A Recursive Construction for Low-Complexity Non-coherent Constellations

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Abstract

It is known that at high signal to noise ratio (SNR), or for large coherence interval (T), a constellations of unitary matrices can achieve the capacity of the non-coherent multiple-antenna system in block Rayleigh flat-fading channel. For a single transmit antenna system, a unitary constellation is simply a collection of T-dimensional unit vectors. Nevertheless, except for a few special cases, the optimal constellations are obtained only through exhaustive or random search, and their decoding complexity is exponential in the rate of the constellation and the length of the coherence interval, T. In this work, we propose a recursive construction method for real-valued single transmit antenna non-coherent constellations, in which a T-dimensional unitary constellation is constructed by using a number of $(T-1)$-dimensional unitary or spherical constellations as its equi-latitude subsets. Comparison of the minimum distances achieved by the proposed constructions with the best known packings in $G(T,1)$ [1] shows that, for practical values of T, the recursive constellations are close to optimal. We also propose a simple low-complexity decoding algorithm for the single-antenna recursive constellations. The complexity of the proposed decoder is linear in the total number of the two-dimensional constituent subsets, which is usually much smaller than the number of the constellation points. Nevertheless, the performance of the suboptimal decoder is similar to the optimal decoder. A comparison of the error rate performance of the recursive constellations with the complex-valued systematic designs of [2] shows that the proposed real-valued constellations have similar performance to the complex-valued systematic designs. The recursive designs also show a significant gain over the low-complexity PSK constellations of [3].

Index Terms

Recursive unitary constellations, non-coherent detection, fading channels, channel coding, wireless communications

I. INTRODUCTION

We consider the problem of of constellation design for non-coherent multiple-antenna communication systems in block Rayleigh flat-fading channels. It is known that a unitary constellation [4] can achieve the capacity of the non-coherent system, provided that either the signal to noise ratio is high, or the coherence interval (in symbol periods) is much larger than the number of transmit antennas. Optimal unitary constellations are the optimal packings in complex Grassmannian manifolds [5]. These packings are usually obtained through exhaustive or random search and their decoding complexity is exponential in the rate of the constellation and the coherence interval (linear in the number of the points in the constellation). In [2], a systematic method for designing unitary space-time constellations has been proposed, however, the resulting constellations still have exponential decoding complexity. A group of low decoding complexity real unitary constellations has been proposed in [3]. These real constellations are optimal when the coherence interval, T, is equal to 2 (symbol periods) and number of transmit antennas, M, is equal to 1. However, the proposed extension to large coherence intervals or multiple transmit antennas does not maintain their optimality.

In this work, we consider the case of a single transmit antenna, and construct real-valued constellations for $T \geq 2$. The packing problem in real Grassmannian manifolds has been studied in [1, 6–8]. Here, we use a recursive approach, in which, a T-dimensional unitary constellation is constructed by using an optimal number of $(T-1)$-dimensional spherical and unitary constellations as its equi-latitude subsets. Spherical constellations (packings on the surface of a T-dimensional sphere) have also been extensively studied (see, e.g., [9] and references therein). In this work, the spherical constellations are also constructed recursively, by using a number of lower dimensional spherical signal sets as their equi-latitude subsets. For the case of spherical constellations, the goal is to maximize the minimum angle between the constellation points, whereas in the case of unitary constellations, the goal is to maximize the minimum angle between subspaces corresponding to the constellation points (which is the design criterion for single antenna unitary constellations [3, 4, 10]).

The rest of this paper is organized as follows. In Section II, we present the system model. In Section III, we explain the proposed recursive construction method for the spherical constellations. In Section IV, we present our proposed construction for unitary constellations based on lower dimensional spherical and unitary constellations. In Section V, we introduce a simple recursive decoding method for the constructed unitary constellations. In Section VI, we present the comparison of the minimum distance and the performance of the proposed constellations with the existing bounds and complex unitary constellations. Finally in Section VII, we summarize the main results of this paper.

II. System Model

The system model for a non-coherent communication system with M transmit and N receive antennas in a block Rayleigh flat fading channel with coherence interval of T symbol periods is given in [10, 11] as

$$X = SH + W, \tag{1}$$

where S is the $T \times M$ matrix of transmitted signals, X is the $T \times N$ matrix of received signals, H is the $M \times N$ matrix of fading coefficients, and W is the $T \times N$ matrix of the additive received noise. Elements of H and W are assumed to be statistically independent, identically distributed circular complex Gaussian random variables from the distribution $\mathcal{CN}(0, 1)$. The power constraint is given by $\frac{1}{T}\mathbb{E}\left\{\text{tr}\left[SS^H\right]\right\} = \frac{1}{T}\sum_{t=1}^{T}\sum_{m=1}^{M}\mathbb{E}\left\{|s_{tm}|^2\right\} = P$, where s_{tm}'s are the elements of the signal matrix S.

