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Using an Optimal Control Model to Simulate Carbon Dioxide-Biomass Interactions

by

Ross S van Wassenhove

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

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ABSTRACT

Using an Optimal Control Model to Simulate Global Carbon Dioxide-Biomass Interactions

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Many studies have proposed expansion of photosynthetic biomass “sinks” as a method of controlling atmospheric carbon dioxide. A defect in most of these studies is that biomass growth is assumed to be linear. In this thesis, optimal control theory is applied to an economic analysis of the interactions between global photosynthetic biomass growth and atmospheric CO2. The model assumes society starts with a fossil fuel resource endowment, which is used to supply energy. One goal is to determine numerically, using a “natural” model of the system, whether a steady state is reached in the interactions between the biosphere and the atmosphere once the use of fossil fuel ceases. The thesis also determines, through a planet-level numeric simulation, optimal biomass and CO2 levels and their associated “shadow prices” that would be required to ensure an efficient outcome in the presence of negative externalities associated with atmospheric CO2 (the “global warming problem”). The optimal solution is tested for sensitivities to changes in parameter values, including a “policy variable” of CO2 “tolerance”. Biological growth is modeled by the logistic function, and CO2 sequestration is based on a non-linear C3 plant CO2 “fertilization” scheme. Alternative model structures to the logistical function are proposed and discussed. The model structure and numerical analysis derives some of its parameters from previous studies and estimations, but mainly the work of Hirofumi Uzawa.
Using an Optimal Control Model to Simulate Global Carbon Dioxide-Biomass Interactions

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I. Introduction

A. Background and Political Involvement

In recent decades, increasing world industrialization, augmented by a rise in population has resulted in the accumulation of what have been called “greenhouse gases” – CO₂ in particular – in the atmosphere. The allegedly potential negative effects (e.g., global warming) of increased CO₂ concentrations have become a controversial topic of domestic and international discussion and politics. In particular, the threat of global warming has been cited as a reason for governments to adopt emissions standards and impose energy taxes. Conferences in Montreal (September 1987), Rome (September 1990), Rio de Janeiro (June 1992), Geneva (February 1994), Kyoto (December 1997), Buenos Aires (November 1998), and most recently, Bonn (October 1999) were dedicated to these ends.

The controversy surrounding these issues revolves around uncertainty about the probability and associated extent of atmospheric warming resulting from increased concentration of greenhouse gases (GHGs) in the atmosphere. Models used to simulate global atmospheric conditions predict severe warming in some areas of the world if CO₂ emissions continue to expand as they have in the recent past. However, in light of recent temperature data and theoretical doubts about the accuracy of the models, uncertainty also exists about the extent of the warming. There is also controversy over the likely consequences should the predicted warming eventuate.

One of the major efforts to study the science of climate change, its impacts, adaptations and mitigations, along with the associated economic and social dimensions, has been the
three-part assessment project of the Intergovernmental Panel on Climate Change (IPCC)\(^1\). The First Assessment Report was completed by Working Group I (WGI) in August 1990 as *Climate Change: The IPCC Scientific Assessment*, and served as the basis of negotiating the UN Framework Convention on Climate Change (FCCC). The report met with much criticism, and the IPCC issued a Supplementary Report in March 1992, timed to coincide with the final negotiations of the June 1992 UN FCCC Conference ("Earth Summit") at Rio de Janeiro. The summary of this report confirmed the essential conclusions of the First Assessment:

1. an observed increase in the global mean temperature during the past 100 years is in broad accord with theoretical model calculations, and
2. a temperature increase of 0.3 °C per decade is expected— as a result of an enhanced greenhouse effect due to human activities that release CO\(_2\) and other greenhouse gases.

According to the World Meteorological Organization (WMO, which is part of the governing body of the IPCC):

“The IPCC completed its first assessment report in August 1990 which indicated with certainty an increase in the concentration of greenhouse gases due to human activity.” The report assisted governments to make important policy decisions in the negotiations and eventual implementation of the UN Framework Convention on Climate Change which was signed by 166 countries at the UN Conference on Environment and Development (Rio de Janeiro, 1992). (italics added)

Thus, the report, most of it based on various models\(^2\) constructed to test the hypothesis of carbon dioxide inducing global atmospheric warming, concluded by correlating human activity to atmospheric GHG concentrations, and GHGs to warming. In particular, it

---

1 See Appendix A for a brief history and purpose of the IPCC and other major climate-study organizations.
2 Davis and Legates (1998) argue that the climate models are unreliable because the computers that operate the models are not fast enough to simulate climate change on sub-global scales.
asserted that past fossil fuel consumption and tropical deforestation were responsible for increased CO₂ stocks in the atmosphere¹.

As a result, unilateral announcements of plans were made by many governments, including US legislative proposals, to stabilize or even cut back emissions of carbon dioxide. Controversy grew as new research was published, and polar extremes were adopted by many special interest groups as positions from which to assess future economic and environmental impacts.

The next major group research publication of the IPCC came as the Second Assessment Report, finished in December 1995⁴. The work was composed of contributions of the three IPCC Working Groups. The report of Working Group I was titled Climate Change 1995: The Science of Climate Change. This report was the cause of even greater controversy, because after the report was approved for publishing by the reviewing scientists and the IPCC, revisions were made before it was submitted for printing. Allegations were made that through these revisions, caveats of uncertainty were removed from the text of Chapter 8. Detection of Climate Change and Attribution of Causes. Further allegations of certain IPCC members fulfilling a political agenda were made public by climate scientists in newspapers, popular journals, and Internet Web sites⁵.

¹ This claim could also allude to chloro-fluorocarbon use. In addition to being labeled as the prime catalyst in the depletion of the stratospheric ozone layer, chloro-fluorocarbons also trap infrared radiation.
⁴ As of February 2000, the IPCC Third Assessment Report (TAR-2000), was issued in draft form to a closed group of “experts”, and its restricted Web site has been removed.
⁵ For example, see the many references at http://www.vision.net.au/~daly and http://www.sepp.org
However, one IPCC contributor claimed these revisions were made to accommodate interpretational differences among the various languages.

Even today, controversy continues with groups at the extremes making claims that their models and data "prove" the conclusions they assert. Political and financial interests have invested large sums in expectation of rewards, or avoiding costs, from legislation aimed at curbing fossil fuel consumption. Given the central role of fossil fuels in the world economy, the consequences of such action become far-reaching. To further explore some of the scientific and economic aspects in this field of study, it is necessary to present the phenomena by which CO₂ may cause temperature variations in the atmosphere, and the possible effects that warming may have on the collection of living organisms, known as the biosphere. The temperature effects are discussed in the next section, while the interactions with the biosphere is treated in more depth in the subsequent model development. as it is a key aspect of this paper.

**B. Radiative Forcing and the Carbon Cycle**

The warming of a glass-enclosed space is caused by the greenhouse effect. An analogy is drawn between this effect and the earth's atmosphere. The effect develops from the following phenomenon. (See Figure 1, page 5 for an illustration of the allocation of the energy balance). A portion of light energy transmitted by the sun toward the earth is not

---

* Stephen Schneider, personal communication 3/23/99

"Machta (1973) reasoned against the analogy, saying glass walls prevent thermal mixing. Schelling (1992) argues this term is technically incorrect, but is too well established in usage to change."
reflected back into space by clouds or surface elements such as snow and ice cover. Of this non-reflected portion, some energy is absorbed directly by the atmosphere, but most passes through. Thus, much of the atmosphere is transparent to this energy, i.e., the light energy is not readily absorbed. The frequencies of solar energy which pass through are in the short-wave part of the electromagnetic spectrum. This energy is then absorbed by the earth and re-radiated into the atmosphere as long-wave infrared terrestrial radiation, much of which is effectively absorbed and trapped by certain gasses, increasing the temperature of the atmosphere.

**Allocation of Earth’s Energy Balance**

![Diagram of Earth's energy balance](image)

*Figure 1.3: The Earth's radiation and energy balance. The net incoming solar radiation of 342 Wm⁻² is partially reflected by clouds and the atmosphere, or at the surface, but 49% is absorbed by the surface. Some of that heat is returned to the atmosphere as sensible heating and most as evapotranspiration that is realised as latent heat in precipitation. The rest is radiated as thermal infrared radiation and most of that is absorbed by the atmosphere which in turn emits radiation both up and down, producing a greenhouse effect, as the radiation lost to space comes from cloud tops and parts of the atmosphere much colder than the surface. The partitioning of the annual global mean energy budget and the accuracy of the values are given in Kiehl and Trenberth (1996).*

**Source:** Trenberth, Houghton, Meira Fliho (1996)

*Figure 1*
The scientific term for the phenomenon is **radiative forcing**. The following diagram (Figure 2, below) shows a similar perspective of the earth's atmospheric distribution of solar energy. The observation can be made that although a significant fraction of the atmospheric radiation is involved with GHGs, there are other atmospheric phenomena that may influence climate and weather.

**Source:** Institut für Raufahrt Systeme, University of Stuttgart. http://www.irs.uni-stuttgart.de  
FIPEX: Oxygen Sensors for Space Applications. Motivation

**Figure 2**
Figure 2 (page 6) shows the radiation and ionic gas distribution among the atmospheric layers by altitude. Again, one may see much more fundamental processes occurring in these layers. The gasses held responsible for the radiative forcing effect are called greenhouse gasses (GHGs). Water vapor is the most significant GHG, while the largest volume anthropogenic GHG is carbon dioxide, a direct by-product of fossil fuel combustion. CO$_2$ is the standard for the index of relative radiative forcing, with a value of one. Smaller volume components of GHGs are methane (CH$_4$), its increase largely a result of increased deforestation and livestock husbandry, and chloro-fluorocarbons (CFCs), used in aerosol propellants and refrigeration. Thus, GHGs act as a one-way filter for certain frequencies of light energy, and each GHG has its own level of radiative forcing. The relative radiative forcing index is 5 for CH$_4$, and 4000-5000$^8$ for CFCs. For a more complete analysis and discussion, see Cline (1991, 1992) or Solow (1991), and the report of IPCC WGI (1996).

Until recently, the carbon cycle of atmosphere-surface exchange was understood only enough to realize that atmospheric CO$_2$ was increasing (with the supposed associated atmospheric warming). Fossil fuel combustion, net deforestation and other land use changes were considered the major sources, while the ocean and the atmosphere were considered the major sinks. Eight years ago, according to Solow (1991), total CO$_2$ emissions from the earth (sources) were about 7.0 GtC/yr, comprised of roughly 5.4 GtC/yr from fossil fuel combustion, and 1.6 GtC/yr from net deforestation and land use

---

$^8$ Estimates vary widely, but there is agreement on the order of magnitude. See Cline (1992).
change. The well-known sinks were the oceans, believed to absorb about 2 GtC/yr. and the atmosphere, believed to absorb about 3.4 GtC/yr. At that time, the total sinks did not account for the other 1.6 GtC/yr. hence the popularly applied term “missing sink” was coined.

A ‘Natural Sinks of CO$_2$’ workshop held during February 1992 in Puerto Rico concluded with two key findings (see Lugo and Wisniewski, 1992). The first key finding was the unanimous decision to discard the long-held assumption that most of the earth’s ecosystems have been in carbon steady state with the atmosphere. Scientists had yet to be able to balance world carbon flux: more anthropogenic carbon is accumulated in oceans and terrestrial ecosystems than leading carbon cycle models predict. Thus, unidentified carbon sinks (the “missing sink”) have stimulated attention to the carbon cycle. Research continued subsequently to identify new carbon sinks. These new sinks included coastal zones, biomass-accreting mature forests, soils, and arid lands, and have been estimated to absorb about 1.6 GtC/yr. completing the balance.

The data were amended following this research, and are reflected in the IPCC report of 1996, presented below in Table 1 (page 9). More recent data (Pocklington, 1998) are given in comparison. Gray (1997) noted that the wide variability in several of the 1980-89 estimates could lead to error propagation. Currently, emissions from fossil fuels are still the largest anthropogenic contribution, and despite all the newly-identified sinks and their absorption, the atmosphere plays the role of the largest immediate sink, acting as a buffer, until oceanic and biomass absorption occur. The lags involved in absorption by
the deep layer of the ocean has been the subject of debate (Daly, 1997; Dietze, 1997),
with estimates ranging from 5 to 500 years.

**Carbon Budget (GtC/yr)**

<table>
<thead>
<tr>
<th>SOURCES OF CO₂</th>
<th>as of mid-1998*</th>
<th>1980-89 avg†</th>
</tr>
</thead>
<tbody>
<tr>
<td>Emissions from fossil fuel combustion and cement production</td>
<td>5.5 ± 0.1</td>
<td>5.5 ± 0.5</td>
</tr>
<tr>
<td>Net emissions from changes in tropical land use</td>
<td>1.8 ± 0.2</td>
<td>1.6 ± 1.0</td>
</tr>
<tr>
<td><strong>TOTAL ANTHROPOGENIC EMISSIONS</strong></td>
<td>7.3 ± 0.3</td>
<td>7.1 ± 1.1</td>
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**PARTITIONING AMONG RESERVIORS**

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<tr>
<td>Storage in the atmosphere</td>
<td>3.2 ± 0.1</td>
<td>3.3 ± 0.2</td>
</tr>
<tr>
<td>Ocean uptake (<em>calculated from models)</em></td>
<td>2.0</td>
<td>2.0 ± 0.8</td>
</tr>
<tr>
<td>Uptake by Northern Hemisphere forest regrowth</td>
<td>0.6 ± 0.1</td>
<td>0.5 ± 0.5</td>
</tr>
<tr>
<td>Inferred sinks (nitrogen fertilization, CO₂ fertilization and climatic effects)</td>
<td>1.5 ± 0.1</td>
<td>1.3 ± 1.5</td>
</tr>
</tbody>
</table>

*Source:* *Pocklington (1998); †IPCC WG I (1996)*

Notes: CO₂ fertilization and climatic effects were estimated at 0 – 2 GtC/yr. nitrogen fertilization at 0.5 ± 1.0 GtC/yr. Northern Hemisphere regrowth is estimated at 0.7 ± 0.2 by IPCC WG II (1996).

**Table 1**

Naturally, the shorter the lag, the more significant the ocean absorption becomes, since it is the largest repository of carbon on the earth. The relationships are now better known, and are diagrammed in **Figure 3** (page 10).

The second key finding of the ‘Natural Sinks of CO₂’ workshop was that a substantial potential exists for carbon storage in these natural systems, namely sequestration of carbon dioxide in rapidly growing, high-carbon storage tropical or Northern Hemisphere trees. The possibility of enhancing absorption of CO₂ in the biosphere is the primary motivation of this paper. While it is known that the ocean is the earth’s largest CO₂ sink, attempts to increase oceanic productivity may have more potential unknown side effects
(see Denman, Hofman, Marchant, 1996). Thus, in the absence of mitigation measures, a possible step to reduction or at least slowing the growth of CO$_2$ is expanding the biota.

