisy codeword c_{SR_1}, and needs an additional $tN(I(X_1;V_1) - I(X_1;Y_1))$ bits to reliably decode the codeword c_{SR_1} using the idea of Slepian-Wolf coding [21]. These additional bits needed to decode c_{SR_1} are provided by a codeword $c_{RD_2} \in \mathcal{C}_{RD_2}$ of rate $\frac{t}{\bar{t}}(I(X_1;V_1)-I(X_1;Y_1))$ jointly transmitted by the source and the relay in the MAC mode. Note that c_{RD_2} communicates no new information, only reiterates information that was already there in the noiseless codeword c_{SR_1}. The codeword c_{RD_2} is decoded by the destination before it attempts to decode the BC mode received codeword c_{SR_1}. For the purpose of decoding c_{SR_1}, the destination can treat it as a codeword c_{SD_1} of a rate $I(X_1;Y_1))$ codebook $\mathcal{C}_{SD_1}$, lower rate because of the side information communicated via c_{RD_2}. To further clarify, we explain with an example of binary codes.

In the binary case, c_{SR_1} is a codeword that satisfies a set of parity check equations that define the codebook $\mathcal{C}_{SR_1}$. This codeword is received at both R and D with different amounts of noise. The relay can recover the codeword from noise by decoding, but the noise at the destination is overwhelming. Now, the destination receives a codeword c_{RD_2} in MAC mode that it can decode. The information bits in c_{RD_2} represent additional parity information for c_{SR_1}. However, there is one important distinction between the parity that characterizes the structure of a binary code, and the additional parity bits that are communicated in MAC mode. Whereas parity in a binary code usually signifies collections of bits in the codeword that sum to zero under GF(2) addition, the additional parity bits communicated in MAC mode instead convey the GF(2) sum of suitably chosen collections of bits. The reason is that the codeword c_{SR_1} has already been transmitted and received in BC mode, and we cannot impose that it should satisfy a new set of parity check equations. However, we can tell the destination the GF(2) sum (which may be 0 or 1) of certain collections of bits. This information, for the purpose of decoding, is no different from a fresh set of parity equations satisfied by the codeword. This last fact also explains the apparent anomaly that two different rate codes $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$ always have the same codeword. Fig. 3 explains the process of decoding c_{SR_1} using side information from c_{RD_2}.

The second part of the information, ν, is also sent in the MAC mode using a codeword $c_{SD_2} \in \mathcal{C}_{SD_2}$ to utilize the remaining capacity of the multiple access channel constituted by the source and relay as the two transmitters and the destination is the receiver. This information is sent by the source alone, since the relay does not have access to new information. The amount of new information is bounded by the capacity region of the multiple-access channel, and is smaller of the $\bar{t}NI(X_2;Y_2|W_2)$ and $\bar{t}NI(X_2,W_2;Y_2) - tN(I(X_1;V_1) - I(X_1;Y_1)$ bits.

It is known [22], [23] that the corner points of the capacity region of the two-user multiple-access channel are achievable by successive decoding (also known as onion peeling, stripping, and superposition coding) with a pair of codes. However, to achieve a general point on the capacity region requires either joint decoding, time-sharing [24] or rate-splitting using at least three codes [25]. We will argue in Section III-B.3 that in our framework, the source and relay rates in MAC mode correspond to a point on the multiple-access capacity region which can be achieved by a pair of single-user codes, one each from the source and the relay.

B. *Code Design Simplifications*

In this section, we simplify the design of the above decode-and-forward code by analyzing the achievable rate R_{DF}. The main observations are three-fold. First, the use of binary codes is near-optimal since the biggest gain from relaying is at low SNRs, where binary signaling performs close to Gaussian signaling. Second, the codebooks used by the source and the relay can be either completely correlated or completely independent of each other, without any significant rate loss as compared to optimally correlated codebooks. Last, the source and relay rates in the MAC mode correspond to a point on the capacity region of the multiple-access channel that can be achieved by a successive interference canceling decoder at the destination.

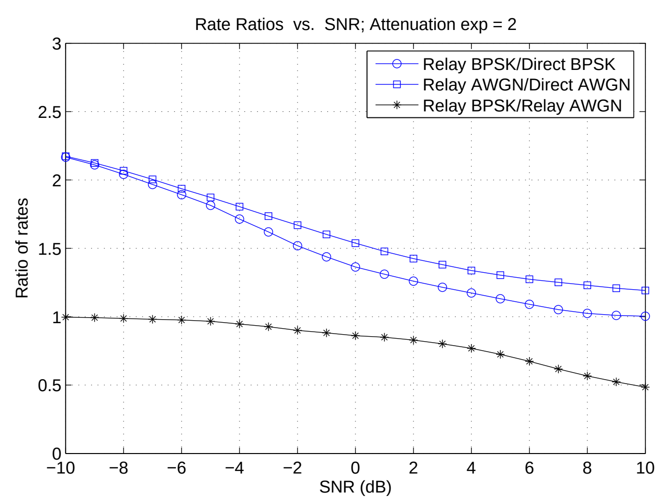

Fig. 4. Ratios of Rates vs. SNR(dB) for direct and relay channels.

1) Binary Codes: At high SNRs, the achievable rate of decode-and-forward relaying on Gaussian channels exceeds that of direct communication only by a constant independent of the SNR (proved in Appendix II). Therefore, at high SNR, the ratio of the decode-and-forward rate to the single-user capacity approaches unity. The relaying gain is, thus, significant only at low SNRs. Fig. 4 compares the rate gain in comparison to direct communication for Gaussian as well as BPSK signaling (rate expressions given in Appendix I) for the relay position of $d = 0.5$. The rate of BPSK relaying is also plotted as a fraction of the rate of AWGN relaying at different SNRs. The conclusion is that relaying is most beneficial in the low to intermediate SNR range, where BPSK achieves a significant fraction of AWGN relay rate. The above observation justifies the use of binary codes for the half-duplex relay channel.