With the above assumptions, the conditional probability density function (pdf) of the received matrix X will be given by the following expression:

$$p(X|S) = \frac{\exp\left\{-\text{tr}\left[\left(I_T + SS^H\right)^{-1} XX^H\right]\right\}}{\pi^{TN}\det^N\left(I_T + SS^H\right)}. \tag{2}$$

It has been shown in [4] that at high signal to noise ratio (SNR) or when $T \gg M$, the capacity of this system can be achieved by using a constellation of unitary (orthonormal) matrices, i.e., signal matrices with the following property

$$S^H S = \frac{TP}{M} I_M. \tag{3}$$

Using (3), and the following identities [12]

$$\det(I + AB) = \det(I + BA) \tag{4}$$

$$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1}DA^{-1}, \tag{5}$$

the conditional pdf in (2) reduces to

$$\begin{aligned} p(X|S) &= \frac{\exp\left\{-\text{tr}\left[\left(I_T - S(I_M + S^H S)^{-1}S^H\right) XX^H\right]\right\}}{\pi^{TN}\det^N\left(I_M + S^H S\right)} && (6) \\ &= \frac{\exp\left\{-\text{tr}\left[\left(I_T - S(I_M + \frac{TP}{M}I_M)^{-1}S^H\right) XX^H\right]\right\}}{\pi^{TN}\det^N\left(I_M + \frac{TP}{M}I_M\right)} && (7) \\ &= \frac{M^N}{\pi^{TN}(M+TP)^N}\exp\left\{-\text{tr}\left[\left(I_T - \frac{M}{M+TP}SS^H\right)XX^H\right]\right\} && (8) \\ &= \frac{M^N\exp\left\{-\text{tr}\left[XX^H\right]\right\}}{\pi^{TN}(M+TP)^N}\exp\left\{\frac{M}{M+TP}\text{tr}\left[SS^HXX^H\right]\right\}. && (9) \end{aligned}$$

In this work, we consider only single antenna constellations. In this case, the transmit matrix S reduces to a $T \times 1$ column vector, and the channel matrix H reduces to a $1 \times N$ row vector. Also, the unitary assumption in (3) implies that all of the constellation vectors lie on the same T-dimensional complex sphere.

Assuming a signal set of size L, $\{S_l\}_{l=1}^L$, and using (9), the Maximum Likelihood (ML) detector for this system will have the following form

$$\widehat{S}_{ML} = \arg\max_{S_l} p(X|S_l) = \arg\max_{S_l} \text{tr}\left[SS^HXX^H\right] = \arg\max_{S_l} \sum_{n=1}^{N} |X_n \cdot S_l|^2, \tag{10}$$

where X_n denotes the nth column of the matrix X, and $\cdot$ denotes the inner product operation. The rate of the above constellation is defined as $\mathcal{R} = \frac{1}{T}\log_2(L)$ b/s/Hz.

The design criterion for unitary constellations has been derived in [4] using the Chernoff bound for the pairwise error probability of this ML detector. In [10], a more general design criterion has been derived for non-coherent constellations based on the KL distance between conditional distributions. For the special case of unitary constellations, the KL-based design criterion of [10] reduces to the same criterion of [4]. It turns out [5, 10] that the unitary non-coherent constellation design problem is equivalent to the packing problem in the Grassmannian manifold $G(T, M)$ [1]. In our case, $M = 1$, and we would like to pack lines a complex T-dimensional space. There is no closed-form solution for the optimal packings in general, and they are usually obtained by exhaustive or random search. In this work, we develop a recursive method for packing lines in a real T-dimensional space (packing in $G(T, 1)$, or designing real T-dimensional unitary constellation). Our construction uses $(T-1)$-dimensional spherical and unitary constellations to construct a T-dimensional unitary constellation. In the next two sections we explain this recursive procedure in greater detail.

III. Recursive Construction of Spherical Constellations

We denote, by $S_T(L)$, the L-point recursively constructed T-dimensional real spherical constellation. The goal is to maximize to the minimum angle between the constellation points. We start with $T = 2$, in which case the optimal constellation is given by

$$S_2(L) = \left\{ \begin{bmatrix} \cos(2\pi l/L) \\ \sin(2\pi l/L) \end{bmatrix} \right\}_{l=0}^{L-1}. \tag{11}$$

For $T > 2$, we construct the constellation by using a number of $(T-1)$-dimensional constellations as latitudes of the T-dimensional constellation. In the following, we explain this procedure with more detail for the case of $T = 3$.

Figure 1 shows a 32-point 3-dimensional constellation, $S_3(32)$, constructed from two $S_2(9)$ and one $S_2(14)$ constellations. The minimum angle between points in the $S_2(9)$ constellations is denoted by β. However, when this constellation is used as a subset of an S_3 constellation, the effective angle between the constellation points is no longer β. The following equation gives the effective minimum intra-subset angle, α,

$$\sin(\frac{\alpha}{2}) = \cos(\theta)\sin(\frac{\beta}{2}), \tag{12}$$

where θ is the latitude of the subset.