The urgency to store CO$_2$ in biomass, alone or in conjunction with any other mitigation strategy, operates under the key assumption that additional amounts of CO$_2$ in the atmosphere will indeed induce atmospheric warming, and that the warming will have net negative effects. IPCC (1996) claimed that the earth’s surface has heated about 0.5°C in the 20$^{th}$ century. Some scientists consider that piece of information as insignificant, as theoretical and empirical studies claim$^9$ that no such trend has occurred or could occur under a 2X CO$_2$ scenario$^{10}$. Temperature data collected from the ocean and at altitude via satellites and balloons in the atmosphere have not shown such a heating trend. At a theoretical level, some scientists believe that the relatively high heat capacity of water, and consequently the ocean, may act as a buffer against warming.

Daly (1997) and Dietze (1997) argue against an atmospheric temperature rise from theoretical thermodynamic considerations of the oceans, and Pocklington (1998) presents 20+ year data from various North Atlantic Ocean stations. Pocklington maintains that “such warming as has occurred since the 19$^{th}$ century is not outside the natural range of climate variation.” Further, consistent with the Daly and Dietze analyses, he states:

The ocean contains about 50 times as much CO$_2$ as the atmosphere and the flux of CO$_2$ from the ocean surface is about 20 times greater than the amount released by the burning of fossil fuels. So the ocean is great in its capacity to produce and absorb CO$_2$ and in its capacity to move heat around the planet.

$^9$ See the various papers at Daly (1997).

$^{10}$ The 2X CO$_2$ scenario is the doubling of atmospheric carbon dioxide concentration, a benchmark used in modeling to test impacts.
As further evidence of temperature measurements, atmospheric temperature from satellites "show no warming trend during the past 19 years – a finding independently validated by weather balloon observations" (Christy and Spencer. 1998). See temperature data on line at various Web sites\textsuperscript{11}, including Daly (1997).

In light of the controversy over this topic, the opinion expressed here is that this assumption can not and should not be taken as axiomatic. Many environmental authorities have, however, been induced by an approach of "better safe than sorry" to embark upon "no regrets" measures, the implementation of which, in theory, have a maximum zero net cost to society. The next section presents more of the key issues involved in the debate and some of the measures to control CO\textsubscript{2} emissions that have already been taken.

\textbf{C. Discussion of Some Issues Relevant to the Possibility of Global Warming}

In 1987, the Montreal Protocol was put in place to ban or limit worldwide use of some of the greenhouse gasses, namely CFCs, which cause depletion of the stratospheric ozone layer, and whose greenhouse effect (radiative forcing) is roughly 4000-5000 times that of CO\textsubscript{2} per unit of emission\textsuperscript{12}. However, the quantities of CFCs produced are proportionally smaller than those of CO\textsubscript{2}, which is expected to double in atmospheric concentration (the 2X CO\textsubscript{2} scenario) in the next century sometime between 2025-2050. According to the IPCC (1996) models, this rise in CO\textsubscript{2} concentration is expected to produce a global

\textsuperscript{11} For example (also see Appendix 1), http://www.cei.org and http://www.cato.org
\textsuperscript{12} See footnote 7, page 7.
warming effect substantial enough to increase the mean temperature in the atmosphere near the surface somewhere in the range 1.5 - 4.5°C, with a "best guess" estimate of 2.0°C\textsuperscript{13}. These warming effects, in contrast to CFCs, will be more difficult to limit, since CO\textsubscript{2} production is a result of worldwide fossil fuel consumption. Its production is fairly evenly distributed between developed and undeveloped parts of the world, and the U.S. alone accounts for about 20% of the world's CO\textsubscript{2} production (IPCC, 1996; Sandler, 1992b; Poterba 1991). Concerns are expressed as to whether the trend is serious enough to warrant abatement and what methods of abatement should be implemented. The recent Kyoto Protocol calls for reduction commitments by the US to 93% of 1990 emission levels. The protocol is worded:

1. The Parties included in Annex I shall, individually or jointly, ensure that their aggregate anthropogenic carbon dioxide equivalent emissions of the greenhouse gases listed in Annex A do not exceed their assigned amounts, calculated pursuant to their quantified emission limitation and reduction commitments inscribed in Annex B and in accordance with the provisions of this Article, with a view to reducing their overall emissions of such gases by at least 5 per cent below 1990 levels in the commitment period 2008 to 2012.

2. Each Party included in Annex I shall, by 2005, have made demonstrable progress in achieving its commitments under this Protocol.

Thus, several major issues are of primary importance in the controversy about atmospheric levels and emission rates of CO\textsubscript{2}, and its incipient effects:

1) is the predicted increase in CO\textsubscript{2} accurate?

2) is the projected magnitude of warming from a given increase of CO\textsubscript{2} a reasonable estimate?

\textsuperscript{13} IPCC (1990) models forecasted a 3.2°C warming using the then "state-of-the-art models".
3) what are the potential impacts (both good and bad) of any global warming that does occur?

4) how will any impacts that do occur be distributed geographically?

5) is control of CO$_2$ output the best way of handling the potential adverse effects?

6) if emission control is cost effective, which, if any, countries have “rights” to produce GHG emissions, and how should the rights be allocated?

7) will agreements provide appropriate incentives to balance the costs of controls against the costs of enhancing sinks? In particular, can large, carbon-fixing biomasses significantly reduce concentrations of CO$_2$ in the atmosphere?

8) are there any direct positive effects of increased CO$_2$ in the atmosphere, such as enhanced agricultural productivity, that could argue against limiting CO$_2$ emissions?

Considering the first issue, it is obvious that given a 40 year upward trend, and barring a major shift away from fossil fuel consumption and into alternative energy development and commercialization, the output of CO$_2$ into the atmosphere seems unlikely to be reduced. Estimates of 2X CO$_2$ were made by the IPCC (1996, 1990) and have become the benchmark scenario for impact studies. Studies have also looked at the impacts of 4X CO$_2$ scenarios (Mahlman 1999).

As regards the second issue, the empirical evidence on timing or magnitude of temperature increases has brought much controversy. Mathematical models (global climate models, or GCMs) have been developed and used to predict temperature patterns based on the phenomena introduced earlier. These models were used as a basis for the
First Assessment Report of the IPCC. Many of the models developed in the late 1980s and early 1990s have since been revised and improved to account for an increasing variable set. The models today used smaller earth grid cells, hoping to increase accuracy. Most of these older models used land and ocean-based weather stations to gather their data, and drew criticism of bias. The critics citing, among other things, urban heat island effects as producing an upward bias to temperature data. As mentioned previously, satellite data, alleged to be less biased in the influence of these effects, has been observed since 1979, and shows less heating than the earth stations.

The major costs of the alleged atmospheric warming would come as a result of the effects of altered climate patterns on agriculture, and the rise in sea level from thermal expansion of the oceans and polar ice sheet melting. Predicting net effects of global warming on sea level changes introduces other difficulties. Apart from thermal expansion of the oceans, there is the issue of induced changes in snowfall as a result of greater H2O in the atmosphere. This effect could expand ice sheets by more than temperature rises decrease them. The predicted consequences of warming come under the third issue, which has been addressed in IPCC WGI and WGIII (1996), (Pearce. Cline. et al. 1996). Fankhauser (1995). Martin (1994). Cline (1992). Nordhaus (1991c). and others. Effects on US income levels are cited there to convey an order of magnitude for the expected impact of global warming. The Nordhaus paper concludes that any CO2-induced climate change14

14Nordhaus (1991c) quips, "The change in temperature while this paper is being read is likely to be greater than the expected change from 1990 to 2090. Few people are likely to notice the CO2 signal among the noisy pandemonium of their daily lives."
will bring about a "combination of gains and losses with no strong presumption of substantial net economic damages". His impact estimate for the US is $6.2 billion, or 0.26% of 1981 US national income. Cline, on the other hand, is less optimistic, and estimates that at the "best guess" warming level, the economic impact for the US will be close to $60 billion, or 1% of 1990 US GDP. Cline claims that intangible losses, particularly species loss and also human disamenity could double that estimate. Cline also stresses that very long-term effects (200+ years) must be considered, since CO₂ levels and its consequent warming may not be stopped at twice its current level.

Chapter 6 of the IPCC WGIII study (Pearce, Cline, et al: 1996) encompasses a compilation and comparison of many studies (including revised versions of some of those mentioned above), which assess the impact of a doubling of the pre-industrial CO₂-equivalent of all greenhouse gasses by industry and major world regions. The damages are monetized in terms of people's willingness to pay (WTP) to secure a benefit, or their willingness to accept compensation (WTA) for a cost. Readers are cautioned prior to accepting these estimates rigorously, since

The level of sophistication of climate change damage analysis is comparatively low. Damage estimates are generally tentative and based on several simplifying assumptions. The degree of uncertainty is high, with respect to both physical impacts and their consequences for social welfare.

The best-guess central estimates of global damage are in the order of 1.5-2.0% of world GNP, including non-market impacts, if 2X CO₂ occurred now.
Issue 4. the geographical distribution of climate change impacts, has been widely studied. For a review of mitigation cost studies, see Hourcade, et al (1996). Geographic adaptation measures are addressed in various chapters of IPCC WG II. For an earlier paper, see Nordhaus (1991c). In general, as might be expected if global warming occurs, low lying coastal areas could be affected by rising a sea level from any melting of polar ice caps, and temperate climates would migrate northward in the Northern Hemisphere.

Issues 5 and 6 have been key topics of debate in the world conferences mentioned above. A “carbon tax” has been proposed as a penalty for CO₂ emissions. Sweden and Finland have carbon taxes ($62/ton and $6.50/ton, respectively) in place that reflect their marginal carbon emissions. Many other European countries have very high taxes on gasoline in particular, but also on other energy sources. While not labeled as “carbon taxes”, they effectively amount to the same thing. A 1990 US Congressional proposal for a carbon tax set rates at $15/ton for coal, $3.25/barrel for oil, and $0.40/MCF for natural gas. In contrast to the Scandinavian countries, the proposed US rates were well below those which a 1990 US CBO study claimed would stabilize 1988 levels of CO₂ emission by 2000. Uzawa (1991) used Calculus of Variations to develop a model that calculates an imputed cost (in dollars per ton of carbon) of atmospheric CO₂ for major emitter countries. The prices range from $4/ton for a small emitter country such as Indonesia, to $150/ton for large emitters such as the US and Japan. The IPCC WGIII study (Pearce, Cline, et al: Chapter 6. 1996) is consistent with this range, using the CO₂- and equivalent GHG-doubling as the basis for estimating marginal damage from emission, ranging from $5 to $125 per tonne of carbon emitted today. As explained later in this paper, the
discount rate applied to future costs and benefits can play a critical role in determining these values. Rates of social time preference (assumed here to be equivalent to the discount rate) on the order of 5% produce values at the low end of this range, while rates of 2% and less obtain values at least an order of magnitude higher.

The model developed and analyzed in this paper attempts to add more scientific and technical details to the existing economic models, and is similar to the Uzawa model. Our model measures cost in real terms (i.e., in terms of foregone consumption) rather than dollars; however, so it is not easy to compare our estimates with Uzawa. In addition, the objective here is to look at potential for paths to steady states in this alternative model structure, using estimated resource stocks and capacities. For further discussion of carbon taxes and their specific analyses, see Pearce (1991), Poterba (1991), Nordhaus (1991b), Brinner et al. (1991), Manne and Richels (1990), Whalley and Wigle (1991).

An alternative proposal closely associated with the carbon tax is a program of tradable permits. In contrast to a carbon tax, this measure would limit the quantity of emissions by limiting the total number of permits. Hypothetically, if governments allowed permits to be bought and sold, the program would create incentives similar to a carbon tax, since the market price of a permit would encourage expenditure on abatement measures up to a marginal cost equivalent to the price. Revenues from such an auction could be used to mitigate adverse economic impacts, e.g., by reducing other taxes, or "tax recycling". Literature on the topic of tradable permits includes Hartley (1997), Piacentino (1994), Rose and Stevens (1993).
The economic implications of carbon taxes or permit auction revenues are in principle similar, but with potentially different results in practice: the set price of a carbon tax would not necessarily control the quantity of emissions, and the set quantity of emission permits would not necessarily control the price. Nordhaus (1992b) presents a comparison of five strategies for controlling GHGs using a specialized model, including a carbon tax and several emission- and climate-stabilization policies. The carbon tax was suggested as “an efficient approach”, whereas the “emission/stabilization approaches would impose significant net economic costs”.

Part of the argument against a worldwide taxing institution to control CO₂ emissions is the hesitance of countries to entrust such an agency with large currency balances and the responsibility of proper re-distribution of these funds. Under some suggested payment mechanisms, billions of dollars worth of tax revenues would flow through such an agency from highly developed CO₂-producing regions to those with minimal development. Other frameworks have proposed self-taxing mechanisms to keep the money internal to each country. Indeed, as mentioned above, certain countries such as Sweden, have self-imposed a carbon tax on industrial emissions. Section IIB will discuss some other aspects of taxing mechanisms.

Another issue brought about under GHG controls is the opportunity costs of competing investments for the control infrastructure. (This argument is also mentioned in Section VAS under selection of an appropriate rate of discount). Nordhaus (1991a) and Warby, et al (1999), argue that there are many higher priority projects in developing countries, such
as literacy and living conditions, that merit investment prior to CO₂ controls. Warby, et al. go further in claiming that the cost of controls for developing countries will fall upon “the taxpayers of those countries whose politicians view the problem as a high priority.” As mentioned above, massive transfers from higher-priority to lower-priority countries will be resisted.

The seventh issue addresses whether countries can justify policing CO₂ output when the enhancement of sinks for CO₂ through afforestation and reforestation might provide a less costly alternative method of control. Many countries have large, national forests which are believed to play a critical part in land-based carbon sequestering, especially tropical forests.¹⁵¹⁶ Historically, governments in tropical areas have leased forestland to foreign livestock interests who in turn clear the trees to enable grazing. If, however, large biomass carbon fixing is feasible, it is necessary to determine the imputed value of a forest’s carbon sequestering potential. Then, if incentives are created for biomass owners to limit biomass eradication, the question arises as to the type and quantity of compensation to be due the landowners (usually federal governments). The compensation might evolve as transfer payments between national governments imposing efficiency losses from taxation and distorting the incentives of recipient governments, and perhaps imposing additional costs as the transfer payments are invested.