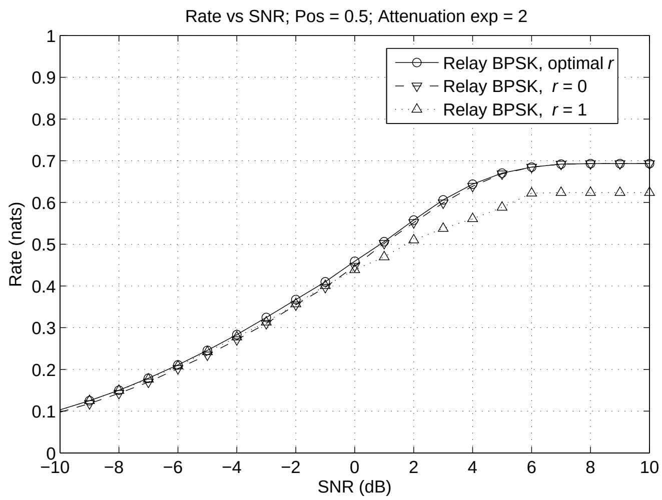

Fig. 5. Achievable rates for the relay channel with $r = 0, 1$ and the optimum value of r.

2) Codebook Correlation: As given in Appendix I, the codebooks used by the source and relay in MAC mode are correlated with a correlation of r in the MAC mode. The correlation stems from the fact the both source and relay have common information since the relay has decoded the ω portion of the total information in the BC mode. Motivated by the fact that is easy to design codebooks for the two extreme cases of $r = 0, 1$, where the source and the relay codebooks are independent and identical respectively, we investigate the suboptimality of using $r = 0, 1$ instead of the optimal correlation. Intermediate values of $r \in (0, 1)$ require introducing a correlation structure which turns out to be challenging for binary input alphabets. The key observation made by us in this context is that the achievable rate is relatively insensitive to r, and in particular, if we can choose the better of $r = 0, 1$, then the rate loss is negligible. This is shown in the plot in Fig. 5. The insensitivity of the achievable rate to r at high SNRs can be shown analytically [26]. From the empirical results of Fig. 5, we are also led to conjecture that $r = 1$, with appropriate power control, approaches the optimum achievable rate at low SNRs. Guided by these observations, we reduce the code design problem to one of designing codes for the two cases of $r = 0, 1$.

As a consequence of the above observation, two distinct coding schemes emerge for $r = 0, 1$ that differ only in MAC mode. For $r = 1$, the source and the relay send identical information in MAC mode, consequently the source sends nothing new and there is no code $\mathcal{C}_{SD_2}$. For $r = 0$, the source and the relay send independent information in MAC mode, meaning that the source sends only new information through the codeword c_{SD_2}, whereas the relay alone sends c_{RD_2} to help the destination in decoding the BC mode codeword c_{SR_1}.

3) Successive Decoding in MAC Mode: In our discussion on the information theoretic relay coding scheme of Section III-A, we mentioned that in MAC mode, the source transmits additional information ν at a rate permitted by the capacity of the multiple-access channel. This rate is the smaller of $\bar{t}NI(X_2; Y_2|W_2)$ and $\bar{t}NI(X_2, W_2; Y_2) - tN(I(X_1; V_1) - I(X_1; Y_1))$ bits. Operation at the corner point of the capacity region corresponds to the equality of the two aforementioned rates, which is the same as equating the two arguments of the $\min$ function in (2)

$$\begin{aligned} & tI(X_1; V_1) + \bar{t}I(X_2; Y_2|W_2) \\ = \; & tI(X_1; Y_1) + \bar{t}I(X_2, W_2; Y_2). \end{aligned} \tag{3}$$

The above equality is easy to prove with the following power constraint

$$\begin{aligned} P_{S_{MAC}} + P_{R_{MAC}} & \leq P, \\ P_{S_{BC}} & \leq P, \end{aligned} \tag{4}$$

which is a special case of (1). When (4) is true, the input distributions, and consequently the mutual information terms are independent of t. Noting that $I(X_2; Y_2|W_2) \leq I(X_2, W_2; Y_2)$, and for decode-and-forward relaying $I(X_1; V_1) \geq I(X_1; Y_1)$, we conclude that the best choice of t equates the two terms in the $\min$ function.

Our empirical evaluations show that (3) holds even for the more general power constraint (1). This also makes sense intuitively, because an inequality in (3) would mean that at least one link carries more information than is necessary to meet the same achievable rate. However, even if the equality does not hold, we can choose rates

$$\begin{aligned} R_{SR_1} &= I(X_1; V_1), \\ R_{SD_1} &= I(X_1; Y_1), \\ R_{SD_2} &= \min\{I(X_2; Y_2|W_2), \\ & \quad I(X_2, W_2; Y_2) - \tfrac{t}{\bar{t}}(I(X_1; V_1) - I(X_1; Y_1))\}, \\ R_{RD_2} &= \tfrac{t}{\bar{t}}(I(X_1; V_1) - I(X_1; Y_1)), \\ R_2 &= R_{SD_2} + R_{RD_2}, \end{aligned} \tag{5}$$

which clearly satisfy

$$\text{Achievable rate} = tR_{SR_1} + \bar{t}R_{SD_2} = tR_{SD_1} + \bar{t}R_2. \tag{6}$$

Note that the coding scheme remains completely unchanged. The above choice of rates guarantees that we operate at a point on the multiple-access capacity region which can be achieved by a pair of single-user codes, one each from the source and the relay. We will use the above notation for the rates (5) to describe our LDPC coding scheme.

IV. LDPC CODE DESIGN

We are now ready to present the binary LDPC coding schemes for $r = 0, 1$. The two schemes differ only in the MAC mode. We first describe the full scheme for $r = 1$, and then explain what is different for $r = 0$. We use the same names for the component LDPC codes as we did in describing the general relay coding scheme in Section III-A, to ensure that the role of each component LDPC code is clear. The reader can also directly apply the interpretation of Fig. 3 to the LDPC coding schemes.

