Exploiting Feedback in Cooperative Relay Networks

by

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Abstract

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Recent results on multiple antenna transmission techniques have shown great potential in their ability to improve the overall performance in fading channels. Despite the promise shown by employing multiple antenna’s, practical implementations may not be feasible due to size and hardware limitations of mobile nodes. Cooperative Coding is a new transmission paradigm which pools together the resources of neighboring nodes in a network to improve performance without requiring multiple antennas at any of the mobile devices.

The power of node collaboration can be seen by considering the relay channel, the simplest cooperative network. Recently, protocols have been developed for the wireless relay channel that allow the network to behave as a virtual multiple antenna system. In this thesis we show that in addition to efficient network protocols, exploiting channel state information can yield even more performance in the relay setting by allowing for temporal power and rate control.

When power control is used for a given transmission rate, minimizing the outage
probability is the appropriate method to maximize performance in the block fading channel. In a relay setting, we derive the optimal power control strategy when the transmitters in the network have perfect knowledge of the network channel state. In practice having perfect channel state knowledge at the transmitters is not possible. In this direction, we derive a power control policy that minimizes the outage probability based on the rate of the feedback link. Interestingly, we observe that only a few bits of feedback are needed to extract much of the gains of the perfect feedback power control policy.

For applications that can support a variable rate of transmission, such as data transfers, the feedback can be used to vary both the transmission rate and power. The appropriate performance metric in this case is throughput. We derive throughput maximizing policies for various cooperative transmission protocols. Once again, we show that with a limited rate of feedback, significant throughput gains are possible in relay networks. Interestingly, we show that simultaneous power and rate adaptation is usually not needed. For small average power constraints, power control is imperative, while for large average powers, rate control is sufficient to achieve a large throughput.

Our results reveal that power and rate adaptation can lead to significant performance improvements. Even a few bits of feedback can lead to large power savings and throughput gains, and as a result, channel state feedback can be readily implemented with minimal communication overhead in next generation protocols.
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Contents

Abstract ii
Acknowledgments iv
List of Illustrations x

1 Introduction 1
  1.1 Relay Channel .............................................. 2
  1.2 Performance Improvements through Feedback ............... 4
  1.3 Related Work .............................................. 5
  1.4 Contributions ............................................... 7
    1.4.1 Constant Rate Transmission ............................. 7
      1.4.1.1 Perfect Channel Knowledge at the Transmitters ..... 7
      1.4.1.2 Limited Feedback at the Transmitters ............. 8
  1.4.2 Variable Rate Transmission ............................. 9
  1.5 Outline .................................................. 11

2 Background 12
  2.1 Relay Network ............................................. 12
  2.2 Fading Channel Model ..................................... 12
  2.3 Performance Metric ....................................... 14
    2.3.1 Outage and Throughput .............................. 14
3 Outage Minimization with Perfect Feedback

3.1 Introduction ................................................. 27
3.2 Outage Minimization with Constant Power Transmission ............... 29
3.3 Optimal Power Control with Perfect CSITR ................................. 31
  3.3.1 General Procedure ......................................... 32
  3.3.2 Power Control for Specific Relaying Protocols ....................... 37
    3.3.2.1 Estimate/Amplify and Forward .......................... 38
    3.3.2.2 Hybrid Protocol and Outage Lower Bound ................. 39
  3.3.3 Analysis and Discussion ................................... 42
3.4 Effect of Practical Constraints on Outage Minimization ..................... 46
  3.4.1 Outage Minimization with Peak Power Constraints ................... 46
  3.4.2 Analysis and Discussion ................................... 52
3.5 Conclusions ................................. 53

4 Power Control with Limited Feedback 56

4.1 Introduction ................................. 56
4.2 Power Control with Finite Rate Feedback ......................... 58
  4.2.1 General Procedure ......................... 59
  4.2.2 Suboptimal Power Control Method ..................... 65
  4.2.3 Lower Bound on Diversity Order ...................... 66
  4.2.4 Analysis and Discussion ...................... 70
4.3 Outage Minimization with No CSIT ............................ 72
4.4 Conclusions .................................. 76

5 Throughput Gains with Limited Feedback 77

5.1 Introduction .................................. 77
5.2 Throughput Maximization based on the Network Channel State ... 80
  5.2.1 Problem Setup .............................. 81
    5.2.1.1 Rate Control Regions .................. 81
    5.2.1.2 Throughput Analysis .................... 82
  5.2.2 Relay Codes ............................... 85
  5.2.3 Multi-hopping ............................. 87
  5.2.4 Direct Transmission ........................ 89
Illustrations

1.1 Layout of the relay network with 3 nodes. The source transmits to the destination, and the relay node assists in the communication process. Communication along the links are corrupted by pathloss along the links in the network and Gaussian noise at the receivers; 4

2.1 Layout of the relay network with 3 nodes. The source transmits to the destination, and the relay node assists in the communication process. Communication along the links are corrupted by pathloss along the links in the network and Gaussian noise at the receivers. 14

2.2 Layout of the relay network with the relay node located along a straight line from the source to the destination. Assuming the fading value is inversely proportional to the distance, then $E[\gamma_0] = 1,$ $E[\gamma_1] = \frac{1}{d^\alpha}$ and $E[\gamma_2] = \frac{1}{(1-d^\alpha)}$. 17

2.3 Probability of outage vs. SNR for direct transmission with constant power transmission (no feedback) and with perfect feedback. The transmission rate is $R = 1$ nats/sec/Hz. 25

3.1 Probability of outage vs. SNR for various relaying protocols using constant power and a rate $R=1$ nats/sec/Hz, $d = 0.5$, and $\alpha = 3$. It is assumed that the source and relay have equal power constraints. 30
3.2 Probability of outage vs. relay distance to source for various relaying protocols using constant power and a rate $R=1$ nats/sec/Hz, and $\alpha = 3$ with an SNR of 10dB. Source and Relay have equal power constraints.

3.3 Probability of outage vs. SNR for $\alpha = 3$, $R=1$ nats/sec/Hz, and $d=0.2$. Decode and forward is near optimal at small source-relay distances.

3.4 Probability of outage vs. SNR for $\alpha = 3$, $R=1$ nats/sec/Hz, and $d=0.95$. Estimate and forward has near optimal behavior at this value of $d$, the source-relay distance.

3.5 Probability of outage vs. relay distance to source for various relaying protocols using constant power and a rate $R=1$ nats/sec/Hz. The SNR is -1dB.

3.6 Comparison of outage performance for Estimate and Forward Protocol with a global network power constraint, and also one with the addition of a peak SNR constraint of 9dB, with $\alpha = 3$, $R=1$ nats/sec/Hz, and $d=0.5$. 
3.7 Outage probability results for Amplify and Forward for different values of the PAPR ratio. Increasing ratios lead to better outage performance. For a given PAPR, the outage curve follows the no peak curve up to a point, then diverges from this curve. In this figure, $\alpha = 3$, $R=1$ nats/sec/Hz, and $d=0.5$. 

3.8 For various values of the relay position $d$, the average value of the source and relay SNR for AF is shown when power control with a sum power constraint is performed. The solid line denotes the source SNR, while the dashed line represents the relay SNR.

4.1 Outage performance vs. SNR for the amplify-and-forward scheme, with $d = 0.5$, $R=1$ nats/sec/Hz and $\alpha = 3$. For the case of 1 feedback bit, the solid line indicates a constant $P_r$, and a dashed line indicates a variable $P_r$. The source and relay are given equal average power constraints.

4.2 Structure of power control regions for a fixed $\gamma_0$. Using $\log_2 L$ bits of feedback, the space of all $(\gamma_1, \gamma_2)$ is divided into $L$ subregions. In region $R_i$, $i \in \{1, ..., L\}$, power level $P_i$ is used.
4.3 Structure of power control regions for a fixed \( \gamma_0 \) and 2 subregions.

The function \( G(\gamma_0, \gamma_1, P_1) \) defines the outage region such that all points lying below this curve require more than power \( P_1 \) to guarantee zero outage. .................................................. 63

4.4 Structure of power control regions for a fixed \( \gamma_0 \) and 2 subregions, using large power approximation. Regions \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are separated by a line, \( C_l(\gamma_0, \gamma_1, P_2) \). Below the dotted line \( C_l(\gamma_0, \gamma_1, P_1) \), the power required to invert the channel is greater than \( P_1 \), so the area below the dotted curve defines the outage probability. .......................... 70

4.5 Effect of more feedback bits on outage performance, for \( d=0.5, \alpha = 3, R=1 \text{ nats/sec/Hz} \) using the AF protocol. The relay in this case transmits with variable power in each time slot, and \( P_s = P_r \). For comparison, the case of constant power transmission is shown, and also the optimal power control policy when perfect CSIT is available. Additionally the performance of a direct transmission system using constant power is shown. .......................... 72
4.6 Savings in power by using the optimal source-relay power ratio vs. equal power among source and relay assuming a rate R=1 nats/sec/Hz and $P_{out} = 10^{-2}$. The $d$-axis represents the relay’s fractional distance between the source and destination. The savings in power corresponds to the reduction in average power that is achieved by using the optimal power ratio versus equal power allocation between the source and relay.

5.1 For a fixed $\gamma_0$, a typical set of rate control regions is shown for $\log_2 L$ bits of feedback in the relay channel. Below the dotted line in $R_1$ indicates the outage region. The contour separating $R_{q-1}$ and $R_q$ denotes all points which can guarantee zero outage while transmitting at a rate $R_q$ and a power $P_q$.

5.2 Rate control region for decode and forward for 1-bit of feedback. The parameter $K/P_2$ determines the contour which separates regions 1 and 2. Rate level $R_1$ is used in region 1 and $R_2$ is used in region 2.

5.3 For the multi-hopping system, rate control regions are shown for 1-bit of feedback. Rate level $R_2$ is used in region 2, and rate level $R_1$ is used in region 1.
5.4 Throughput versus outage probability for system with a constant SNR = 10dB and with a distance parameter of d=0.5. The performance of the decode and forward (DF) protocol is compared to that of direct transmission (DT) for different sizes of the rate control codebook.

5.5 For an SNR of 10dB, the throughput of decode and forward is shown as a function of the number of feedback bits for a source-relay distance of d = 0.5. Adapting both the rate and power leads to increases over constant power transmission, with the gains most evident as the feedback rate increases. Power control alone leads to good performance for small average powers.

5.6 For an SNR of −5dB, the throughput of decode and forward is shown as a function of the number of feedback bits for a source-relay distance of d = 0.5. Adapting both the rate and power leads to increases over constant power transmission, with the gains most evident as the feedback rate increases. The performance is further from the combined rate/power control limited at high average powers.

5.7 Throughput of Decode and Forward for SNR=2dB. At this SNR, power and rate control have similar performance, with the power control algorithm having a slightly higher throughput than rate control.
5.8 For a source-relay distance of (a) $d = 0.5$, and (b) $d = 0.25$, the %
gain in throughput of various protocols over constant rate direct
transmission is shown as a function of SNR. The protocols shown are
direct transmission (DT), Decode and Forward (DF), and
multi-hopping (MH). In all cases, the transmit power is constant. . . 98

5.9 The effect of individual energy sources for the source and relay with
Decode and Forward (DF) and selection relaying (SR) are shown in
comparison to direct transmission (DT) for a source-relay distance of
d = 0.5 and constant power transmission for (a) no feedback and (b)
1 bit of feedback for rate control. . . . . . . . . . . . . . . . . . . . . 99

5.10 For a constant arrival rate of $R = 0.5nat/sec/Hz$, the (a) normalized
throughput and (b) outage probability is shown for Decode and
Forward various protocols over constant rate as a function of SNR.
The results are shown for the case of no feedback, and also for delay
bounds of $V = 1, 2, 3, 4$. The power is constant whenever the rate is
non-zero. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 103

5.11 For an arrival rate of $R = 0.5nat/sec/Hz$, and a source-relay distance
of (a) $d = 0.2$ (b) $d = 0.5$, the normalized throughput as a function of
SNR is shown for various cooperative transmission schemes. Results
are shown for $V = 2$ and with constant power whenever the rate is
non-zero. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 106
5.12 For an arrival rate of $R=0.5 \text{ nats/sec/Hz}$, the normalized throughput is shown for $V = 1$ as a function of SNR for decode and forward (DF) and direct transmission (DT) with perfect feedback, and with no power control.

5.13 The effect of having a fixed number of feedback bits for power/rate control is more useful for higher delays. Outage results are shown for $V = 1, V = 2$ for 2-bits of feedback using DF and DT.
Chapter 1

Introduction

The increasing prevalence of mobile devices and need for wireless information access has led to more demands on system designers to provide a higher throughput and improve battery longevity. Wireless channels differ from their wired counterparts due to a phenomenon known as fading. As a result, techniques and algorithms from wired systems cannot always be directly applied to wireless scenarios.

Various methods are employed to exploit the random fading effect in wireless systems, and one such method is through increasing the diversity of the transmission. Diversity gains result when an information sequence is passed through multiple, independent realizations of the channel. One method to achieve spatial diversity gains is by using multiple transmit antennas. Performance is improved due to the increased likelihood of one of the data streams experiencing a good channel condition. Despite the promise shown by multiple antennas in mitigating the effects of fading, increasing the number of transmit antennas on small mobile devices is often impractical as a result of size and hardware complexity constraints.

To meet the demands of increased spectral and power efficiency without increasing the size of mobile nodes, fundamentally new paradigms are needed to improve performance. A technique which is particularly useful in a network of nodes is to have
nodes cooperatively send their data. Cooperation provides a method of achieving spatial diversity without the need for multiple antenna’s at the mobile nodes. Furthermore, utilizing cooperative techniques leads to a higher throughput than having users transmit directly to the destination in a point-to-point fashion [1]. However, due to the relative infancy of network information theory, especially in wireless applications, none of the current proposed network coding schemes have fully exploited the true benefits of node cooperation.

The cooperation paradigm is certainly useful in the communication between handsets and base stations. Cooperation can lead to improved battery life and a higher throughput, thereby enabling high data rate multimedia applications. Although useful in the cellular context, conceptually cooperation can be applied in more general settings. Two immediate applications which can have improved performance from collaborating nodes are ad-hoc and sensor networks [2].

1.1 Relay Channel

The simplest example of a cooperative network is the relay channel, which was first introduced in [3]. Relaying occurs where there is a source-destination pair and a helper node that assists the source in transmission to the destination. An example of a relay system is shown in Figure 1.1. Although the concept of relaying is more than 30 years old, there are still quite a few open problems regarding this channel. For example, in general the capacity of the relay channel is unknown even for the case of
Gaussian channels. As a result, most of the research efforts have focused on finding efficient protocols that lead to lower bounds on the capacity. As a second example, consider channel coding. In direct transmission, techniques such as Turbo\cite{4} and Low-Density Parity Check\cite{5} (LDPC) coding exist that are nearly capacity achieving. In the relay setting, code design principles from direct transmission cannot be directly applied, and research efforts are ongoing to find powerful channel codes that approach the lower bounds on capacity for the relay channel \cite{6, 7}.

Fading channels is an area where relaying has shown great potential. Research efforts in the fast fading environment have generally been in the form of protocols that lead to lower bounds on the capacity. In the block fading channel, results are also shown in terms of outage probability, which provides a lower bound to the frame error rate for coded systems \cite{8}. Much like the case of multi-antenna transmission, results are often quoted in terms of the diversity gains achieved by a particular coding protocol\(^*\). Recent results have shown that low complexity relay protocols can achieve full diversity and exhibit an outage performance similar to a multiple antenna system \cite{9}. The relay node, through its spatial separation with the transmitter, transmits its signal through a channel that is independent of the source-destination channel, which

\(^*\)In this work, we consider a general definition of diversity. A diversity order of \(d\) is obtained if for some constant \(C\) and power \(P\), the outage behaves as \(\frac{C}{P^d}\). Increased diversity can be obtained by adding antenna’s and/or relays to the system. As shown later, diversity can also be increased through feedback.
in effect creates a virtual transmit antenna array.

1.2 Performance Improvements through Feedback

Relaying, like multiple antenna systems, provides a powerful way to exploit the effects of fading in wireless systems. Another such method, which is the focus of this thesis, is to further improve performance in fading relay channels by exploiting feedback of the channel state information when it is available. Figure 1.1 demonstrates the feedback process in relay networks. The destination is assumed to have a perfect measure of the quality of all the links in the network. The destination can then use this perfect estimate to decide the power and rate for the source node (S) and the relay node (R). The source and relay nodes, upon reception of the feedback information, transmit
with the appropriate parameters to maximize the performance.

The feedback can be used to improve performance in two ways. When the source has applications that transmit data at a constant rate, the feedback is used to minimize the outage probability, which closely correlates with the frame error rate in blocking fading channels, by using temporal power control. On the other hand, when the source application allows for variable rate transmission, such as a file transfer, the feedback is used to maximize the effective data rate, or throughput.

In both constant rate and variable rate applications, the performance improvement with feedback is dependent on the degree of channel knowledge at the source and relay. Limits of performance can be derived when the source and relay have perfect knowledge of the channel state information in the network. However, in practice the feedback link to the transmitters will be limited. As a result, the perfect feedback scenario represents a bound on the performance of rate and power control algorithms with a limited rate feedback link.

1.3 Related Work

Our work is concerned with maximizing performance in a wireless system based on the amount of feedback available. The use of feedback in cooperative systems is a relatively new area, however, related studies have been performed for direct transmission. We highlight some of the most similar work below.

For direct transmission, in the limit of perfect feedback, the work of [10] deals
with the problem of minimizing the information outage probability in the limit of perfect channel knowledge at the transmitter and receiver. Similar results have also been derived [11] for the case of multiple antenna systems. The basic idea behind [11, 10] is to use power control to minimize the outage probability. The solution involves guaranteeing zero-outages for a subset of the channel states, and declaring outages and transmitting with no power for channel states that are too prohibitive in terms of the required power to invert the channel effects [11].

For multiple-antenna systems, the practical problem of outage minimization with a finite feedback link has been explored recently. When a long-term power constraint is available, [12] constructs a power control algorithm based on the rate of the feedback link. The work of [13] discusses the concept of feedback diversity, and shows how increasing feedback bits leads to gains similar to multiple-antenna systems. When a short-term power constraint is available, the work of [14] constructs optimal beamformers to reduce the outage probability based on the rate of the feedback link. The authors in [15] looked at increasing the data rate through power and rate adaptation with limited feedback for a single antenna direct transmission system.

For the fading relay channel, the majority of the proposed work has dealt with searching for efficient protocols, and the work generally assumes a constant transmit power. The results of [9] show how in the relay channel, efficient full-diversity achieving protocols are available that use repetition coding. Furthermore, [16, 17] discuss methods to calculate the diversity order and provide large power asymptotic results
regarding the outage probability of the proposed relaying methods.

1.4 Contributions

Our contributions can be categorized based on the constraints of the source in the network. We first consider the case where the source transmits with a constant rate over all time. This type of model is valid in applications such as voice, where the desired system metric is a low probability of error for a given transmission rate. On the other hand, for applications such as file transfers, the transmission rate may be adapted over time based on the channel conditions. The adaptation can lead to a larger throughput.

1.4.1 Constant Rate Transmission

Assuming transmission at a fixed rate, we consider the use of power control to minimize the outage probability in cooperative networks. The gains from power control can also be understood by considering a fixed performance level, and observing the power savings achieved through temporal power adaptation. Our contributions in this regard are outlined next.

1.4.1.1 Perfect Channel Knowledge at the Transmitters

- For a given rate of transmission, we derive the optimal outage minimizing power control policy assuming a sum power constraint on the source and relay for any
relay protocol when perfect channel knowledge is available to the transmitters. The procedure is general, and we show how to apply the results to various relay coding protocols.

- When perfect feedback is available, but power control not possible, we show that gains can still be achieved using phase control. By having the source and relay adjust their phases to allow for coherent combining at the destination, significant improvements are achieved over direct transmission. Furthermore, a hybrid between the decode and forward (DF) [18] and estimate and forward (EF) [19] protocols nearly achieves the universal lower bound on outage probability.

- We derive the optimal power control policy with perfect feedback and a peak power constraint on the source and relay. This constraint is very practical, as power amplifier non-linearities and FCC regulations can limit the instantaneous peak transmit power. For small average power constraints, the peak limitation does not degrade the outage performance. However, significant performance degradation occurs for large average power constraints.

1.4.1.2 Limited Feedback at the Transmitters

- Having perfect channel state knowledge at the transmitters may not be feasible in practice. In this direction, we derive a power control policy based on the rate of the feedback link. We construct a low complexity, sub-optimal procedure
that performs very well for large average power constraints. Our analysis shows that with only a few bits of feedback, most of the gap to the perfect feedback power control policy can be achieved.

• We show that with the use of one bit of feedback, the diversity order of the amplify and forward (AF) [9] transmission scheme doubles to four. The increased diversity order leads to a performance comparable to a system with 4 transmit antenna’s and constant power transmission on each antenna.

• When no feedback is available, we show the outage minimizing spatial power allocation policy for the AF protocol. Interestingly, we demonstrate that the source power must always exceed the relay power to minimize the outage. Furthermore, the performance loss from using equal source and relay powers is minimal for small source-relay distances.

1.4.2 Variable Rate Transmission

For applications that can support a variable rate of transmission, the use of feedback from the destination can be used to temporally adapt both the transmission rate and power. The rate of transmission is increased for good channel conditions, whereas the rate is decreased in poor channels to reduce the likelihood of a decoding error. Our contributions are outlined next.
• We derive a rate control procedure based on the limited capacity of the feedback link. The algorithm is general, and can be applied to any network code. We explicitly demonstrate results for the DF protocol, multi-hopping, selection relaying [9] and direct transmission.