¹⁵Reis and Margulis (1991) discuss studies on deforestation in central and South American rain forests.
¹⁶Gillis (in Repetto and Gillis, 1988) studied deforestation in West Africa, Malaysia, and Indonesia.
To explore the potential flow of payments, it is necessary to determine the primary countries and regions involved in the problem. The major CO₂ producers in total volume are: U.S., the former Soviet Bloc (which would now be represented by Russia, Ukraine, and Eastern Europe), the European Community (EC), and eastern Asia (Japan, Korea, and China). Many countries share the major biomass locations: Russia, central and South America, southeastern Asia, central Africa, and North America.

Enhancing biomass sinks depends on whether the alleged potentially damaging warming trend can be slowed or reversed via increasing the biomass of carbon-fixing matter, or what is commonly referred to as "carbon sequestering". One way to increase the biomass is by forestation. This process includes afforestation, or planting trees on a site that has not been in forest for 50 years or more, and re-forestation, which is planting trees on a site that was recently in forest. A number of studies have proposed using trees for removing CO₂ from the atmosphere (see Nordhaus. 1991b for a survey). One study in particular (Schroeder, 1992) tabulates carbon storage quantity data for most short rotation (i.e., growth to maturity) tropical trees. Schroeder also discusses several feasibility assessments and the sequestering potential of these specific tree species. Other studies (e.g., Grübner. et al. 1993: Marland and Marland. 1992: Sampson. 1992: Sedjo and Solomon. 1989) have estimated overall quantities of CO₂ that potentially may be stored on the earth from forestation and other land management activities. Results of these analyses are used later in this work for a planetary application of a dynamic optimal control model to determine long-term CO₂–biomass steady state potential.
In addition to the costs of forestation and management, however, one must consider another cost of reducing CO$_2$ in the form of foregone consumption resulting from limited biomass extractions netted against the gains in utility from less of the supposed effects from global warming. Uzawa (1991) analyzed this imputed price of atmospheric CO$_2$ in dynamic, single and multi-country models. He modified the model to include carbon sequestering and ocean absorption. Nordhaus (1991b) analyzes the option of subsidizing productive uses of wood that effectively sequester carbon for long periods of time. A possibility for this suggestion is mentioned in Section VD. Suggestions for Further Research and Analysis. A similar discussion could be extended to boosting ocean absorption of CO$_2$ via "iron fertilization" as another method to enhance a CO$_2$ sink. Roughly one-half of the CO$_2$ is absorbed in ocean waters, but as mentioned above, there exist potential side effects of this process.\textsuperscript{17}

To summarize, to answer in the affirmative the question of whether energy taxes should be imposed, the following decisions must be made:

1) whether warming will occur

2) whether the negative effects of warming outweigh the positive effects and other positive effects of CO$_2$ apart from any effect on warming

3) whether the causes are cheaper to treat than the effects, taking account also of the value of any foregone benefits from reducing CO$_2$ output

4) whether among causes, reducing CO$_2$ (not other GHGs) is the cheapest option (foregoing the benefits of CO$_2$ must be considered as part of the cost)

5) whether decreasing CO$_2$ output is cheaper than increasing CO$_2$ sinks

\textsuperscript{17}A geo-engineering option of increasing CO$_2$ absorption is "fertilizing" the oceans with iron particles, but damage to plankton is a possible externality. See Cline (1992) and Nordhaus (1991b) for options.
6) whether the cost of taxing is less than the cost of warming

The primary focus of this paper is twofold. It addresses whether additional, large amounts of carbon-fixing biomass can sequester an amount of atmospheric carbon large enough to compensate for the gigatonne additions to it made by fossil fuel consumption. Additionally, the model looks at a fuel switching scenario, one where biomass replaces fossil fuels (particularly as a transport fuel) at a certain price. The scenario we have in mind is that eventually oil and its derivatives will be replaced by methanol and other biologically derived fuels for providing transportation services. It is likely that the other major energy use (electricity) will use a non-biological fuel source (perhaps direct conversion of solar energy) in due course, but we ignore this issue for simplicity.

Technically, one could view our analysis as implicitly assuming energy use for electricity generation is separable from energy use for transportation services. Optimal control models, widely used in dynamic applications of economics will be used to determine a steady state and an optimal path.

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18 The cost of energy taxes depends on the factor shares and the degree of substitution between energy and other factors of production, as energy is primarily an input to production. It also depends on how much adjustment time is allowed before taxes are imposed. With little adjustment time, substantial losses will occur from making capital obsolete. With a larger adjustment period, existing capital can be allowed to depreciate, being replaced by new capital that produces less CO2 per unit of energy input.
II. Economic Foundations

A. Applicable Model Structures

The following sections discuss selected works in growth economics, natural resource economics and public economics on which this paper is based. Some alternative models for the ecosystem, such as the predator-prey model, and alternative models for the welfare system, such as game theoretic models, are also discussed.

1. Economic Growth Models

We shall model the global warming problem using an infinite horizon, optimal growth model, sometimes referred to as the “Ramsey problem” (Ramsey, 1928). The growth path of the one-sector model economy was made more general by Solow (1956), who alleviated the Harrod-Domar (Harrod, 1939; Domar, 1946) “knife-edge” instability by suggesting simple hypotheses for growth in the supply of the factors of production. Similar models, leading essentially to the same conclusions, were developed independently by Swan (1956) and Meade (1961). This type of analysis was expanded to dynamic two-sector growth models involving capital accumulation by Uzawa (1964). Preferences in growth models were analyzed by Cass (1965) and Koopmans (1964). Later, optimal control models were developed to include a pollution externality (Forster: 1973, 1977) and environmental and natural resources in general (Smith, 1977). A more recent optimal control model without biomass presented by Götttinger (1992), includes CO₂ removal via a linear, “natural” process and other, anthropogenic non-linear methods, such as emission scrubbing.
William Nordhaus of Yale University has probably undertaken the most extensive analysis of the impact of climate change using the growth theory framework. His work on CO\textsubscript{2} modeling began in the mid 1970s and continued in the mid 1990s. Nordhaus (1992a,b; 1993a) presented the DICE (Dynamic Integrated Climate-Economy) model of climate change. In the author’s own words:

The DICE model is designed to choose levels of investment in tangible capital and in GHG reductions that maximize a social-welfare function that is the discounted sum of the utilities of per capita consumption.

Nordhaus (1993b) later enhanced the DICE model to consider an aggregate global economy.

Lewis and Seidman (1995) investigated global warming in an optimal growth framework by adapting Nordhaus’ DICE model and performing sensitivity analysis not included in the Nordhaus model. They utilize an optimal control program in FORTRAN to begin at steady state values for the savings and emissions rates and iterate until a maximum is achieved for a welfare function over a 100-decade horizon.

Hirofumi Uzawa (1991) presented an optimal control model at the 1990 Rome conference on global warming to evaluate key issues of shadow pricing of CO\textsubscript{2} (i.e., to determine the optimal level for a carbon tax) in the policy debate. This work was presented as part of a workshop at Lusaka, Zambia, in 1991 and later refined (Uzawa, 1993). The work presented in this paper somewhat resembles Uzawa (1991. 1993). His model is analyzed in Section IIIB, with direct comparisons made where appropriate.
2. Exhaustible Resource Exploitation Models

a. Renewable Resources: Biological Growth and Population Dynamics

Renewable resources which can be exploited must rely upon natural or augmented growth to sustain a population. While such a resource could in principle thus maintain a virtually infinitely-lived stock, over-exploitation could also result in exhaustion. Growth models for these resources date back at least to the 19th century, when industrial demand for wood rose to meet increasing mass production of consumption goods. The most popular and widely applied resource renewal used in these models is the logistic function.

i. The Logistic Function and Variants

The logistic equation is essentially one of several biomass response functions in a homogeneous landscape of biomass. Shugart (1984. Chapter 6) analyzes four predominant models for biomass growth, one of which is the logistic function. These models are described below, and are widely used in the field of population dynamics. The logistic equation was formulated as an early development in population theory by Verhulst (1838), who mathematically attempted to describe population growth in a restricted environment. According to Harper (1977), the work was almost completely ignored until Pearl and Reed (1920) derived essentially the same formula independently. Both are given credit and the usual name is the Verhulst-Pearl logistic equation. The basic differential form is:

$$\dot{R}(t) = gR(1 - R)$$

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1Exhaustible resources, such as biomass or fossil fuels, can theoretically be completely depleted in a relevant time frame (if they are over-exploited), unlike non-exhaustible resources, such as solar, wind, or geothermal energy.
and its general, integrated form is:

\[ R(t) = \frac{a}{1 + be^{-at}} \]

where \( g \) is the net intrinsic growth rate, \( a \) and \( b \) are environmental parameters, with \( a \) usually representing the capacity or maximum size (often, as above, normalized to unity). while \( b \) usually represents the fraction of remaining possible growth.

The two principal variants of the logistic equation, the Gompertz (1825) and the von Bertalanffy (1960) equations, have the common features of a linear quadratic function: the integrals have an initial growth of exponential proportions, followed by an inflection point, then damped, or concave, growth to an asymptotic value of capacity. These variants were originally developed to describe biological growth in small animal populations or single, larger animals, but have also been applied for growth in plants. To illustrate and contrast the variants, the capacity parameter \( a \) (changed to \( A \) later in the growth model used in this paper) in the logistic equation is scaled to equal unity, and the biomass stock \( R \) normalized to range between 0 and \( a \).

The Gompertz equation in differential form appears as,

\[ \dot{R} = g(1 - R \ln R) \]

and was developed in the early 19th century for actuarial purposes. Its primary feature is that the growth rate undergoes an exponential retardation, no matter what part of the growth curve is considered. Its general, integrated form is

\[ R(t) = ae^{-be^{-at}} \]
The von Bertalanffy equation, in contrast, takes the differential form.

\[ \dot{R} = gR^K(1 - R^K) \]

which fits the age-size profile for animal growth. The integrated form of the von Bertalanffy equation.

\[ R = a(1 - be^{-rt})^3 \]

is sometimes confused with the "monomolecular" equation, (similar to the concave growth equation described below; also see Manawar, 1945), which has the integrated form:

\[ R = a(1 - be^{-rt}) \]

where \( a \) and \( b \) are related as above to the growth capacities. and \( g \) is again the growth rate. Both have similar graphs, but the monomolecular formula has no inflection point. and was used to model growth in weight, whereas the von Bertalanffy was initially used for growth in linearly-scaled size. Thus, the cubic form of the von Bertalanffy equation accounts for linear-scaled growth in three dimensions.

Some applications were developed in order to choose the growth function of best fit for specific plants. Zavitovski and Stevens (1972), using the method of Ricklefs (1967) concluded that for a moderate growth rate tree such as in red alder (Alnus rubra) communities\(^2\), the von Bertalanffy equation best fit the growth pattern with \( g = 0.09 \). The growth rate used in the analysis of the model of this paper is more conservatively estimated at \( g = 0.08 \). Fresco (1973), using the method of Erkelens (1968) found that the

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\(^2\) 119 trees, ranging in age from 1 to 65 years, were felled from 20 stands selected randomly from 58 stands. The trees were dried and weighed. Roots from 28 trees were extracted, dried and weighed.
logistic function fit growth data for various smaller plants and estimated their $g$ and $a$
parameters.

To compare forms and illustrate graphical differences, the four biologic accumulation
functions described above are presented in Table 2 below and graphed with their time
derivatives (growth rate functions) in Figure 4 (page 30). The graphs were developed
using MathCAD® version 8 (worksheets are shown in Appendix D). Note the
progressive reduction in time to both maximum growth rate and maximum biomass for
the four curves.

### Logistic Function and Variants

<table>
<thead>
<tr>
<th>Logistic</th>
<th>Gompertz</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R(t) = \frac{A}{1 + \left( \frac{A}{R_0} - 1 \right) e^{-\mu t}}$</td>
<td>$R(t) = R_0 e^{-\mu t}$</td>
</tr>
<tr>
<td><strong>von Bertalanffy</strong></td>
<td><strong>Concave</strong></td>
</tr>
<tr>
<td>$R(t) = \left[ A^{\frac{r_0}{K}} - \left( A^{\frac{r_0}{K}} - R_0^{\frac{r_0}{K}} \right) e^{-\mu t} \right]^3$</td>
<td>$R(t) = A + (R_0 - A)e^{-\mu t}$</td>
</tr>
</tbody>
</table>

Table 2

### ii. Alternatives to the Logistic function

As mentioned above, a growth model similar to, but simpler than the logistic function is
the concave function

$$\dot{R} = g - \frac{g}{A} R$$
Graphs of the Logistic Function and Variants

Graphs of Derivatives of Logistic Function and Variants

Figure 4
which implies a constant input of biomass less the loss of biomass that is a constant proportion of the biomass. This formulation would monotonically increase the biomass to the asymptote with no inflection. Note the proportional growth formula of the logistic function has the identical form:

$$\frac{\dot{R}}{R} = g - \frac{g}{A} R$$

The principal alternatives to the monotonic logistic or concave functions arise from observations that the biomass of a forest may overshoot the asymptotic value, then either gradually return to the asymptote or oscillate about it in a damped manner. These characteristics are generated by a time-lagged logistic curve involving synchrony of the mortality of the first generation of trees.

The overshooting function is represented by

$$\dot{R} = gR \left( 1 - \frac{1}{A} R_{r-\tau} \right)$$

where $R_{r-\tau}$ indicates the phytomass at some earlier time $t - \tau$. The damped oscillatory function is represented by

$$\dot{R} = f(R_{r-\tau}) gR \left( 1 - \frac{1}{A} R_{r} \right)$$

iii. The Predator-Prey Model

There is one final model of the interaction between biomass and CO$_2$ that we shall discuss. It is an adaptation of the well-known Lotka (1925)-Volterra (1931) equations.
which model the non-linear relationship between predator and prey. The "predator" would be the biomass, since it must "feed" and therefore reduce the population of the "prey", or CO$_2$. Similarly, in the absence of a "predator", the prey would grow unchecked (CO$_2$ would be produced by the economic activity of fossil fuel combustion and/or deforestation). The system of equations for this adaptation are:

$$\dot{R} = -cR + gRV$$

$$\dot{V} = \beta V - \mu RV$$

where $V$ is CO$_2$ present in the atmosphere, and $R$ is terrestrial photosynthetic biomass, and the other symbols are parameters.