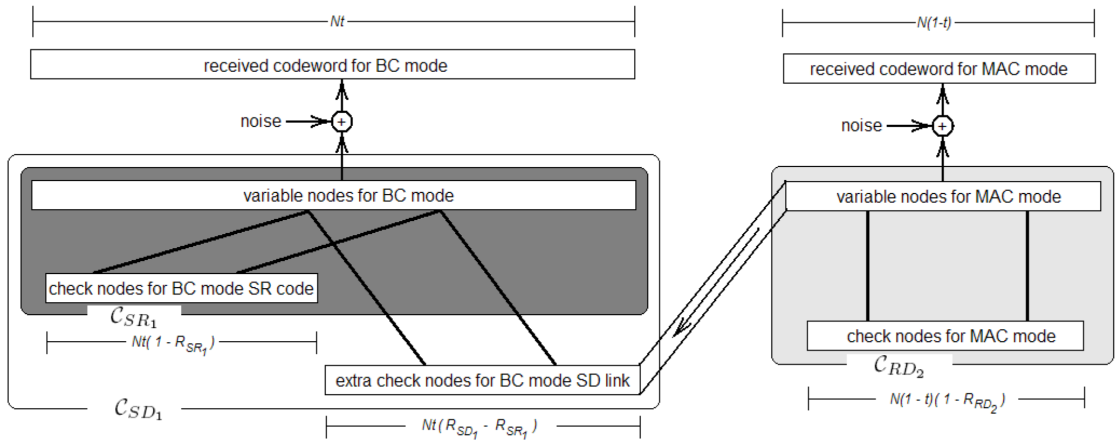

Fig. 6. LDPC code structure for $r = 1$.

A. LDPC code design for $r = 1$

In the BC mode, the source S uses an LDPC code $\mathcal{C}_{SR_1}$ with an $Nt \times Nt(1 - R_{SR_1})$ parity check matrix to transmit information. The relay decodes this codeword. The destination D is unable to decode this codeword, however, it stores it for future decoding. In the MAC mode, S and R use the BC mode codeword as the basis for cooperation. Both nodes multiply the BC mode codeword with an $Nt \times Nt(R_{SR_1} - R_{SD_1})$ matrix to generate $Nt(R_{SR_1} - R_{SD_1})$ additional parity bits (side information to help the destination). These additional parity bits are then channel coded using an LDPC code $\mathcal{C}_{RD_2}$ with an $N\bar{t} \times N\bar{t}(1 - R_{RD_2})$ parity check matrix and transmitted from both S and R in a phase synchronized manner to the destination. The bits communicated by $\mathcal{C}_{RD_2}$, in addition to the parity bits of the original code $\mathcal{C}_{SR_1}$, form a code $\mathcal{C}_{SD_1}$ of lower rate R_{SD_1} that is decodable by D. Fig. 6 shows the LDPC code structure for $r = 1$.

B. LDPC code design for $r = 0$

The BC mode remains unchanged. In the MAC mode, the source and the relay transmit independent information. The relay first generates additional parity bits from the BC mode codeword by multiplying with a $Nt \times Nt(R_{SR_1} - R_{SD_1})$ matrix (same as for $r = 1$). It then uses an LDPC code $\mathcal{C}_{RD_2}$ with an $N\bar{t} \times N\bar{t}(1 - R_{RD_2})$ parity check matrix to encode and transmit the additional parity information in MAC mode. The information carried by this codeword enables decoding of the BC mode codeword at the end of MAC mode. The source, in the MAC mode, uses an LDPC code $\mathcal{C}_{SD_2}$ with an $N\bar{t} \times N\bar{t}(1 - R_{SD_2})$ parity check matrix to send new information to the destination. At the end of the MAC mode, D uses successive decoding to recover both the additional parity information and the new source information. In other words, information received over the channel with better SNR is decoded first, treating both noise and other information as noise. After decoding, the better signal is subtracted out and the remaining information is then decoded in the presence of noise alone. Finally, the additional parity bits received in the MAC mode are used to decode the received BC mode codeword at D. Fig. 7 shows a block diagram of the overall coding scheme for $r = 0$.

V. Code Profile Optimization

The main design challenge in the proposed coding schemes is that of designing the profiles of the two codes $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$ jointly. We start with a brief introduction to LDPC codes, following which, we will present a solution to the joint code profile optimization problem.

A. Introduction to LDPC codes

A binary LDPC code is a linear block code with a sparse binary parity-check matrix. This $n \times m$ parity check matrix can be equivalently represented by a bipartite graph with n variable nodes corresponding to rows (bits in the codeword) and m check nodes corresponding to columns (parity check equations). A one in a certain row and column of the parity check matrix corresponds to a single edge between the corresponding variable and check node in the graph, whereas a zero indicates no edge.

A regular LDPC code is one in which all nodes of the same type (variable or check) have the same degree. In an irregular code, there is no such constraint. An LDPC code is characterized by its variable and check degree distributions (or profiles) $\lambda = [\lambda_2 ... \lambda_{d_v}]$ and $\rho = [\rho_2 ... \rho_{d_c}]$ respectively, where λ_i (ρ_i) denotes the fraction of *edges* connected to a variable (check) node of degree i, and d_v (d_c) is the maximum number of edges connected to any variable (check) node. All LDPC codes with the same λ and ρ are said to belong to the same *ensemble*. In their seminal work [27], Richardson and Urbanke showed that nearly all codes of long block length in an ensemble have almost identical error performance, thereby justifying the representation of entire ensembles instead of individual codes. An equivalent, and often more convenient representation of LDPC code profiles