Assuming a constellation of K levels (latitudes), we define the intra-subset and inter-subset distances by

$$d_{intra}(k) = \sin^2(\alpha_k), \quad \text{for } k = 1, \cdots, K, \tag{13}$$

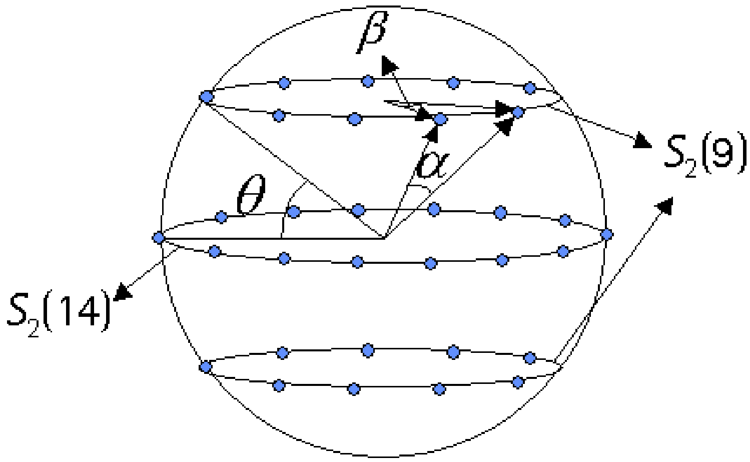


Fig. 1. A three-level three-dimensional real spherical constellation

and

$$d_{inter}(k, k') = \sin^2(\theta_k - \theta_{k'}), \quad \text{for } k, k' = 1, \cdots, K, \tag{14}$$

where α_k is the effective minimum intra-subset angle of the kth subset, and θ_k and $\theta_{k'}$ are the latitudes of the subsets k and k', respectively.

Next, similar to the approach in [10], we simplify the optimization problem by only maximizing the minimum of the intra-subset and inter-subset distances, instead of maximizing the minimum distance between all pairs of constellation points. For this, we solve the following optimization problem:

$$\underset{1\le K\le L, \sum_{k=1}^{K} l_k = L, -\frac{\pi}{2}\le\theta_1<\cdots<\theta_K\le\frac{\pi}{2}}{\text{maximize}} \quad \min\left\{\{d_{intra}(k)\}_{k=1}^{K}, \{d_{inter}(k, k+1)\}_{k=1}^{K-1}\right\}, \tag{15}$$

where l_k denotes the number of the points in the kth subset.

Here K and $l_1, \cdots, l_K$ are discrete variables, whereas $\theta_1, \cdots, \theta_K$ are continuous variables, and the optimization problem can be solved using a method similar to the one explained in [10]. For a given choice of K and $l_1, \cdots, l_K$, satisfying the specified constraints, we have a continuous optimization problem which can be solved numerically to find the optimal values for $\theta_1, \cdots, \theta_K$. Also, the following lemma can be used to further restrict the search domain.

Lemma 1: The solution of the above optimization problem can always be rearranged such that the values of $l_1, \cdots, l_K$ satisfy the following extra constraint

$$l_k \geq l_{k-1} \text{ or } l_k \geq l_{k+1}, \quad \text{for } k = 2, \cdots, K-1. \tag{16}$$

Proof: Let's assume that for some k, where $1 < k < K$, we have $l_k < l_{k-1}$ and $l_k < l_{k+1}$. Obviously, $|\theta_k| < |\theta_{k-1}|$ or $|\theta_k| < |\theta_{k+1}|$. Without loss of generality, let's assume that $|\theta_k| < |\theta_{k+1}|$. Now, since the actual intra subset distance, β_k, is a decreasing function of l_k, from $l_k < l_{k+1}$ we have $\beta_k \geq \beta_{k+1}$. Also, since the effective intra subset distance, α_k, is an increasing function of β_k, and a decreasing function of $|\theta_k|$, from $\beta_k \geq \beta_{k+1}$ and $|\theta_k| < |\theta_{k+1}|$, we have $\alpha_k > \alpha_{k+1}$, or

$$\alpha_{k+1} = \min \{\alpha_k, \alpha_{k+1}\}. \tag{17}$$

Now, if we remove one point from the $(k+1)$st subset and add it to the kth subset, and denote the new subset sizes by l'_k and l'_{k+1}, we will have $l'_k = l_k + 1$ and $l'_{k+1} = l_{k+1} - 1$. We do not change the latitudes of the subsets, so we have $\theta'_k = \theta_k$ and $\theta'_{k+1} = \theta_{k+1}$. We also denote the new actual intra subset distances by β'_k and β'_{k+1}, and the new effective intra subset distance by α'_k and α'_{k+1}. From $l'_{k+1} = l_{k+1} - 1$ we have $\beta'_{k+1} \geq \beta_{k+1}$, and since $\theta'_{k+1} = \theta_{k+1}$, we have