It is instructive to point out the similarities and differences between this system of interaction and the ones previously discussed. With regard to the equation for $\dot{R}$, which describes biomass stock evolution, the term $+gRV$ has the factor $V$ for the stock of CO$_2$. Technically, carbon dioxide is required by photosynthetic biomass to survive, and indeed, carbon is the basis of life on earth. Biomass would disappear if the gas is not present in sufficient quantity, i.e., if $V = 0$. Hence, in this sense the equation for the rate of change in biomass is more accurately formulated by the predator model. However, it is unlikely that insufficient quantities of CO$_2$ will threaten the earth's flora. It is more likely that excessive quantities of CO$_2$ could poison plant life. But again, even at pessimistic expected future rates of unchecked carbon dioxide production, this latter type of poisoning is unlikely for centuries, if at all$^3$. Thus, $V$ should be included in a growth model for biomass, since it will not likely go to zero. Mature biomass will decay as

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$^3$ Before carbon was "trapped" in fossil fuel deposits, it was part of a functioning and productive biosphere.
\(-cR\), even if \(V\) does not vanish, but the equation can be thought of as net growth, which is limited by natural constraints such as water, temperature, and nutrients. Therefore, a growth-dampening factor, such as a quadratic reduction term (e.g., \(-gR^2/A\) in the logistic equation), should also be present to gradually dampen or reduce the linear rate of growth. This term is present in the models used in this work and similar terms appear in the variants discussed above.

With respect to the equation for \(\dot{V}\), the evolution of CO\(_2\) stock, a term similar to \(-\mu RV\) is used in the models of this work. The term \(\beta V\), however, must be reinterpreted, since carbon dioxide does not grow unchecked “naturally” in the absence of its biomass “predator”. As mentioned above, economic activity in the forms of fossil fuel burning and biomass consumption produces the CO\(_2\). Without these latter producers, CO\(_2\) stock would remain fairly constant, except for massive geologic phenomena such as tectonic plate movements that may have resulted in volcanic eruptions, releasing CO\(_2\) in pre-human times.

The key feature of the predator-prey model is the well-known phase diagram, tracing out a loop in state space. The loop is one type of solution for non-linear systems, resulting in cyclical variations for stocks of state variables. The time trace of the state variables are sinusoidal, and if growth rates are different, out of phase. Without extending the work in this paper to model and simulate the pre-industrial or pre-human carbon cycle, it is worth speculating that at least a close variant of the predator-prey paradigm would represent the carbon cycle in that era.
b. Non-renewable Resources

The key distinction for non-renewable resources is that in their natural, undisturbed state, their growth rate is zero. Thus, according to Dasgupta and Heal (1979)

In short, the intertemporal sum of the services provided by a stock of an exhaustible resource is finite. [Italics theirs]

They use for clarity the example of land, which can, if well managed, provide “an unbounded sum of services over time”. Thus, the quantity of land, though finite, need not be exhaustible. This paper uses a mix of resources, a renewable one of terrestrial biomass, and a non-renewable one of fossil fuels.

The exploitation of non-renewable resources must then involve an extraction function, whose integral cannot exceed the initial endowment or stock of the resource. For example, if \( R_0 \) is the initial stock of the resource, and \( h(t) \) is the extraction function,

then \( \int_0^t h(t) \, dt \leq R_0 \).

One of the key differences between infinite-lived models of renewable and non-renewable resources is that in the case of a renewable resource, the existence of a steady state is possible. For the non-renewable resource, by contrast, one would surmise

\( \lim_{t \to \infty} h(t) = 0 \), and thus the shadow price of the resource would approach infinity.

3. Game Theoretic Models

It is possible that interaction between agents may also play a critical role in the potential problem of possible global warming. A dynamic game model could be used to capture
these interactions. In an oligopolistic market, a small number of agents act non-cooperatively and affect output or prices of each other. Each agent has to make assumptions regarding the behavior of others, i.e., conjectural variations, and a behavioral strategy is formulated. Thus, the rival agents are playing a game. Negative externalities can be modeled as game theoretic problems because the outcomes are a result of the actions of two or more parties. However, non-optimal outcomes such as the “Prisoner’s dilemma” are possible (Sandler, 1992b).

Two types of game theoretic models can be used to examine a system of this type with more than one party. The first is the closed loop or feedback equilibrium. This type of strategy assumes that every country at each point in time receives information on CO₂ emissions and carbon sequestration in all countries. Reaction functions can then be constructed to evaluate future actions. International environmental agreements have been modeled in this context by Barrett (1992, 1994) and Hoel (1992), and discussed more generally in Sandler (1992a). Proceedings of a conference of environmental economists at Freudenberg, Germany in November 1990 concerning this topic are collected in Pethig (1990). Military conflict and arms buildups can also use the closed loop strategy. Applicable previous work in this area is due to Richardson (1960), Brito (1972), and Simaan and Cruz (1975). The second model strategy is the open loop equilibrium. Here, countries cannot monitor emissions or removals in other countries. Strategies are designed for all future time and implemented regardless of the actions of other countries. A model such as the one posed in this paper could be used for such a strategy.
In modeling such a game, it becomes necessary to determine whether open or closed loop strategies are both convenient and applicable. The easier and less interesting open loop is essentially a “one-shot” system. Choices of allowable pollution levels would be made, the system evolves, and finally utility comparisons are made at the steady state. Thus, the logical choice is to use the closed loop strategy, since it allows for strategy changes after the start of the system, i.e., countries may react and adjust their investment levels to compete for the “use” of the atmosphere.

B. Externalities in the Use of Common Property Resources

An externality occurs whenever a decentralized economy has insufficient incentives to create a potential market in some commodity and where, as a result, the market equilibrium is Pareto inefficient\(^4\). In other words, externalities occur when economic agents don’t bear all the consequences of their actions. For example, if someone disposes toxic waste, such as paint, in their neighbor’s yard, instead of paying for proper disposal, the act lowers the total cost of painting, but reduces the neighbor’s property value. The neighbor’s yard, however, is not common property. The neighbor could prosecute to recover damages, and thereby internalize the externality. A more applicable example is disposing of CO\(_2\) in the atmosphere, eventually inducing damage from warming\(^5\). The CO\(_2\) producer’s cost of disposing of the harmful CO\(_2\) is reduced, and public and private parties may suffer the effects of warming.

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\(^4\) Dasgupta and Heal (1979)

\(^5\) For the purpose of exposition and application of this theory one must allow the assumption discussed previously that increasing CO\(_2\) stocks will lead to atmospheric warming that is more harmful on net that any other benefits that might result from increased CO\(_2\) in the atmosphere.
Pigou (1920) studied environmental problems involving certain types of externalities. His approach to mitigating their effects was to invoke the authority of the government to impose a tax, the “Pigouvian tax”, on whomever generates the externality. This tax was to emerge as an instrument to equalize private and social marginal costs. Under a system of Pigouvian taxes in the examples mentioned above, a tax could be placed on the disposer. (Direct regulation in the form of a lawsuit or fine is also possible.) The carbon tax mentioned above may be an example of a Pigouvian tax.

If global warming occurs\(^6\), and no mechanism is enacted to mitigate the effects, it is argued that the social costs of CO\(_2\)\(^7\) and other GHG emissions will be greater than those borne by the producer. The net effect, then, is a negative externality. Nordhaus (1991a) follows this line:

> The economics of the greenhouse effect is a classic case of a public good, in which emissions of GHGs involve a global externality.

Thus, the carbon tax, in the Pigouvian sense, “internalizes” the externality produced by CO\(_2\) emissions. In the model presented in this paper, we look at an optimal solution assuming the externality could be internalized.

Ronald Coase (1960) critiqued the Pigouvian idea of a tax on externalities with an analysis based on property rights and transaction costs, arguing that the presence of an externality does not provide an a priori case for government intervention. If transactions

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\(^6\)To support the position taken here, CO\(_2\) can be taken as a proxy for any net negative externality introduced into the atmosphere.

\(^7\)Hall (1990) attempts to quantify the costs of externalities generated in domestic energy consumption, including the cost of delaying global warming.
costs are zero\(^8\), the Pigouvian remedy is not necessary, because there would be no
efficiency differences among different systems of property rights. If the paint disposer's
neighbor has a right to demand compensation, the disposer would be willing to pay the
neighbor any amount up to the cost of proper disposal, and both would be better off. If
the disposer has a right to dispose, the neighbor would be willing to pay the disposer any
amount up to the damage done to the yard not to dispose on his property and both would
be better off. Thus, with no transaction costs, bargaining among agents over allowable
levels of externalities can achieve efficiency without government intervention. After the
initial assignment, the rights will be traded to the party who values them the most, and the
problem is eliminated at lowest cost, without Pigouvian taxes. When bargaining or
transaction costs are high, Coase argues that the allocation of property rights has
efficiency consequences. When there are transaction costs, he suggests property rights
should be assigned in a manner that will minimize these costs. He further suggests that
the common law has in practice tended to follow this principle.

The case of responsibility and remedy for the externality of atmospheric warming as a
result of CO\(_2\) production from fossil fuel consumption would certainly appear to be one
case where transaction costs are high. With so many parties involved, it is unlikely that
transaction costs are zero, making bargaining extremely costly and, for all practical
purposes, impossible. Buying permission from other countries would also provide an
incentive for one country to hold out, hoping for a larger share\(^9\). In the case of CO\(_2\),

\(^8\) Coase maintained, however, that “This is, of course, a very unrealistic assumption.”
\(^9\) Quiggin (1988) also asserts shortcomings in the bargaining process, arguing that if rights to pollute are
possessed by all countries, and a non-polluter pays a polluter to stop polluting, then non-polluters have an
incentive to pollute. This case, and its converse (where non-polluters are compensated), he claims, explains
why Coasian bargaining is rare in practice.
emissions to the atmosphere, it is also unclear whether definite and enforceable property rights could be devised. After all, CO₂ is a colorless, odorless gas, readily absorbed into the atmosphere and biosphere, making cheating very hard to detect.

No property rights have been assigned to the atmosphere: therefore it is not a private good.¹⁰ It could be characterized as a “common property resource”,¹¹ or alternatively as a “public good”. Stiglitz (1986) characterized a common property resource as one to which access is unrestricted, which would imply the atmosphere is a common property resource. His characterization of a public good was that the marginal cost to exclude a person from usage is high and the marginal cost to provide the good to a person is low. Thus, the atmosphere has many characteristics of a public good, and a common property resource. Likewise, the effect of CO₂ emissions is viewed by some as a public “bad”.

As mentioned above, one would apply the Coase arguments to the atmosphere and assign property rights to one individual or institution. The governing body would attempt to regulate and monitor CO₂ emissions. However, it will be in an individual polluter’s best interest to ignore the impact of their pollution on the quality of the atmosphere and on others, and then enforcement becomes a global policing problem. Thus, assignment of property rights may not lead to an efficient outcome. Other examples are analyzed in Ostrom (1990). She explores the conditions under which common property resource problems have been successfully or unsuccessfully solved. One such successful

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¹⁰ In many cases, the atmosphere is in constant motion, so the concept of private gaseous atmosphere is of minimal use. A case could be made, however, for predictable migratory climate patterns.
¹¹ Variations on this term are: “common property”, “commons”
institution discussed is a voluntary organization rather than a coercive state, implying a cooperative behavioral strategy. However, even though members of such an organization have special interests in common like the reduction of global pollution, they are reluctant to contribute voluntarily to the cost of pollution control because they hope for a free ride.

The model in this work could be extended to include additional countries. A requirement would be an agreement established for acceptable CO$_2$ emissions. Next, an agreed-upon analysis for each country would determine its CO$_2$ sequestering capability based on biomass stock and growth. Then, its steady-state potential would be determined based on an allowable CO$_2$ emission path. If the allowable emission pattern exceeds its observed pattern, a country would receive transfer payments in the form of a carbon tax generated from net emitters. One of the key problem areas would be to establish the value of CO$_2$ emissions based upon some acceptable standard measure of impact on utility.

The effects of stock externalities in this type of model have been investigated. A case in which the externality inhibited the growth of the renewable resource was analyzed in Siebert (1982). Stability of this type of equilibrium in a case where an externality is present was analyzed in Brito and Intriligator (1987). There, in a case where a Pigouvian tax is levied on the consumption of an externality-producing good, the stability of equilibria depend on the substitutability of the good producing the externality with respect to the externality. If they are gross substitutes, then the Pigouvian tax equilibrium is stable. Further, a sub-unitary elasticity between them is both necessary and sufficient for a stable optimal equilibrium. at the optimal equilibrium level of the externality.
C. Multi-Country Game Models

The model used in this work could be also modified to investigate any effects on the optimal path of any one country when a second country is introduced. Alternatively, a model like the one we examine could be applied to several countries in succession to analyze the potential for achieving equilibrium where the combined stock of CO₂ affects all entities. In that case, the model becomes a non-cooperative game with strategic behavior. Treated as a differential game, it could generate various outcomes depending on initial parameters such as marginal utilities and stocks of resources. The application of these cases might simulate bordering countries with different concerns about pollution and different stocks of resources. Uzawa (1991) used a similar approach, introducing money in a consistent unit via the country’s GDP. A similar model using investment as a control variable in a three-party forest sector differential game is in Kivijärvi and Soismaa (1991). Their three sectors are: the forest industry, the timber industry, and society, where society taxes the timber industry. Their goal was to determine Nash equilibrium levels of investment and harvest strategies, and forest tax policies. Other dynamic games of transboundary pollution using a similar model structure are: 1) Kaitala, Pohjola, and Tahvonen (1991), using an acid rain model without capital or capital that reduces an externality, and 2) List and Mason (1997) addressing the issue of whether environmental regulation for these pollutants be established locally or centrally.

A goal in a game model would be to examine the results of strategic behavior in two ways. The first will have both countries using the same technology where the biomass represents a capital stock which can produce energy or consumption goods and reduce
CO₂ pollution. The countries share the atmosphere, and thus each country’s utility is affected by the other’s consumption pattern. The countries will act as myopic Cournot/Nash players in an open loop strategy, taking each other’s consumption as given and unchangeable, and the object will be to determine the conditions which will generate an equilibrium. Brito and Intriligator (1995) proposed that in an arms race, an equilibrium point is stable if each country attempts to behave in a myopically optimal manner.

In the second, more complicated strategic behavior would be introduced, similar to the arms race models presented in Brito (1972) and Simaan and Cruz (1975). The case would use a closed loop or feedback strategy, where each country uses information about current pollution and biomass stock levels to predict future levels.
III. Our Model Description and Comparison

A. Our General Model

A dynamic model of optimal natural resource management for a single country (or the world as a whole) is proposed here and analyzed in Section IV. This type of model has a well-known application as the benevolent social planner's problem, where the social planner can augment an endowment of a capital stock through investment or reduce it through consumption. In the model presented here, endowments for two resources are present: a non-renewable resource endowment of fossil fuels, and a renewable resource endowment of terrestrial biomass. The biomass can be augmented by its natural growth process or consumed. It may be consumed as two distinct types: for non-energy purposes as food consumption, and for energy production purposes. The endowment of fossil fuels may also be consumed in two ways: for non-energy purposes in chemical production, and for energy production purposes. The consumption of either resource endowment for energy purposes produces carbon dioxide, which may be perceived as a "bad". Biomass use for non-energy purposes also produces CO₂. Biomass also plays the role of reducing atmospheric CO₂ via photosynthesis by carbon sequestration. The ocean also has a CO₂-reducing role by its natural absorption process.