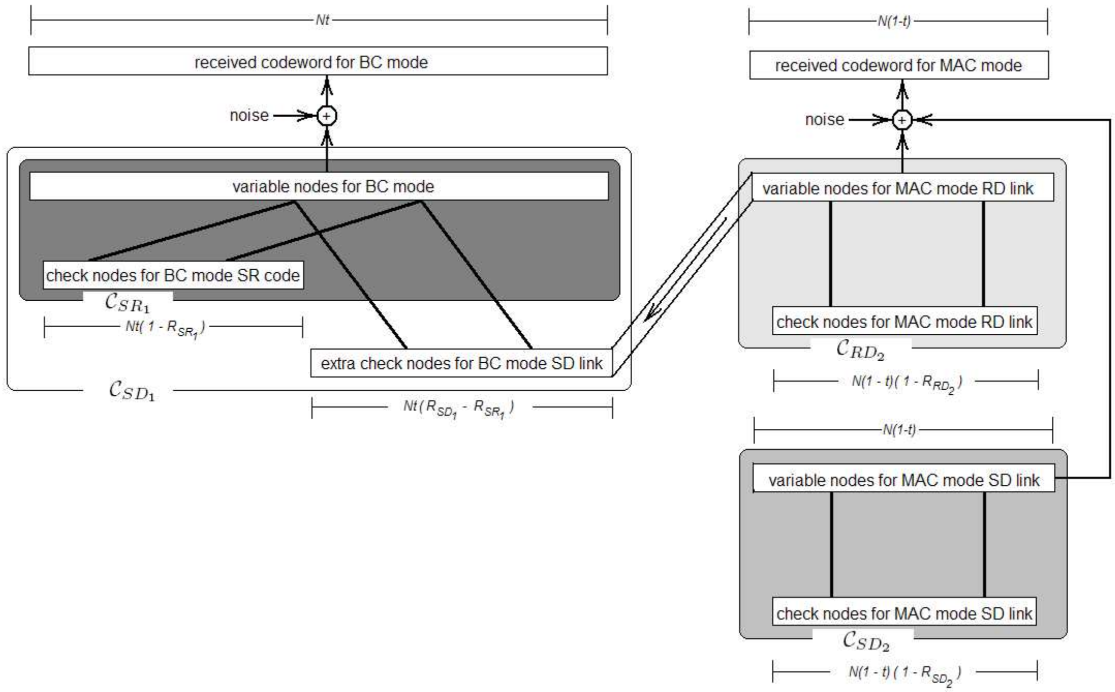

Fig. 7. LDPC code structure for $r = 0$.

uses generating functions

$$\lambda(x) = \sum_{i=2}^{d_v} \lambda_i x^{i-1} \quad (7)$$

$$\rho(x) = \sum_{i=2}^{d_c} \rho_i x^{i-1} \quad (8)$$

The design rate of an ensemble is given in terms of $\lambda(x)$ and $\rho(x)$ by

$$R = 1 - \frac{m}{n} = 1 - \frac{\int_0^1 \rho(x)dx}{\int_0^1 \lambda(x)dx} \quad (9)$$

Although it is common to represent the variable and check degree distributions from an edge perspective as above, an equivalent representation from a node perspective is sometimes more convenient. To distinguish, we will use the superscript N for node, and call $\lambda^N(x)$ the variable *node* degree distribution and $\lambda(x)$ the variable *edge* degree distribution.

$$\lambda_i^N = \frac{\lambda_i / i}{\int_0^1 \lambda(x)dx} \quad (10)$$

LDPC codes can be decoded by a variety of message passing algorithms, of which we will consider only belief propagation [27]. In a message passing decoder, messages are passed iteratively from variable node to check node and back until decoding is successful or a maximum number of iterations is reached. In belief propagation, the message sent on an edge represents the a posteriori log likelihood ratio (LLR) of the variable bit connected to that edge based on received messages. The belief propagation algorithm calculates the a posteriori LLRs for each variable exactly if the graph is cycle free. Although all interesting parity check matrices have cycles, if the length of the smallest cycle is greater than $2l$ then the LLRs produced by belief propagation are exact up to l iterations. For the sake of analysis, we will assume the graph to be free of cycles that compromise the accuracy of belief propagation.

Using the message update rules at variable and check nodes, it is possible to track message densities and predict the number of erroneous bits at different iterations - a process known as density evolution. Gallager [28] was the first to observe that LDPC codes demonstrate a threshold phenomenon, i.e. in the limit of infinite block length and infinite iterations, LDPC codes within a given ensemble can be decoded with probability 1 if and only if the noise variance is below a threshold. This threshold can be numerically determined by density evolution for a broad class of channels including the BEC, BSC and BAWGN channels [27]. Consequently, it is possible to discover good codes using density evolution, i.e. search for code profiles for a given design rate, which will yield a high noise threshold.

Unfortunately, tracking entire densities over thousands of iterations is computationally intensive. To reduce the computational burden of threshold determination, Chung et al. [1] showed that by approximating the message densities as Gaussians, a fair approximation to density evolution is obtained. Gaussian approximation, therefore reduces the infinite dimensional problem of tracking entire densities to a one-dimensional problem of tracking means. We will use the Gaussian approximation to find good code profiles and to obtain approximate thresholds for relay channel codes.

A Gaussian is completely specified by its mean and variance. Moreover, as pointed out in [29], message densities satisfy a *symmetry* condition in density evolution given by $f(x) = f(-x)e^x$, where $f(x)$ is the density of the LLR message. For a Gaussian density with mean m and variance σ^2, this condition yields the relation $\sigma^2 = 2m$, which means that we only need to track the mean at every iteration. For brevity, we will present the procedure for determining the threshold without the steps in its derivation, which can be found in [1]. For a given $\rho(x)$ and $\lambda(x)$, and given channel noise variance σ_c^2, decoding succeeds in the limit of infinite blocklength and infinite iterations iff

$$r > h(s,r) \qquad \forall r \in (0, \phi(s)) \tag{11}$$

where

$$\begin{aligned} h(s,r) &= \sum_{i=2}^{d_v} \lambda_i h_i(s,r) \qquad (12) \\ h_i(s,r) &= \phi\Bigg(s + (i-1)\sum_{j=2}^{d_c} \rho_j \phi^{-1}\big(1-(1-r)^{j-1}\big)\Bigg) \end{aligned}$$

with $s = 2/\sigma_c^2$, where σ_c^2 is the channel noise variance, and

$$\phi(x) = \begin{cases} 1 - \frac{1}{\sqrt{4\pi x}} \int_R \tanh\frac{u}{2} e^{-\frac{(u-x)^2}{4x}} du & \text{, if } x > 0 \\ 1 & \text{, if } x = 0 \end{cases} \tag{13}$$

For numerical purposes, we use the following approximations [1] for the function ϕ.

$$\phi(x) \simeq \begin{cases} e^{(-0.4527x^{0.86}+0.0218)} & \text{if , } x \in [0,10] \\ \sqrt{\frac{\pi}{x}} e^{-\frac{x}{4}} \left(1 - \frac{20}{7x}\right) & \text{, if } x > 10 \end{cases} \tag{14}$$

We use (11) to find the largest σ_c^2 for which decoding is successful.