$$\alpha'_{k+1} \geq \alpha_{k+1}. \tag{18}$$

Also, since we had $l_k < l_{k+1}$, from $l'_k = l_k + 1$ we have $l'_k \leq l_{k+1}$. This implies that $\beta'_k \geq \beta_{k+1}$, and since $|\theta'_k| = |\theta_k| < |\theta_{k+1}|$, we have

$$\alpha'_k \geq \alpha_{k+1}. \tag{19}$$

Equations (18) and (19) imply that

$$\min \left\{\alpha'_k, \alpha'_{k+1}\right\} \geq \alpha_{k+1}. \tag{20}$$

From (17) and (20), we have

$$\min \left\{\alpha'_k, \alpha'_{k+1}\right\} \geq \min \{\alpha_k, \alpha_{k+1}\}. \tag{21}$$

Since the other intra subset distances and also the inter subset distances are not affected by this rearrangement of the constellation points, we conclude that the overall minimum distance of the constellation does not decrease by this rearrangement. Obviously we can continue this procedure until $l_k \geq l_{k+1}$. ■

For the case of $T > 3$, we use the same procedure as explained above, with the difference that instead of S_2 constellations, we use S_{T-1} constellations as the subsets, and construct the constellations recursively.

IV. Recursive Construction of Unitary Constellations

In this section, we recursively construct T-dimensional unitary constellations of size L, which we will denote by $U_T(L)$. The goal is to maximize the minimum angle between the subspaces corresponding to the constellation points. As in the previous section, we start with $T = 2$, in which case the optimal constellation is given by

$$U_2(L) = \left\{ \begin{bmatrix} \cos(\pi l/L) \\ \sin(\pi l/L) \end{bmatrix} \right\}_{l=0}^{L-1}. \quad (22)$$

This is the same constellation proposed in [3] for the case of single antenna system with $T = 2$. We notice that the points in $U_2(L)$ cover only one-half of the two-dimensional unit circle. The reason, as mentioned earlier, is that we want to maximize the minimum angle between subspaces corresponding to the constellation points, and not the angle between constellation points themselves.

For $T > 2$, we construct the constellation by using a number of $(T-1)$-dimensional spherical and unitary constellations as latitudes of the T-dimensional constellation. In the following, we explain this procedure with more details for the case of $T = 3$.

The three-dimensional unitary constellation, U_3, is obtained by using an optimal number of U_2 or S_2 constellations as equi-latitude subsets of a three-dimensional hemisphere. Figure 2 shows a case in which $U_2(7)$ and $S_2(9)$ constellations are used to construct $U_3(16)$. Notice that, unlike the case of spherical constellations of the previous section, here we do the packing only in one hemisphere. Furthermore, if a subset has a latitude of zero, we use a unitary constellation for that subset, otherwise, we use a spherical constellation. The minimum angle between points in the S_2 constellation is denoted by β. However, similar to the case of spherical constellations in the previous section, when this constellation is used as a subset of a U_3 constellation, the

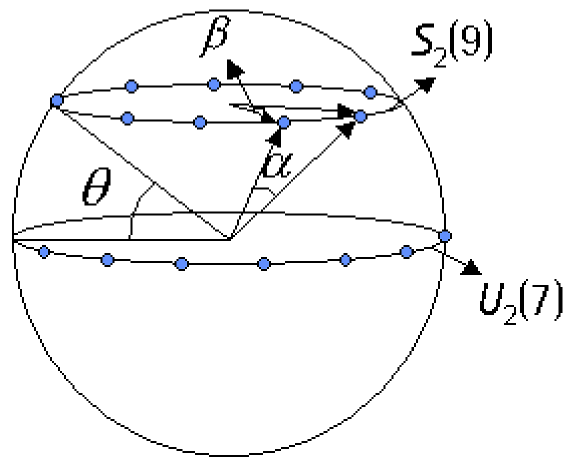


Fig. 2. A two-level three-dimensional real unitary constellation

effective angle between the constellation points is no longer β, but is given by (12), where θ is the latitude of the subset.