This one-country, two-sector model structure will use growth model analyses similar to those developed by Uzawa (1964) and Cass (1965), but differs from these analyses in two ways: natural augmentation, or "growth" of one of the resources, and the non-substitutability of resources in one of the sectors (consumption of fossil fuels and biomass
for non-energy purposes). Some other models—referred to colloquially as "putty-clay" models—do not allow one resource to substitute for another in the production of capital. Non-substitutability is characteristic of most resource conversion processes, which are primarily irreversible. The paradigm usually involves a capital stock which, once having been created from a resource stock (e.g., harvested trees made into building materials), cannot be converted back into the original resource. Irreversibility in capital theory has been explored generally in Arrow (1968), in a Ramsey model by Arrow and Kurz (1970a,b), and with a more specific application to renewable resources in Clark, Clarke, and Munro (1979).

The renewable resource endowment (terrestrial photosynthetic biomass) can be harvested directly for consumption in either the energy sector or the food sector, i.e., as a transportation fuel (say, methanol) or as an agricultural crop. A linear harvesting function will be used to describe the usual process in which one or more control variables change the stock of a state variable. Linear control processes, commonly referred to as "bang-bang" control, typically result in extreme values for control variables, and "jumps" in the state variables. However, to maintain regularity, these piece-wise continuous functions are usually constrained. A quadratic control process, in contrast, by its property of possessing an internal maximum over the convex set of admissible controls, tends to generate an internal optimal control path. In the interest of simplicity, however, linear controls are used in this model. Thus, the flow of utility is gleaned from the stocks of the
resources via a linear (quasi-concave), cardinal utility function. Biomass grows via a non-linear production function, the widely used logistic function.

Consumption of either good, fossil fuel or biomass (directly, or from a derived product such as methanol), for energy purposes generates the greenhouse gas carbon dioxide via combustion. As with many global warming studies, we assume the stock of CO₂ acts as an economic "bad", directly reducing utility. For simplicity, this effect is assumed to be linear. The reduction in utility resulting from the generation of CO₂ and its subsequent alleged atmospheric warming, is at least partly offset by the benefits obtainable from the stock of the biomass, via the feedback mechanism of carbon sequestration. In addition, the build-up of CO₂ is partially controlled by ocean absorption. The additional mechanism of carbon dioxide fertilization acts as a compound effect, which tends to offset the potential negative effect of global warming¹. Thus, the extraction of fossil fuels or the harvest of biomass generates a flow of utility when they are consumed, possibly offset by a flow of CO₂, while the growth of biomass and the absorption of CO₂ by biomass and oceans also determine the stock and flow of CO₂.

In the centrally planned, one country model, the planner myopically chooses, via the optimal initial shadow prices², the resulting optimal transformation pattern for the

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¹ In a multi-country model, CO₂ could be viewed as a stock externality (described in Section IIB) with positive (fertilization) and negative (global warming) consequences.

² Since money is not used in our model, this imputed price is in terms of lost utility units, or utils, not a monetary unit such as dollars. If one were willing to specify a particular utility of consumption function, such as $C^{1/\gamma - 1}/\gamma$, one could translate marginal utilities into proportions of aggregate consumption. Thus, for $U = C^{1/\gamma - 1}/\gamma$, $\frac{U'}{C'} = \gamma$ and a 1% change in marginal utility would correspond to a $1/\gamma$ % change in per capita consumption. A common value taken for $\gamma$ is $\gamma = 2$. 
renewable resource. No explicit markets exist. Optimal prices, with trajectories for CO₂, biomass, and fossil fuel may be then determined. A steady state will be sought, and where possible, exact optimal solution paths will be given. Since we have three state and three co-state variables, it will only be possible to determine optimal trajectories using numerical solutions.

Comparable models involving biomass growth that sequesters carbon include the analyses of Uzawa (1991, later revised to Uzawa, 1993) and Marland and Marland (1992), and are more somewhat related to the one presented in this paper. The Uzawa works are functionally similar to the models presented within and merit separate discussion and comparison. Because of their similarity, they are analyzed in detail in Section IIIB. The Marland and Marland (1992) work is similar but differs in that it uses a discrete, concave growth function (described in Section IIIB3 above) which accumulates to an asymptote, but has no inflection. Sample sequestrations are used instead of a functional form. No dynamic analysis is made. The goal of their analysis is to determine the policy for a given biosystem, depending on growth potential and initial conditions. Thus, the recommendation for a large, old tree biomass with slow or no growth potential can be only to protect the existing forest. Conversely, where high productivity can be expected, the most effective strategy is to manage the forest for a harvestable crop.

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1 Our model is aimed at determining an optimal policy, not predicting an equilibrium outcome.

1. Overview

The purpose of Uzawa (1991, later revised to Uzawa 1993\(^1\)) is similar to the work in this paper, and is best expressed in his own words:

This chapter concerns itself with understanding the mechanisms through which the phenomenon of global warming occurs, with reference to the Pacific Rim. and proposes an incentive scheme that may be effective in restoring the atmospheric equilibrium.

He also states, in motivation for the work:

A number of policy proposals have been advanced in recent years to arrest the process of atmospheric warming, with particular impacts upon industrial and agricultural activities in the Pacific Rim countries…

The policy proposals have been mostly formulated in terms of quantity constraints concerning the production or use of greenhouse gases and are implemented through governmental intervention into the activities of individual citizens or business establishments. … Since the evidence concerning the causal relationships between emissions of greenhouse gases and increases in the average global surface temperature involves a significant degree of uncertainty, the result of international or regional negotiations tends to be influenced by factors of a largely political nature that have little or no relevance to the substantive issues involved.

He voices concern into the potential implications of such governmental intervention, saying:

Any attempt by the governments to intervene in the activities of private citizens or business firms for the purpose of implementing an international agreement would destabilize the environmental equilibrium, as well as encourage despotic behavior.

He proposes:

…an alternative system will be introduced effectively to stabilize atmospheric temperature whereby governmental intervention will be held down to a minimum.

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\(^1\) Uzawa (1993) is a rework of Uzawa (1991) in "Proceedings of the Workshop on Solar Radiation, Environment and Climate Change" at Lusaka, Zambia 22-27 July 1991. The later work uses a slightly different approach, which involves separate maximization problems for production and consumption. Consumption is now constrained by production. The results are identical except for separating the production vector, obtained from maximizing national product, from the consumption vector, obtained from maximizing national income.
A caveat is given as:

The problem is how to define the concept of imputed price and how to compute the imputed price for each greenhouse gas and for an acre of each type of land forest.

His work is developed within the framework of the theory of imputation for dynamic circumstances by Ramsey (1928), Cass (1965) and Koopmans (1965). His models are not concerned with arriving at exact levels of imputed prices, but

...an approximation for exact levels of imputed prices, which, however, guarantee the long-run optimality of the process of accumulation.

Finally, the difference between his model and others is that:

Our approach differs from the more standard theory of optimum growth and imputation in that attention is paid to equity issues during the transient process. One of the more important results obtained in this study is that imputed prices of greenhouse gases and land forests in each country involved are determined in such a way that the ratios of imputed prices to the per-capita level of national income in each country take the same values for all countries in the Pacific Rim.²

The motivation is that such a price could provide the basis for a carbon tax on CO₂ emissions. As will be seen later. Uzawa obtains a ratio applied across-the-board worldwide in the form of emissions per capita for atmospheric carbon emissions. This ratio is multiplied by per capita national income (GNP) to arrive at an imputed price (in $/t) for each unit of emission. The assessment calculation is completed by multiplying the imputed price by the amount of carbon-based gases that the country emits. Thus, each emitter country would be required to pay the same per capita fraction of its per capita national income for a carbon tax, and receive a per capita carbon subsidy on

² Thus, for the Uzawa analysis, \( U' / C = k_0 \), constant, implying \( U(\cdot) \) is quadratic, or, in comparison to the footnote above (p. 45), \( \gamma = -1 \)
re forestation. The imputed prices for a country are defined relative to the level of national income at constant prices. Per Uzawa:

The function of the imputed price is twofold. The imputed price is used as an instrument through which efficient and equitable allocations of scarce resources, both private and public, may be attained. At the same time, it serves as the guideline that will guarantee that the allocative mechanism in terms of imputed price is optimum from the dynamic point of view.

The funds collected from such a tax could accumulate in an aggregate transfer fund. The funds would represent a transfer from a high CO₂ emitting country to a lower CO₂ emitting country. Uzawa does not describe the mechanism for monitoring compliance.

The first model for one country simply assumes excess CO₂ levels in the atmosphere decrease naturally, e.g., via an ocean absorption process. The second model for one country incorporates a linear difference of afforestation and deforestation (biomass) activities, where the biomass simply serves the function of reducing pollution.

Scarce resources are used in these biomass activities. Unlike the model we consider, the CO₂ fertilization process is not modeled, and the natural growth of biomass is not taken into account. The model for multiple countries scales up the single country model assuming the marginal utilities and disutilities are identical (and proportional to aggregate per capita consumption) across countries, then calculates the carbon tax as the imputed price of CO₂ per unit of per capita income.
The analysis relies on the turnpike theory\(^1\) to approximate the optimal solution of an optimal control problem in two state variables. Uzawa (1993) describes the process:

In the analysis below, the central role is played by the concept of imputed price in the dynamic context, as developed by Ramsey (1928), Koopmans (1965), and Cass (1965), and effectively explored by Måler (1974) in the present context. However, the models of global warming to be introduced in the present paper involve two or more state variables (it is, in general, impossible to find exactly optimum solutions) and we will be, in the following, satisfied by approximations for exact optima, which, however, will approach the long-run stationary state identical with the case of the exact optima.

One would expect this tendency to be enhanced when the time horizon is extended to infinity. Uzawa cites Måler (1974) and Nordhaus (1980, 1982) as prior studies exploring this method. Måler (1985) expanded his earlier treatment. Haurie and Hung (1977) deal rigorously with some fundamental problems in this area.

Using this technique, the imputed price of atmospheric CO\(_2\) is used to obtain the allocation of scarce resources that is optimum from a static point of view. By analogy with Cass (1965) and Koopmans (1965), the current imputed prices are sought on the optimal path and these are used to approximate the required imputed prices for a saddle path. The emissions are charged at this imputed price at each moment in time. The price is then adjusted over time to optimize the time path of consumption and stock of CO\(_2\). The models are shown to be dynamically stable, and can thus use the turnpike property.

---

\(^1\) Måler (1974, p 104) acquaints the reader: "Turnpike theory asserts that given an arbitrary optimal program of capital accumulation, this program will have a tendency to approach the steady state in the long run. Even if the initial capital stocks and the terminal capital requirements are different from the steady state configuration, the optimal program will spend most of the time, if the horizon is long enough, in a small neighborhood of the steady state. The importance of this theory lies in the fact that it allows us to approximate an optimal program by a program that steers the economy toward the steady state and that it allows us to analyze the effects of parameter changes by analyzing the effects on the steady state."
2. **Uzawa Model Analysis**

a. **without Forests**

The theoretical analysis begins with discussion and formulation of the simplest CO$_2$ accumulation process. The change in the atmospheric level of CO$_2$ is written as

$$
\dot{V}_i = v_i - \mu V_i
$$

(3.1)

where, at each moment $t$, the CO$_2$ accumulated in the atmosphere is denoted by $V_i$, and $v_i$ is the rate at which CO$_2$ is emitted into the atmosphere, which in turn is the dot product of the vector$^1$ of consumption $x = (x_1, \ldots, x_n)$ for $n$ consumption goods and $u = (a_1, \ldots, a_n)$ the vector representing the amount of CO$_2$ generated from production of each commodity consumed:

$$
v_i = a_i
$$

(3.2)

The term $\mu V_i$ represents the rate at which CO$_2$ is absorbed by the oceans, where $\mu$ is a positive constant.

A utility measure for the various economic activities is then established, represented by a preference relation over the set of feasible final consumption vectors. The preference relation is "assumed to satisfy all the conditions required for the standard analysis". The measure uses an inverse indirect utility function index $y(x)$, which describes the minimum expenditure necessary to achieve a certain utility $u(x)$ or level of satisfaction, which is at least as high as the utility provided by the consumption vector $x = (x_1, \ldots, x_n)$.

---

$^1$ At the loss of some generality and the gain of much convenience, after discussing the vector case analyzed by Uzawa, we will restrict our own model to use only scalars.
given a price vector \( p^0 = (p^0_1, ..., p^0_n) \). The notation for this utility measure is represented by:

\[
y = \min \left\{ \sum_i p^0_i x'_i, x' = (x'_1, ..., x'_n) = (x_1, ..., x_n) \right\}
\]

(3.3)

The function \( y(x) \) is deemed by Uzawa as a "standard utility function"².

The utility measure is affected negatively by a factor, called an impact index, related to the amount of CO₂ in the atmosphere. The level of CO₂, \( V \), does not enter the goods production function explicitly. For convenience, the utility index is deemed separable, so the criterion becomes \( y(x, V) = u(x)\phi(V) \). It is assumed \( \phi'(V) < 0 \), so the impact of CO₂ is to reduce utility, and this effect is realized through the multiplicative factor \( \phi(V) \). The discounted present value (\( \delta \) is the discount rate) of utility:

\[
\int_0^\infty y(x_t, V_t) e^{-\delta t} dt = \int_0^\infty u(x_t)\phi(V_t) e^{-\delta t} dt
\]

(3.4)

is then maximized among all feasible time paths.

The next step constructs the formula for the theoretical imputed (shadow) price of CO₂.

The construct uses the condition of the Maximum Principle, which states that the change in the Hamiltonian with respect to a state variable is the time path of its shadow price.