• We show that for a finite rate of feedback, it is best to select transmission rates and powers such that the outage probability is relatively high. Small transmission rates lead to a minimal probability of a frame error, but a small throughput. On the other hand, large transmission rates lead to larger frame error rates. Throughput maximization requires a balancing of the transmission rate to guarantee the maximum spectral efficiency.

• We show that both power and rate control are usually unnecessary for throughput maximization. When the average power constraint is large, rate control with constant power is nearly optimal, especially for small rates of the feedback link. On the other hand, for small average power constraints, rate control is unnecessary, and power control with rate selection suffices.

• We show the power of relay coding over direct transmission. Decode and forward is shown to have tremendous throughput gains over direct transmission, with the largest percentage gains occurring at small average power constraints. Furthermore, DF has large gains over multi-hopping, suggesting the importance collaboration between the source and relay transmissions. We also show that for
small to moderate average power constraints, having a half-duplex constraint on the relay node does not lead to a significant loss in performance compared to full-duplex relaying.

1.5 Outline

The thesis is organized as follows. Chapter 2 overviews relevant background information and notation. Next in Chapter 3 the optimal power control policy for a relay code is outlined for the case of perfect channel knowledge at the transmitters. Chapter 4 considers the case of finite rate feedback and power control. Chapter 5 looks at variable rate applications, and methods to maximize the throughput for a given rate of the feedback link. Chapter 6 provides concluding remarks and avenues for future research.
Chapter 2

Background

2.1 Relay Network

Consider the relay network in Figure 2.1, with one relay node and one source-destination pair. The relay assists in the communication of data between the source and the destination, and it does not produce its own data. It is assumed that link $i$ in the network is attenuated by fading coefficient $h_i$, where $i \in \{0, 1, 2\}$. At both the source and relay, the received signal is corrupted by additive white Gaussian noise with zero mean and unit variance. The received signal at the relay is $y_1 = h_0 x_1 + z_1$, where $x_1$ is the relay input and $z_1$ is the noise at the relay. At the destination, the received signal is $y = h_1 x_1 + h_2 x_2 + z$, where $x_2$ is the input signal at the relay, and $z$ is the noise at the destination. Direct transmission can be considered a special case where the relay is not present, or does not participate in the communication process. In this case, the transmit signal is $x_1$ and the received signal at the destination is $y = h_1 x_1 + z$.

2.2 Fading Channel Model

As implied by the receive signal equations for the relay channel, the fading is represented by a multiplicative gain on the transmit signal. This type of fading is referred
to as block fading. The fading is characterized by the fact that the time scale of the channel changing is larger than a time slot. In a time slot, a packet or frame of length $N$ symbols is transmitted, and each of the transmitted symbols undergo the same fading value. The length of time over which the channel is assumed constant is referred to as the *coherence time*. In our model, the coherence time is $N$ symbols.

We assume communication where line of sight is not present. Under this assumption, the magnitudes of the fading coefficients are assumed to follow a Rayleigh distribution [20]. In the sequel, we will denote $\gamma_0 = |h_0|^2$, $\gamma_1 = |h_1|^2$ and $\gamma_2 = |h_2|^2$. The *network channel state* is defined by the 3-tuple $\gamma = (\gamma_0, \gamma_1, \gamma_2)$, where $\gamma_i$ follows an exponential distribution with mean $\lambda_i$, $i \in \{0, 1, 2\}$. The distribution for link $i$ in the network is expressed as

$$f(\gamma_i) = \frac{1}{\lambda_i} e^{-\gamma_i/\lambda_i}, \quad (2.1)$$

and

$$F(\gamma_i) = 1 - e^{-\gamma_i/\lambda_i} \quad (2.2)$$

the PDF and CDF, respectively. The parameter $\lambda_i$ captures the pathloss across link $i$ in the network, which is a function of the length of the link, and the pathloss exponent $\alpha$; typically, $\alpha$ lies in the range $(2, 5)$.

To consider the effect of the relay nodes positioning, we use the model shown in Figure 2.2. We assume that the distance between the source and relay is one unit, and the relay is located in a line between the source and destination. The parameter...
Figure 2.1: Layout of the relay network with 3 nodes. The source transmits to the
destination, and the relay node assists in the communication process. Communication
along the links are corrupted by pathloss along the links in the network and Gaussian
noise at the receivers.

\[d\] represents the distance from the source to the relay, and \(1 - d\) is the distance
from the relay to the destination. The mean value of the fading distribution for the
source-relay link is consequently \(\lambda_0 = \frac{1}{d^\alpha}\) and for the relay-destination link we have
\(\lambda_2 = \frac{1}{(1-d)^\alpha} .\)

### 2.3 Performance Metric

#### 2.3.1 Outage and Throughput

A practical analysis tool for the block fading environment is the outage probability [8],
which for large blocklengths, serves as a lower bound to the frame error rate, making
it useful for the analysis of coded systems. Outage probability is the probability that
the instantaneous achievable rate of the channel is less than the transmission rate,

\[
P_{out}(R, P_{avg}) = \text{Prob}[R > R_{gen}(\gamma), P_s(\gamma), P_r(\gamma)],
\]  

(2.3)
where $R_{\text{gen}}$ is the instantaneous achievable rate of the transmission protocol used. In (2.3), $P_s(\gamma)$ is the transmit power of the source, $P_r(\gamma)$ is the transmit power of the relay and $R$ is the attempted rate of transmission. Note that in (2.3) the source and relay powers have been written as functions of the instantaneous network channel state $\gamma$ to show that power control is possible when information regarding the network channel state is available to the transmitters. In (2.3), the average power constraint is $P_{\text{avg}},$ which is dependent on the type of power constraint in the network (see Section 2.3.2).

Due to outage events, the effective data rate is less than the attempted rate of transmission. Given source and relay power allocation policies of $P_s(\gamma)$ and $P_r(\gamma),$ respectively, the throughput can be expressed as

$$T = \int_\gamma R(\gamma, P_s(\gamma), P_r(\gamma)) \cdot \mathcal{I}_F\{R(\gamma, P_s(\gamma), P_r(\gamma)) < R_{\text{gen}}(\gamma, P_s(\gamma), P_r(\gamma))\} f(\gamma) d\gamma,$$

(2.4) where $R_{\text{gen}}(\cdot)$ is the achievable rate of the transmission scheme, $f(\gamma)$ is the distribution function of the network channel state, $\mathcal{I}_F\{\cdot\}$ is the indicator function, and $R(\gamma, P_s(\gamma), P_r(\gamma))$ denotes the fact that the in general, the transmit power and the attempted transmission rate can be a function of the channel state.
2.3.2 Power Constraints

The outage and throughput metrics are calculated for a given average power constraint in the network. The transmit power need not be constant, and can be adapted over time to improve performance. The degree to which power adaptation takes place depends on the amount of feedback available at the transmitters. If the source and relay have individual power constraints of $P_s$ and $P_r$, respectively, a power control policy satisfies a long term power constraints if $E[P_s(\gamma)] \leq P_s$ and $E[P_r(\gamma)] \leq P_r$. If channel state information is unavailable, then power adaptation is not possible. In this case the source transmits with a constant power $P_s$ and the relay with a constant power $P_r$.

If the source and relay are allowed to pool their power resources together, then the power adaptation can be performed using a sum power constraint. Assuming the sum of the source and relay power is $P_s + P_r$, the transmission policy satisfies the sum power constraint if $E[P_s(\gamma) + P_r(\gamma)] \leq P_s + P_r$. By allowing for a sum power constraint, more feasible power control policies are available and as a result the performance obtained with a sum power constraint will always exceed that of individual power constraints.

Referring back to the definition of outage probability in (2.3), if individual power constraints are utilized, then $P_{\text{avg}} = (P_s, P_r)$. This means that the outage probability is defined subject to individual average power constraints on the source and relay. On
Figure 2.2: Layout of the relay network with the relay node located along a straight line from the source to the destination. Assuming the fading value is inversely proportional to the distance, then $E[\gamma_0] = 1$, $E[\gamma_1] = \frac{1}{d^\alpha}$ and $E[\gamma_2] = \frac{1}{(1-d)^\alpha}$.

The other hand, if the source and relay pool their resources and a sum power constraint is used, then $P_{\text{avg}} = P_s + P_r$. This means that the mean value of the source and relay power must satisfy $P_{\text{avg}}$.

### 2.4 Relaying Protocols

The achievable rate [10], the highest reliable data rate for a codeword, is determined by maximizing the mutual information of a given relay transmission protocol for a given channel state. We assume that Maximum a posteriori (MAP) detection is used at the receiver[21]. The achievable rate is an asymptotic quantity that is achieved as the blocklength $N \to \infty$. Assuming a Gaussian noise process and perfect channel state knowledge at the receivers, we next highlight some important network coding protocols.
2.4.1 Half-Duplex Relays

If the nodes are 'cheap'[22], then transmission and reception simultaneously in the same frequency band is not possible. In this case, a practical transmission protocol is the amplify and forward (AF) technique, developed in [9]. Given a source with average power $P_s$ and a relay with average power $P_r$, the achievable rate of the AF transmission protocol is [9]

$$R_{AF}(\gamma, P_s, P_r) = \frac{1}{2} \log \left( 1 + 2\gamma P_s + \frac{4\gamma_0 P_s \gamma_0 P_r}{1 + 2P_s \gamma_0 + 2P_r \gamma_2} \right).$$

Amplify and forward operates in two phases. A time slot of $N$ symbols is divided into two equal portions, each of length $N/2$ symbols. For the first half of the time slot, the source broadcasts its transmit signal to both the destination and the relay. In the second half of the slot, the source remains silent, and the relay forwards the signal it received from the source to the destination. Prior to forwarding, the relay scales the signal to meet its power constraint. The scaling assumes knowledge of the source-relay link parameter $\gamma_0$, which is a valid assumption for the relay.

Note that in (2.5), since each transmitter sends data for half the time slot, the source uses power $2P_s$ and the relay uses power $2P_r$ to guarantee an average power of $P_s + P_r$. The AF protocol was shown to achieve a diversity order of two in the fading channel with constant power transmission. In (2.5) the transmit powers in general can be functions of the channel state $\gamma$. However, for clarity of exposition, we have removed references to the power being a function of the channel state in the
Another recently proposed 'cheap' protocol is selection relaying (SR). Given a source with average power $P_s$ and a relay with average power $P_r$, the achievable rate of SR is [9]

$$R_{SR}(\gamma, P_s, P_r) = \begin{cases} R_{SR,1}, & \text{if } \frac{1}{2} \log(1 + P_s \gamma_0) > R \\ R_{SR,2}, & \text{if } \frac{1}{2} \log(1 + P_s \gamma_0) \leq R, \end{cases}$$

where $R_{SR,1} = \frac{1}{2} \log(1 + P_s \gamma_1 + P_r \gamma_2)$, $R_{SR,2} = \frac{1}{2} \log(1 + 2P_s \gamma_1)$ and $R$ is the attempted rate of transmission. Additionally, $P_s$ is the source power, and $P_r$ is the power of the relay. For a given rate of transmission, if the source-relay link results in an outage, then the relay remains silent in the second half of the time slot, and the source repeats its transmission to the destination. However, if the relay is able to decode the source’s transmission, then in the second time slot, the relay re-transmits the packet to the destination and the source remains silent.

### 2.4.2 Full-Duplex Relays

For nodes that can transmit and receive simultaneously, protocols with higher achievable rates are available. The limits of communication on the relay channel are defined by the cut-set upper bound(UB)[21]. When no channel side information is available at the transmitters, the upper bound on the achievable rate is

$$R_{UB}(\gamma, P_s, P_r) = \max_{0 \leq |\rho| \leq 1} \min \{ \log(1 + (1 - |\rho|^2)(\gamma_0 + \gamma_1)P_s),$$

$$\log(1 + \gamma_1 P_s + \gamma_2 P_r + 2|\rho| \cos(\angle h_1 - \angle h_2) \sqrt{\gamma_1 \gamma_2 P_s P_r}) \}.$$
The parameter $\rho$ controls the correlation between the input of the source and relay channels. Note that in general, when channel state is not available at the transmitters, then $\rho$ is a constant and cannot be adjusted based on channel state. However, the outage minimizing constant can be obtained based on the statistics of the network channel states. The achievable rate of (2.6) is an upper bound, and no coding schemes have been found that have an achievable rate of $R_{UB}$.

If the channel state $h = (h_0, h_1, h_2)$ is known to all nodes, then transmitters can perfectly align their signals at the receiver of the destination node. More specifically, we can completely ignore all the phases values of the constants $h_1$, $h_2$, $h_0$ by multiplying the transmitted signals $X_1$, $X_2$ by the values $e^{-j\angle h_1}$, $e^{-j\angle h_2}$ and the relay received signal $Y_1$ by the value $e^{j\angle h_1+j\angle h_0}$. Also, $\rho = \rho(h)$ can also be optimally found, therefore the upper bound in the case of transmitter channel knowledge is given by

$$R_{UB}(\gamma, P_s, P_r) = \max_{0 \leq \rho \leq 1} \min\{\log(1 + (1 - \rho^2)(\gamma_0 + \gamma_1)P_s),$$

$$\log(1 + \gamma_1P_s + \gamma_2P_r + 2\rho\sqrt{\gamma_1\gamma_2P_sP_r})\}.$$

(2.7)

In terms of achievable schemes, under the assumption that the relay fully decodes the transmission from the source, an achievable rate was derived in [18] and shown to be

$$R_{DF}(\gamma, P_s, P_r) = \max_{0 \leq |\rho| \leq 1} \min\{\log(1 + (1 - |\rho|^2)\gamma_0P_s),$$

$$\log(1 + \gamma_1P_s + \gamma_2P_r + 2\rho\cos(\angle h_1 - \angle h_2)\sqrt{\gamma_1\gamma_2P_sP_r})\}.$$  

(2.8)

Once gain, in (2.8), if channel state is not available at the transmitters, then the
outage minimizing $\rho$ is chosen based on the channel statistics and used for all time. Similar to the case of the upper bound on the capacity, when channel knowledge is available to the transmitters, then the decode and forward achievable rate is reduced to

$$R_{DF}(\gamma, P_s, P_r) = \max_{0 \leq \rho \leq 1} \min \{ \log(1 + (1 - \rho^2)\gamma_0 P_s), \log(1 + \gamma_1 P_s + \gamma_2 P_r + 2\rho\sqrt{\gamma_1 \gamma_2 P_s P_r}) \}. \quad (2.9)$$

The Markovian decode and forward scheme is limited by the fact that the relay fully decodes, which would result in poor performance when the source-relay link is in a deep fade. Recently, a new transmission protocol called estimate and forward [19] was proposed, and it relies on the principle that the relay uses a statistical estimate of the signal from the source instead of fully decoding. The achievable rate of this scheme is

$$R_{EF}(\gamma, P_s, P_r) = \log \left( 1 + P_s \gamma_1 + \frac{P_s \gamma_0 P_r \gamma_2}{1 + P_s \gamma_0 + P_s \gamma_1 + P_r \gamma_2} \right). \quad (2.10)$$

The interesting point regarding this transmission scheme is that using estimate and forward allows for an achievable rate that always exceeds direct transmission.

It was shown in [19], that both EF and DF have their utilities in certain instances of the network channel state. When the relay is positioned near the source, then the DF protocol performs well, while the EF protocol has a high achievable rate for relays that are close to the destination. With this in mind, we propose an adaptive protocol which chooses the protocol with the larger achievable rate in each time slot. The
proposed transmission scheme, which is a hybrid between the estimate and forward and the decode and forward transmission schemes, has an achievable rate as follows

\[ R_{HB}(\gamma, P_s, P_r) = \max\{R_{EF}(\gamma, P_s, P_r), R_{DF}(\gamma, P_s, P_r)\}. \] (2.11)

Interestingly, it will be seen that the hybrid performs well in terms of outage performance in cases where EF and DF perform poorly.

### 2.4.3 Other Transmission Protocols

To assess the performance of the described relay codes, two non-cooperative benchmarks will be used. First is the case of direct transmission, which has an achievable rate of,

\[ R_{DT}(\gamma_1, P_s) = \log(1 + P_s \gamma_1). \] (2.12)

In comparing direct transmission with cooperative schemes, the total power consumption of each protocol must be considered. Cooperative techniques have two power supplies, one at the source and the other at the relay. To fairly compare against direct transmission, for most of this work we consider the case where the total network power (i.e. source plus relay power) is equated with the power used in direct transmission.

Another transmission protocol worth comparing against relaying is data forwarding or multi-hopping. In this scheme a relay node is present that receives information from the source and decodes the message. The relay then transmits the decoded
information using an optimal channel code towards the destination. The destination only utilizes the message from the relay. A decoding error at the relay will clearly lead to an error at the destination. The achievable rate of this cascaded system is simply the minimum of the capacity of the individual links in the network, i.e.

\[ R_{MH}(\gamma, P_s, P_r) = \min\{\log(1 + P_s \gamma_0), \log(1 + P_r \gamma_2)\}. \] (2.13)

2.5 Direct Transmission Analysis

In Chapter 3 we consider outage minimization in a relay network when perfect channel state knowledge is available at the source and relay, and Chapter 4 addresses the practical constraint of outage minimization with a limited rate feedback link. In Chapter 5, we derive a throughput maximization policy for relay networks based on the rate of the feedback link and on constraints imposed by the source. To better understand these solutions for relay networks, it is instructive to review some well-known results for direct transmission. We next consider the outage minimization and throughput maximization process for a direct transmission system in the limit of perfect feedback.

2.5.1 Outage Minimization for Direct Transmission

Given a source transmitting at a rate \( R \) nats/sec/Hz, the objective is to minimize the outage probability subject to a long term constraint on the average power,

\[ E_{\gamma_1}[P_s(\gamma_1)] \leq P_{\text{avg}}. \]
Since perfect feedback of $\gamma_1$ is available at the source, the power can be adapted for each channel state to minimize the outage probability while satisfying the long term constraint on the average power. The optimal solution for $P_s(\gamma_1)$ turns out to be \[10\],

\[
P_s(\gamma_1) = \begin{cases} 
P^*_s(\gamma_1), & \text{if } P^*_s(\gamma_1) \leq p^* \\ 0, & \text{if } P^*_s(\gamma_1) > p^* \end{cases} \tag{2.14}
\]

where

\[
P^*_s(\gamma_1) = \frac{e^{R} - 1}{\gamma_1}.
\]

$P^*_s(\cdot)$ is the power required to invert the instantaneous effects of the channel. The optimal outage minimization solution is to guarantee zero outages for a subset of channel states, and to declare outage and not transmit with any power for channel states below some threshold. The power threshold $p^*$ is chosen to satisfy the long term power constraint,

\[
\int_\zeta^\infty \frac{e^R - 1}{\gamma_1} f(\gamma_1) d\gamma_1 = P_{\text{avg}},
\]

where $\zeta = \frac{e^R - 1}{p^*}$ and $f(\gamma_1)$ is the probability distribution function of the source-destination channel gain $\gamma_1$. The resulting outage probability is then $P_{\text{out}} = \text{Prob}[\gamma_1 \leq \zeta]$.

When constant power transmission is used, plotting the outage probability as a function of the SNR on a log scale results in a constant decay rate of the outage curve, as can be seen in Figure 2.3. When power control is used with perfect feedback, the decay rate becomes exponential. This leads to significant power savings over direct
transmission at a target outage probability [10]. At an outage probability of $10^{-2}$, having a perfect estimate of the feedback link leads to an SNR saving of more than 14 dB.

2.5.2 Throughput Maximization for Direct Transmission

Maximizing the throughput for direct transmission depends on the amount of feedback available to the transmitter. We next consider the case with perfect feedback.

If the channel state is adapted for each channel state, for constant power transmission, the throughput can be expressed as

$$T = \int_{0}^{\infty} \log(1 + P_s \gamma_1) d\gamma_1.$$

Since constant power transmission is used, the adaptation can be done without any
outages. When power control is performed in addition to rate control, a higher throughput is possible. If the power is adapted along with the rate for each channel state, then the optimal solution to maximize the throughput is to perform water-filling [23], which leads to a power adaptation of

$$ P_s(\gamma_1) = \begin{cases} \frac{1}{\lambda} - \frac{1}{\gamma_1}, & \text{if } \gamma_1 \geq \lambda \\ 0, & \text{if } \gamma_1 < \lambda. \end{cases} \quad (2.15) $$

The cut-off parameter $\lambda$ is chosen to satisfy the long term power constraint

$$ \int_{\lambda}^{\infty} \left( \frac{1}{\lambda} - \frac{1}{\gamma_1} \right) f(\gamma_1) d\gamma_1 = P_{avg}. $$

We see that for poor channel conditions, transmission is shut off and an outage is declared, whereas in better channel conditions the rate and power are increasing functions of $\gamma_1$. The throughput of such a scheme is then

$$ T = \int_{0}^{\infty} \log(1 + P_s(\gamma_1) \cdot \gamma_1) f(\gamma_1) d\gamma_1 = \int_{\lambda}^{\infty} \log \left( \frac{\gamma_1}{\lambda} \right) f(\gamma_1) d\gamma_1. $$

Clearly any finite rate of feedback will lead to a throughput that is upper-bounded by the perfect feedback limit. The objective is to try to achieve significant gains with as few bits as possible.
Chapter 3

Outage Minimization with Perfect Feedback

3.1 Introduction

The capacity of the relay channel has been an open problem since it was first proposed in [3] more than 30 years ago. Recently, efforts have been made to develop network codes which close the gap to the upper bound on the achievable rate [19]. Interestingly, in the fast fading channel, the exact capacity can be calculated for special cases of the relay nodes position with respect to the source and destination [24]. However, in situations where the transmitters are slow moving, such as indoor wireless networks, the optimal network coding technique is still unknown.