Finding the optimum degree distribution for a given rate is a search for the $(\lambda(x), \rho(x))$ pair that yields the largest noise threshold. For single-user codes, it has been empirically observed that good check node distributions are concentrated, i.e. all parity check nodes should have equal or nearly equal degrees. This has also been argued through analysis in [30], [1]. The variable node distribution, on the other hand, is not as easily characterized. Therefore, the search for a good $(\lambda(x), \rho(x))$ pair for a single user code is carried out in practice by choosing several concentrated $\rho(x)$ and finding the best $\lambda(x)$ for each, followed by picking the $\rho(x)$ for which the best noise threshold was obtained. An advantage of this approach is that the best $\lambda(x)$ for a given $\rho(x)$ can be found by linear programming. The following is a brief description of this procedure.

We are interested in finding $\lambda(x)$ which maximizes the noise threshold σ_c^2 for a given rate and $\rho(x)$. Instead, we solve the equivalent problem of finding $\lambda(x)$ which will maximize the rate for a given noise variance and $\rho(x)$, and slowly increasing the variance until the design rate is barely achievable. From (9), we see that for fixed $\rho(x)$, maximizing rate is equivalent to maximizing $\int_0^1 \lambda(x)dx = \sum_{i=2}^{d_v} \frac{\lambda_i}{i}$. The constraints are $\lambda(1) = 1$, and (11), which are both linear. This linear program with inequality constraints can be solved quickly and accurately using available optimization tools.

We conclude our brief summary of LDPC codes, density evolution and its Gaussian approximation here. The interested reader can find detailed discussions on these in [28], [31], [30], [27], [29], [1].

B. Relay code profile optimization

The novel challenge in the context of half-duplex relaying is identical for both correlations $r = 0$ and $r = 1$, and it requires building two LDPC codes $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$ that are both excellent single-user codes of disparate rates R_{SR_1} and R_{SD_1} respectively, such that the bipartite graph of $\mathcal{C}_{SR_1}$ is a subgraph of $\mathcal{C}_{SD_1}$. The LDPC codes are represented by rectangles with rounded edges in Fig. 6 and Fig. 7. Both $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$ must produce codewords of the same length tN[3]. Therefore, $\mathcal{C}_{SR_1}$ has $tN(1-R_{SR_1})$ parity check nodes, and $\mathcal{C}_{SD_1}$ has $tN(1-R_{SD_1})$ parity nodes that are a superset of those corresponding to $\mathcal{C}_{SR_1}$. As a result, the check degree distributions must satisfy a relationship given by the following simple theorem.

Theorem 5.1: If $\mathcal{C}_{SR_1}$ has a check node degree distribution $\rho^N_{SR_1}(x) = \sum_{i=2}^{d_{c1}} \rho^N_{SR_1,i} x^{i-1}$, and $\mathcal{C}_{SD_1}$ has a check node degree distribution $\rho^N_{SD_1}(x) = \sum_{i=2}^{d_{c2}} \rho^N_{SD_1,i} x^{i-1}$, then the following relationships must hold

$$\begin{aligned} &\frac{(1-R_{SR_1})}{(1-R_{SD_1})} \rho^N_{SR_1,i} \leq \rho^N_{SD_1,i} \\ &\forall i = 2,3,...,\max(d_{c1}, d_{c2}) \end{aligned} \tag{15}$$

Proof: *The number of check nodes of degree i in $\mathcal{C}_{SR_1}$ is $Nt(1-R_{SR_1})\rho^N_{SR_1,i}$, whereas the number of check nodes of degree i in $\mathcal{C}_{SD_1}$ is $Nt(1-R_{SD_1})\rho^N_{SD_1,i}$. Since $\mathcal{C}_{SD_1}$ is formed by adding check nodes to $\mathcal{C}_{SR_1}$, the latter must equal or exceed than the former, which gives us the result.* ◊

The relationship between variable nodes and their degree distributions is somewhat more involved. We state it in the following theorem.

Theorem 5.2: If $\mathcal{C}_{SR_1}$ has a variable node degree distribution $\lambda^N_{SR_1}(x) = \sum_{i=2}^{d_{v1}} \lambda^N_{SR_1,i} x^{i-1}$, and $\mathcal{C}_{SD_1}$ has a variable node degree distribution $\lambda^N_{SD_1}(x) = \sum_{i=2}^{d_{v2}} \lambda^N_{SD_1,i} x^{i-1}$, then the following relationships must hold

$$\begin{aligned} &\sum_{i=j}^{\max(d_{v1},d_{v2})} \lambda^N_{SR_1,i} \leq \sum_{i=j}^{\max(d_{v1},d_{v2})} \lambda^N_{SD_1,i} \\ &\forall j = 2,3,...,\max(d_{v1}, d_{v2}) \end{aligned} \tag{16}$$

Proof: *The proof is a consequence of Hall's marriage theorem. The statement of the Hall's marriage theorem is as follows: Let $S = \{S_1, S_2, ..., S_m\}$ be a finite collection of finite sets. Then there exists a system of distinct representatives of S if and only if the following condition holds for any $T \subseteq S$*

$$|\cup T| \geq |T| \tag{17}$$

where $|.|$ represents cardinality.

Let us number the variable nodes $1, 2, ..., m$ in arbitrary order. Let d_i be the degree of node i in $\mathcal{C}_{SR_1}$, and let S_i denote

[3] We will ignore rounding effects in obtaining codewords of integer length.

the set of variable nodes in $\mathcal{C}_{SD_1}$ with degree $\geq d_i$. Note that in this way, if $d_i = d_j$, then $S_i = S_j$. When more check nodes and their connecting edges are added to $\mathcal{C}_{SR_1}$ to form $\mathcal{C}_{SD_1}$, then a variable node of degree k in $\mathcal{C}_{SR_1}$ becomes a node of degree $\geq k$ in $\mathcal{C}_{SD_1}$. In other words, node i in $\mathcal{C}_{SR_1}$ would be mapped to a node in S_i. Since both $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$ have the same variable nodes, there must exist a one-one mapping between variable node degrees in the two graphs. In other words $S = \{S_1, S_2, ..., S_m\}$ must have a system of distinct representatives. The statement of our theorem now follows from (17). $\diamondsuit$

As before, we have the constraints

$$\lambda_{SR_1}(1) = \lambda_{SD_1}(1) = 1 \tag{18}$$

An apparent difference from the single-user case is that now we have two equivalent noise variances that are related by the relative channel strengths.

$$\sigma^2_{SD_1} = \frac{\gamma_{SR}}{\gamma_{SD}} \sigma^2_{SR_1} \tag{19}$$

where $\sigma^2_{SD_1}$ is the noise variance for $\mathcal{C}_{SD_1}$ and $\sigma^2_{SR_1}$ is the noise variance for $\mathcal{C}_{SR_1}$. We treat only $\sigma^2_{SD_1}$ as a variable, since it completely determines $\sigma^2_{SR_1}$.