²Uzawa (1993) gives specific properties as strictly positive, and increasing at a decreasing rate, i.e., \( u(c) > 0 \), \( u'(c) > 0 \), \( u''(c) < 0 \), \( \forall c > 0 \). A function such as \( \ln(x) \) would serve as an example. This function depends on the vector \( p \), as well as the vector \( x \). For a textbook discussion, see Varian (1992).
No Hamiltonian is presented, so one is reconstructed here based on the Hamiltonian that would generate such a price formula. The formula built by Uzawa for the imputed price reverses convention and treats atmospheric carbon dioxide as if it is a negative marketable asset (rather than a positive asset with a negative shadow price, since it is a “bad”), subject to the usual dividends, capital gains, depreciation, and opportunity cost. Thus, additional CO₂ will reduce utility through the impact index described above, but its price will become more positive. The precise formula for the time path of the imputed price of CO₂ is given as

$$\frac{\dot{p}}{p_t} = (\delta + \mu) + \frac{u(x_t)\phi'(V_t)}{p_t}$$  \hspace{1cm} (3.5)$$

which would result as a Pontryagin Maximum Principle condition from the continuous time, current value Hamiltonian

$$H(t) = u[x(t)]\phi[\dot{V}(t)] + p(t)[\mu \dot{V}(t) - v(t)]$$  \hspace{1cm} (3.6)$$

where, according to the Maximum Principle for a current value Hamiltonian

$$\dot{p} = \delta p + \frac{\partial H}{\partial \dot{V}}$$  \hspace{1cm} (3.7)$$

Uzawa then equates his “heuristic formula” for the imputed price $p_t$ to the steady-state formula (i.e., where $\lim_{t \to +\infty} \dot{p} = 0$), using the turnpike property to assert the asymptotic convergence of the heuristic formula to the precise formula for the co-state variable, as the system approaches equilibrium:

$$p_t = \frac{1}{\delta + \mu} u(x_t)[-\phi'(V_t)]$$  \hspace{1cm} (3.8)$$

Similarly for state variable $V_t$ (i.e., where $\lim_{t \to +\infty} \dot{V} = 0$).
\[ x_t = \frac{\mu}{a} V_t \]  

(3.9)

Applying the technique described in the overview, the effect of the imputed price on the allocation of scarce resources results from choosing consumption, \( x_t \), to maximize imputed national income:

\[ y_t(x,t,V_t) - p_t v_t = u(x_t) \phi(V_t) - p_t v_t \]  

(3.10)

Noting that \( v_t = ax_t \), a static maximization w.r.t. \( x_t \) would yield the first order condition:

\[ u'(x_t) \phi(V_t) = p_t a \]  

(3.11)

A new variable is introduced that essentially normalizes the imputed price to remove the effect of the impact index. Hence, the imputed price becomes the imputed price of atmospheric carbon dioxide per impact index:

\[ \lambda_t = \frac{p_t}{\phi(V_t)} \]  

(3.12)

Now the approximate imputed price per impact index (3.8) is written so as to normalize income (recall \( y(x,V) = u(x) \phi(V) \)) in order to introduce another variable \( \theta_t \), (explained later):

\[ \frac{p_t}{y_t} = \frac{p_t}{u(x_t) \phi(V_t)} = \frac{\lambda_t}{u(x_t)} = \frac{1}{\delta + \mu} \left[ -\frac{\phi'(V_t)}{\phi(V_t)} \right] \]  

(3.13)

Imputed national income per impact index can now also be rewritten as:

\[ \frac{y_t}{\phi(V_t)} - \frac{p_t v_t}{\phi(V_t)} = u(x_t) - \lambda_t v_t \]  

(3.14)

---

1 Imputed real national income is the maximand used in the 1993 paper, which takes into account the CO₂ absorbed by the ocean: \( N y_t = p_t (v_t - \mu V_t) \). The first order condition used to determine \( x_t \), which is the vector of production of consumption goods, is identical to the case analyzed here.
which, when maximized in this form, gives the first order condition:

$$u'(x_c) = \lambda \alpha$$

(3.15)

Thus, the imputed price per impact index is the marginal utility of consumption, and national income per impact index is seen as the utility derived from consumption, less the amount paid for generating CO$_2$, and the level of CO$_2$ is in turn related to consumption.

Comparative statics are now employed to examine the sensitivity of the solution to various exogenous variables. An increase in the imputed price $\lambda_i$ implies an increase in the marginal utility of consumption, which in turn implies a decrease in the level of consumption, and therefore utility, given the properties of the utility function mentioned above. The decrease in consumption is also associated with a decrease in the rate of anthropogenic increase in the atmospheric level of carbon dioxide, since $v_i = \alpha x_i$. Then, since the utility of consumption decreases when the imputed price per impact index increases, a new term, defined as $\theta_i = \lambda_i / u(x_c)$, increases even more, giving rise to the positive first and second derivatives in the relationship depicted as the BB curve in quadrant II of **Figure 5** (page 57).

Uzawa uses $\theta_i$ in the multi-country analysis to normalize the imputed price $p_i$ per level of national income $v_i$ (per capita, evaluated at constant prices). At this point, the specific form for the impact index can be introduced as:

$$\phi(V) = (\hat{V} - V)^\beta$$

(3.16)
where \( \beta \in (0, 1) \) is an intensity parameter, and \( \hat{V} \) is a threshold for CO\(_2\). or in Uzawa's words

> a certain critical level of the atmospheric concentration of carbon dioxide, beyond which drastic changes in environmental conditions brought about by an increase in atmospheric temperature are feared to exert serious, irrevocable damage on human life on the earth.\(^4\) \(^5\) \(^6\)

One may see that \( V'_i = \hat{V} \) implies utility is zero, so the costs of CO\(_2\) are so high that they completely offset the utility obtained from consumption of market goods. This implication is also seen as \( V'_i \to \hat{V} \) in Figure 5.

Using (3.16), the last term in (3.13) can be converted by the equality.

\[
\left[ -\frac{\phi'(V'_i)}{\phi(V'_i)} \right] = \frac{\beta}{\hat{V} - V'_i}
\]

which, within a constant of \( 1/(\delta + \mu) \), is described by the AA curve of quadrant I in Figure 5, and has the properties of positive first and second derivatives. Also from Figure 5, one may see that for a given level \( V'_i \) of CO\(_2\), the imputed price per impact index \( \lambda_i \) is determined. Then, when \( \lambda_i \) is transformed to quadrant IV, the CC curve is determined. The CC curve depicts the relationship between CO\(_2\) and the imputed price \( \lambda'_i \): increasing CO\(_2\) implies an increasing imputed price.

---

\(^4\) See Uzawa (1991), p 298
\(^5\) Nordhaus (1982, p. 243) states a caveat in his own model to an identical simplifying assumption of a saturating CO\(_2\) concentration where "the cost becomes catastrophic" by parenthetically remarking, "it should be emphasized that this is an unrealistic assumption".
\(^6\) One may apply the citation above and see a mathematical pitfall if \( 0 < \beta < 1 \) and \( i' > i'' \).
Recalling $\theta_i = \frac{\lambda_i}{u(x_i)}$, the following relationship results between $\theta_i$ and $V_i$:

$$\theta_i = \frac{\lambda_i}{u(x_i)} = \frac{p_i}{u(x_i)\phi(V_i)} = \frac{p_i}{y_i} = \frac{1}{\delta + \mu} \left( \frac{\beta}{\hat{V} - V_i} \right)$$  \hspace{1cm} (3.18)

In order to obtain a per capita figure for the carbon tax calculation, Uzawa modifies (3.18) with a factor $N$ to reflect world population:

$$\theta_i = \frac{1}{\delta + \mu} \left( \frac{\beta}{\hat{V} - V_i} \right) N$$  \hspace{1cm} (3.19)
To arrive at an assessment for CO$_2$ emissions per country per capita per tonne of carbon, the following parameters are used:

\[
\begin{align*}
\mu &= 0.04 \\
\delta &= 0.05 \\
\beta &= 0.1 \\
\dot{N} &= 1200 \text{ GtC} \\
N &= 720 \text{ GtC} \\
N &= 5.2 \text{ billion people}
\end{align*}
\]

Thus, \( \theta_i = \frac{1}{0.05 + 0.04} \left( \frac{0.1}{1200 - 720} \right)^{5.2} \approx 0.012 \approx 0.01 \frac{\$}{\text{tonne}} \frac{\text{tonne}}{\text{person}} \). This measure is multiplied by the per capital national income to determine a $/tonne carbon tax. The carbon tax is then multiplied by the net annual increase of atmospheric carbon per capita to arrive at a per capita assessment. For the U.S., per capita income was assumed near $15000. The imputed price (carbon tax) is then $150 per tonne. Next, since the U.S. contributes 4.0 tonnes equivalent carbon (for greenhouse gasses: carbon dioxide, methane, and chloro-fluorocarbons) per capita annually to the atmosphere, an assessment to the U.S. would be $600 per capita. Estimating the U.S. population near 270 million people, the total cost for its contribution to atmospheric carbon would be $162 billion. This figure is roughly half of the U.S. defense budget. The results for other countries are tabulated in Uzawa (1991, 1993)

To complete the analysis of the first case in Uzawa (1991), a two-dimensional phase diagram may be constructed in co-state-state space or control-state space. The key to this procedure is to determine the level of economic activities \( x_i \), when the imputed price per
impact index $\lambda_i$ is known, independently of the level of carbon dioxide, $V_i$. However, a relationship between $\lambda_i$ and $V_i$ is necessary before $x_i$ can be determined. The value for $\theta_i$ is equal to $\lambda_i / u(x_i)$, which is $u'(x_i) / u(x_i)$. Since $u'(x) / u(x)$ is monotonic, it can be inverted to determine the level of economic activities $x_i$, given $\theta_i$ and $a$. The value $\lambda_i$ can then be found when the specific form of the expression $u'(x_i) = \lambda_i a$ is known and the value for $p_i$ can be found from $\lambda_i = p_i / \phi(V_i)$. Thus, a relationship can also be developed between $x_i$ and $V_i$ by substituting for $\lambda$ in (3.18) and rearranging:

$$\frac{u(x_i)}{u'(x_i)} = \frac{\delta + \mu}{a \beta} (\dot{V} - V)$$

(3.20)

Qualitatively, since the LHS of (3.20) is positive and increasing in its argument, the slope of the equilibrium relationship between $x_i$ and $V_i$ is negative. The system also contains a dynamic constraint (3.1) which describes the evolution of CO$_2$ in the atmosphere but its stationary relationship between $x_i$ and $V_i$ is positive:

$$x_i = \frac{\mu}{a} V_i$$

(3.21)

Solving the two equations in the two unknowns $x_i$ and $V_i$ will yield the stationary values for the optimal steady-state consumption rate and CO$_2$ stock, from which all other variables can be solved. The two equations (3.21)-(3.22) are graphed as isocones in the phase system as Figure 6 (page 60). Trajectory motions are not depicted, but the motion in $V_i$ using (3.1) is described.
The motion in $x_i$ is more complicated and can be described by the first-order non-linear differential equation, obtained by differentiating (3.11) with respect to time and substituting that result into (3.5):

$$\frac{u''(x_i)}{u'(x_i)} \dot{x} + \left[ \frac{\phi'(V_i)}{\phi(V_i)} \right] \frac{u(x_i)}{u'(x_i)} + x_i = (\delta + \mu) + \frac{\phi'(V_i)}{\phi(V_i)} V_i \quad (3.22)$$

Phase Diagram of Emission Rate and CO$_2$ Stock

Figure 6

The equilibrium is dynamically stable for perturbations in $V_i$ if the optimal trajectory for the control in $x_i$ is followed. The stability uses the turnpike property described above.

In a neighborhood of the steady state, any perturbation can be restored to equilibrium by directing the system, using the control variable, in the direction of the steady state. The motion toward equilibrium in this neighborhood will be monotonic.
b. Uzawa Model Analysis with Forests

Uzawa then extends the analysis by including forests, represented by a state variable $R$, as an additional carbon sink into the CO$_2$ equation of motion:

$$\dot{V}_r = v_r - \mu V_r - \gamma R$$  \hspace{1cm} (3.23)

Thus, the acreage of forests augments the absorption of CO$_2$ by the ocean. Unlike the model we shall examine, natural growth of the biomass of forests is not included in the process. Neither does the forest stock feed back into the production function ultimately to generate utility. The only function of the stock of forests is to sequester CO$_2$, and its stock can only be increased by net forestation activities, which use constrained resources. The benefit of CO$_2$ fertilization is also not included as was suggested by Rosenberg (1991) in the commentary to the Uzawa (1991) paper.

The biomass change function $\dot{R}_r$ (measured in total acreage of forest in a region), evolves according the difference between control variables $r$, the annual rate of increase of acreage of forest due to the activities of afforestation (which could also include reforestation), and $s$, the acreage of forest lost due to various economic activities during the year $t$:

$$\dot{R}_r = r_r - s_r$$  \hspace{1cm} (3.24)

where $r$ and $s$ are the dot products of vectors$^7$ of forestation and deforestation

$$r_r = mz_r$$  \hspace{1cm} (3.25)

$^7$ Again, for convenience in analysis, we will use scalars rather than vectors.
\[ s_i = bx_i \quad (3.26) \]

and \( m = (m_1, \ldots, m_F) \) is the rate of increase in the forest acreage when afforestation activity \( f = 1, \ldots, F \) is operated at the unit level, and \( z = (z_1, \ldots, z_F) \) is the vector of afforestation activities for the year \( t \). Similarly, \( b = (b_1, \ldots, b_n) \) denotes a vector specifying the acreage of forest lost by economic activities, and as before, \( x_i = (x_{i1}, \ldots, x_{in}) \) is a vector of consumption activities. The forestation activities have a resource constraint

\[ Bx_i + Mz_i \leq K \cdot x_i \geq 0 \quad (3.27) \]

where \( M \) stands for the matrix specifying the resource requirements for afforestation activities, and \( K = (K_1, \ldots, K_F) \) are the available quantities of scarce resources, which are in turn denoted by \( l = 1, \ldots, L \). Thus, two additional imputed prices are needed. One values the opportunity cost of not having more forest. The other values the opportunity cost of not having more resources for forest activities.

A new, current value Hamiltonian can now be expressed as:

\[ H(t) = u[x(t)]\phi[V(t)] + p(t)[\mu V(t) + \gamma R(t) - v(t)] + q(t)[mz(t) - bx(t)] \quad (3.28) \]

but incorporating the control constraint (3.27) requires the formulation of the Lagrangian:

\[ L(t) = u[x(t)]\phi[V(t)] + p(t)[\mu V(t) + \gamma R(t) - v(t)] + q(t)[mz(t) - bx(t)] + h(t)[K - mz(t) - bx(t)] \quad (3.29) \]
where \( p_t \) is determined as discussed in the previous section, and the precise formula for motion of the new co-state variable \( q_t \) can be interpreted as the change in the imputed price of forest according the Pontryagin Maximum Principle*: 

\[
\dot{q} = \delta q - \frac{\mathcal{L}}{\mathcal{R}} = \delta q - \gamma p 
\]

(3.30)

This change in \( q \) has to equal the discount rate times the current price (so \( R \) yields the "market return" on resources) minus the value of its CO\(_2\) reduction in terms of the imputed price \( p_t \). However, Uzawa again focuses on the steady state, where \( \dot{q} = 0 \). viz., 

\[
q_t = \frac{\gamma}{\delta} p_t 
\]

(3.31)

The new expression for imputed national income now includes the amount of benefit received from increasing the acreage of forest, i.e., the value of CO\(_2\) reduced from the atmosphere netted against the value produced:

\[
y_t = p_t v_t + q_t (r_t - s_t) 
\]

(3.32)

The maximization of this income expression is subject to the inequality constraint on the resources available for economic (deforestation) and forestation activities. Thus, a static optimization could use a Lagrangian of the form:

\[
u(x_t) \phi(V_t) - p_t ax_t + q_t (mz_t - bx_t) + h_t (K - mz_t - bx_t) \]

(3.33)

where \( h_t \) is the Lagrange multiplier corresponding to the shadow (imputed) price of scarce resources used in these forestry activities. The first order necessary conditions are:

\[
u'(x_t) \phi(V_t) = ap_t + bq_t + bh_t 
\]

(3.34)

\[
m(q_t - h_t) = 0 \quad q_t \geq 0 \quad h_t \geq 0 
\]

(3.35)

* See Pontryagin (1962) and Hestenes (1966).
\[ h_i(K - mz_i - bx_i) = 0, \ h_i \geq 0, \ K \geq mz_i - bx, \]  

(3.36)

Observe that since \( m > 0 \), \( q_i = h_i > 0 \), which means that if reforestation activity \( i \) is occurring despite a positive resource cost, forest resources must have a positive shadow value \( q \). Also, the resource constraint (3.27) must be binding, i.e., \( K = mz_i + bx_i \). Any feasible path without this condition could not be optimal, because an improvement could be made in the Hamiltonian by increasing \( z_i \) at zero cost. Another indication that the constraint must bind is the linearity of the Hamiltonian in \( z_i \). Hence, the Hamiltonian is maximized when \( z_i \) takes its maximum value.