The objective of this chapter is to consider the block fading environment and derive the performance limits in the relay channel when the transmitters have perfect channel state knowledge. First, if the source and relay must transmit with a constant power, then with the use of channel state information, they can allocate for phase offsets to ensure that the signals at the receiver add coherently. Additionally, for different channel states, the source and relay can modify the correlation between their transmitted signals to further reduce the outage. Under these assumptions, a hybrid between two known network codes is shown to be sufficient to approach the lower bound on outage probability.
Second, we derive the optimal power control strategy for any relay code, and compare the performance against the lower bound on outage probability. It is shown that the decode-forward (DF) protocol, which exhibits optimal performance for special cases of the relay nodes position in the ergodic environment, is nearly optimal when the relay is positioned close to the source. Another relay code, known as estimate-forward (EF), is shown to be robust to the position of the relay node, and outperforms decode-forward when the relay moves toward the destination. Our results also outline the power of network coding, as we show that relaying can significantly outperform multiple antenna transmission, with the gains being most evident when the relay is midway between the source and destination.

Third, we a practical constraint on the power policies of the source and relay. Due to the effects of amplifier nonlinearities, the peak power would often be limited at the source and relay to avoid excessive distortion in the transmit signal. Furthermore, peak limitations imposed by FCC regulations will limit the range of powers available in the optimal power control strategy. In this direction, we derive the optimal power control policy when the source and relay have peak power limitations.

The rest of the chapter is organized as follows. Section 3.2 investigates the outage performance of the relay protocols under the assumption of constant power transmission. Section 3.3 describes the optimal power allocation when the perfect network channel state is available to the transmitters and power control is performed. Section 3.4 considers the effect of a peak power limitation on the outage minimization
process, and Section 3.5 concludes the chapter.

3.2 Outage Minimization with Constant Power Transmission

In a relay system, even when the source and relay are restricted to transmit with a constant power in each time slot, methods exist to reduce the outage probability, which is not the case for a direct transmission system. When channel state information is available to the source and relay, two optimizations can be performed. First, the phase at the source and relay can be corrected, such that the signals at the destination node combine coherently. Second, for some network coding protocols, the correlation between the signals from the source and relay can be adjusted to maximize the rate in each channel state. Both of these optimizations can be performed while performing constant power transmission.

Figure 3.1 shows the outage probability results for the case of $d = 0.5$, where $d$ is the distance parameter from Figure 2.2, with the source and relay able to use phase correction and select the optimal $\rho$ for each transmission. It is assumed that the source and relay both have the same average power constraint, and the plots are versus the network SNR (source and relay power) for comparison. The results for the direct link system using the same power as the total power in the relaying system is shown as a baseline for comparison. Since the decode and forward protocol suffers from decoding errors at the relay, it also has the same diversity order as direct link transmission, although a better coding gain. The amplify and forward protocol has a second order
Figure 3.1: Probability of outage vs. SNR for various relaying protocols using constant power and a rate R=1 nats/sec/Hz, d = 0.5, and α = 3. It is assumed that the source and relay have equal power constraints.

diversity, yet it has poor performance at low powers. This protocol suffers from the fact that the source and relay remain idle for half of each transmission slot. The estimate and forward is also seen in the figure, and it has a 2.5dB advantage over amplify and forward. Amazingly, the hybrid protocol is shown in the figure to closely follow the outage lower bound (from (2.6)) for a relay channel with constant power. This confirms that from an outage perspective with constant power transmission, the hybrid protocol is sufficient to approach the fundamental limits.

Using a distance of d = 0.5 indicates that the relay is midway between the source and relay. However, it is interesting how robust the transmission schemes are to the relay node’s position. In Figure 3.2, for a fixed SNR of 10dB, the outage probability is shown as a function of d. The amplify and forward protocol performs well when
the relay is located midway between the source and destination, but performance degrades when $d$ is small or large. For a good source-relay link, which occurs when $d$ is small, the decode and forward protocol performs well, as the chance of outage on this link is low. As the relay moves towards the destination, the performance degrades substantially. The estimate and forward protocol exhibits almost the opposite behavior. For relays that are closer to the source, it performs poorly, while as the relay-destination link reduces, the performance approaches the optimal solution. Finally, the hybrid protocol is also shown in Figure 3.2, and it can be seen that throughout nearly the entire range of $d$, the outage performance follows closely to the lower bound on outage probability. This indicates that the hybrid protocol is robust to node positioning. The power of the HB protocol becomes clearer by looking at distance $d=0.33$, where both the EF and DF protocols are far from the lower bound, but the HB still closely follows the bound. By using a combination of the DF and EF protocols, the HB protocol can compensate for their mutual weakness.

3.3 Optimal Power Control with Perfect CSITR

In this section, we derive the optimal power control policy for any relay code when perfect feedback is available to both the source and relay. The solution has similarities to that of a two-block fading channel[10], with one important difference. The source and relay power are not independent, since when the source power is zero, the relay power must also be zero. On the other hand, in the two-block fading channel, each
Figure 3.2: Probability of outage vs. relay distance to source for various relaying protocols using constant power and a rate $R=1$ nats/sec/Hz, and $\alpha = 3$ with an SNR of 10dB. Source and Relay have equal power constraints.

3.3.1 General Procedure

When the network channel state is available at the source and relay, outage minimization with power control can provide significant performance improvements, as will be seen next. Given a network channel state of $\gamma$ that is perfectly measured at the destination, based on the power control, the source and relay are instructed to transmit with powers $P_s(\gamma)$ and $P_r(\gamma)$, respectively. Assuming a generic transmission protocol with an achievable rate of $R_{gen}(\gamma, P_s(\gamma), P_r(\gamma))$, the outage probability becomes

$$\text{Prob}(R_{gen}(\gamma, P_s(\gamma), P_r(\gamma)) < R) = E_\gamma[I_F\{R_{gen}(\gamma, P_s(\gamma), P_r(\gamma)) < R\}], \quad (3.1)$$
where \( I_F(\cdot) \) is the indicator function. To obtain significant reductions in outage probability, the minimization is done with respect to an average long term sum power constraint, meaning that

\[
E_\gamma [P_s(\gamma) + P_r(\gamma)] \leq 2P_{\text{avg}}.
\] (3.2)

The network power optimization problem involves the minimization of outage subject to a sum power constraint with two variables \( P_s \) and \( P_r \), which seems intractable. However, we next show that this problem can be turned into a single variable optimization problem which allows us to use the same idea of outage minimization used for the single link fading channel [10].

**Theorem 3.3.1.** The optimal power allocation that minimizes the outage for a relaying protocol with achievable rate \( R_{\text{gen}}(\gamma, P_s(\gamma), P_r(\gamma)) \) under a long term power constraint is

\[
P_{\text{LT}}(\gamma) = \begin{cases} 
P^*_s(\gamma), & \text{with probability } 1, \text{ if } P^*_s(\gamma) < p^* \\
P^*_s(\gamma), & \text{with probability } w_0, \text{ if } P^*_s(\gamma) = p^* \\
0, & \text{with probability } 1 - w_0, \text{ if } P^*_s(\gamma) = p^* \\
0, & \text{with probability } 1, \text{ if } P^*_s(\gamma) > p^*.
\end{cases}
\] (3.3)

Here \( P^*_s \) is the solution to \( T(\gamma, P_{st}(\gamma)) = R, w_0 \in (0, 1), \) and

\[
T(\gamma, P_{st}(\gamma)) = \max_{P_s(\gamma), P_r(\gamma)} \{ R_{\text{gen}}(\gamma, P_s(\gamma), P_r(\gamma)) : P_s(\gamma) + P_r(\gamma) \leq 2P_{st}(\gamma) \},
\] (3.4)

where \( P_s(\gamma) \) is the instantaneous source power and \( P_r(\gamma) \) is the instantaneous relay...
power. Furthermore, \( p^* \) is chosen such that the average power constraint is satisfied.

**Proof:** In general, the procedure for minimizing the outage probability under a long term constraint involves two steps, as was shown in [10]. The first step requires the solution to a short term power allocation, which minimizes the network power and guarantees zero outage in each network channel state while transmitting at the target spectral efficiency. However, as will be seen, the key difference with network power control and traditional power control is in the solution of the short term power, where an additional optimization must be performed to maximize the achievable rate by finding the optimal values of \( P_s \) and \( P_r \) given a constraint on their sum. The second step involves finding a cutoff region that modifies the short term power allocation to shut off transmission in poor channel conditions. The cutoff region is determined to satisfy the average sum power constraint. Next, we describe the algorithm in more detail.

The minimization of (3.1) with a long-term sum power constraint requires the solution of a short term power constraint \( P_s(\gamma) + P_r(\gamma) = 2P_{st}(\gamma) \) which is a function of the current network state \( \gamma \). Given the solution of the short term power allocation, the optimal power allocation with a long term constraint has the following structure
The minimized outage probability is then $P_{\text{out}}(\gamma, R) = E_{\gamma}[1 - w(\gamma)]$. To obtain $P^{*}_{st}(\gamma)$, we find a power control policy with minimum power that guarantees zero outage at transmission rate $R$. Note that for a given $P_{st}(\gamma)$, infinitely many choices exist such that $P_s(\gamma) + P_r(\gamma) = 2P_{st}(\gamma)$. However, based on Proposition 3 in [10], under a short-term power allocation, the optimal policy is to maximize $R_{\text{gen}}(\gamma, P_s(\gamma), P_r(\gamma))$ for a given $P_{st}$. If we let

$$T(\gamma, P_{st}(\gamma)) = \max_{P_s(\gamma), P_r(\gamma)} \{ R_{\text{gen}}(\gamma, P_s(\gamma), P_r(\gamma)) : P_s(\gamma) + P_r(\gamma) \leq 2P_{st}(\gamma) \}$$

(3.6)

then for a fixed $P_{st}(\gamma)$, (3.6) can be written as

$$T(\gamma, P_s(\gamma)) = \max_{P_s(\gamma)} \{ R_{\text{gen}}(\gamma, P_s(\gamma), 2P_{st}(\gamma) - P_s(\gamma)) : 0 \leq P_s(\gamma) \leq 2P_{st}(\gamma) \}.$$

(3.7)

The maximization can be done over the single variable $P_s(\gamma)$. Note that with the use of (3.7), the original optimization over two variables $P_s(\gamma)$ and $P_r(\gamma)$ is now turned into a single variable maximization over $P_s(\gamma)$ given the sum power constraint $2P_{st}(\gamma)$.

Now, for any $P_{st}(\gamma)$, the optimal short term power allocation between the source and relay is known that maximizes the achievable rate. Since the short term allocation is simply a function of $P_{st}$, we can use the same procedure as [10] in finding the optimal solution. The next step is to determine the optimal value of $P^{*}_{st}(\gamma)$. 

\[ P_{LT}(\gamma) = \begin{cases} 
  P^{*}_{st}(\gamma), & \text{with probability } w(\gamma) \\
  0, & \text{with probability } 1 - w(\gamma).
\end{cases} \]
The optimal short term power, $P^*_{st}(\gamma)$, can be obtained as the solution to

$$P^*_{st}(\gamma) = \min_{P_{st}(\gamma)} T(\gamma, P_{st}(\gamma)) \geq R. \tag{3.8}$$

Note that since $T(\gamma, P_{st}(\gamma))$ is monotonically increasing in $P_{st}$, then $P^*_{st}$ exists. This corresponds to the minimum power for zero outage, given that the source and relay power are optimally calculated to maximize the achievable rate.

The policy $P^*_{st}(\gamma)$ guarantees zero outage for each channel state $\gamma$. To ensure that the long term power constraint is satisfied, a weighting function $w(\gamma)$ is found as the solution to

$$\max_w \{E[P^*_{st}(\gamma)w(\gamma)] \leq P_{avg}, 0 \leq w(\gamma) \leq 1\}. \tag{3.9}$$

Following the discussion of [10], the optimal value of $w(\gamma)$ has the form

$$w(\gamma) = \begin{cases} 
1, & \text{if } P^*_{st}(\gamma) < p^s \\
w_0, & \text{if } P^*_{st}(\gamma) = p^s \\
0, & \text{if } P^*_{st}(\gamma) > p^s
\end{cases} \tag{3.10}$$

with $w_0 \in (0, 1)$ and

$$p^s = \sup \left\{ p : \int_{R(p)} P^*_{st}(\gamma) dF(\gamma) < P_{avg} \right\}, \tag{3.11}$$

and

$$R(p) = \{\gamma : P^*_{st}(\gamma) < p\}. \tag{3.12}$$

Note that for a continuous distribution of $\gamma$, $F(\gamma)$, Prob($P^*_{st} = p^s$) is a set of measure zero [10], so any $w_0 \in (0, 1)$ suffices. The result then follows. ■
From the proof of Theorem 3.3.1, it can be seen that the region $\mathcal{R}(p)$ is the set of all network channel states $\gamma$ that require $P_{st}^*(\gamma) < p$. The outage region can be interpreted as a volume in the 3-D space of all $(\gamma_0, \gamma_1, \gamma_2)$ that requires more power than $p^*$ to invert the channel effects.

To summarize, minimizing the outage and satisfying the average sum power constraint involves first solving a short term power allocation problem that completely inverts the effects of the channel, and this is followed by a cutoff power that guarantees the sum average power constraint. By using (3.7), the best allocation of power between the source and relay is determined for any given network channel state. The power allocation procedure for specific protocols is similar, except the solution of $T(\gamma, P_{st}(\gamma))$ varies depending on the form of the achievable rate. In the next section, an analytical expression $T(\gamma, P_{st}(\gamma))$ will be found for the protocols of interest in this work.

### 3.3.2 Power Control for Specific Relaying Protocols

We next focus the results of Section 3.3.1 to specific relay protocols. We will study the form of $T(\gamma, P_{st}(\gamma))$ for the EF, AF, and Hybrid Protocols. We limit our study to $T(\gamma, P_{st}(\gamma))$ since the solution to the optimal power allocation for each of these protocols is similar, except for the form of $T(\gamma, P_{st}(\gamma))$. In general the power functions $P_s, P_r, P_{st}$, which were defined in Section 3.3.1, are functions of $\gamma$. However, for simplicity of presentation, we drop the reference to $\gamma$ in this section.
3.3.2.1 Estimate/Amplify and Forward

To solve for the optimal power allocation using the estimate and forward protocol, two steps need to be performed. First, the function $T_{EF}(\gamma, P_{st})$ must be obtained, by using (3.6). For the estimate and forward technique, using $R_{EF}(\gamma, P_s, P_r)$, the function $T_{EF}(\gamma, P_{st})$ can be written as

$$T_{EF}(\gamma, P_{st}) = \max_{P_s} \left\{ \log \left( \frac{1 + \gamma_1 P_s + \frac{P_s \gamma_0 (2P_{st} - P_s)) \gamma_2}{1 + P_s \gamma_0 + (2P_{st} - P_s) \gamma_2} \right) : 0 \leq P_s \leq 2P_{st} \right\}, \quad (3.13)$$

where we have replaced $P_r$ with $2P_{st} - P_s$. The maximizing source power can be computed by solving for $P_s$ in the following

$$\frac{\partial R_{EF}(\gamma, P_s, 2P_{st} - P_s)}{\partial P_s} = 0,$$

where $P_s \in (0, 2P_{st})$. After solving for $P_s^*$, the form of $T_{EF}(\gamma, P_{st})$ is then known. This gives the maximum achievable rate given that the constraint on the power is $P_s + P_r = 2P_{st}$.

Next, assuming a transmission rate $R$, we need to solve for the minimum value of $P_{st}$ which satisfies the rate requirement, which is shown in (3.8). However, since the achievable rate is monotonically increasing in $P_{st}$, the minimum power occurs when the $R$ is met with equality. That is,

$$T_{EF}(\gamma, P_{st}^*) = R. \quad (3.14)$$
Once $P_{st}^*$ is known, then the cutoff power from (3.11) is determined that ensures the sum power constraint is met. This cutoff point, in conjunction with $P_{st}^*$ determines the optimal power allocation. For the amplify and forward protocol, the same procedure is performed, except now $R_{AF}(\gamma, P_s, P_r)$ is used.

3.3.2.2 Hybrid Protocol and Outage Lower Bound

For the hybrid protocol, a simplification can be done to solve for $T_{HB}(\gamma, P_{st})$, which can be expressed as

$$T_{HB}(\gamma, P_{st}) = \max_{P_s, P_r}\{\max_{P_s}\{R_{EF}(\gamma, P_s, P_r), R_{DF}(\gamma, P_s, P_r)\} : P_s + P_r \leq 2P_{st}\}.$$ 

Note that in the above, the outer 'max' operation maximizes over all power control policies with average source power $P_s \in (0, 2P_{st})$. The inner max operation selects the largest value between the estimate and forward and the decode and forward protocols for a given $P_s$. Because the inner max is simply selecting between two choices, the two operations can be interchanged without affecting the solution. As a result, we have

$$T_{HB}(\gamma, P_{st}) = \max_{P_s}\left\{\max_{P_s}\{R_{EF}(\gamma, P_s, 2P_{st} - P_s) : 0 \leq P_s \leq 2P_{st}\}, \max_{P_s}\{R_{DF}(\gamma, P_s, 2P_{st} - P_s) : 0 \leq P_s \leq 2P_{st}\}\right\}$$

$$= \max\{T_{EF}(\gamma, P_{st}), T_{DF}(\gamma, P_{st})\}.$$  \hspace{1cm} (3.15)

Note that in (3.15), all references to $P_r$ have been replaced with $2P_{st} - P_s$, allowing for the maximization to be done over the single variable $P_s$. Maximizing $T_{EF}(\gamma, P_{st}(\gamma))$
has already been discussed, so we focus on the decode and forward protocol.

Recall that when channel state is available at the transmitters, with phase cancellation, we have that

$$R_{DF}(\gamma, P_s, P_r) = \max_{0 \leq \rho \leq 1} \min\{\log(1 + (1 - \rho^2)\gamma_0 P_s), \log(1 + \gamma_1 P_s + \gamma_2 P_r + 2\rho\sqrt{\gamma_1 \gamma_2 P_s P_r})\}. \quad (3.16)$$

The solution for $T_{DF}(\gamma, P_{st})$ depends on the values of $\gamma$, $P_s$ and $P_r$. We discuss the solution based on three intervals.

First, by equating the two terms inside the minimum, it is apparent that if $\gamma_0 < \gamma_1 + \gamma_2 \frac{P_r}{P_s}$, then $\log(1 + (1 - \rho^2)\gamma_0 P_s) \leq \log(1 + \gamma_1 P_s + \gamma_2 P_r + 2\rho\sqrt{\gamma_1 \gamma_2 P_s P_r})$ for any $\rho$, and as a result

$$R_{DF}(\gamma, P_s, P_r) = \log(1 + \gamma_0(1 - \rho^2)P_s). \quad (3.17)$$

However, since $\rho \in (0, 1)$, it is sufficient to write the maximizing value of $R_{DF}$ as

$$R_{DF}(\gamma, P_s, P_r) = \log(1 + \gamma_0 P_s). \quad (3.18)$$

Since $\gamma_0 < \gamma_1 + \gamma_2 \frac{P_r}{P_s}$ and $P_r = 2P_{st} - P_s$, we have that $P_s \frac{\gamma_1}{\gamma_2 + \gamma_0 - \gamma_1} < P_s < 2P_{st} \frac{\gamma_2}{\gamma_2 + \gamma_0 - \gamma_1}$. Maximizing $R_{DF}$ is equivalent to setting $P_s^* = \frac{2P_{st}\gamma_2}{\gamma_2 + \gamma_0 - \gamma_1}$. This leads to

$$T_{DF}(\gamma, P_{st}) = \log \left(1 + \frac{2P_{st}\gamma_0\gamma_2}{\gamma_2 + \gamma_0 - \gamma_1}\right). \quad (3.18)$$

Second, if $\gamma_0 < \gamma_1$, then no additional restriction is placed on $P_s$, since in this case direct transmission is superior to the decode and forward scheme, it is sufficient to
set $P^*_s = 2P_{st}$. This leads a maximum rate of

$$T_{DF}(\gamma, P_{st}) = \log(1 + 2P_{st} \gamma_1).$$  \hfill (3.19)

The final interval occurs if $\gamma_0 > \gamma_1 + \gamma_2 \frac{P_r}{P_s}$, then (3.16) is simplified by noting that the minimum occurs when $\log(1 + (1 - \rho^2)\gamma_0 P_s) = \log(1 + \gamma_1 P_s + \gamma_2 P_r + 2\rho \sqrt{\gamma_1 \gamma_2 P_s P_r})$, which corresponds to the optimal value of $\rho$ of

$$\rho^* = \frac{\sqrt{\gamma_2 \gamma_1 P_s P_r}}{P_s \gamma_0} + \frac{\sqrt{\gamma_2 \gamma_1 P_s P_r - P_s \gamma_0(P_s \gamma_1 + P_r \gamma_2 - P_s \gamma_0)}}{P_s \gamma_0}. \hfill (3.20)$$

This leads to an achievable rate of

$$R_{DF}(\gamma, P_s, P_r) = \log(1 + \gamma_0 (1 - \rho^* P_s)). \hfill (3.21)$$

The maximizing $P_s$ for this achievable rate must now be found. Equivalently, it is sufficient to maximize $\gamma_0 (1 - \rho^* P_s)$, since (3.21) is monotonically increasing with $P_s$. We then must solve for $P^*_s$ in,

$$\gamma_0 (1 - \rho^* P_s) - 2\gamma_0 P_s \rho^* \frac{\partial \rho^*}{\partial P_s} = 0. \hfill (3.22)$$

This leads to a quadratic solution for $P_s$ with solutions

$$P^*_s \in \left\{ \frac{2\gamma_2 P_{st}}{\gamma_1 + \gamma_2}, \frac{2P_{st}(\gamma_0 \gamma_1 + 2\gamma_1 \gamma_2 + \gamma_2^2)}{\gamma_1 + \gamma_2} \right\}, \hfill (3.23)$$

From, (3.23), the solution which lies in the range $0 < 2P_{st}$ and gives the largest achievable rate is taken. If none of the solutions lie within this range, then the maximum rate occurs at the endpoint, when $P^*_s = 2P_{st}$. Given the value of $P^*_s$, then the maximum rate of $R_{DF}$ is expressed as

$$T_{DF}(\gamma, P_{st}) = \log(1 + \gamma_0 (1 - \rho^* P^*_s)). \hfill (3.24)$$
Figure 3.3: Probability of outage vs. SNR for $\alpha = 3$, $R=1$ nats/sec/Hz, and $d=0.2$. Decode and forward is near optimal at small source-relay distances.