The set of inequality constraints (11) must be satisfied for both $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$, as in the single-user case.

With the above constraints, it is possible to perform density evolution and search for good degree distributions for $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$ for a given pair of rates. However, this search is computation intensive, more so because now we must find a pair of codes instead of one, making the search space much larger. In the next section we modify the Gaussian approximation procedure [1] to find codes for the relay channel with less complexity.

C. Gaussian Approximation for the Relay Channel

Our aim is to use the Gaussian approximation to pose the search for good code profiles as a linear program using the means of the messages instead of entire densities. However, there are several important differences with the single-user case, which make this problem interesting and challenging.

First, usually the rates of $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$ are significantly different. Consequently, the average check degrees for the two codes to be individually good are also not close. As a result, we do not have the luxury of assuming that ρ_{SR_1} and ρ_{SD_1} are both concentrated, which was the case for single-user LDPC codes. However, in this paper, we will search for good codes with ρ_{SR_1} concentrated at a single degree j and ρ_{SD_1} supported on two degrees i and j only with $i < j$.

$$\rho_{SR_1}(x) = x^{j-1} \tag{20}$$

$$\rho_{SD_1}(x) = \frac{ax^{i-1} + bx^{j-1}}{(a+b)} \tag{21}$$

$$a = i(R_{SR_1} - R_{SD_1})$$

$$b = j(1 - R_{SR_1})$$

There is no theoretical justification for reducing our search space in this way, but the thresholds of the codes that we discover in this way indicate that good code pairs can be obtained with the above constraint.

Second, (16) is not linear in the variable edge degrees but is linear in the node degrees. Fortunately, the other constraints remain linear when stated in terms of node degrees instead of edge degrees. Therefore, it is natural to pose the code profile optimization in terms of the distribution of variable node degrees.

Third, for single-user LDPC codes, the optimization was easily posed as a rate maximization problem. But now, there are two code rates. To tackle this, we fix one of the LDPC code rates to its design rate (we fix R_{SR_1}), and maximize the other (R_{SD_1}). By fixing R_{SR_1}, we obtain the following equality constraint.

$$R_{SR_1} = R_{SR_1} = 1 - \frac{\int_0^1 \rho_{SR_1}(x)dx}{\int_0^1 \lambda_{SR_1}(x)dx} \tag{22}$$

In the cause of clarity, we summarize all the constraints and the objective of our linear program below. We assume that the check degree distributions are given by (20) and (21). Our objective is to find the maximum noise variance $\sigma^2_{SD_1}$, for which the achievable R_{SD_1} equals or exceeds R_{SD_1}. As in the single-user case, we equivalently maximize the rate R_{SD_1} for a given $\sigma^2_{SD_1}$. Since,

$$R_{SD_1} = 1 - \frac{\int_0^1 \rho_{SD_1}(x)dx}{\int_0^1 \lambda_{SD_1}(x)dx} \tag{23}$$

$$= 1 - \left(\int_0^1 \rho_{SD_1}(x)dx\right)\left(\sum_{i=2}^{d_v} i\lambda_i^N\right), \tag{24}$$

our objective of maximizing R_{SD_1} is equivalent to that of minimizing $\sum_{i=2}^{d_v} i\lambda_i^N$, where d_v is the maximum allowed variable node degree. The following are the constraints, stated in terms of variable *node* degrees.

$$\lambda^N_{SR_1}(1) = \lambda^N_{SD_1}(1) = 1 \tag{25}$$

Relation (11) imposes the following constraints

$$\sum_{i=2}^{d_v} i\lambda^N_{SR_1,i}(h_i(s,r) - r) < 0 \quad \forall r \in (0, \phi(s)) \tag{26}$$

$$\sum_{i=2}^{d_v} i\lambda^N_{SD_1,i}(h_i(s,r) - r) < 0 \quad \forall r \in (0, \phi(s)). \tag{27}$$

Equation (16) relating $\lambda^N_{SR_1}(x)$ and $\lambda^N_{SD_1}(x)$ must be satisfied. Finally, (22) yields the following constraint

$$R_{SR_1} = 1 - \left(\int_0^1 \rho_{SR_1}(x)dx\right)\left(\sum_{i=2}^{d_v} i\lambda^N_{SR_1,i}\right) \tag{28}$$

VI. Numerical results

The numerical results are presented in two parts - asymptotic noise thresholds of the codes, and BER simulation results.

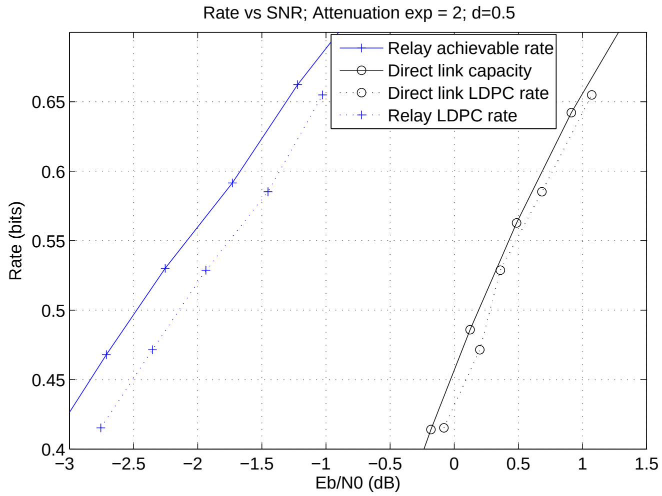

Fig. 8. Rate vs. E_b/N_0 (d=0.5,d_v=25). Theoretical limits and LDPC performance based on thresholds.