The last step before describing the system in a two dimensional state phase space is to find an expression which will allow the determination of \( q_i \) as a function of \( V_i \).

Introduce \( \eta_i = q_i / y_i \) as the ratio of imputed price of forests per unit of national per capita income. Substituting (3.31) into the RHS of (3.18) gives the desired result:

\[ \eta_i = \frac{\gamma}{\delta(\delta + \mu)} \frac{\beta}{V_i} \]  

(3.37)

The term \( \eta_i \) is the imputed price of forest per impact index per income level, which, described more completely, can be written:

\[ \eta_i = \frac{q_i}{y_i} = \frac{q_i}{u(x_i)\phi(V_i)} = \frac{\gamma}{\delta(\delta + \mu)} \frac{\beta}{V_i - V_i} \]  

(3.38)

At this point, one may determine the imputed price of reforestation \( q \) by simply multiplying the imputed price of atmospheric carbon times the factor \( \gamma / \delta \). Similarly,
\( \eta_i = \theta, \gamma / \delta \). Recall that \( \gamma \) is a carbon "absorption" factor for forests, and \( \delta \) is the discount rate ( \( \delta = 0.05 \)). Uzawa distinguishes between temperate forests, with \( \gamma = 5 \) GtC/ha\(^9\)/yr, and tropical forests, with \( \gamma = 15 \) GtC/ha/yr. The per capita imputed prices per hectare are thus 100 and 300 times the per capita imputed prices per tonne of atmospheric carbon for temperate and tropical forests, respectively. For the U.S., whose net annual reforestation is 1.6 million hectares, the assessment would be:

\[
$150/\text{tonne} \times 100 \times 1.6 \text{ million hectares} = $24 \text{ billion}
\]

The sequence of calculations to determine \( q_i \) from \( V_i \) is similar to the above method for determining \( p_i \) from \( V_i \). For a given level \( V_i \) of CO\(_2\), the imputed price per impact index per income level \( \eta_i \) is determined from the RHS of (3.38). However, the level of economic activities \( x \), cannot be determined directly from inverting the LHS in terms of \( x \), since \( u'(x_i) \) does not simply equal \( \lambda_i \). The marginal utility must now equate to the sum of both shadow prices. The values for \( x \), and \( z \), are determined from the binding scarce resource constraint (3.27).

The general nature of the problem presented by Uzawa precludes specific and explicit solutions. However, since the system is autonomous, phase diagram analysis is possible if we reduce the number of variables to two. Thus, the state-state space, rather than the more commonly used phase diagram of price-state space, was chosen to exemplify the

\footnote{The hectare (ha) is an international unit of area measurement, which converts to 2.471 acres.}
dynamic behavior in the neighborhood of the steady state. The equilibrium values can be found using the following method. At steady state:

\[ \dot{p}_i = 0 \Rightarrow p_i = \frac{1}{\delta + \mu} [-\phi'(V_i)u(x_i)] \]  

(3.8)

\[ \dot{q}_i = 0 \Rightarrow q_i = \frac{\gamma}{\delta} p_i \]  

(3.31)

\[ \dot{V}_i = 0 \Rightarrow V_i = R_i(ax_i - \mu V_i) \]  

(3.39)

\[ \dot{R}_i = 0 \Rightarrow mz_i = bx_i \]  

(3.40)

The AA line in Figure 7 (page 67) represents the steady state stock of CO₂, which can be determined by substituting (3.8) into the first order condition and using \( q = h = \frac{\gamma}{\delta} p \):

\[ u'(x) \phi(V) = p \left[ \frac{a \delta + 2b \gamma}{\delta} \right] \]

\[ = \frac{1}{\delta(\delta + \mu)} [-\phi'(V)u(x)] \left[ a \delta + 2b \gamma \right] \]

(3.41)

Re-arranging results in

\[ \frac{u(x_i)}{u'(x_i)} = \frac{\delta(\delta + \mu)}{(a \delta + 2b \gamma) \phi'(V_i)} \]

(3.42)

and using the functional form \( \phi(V_i) = (\dot{V} - V_i)^\beta \) prescribed by Uzawa, the expression becomes

\[ \frac{u(x_i)}{u'(x_i)} = \frac{\delta(\delta + \mu)}{\beta(a \delta + 2b \gamma)} (\dot{V} - V_i) \]

(3.43)
CO$_2$ and Forest Stock Converge to Steady State

![Graph showing equilibrium values and steady-state conditions.]

**Figure 7**

The equilibrium value for $x_i$ can be determined from the constraint (3.27) and the steady state condition for forests (3.40)

$$z_i = \frac{K}{2m} \quad \text{and} \quad x_i = \frac{K}{2b} \quad (3.44)$$

Thus, the value for $V_i$, the equilibrium stock of CO$_2$, is determined when $u(x_i)$ is known.

The BB line in **Figure 7** is the graph of the steady-state pairs $(R_i, V'_i)$ for which the steady-state condition (3.39) holds. The negative slope can be explained as was done in the simpler model. An increase in CO$_2$ stock results in a decrease in consumption of $x_i$ and the emission rate then is less than the amount absorbed by the ocean, i.e., $v_i < \mu V'_i$, which implies, by (3.39), that the stock of forest is decreased below its previous value.
The motions in the neighborhood of the steady state brought about by perturbations such as these can now be explained by the following phase space analysis. Using (3.23), it can be seen that points NE of the BB line indicate
\[ \frac{1}{\gamma} (ax_r - \mu V_r) > 0 \Rightarrow \dot{V}_r < 0 \]
and conversely, $\dot{V}_r > 0$ for points SW of the BB line. Points East of the AA line can be described by using (3.41), which can be rewritten for convenience (by normalizing the impact index) as
\[ u'(x_r) = \lambda_i \left( \frac{a\delta + 2b\gamma}{\delta} \right) \]
(3.45)
As previously discussed, an increase in the stock of CO$_2$ decreases utility, and increases the marginal utility of both $V$ and $x$. Thus, since the constraint (3.27) is binding, a decrease in consumption $x_r$ must be offset by an increase in forestation $z_r$, and thus $\dot{R} > 0$. To see the effect mathematically, differentiate (3.45) w.r.t. $x_r$ to get
\[ \frac{dx_r}{d\lambda_i} = \left( \frac{a\delta + 2b\gamma}{\delta} \right) \frac{1}{u''(x_r)} < 0 \]
\[ \frac{d(r_r - s_r)}{d\lambda_i} = \frac{d(mx_r - bx_r)}{d\lambda_i} = -2b \frac{dx_r}{d\lambda_i}, > 0 \]
The converse holds for points West of the AA line. The foregoing analysis confirms the descriptions of motions presented by Uzawa (1991, 1993). The various trajectories depicted in Figure 7 are for different initial conditions of the state variables $(R_0, V_0)$. Initial points North of BB and East of AA must move in a NW direction. Trajectories starting in this region will not reach equilibrium if their slope remains negative near
steady state. and must cross the AA line or converge to equilibrium with zero slope
\( (d\dot{R}_i/d\dot{V}_i = 0 \text{ at } \dot{V}_i = 0) \). Thus, where the initial forest stock is in excess of the long run steady state, the forest will sequester CO\(_2\) below its equilibrium level. Only optimal paths within a critical range of levels of forests NE of BB will monotonically converge to the stationary state, since trajectories crossing the BB line will have infinite slope
\( (d\dot{R}_i/d\dot{V}_i \to \infty \text{ at } \dot{R}_i = 0) \). All other trajectories will have at least one sign change in the system described by (3.23) and (3.24).

The equilibrium in the figure is an unstable focus, or more commonly known as a spiral. The roots of the characteristic equation \( \sigma^2 - \text{tr } B\sigma + |B| = 0 \) must be complex with negative real parts, which can only occur if the determinant is positive. To show this qualitative property, a Jacobian matrix of the signs of the derivatives is formed from the state equations (3.23) and (3.24), and the associated properties

\[
\dot{V}_i = v_i - \mu \dot{V}_i - \gamma R_i \tag{3.23}
\]

\[
\dot{R}_i = r_i - s_i \tag{3.24}
\]

which has been shown above to be a system in state variables \((R_i, V_i)\)

\[
\ddot{V}_i = [ax_i(V_i) - \mu \dot{V}_i] - \gamma R_i
\]

\[
\dot{R}_i = [mz_i(V_i) - hx_i(V_i)] - 0 R_i
\]

The qualitative Jacobian can be constructed based only on signs of the derivatives, which were described in the text analyzed above:
\[
B = \begin{bmatrix}
\frac{dV}{dV} & \frac{dV}{dR} \\
\frac{dV}{dR} & \frac{dR}{dR}
\end{bmatrix} = \begin{bmatrix}
\left( \frac{dx}{dV} - \mu \right) & -\gamma \\
0 & +0
\end{bmatrix} \quad \text{or} \quad \begin{bmatrix}
+0 & -
\end{bmatrix} > 0
\]

With a positive determinant, \(|B| > 0\), and a negative trace, \(\text{tr} \ B < 0\), the roots are calculated with the quadratic solution

\[
\sigma = \text{tr} \ B \pm \frac{\sqrt{(\text{tr} \ B)^2 - 4|B|}}{2}
\]

The roots will be complex with negative real part if \(0 < (\text{tr} B)^2 < 4|B|\), leading to the spiral type curves depicted in Figure 7.
IV. Our Model Development

A. General Model

The general model is similar to that used in Uzawa (1991), but differs in several ways. The first major difference is the method. Uzawa used the steady-state values of carbon dioxide and forest acreage as a proxy for the theoretical exact imputed price path. Then, by showing that the steady states were stable (e.g., not saddle points), he could apply turnpike theory\(^1\) in justifying these prices. It is our opinion that the prices were not necessarily intended to substitute for the optimal price path, but simply as a "heuristic" value on which to base a carbon tax or (de)forestation (tax)/subsidy. The second major difference is the application. Using GNP per capita as a proxy for utility, Uzawa introduced money into the model to arrive at a monetary measurement for a carbon tax/forestation subsidy. The tax was based on the shadow price of carbon in the atmosphere, applied across all countries. As money is not introduced in our model, prices developed in our model are scaled in utility units. The third major difference was that since no specific utility function was used. Uzawa made no attempt to solve the problem explicitly or numerically and thus did not determine the time paths for any of the variables, nor steady-state values for the state variables.

Other differences were structural. Uzawa used a non-linear form of the "disutility" function in the maximand similar to (4.1) as \( \phi(V) = (\hat{V} - V)^{\beta} \) compared to our linear

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\(^1\) See Section IIIB for more information. Per Uzawa (1991, p 301): "In fact, we can easily show that the heuristic solution for the imputed price of atmospheric carbon dioxide gives us a close approximation for the true price, and, for any practical purposes, the heuristic solution for the imputed price may be used to achieve a dynamically optimum allocation of scarce resources."
form $-\tau(V - \hat{V})$. More discussion on this difference is given in the next section. The other difference is in the biomass growth function. We use a logistic function to represent biomass growth, which, if undisturbed, is constrained to a maximum. Uzawa uses a linear function that has a general resource constraint.

The single country model posed for this paper is the infinite horizon form

$$\max_{h(t)} W = \int_0^a \{u[h(t)] - \tau[V(t) - \hat{V}]\} e^{-\delta t} dt$$  \hspace{1cm} (4.1)

subject to

$$\dot{R}(t) = f[V(t), R(t)] - h(t)$$  \hspace{1cm} (4.2)

$$\dot{V}(t) = v(t) - \Omega[V(t), R(t)]$$  \hspace{1cm} (4.3)

$v(t)$ represents goods production, which also produces a weight $\hat{V}(t)$ in GtC (gigatonnes carbon) of “excess” CO$_2$. The term $\hat{V}$ refers to the pre-industrial estimate of CO$_2$ concentration of 280 ppm. The “excess” CO$_2$ is absorbed via mechanism $\Omega[V(t), R(t)]$ of (4.3) into the three major carbon sinks, which, for simplicity we have modeled as two: the ocean and the atmosphere $V(t)$, and the terrestrial biomass $R(t)$. Biomass change is regulated by mechanism (4.2), which contains a growth function $f[V(t), R(t)]$, and a consumption flow $h(t)$. The flows $h(t)$ and $v(t)$ may share components, but $v(t)$ will include fossil fuels. To ensure convergence of the integral, and the existence of a steady state (Seierstad. 1987), the discount rate, or the social rate of time preference $\delta$, must be strictly positive, and $u(t)$ bounded from above. The parameter $\delta$ could be based on a different valuation.
i.e., an alternative investment opportunity cost. Discussion of the numerical value for the discount rate and other parameters is deferred until Section VA.

In the objective function (4.1), maximization with respect to consumption will occur over a quasi-concave utility function $u(\cdot)$ that must be at least piece-wise continuous. However, for convenience we strengthen this requirement to be continuous and differentiable everywhere, with non-negative and non-increasing marginal utility of consumption. i.e., $u' \geq 0$, $u'' \leq 0$. Since we shall assume $u'(x) \to +\infty$ as $x \to 0$, biomass consumption $h(t)$ must always be positive. Specific functional forms for utility and CO₂ mechanisms will be chosen for their applicability and convenience when numerical solutions are sought in the next section (Section IVB).