Based on the discussion for the hybrid protocol, the solution of $T_{UB}(\gamma, P_{st})$ for the cut-set bound on outage probability is now trivial, as it has a similar form to the decode and forward protocol, with the term $\gamma_0$ replaced by $\gamma_0 + \gamma_1$. Otherwise, the procedure to solve for $P_s^*$ follows directly from that of the decode and forward protocol.

3.3.3 Analysis and Discussion

In Figure 3.3, the outage probability with power control at the source and relay is shown. The relay is assumed to be at a distance of $d = 0.2$, which allows for a good source-relay link. The pathloss exponent is $\alpha = 3$. The tremendous gains of performing optimal power allocation over the fading channel are seen. The lower
bound on outage probability is shown, and it can be seen that the hybrid protocol and the decode and forward protocol closely follow the lower bound.

The estimate and forward protocol, on the other hand, is approximately 2dB away from the lower bound. This result makes sense with what was seen for the scenario of phase cancellation, where for small values of $d$, decode and forward and the hybrid protocol closely followed the lower bound. For large distances, however, the rate of the DF protocol degrades substantially compared to the outage lower bound. On this same figure, the outage results for the amplify and forward protocol are also seen, and it is seen that there is a tremendous loss in performance by performing such a 'cheap' transmission protocol.

Figure 3.4 shows the outage probability results when $d = 0.95$, which implies that the relay is closer to the destination than it is to the source. In this case, it can be seen that once again, the hybrid protocols performance follows closely that of the outage lower bound. In [24], the estimate and forward protocol was shown to be almost capacity achieving for large values of $d$, while the performance of the DF protocol dramatically declined for increasing $d$. This trend is also seen from in the case of optimal power control. Here, the performance of the DF scheme is approximately 1.5dB away from the outage lower bound, while the EF protocol closely follows the lower bound.

In Figure 3.5, for a fixed sum power, the outage is shown as a function of the relay’s positioning with respect to the source and destination. Amplify-forward (AF)
Figure 3.4: Probability of outage vs. SNR for $\alpha = 3$, $R=1$ nats/sec/Hz, and $d=0.95$. Estimate and forward has near optimal behavior at this value of $d$, the source-relay distance.

and estimate-forward (EF) perform better as the relay moves towards the destination, and behave similar to a system with multiple receive antenna’s. AF suffers from a rate loss due to the time-orthogonality constraint, which is reflected in the gain of EF over AF. Decode-forward (DF) performs close to the lower bound on outage probability when the source-relay distance is small. Interestingly, the hybrid between EF and DF performs better than both, which is most evident at $d = 0.8$. Also shown in this figure is a 2x1 multiple antenna system, which performs the same as DF at the extremes (i.e. $d = 0$ and $d = 1$). On the other hand, DF significantly outperforms the 2x1 system when the relay is positioned between the source and relay, which justifies the use of relaying in favor of multiple transmit antenna’s.
Figure 3.5: Probability of outage vs. relay distance to source for various relaying protocols using constant power and a rate R=1 nats/sec/Hz. The SNR is -1dB.

An interesting point regarding Figure 3.5 is the performance of a multi-hop system, in which the source and relay do not perform collaboration in their communication to the destination. Clearly, this protocol has its best performance when the relay is midway between the source and destination, which allows for strong source-relay and relay-destination links. However, it is apparent that using 'expensive' relay techniques such as EF and DF clearly outperform multi-hopping. This fact justifies the use of collaborative communication between the source and relay.
3.4 Effect of Practical Constraints on Outage Minimization

In our formulation of the optimal power control policy, we make a critical assumption that in practical scenarios may not exist. In real systems, the source and relay cannot transmit an unbounded amount of energy in a particular time slot, even if the power control policy meets the average power constraint. To take this point into account, we next describe the optimal power control policy with peak power constraints.

3.4.1 Outage Minimization with Peak Power Constraints

The optimal power control policy described earlier assumes that for channel states lying within $\mathcal{R}(p^*)$, both the source and relay can use an unbounded amount of power to invert the effects of the channel. In practice, however, due to limitations imposed by the power amplifier, transmission would be peak power limited at both the source and relay. We next consider the optimal power control policy under both individual peak constraints, and a sum power constraint.

Theorem 3.4.1. The optimal power allocation that minimizes the outage for a relaying protocol with achievable rate $R_{\text{gen}}(\gamma; P_s(\gamma), P_r(\gamma))$ under long term average and
peak power constraints is

\[
P_{LT}(\gamma) = \begin{cases} 
  P^*_{st}(\gamma), & \text{with probability 1, if } \gamma \notin \mathcal{G}(P^\text{max}_s, P^\text{max}_r) \text{ and } P^*_{st}(\gamma) < p^* \\
  P^*_{st}(\gamma), & \text{with probability } w_0, \text{ if } \gamma \notin \mathcal{G}(P^\text{max}_s, P^\text{max}_r) \text{ and } P^*_{st}(\gamma) = p^* \\
  0, & \text{with probability 1 - } w_0, \text{ if } \gamma \notin \mathcal{G}(P^\text{max}_s, P^\text{max}_r) \text{ and } P^*_{st}(\gamma) = p^* \\
  0, & \text{with probability 1, if } \gamma \in \mathcal{G}(P^\text{max}_s, P^\text{max}_r) \\
  0, & \text{with probability 1, if } \gamma \in \mathcal{G}(P^\text{max}_s, P^\text{max}_r) \\
\end{cases}
\]

for some subset \( \mathcal{G}(P^\text{max}_s, P^\text{max}_r) \subset R^3_+ \) and \( w_0 \in (0, 1) \). Here \( P^*_{st} = \frac{1}{2}(P^*_s + P^*_r) \), where

\[
P^*_{k} = \min \{ \hat{P}_k(\eta^*), P^\text{max}_k \}, \quad k \in \{ s, r \}, \quad \text{and } \hat{P}_k(\eta^*) \text{ is found from the solution of}
\]

\[
R_{\text{gen}}(\gamma, \min \{ \hat{P}_s(\eta^*), P^\text{max}_s \}, \min \{ \hat{P}_r(\eta^*), P^\text{max}_r \}) = R \tag{3.25}
\]

Furthermore, \( P^\text{max}_s \) is the source peak power, \( P^\text{max}_r \) is the relay peak power, and \( p^* \) is chosen such that the peak and average power constraints are satisfied, and \( \eta^* \) is chosen to satisfy the rate constraint.

**Proof:** Let the subset \( \mathcal{G}(P^\text{max}_s, P^\text{max}_r) \) define all possible channel states in \( R^3_+ \) for which transmitting at both peak powers does not allow zero outage at a transmission rate \( R \),

\[
\mathcal{G}(P^\text{max}_s, P^\text{max}_r) = \{ \gamma : R_{\text{gen}}(\gamma, P^\text{max}_s, P^\text{max}_r) < R \}.
\]

Next, define

\[
\mathcal{R}(p) = \{ \gamma \notin \mathcal{G}(P^\text{max}_s, P^\text{max}_r) : P^*_{st} < p \}.
\]
as the set of channel states not in \( \mathcal{G}(P_s^{max}, P_r^{max}) \) that use the optimal power allocation strategy \( P_{st}^* \).

Based on the above definitions, when \( \gamma \notin \mathcal{G}(P_s^{max}, P_r^{max}) \), the optimal power allocation policy under peak and average power constraints for

\[
P_{LT}(\gamma) = \begin{cases} 
P_{st}^*(\gamma), & \text{with probability 1, if } P_{st}^*(\gamma) < p^* \\
P_{st}^*(\gamma), & \text{with probability } w_0, \text{ if } P_{st}^*(\gamma) = p^* \\
0, & \text{with probability } 1 - w_0, \text{ if } P_{st}^*(\gamma) = p^* \\
0, & \text{with probability 1, if } P_{st}^*(\gamma) > p^* 
\end{cases}
\]  

(3.26)

where \( w_0 \in (0, 1) \) and

\[
p^* = \sup \left\{ p : \int_{\mathbb{R}} P_{st}^*(\gamma)dF(\gamma) < P_{avg} \right\}.
\]  

(3.27)

Next, we need to find the form of \( P_{st}^* \), which minimizes the power while transmitting at rate \( R \) and with peak power constraints \( P_s^{max} \) and \( P_r^{max} \). Based on the discussion outlined in the proof of Theorem 3.3.1, we seek to solve

\[
\min_{P_s, P_r} \{ P_s + P_r : R_{gen}(\gamma, P_s, P_r) = R, 0 \leq P_s \leq P_s^{max}, 0 \leq P_r \leq P_r^{max} \}.
\]  

(3.28)

The above has similarities with a 2-block fading channel with peak and average power constraints, and can be solved in a similar manner as was done in [25]. The following Lagrangian equation is setup

\[
J = \sum_{k \in \{s,r\}} P_k - \sum_{k \in \{s,r\}} \psi_k P_k + \sum_{k \in \{s,r\}} \mu_k(P_k - P_k^{max}) + \eta(R_{gen}(\gamma, P_s, P_r) - R)
\]  

(3.29)
The objective function is clearly convex, as well as the set of feasible points, and as a result the optimal power allocation strategy \((P^*_s, P^*_r)\), and the corresponding \((\psi^*_k, \mu^*_k, \eta^*)\) satisfy the KKT conditions [26]

\[
\begin{align*}
    P^*_k &\geq 0 \\
    P^*_k &\leq P^*_{k}^{\text{max}} \\
    R_{\text{gen}}(\gamma, P^*_s, P^*_r) &= R \\
    \psi^*_k &\geq 0 \\
    \mu^*_k &\geq 0 \\
    \psi^*_k P^*_k &= 0 \\
    \mu^*_k (P^*_k - P^*_{k}^{\text{max}}) &= 0 \\
\end{align*}
\]

\[
\frac{\partial J}{\partial P^*_k} = 1 - \psi^*_k + \mu^*_k + \eta^* \frac{\partial R_{\text{gen}}(\gamma, P^*_s, P^*_r)}{\partial P^*_k} = 0
\]

For the relay protocols discussed in this work, it can be verified that solutions of the form

\[
P^*_k = \min\{\hat{P}_k(\eta^*), P^*_{k}^{\text{max}}\}, \tag{3.30}
\]

satisfy the KKT conditions [25], where \(\eta^*\), \(\hat{P}_s(\eta^*)\) and \(\hat{P}_r(\eta^*)\) are found from the solution of

\[
R_{\text{gen}}(\gamma, \min\{\hat{P}_s(\eta^*), P^*_{s}^{\text{max}}\}, \min\{\hat{P}_r(\eta^*), P^*_{r}^{\text{max}}\}) = R \tag{3.31}
\]

and \(\eta^*\) is the Lagrange multiplier chosen to meet the rate constraint. This is in fact quite similar to the power control policy of Theorem 3.3.1. The only difference in
the short term power inversion process is in the derivation of \( P_{st}^* (\gamma) \). Rather than allowing for any \((P_s^*, P_r^*)\), there is now a maximum value on these quantities. At most one of these values may reach a peak, and then the problem reduces to searching for one power level that meets the rate constraint.

Functionally, the optimization procedure is similar to that of the case where no peak power constraints exists. The major difference is the region \( \mathcal{G}(P_s^{\text{max}}, P_r^{\text{max}}) \), which increases the size of the outage region (where transmission is shut off). In terms of the short term power allocation process, to incorporate the peak power constraint, first the unbounded zero outage power is found, which is identical to that
Figure 3.7: Outage probability results for Amplify and Forward for different values of the PAPR ratio. Increasing ratios lead to better outage performance. For a given PAPR, the outage curve follows the no peak curve up to a point, then diverges from this curve. In this figure, $\alpha = 3$, $R=1$ nats/sec/Hz, and $d=0.5$.

operations performed in Section 3.3.2. However, it is possible that at most one of the powers, $\hat{P}_k(\eta^*)$, $k = s$ or $r$, could violate the peak power constraint. In such a case, then power $\hat{P}_k(\eta^*)$ would be set to the maximum $P_{\text{max}}^k$, and then $\hat{P}_i(\eta^*)$, $i \neq k$, can be easily solved for. The individual power allocations for the source and relay are not uniform, however, which is the case for K-block transmission on the single link fading channel. The source power is always non-zero in the solution to the short-term power constraint, whereas the relay power can become zero.
3.4.2 Analysis and Discussion

Figure 3.6 shows the effects of a peak power constraint on system performance. For small SNR’s, the peak limited system performs the same as the peak unconstrained system. However, as the SNR becomes large, the peak limitation becomes prominent, and in fact after a certain average power the outage remains constant. This is expected as the optimal power control solution with higher sum power allows for a larger region of states for which the channel inversion process is possible. However, due to the peak power limitation, these states will not be useable, and as a result the outage will remain constant. Figure 3.7 shows effect of a peak to average power ratio on the outage performance. When the PAPR=∞, then this corresponds to the case with no peak power constraint. When the peak to average power ratio is finite, then for a small average power constraint the peak constraint does not effect the outage probability. However, as the average power constraint increases, the peak constraint effects the outage probability and leads to a finite diversity order. The diversity of the power control policy with a finite PAPR is equal to the diversity of the same protocol with constant power transmission.

The PAPR results can be easily understood by considering direct transmission. In direct transmission, without a peak constraint, the channel inversion process leads to a power allocation of \( P^* = \frac{1}{\gamma_1} \) for a given channel state \( \gamma_1 \). A cutoff value \( s^* \) is chosen such that if \( \gamma_1 < s^* \), then an outage is declared. Recall that the cutoff value is
chosen to meet the long-term power constraint. The outage probability then becomes $1 - e^{-s^*}$. The cutoff value $g^* = 1/s^*$ represents a peak value on $\frac{1}{\gamma_1}$. For a peak power constraint of $\kappa$, the outage curve follows the no-peak power case as long as $p^* \geq \kappa$.

When the average power constraint increases, $p^*$ decreases and becomes less than $\kappa$. In this case, $\kappa$ will determine the outage probability, which will remain constant at $P_{out} = 1 - e^{-\frac{1}{\kappa}}$. When a peak to average power constraint is used, when $g^*$ is less than $\kappa$, the outage will once again be a function solely of the peak value $\kappa$. However, defining the PAPR as $\zeta = \frac{\kappa}{P}$, the outage will decrease as a function of $\zeta \cdot P$. Clearly, for the case of direct transmission the outage probability decreases with a diversity order of one. The coding gain over constant power transmission is proportional to $\zeta$, which is the peak to average power ratio. A similar argument holds for relaying.

### 3.5 Conclusions

In this chapter, we have analyzed the outage performance of different relaying protocols for the fading channel with side information both at the transmitters and the receiver. The main contributions of this chapter are twofold. First, having side information at the source and the relay provides a tremendous gain which can be exploited by having feedback. Moreover, even a finite rate of feedback can significantly improve the performance and lower the outage probability [27]. Second, by using a hybrid of the coding protocols discussed in [19] the universal lower bound on the outage probability can be almost achieved.
Figure 3.8: For various values of the relay position $d$, the average value of the source and relay SNR for AF is shown when power control with a sum power constraint is performed. The solid line denotes the source SNR, while the dashed line represents the relay SNR.

This work reveals that side information at the transmitters for the relay channel is more crucial than that of the single link channel. The side information at the transmitter can be used to devise coding schemes which perform better even if power variation is not allowed at the transmitter. However, in a single link side information at the transmitter is useless if transmitting power is constant. Additionally, in this chapter, we derived the optimal power allocation for different relaying protocols when power control is allowed. It is imperative to note that by exploiting power control the outage probability decays much faster.

A major assumption in this chapter is the presence of perfect feedback at both
the source and relay. In reality, having a perfect estimate is not feasible, due to the
finite capacity of the feedback links. In Chapter 4, a power control policy is devised
for the scenario where a finite rate feedback link exists.
Chapter 4
Power Control with Limited Feedback

4.1 Introduction

Chapter 3 discussed the fundamental limits in the block fading relay channel under the assumption of perfect channel knowledge at the transmitter. Adapting the power over time leads to significant power savings for a target error performance level. Furthermore, significant gains over direct transmission are possible with the optimal power control policy. However, in practice having a perfect measure of the instantaneous channel state is not practical, and methods must be developed to reduce the outage probability based on the degree of channel state knowledge at the transmitters.

The objective of this chapter is to investigate methods to approach the performance limits in the fading relay channel under different assumptions of network channel state information at the transmitters (CSIT). The first performance limit considered is the one defined by the optimal power control policy when perfect network channel state information is available at the source and relay. However, in practice, having a perfect channel estimate at the transmitters is impractical, especially in network scenarios. Hence, we consider the effect of finite rate feedback links. We derive a power control policy based on the rate of the feedback link, and we show how it can
be used to approach the perfect feedback power control limit. Second, when channel state information is unavailable to the transmitters, we find the optimal performance limit for a given protocol and provide a simple method to approach this limit.

To approach the performance of the optimal CSIT power control algorithm, we describe a power control procedure based on a limited feedback channel that is extendable to any number of feedback bits. Interestingly, we see that with just one or two bits of power control information, the finite rate feedback algorithm can overcome most of the performance gains that the optimal CSIT power control policy achieves over constant power transmission. Furthermore, we show a simple power control policy, where equal average power is given to each power control subregion. This practical policy allows for efficient computation of the power control regions, and is easily extendable to any rate of the feedback link. Our results are general and can be extended to many relay coding protocols. However, we show results based on the amplify-and-forward(AF) protocol[9], which is an attractive network code due to its simplicity and ability to achieve full diversity. For the AF technique, through an analysis of the outage probability, we are able to show that the use of a feedback bit doubles the diversity order over constant power transmission. The effect of the increased diversity order is a significant savings in power over constant power transmission for a target frame error rate. Such power savings are of particular importance in systems requiring energy efficiency, such as ad-hoc and sensor networks [28]. It is therefore imperative that next-generation network protocols utilize feedback to enable
power control, as it will result in significant battery life improvements.

The second performance limit considered occurs when no channel state knowledge is available to the transmitters. When no CSIT is available, then temporal power control is not possible. However, based on the statistics of the links in the network, the source and relay are able to determine the fraction of the total available power with which to transmit. For the AF protocol, we derive the optimal spatial power allocation between the source and relay. Interestingly, it is seen that for relays positioned close to the source, which is a scenario where relaying becomes feasible, equal power allocation between the source and relay is close to optimal. As a result, in the absence of CSIT, there is minimal power savings from using spatial power allocation at the transmitters. Our work suggests that to obtain large performance improvements over constant power transmission, it is imperative to have feedback for each realization of the channel state to allow for temporal power control.

The rest of the chapter is organized as follows. Section 4.2 investigates the outage performance of the relay protocol under the assumption that limited channel state information is available to the transmitters. Section 4.3 looks at the case of no transmitter channel state information, and Section 4.4 provides concluding remarks.

### 4.2 Power Control with Finite Rate Feedback

In this section, we derive a power control algorithm for the relay channel that uses limited feedback. First, we outline the general procedure, and then we present a
Figure 4.1: Outage performance vs. SNR for the amplify-and-forward scheme, with $d = 0.5$, $R=1$ nats/sec/Hz and $\alpha = 3$. For the case of 1 feedback bit, the solid line indicates a constant $P_r$, and a dashed line indicates a variable $P_r$. The source and relay are given equal average power constraints.

low complexity suboptimal solution. The low complexity solution has the property that it can be easily extended to an arbitrary number of feedback bits. For the case of one feedback bit, an approximation to the outage probability is developed, and the diversity gain for the AF protocol is shown to double over constant power transmission.

4.2.1 General Procedure

Consider the destination, which has a perfect estimate of the network channel state $\gamma$. Given $M$ bits of feedback, the space defined by all possible sets of $\gamma$ is quantized
into $L = 2^M$ regions. For the network channel state $\gamma$, the region is a volume in the
space defined by all positive $(\gamma_0, \gamma_1, \gamma_2)$. The destination, upon measuring the channel
state, selects a power-tuple $P_q = (P_{s,q}, P_{r,q})$, from a power control codebook $C$, of
size $L$, where $q \in \{1, 2, ..., L\}$. The index $q$ to the selected power-tuple is transmitted
to both the source and relay through a noiseless feedback link. It is assumed that
both the source and relay have copies of $C$. Upon reception of the index $q$, the source
transmits with power $P_{s,q}$ and the relay with power $P_{r,q}$.

The elements of $C$ are chosen to maintain the average power constraints of both the
source and relay. We consider the case where both the source and relay have individual
average power constraints. The power control policy described by Theorem 3.3.1
involves outage minimization with a sum power constraint, and serves as a lower
bound to the outage for the case of individual power constraints on the source and
relay. As a result, even as $L \to \infty$, the power control policy of Theorem 3.3.1
will provide a lower bound on the outage probability of the developed power control
algorithm.