Assuming a constellation of K levels (latitudes), we also define the intra-subset and inter-subset distances as in the previous section, i.e., by Equations (13) and (14) , where α_k is the effective intra-subset angle of the kth subset, and θ_k and $\theta_{k'}$ are the latitudes of the subsets k and k', respectively. Moreover, if $\theta_1 \neq 0$, we define the following additional intra-subset distance for the first subset,

$$d_0 = \sin^2(2\theta_1). \tag{23}$$

Next, similar to the spherical constellations in the previous section, we simplify the optimization problem by only maximizing the minimum of the intra-subset and inter-subset distances, instead of maximizing the minimum distance between all pairs of constellation points. For this, we solve the following two optimization problems

$$\underset{1\leq K\leq L,\sum_{k=1}^{K} l_k=L, 0=\theta_1<\cdots<\theta_K\leq\frac{\pi}{2}}{\text{maximize}} \quad \min\left\{\{d_{intra}(k)\}_{k=1}^{K}, \{d_{inter}(k,k+1)\}_{k=1}^{K-1}\right\}, \tag{24}$$

and

$$\underset{1\leq K\leq L,\sum_{k=1}^{K} l_k=L, 0<\theta_1<\cdots<\theta_K\leq\frac{\pi}{2}}{\text{maximize}} \quad \min\left\{d_0, \{d_{intra}(k)\}_{k=1}^{K}, \{d_{inter}(k,k+1)\}_{k=1}^{K-1}\right\}, \tag{25}$$

where l_k denotes the number of the points in the kth subset. Each subset is a spherical subset of type S_2, except for the first subset in the case of $\theta_1 = 0$, which is a unitary subset of type U_2.

The optimization problems in (24) and (25) can be solved using a method similar to the optimization in the previous section for the spherical constellations. Furthermore, the following lemma can be used to further restrict the domain of search for the discrete variables in the optimization.

Lemma 2: The solution of the above optimization problem can always be rearranged such that the values of $l_1, \cdots, l_K$ satisfy the following extra constraint

$$\begin{aligned} 2l_1 \geq l_2 \geq \cdots \geq l_K, \quad \text{if } \theta_1 = 0, \text{ or} \\ l_1 \geq l_2 \geq \cdots \geq l_K, \quad \text{if } \theta_1 > 0, \end{aligned} \tag{26}$$

Proof: The proof is similar to the proof of Lemma 1. The details are omitted here for brevity. ■

After solving the maximin problems in (24) and (25), we compare the minimum distances of the two optimum constellations, and choose the one with the larger minimum distance as the unitary constellation $U_3(L)$.

For the case of $T > 3$, we use the same procedure as explained above, with the difference that instead of U_2 and S_2 constellations, we use U_{T-1} and S_{T-1} constellations as the equi-latitude subsets, and construct the constellations recursively. Each of these constellations are parameterized by the values of K, θ_k and l_k parameters, $\{K, \theta_1, \cdots, \theta_K, l_1, \cdots, l_K\}$.

Unfortunately, the proposed recursive construction does not easily extend to the case of multiple transmit antennas. In [1], it has been shown that the representation of the M-dimensional subspaces of the T-dimensional space (i.e., elements of $G(T, M)$) by their projection matrices, gives an isometric embedding of $G(T, M)$ into a sphere of radius $\sqrt{M(T-M)/T}$ in $\mathbb{R}^D$ where $D = \begin{pmatrix} T+1 \\ 2 \end{pmatrix} - 1$. If this mapping was bijective, we could use our recursive method to construct vector constellations in $\mathbb{R}^D$, and then map them back to $G(T, M)$. However, the above mapping is not bijective, and arbitrary points of the mentioned sphere in $\mathbb{R}^D$ do not necessary correspond to elements of $G(T, M)$. Investigation of isometries between $G(T, M)$ and spheres in a higher dimensional space is an interesting direction for future research in this area and can result in a natural extension of the results of this work to multiple transmit antenna systems.

V. DECODING OF THE RECURSIVE UNITARY CONSTELLATIONS

For the case of a single antenna system, the ML decoding in (10) is equivalent to maximizing the absolute value of the inner product of the received vector and the constellation points,

$$\widehat{S}_{ML} = \arg\max_{S_l} |X \cdot S_l| \,. \tag{27}$$

The optimal detector usually requires L inner product computations and $L-1$ comparisons, resulting in a computational complexity which grows linearly with L (exponentially with $\mathcal{R}$ and T). The so-called PSK constellations of [3] can be decoded with a computational complexity which is independent of L. However, as also mentioned earlier, the PSK constellations lose their optimality and present a poor performance for $T > 2$.