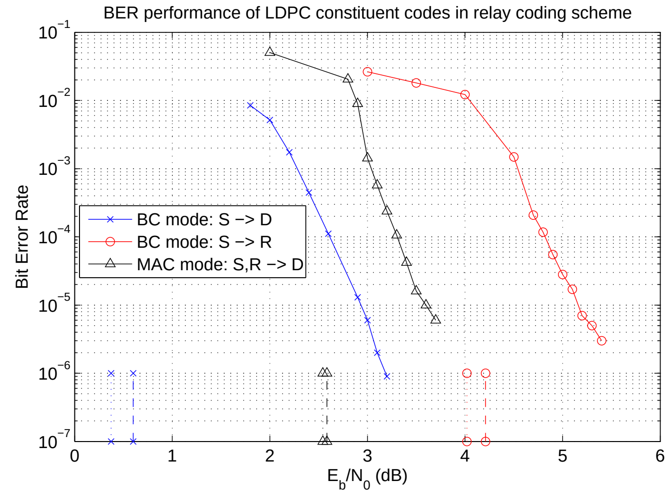

Fig. 9. BER vs. E_b/N_0 for LDPC codes which take part in the relay coding scheme ($P = 0dB, r = 1$). The vertical lines indicate theoretical limits and noise thresholds of the codes.

A. Asymptotic noise thresholds

For comparison with theoretical bounds, we present asymptotic noise thresholds for the entire coding scheme. These thresholds are a function of the thresholds of all the constituent LDPC codes in the relay coding scheme, and capture the total power necessary to achieve negligible error probability in the limit of infinite blocklength and infinite iterations. The profiles and thresholds of individual codes are obtained as follows. The code $\mathcal{C}_{RD_2}$ is a single-user code when $r = 1$, for which we use a good degree profile from [32]. We make use of single-user codes in the MAC mode $r = 0$ also. Although these codes are not optimized for the multiple-access channel, their performance in the presence of MAC interference is close to their single-user noise threshold, indicating that it is acceptable to use single-user codes in MAC mode (see Fig. 10). For the codes $\mathcal{C}_{SR_1}$ and $\mathcal{C}_{SD_1}$, we jointly optimize the degree profiles using the Gaussian approximation of density evolution outlined in Section V-C.

We plot noise thresholds for the $r = 1$ coding scheme in Fig. 8. For a maximum variable node degree $d_v = 25$, the thresholds are less than 0.4 dB from the decode-and-forward bound. For $d_v = 100$, the gap reduces further to approximately 0.1 dB. A large variable node degree is necessary for obtaining noise thresholds close to capacity if $\mathcal{C}_{SD_1}$ has a low rate. The reader is reminded that the thresholds of some of the component codes have been calculated using the Gaussian approximation, therefore they are not precise. Since it has been shown that the error in using the Gaussian approximation is small [1], we are convinced that the profiles are asymptotically good. However, it is hard to accurately estimate the gap to the theoretical limit, since the gap is of the same order as the approximation error.

Similar results are obtained for $r = 0$, since the essential code profile optimization problem remains same as in the case of $r = 1$.

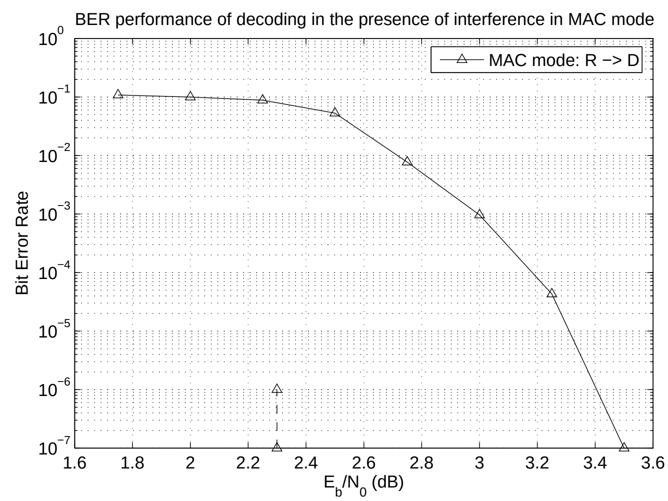

Fig. 10. Performance of the MAC code $\mathcal{C}_{RD_2}$ with interference from $\mathcal{C}_{SD_2}(P = -1dB, r = 1)$. The vertical line indicates the single-user noise threshold of the code.

B. BER performance of randomly generated codes

For BER simulations, the component codes are of blocklength 10^5, decoding uses 20 iterations, and we limit the maximum variable node degree d_v to 25. The codes are randomly generated from their profiles, and no cycle removal is performed with the exception of removing double edges between node pairs. An error at the destination occurs when there is an error in decoding any of the constituent codes.

Fig. 9 shows BER vs. EbN0 plots for each of the three constituent codes that were designed for an SNR of -1dB. With 20 iterations, the gap to the asymptotic threshold is over 2dB for the code $\mathcal{C}_{SD_1}$, whereas it is less than a dB for single-user codes of comparable profiles. The BER performance improves significantly when the number of iterations is increased to

1000.[4]

The reason for the high BER for few iterations is as follows. Since the graph corresponding to $\mathcal{C}_{SR_1}$ is a subgraph of that of $\mathcal{C}_{SD_1}$, every parity check node of $\mathcal{C}_{SR_1}$ is also a parity check node of $\mathcal{C}_{SD_1}$. Because $\mathcal{C}_{SR_1}$ is a higher rate code, its check nodes have higher degree, and these high degree check nodes are inherited by $\mathcal{C}_{SD_1}$. A consequence of having check nodes of high degree in the low rate code $\mathcal{C}_{SD_1}$ is that the convergence to the asymptotic threshold is slow. When the degree of a check node is much higher than the optimal degree for a given rate and d_v, the node receives too much noisy information, and is unable to reliably estimate the values of its neighboring variable nodes. Consequently it sends incorrect messages nearly as often as it sends correct ones. Since these check nodes have large degrees, a significant fraction of the edges in the belief propagation network carry unreliable information in the initial iterations, which slows down convergence.