Consider the power control function $S : \mathbb{R}_+^3 \to \mathbb{R}_+^2$, which maps the current
channel state $\gamma \in \mathbb{R}_+^3$ to a codebook element $P_q \in \mathbb{R}_+^2$. To satisfy the average power
constraint, $E[\gamma][S(\gamma)] \leq (P_s, P_r)$ must hold on a per element basis. The objective of
the power control algorithm is to find a $S(\gamma)$ that minimizes the outage probability
while meeting the power constraint. In general, the elements of $P_q$ can differ, chosen
to meet individual power constraints of the source and relay. To simplify the analysis,
we impose one of two possible restrictions on $P_{r,q}$. The first restriction is where the relay transmits with a constant power $P_r$ in each time slot. This leads to a power-tuple of $P_q = (P_{s,q}, P_r)$. The second restriction is where the relay takes a similar action as the source, depending on its power constraint. That is, if $P_r = \eta P_s$, then we impose a constraint on power-tuple $q$ as $P_q = (P_{s,q}, \eta P_{s,q})$. We will show later that the second form of the power-tuple allows for an increase in performance over using a constant relay power. The results presented next are applicable to both scenarios.

Given $M$ bits of feedback, the space defined by all $(\gamma_0, \gamma_1, \gamma_2)$ will be divided into $L = 2^M$ subregions $R_q$, $q \in \{1, 2, \ldots, L\}$. If the instantaneous value of $\gamma$ falls into region $R_q$, the destination indicates to the source and relay to use power-tuple $P_q$. The power levels $(P_{s,q}, P_{r,q})$, $q \in \{1, 2, \ldots, L\}$ are chosen to satisfy the long term power constraint, i.e.,

$$
(P_s, P_r) = \left( \sum_{q=1}^L P_{s,q} \int_{R_q} f(\gamma) d\gamma, \sum_{q=1}^L P_{r,q} \int_{R_q} f(\gamma) d\gamma \right),
$$

(4.1)

where $f(\gamma)$ is the joint probability distribution of the network channel state $\gamma$.

In Figure 4.2, for the amplify-and-forward protocol and for a given $\gamma_0$, the power control regions $R_q$ are shown. Although we have shown the space of $(\gamma_1, \gamma_2)$ for a given $\gamma_0$, changing $\gamma_0$ changes the position of $\mu_q$. The power control regions are in fact volumes in the space defined by $\gamma$, where for any particular $\gamma_0$, a cross-section of the 3-D space is similar to that shown in Figure 4.2.

One key feature of the power control regions is that in region $R_q$, $q \geq 2$, the
assigned power $P_q$ is the minimum required to guarantee zero outage for any point in the region. This is a fundamental property of all optimal finite rate feedback power control algorithms [12]. With this property in mind, assuming a relaying protocol with achievable rate $R_{gen}$ and transmitting at a constant rate $R$, power level $P_{s,q}$ is the solution to

$$R_{gen}(\gamma, P_{s,q}, x) = R.$$  \hspace{1cm} (4.2)

Note that in (4.2), $x = P_r$ when the relay power is always constant, and $x = \eta P_{s,q}$ when the relay also adapts its power.

From Figure 4.2 observe that for a fixed $\gamma_0$, the boundary between $\mathcal{R}_q$ and $\mathcal{R}_{q+1}$
Figure 4.3: Structure of power control regions for a fixed $\gamma_0$ and 2 subregions. The function $G(\gamma_0, \gamma_1, P_1)$ defines the outage region such that all points lying below this curve require more than power $P_1$ to guarantee zero outage.

is separated by a curve $G(\gamma_0, \gamma_1, P_{q+1})$. This curve is found by solving for $\gamma_2$ in (4.2). Any $(\gamma_1, \gamma_2)$ along this curve requires exactly powers $P_{q+1}$ for zero outage, while any other points in $R_{q+1}$ require instantaneous source and relay powers less than $P_{s,q}$ and $P_{r,q}$, respectively, for zero outage. We state this formally in the following theorem.

**Theorem 4.2.1.** For the amplify-and-forward protocol, any points lying below the curve $G(\gamma_0, \gamma_1, P_q)$ require source and relay powers greater than $P_{s,q}$ and $P_{r,q}$, respectively, to guarantee zero outage. Furthermore, any points lying above this curve require source and relay powers less than $P_{s,q}$ and $P_{r,q}$, respectively, to guarantee zero outage.
**Proof:** See Appendix A.

As a result of Theorem 4.2.1, the entire region $\mathcal{R}_{q+1}$ has no outages. This property holds for all $\mathcal{R}_q$, $q \in \{2, ..., L\}$. Based on the power constraint, however, a portion of $\mathcal{R}_1$ would be in outage. Therefore, calculating the outage probability reduces to an analysis of region $\mathcal{R}_1$.

Region $\mathcal{R}_1$ uses power-tuple $P_1$ corresponding to a source power of $P_{s,1}$ and a relay power of $P_{r,1}$. The outage probability is the probability that the source power required to invert the channel is greater than $P_{s,1}$. Note that our analysis stems from the properties of the source power, as the relay power is either constant or a scaled version of the source power. Defining $P^*$ as the minimum power to guarantee zero outage for network channel state $\gamma$, then $P^*$ can be written as the solution of

$$R_{gen}(\gamma, P^*, x) = R,$$

where $x = P_s$ when the relay transmits with constant power, or else $x = \eta P^*$ when the relay also adapts its power. With the solution to $P^*$ in hand, the outage probability can be expressed as

$$P_{out} = \int_{\gamma : P^* \geq P_{s,1}} f(\gamma) d\gamma.$$  

(4.3)

Different relaying protocols will have different solutions of $G(\gamma_0, \gamma_1, P_q)$. Considering the amplify-and-forward protocol, solving for $\gamma_2$ leads to the following

$$G_{AF}(\gamma_0, \gamma_1, P_q) = \frac{K(1 + P_{s,q}\gamma_0) - P_{s,q}\gamma_1(1 + P_{s,q}\gamma_0)}{P_{r,q}(-K + P_{s,q}\gamma_1 + P_{s,q}\gamma_0)},$$  

(4.4)
where \( K = e^{2R} - 1 \). It can be easily verified that \( \mu_q = K / P_{s,q} - \gamma_0 \) for \( q \in \{2, ..., L\} \), and \( \pi_q = K / P_{s,q} \) for \( q \in \{1, ..., L\} \), where \( \pi_q \) and \( \mu_q \) are defined in Figure 4.2. The power control regions for variable \( \gamma_0 \) can be visualized by considering the effect of \( \gamma_0 \) on \( \mu_q \) and also in the form of \( G_{AF}(\cdot) \).

### 4.2.2 Suboptimal Power Control Method

In general, solving the regions \( \mathcal{R}_q \) and the associated power levels is computationally complex. However, for a more efficient approach, we consider a method similar to [13], where equal total power is allocated to each subregion. For the case of multiple antenna systems with finite rate feedback, this technique was shown to be a good solution and close to optimal for large powers and for increasing bits of feedback. The power of this method is that, instead of jointly solving for the power control levels, they can be found in a successive fashion, which makes this algorithm amenable to a large number of feedback bits. The procedure is described next.

First, the power levels \((P_{s,L}, P_{r,L})\) are solved by noting that

\[
\frac{P_s}{L} = P_{s,L} \int_{R_L} f(\gamma) d\gamma.
\]

The solution determines the power levels \((P_{s,L}, P_{r,L})\) and the region boundary \( G(\gamma_0, \gamma_1, P_L) \). Once region \( L \) has been solved, then region \( L - 1 \) can be found. This process is continued until power level \( P_1 \) has been found. Solving power level \( P_q \) requires knowledge of \( P_{q+1} \), and by the simplifying assumption that \( \frac{P_q}{L} = P_{s,q} \int_{R_q} f(\gamma) d\gamma \). Note that we
have used the total power in each region as $P_s/L$, since the power level for each region $\mathcal{R}_q$ corresponds to the transmit power of the source, and the relay can either transmit with a constant $P_r$ or a variable power related to the source power. In either case, the relay’s action is reflected in the algorithm by the form of $G(\gamma_0, \gamma_1, P_q)$ and hence in the solution of regions $\mathcal{R}_q$. In Section 4.2.4, we will see how using this suboptimal technique with just a few power levels leads to tremendous savings in power at a target outage probability over constant power transmission.

### 4.2.3 Lower Bound on Diversity Order

It was seen in [9] that the amplify-and-forward protocol transmitting at constant power has a diversity order of two compared to a first order diversity for the single antenna direct transmission system. We next show that for the case of one bit of feedback, the diversity gain doubles from two to four for the AF protocol.

To show the behavior of 1-bit of feedback for the amplify-and-forward protocol, we first consider the effect of the source-relay fading value, $\gamma_0$. It should be noted that even in the case of a Gaussian source-relay link with a fixed $\gamma_0$, amplify-and-forward still exhibits a second order diversity. This can be shown rigorously by analyzing the asymptotic behavior of the exponential distribution.

**Theorem 4.2.2.** For the amplify-and-forward protocol with a fixed $\gamma_0$, and random $\gamma_1$ and $\gamma_2$, transmitting at a constant power leads to a second order diversity.

**Proof:** See Appendix B.
Aside from Theorem 4.2.2, the fact that a fixed value of $\gamma_0$ does not effect the diversity can also be understood by observing that the destination node still sees two independent copies of the information, through the random source-destination and the relay-destination links. With this in mind, we state the following theorem.

**Theorem 4.2.3.** For the amplify-and-forward protocol, as $P_r = P_{avg}$ increases, the optimal one bit network power control offers at least a diversity order of four.

**Proof:** Assume that the source-relay link is Gaussian, with a fixed source-relay link gain of $\gamma_0$. As was proven in Theorem 4.2.2, this assumption does not affect the diversity order analysis. Additionally, it is assumed that the relay simply transmits with power $P_r$ in each time slot, as we are seeking a lower bound for diversity order.

The analysis for outage probability that follows assumes large values of SNR. Under such a scenario, the hyperbola shown in Figure 4.3 intersects the $\gamma_2$ axis. To compute a lower bound to the outage probability under such a scenario, we approximate the hyperbola as a triangle, as seen in Figure 4.4. The concavity of the hyperbola guarantees that the approximate outage analysis will be a lower bound, since $P_{s,1} \geq P_{s,2}$ for large power constraints. Looking in $R_1$, a line defined as $C_l(\gamma_0, \gamma_1, P_1) = \delta_{out} - \gamma_1 \delta_{out} / \gamma_{out}$ defines the outage region. Also, in this figure the line $C_l(\gamma_0, \gamma_1, P_2) = \delta_B - \gamma_1 \delta_B / \gamma_B$ is the boundary between $R_1$ and $R_2$. Note that $\gamma_{out} = K / P_{s,1}$ and points below this curve are assumed to be in outage. Also, $\delta_{out}$ is found by setting $\gamma_1 = 0$ in $G_{AF}(\gamma_0; \gamma_1; (P_{s,1}, P_r))$, and $\delta_B$ is found by solving for $\gamma_2$ in $G_{AF}(\gamma_0; 0; (P_{s,2}, P_r))$. The
outage probability can be written as
\[
P_{\text{out}} = \int_{\gamma_1=0}^{\gamma_{\text{out}}} \int_{\gamma_2=0}^{C_I(\gamma_0, \gamma_2)} f_{\gamma_1, \gamma_2}(\gamma_1, \gamma_2) d\gamma_1 d\gamma_2. \tag{4.6}
\]

Denoting the probability that the network state \((\gamma_1, \gamma_2)\) is in region \(R\) as \(\Delta_2\), we can then write
\[
\Delta_2 = \frac{1}{\lambda_1 \lambda_2} \int_{\gamma_1=0}^{\gamma_0} e^{-\frac{\gamma_1}{\lambda_1}} \int_{\gamma_2=0}^{\gamma_2} e^{-\frac{\gamma_2}{\lambda_2}} d\gamma_1 + \frac{1}{\lambda_1 \lambda_2} \int_{\gamma_1=0}^{\gamma_0} e^{-\frac{\gamma_1}{\lambda_1}} \int_{\gamma_2=C_I(\gamma_0, \gamma_1, \gamma_2)}^{\gamma_2} e^{-\frac{\gamma_2}{\lambda_2}} d\gamma_2 d\gamma_1. \tag{4.7}
\]

Using the second order Taylor approximation to the exponential function, \(e^{-x} \approx 1 - \frac{x^2}{2}\), it can be shown that \(\Delta_2 \approx 1 - \frac{\delta_{0, \gamma_0}}{2\lambda_1 \lambda_2}\). Using similar arguments, it can be shown that \(P_{\text{out}} = \frac{\delta_{\text{out}} \gamma_{\text{out}}}{2\lambda_1 \lambda_2}\), where \(\gamma_{\text{out}} = K/P_{s,1}\). Therefore, we have the following approximation for the outage probability for large \(P_{s,1}\)
\[
P_{\text{out}} = \frac{\delta_{\text{out}} \gamma_{\text{out}}}{2\lambda_1 \lambda_2} = \frac{K^2 (1 + P_{s,1} \gamma_0)}{2P_r P_{s,1} (P_{s,1} \gamma_0 - K) \lambda_1 \lambda_2} \approx \frac{K^2}{2P_r P_{s,1} \lambda_1 \lambda_2}. \tag{4.8}
\]

To complete the analysis, we need to find \(P_{s,1}\) as a function of \(P_{\text{avg}}\). Using the power constraint on region \(R_1\), we can show that \(P_{s,1}(1 - \Delta_2) = P_{\text{avg}}/2\), which leads to \(P_{s,1} = \frac{P_{\text{avg}} P_r P_{s,2} \lambda_1 \lambda_2 (P_{s,2} \gamma_0 - K)}{K^2 (P_{s,2} \gamma_0 + 1)}\). With this in mind, the outage probability is now rewritten as a function of \(P_{s,2}\) as
\[
P_{\text{out}} \approx \frac{K^4 (P_{s,2} \gamma_0 + 1)}{2\lambda_1 \lambda_2^2 P_r^2 P_{\text{avg}} P_{s,2} (P_{s,2} \gamma_0 - K)}. \tag{4.9}
\]

Clearly, (4.9) has a fourth order decay with respect to power \(P_{\text{avg}}\) as long as \(P_{s,2}\) is a linear function of \(P\) and when \(P_r = P_{\text{avg}}\). Next, \(P_{s,2}\) is found as a function of \(P\).

Using the fact that we are considering an algorithm where each power control region
has the same total power, i.e., $P_{s,2} \cdot \Delta_2 = \frac{P_{\text{avg}}}{2}$, $P_{s,2}$ is the solution to the following

$$P_{s,2} \approx \frac{P_{\text{avg}}}{4} + \frac{K}{2\gamma_0} + \frac{\sqrt{(P_{\text{avg}}/\gamma_0)^2 + 2K^2/(\lambda_1 \lambda_2) - 4P_{\text{avg}} K/\gamma_0 + 4K^2/\gamma_0^2}}{4}.$$ 

As a result, since $P_{s,2}$ depends on $P_{\text{avg}}$, substitution into (4.9) leads to a fourth order decay of the outage probability as a function of the power. ■

From this result, it is clear that, with the use of just one feedback bit, the diversity order has doubled from two to four. A similar effect was seen in [13], for the case of direct transmission, where the decay in outage probability was proportional to the number of elements in the power control codebook. An additional point of interest is the effect of the mean values of the fading links $\lambda_1$ and $\lambda_2$. Increased values of these parameters lead to a decrease in the outage probability. However, the diversity order is still four. For the case of constant power transmission using the AF protocol, changes in $\lambda_1$ and $\lambda_2$ also do not effect the diversity order [9].

In [9], the authors explored the use of one feedback bit to improve system performance and proposed a technique known as incremental relaying. This relay protocol makes more efficient use of the available degrees of freedom by using a feedback bit to indicate the success/failure of the source transmission to the destination and only relaying when the source transmission leads to a decoding failure. This results in gains over traditional AF by increasing the rate for good source-relay conditions. However, it is shown in [9] that such a technique only provides a diversity order of 2. Our work reveals that it is more efficient to use the feedback information for power control,
Figure 4.4: Structure of power control regions for a fixed $\gamma_0$ and 2 subregions, using large power approximation. Regions $\mathcal{R}_1$ and $\mathcal{R}_2$ are separated by a line, $C_l(\gamma_0, \gamma_1, P_2)$. Below the dotted line $C_l(\gamma_0, \gamma_1, P_1)$, the power required to invert the channel is greater than $P_1$, so the area below the dotted curve defines the outage probability.

which can lead to tremendous power savings over incremental relaying at the same transmission rate.

### 4.2.4 Analysis and Discussion

In Figure 4.1, power control with one bit of feedback for the AF protocol is shown. With just one bit of feedback, fourth order diversity is obtained, compared to a second order diversity for constant power transmission. At an outage probability of $10^{-2}$, there is approximately 5dB of SNR savings with just one feedback bit. Furthermore, we observe that, at this same outage probability, one bit of feedback substantially
reduces the gap to the optimal power control strategy. This motivates the need for future network protocols to allocate a few bits in feedback packets to allow for power control.

Recall that two possibilities were described for the action of the relay. First, the relay transmits with a constant power in each time slot. Second, the relay takes the same action as the source (when they have the same average power constraints). In Figure 4.1, for the case of 1-feedback bit, the gains of using a variable relay power are also shown. We see that there is a small gain from performing this type of power adaptation at the relay.

In Figure 4.5, for the amplify-and-forward protocol, the effect of increasing feedback bits is shown. Constant power transmission is compared to the proposed power control strategy with 2 power levels (1 bit feedback). Additionally, the gain from adding one more power control level is shown, and we see that, for small outage probabilities, much of the gap to the optimal power control strategy has been bridged. This suggests that only a few bits of feedback are necessary to extract large savings in power, and further increases in the feedback rate offer diminishing returns. Also shown in the figure is the performance of a direct transmission system using the same total power as AF and transmitting at the same rate. Clearly, direct transmission offers only a first order diversity, whereas AF has double this diversity, which translates into large power savings.
Figure 4.5: Effect of more feedback bits on outage performance, for $d=0.5$, $\alpha=3$, $R=1$ nats/sec/Hz using the AF protocol. The relay in this case transmits with variable power in each time slot, and $P_s = P_r$. For comparison, the case of constant power transmission is shown, and also the optimal power control policy when perfect CSIT is available. Additionally the performance of a direct transmission system using constant power is shown.

4.3 Outage Minimization with No CSIT

In the previous sections, the potential gains of using the optimal power control strategy were seen, and also the effects of limited feedback on outage minimization. We observed that only a few bits of feedback are needed to bridge much of the gap to the optimal CSIT power control algorithm. We next consider the case where the transmitters have no channel state information (CSIT) and thus cannot perform temporal power control.

Even though the transmit powers for the source and relay are fixed, the outage
probability can be minimized by determining the optimal fraction of the total power to be allocated to the source and relay. In each time slot, we have that \( P_s + P_r = 2P_{\text{avg}} \).

The objective is to find a \( \kappa \in (0, 1) \) such that the outage probability is minimized given that \( P_s = 2P_{\text{avg}}\kappa \) and \( P_r = 2P_{\text{avg}}(1 - \kappa) \). In addition to the derivation of the optimal source-relay power ratio \( \kappa^* \), we will see how the practical choice of using equal power at the source and relay performs close to optimal for many cases of interest.

Next, we consider the performance of the amplify-and-forward protocol for the case of constant power transmission.

Consider the amplify-and-forward protocol and an optimal source-relay power ratio \( \kappa^* \). The achievable rate is

\[
R_{\text{AF}}(\gamma, P_s, P_r) = \frac{1}{2} \log \left( 1 + 2\gamma_1 P_{\text{avg}}\kappa^* + \frac{4\gamma_2 P_{\text{avg}}^2 \kappa^* \gamma_0 (1 - \kappa^*)}{1 + 2P_{\text{avg}} \kappa^* \gamma_0 + 2P_{\text{avg}} (1 - \kappa^*) \gamma_2} \right).
\]

We next characterize the outage probability for the amplify-and-forward protocol in the limit for large powers and for a given \( \kappa \).

**Lemma 4.3.1.** As the average power \( 2P_{\text{avg}} \) becomes large, the outage probability of the amplify-and-forward protocol can be approximated as

\[
P_{\text{out}} \approx \left( e^{2R} - 1 \right) \left( \frac{1}{2\sqrt{2P_{\text{avg}}}} \right)^2 \left[ \frac{1}{\kappa \lambda_1} \cdot \frac{(1 - \kappa) \lambda_2 + \kappa \lambda_0}{\kappa(1 - \kappa) \lambda_0 \lambda_2} \right], \quad (4.10)
\]

where \( \lambda_i, i \in \{0, 1, 2\} \) is the mean value of the fading for link \( i \) in the relay network and \( 0 \leq \kappa \leq 1 \) allocates power between the source and relay.

**Proof:** The proof is based on asymptotic analysis of the exponential distribution, which was described in [9]. The total network power is \( 2P_{\text{avg}} \). Based on [9], it can be
shown that
\[
\lim_{s \to \infty} s \cdot \text{Prob}[s\kappa \gamma_1 < t] = \frac{t}{\lambda_1 \kappa} = f(t),
\]
and
\[
\lim_{s \to \infty} s \cdot \text{Prob}[f(s\kappa \gamma_0, s(1 - \kappa) \gamma_2) < t] = \frac{t}{\lambda_0 \kappa} + \frac{t}{\lambda_2 (1 - \kappa)} = g(t),
\]
where \( f(x, y) = \frac{x - y}{x + y + 1} \). In the above formulations, \( s\kappa \) is the total power used by the source and \( s(1 - \kappa) \) is the power used by the relay. Using Theorem 1 from [17],
\[
\lim_{s \to \infty} s^2 \cdot \text{Prob}[s\kappa \gamma_1 + f(s\kappa \gamma_0, s(1 - \kappa) \gamma_2) < t] =
\int_0^t g(t - x) \cdot f'(x) dx =
\frac{t^2}{2 \kappa \lambda_1} \left[ \frac{1}{\kappa \lambda_0 + \frac{1}{\lambda_2 (1 - \kappa)}} \right].
\]
Substituting \( s^2 = 4P_{\text{avg}}^2 \) and \( t = e^{2R} - 1 \), the result follows.