Since, for $T = 2$, the recursive constellations are the same as PSK constellations, we can use the low complexity decoding algorithm of [3]. However, that simple decoding algorithm does not easily extend to the case of recursive constellations with $T > 2$. In this section, we present a suboptimal decoding algorithm for the recursive constellations with a computational complexity which is linear only in the number of two-dimensional constituent subsets of the constellation. For this, we first convert the complex received vector into a real vector. We denote the T-dimensional received vector by $R = (R_1, \cdots, R_T)$. From this vector, we construct a real vector, $\widetilde{R}$, as follows,

$$\begin{aligned} \widehat{t} &= \arg\max_{t\in\{1,\cdots,T\}} |R_t|, \\ \widetilde{R} &= \Re\left\{R \exp(-j\angle R_{\widehat{t}})\right\}. \end{aligned} \tag{28}$$

Here, we have used the phase of the strongest element of the received vector as an estimate of the phase of the channel. Since the elements of the transmitted vector are real, the non-zero phase of the elements of the received vector are only due to the channel and the additive noise. For the strongest element, the contribution of the additive noise is small and the whole phase value can approximately be attributed to the channel. This is a very simple estimator, and one could probably use a better estimator instead. Nevertheless, as we will see in Section VI, the performance of the suboptimal decoder using this estimate is very close to the performance of the optimal decoder. Notice that no training or pilot symbols are used for the above phase estimation and cancellation. Furthermore, the magnitude of the channel is still unknown. Therefore, the receiver is still a non-coherent receiver.

Next, we derive the decoder structure using $\widetilde{R}$ as the real received vector. For that, we first parameterize each constellation point by $(\phi_1, \cdots, \phi_{T-1})$, where ϕ_1 is the angle of the constellation point inside its corresponding U_2 or S_2 constellation (i.e., $(l-1)\pi/l_k$ or $(l-1)2\pi/l_k$), ϕ_2 is the latitude, θ, of that U_2 or S_2 subset inside its corresponding U_3 or S_3 subset, and so on. For example, for $T = 4$, the lth constellation point is given by

$$S_l = (\cos(\phi_{l3})\cos(\phi_{l2})\cos(\phi_{l1}), \cos(\phi_{l3})\cos(\phi_{l2})\sin(\phi_{l1}), \cos(\phi_{l3})\sin(\phi_{l2}), \sin(\phi_{l3}))\,. \quad (29)$$

First, we consider the case of $T = 2$.

A. T = 2

Using (22) and (27), the ML detection reduces to the following maximization problem

$$\begin{aligned} \widehat{l} &= \arg\max_{l=0,\cdots,L-1} \left|\cos\left(\tfrac{\pi l}{L}\right)\widetilde{R}_1 + \sin\left(\tfrac{\pi l}{L}\right)\widetilde{R}_2\right| \\ &= \arg\max_{l=0,\cdots,L-1} \rho_1 \left|\cos\left(\tfrac{\pi l}{L} - \psi_1\right)\right|, \end{aligned} \quad (30)$$

where

$$\begin{aligned} \rho_1 &= \sqrt{\widetilde{R}_1^2 + \widetilde{R}_2^2}, \text{ and} \\ \psi_1 &= \arctan\left(\tfrac{\widetilde{R}_2}{\widetilde{R}_1}\right). \end{aligned} \quad (31)$$

Therefore, the detection can be done in one angle computation and one quantization operation.

B. $T > 2$

Again, we explain the case of $T = 3$ with more detail. For this case, we have

$$\begin{aligned} \widehat{l} &= \arg\max_{l=1,\cdots,L} \left|\cos(\phi_{l2})\left(\cos(\phi_{l1})\widetilde{R}_1 + \sin(\phi_{l1})\widetilde{R}_2\right) + \sin(\phi_{l2})\widetilde{R}_3\right| \\ &= \arg\max_{l=1,\cdots,L} \left|\cos(\phi_{l2})\rho_1\cos(\phi_{l1} - \psi_1) + \sin(\phi_{l2})\widetilde{R}_3\right| \\ &= \arg\max_{l=1,\cdots,L} \rho_2(\phi_{l1})\left|\cos(\phi_{l2} - \psi_2(\phi_{l1}))\right|, \end{aligned} \quad (32)$$

where ρ_1 and ψ_1 are as in (31), and

$$\begin{aligned} \rho_2(\phi_{l1}) &= \sqrt{\rho_1^2\cos^2(\phi_{l1} - \psi_1) + \widetilde{R}_3^2}, \text{ and} \\ \psi_2(\phi_{l1}) &= \arctan\left(\tfrac{\widetilde{R}_3}{\rho_1\cos(\phi_{l1} - \psi_1)}\right). \end{aligned} \quad (33)$$

Now, as mentioned earlier, ϕ_{l2} is the latitude of the constellation point and takes its value from the set $\{\theta_1, \cdots, \theta_K\}$. For any fixed value of ϕ_{l2}, the objective function in problem (32) is of the form $f(x) = |ax + b|$, with $a = \cos(\phi_{l2})$, $b = \sin(\phi_{l2})\widetilde{R}_3$, and $x = \rho_1 \cos(\phi_{l1} - \psi_1)$. Since $f(x)$ takes its maximum when x takes its either maximum or minimum value, we need to find only the two values of ϕ_{l1} for which x is maximized or minimized. Therefore, we can solve the above maximization problem in the following two steps:

1) *Point-in-Subset Decoding*: For each subset (θ_k), find those values of ϕ_1 for which x is maximized or minimized, i.e.,

$$\begin{aligned} \widehat{\phi}_1^M(\theta_k) &= \arg \max_{\phi_1;\phi_2=\theta_k} \cos(\phi_1 - \psi_1), \\ \widehat{\phi}_1^m(\theta_k) &= \arg \min_{\phi_1;\phi_2=\theta_k} \cos(\phi_1 - \psi_1). \end{aligned} \tag{34}$$

This can be done in one phase calculation and two quantization operations.