Note that the high degree check nodes in $\mathcal{C}_{SD_1}$ exist only for the decodability of $\mathcal{C}_{SR_1}$. BER simulations indicate that for few iterations, it is better to ignore the messages from these nodes. The rate loss that results is negligible if the fraction of high degree check nodes is small. In other words, when the rate of $\mathcal{C}_{SR_1}$ is high compared to that of $\mathcal{C}_{SD_1}$ and the number of iterations is small, it is preferable to deliberately ignore the fact that the received signal is a codeword for $\mathcal{C}_{SR_1}$, and to decode with the additional parity bits received in the MAC mode alone. Even if we do not use the high degree check nodes corresponding to $\mathcal{C}_{SR_1}$ in belief propagation decoding of $\mathcal{C}_{SD_1}$, we can still use them to add error detection capability for which CRC bits are separately allocated in many real systems.

For the case of $r = 0$, the BER performances of the component codes are again similar to that of the $r = 1$ codes. The only difference is that there are two codes $\mathcal{C}_{SD_2}$ and $\mathcal{C}_{RD_2}$ in MAC mode, and the latter is decoded treating the former as interference. Fig. 10 shows the BER performance of decoding $\mathcal{C}_{RD_2}$ in the presence of interference. The interference is modeled by a random bit sequence. Once $\mathcal{C}_{RD_2}$ has been decoded and subtracted out, the performance of $\mathcal{C}_{SD_2}$ will be the same as that of a single-user code in Gaussian noise.

VII. Conclusions and Future Directions

We have presented LDPC code designs for decode-and-forward relaying. Codes that perform better than the ones presented in this paper can be found with additional computational resources, however, the procedure for finding better codes will remain unchanged. Our solution can be extended to higher modulation schemes for decode-and-forward relaying at higher SNRs. An extension to multiple relays is also possible. A combination of different modulation schemes for the various links is likely to yield best results with respect to the performance-complexity tradeoff.

[4]Note to reviewers: Simulations are currently being carried to show this. Initial results for up to 200 iterations are encouraging, but due to the limits of our computational resources, we are unable to furnish complete results at this time. They will be included in the final version of the paper.

Appendix I
Achievable rates of Gaussian and BPSK Gaussian relay channels

For a Gaussian relay channel, the achievable rate is known to be [8]

$$R_G = \sup_{\Theta, 0\le t, r\le 1} \min\{tC(P_{SR}) + \bar{t}C((1-r^2)P_{SD_2}), \\ tC(P_{SD_1}) + \bar{t}C(P_{SD_2} + P_{RD} + 2r\sqrt{P_{SD_2}P_{RD}})\}. \quad (29)$$

where r is the correlation between the source and relay signals in the MAC mode, $C(x) = \frac{1}{2}\log(1+x)$ is the capacity of a Gaussian link, and the following are notations for received power.

$$P_{SR} = P_{S_{BC}}\gamma_{SR} \quad , \quad P_{SD_1} = P_{S_{BC}}\gamma_{SD} \; , \\ P_{RD} = P_{R_{MAC}}\gamma_{RD} \quad , \quad P_{SD_2} = P_{S_{MAC}}\gamma_{SD} \; . \quad (30)$$

We also present the achievable rate of decode-and-forward relaying using BPSK modulation over an AWGN relay channel. The rate is given by (2) and the following mutual information terms [33]

$$\begin{aligned} I(X_1;V_1) &= H(V_1) - H(Z) \\ I(X_1,Y_1) &= H(Y_1) - H(Z) \\ I(X_2;Y_2|W_2) &= \frac{1}{2}\left\{\sum_{b_2=\pm1} H(Y_2|W_2=b_2)\right\} - H(Z) \\ I(X_2,W_2;Y_2) &= H(Y_2) - H(Z) \qquad (31) \end{aligned}$$

where

$$f_{V_1}(v_1) = \sum_{b_1=\pm1} f_{X_1}(b_1) f_Z(v_1 - b_1\sqrt{P_{SR}}) \quad (32)$$

$$f_{Y_1}(y_1) = \sum_{b_1=\pm1} f_{X_1}(b_1) f_Z(y_1 - b_1\sqrt{P_{SD1}}) \quad (33)$$

$$f_{Y_2}(y_2) = \sum_{b_1=\pm1}\sum_{b_2=\pm1} f_{X_2,W_2}(b_1,b_2) \\ f_Z(y_2 - b_1\sqrt{P_{SD2}} - b_2\sqrt{P_{RD}}) \quad (34)$$

$$f_{Y_2|W_2}(y_2|b_2) = \sum_{b_1=\pm1} f_{X_2|W_2}(b_1|b_2) \\ f_Z(y_2 - b_1\sqrt{P_{SD2}} - b_2\sqrt{P_{RD}}) \quad (35)$$

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{z^2}{2\sigma^2}\right) \quad (36)$$

where $H(.)$ is the entropy function. In the above, $f_{X_1}(-1) = f_{X_1}(1) = \frac{1}{2}$, and the optimal input distribution $f_{X_2,W_2}(b_1,b_2)$ is

$$f_{X_2,W_2}(1,1) = f_{X_2,W_2}(-1,-1) = \frac{1+r}{4} \\ f_{X_2,W_2}(1,-1) = f_{X_2,W_2}(-1,1) = \frac{1-r}{4} \quad (37)$$

Appendix II
Relationship between relay and single-user rates at high SNR

We prove that the achievable rate of decode-and-forward relaying at high SNR exceeds the capacity of the direct link by at most a constant independent of the SNR. In the high

SNR regime, the capacity function $C(x)$ approaches $\frac{1}{2}\log(x)$. Therefore, in the limit

$$\begin{aligned}
&\lim_{P\to\infty} R_G \\
&= \sup_{\Theta, 0\le t,r\le 1} \min\{\frac{t}{2}\log(P_{SR}) + \frac{\bar{t}}{2}\log((1-r^2)P_{SD_2}), \\
&\quad \frac{t}{2}\log(P_{SD_1}) + \frac{\bar{t}}{2}\log(P_{SD_2} + P_{RD} + 2r\sqrt{P_{SD_2}P_{RD}})\} \\
&\le \sup_{\Theta, 0\le t\le 1} \frac{t}{2}\log(P_{SR}) + \frac{\bar{t}}{2}\log(2(P_{SD_2}+P_{RD})) \qquad (38)
\end{aligned}$$

where the notation is the same as in Appendix I. The last expression is easily maximized with respect to t subject to the power constraint Θ defined in (1), and the maximum equals $\frac{1}{2}\log(P) + \textbf{constant terms}$.

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