We next investigate how the optimal source power \( 2P_{\text{avg}} \kappa^* \) is a function of the position of the relay. To do this, we consider again the scenario where the source and destination are one unit apart, and the relay is a distance \( 0 \leq d \leq 1 \) from the source. Given a pathloss exponent \( \alpha \), this leads to \( \lambda_1 = 1 \), \( \lambda_0 = \frac{1}{d^\alpha} \) and \( \lambda_2 = \frac{1}{(1 - d)^\alpha} \). To find the optimal source-relay power ratio, it suffices to minimize the outage probability of (4.10) over all \( \kappa \). Performing the minimization, the optimal value of \( \kappa \) is
\[
\kappa^* = \frac{(1 - d)^\alpha - 4d^\alpha + \sqrt{(1 - d)^{2\alpha} + 8(1 - d)^\alpha d^\alpha}}{4(1 - d)^\alpha - 4d^\alpha}. \tag{4.11}
\]
An interesting property of the power ratio is that the solution is independent of the network power constraint \( 2P_{\text{avg}} \). Another interesting point is that the solution
Figure 4.6 : Savings in power by using the optimal source-relay power ratio vs. equal power among source and relay assuming a rate $R=1$ nats/sec/Hz and $P_{\text{out}} = 10^{-2}$. The $d$-axis represents the relay’s fractional distance between the source and destination. The savings in power corresponds to the reduction in average power that is achieved by using the optimal power ratio versus equal power allocation between the source and relay.

$\kappa^* \geq 0.5$, meaning that the relay should never transmit with more power than the source. In Figure 4.6, the savings in power by using the optimal power ratio of (4.11) are seen. By using the optimal ratio, up to 3dB is saved over equal power allocation between the source and relay. However, we see that for small distances ($d < 0.5$), the gains of using the optimal ratio are minimal. As a result, when the source is close to the destination, using equal power for the source and relay is a good strategy.
4.4 Conclusions

In this chapter, we have analyzed power control methods to approach the fundamental limits in the fading relay channel for varying degrees of side information at the source and relay. When perfect side information is available at the transmitters, significant energy savings over constant power transmission can be obtained through optimal power control. However, we showed that only a few bits of feedback are sufficient to achieve most of the gains of the optimal CSIT power control policy. This result suggests the importance of designing protocols that incorporate feedback in future wireless networks, as even limited amounts of feedback will translate to significant increases in battery life for mobile nodes.

We also analyzed the case where no side information was available at the transmitters. Interestingly, transmitting with equal power at the source and relay is close to optimal, especially for relays positioned close to the source. This hints at the importance of the relay’s contribution in improving system performance, as the power of the relay needs to be similar to the source’s power in order to minimize the outage.
Chapter 5

Throughput Gains with Limited Feedback

5.1 Introduction

The work described in the previous chapters has focused on outage minimization subject to a power constraint on the transmitters. We have looked at both peak and average power constraints, and also considered the practical problem of having a limited feedback channel. Up to this point, the source applications for which the derived algorithms are appropriate have been ones that transmit at a fixed rate over all time. In this case, the outage probability is the appropriate metric to minimize. However, in applications that can support a variable rate of transmission such as data transfer, one is more concerned with maximizing the overall throughput to allow as much correctly decoded data packets as possible at the destination.

Typically, the performance of systems in the block fading network environment can be characterized by the outage probability [8] and the delay-limited capacity [29]. However, due to outage events, the effective data rate, or throughput, is less than the attempted transmission rate. Higher transmission rates leads to a higher frame error rate (FER), and low transmission rates translate to low FER’s. The FER, which serves as a tight lower bound to the outage probability [30], needs to be balanced with the transmission rate to maximize the throughput. We consider the practical
problem of throughput maximization, which reflects end-to-end performance more accurately than the rate of transmission.

In the network setting, we consider different possible methods of transmission: relay coding, data forwarding, and direct transmission. We derive throughput maximization algorithms based on properties of the transmit buffer. First, assuming that the source always has enough information bits to send, we outline the optimal rate control procedure which maximizes the system throughput for the above mentioned transmission techniques. The analysis assumes that the rate control process is solely a function of the network channel state with no finite backlog effects. Interestingly, we observe that for a finite rate of feedback there is an optimal outage operating point that maximizes the throughput which, for a small rate of feedback, can be greater than 10%. This suggests that minimizing the outage probability is unnecessary to maximize the throughput. In fact, for a finite rate of feedback, it is better to increase the coding rate and allow for some frame errors. Furthermore, we show that for a limited rate of feedback, it is unnecessary to temporally vary both the rate and power. For small average power constraints, power control suffices to maintain a high throughput. On the other hand, for large average power constraints, adapting the transmission rate while using a constant power is nearly as useful as a transmission employing a variable rate and variable power.

Second, we consider the scenario where the buffer size is limited, and supplied by packets at a constant rate. Outages can occur in a network from both decoding errors
at the receiver and dropped packets at the source resulting from buffer overflows. We
show the gains in throughput achievable by increasing the buffer size and by having
channel state information available at the transmitters. Our results demonstrate that
transmission with the aid of a relay node leads to a large increase in throughput over
direct transmission. Also, the effect of power control in conjunction with rate control
is shown to provide significant gains in throughput over constant power transmission.
Much like the case of ergodic capacity, temporal power adaptation provides most of
its gains over constant power transmission for small average power constraints.

Our results suggest the power of the relay transmission paradigm. Decode and
Forward [18], one of several cooperative protocols considered in this work, is shown to
have significant gains in throughput over both data forwarding and direct transmis-
sion. We show that even a low complexity relay protocol such as selection relaying [9]
can achieve large throughput gains over direct transmission for many practical cases
of interest. Our analysis reveals that the true benefits of relaying become evident in
wireless networks when feedback regarding channel state information is available at
the transmitters. Even limited amounts of feedback lead to large gains in throughput.
As a result, next generation relay protocols should incorporate channel state feedback
for throughput maximization in order to realize the true benefits of node cooperation.

The remainder of this chapter is as follows. Section 5.2 outlines the rate con-
trol procedure assuming an infinite supply of source data. Section 5.3 analyzes the
throughput when the source rate is limited, and the queue size is finite. Section 5.4
Figure 5.1: For a fixed $\gamma_0$, a typical set of rate control regions is shown for $\log_2 L$ bits of feedback in the relay channel. Below the dotted line in $\mathcal{R}_1$ indicates the outage region. The contour separating $\mathcal{R}_{q-1}$ and $\mathcal{R}_q$ denotes all points which can guarantee zero outage while transmitting at a rate $R_q$ and a power $P_q$.

provides concluding remarks.

5.2 Throughput Maximization based on the Network Channel State

In this section, we describe a rate control algorithm for cooperative relay networks which maximizes the throughput for a limited feedback channel. First, the general problem setup is described, and then the analysis is refined for relay codes, multihopping and direct transmission. The transmission policy assumes packet fragmentation is possible up to arbitrary precision [31]. Furthermore, we assume that there is an infinite backlog of data at the source. These assumptions together imply that the transmission policy is a function solely of the network channel state.
5.2.1 Problem Setup

5.2.1.1 Rate Control Regions

Consider a network of 3 nodes with a source communicating to a destination with the possible assistance of a relay node. We assume that the destination has a perfect estimate of the network channel state $\gamma$. Given $M$ bits of feedback, the space defined by all possible $\gamma$ is quantized into $L = 2^M$ regions $\mathcal{R}_q, q \in \{1, 2, \ldots, L\}$. Upon measuring the channel state, if $\gamma \in \mathcal{R}_q$, the destination selects a rate level $R_q$ from a rate control codebook $\mathcal{C}$ and a power $P_q$ from a power control codebook $\mathcal{P}_q$, where $q \in \{1, 2, \ldots, L\}$. The index $q$ is transmitted to both the source and relay through a noiseless feedback link. It is assumed that both the source and relay have copies of $\mathcal{C}$ and $\mathcal{P}$. Upon reception of the index $q$, the source and relay then transmit with rate $R_q = \mathcal{C}(q)$ and power $P_q = \mathcal{P}(q)$.

An example of the $L$ rate control subregions is shown in Figure 5.1 for a given value of $\gamma_0$. The rate control regions can be visualized as three dimensional volumes, where a cross section of the regions at $\gamma_0$ is represented in Figure 5.1. One key feature of the rate control regions is that in region $\mathcal{R}_q, q \geq 2$, for a given power, the rate assigned is the maximum that guarantees zero outage for any point in the region. This is a fundamental property of all optimal finite rate feedback algorithms [12]. In $\mathcal{R}_1$, a subset of the channel states cannot support a rate of $R_1$ for a finite average power constraint, and channel states belonging to this subset would result in
an outage event.

Given that all regions except $R_1$ have a zero outage condition, the boundary between regions $R_{q-1}$ and $R_q$ is separated by a contour, $G(\gamma, P_q, R_q)$ [27], which represents all $(\gamma_0, \gamma_1, \gamma_2)$ such that

$$R_{\text{gen}}(\gamma; P_q, P_q) = R_q,$$

(5.1)

where $R_{\text{gen}}(\cdot)$ is the achievable rate of a generic transmission protocol. Any points lying along the contour can support rate $R_q$ and guarantee zero outage while transmitting with power $P_q$. Any points lying above the contour can support a higher rate than $R_q$ while maintaining zero outage, while points below the contour require a smaller rate than $R_q$ to invert the effect of the channel. The outage region is a subset of $R_1$. For a given $\gamma_0$, the outage region is denoted by any points lying below the dotted curve in Figure 5.1. Any points in this region require a rate less than $R_1$ to invert the effect of the channel. The outage probability in the general setting of time-varying $(\gamma_0, \gamma_1, \gamma_2)$ is expressed as

$$\int_{\gamma: R_{\text{gen}}(\gamma; P_q, P_q) \leq R_1} f(\gamma) d\gamma,$$

(5.2)

where $f(\gamma)$ is the joint distribution of the network channel state $\gamma$.

5.2.1.2 Throughput Analysis

Due to the effect of outages, the effective data rate will be less than the attempted rate of transmission. Given $L$ rate control regions, with outages occurring in $R_1$, the
throughput can be expressed as

\[ T = R_1(\Delta_1 - \epsilon) + \sum_{q=2}^{L} R_q \Delta_q, \]

where \( \epsilon \) is the outage probability defined in (5.2) and

\[ \Delta_q = \int_{R_q} f(\gamma) d\gamma. \]

The objective is to maximize the throughput subject to a long term power constraint of

\[ \sum_{q=1}^{L} P_q \Delta_q \leq P. \]

For a given outage probability, the throughput can be maximized by finding appropriate rates and powers for the codebooks \( \mathcal{C} \) and \( \mathcal{P} \), which involves searching for a set of \( L \) contour functions. For variable rate transmission with a constant transmit power (VRCP), as the rate of the feedback link increases, the outage probability can be made arbitrarily small. In the limit with perfect feedback, the throughput can be expressed as

\[ T = \int_{\gamma} R_{gen}(\gamma, P, P) f(\gamma) d\gamma. \]

(5.3)

Note that (5.3) is achieved without outages, as the rate is adapted for each and every channel state. It also serves as an upper bound to the rate control finite rate feedback procedure whenever constant power transmission is assumed.

For constant rate transmission with variable power (CRVP), throughput maximization involves two steps. First, given the rate of the feedback channel and a
fixed transmission rate, the outage probability is minimized using the power control strategies in [27]. Second, rate selection is performed to find the transmission rate that leads to the largest throughput. When perfect feedback is available, the power allocation procedure resembles channel inversion for all but a subset of the channel states. In poor channel conditions, transmission is shut off and an outage is declared to meet the long-term power constraint [32].

Variable rate transmission with power control (VRVP) leads to the largest possible throughput given a long-term power constraint. In the limit of perfect feedback, the optimal solution involves power control based on the concept of 'water-filling' [23]. In this case, transmission is shut off in poor channel conditions, and, in better channel conditions, increased rate and power are used to maximize the throughput. The power allocation strategies for constant rate and variable rate transmission are quite different. In constant rate transmission, more power is allocated for poor channel states to reduce the risk of an outage. In variable rate transmission, increased power is used for better channels to maximize the throughput.

It will be seen that, even with a few bits of feedback, using VRVP to maximize the throughput leads to tremendous gains over the case of constant rate/constant power transmission (CRCP). Interestingly, we will show that in some domains of operation, it suffices to use CRVP and in other cases using VRCP is sufficient to achieve large throughput gains over CRCP transmission.
5.2.2 Relay Codes

To apply the throughput analysis to relay codes, the rate control regions must be derived for a given protocol. For the Decode and Forward protocol, the rate control regions are shown in Figure 5.2 for the case of 1 bit of feedback. With 1 bit of feedback, there are two rate control regions. In $R_1$, a rate of $R_1$ and a power of $P_1$ is used, and in $R_2$, a rate of $R_2$ and a power of $P_2$ is used. The boundary between $R_1$ and $R_2$ is separated by $G(\gamma, P_2, R_2)$, which is the shaded surface in the figure, and is obtained by solving (5.1) with $R_{gen}()$ substituted by the achievable rate expression for Decode and Forward. The contour $G(\gamma, P_2, R_2)$ is characterized by the term $K/P_2$ in Figure 5.2, where $K = e^{R_2} - 1$.

In Figure 5.2, we see that as long as $\gamma_0 < K/P_2$, then the channel state lies in...
and as a result the possibility of an outage exists, irrespective of the value of $\gamma_1$ or $\gamma_2$. This emphasizes that the performance of Decode and Forward is strongly influenced by the source-relay link. The outage region for Decode and Forward is a subset of $\mathcal{R}_1$, and its boundary is a contour, similar to the boundary between $\mathcal{R}_1$ and $\mathcal{R}_2$ shown in Figure 5.2, except the outage region contour is parameterized by $\frac{e^{\gamma_1} - 1}{P_1}$.

The following theorem formally states that the boundary between $\mathcal{R}_1$ and $\mathcal{R}_2$ shown in the figure is a contour based on the definition in (5.1).

**Theorem 5.2.1.** For the rate control contour of Decode and Forward separating $\mathcal{R}_1$ and $\mathcal{R}_2$ with 1 bit of feedback, any points lying above this contour can support a rate larger than $R_2$ and still guarantee zero outage, while points below this contour require a rate less than $R_2$ to guarantee zero outage.

**Proof:** See Appendix C

In the derivation of the rate control region for decode and forward, we have assumed that $\rho = 0$, implying that the source and relay transmit signals are uncorrelated. Adapting $\rho$ along with the rate and power would lead to a gain over assuming a constant $\rho$ over all time. However, we next show that for the case of constant rate and power transmission, selecting a value of $\rho = 0$ is optimal.

**Theorem 5.2.2.** For decode and forward relaying in the absence of channel state information, selection a source-relay correlation value of $\rho = 0$ maximizes the throughput.
Proof: See Appendix D

The rate control region for selection relaying resembles that of Decode and Forward, and is described next. Consider the case of 1 bit of feedback. A key difference between SR and DF is that the contour $G(\gamma, P_2, R_2)$ is now defined by $K = e^{2R_2} - 1$. The factor of two in the exponent is to compensate for the loss associated with half-duplex relaying. In SR, when $\gamma_0 \geq \frac{K}{P_2}$, the contour $G(\gamma, P_2, R_2)$ is identical to that of DF, except that $K = e^{2R_2} - 1$. When $\gamma_0 < \frac{K}{P_2}$, the contour $G(\gamma, P_2, R_2)$ is a cube defined by all $\gamma$ such that $\gamma_0 = \frac{K}{P_2}$ and $\gamma_1 = \frac{K}{2P_2}$. The structure of the contour can be verified by substituting the achievable rate of selection relaying for $R_{\text{gen}}(\cdot)$, and solving (5.1).

5.2.3 Multi-hopping

For multi-hopping, the throughput maximization procedure is similar to that of Decode and Forward, except now the rate control contours are defined by the space of all $(\gamma_0, \gamma_2)$ rather than $(\gamma_0, \gamma_1, \gamma_2)$. The rate control contour for multi-hopping is shown in Figure 5.3 for 1 bit of feedback. In region $\mathcal{R}_2$, rate $R_2$ and power $P_2$ are used, and in region $\mathcal{R}_1$, rate $R_1$ and power $P_1$ are used. Outages only occur in $\mathcal{R}_1$, and they correspond to all channel states that require a smaller rate than $R_1$ to invert the effects of the channel. The contour $G(\gamma, P_2, R_2)$ is characterized by the parameter $K/P_2$, where $K = e^{R_2} - 1$. For constant rate transmission, closed form results are possible for multi-hopping.
Theorem 5.2.3. For multi-hopping with the source and relay transmitting with a constant power $P$ and at a constant rate, the maximum throughput is

$$T = (1 - \epsilon^*) \log \left( 1 - P \log(1 - \epsilon^*) \frac{\lambda_2 \lambda_0}{\lambda_2 + \lambda_0} \right),$$  \hspace{1cm} (5.4)

where

$$\epsilon^* = 1 - e^{-K P \lambda_2^2} - \frac{\lambda_0}{\lambda_2 + \lambda_0},$$  \hspace{1cm} (5.5)

and $C = \frac{\lambda_2 \lambda_0}{\lambda_2 + \lambda_0}$. The function $W(\cdot)$ is known as the Lambert-$W$ function, which satisfies $W(x) e^{W(x)} = x$.

**Proof:** For the case of constant rate transmission, only one rate $R_1$ and one power $P$ is used for all channel states. The outage region is characterized by a contour, which subdivides $R_1$ into two subregions. Channel states lying below the contour are always in outage, while points lying above the contour are never in outage.

The structure of the outage contour can be understood by considering the rate control region for 1 bit of feedback shown in Figure 5.3. The boundary between $R_1$ and $R_2$ is a contour, it and represents all channel states that require a rate $R_2$ to guarantee zero outage. In much the same way, the outage region contour is a surface that represents all channel states that require a rate $R_1$ to guarantee zero outage. Any points lying below this contour require a smaller transmission rate than $R_1$ to invert the effects of the channel, and there is an outage for those states. The area of the outage region can be expressed as

$$\epsilon = e^{-\frac{K}{\lambda_2^2}} \left( 1 - e^{-\frac{K}{\lambda_0}} \right) + 1 - e^{-\frac{K}{\lambda_2^2}},$$
Figure 5.3: For the multi-hopping system, rate control regions are shown for 1-bit of feedback. Rate level $R_2$ is used in region 2, and rate level $R_1$ is used in region 1.

where $K = e^{R_1} - 1$. Solving for the rate leads to

$$R_1 = \log \left( 1 - P \log(1 - \epsilon) \frac{\lambda_2 \lambda_0}{\lambda_2 + \lambda_0} \right).$$

Maximizing the throughput $(1 - \epsilon) R_1$ with respect to $\epsilon$ leads to (5.5).

As the source-relay distance decreases, then $\lambda_0 \to \infty$, and $\lambda_2 \to 1$. This scenario corresponds to the throughput of direct transmission, which is less than multi-hopping for any given $\epsilon$, under the relay model shown Figure 2.2.

5.2.4 Direct Transmission

The analysis for direct transmission is simplified in that instead of a network channel state, only the fading parameter $\gamma_1$ is used. Furthermore, rather than a rate control contour, a threshold is used to separate two subregions. A similar approach for direct transmission was taken in [33], however in this section we consider throughput as a
function of the outage probability, which in the results section leads to interesting insight with regards to optimal outage operating points and the importance of rate selection.

We next proceed to define the throughput maximization problem for direct transmission, and consider the solution for some special cases. The space defined by all $\gamma_1$ is divided into $L$ subregions, with each subregion corresponding to a rate of transmission. Rate control value $R_q$ is chosen if $\beta_{q-1} \leq \gamma_1 \leq \beta_q$. As noted earlier, the outage region is always a subset of $\mathcal{R}_1$. The throughput with limited feedback at an outage probability of $\epsilon$ is expressed as,

$$T = R_1 \int_{\beta_0}^{\beta_1} f(\gamma_1) d\gamma_1 + \sum_{q=2}^{L} R_q \int_{\beta_{q-1}}^{\beta_q} f(\gamma_1) d\gamma_1,$$

(5.6)

where $R_q$ is the rate of transmission in region $q$, $\beta_q = \infty$, and $\beta_0$ is the cutoff value chosen such that all channel gains lying below this cutoff will be in outage.

The throughput maximization of (5.6) is done over all power allocations which satisfy $\sum_{q=1}^{L} \Delta_q P_q \leq P$, where $\Delta_q = \int_{\beta_{q-1}}^{\beta_q} f(\gamma_1) d\gamma_1$ denotes the probability of the channel state lying in region $\mathcal{R}_q$. For Rayleigh fading this probability can be expressed as $\Delta_q = e^{-\beta_{q-1}} - e^{-\beta_q}$. If we rewrite the throughput as $T = \sum_{q=1}^{L} \left( e^{-\beta_{q-1}} - e^{-\beta_q} \right) R_q$, then to find the optimal value of $\beta_q$, we need to solve $\frac{\partial T}{\partial \beta_q} = 0$, which leads to

$$e^{-\beta_q} \log(1 + P_q \beta_q) + \frac{(e^{-\beta_q} - e^{-\beta_{q+1}}) P_{q+1}}{1 + P_{q+1} \beta_q} - e^{-\beta_q} \log(1 + P_{q+1} \beta_q).$$

(5.7)

Therefore, threshold $\beta_q$ will be a function of thresholds $\beta_{q-1}$ and $\beta_{q+1}$. We know that $\beta_0$ is fixed for a given outage probability $\epsilon$, and that $\beta_L = \infty$. As a result, for a few
special cases of the rate of the feedback link, closed form results are possible.