2) *Subset Decoding*: Find the best value of ϕ_2 (i.e. the best subset) by comparing the best points from each subset

$$\widehat{\phi_2} = \arg \max_{\phi_2 \in \{\theta_1, \cdots, \theta_K\}} \max \left\{ \rho_2 \left(\widehat{\phi}_1^M(\phi_2) \right) \left| \cos \left(\phi_2 - \psi_2 \left(\widehat{\phi}_1^M(\phi_2) \right) \right) \right|, \rho_2 \left(\widehat{\phi}_1^m(\phi_2) \right) \left| \cos \left(\phi_2 - \psi_2 \left(\widehat{\phi}_1^m(\phi_2) \right) \right) \right| \right\}, \tag{35}$$

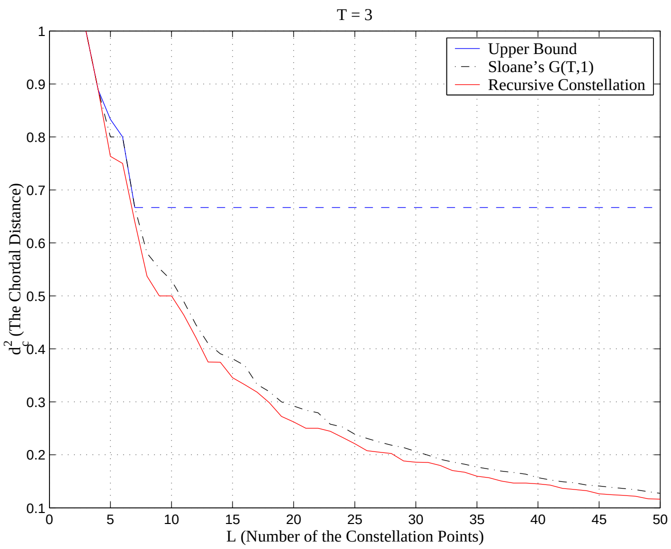
which can be done in $2K$ distance calculations followed by $2K - 1$ comparisons.

As we see, the complexity of the proposed algorithm is linear in K, which is usually much smaller than L. Comparing this decoding complexity with the complexity of the optimal decoding of the unitary constellations, which is linear in L, we see that using the recursive constellations together with the proposed decoding algorithm can save a significant amount of computation at the receiver.

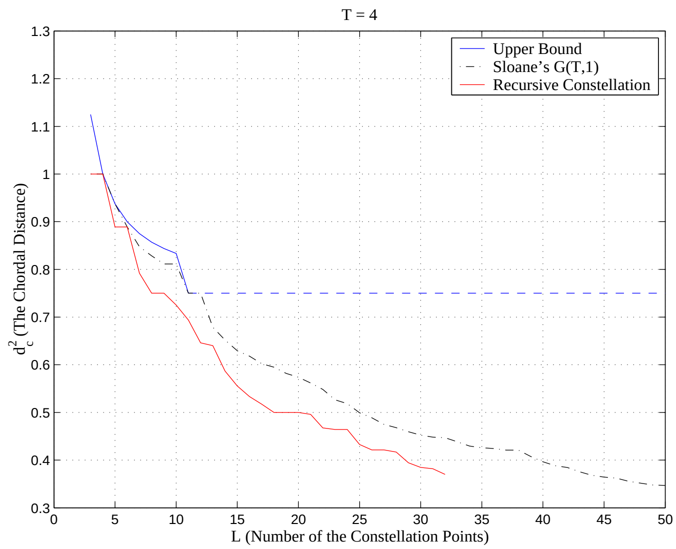
For the case of $T > 3$, the procedure is very much like the case of of $T = 3$, with the difference that each 2-dimensional subset is determined by more one latitude angle. The details are omitted here for brevity. The decoding complexity, in either case, will be linear in the number of two-dimensional constituent constellations.