For the case of 1 bit of feedback, it’s not hard to verify that the objective function $T$ is convex and as such a unique maximum exists. Additionally, we know that $\beta_2 = \infty$, and threshold level $\beta_1$ can be solved for explicitly as

$$\beta_1 = \frac{1}{\mathcal{W}\left(\frac{P_2}{1 - P_1 \log(1 - \epsilon)}\right)} - \frac{1}{P_2},$$

where $\mathcal{W}$ is the Lambert-W function. The throughput of this scheme is

$$T = (1 - \epsilon - e^{-\beta_1}) \log(1 - P_1 \log(1 - \epsilon)) + e^{-\beta_1} \log(1 + P_2 \beta_1).$$

(5.9)

As another special case, consider the scenario when $M = 0$, i.e., when there is no feedback. In this case, a constant rate and power is used for each time instant. The solution can be obtained by setting $\lambda_2 = 1$ and $\lambda_0 = \infty$ in (5.4) for multi-hopping. As a result, we obtain the following throughput expression

$$T = \log(1 - P \log(1 - \epsilon)) \cdot (1 - \epsilon).$$

(5.10)

Interestingly, for any outage probability that does not equal 0 or 1, the throughput is greater than zero. The same cannot be said about the delay-limited capacity, which is zero for the block Rayleigh fading channel.

### 5.2.5 Analysis and Results

Figure 5.4 shows the throughput versus outage results for Decode and Forward and direct transmission. In this figure, constant power transmission is used in each rate
control subregion. We assume that the total network signal to noise ratio (SNR) is $10\,dB$ for the relay coding case, which includes power from both the source and relay. For direct transmission, to fairly compare against relaying, we use a source power that equals the total network power used in the relay case. Additionally, we assume that the source-relay distance is $d = 0.5$. This implies that the relay is midway between the source and destination. We see that for decreasing outage probabilities, both Decode and Forward and direct transmission have a decreasing throughput. This result confirms that for Rayleigh block fading the delay-limited capacity is zero, meaning that a non-zero throughput is not possible when error free communications is desired. However, we observe that a large throughput is in fact possible, which
suggests that delay-limited capacity is not a good measure of performance in block fading channels.

Observing Figure 5.4, it is apparent that for small outage probabilities, the throughput is always non-zero for both Decode and Forward and direct transmission when feedback is incorporated. The maximum throughput for Decode and Forward increases by more than 33% with the use of 2 rate control levels. Furthermore, the gap to the perfect feedback rate control limit is substantially reduced with the addition of just one bit of feedback. Interestingly, from Figure 5.4, we see that with one feedback bit, Decode and Forward has a larger maximum throughput than direct transmission with perfect feedback.

Power control in conjunction with finite rate feedback has shown great promise in the relay channel for outage reduction with fixed rate transmission [27]. When the rate and power are adapted over time, in the limit of perfect feedback, performance is upper-bounded by the ergodic capacity of the channel, which can be achieved with a water-filling based power control algorithm [23]. Figure 5.5 demonstrates the effect of a combined power/rate control (VRVP) procedure in comparison with variable rate/constant power (VRCP) and constant rate/variable power (CRVP) throughput maximization at an SNR of 10dB. Adapting both the rate and power leads to a larger throughput than VRCP. However, we see that the gains over VRCP are not tremendous for large average power constraints. For example, using 1-bit of feedback, the throughput gains over 0 feedback bits are approximately 28% at an SNR of 10dB.
for both VRCP and VRVP. Furthermore, both VRCP and VRVP achieve diminishing returns with increasing bits of feedback.

Also shown in Figure 5.5 is the throughput when a constant rate is transmitted and the feedback bits are used solely for power control (CRVP). In this case, rather than using multiple transmission rates, one rate is selected that maximizes the throughput. We see that CRVP leads to a reduced throughput over techniques which use a variable rate. Rate adaptation is essential at large average powers. On the other hand, for small power constraints, power adaptation is imperative. This is seen by comparing Figure 5.5 and Figure 5.6. When the SNR is $10dB$, VRCP is nearly as useful as VRVP. However, for smaller average power constraints, power control is essential to maximize the throughput. In fact, as shown in Figure 5.6, transmitting at a constant rate and using power control (CRVP) significantly outperforms VRCP at an SNR of $-5dB$. At an SNR of $2dB$, the performance of VRCP and CRVP are almost similar, with CRVP having a modest throughput gain over VRCP, as can be seen from Figure 5.7.

Another interesting conclusion that can be drawn is that for a low rate feedback link, simultaneous power and rate adaptation is unnecessary. For small SNR’s, power control and rate selection (CRVP) is nearly optimal for 1 bit of feedback. On the other hand, for large SNR’s, rate control with constant power transmission (VRCP) is nearly optimal when 1-bit of feedback is used.
Figure 5.5: For an SNR of 10dB, the throughput of decode and forward is shown as a function of the number of feedback bits for a source-relay distance of $d = 0.5$. Adapting both the rate and power leads to increases over constant power transmission, with the gains most evident as the feedback rate increases. Power control alone leads to good performance for small average powers.

5.2.5.1 Optimal Outage Operating Point

Observing Figure 5.4 once again for the case of Decode and Forward with constant rate transmission, we see that the throughput is maximized at a large outage probability, greater than 10%. For increasing amounts of feedback, the optimal outage probability decreases, and only in the limit of perfect feedback does it equal zero. The result suggests that, for a finite rate of feedback, it is unnecessary to minimize the outage probability when constant power transmission is used. Rather, the coding rates should be appropriately chosen to meet the throughput maximizing outage probability, which can be quite high.
Figure 5.6: For an SNR of $-5 dB$, the throughput of decode and forward is shown as a function of the number of feedback bits for a source-relay distance of $d = 0.5$. Adapting both the rate and power leads to increases over constant power transmission, with the gains most evident as the feedback rate increases. The performance is further from the combined rate/power control limited at high average powers.

When power control is utilized with rate selection to maximize the throughput, the outage probability can still be quite large. For example, for direct transmission with perfect feedback and power control, the maximum throughput at $SNR = 10 dB$ is 1.84 nats/sec/Hz, at an outage probability of approximately 25%. This emphasizes that high outage operating points are necessary whenever power control is employed. It is better to increase the coding rate, and allow for frame errors to maximize the long-term throughput. Interestingly, for a given rate of feedback, rate control leads to a smaller optimal outage operating point than power control.

For the case of direct transmission with constant power transmission, explicit results can be found for the case of constant rate transmission. Maximizing the
Figure 5.7: Throughput of Decode and Forward for SNR=2dB. At this SNR, power and rate control have similar performance, with the power control algorithm having a slightly higher throughput than rate control.

The throughput of (5.10) over all \( \epsilon \) leads to the following optimal operating point

\[
\epsilon^* = 1 - e^{-\left(\frac{1}{\mathcal{W}(e)} - \frac{1}{\mathcal{W}'}(e)\right)},
\]

where \( \mathcal{W} \) is the Lambert-W function. For example, at an SNR of 10dB, the throughput is maximized when the outage probability is approximately 20%.

### 5.2.5.2 Comparison of Cooperative Techniques

We next compare results for various collaborative communication techniques. Figure 5.8(a) shows the percentage gain of various protocols over constant rate direct transmission when the source-relay distance \( d = 0.5 \) for constant power transmission. Multi-hopping has the smallest gains over direct transmissions. Furthermore, the gains over direct transmission are diminishing with respect to power. Decode and
Figure 5.8: For a source-relay distance of (a) $d = 0.5$, and (b) $d = 0.25$, the % gain in throughput of various protocols over constant rate direct transmission is shown as a function of SNR. The protocols shown are direct transmission (DT), Decode and Forward (DF), and multi-hopping (MH). In all cases, the transmit power is constant.

Forward has large gains over both direct transmission and multi-hopping. Adding feedback increases the gains over constant rate direct transmission for all the protocols. The gains of cooperative communication are quite apparent when feedback is considered. For an SNR of $5dB$, Decode and Forward with 1 bit of feedback has a 115% gain over constant rate direct transmission, whereas direct transmission with 1 bit of feedback has a 33% gain over constant rate direct transmission. As the SNR increases, the gains over direct transmission are diminishing. However, for a practical range of SNR’s, tremendous gains in throughput are seen with the rate control procedure in combination with relay coding.

When considering a smaller source relay distance, such as $d = 0.25$ depicted in Figure 5.8(b), we see that the gap in performance between multi-hopping and Decode and Forward increases. For small source-relay distances, it has been observed
Figure 5.9: The effect of individual energy sources for the source and relay with Decode and Forward (DF) and selection relaying (SR) are shown in comparison to direct transmission (DT) for a source-relay distance of $d = 0.5$ and constant power transmission for (a) no feedback and (b) 1 bit of feedback for rate control.

that Decode and Forward is nearly optimal in the case of outage minimization with power control [32] and for the case of ergodic capacity [24]. For a value of $d = 0.25$, multi-hopping has a small gain over direct transmission, but cooperative coding clearly surpasses multi-hopping. Multi-hopping gains over direct transmission from strong source-relay and relay-destination links. As the source-relay distance decreases, there is an almost noiseless source-relay link. However, the relay-destination average channel gain approaches that of the source-destination, making multi-hopping only marginally superior to direct transmission.

5.2.5.3 Individual Power Constraints on the Source and Relay

In comparing relaying to direct transmission, the results to this point have used a sum power constraint on the source and relay and equated this with the power used in direct transmission. Next, we consider another practical scenario in which we compare
a system where the source always has the same power for both direct transmission and relaying. In relaying schemes, the relay has its own individual energy source, and we assume $P_r = P_s$. The results shown in Figure 5.9(a) and Figure 5.9(b) demonstrate the gains of relaying in this scenario. The performance of direct transmission is shown for constant rate transmission in comparison with selection relaying and Decode and Forward in in Figure 5.9(a). We see that, despite the rate loss from the source not transmitting data over the entire time slot, selection relaying has a gain of $4\, \text{dB}$, and Decode and Forward a gain of $7\, \text{dB}$ over direct transmission at a spectral efficiency of $1\, \text{nat/sec/Hz}$. As the SNR increases, selection relaying eventually performs worse than direct transmission. However, for a practical range of SNR’s, it is superior to direct transmission. In the low power regime, the spectral efficiency is proportional to the average received power. Due to the better channels in the source-relay and relay-destination links, the use of selection relaying leads to a large gain over direct transmission in the power limited regime. Figure 5.9(b) shows a similar trend for the case of 1 bit of feedback used for rate control.

### 5.3 Impact of Finite Queue Backlogs on Throughput

Controlling the transmission rate over time can lead to significant improvements in throughput. In practice, the transmission rate is not only a function of the channel state, but also of the amount of data available in the transmission buffer. In periods of poor channel conditions, transmitting at low rates leads to an increased backlog
in the transmit buffer. When the backlog exceeds the buffer size, dropped packets result and this leads to another source of outage in a network. Transmitting at a higher rate is possible only if the data backlog is sufficient to allow for large transmit spectral efficiencies. In this section, we study the interplay between finite backlogs, maximum buffer sizes and throughput.

### 5.3.1 Transmission and Queueing Model

The network channel state $\gamma$ is assumed constant for $T_c$ consecutive symbols, and varies independently from one time slot to the next. For large values of $T_c$, the achievable rates of Section 2.4 serves as a good indicator of performance for coded systems [8]. Hence, given a source power of $P_s$ and a relay power of $P_r$, an outage event occurs if the attempted rate of transmission is larger than $R_{gen}(\gamma, P_s, P_r)$, where $R_{gen}(\cdot)$ is the conditional mutual information of a generic network transmission protocol. This implies that an outage occurs if the number of information bits in a packet exceeds $T_c \cdot R_{gen}(\cdot)$. For the remainder of this section, we assume that $P = P_s = P_r$ for simplicity of exposition and that $T_c$ is large enough so that the achievable rate serves as a good indicator of performance.

A constant rate source provides a single packet of size $R \cdot T_c$ information bits in each time slot into a first come first serve queue of size $V$ packets located at the source node. The buffer state at time $t$ is $q_t \leq V$, which is the number of buffered packets. In state $q_t$, the source may transmit any of $r_t = \{0, 1, \ldots, q_t\}$ packets.
This implies a packet integrity constraint [31], meaning that multiple packets may be transmitted in a particular time slot, but packets may not be fragmented over multiple time slots. The transmission rate, $r_t$, and power are determined by the amount of feedback available from the destination, the current buffer state, and the chosen network coding protocol. The relay node is assumed to transmit with the same rate and power as the source and is not required to buffer any of the source’s data. With the use of feedback, the transmission rates can be chosen to avoid outage events at the destination. In poor channel states, no transmission occurs and the data is buffered. This is a similar model to that used in [34].

The source and relay have $V$ power codebooks $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_V$, where

$$\mathcal{P}_k = \{P_{k,0}, P_{k,1}, P_{k,2}, \ldots, P_{k,k}\}$$

and $P_{k,0} = 0$. Power level $P_{k,i}$ is used when the buffer size $q_t = k$ and the transmission rate is $iR$, $i \in \{0, 1, \ldots, k\}$. The decision as to which rate and power to use is determined by the destination, which has a perfect measure of the network channel state $\gamma$. If the destination feeds back index $i$ to the transmitters when $q_t = k$, then the source and relay are to transmit with rate $iR$ and power $\mathcal{P}_k(i) = P_{k,i}$ in the current time slot.

It is assumed that the feedback link is noiseless, and as a result the source and relay are always able to correctly infer the current transmission rate and power to be used. In order to calculate the appropriate rate and power allocation, the relay and
Figure 5.10: For a constant arrival rate of $R = 0.5 \text{nat/sec/Hz}$, the (a) normalized throughput and (b) outage probability is shown for Decode and Forward various protocols over constant rate as a function of SNR. The results are shown for the case of no feedback, and also for delay bounds of $V = 1, 2, 3, 4$. The power is constant whenever the rate is non-zero.

destination must be aware of the current source buffer state. The relay and destination only need the starting buffer state, which can be discovered in the initialization phase of the communication process. The protocol overhead to communicate the initial buffer size is $\log_2 V$ bits. Since the destination calculates the transmission rate for the source and relay to use, and since the arrival rate to the source is constant, the destination will be able to keep track of the state of the source buffer without any additional protocol overhead. Similarly, the relay will have the initial buffer state, and throughout the communication process it can maintain an update of the state of the source buffer when it receives feedback from the destination.
5.3.2 Throughput Maximization

Given the probability of being in buffer state \( k \) is \( s_k, k \in \{1, \ldots, V\} \), the long-term throughput can be expressed as

\[
T = \sum_{k=1}^{V} s_k \sum_{i=0}^{k} i \cdot \Delta_{k,i} \cdot R, \tag{5.12}
\]

where for a given power control policy, \( \Delta_{k,i} \) is the probability of transmitting \( i \) packets error-free while in buffer state \( k \). Let \( R_{k,i} \) denote all channel states that lead to the transmission of \( i \) packets in buffer state \( k \). In the transmission scenario described above, outages occur if a packet is dropped at the source due to buffer overflow. Since it is assumed that exactly one packet arrives at the buffer in each time slot, overflow can only occur while \( q_t = V \), and zero packets are transmitted in the current time slot. The blocking probability is expressed as

\[
\Pi = s_V \cdot \Delta_{V,0}. \tag{5.13}
\]

When feedback is unavailable, then the source will always transmit one packet in each time slot. Consequently, there will be no dropped packets, but outages will occur due to decoding errors at the destination. When feedback is available, channel outages can be removed by buffering, but outages still result due to dropped packets. In the case of dropped packets, the outage probability is obtained from (5.13).

It is not hard to verify that the buffer state \( q_t \) forms a stationary Markov chain with \( V \) possible states [34]. Defining the state transition probability as \( p_{ij} = Pr[q_{t+1} =
\( \text{if } q_t = i \), and \( \mathbf{C} = [\mathbf{p}_{ij}] \), then the stationary buffer state probabilities can be found as the solution to
\[
\mathbf{C} \mathbf{s} = \mathbf{s},
\]  
(5.14)

where \( \mathbf{s} = [s_1 s_2 \ldots s_V]^t \). We seek to maximize the throughput in (5.12) for a given average power constraint
\[
\sum_{k=1}^{V} s_k \sum_{i=0}^{k} P_{k,i} \Delta_{k,i} \leq P, 
\]  
(5.15)

where \( P_{k,i} \) is the power used when the buffer state \( q_t = k \) and the transmission rate \( r_t = i \). The value of \( P_{k,i} \) can be optimized to maximize the throughput. Alternately, the power can also be constant whenever it is greater than zero. This implies that \( P_{k,i} = P_{j,m}, \forall i, m \neq 0 \). We will show results for both variable and constant \( P_{k,i} \) in Section IV.D.

The rate probabilities \( \Delta_{k,i} \) are dependent on the fading distribution and the transmission protocol. For Decode and Forward, \( \Delta_{k,i} \) is derived by considering the rate control contour of Figure 5.2. Given a rate \( iR \) and power \( P_{k,i} \), a contour \( G(\gamma; iR, P_{k,i}) \) exists with defining parameter \( \psi_{k,i} = (e^{iR} - 1)/P_{k,i} \). The rate probability \( \Delta_{k,i} \) is the probability of a channel state lying between the contours with parameters \( \psi_{k,i} \) and \( \psi_{k,i+1} \). Let
\[
\Psi_{k,i} = 1 - \left[ \left( 1 - e^{-\psi_{k,i}/\lambda_1} - e^{-\psi_{k,i}/\lambda_2} \left( 1 - e^{-\psi_{k,i}/(\lambda_1 - \lambda_2)} \right) \frac{\lambda_2}{\lambda_2 - \lambda_1} \right) e^{-\psi_{k,i}/\lambda_0} + 1 - e^{-\psi_{k,i}/\lambda_0} \right],
\]
where $\psi_{k,i} = (e^{iR} - 1)/P_{k,i}$. Then
\[
\Delta_{k,i} = \Psi_{k,i} - \Psi_{k,i+1}
\]
for $i = \{0, 1, \ldots, k\}$. Furthermore, $\Psi_{k,0} = 1$ and $\Psi_{k,k+1} = 0$. The value of $\Psi_{k,i}$ is in fact the probability of lying above the contour $G(\gamma, iR, P_{k,i})$ of Figure 5.2 which is characterized by the parameter $\frac{e^{iR} - 1}{P_{k,i}}$. Similar expressions can be obtained for direct transmission, multi-hop and selection relaying. Maximizing the throughput of (5.12) is analytically intractable for the relay channel. We will therefore resort to numerical optimization techniques.

### 5.3.3 Increasing Feedback Bits

As described previously, all nodes in the network have the ability to track the buffer state of the source. The feedback bits are used to determine which power and rate
are to be used by the source and relay. The power codebook chosen is determined by
the current buffer state. The source and relay then use the index received from the
destination in the selected power control codebook to decide which power to send with.
As a result, the rate of feedback required is \( \log_2(V+1) \) bits, which corresponds to the
size of power codebook \( \mathcal{P}_V \). All other power codebooks have less than \( V+1 \) elements,
and therefore this feedback rate is sufficient to index into any power codebook. The
transmission rate to use is easily determined by noting that the index into the power
codebook also corresponds to the number of packets to be transmitted. In other
words, received index \( i \in \{0, 1, \ldots, k\} \) in buffer state \( k \) corresponds to \( i \) packets
transmitted in the current time slot.

It was shown in [34] that further performance improvements are possible through
the use of additional feedback bits for power control. Given a buffer state of \( q_t = k \), then the source and relay use power control codebook \( \mathcal{P}_k \) which has elements
\( \{P_{k,0}, P_{k,1}, P_{k,2}, \ldots, P_{k,k}\} \). Power \( P_{k,i} \) corresponds to a transmission rate of \( iR \). It is
possible to include additional power control levels for any particular rate of transmis-
sion. For example, if additional power levels of \( P_a \) and \( P_b \) are added to codebook \( k \)
in the following manner

\[
\mathcal{P}^*_k = \{P_{k,0}, P_{k,1}, P_a, P_{k,2}, \ldots, P_{k,k}, P_b\},
\]

then the transmission rate of \( R \) now has two potential power levels of \( P_{k,1} \) or \( P_a \), and
similarly transmission rate \( kR \) also has two power levels. The index for this codebook
must now range from 0 to $k + 2$. In general the rate of the feedback link will be 
$\log_2 X$, where $X$ is the size of the largest power codebook in the set $\mathcal{P}_1, \mathcal{P}_1, \ldots, \mathcal{P}_V$.

5.3.4 Analysis and Results

We next consider the performance of the proposed rate and power control strategy. The objective is to maximize the throughput, which can never exceed $R$, the arrival rate of the source. The throughput is dependent on many factors such as the size of the source buffer, the number of feedback bits, and whether or not power control is utilized. We will consider each of these factors.

Figure 5.10(a) shows the normalized throughput for Decode and Forward as a function of SNR for different buffer sizes. In this figure, constant power transmission is used whenever data is transmitted. This implies that $P_{k,i}$ is a constant for all $i$ and $k$ in (5.15). As a reference, the performance of a system with no feedback or queuing is shown. In this transmission scheme, one packet is always transmitted, and outage events only occur at the destination since the source is always transmitting at a fixed rate $R$. Next, when $V = 1$, rate control is performed with the use of feedback and no data is sent for poor channels. When $V = 1$, the transmission rate is either 0 or 1 packet, and the source can buffer at most one packet. There is a large gain in throughput over constant rate transmission from the additional power savings achieved by not transmitting during poor channel states. However, for large SNR’s, both the throughput and the outage probability (shown in Figure 5.10(b))
converge for \( V = 1 \) and the no-feedback case. Using an increased buffer size allows for more variation in the transmission rates and an increased throughput. This is evident from the gains in throughput with \( V = 2 \). Increasing \( V \) leads to diminishing returns, but it is clear that the addition of delay through the use of the feedback allows for a reduced outage probability and a significantly higher throughput. In the limit of \( V = \infty \) packets are never dropped and the normalized throughput will be 1 in the range of SNR’s shown in Figure 5.10(a).

Given a fixed arrival rate of \( R \), and a channel with ergodic capacity \( C_e \), as \( L \to \infty \), the outage probability approaches 0 if \( R < C_e \) [34]. On the other hand, if \( R > C_e \), then the outage probability approaches \( 1 - \frac{C_e}{R} \). Consequently, for large delays, at high average powers the throughput is bounded above by \( R \), since the outage probability approaches zero. On the other hand, for small average powers, we have \( R > C_e \) and the throughput is bounded above by \( C_e \).

The outage probability is seen in Figure 5.10(b), and shows an interesting result. When feedback is unavailable, then the outage probability is solely a function of the channel as a transmission rate of \( R \) is used in each time slot. When feedback is available, then channel outages can be avoided, and the outage probability results depict the blocking probability defined in (5.13). For \( V = 1 \) and for no-feedback, a first order diversity* behavior is experienced with Decode and Forward. Furthermore,

\*In this work, we consider a more general definition of diversity. A diversity order of \( d \) is obtained if for some constant \( C \) and power \( P \), the outage behaves as \( \frac{C}{P^d} \). This is somewhat different from the
for constant power transmission, additional levels of delay increase the diversity. From this figure, it is apparent that for a buffer size of $V$, the diversity order is also $V$. This suggests significant power savings with increasing delay for a target probability of buffer overflow.

Figures 5.11(a) and 5.11(b) shows throughput results for Decode and Forward in comparison with other network transmission schemes. Figure 5.11(a) considers a source-relay distance of $d = 0.2$, which implies a strong source-relay channel gain. In this case, Decode and Forward is seen to have a large throughput gain over multi-hop and selection relaying. Multi-hopping performs about the same as direct transmission traditional notion of diversity, which is obtained through the reception of independent paths (i.e. multiple transmit/receive antennas) of the data.
in this case. For this source-relay distance, there are hardly any decoding errors at the relay. Moreover, the relay-destination channel gain has a mean value that is very close to that of the source-destination link. As a result, multi-hopping and direct transmission have comparable performance. Selection relaying gains over direct transmission at large average powers, and for small power constraints it has a similar performance as direct transmission. When the source-relay distance is \( d = 0.5 \), as in Figure 5.11(b), we see that Decode and Forward once again outperforms selection relaying and multi-hop, but multi-hop performs quite well in this scenario. When \( d = 0.5 \), multi-hop performs at its best as it relies heavily on strong source-relay and relay-destination links. Despite the fact that selection relaying is half-duplex and relies on a repetition code at the relay node, it achieves large throughput gains over direct transmission for \( d = 0.5 \).

The addition of power control in addition to rate control can lead to significant reductions in outage probability, and gains in throughput. The use of power control is shown in Figure 5.12 for \( V = 1 \) and for both direct transmission and Decode and Forward. In this case, we assume that there is a perfect amount of feedback so that the optimal power control strategy can be performed, which amounts to channel inversion for channel states in which the transmission rate is non-zero [10, 32]. We see that, for both direct transmission and Decode and Forward, tremendous gains are possible with power control [23]. When a limited feedback channel is available, the gains will be reduced accordingly.
Figure 5.13: The effect of having a fixed number of feedback bits for power/rate control is more useful for higher delays. Outage results are shown for $V = 1, V = 2$ for 2-bits of feedback using DF and DT.

5.4 Conclusions

In this chapter, we have analyzed the throughput performance of different network transmission protocols in the fading channel when a finite rate of feedback is provided to the transmitters. We derived the optimal rate control algorithm that maximizes the throughput. Our work leads to some interesting design guidelines for next generation network protocols. First, only a few coding rates are necessary to maximize the throughput, which is useful as it shows that throughput gains are achievable with minimal additional hardware complexity at the transmitter. Second, the selected transmission rates should be large enough to allow for a moderate amount of frame errors to maximize the long-term throughput. Third, cooperative techniques such as Decode and Forward can give significant performance improvements over tradi-
tional point-to-point communications, and should therefore be implemented in next generation protocols.

We also looked at the use of simultaneous rate and power adaptation. An interesting conclusion is that simultaneous rate and power adaptation is not needed. When the average power constraint is small, using one rate control codebook is sufficient as long as power control is employed. On the other hand, for systems operating in the high power regime, power control is unnecessary.
Chapter 6
Conclusions and Future Work

6.1 Conclusions

The intense research focus by many groups on cooperative communications is with merit, as node collaboration has the potential to provide significant performance improvements in networks. Relaying, a simple example of the cooperation paradigm, has shown the potential for tremendous gains over direct transmission, especially in the context of fading channels.

By utilizing a relay, a virtual multiple antenna system is created. Relaying leads to similar performance as a multiple transmit/single receive antenna system in the block fading channel in terms of outage and diversity. In this thesis, we showed that the true potential of relaying becomes evident with the use of feedback. Feedback can be used for different purposes based on requirements at the source. If the source transmits at the same rate irrespective of the channel conditions, then the optimal strategy is to minimize the outage probability. On the other hand, if adaptive rate transmission is allowed, then throughput maximization should be the goal.

Our analysis has resulted in several interesting conclusions. With constant rate transmission, the outage probability is minimized by adapting the transmit power both spatially and temporally. We saw that:
• The optimal power control policy with perfect feedback involves a 'channel-inversion' type procedure where transmission is shut off for a subset of channel states which require large amounts of power to invert the channel effects.

• Much like in the case of ergodic-capacity, decode and forward is nearly optimal for strong source-relay links in the block fading environment. In fact, a hybrid between decode and forward and estimate and forward nearly approaches the universal lower bound on outage probability for a wide array of channel conditions.

• Power control with finite rate feedback quickly approaches the limit defined by the perfect feedback power control policy. In fact, only a few bits of feedback are really needed to achieve significant gains over constant power transmission.

• Feedback can be used to achieve diversity gains similar to spatial diversity. For example, with 1 feedback bit, the diversity order of amplify and forward doubles to four, which leads to similar outage behavior as a 4x1 multiple antenna system. For direct transmission, the use of 1 feedback bits results in only a second order diversity.

• When feedback is unavailable, equal power allocation between the source and relay is nearly optimal for a wide range of network geometries.

When feedback is available and the transmission rate is not constrained to be
constant, we showed results for throughput maximization. Here we saw that:

- For a limited rate of feedback, the throughput is maximized for a relatively high outage operating point (i.e. greater than 10%). This suggests using higher rate codes and allowing for frame errors to maximize the effective data rate. As the rate of the feedback link increases, the optimal outage operating point decreases.

- It is unnecessary to temporarily adapt both the rate and power, especially for a small amount of feedback. For small average power constraints, power control with rate selection is sufficient. On the other hand, for large average power constraints, adapting the transmission rate is imperative, and the addition of power control does not provide any significant throughput gain.

- Cooperation provides large throughput gains over direct transmission. Even simple protocols such as selection decode and forward can achieve gains over direct transmission in the low power regime.

The results motivate the need for next generation network protocols to incorporate feedback regarding channel state information. Even limited amounts of feedback can lead to significant gains. These gains can translate into significant improvements in battery life for a target performance level. Our results suggest that although relaying has shown excellent potential in the fading channel in the case of no transmitter
channel knowledge, its true power becomes evident when feedback is available and
temporal adaptation of rate and power are performed.

6.2 Future Work

Our analysis focuses on a network with three nodes: one source-destination pair, and a relay node. In a more general setting, the use of additional relays in the cooperation process leads to a larger achievable rate. An interesting protocol for a multi-relay network was proposed in [16]. A set of relay nodes (the ’decoding set’) comprises all relays that were able to successfully decode the source transmission. In the second phase of transmission, the relays from the decoding set use a space-time structure to transmit to the destination. This is one example of a multi-relay network. The potential throughput gains of such a system needs to be quantified, and also appropriate rate and power control strategies need to be derived. There is clearly a tradeoff in using more relay nodes to increase the throughput at the expense of increasing the interference with neighboring communications.

The half-duplex constraint is a practical restriction on the relay terminals. The results presented in this work for half-duplex relays involved primarily the amplify and forward and selection relaying protocols. These protocols, while simple to implement, impose an additional time orthogonality constraint. The time-orthogonality reduces the achievable rate. Furthermore, the above mentioned protocols utilize repetition coding at the relay nodes, which also leads to a throughput loss. The half-duplex
protocol described in [22] does not impose a time orthogonality constraint. As a result, it is expected that for throughput maximization, the ‘cheap’ decode-forward [22] protocol will have a throughput behavior that more closely follows that of full-duplex relaying. Power and rate control analysis is an area of future work for more advanced half-duplex protocols.

In the work described in this thesis, feedback is used to improve performance either by outage minimization or throughput maximization. A crucial assumption in relates to the feedback link. First, we assume that the feedback transmission is error-free. This may be practical for limited rates of feedback, but for large feedback rates, errors would be expected. One area of future work is to construct power and rate control algorithms that maximize throughput with errors in the feedback link. Second, we assume the feedback information sent from the destination is instantaneously received at the transmitters without delay. However, in a practical system, the transmission delay may lead to ‘outdated’ quantized channel information at the transmitters. The addition of feedback delay leads to an interesting problem of determining the optimal throughput maximization algorithm for a given delay.
Appendix A

Proof of Theorem 4.2.1

Note that $\mathcal{R}_{AF}(\gamma, P_{s,q}, x)$ is a monotonically increasing function of $P_{s,q}$, for $x \in \{P_r, \eta P_{s,q}\}$. This can be verified by confirming that $\frac{\partial R_{AF}(\gamma, P_{s,q}, x)}{P_{s,q}} > 0$. Recall that $G(\cdot)$ is found by solving for $\gamma_2$ in $\mathcal{R}_{gen}(\gamma, P_{s,q}, x) = R$. For any $\gamma_a = (\gamma_0, \gamma_1, \gamma_2)$ lying on $G(\cdot)$, we will next show that for any $\gamma_b = (\gamma_0 - \epsilon_0, \gamma_1 - \epsilon_1, \gamma_2 - \epsilon_2)$, $\mathcal{R}_{AF}(\gamma_b, P_{s,q}, x) < \mathcal{R}_{AF}(\gamma_a, P_{s,q}, x)$. As a result, since $\mathcal{R}_{AF}(\cdot)$ is monotonically increasing in the source power, to transmit at rate $R$ with new power $P_{b_{s,q}}$, an increase in source power ($P_{b_{s,q}} > P_{s,q}$) is necessary to guarantee $\mathcal{R}_{AF}(\gamma_b, P_{b_{s,q}}, x) = R$, where $x_b \in \{P_r, \eta P_{b_{s,q}}\}$.

For AF, consider $y(\gamma) = P_{s,q}\gamma_1 + \frac{P_{s,q}\gamma_0 x\gamma_2}{1 + P_{s,q}\gamma_0 + x\gamma_2}$. Let $y_1(\gamma) = 1 + P_{s,q}\gamma_1$, and $y_2(\gamma) = \frac{P_{s,q}\gamma_0 x\gamma_2}{1 + P_{s,q}\gamma_0 + x\gamma_2}$. For $y_1(\gamma)$, clearly $1 + P_{s,q}\gamma_1 > 1 + P_{s,q}(\gamma_1 - \epsilon_1)$. For $y_2(\gamma)$, it needs to be shown that

$$\frac{P_{s,q}\gamma_0 x(\gamma_2 - \epsilon_2)}{1 + P_{s,q}(\gamma_0 - \epsilon_0) + x(\gamma_2 - \epsilon_2)} < \frac{P_{s,q}\gamma_0 x\gamma_2}{1 + P_{s,q}\gamma_0 + x\gamma_2}. \quad (A.1)$$

After some manipulation, this can be rewritten as

$$(1 + P_{s,q}) \left( \frac{\gamma_0}{\epsilon_0} - 1 \right) + (1 + x) \left( \frac{\gamma_2}{\epsilon_2} - 1 \right) > -1. \quad (A.2)$$

Since all fading elements are positive, then $\epsilon_i \leq \gamma_i$. As a result, (A.2) is always satisfied. We have then shown that $\frac{1}{2} \log(y(\gamma_0 - (\epsilon_0, \epsilon_1, \epsilon_2))) < \frac{1}{2} \log(y(\gamma_0))$ which corresponds to $\mathcal{R}_{AF}(\gamma_b, P_{s,q}, x) < \mathcal{R}_{AF}(\gamma_a, P_{s,q}, x)$. Consequently, an increase in source
power is required to guarantee zero outage for any channel state lying below the curve $G(\cdot)$.

Proving the second part of the theorem, that points lying above $G(\gamma_0, \gamma_1, \mathbf{P}_q)$ requires a source power less than $P_{s,q}$ and a relay power less than $P_{r,q}$, is straightforward. By following similar steps as above, except now setting $\gamma_b = (\gamma_0 + \epsilon_0, \gamma_1 + \epsilon_1, \gamma_2 + \epsilon_2)$, the result follows. ■
Appendix B

Proof of Theorem 4.2.2

Consider the function \( g(P_{\text{avg}}, \gamma_0, \gamma_2) = \frac{P_{\text{avg}}^2 \gamma_0 \gamma_2}{1 + P_{\text{avg}} \gamma_0 + P_{\text{avg}} \gamma_2} \). We first will find

\[
Pr[g(P_{\text{avg}}, \gamma_0, \gamma_2) < t].
\]

This can be re-written as \( Pr[1/g(P_{\text{avg}}, \gamma_0, \gamma_2) > 1/t] \). Next, note that

\[
Pr\left[\gamma_2 < \frac{t(1 + P_{\text{avg}} \gamma_0)}{P_{\text{avg}}(P_{\text{avg}} \gamma_0 - t)}\right] = 1 - \exp\left(-\frac{t(1 + P_{\text{avg}} \gamma_0)}{\lambda_2 P_{\text{avg}}(P_{\text{avg}} \gamma_0 - t)}\right).
\] (B.1)

Next, using the fact that \( e^{-x} = 1 - x + x^2/2 + ... \), we can approximate \( \lim_{P_{\text{avg}} \to \infty} (P_{\text{avg}} \cdot Pr[g(P_{\text{avg}}, \gamma_0, \gamma_2) < t]) \) as

\[
\lim_{P_{\text{avg}} \to \infty} (P_{\text{avg}} \cdot Pr[g(P_{\text{avg}}, \gamma_0, \gamma_2) < t]) = \lim_{P_{\text{avg}} \to \infty} \frac{P_{\text{avg}} t(1 + P_{\text{avg}} \gamma_0)}{\lambda_2 P_{\text{avg}}(P_{\text{avg}} \gamma_0 - t)} = \frac{t P_{\text{avg}} \gamma_0}{\lambda_2 (P_{\text{avg}} \gamma_0 - t)} = \frac{t}{\lambda_2}. \] (B.2)

Next, consider \( Pr[P_{\text{avg}} \gamma_1 < t] \). It can be verified that

\[
\lim_{P_{\text{avg}} \to \infty} (P_{\text{avg}} \cdot Pr[P_{\text{avg}} \gamma_1 < t]) = \lim_{P_{\text{avg}} \to \infty} (P_{\text{avg}} \cdot (1 - e^{-\frac{t}{P_{\text{avg}} \gamma_1}})) = \frac{t}{\lambda_1}. \] (B.3)

Let \( a(t) = t/\lambda_1 \), and \( b(t) = t/\lambda_2 \). Using Theorem 1 from [17],

\[
\lim_{P_{\text{avg}} \to \infty} P_{\text{avg}}^2 \cdot Pr\left(P_{\text{avg}} \gamma_1 + \frac{P_{\text{avg}}^2 \gamma_0 \gamma_2}{1 + P_{\text{avg}} \gamma_0 + P_{\text{avg}} \gamma_2} < t\right) = \int_0^t a(t-x)b'(x)dx = \frac{t^2}{2\lambda_1 \lambda_2}. \] (B.4)
Using the fact that $t = e^{2R} - 1$, for large $P$, we have that

$$
Pr \left( P_{\text{avg}} \gamma_1 + \frac{P_{\text{avg}}^2 \gamma_0 \gamma_2}{1 + P_{\text{avg}} \gamma_0 + P \gamma_2} < e^{2R} - 1 \right) \approx \frac{(e^{2R} - 1)^2}{2P_{\text{avg}}^2 \lambda_1 \lambda_2}.
$$

(B.5)

Clearly, there is a second order decay of the outage with respect to power for this case. As a result, a deterministic $\gamma_0$ does not affect the diversity order.
Appendix C

Proof of Theorem 5.2.1

Consider the rate control region for decode and forward, shown in Figure 5.2. This figure contains a surface which separates regions 1 and 2, for the case of 1-bit of feedback. The achievable rate for decode and forward is given by

\[ R = \min\{R_a, R_b\}, \quad (C.1) \]

where \( R_a = \log(1 + P_2 \gamma_0) \), and \( R_b = \log(1 + P_2 \gamma_1 + P_2 \gamma_2) \). Along the contour, since \( K = e^{R_2} - 1 \). 

Consider the case where \( \gamma_0 = K/P_2 \), where \( K = e^{R_2} - 1 \), and \( \gamma_1 > K/P_2 \). Then the achievable rate can be expressed as

\[ R = \min\{R_2, R_b\} = R_2, \]

for any value of \( \gamma_2 \). Furthermore, any \( \psi \) decrease in \( \gamma_0 \) leads to an achievable rate of \( \log(1 + P_r(\gamma_0 - \psi)) < R_2 \), since in this case \( R_a \) is active in the expression for the achievable rate of decode and forward in (C.1).

Consider the case where \( \gamma_0 > K/P_2 \), and \( \gamma_1 + \gamma_2 = K/P_2 \), which also lies along the surface. In this case, the achievable rate is expressed as

\[ R = \min\{R_a, R_2\} = R_2. \]
Any $\psi$ decrease in $\gamma_2$ leads to an achievable rate of $\log(1 + P_r(\gamma_2 - \psi) + P\gamma_1) < R_2$. The same trend holds for a decrease in $\gamma_1$.

Finally, consider where $\gamma_0 = K/P$ and $\gamma_1 + \gamma_2 = K/P_2$. Here we have $R = R_a = R_b = R_2$. Any decrease in $\gamma_0$ will lead to an achievable rate of $R = \log(1 + P_r(\gamma_0 - \psi)) < R_a = R_2$. As a result, the rate required for zero outage is less than $R_2$. Similarly, when $\gamma_0 = K/P_2$ and either $\gamma_1$ or $\gamma_2$ is decreased by $\psi$, then $R = \log(1 + P_r(\gamma_1 - \psi) + P_2\gamma_2) < R_b = R_2$. This implies that any point lying below the triangular portion of the contour requires a rate less $R_2$ to invert the effects of the channel.

As a result, any points lying on the surface require a rate of $R_2$ to invert the channel effects, while points lying below the contour require a rate less than $R_2$ to maintain zero outage. Similarly, it can be shown that points above the contour can support a higher rate than $R_2$ and still maintain zero outage. The surface is then a contour with rate $R_2$, and separates $\mathcal{R}_1$ and $\mathcal{R}_2$. 
Appendix D

Proof of Theorem 5.2.2

Consider (2.8), and to indicate the dependence on the correlation $\rho$, denote it by $R_{DF}(\rho)$. Since channel state information is unavailable at the transmitters, the outer maximization is not present, and the achievable rate is

$$R_{DF}(\rho) = \min \left\{ \log (1 + (1 - |\rho|^2)\gamma_1 P_s), \right. \\
\left. \log (1 + \gamma_1 P_s + \gamma_2 P_r + 2 \Re\{\rho e^{j\angle h_1 - j\angle h_2}\} \sqrt{\gamma_1 \gamma_2 P_s P_r}) \right\}, \quad (D.1)$$

At a transmission rate $R$, the outage probability as a function of $\rho$ can be expressed as

$$P_{out}(\rho) = 1 - E[I_F(R_{DF}(\rho))], \quad (D.2)$$

where $I_F(R_{DF}(\rho))$ is the indicator function defined as

$$I_F(R_{DF}(\rho)) = \begin{cases} 
1, & R_{DF}(\rho) > R \\
0, & R_{DF}(\rho) \leq R. 
\end{cases} \quad (D.3)$$

When no channel state information is available at the transmitters, the throughput is simply $R \cdot (1 - P_{out}(\rho))$. Therefore for any given $R$, maximizing the throughput requires a minimization of the outage probability.

The indicator function (D.3) operates on a convex set, and as a result, $g(\rho) = I_F(R_{DF}(\rho))$ is log-concave [26]. Note that $P_{out}(\rho) = 1 - E[g(\rho)]$, and is then log-convex. Considering $R_{DF}(\rho)$, it should be noted that the first log term is independent
of the phase of $\rho$. Moreover, since $\angle h_1$ and $\angle h_2$ are uniformly distributed on $[0, 2\pi)$, in the calculation of $P_{out}$ the second log term of $R_{DF}(\rho)$ would also be independent of the phase of $\rho$. Consequently, we have that $P_{out}(\rho) = P_{out}(-\rho)$. The log-convexity of $P_{out}$ implies that

$$\frac{\log P_{out}(\rho) + \log P_{out}(-\rho)}{2} \geq \log P_{out}(0),$$

which leads to the fact that

$$\sqrt{P_{out}(\rho) \cdot P_{out}(-\rho)} \geq P_{out}(0).$$

Also, based on the arithmetic-geometric mean inequality

$$\frac{P_{out}(\rho) + P_{out}(-\rho)}{2} \geq \sqrt{P_{out}(\rho) \cdot P_{out}(-\rho)},$$

and as a result

$$P_{out}(\rho) = \frac{P_{out}(\rho) + P_{out}(-\rho)}{2} \geq P_{out}(0).$$

Therefore, having $\rho = 0$ would minimize the outage probability and hence maximize the throughput.
Bibliography


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