VI. Performance Comparison

In this section, we compare the performance of the proposed constellations with the known bounds and unitary constellations. Figure 3 compares the minimum *chordal* distances of the



$T = 3$



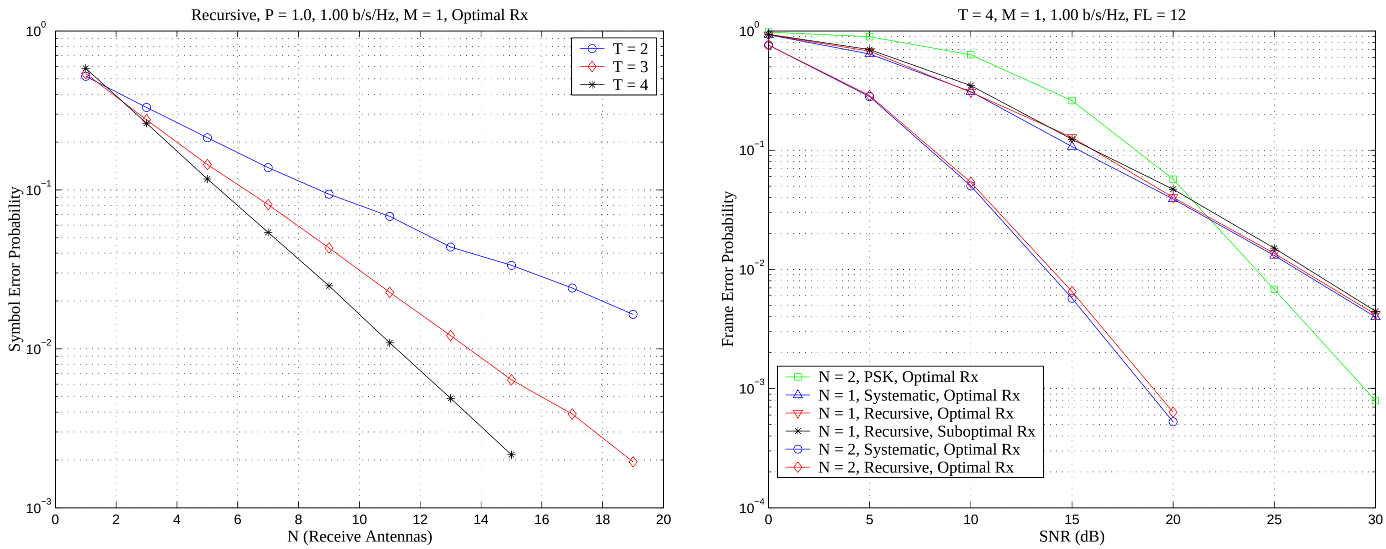
$T = 4$

Fig. 3. Comparison of the minimum chordal distances of the recursive constellations with the best known packings [1].

recursive constellations with the minimum distances of the best known packings in $G(T, 1)$ [1], for $T = 3$ and $T = 4$. As we see, the recursive constellations are very close to the optimal packings for $T = 3$. As T increases, the distance between the two constellations become larger, however, for practical values of T (i.e., the values for which the constellation size, $L = 2^{\mathcal{R}T}$ is in practical range), the two packings are reasonably close. The upper bound in Figure 3 is the Rankin bound for the packings in the Grassmannian manifolds, given in [1].

Next, in Figure 4(a), we present the error rate performance of the recursive constellations of rate 1 b/s/Hz for $T = 2$, 3, and 4, as a function of number of receive antennas. The SNR is kept at 0 dB for all curves. As we see, by designing the constellations over longer coherence intervals (if available), we can significantly improve the performance of the system and achieve a substantial gain in terms of number of receive antennas. The number of the two-dimensional constituent subsets of the above 4, 8, and 16-point constellations is 1, 2, and 4 respectively.

Finally, in Figure 4(b), we compare the performance of the recursive constellations with the systematic constellations of [2] and the PSK constellations of [3]. As we see, in both cases of one and two receive antennas, the real recursive constellations achieve a similar performance to the complex systematic unitary designs. Furthermore, the recursive constellations achieve a



(a) (b)

Fig. 4. Performance comparison of (a) recursive constellations with different values of T vs. N (receive antennas) at 0 dB, and (b) recursive constellations with systematic unitary constellations of [2] and PSK constellations of [3] for $N = 1$ and 2, vs. SNR

significant gain over the low-complexity PSK designs of [3]. We also observe that the suboptimal decoding method proposed in this paper has a performance very close to the optimal decoder, while significantly reducing the computational complexity. As mentioned above, the number of the subsets in the 16-point recursive constellation is only 4 (with one of them having only one point), which results in a lower decoding complexity as compared to the systematic unitary constellation.

VII. Conclusions

We considered the constellation design problem for non-coherent communication in flat Rayleigh fading channels. We proposed a recursive construction for the real single antenna constellations. Comparison of these constellations with the best known packings in $G(T, 1)$ shows that these constellations are close to optimal packings for practical values of T. We also proposed a low-complexity decoding algorithm for the recursive constellations. Our simulation results show that the performance of the proposed suboptimal decoder is very close to the performance of the optimal decoder. The recursive constellations show a significant gain over the low-complexity

PSK constellations of [3], and similar performance to the complex systematic unitary designs of [2] which have higher decoding complexity.

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