The basic method of analysis parallels the analysis of Uzawa presented in the previous chapter. The marginal utility of the control variable is then related to the marginal valuation (shadow price) of the state variable. The first order condition is substituted into the dynamic equations implied by the Maximum Principle conditions to yield a system of simultaneous dynamic equations in control-state space and shadow price-state space. The results can be graphed, as phase diagrams, enabling a visualization of the trajectories, or as relationships between key variables.
B. Specific Model

1. Development

A more explicit form of the optimal control problem is proposed as follows with the assignment of explicit forms and discussion of some qualitative properties:

\[ u[x_i(t), y_i(t)] = \alpha_1 \ln x_i + \alpha_2 \ln y_i + \ln(\phi x_2 + y_2) \quad (4.4) \]

\[ i \in \{1, 2\} \]

\[ \dot{y}_1 = -y_1 - y_2 \quad (4.5) \]

\[ \dot{R} = gR - \frac{g}{A} R^2 + y_1 R(V - \tilde{V}) - x_1 - x_2 \quad (4.6) \]

\[ \dot{V} = -y_2 R(V - \tilde{V}) - \mu(V - \tilde{V}) + ax_1 + x_2 + by_1 + y_2 \quad (4.7) \]

We shall assume a non-negativity constraint for biomass and point out that it is bound from above implicitly through its growth function contained in equation (4.6).

\[ R(t) \geq 0 \text{ and } \lim_{t \to \infty} R(t) = A \]

The control variable \( h(t) \) is bounded below:

\[ h(t) \in [0, \infty) \]

while the time integral of \( h \) is bounded by the stock of biomass\(^2\):

\[ \int_0^\sigma h(t)dt \leq R_0 + \int_0^\sigma R(t)dt \]

Comparing the specific model to that of Uzawa (with forests, see Section III.B2), one difference is noted in the utility function. He assumed utility depends on a vector

\(^2\) It is probably more reasonable in this context to think of \( R \) as general biomass rather than trees alone.
of activity levels for the entire economy, a fraction of which result in carbon dioxide emissions. Our utility function groups the activities into specific categories of entities that enter the carbon cycle. The control variables $x_i$ are derived from surface biomass (plants, etc) and $y_i$ are derived from hydrocarbon deposits (fossil fuels):

$x_1$ is derived directly from biomass for human energy. It could be thought of as food and clothing.

$x_2$ is derived directly from biomass for energy production. The most obvious form is charcoal or fuel wood, but these are not competitive in developed countries with fossil fuel as a major energy source and are likely to remain so. More realistically, we are thinking of methanol or other as yet to be developed – carbon-based, but naturally derived, fuel sources.

$y_1$ is a non-energy derivative from fossil fuels used to produce material such as plastics and other petrochemicals.

$y_2$ is a derivative from fossil fuels for energy production. It is the oil, coal, and gas extracted from the earth that is used to produce energy.

From the structure of the utility function (4.4), the control variables $x_2$ and $y_2$ must be perfect substitutes. From the perspective of energy production the source of the energy does not matter – only the amount produced. The relative energy value of the sources is determined by the coefficient $\phi$. Since the alternative energy sources are perfect substitutes, only one of them will be non-zero: energy is consumed constantly but will be obtained from the cheapest natural resource endowment.

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1 It is perhaps more likely that future non-fossil fuel sources will be based on nuclear or solar technologies (particularly in so far as electricity generation is concerned) and may not involve carbon at all. Allowing for this additional category of energy sources would complicate the model, and take the focus away from the carbon cycle. Nevertheless, it is an issue that may be worth investigating in future models.
By contrast with energy production, the control variables \( x_i \) and \( y_i \) are not perfect substitutes. People do not eat petrochemicals – although clothing and other insulating fibers derived from oil and coal do compete with natural products such as cotton or wool. Energy is also a significant input to agriculture. The relative importance of \( x_i \) and \( y_i \) in the utility function is indicated by their respective coefficients \( \alpha_i \) and \( \alpha_2 \). Note that both non-energy consumption controls must be positive, since \( \ln x_i \) or \( \ln y_i \to -\infty \) as \( x_i \to 0 \) or \( y_i \to 0 \).

The state variable equations (4.5)-(4.7) describe the evolution of the carbon exchange cycle between the atmosphere and biosphere. They differ from those used by Uzawa in using a growth function, and by including carbon “fertilization” factors, to describe the evolution of biomass (4.6). The carbon fertilization factor also influences the evolution of the \( \text{CO}_2 \) content of the atmosphere (4.7). Consumption of fossil fuels (4.5) for any use depletes the stock of hydrocarbons and adds to atmospheric \( \text{CO}_2 \) stock via (4.7). The coefficient \( b \) allows the different uses of fossil fuels to have different impacts on \( \text{CO}_2 \) emissions. Consumption of biomass depletes recent surface biota via mechanism (4.6), and this depletion also adds to the stock of atmospheric \( \text{CO}_2 \). However, \( \text{CO}_2 \) is also sequestered via photosynthesis, which is represented by the logistic function, while \( \text{CO}_2 \) “fertilization” is characterized by the term \( +\gamma_1 R(V - \dot{V}) \) in (4.6), with its counterpart term \( -\gamma_2 R(V - \dot{V}) \) in (4.7). Finally, ocean absorption of atmospheric \( \text{CO}_2 \) is proposed as a steadily \( \text{CO}_2 \)-depleting
function, \(-\mu(V - \hat{V})\) in (4.7). Further development and explanation of these concepts are set forth in the following section.

Since \(\ln x\) is concave and monotonically increasing, and the sum of concave functions is also concave, all output is consumed in a non-satiating manner. Also, for purposes of exposition, let \(c(t) = \phi(x_2(t) + y_2(t))\). The terms will be interchanged where convenient. With these specifications, the problem now can be stated as (the time arguments are often omitted for brevity):

\[
\max_{t, x_0} \int_0^{-x_0} [\alpha_1 \ln x_1 + \alpha_2 \ln y_1 + \ln c - \tau(V - \hat{V})] e^{-\delta t} dt
\]

(4.8)

where \(x_1, y_1, c > 0\)

\[
\hat{F} = -y_1 - y_2
\]

(4.5)

where \(F(0) = F_0, F(t) \geq 0\)

\[
\hat{R} = gR - \frac{g}{A} R^2 + \gamma_1 R(V - \hat{V}) - x_1 - x_2
\]

(4.6)

where \(R(0) = R_0, R(t) \geq 0\)

\[
\hat{V} = -\gamma_2 R(V - \hat{V}) - \mu(V - \hat{V}) + ax_1 + x_2 + by_1 + y_2
\]

(4.7)

where \(V(0) = V_0, V(t) \geq 0\)

with \(\delta > 0\) the discount rate or social rate of time preference. \(A > 0\) is biomass capacity, usually referred to as the environmental carrying capacity or the saturation level. \(g > 0\) the net proportional growth rate for the biomass. The parameter \(g\) is more elaborately described as the birth rate less mortality rate, which is usually
intended to stand for the "intrinsic" growth rate, i.e., the rate which would occur if no environmental constraints were present. This rate is usually measured as a percentage or similar fractional measure. The physical unit of measurement can be weight, say, in tons or grams of carbon. $\gamma_1$ and $\gamma_2$ are parameters which specify the increase in the rate of carbon dioxide conversion to biomass (and symmetrical reduction of $V$) when the concentration of carbon dioxide is increased slightly, i.e., the CO$_2$ "fertilization" parameter. $\tau$ measures "disutility" resulting from CO$_2$ in the atmosphere, reflecting the possibility of rises in the ocean and other negative effects of atmospheric warming. $\alpha_1, \alpha_2, a, b, \phi$ are parameters which will be specified to reflect the existing world fuel mix and associated substitutions.

Resource constraints can also be specified. The stock of CO$_2$, $V$, cannot be negative: $V(t) \geq 0$. Fossil fuels, $F$, are a non-renewable resource. Hence the accumulated extraction is limited to the initial stock, $F_0$, by

$$\int_0^\kappa [y_1(t) + y_2(t)] dt \leq F_0$$

where $\kappa$ is the (possibly infinite) date of exhaustion of fossil fuel deposits\(^1\). The extraction or harvesting of biomass, $R$, a renewable resource, is limited by its initial stock, $R_0$, plus its accumulated growth:

$$\int_0^T [x_1(t) + x_2(t)] dt \leq R_0 + \int_0^T R(t) dt \quad \forall T \geq 0$$

\(^1\) Since $\ln y_1 \to -\infty$ as $y_1 \to 0$, the resource will only be depleted asymptotically.
2. Solution Development

The present value Hamiltonian, $H$, for the problem is

$$H = \left[ \alpha_1 \ln x_1 + \alpha_2 \ln y_1 + \ln(\phi x_2 + y_2) - \tau(V - \tilde{V}) \right] e^{-\alpha t}$$

$$+ m_1 [-\gamma_2 R (V - \tilde{V}) - \mu(V - \tilde{V}) + ax_1 + x_2 + by_1 + y_2]$$

$$+ m_2 [gR - \frac{G}{A} R^2 + \gamma_1 R(V - \tilde{V}) - x_1 - x_2]$$

$$+ m_3 (-y_1 - y_2)$$

(4.9)

The autonomous, current value Hamiltonian $H$, or instantaneous utility, will then be

$$H = \left[ \alpha_1 \ln x_1 + \alpha_2 \ln y_1 + \ln(\phi x_2 + y_2) - \tau(V - \tilde{V}) \right]$$

$$+ p [-\gamma_2 R (V - \tilde{V}) - \mu(V - \tilde{V}) + ax_1 + x_2 + by_1 + y_2]$$

$$+ q [gR - \frac{G}{A} R^2 + \gamma_1 R(V - \tilde{V}) - x_1 - x_2]$$

$$+ r (-y_1 - y_2)$$

(4.10)

with the substitutions $p = m_1 e^{-\alpha t}$, $q = m_2 e^{-\alpha t}$, and $r = m_3 e^{-\alpha t}$ for the current value co-state variables. A Lagrangian is formed using the control variable constraints:

$$L = \left[ \alpha_1 \ln x_1 + \alpha_2 \ln y_1 + \ln(\phi x_2 + y_2) - \tau(V - \tilde{V}) \right]$$

$$+ p [-\gamma_2 R (V - \tilde{V}) - \mu(V - \tilde{V}) + ax_1 + x_2 + by_1 + y_2]$$

$$+ q [gR - \frac{G}{A} R^2 + \gamma_1 R(V - \tilde{V}) - x_1 - x_2]$$

$$+ r (-y_1 - y_2) + v x_1 + w y_2$$

(4.11)

The maximum principle assumes that the optimal controls $x_1(t)$ and $y_1(t)$ maximize the current increase in assets, that is, the cumulative "dividend" from consumption (the first term on the RHS of (4.10)) and the increase in the values of the capital and resource stocks. The first order and associated Kuhn-Tucker conditions are then

$$\frac{\partial L}{\partial x_1} = \frac{\alpha_1}{x_1} - q + ap = 0 \quad \text{or} \quad x_1 = \frac{\alpha_1}{q - ap}$$

(4.12)
\begin{align}
\frac{\partial L}{\partial x_2} &= \frac{\phi}{\phi x_2 + y_2} - q + p + v = 0 \quad \text{(4.13)}
\end{align}

\begin{align}
x_2 v &= 0, \quad x_2 \geq 0, \quad v \geq 0
\end{align}

\begin{align}
\frac{\partial L}{\partial y_1} &= \frac{\alpha_2}{y_1} - r + b p = 0 \quad \text{or} \quad y_1 = \frac{\alpha_2}{r - b p} \quad \text{(4.14)}
\end{align}

\begin{align}
\frac{\partial L}{\partial y_2} &= \frac{1}{\phi x_2 + y_2} - r + p + w = 0 \quad \text{(4.15)}
\end{align}

\begin{align}
y_2 w &= 0, \quad y_2 \geq 0, \quad w \geq 0
\end{align}

\begin{align}
\frac{\partial L}{\partial v} &= x_2 \geq 0 \quad \text{and} \quad \nu x_2 = 0 \quad \text{(4.16)}
\end{align}

\begin{align}
\frac{\partial L}{\partial w} &= y_2 \geq 0 \quad \text{and} \quad \nu y_2 = 0 \quad \text{(4.17)}
\end{align}

The maximum principle indicates time paths of co-state variables as

\begin{align}
\dot{p} &= \dot{\phi} - \frac{\partial L}{\partial v} = p[\delta + \gamma_2 R + \mu] - q[y_1 R] + r \quad \text{(4.18)}
\end{align}

\begin{align}
\dot{q} &= \dot{\phi} - \frac{\partial L}{\partial R} = p[y_1 (V - \dot{V})] + q\left[\delta - g\left(1 - \frac{2R}{A}\right) - y_1 (V - \dot{V})\right] \quad \text{(4.19)}
\end{align}

\begin{align}
\dot{r} &= \delta r - \frac{\partial L}{\partial F} = \dot{\delta} r \quad \text{(4.20)}
\end{align}

along with the recovery of the dynamic state transition equations

\begin{align}
\frac{\partial L}{\partial p} = \dot{V} &= -\gamma_2 R (V - \dot{V}) - \mu (V - \dot{V}) + \alpha x_1 + x_2 + b y_1 + y_2 \quad \text{(4.7)}
\end{align}

\begin{align}
\frac{\partial L}{\partial q} = \dot{R} &= g R - \frac{g}{A} R^2 + \gamma_1 R (V - \dot{V}) - x_1 - x_2 \quad \text{(4.6)}
\end{align}

\begin{align}
\frac{\partial L}{\partial r} = \dot{F} &= -y_1 - y_2 \quad \text{(4.5)}
\end{align}
and transversality conditions that dictate worthless present value of stocks as $t \to +\infty$

$$\lim_{t \to +\infty} m_t V = e^{-\alpha} p(V - \bar{V}) = 0 \quad (4.21)$$

$$\lim_{t \to +\infty} m_t R = e^{-\delta} qR = 0 \quad (4.22)$$

$$\lim_{t \to +\infty} m_t F = e^{-\alpha} rF = 0 \quad (4.23)$$

First order conditions (4.12)-(4.17) are the values (in terms of shadow prices) of the marginal contributions of consumption variables (controls) along the optimal path. Equations (4.18)-(4.20) describe classic arbitrage conditions, where total return, composed of the rate of capital gains (price appreciation) and yield (rate of biomass growth), must compete with alternative investments (or the social rate of time preference). The transversality conditions (4.21)-(4.23) insure vanishing present value (constant current value) shadow prices at steady state.

The additive separability in the model (4.8) structure, and a simplifying assumption on the contribution of non-energy uses of fossil fuels, induces a partitioning of the system to allow a steady state for biomass and CO$_2$, while fossil fuels are continuously depleted. The reasonable assumption is made that non-energy uses of fossil fuels (which typically require carbon to remain in the manufactured product) do not contribute to atmospheric CO$_2$; thus, the coefficient $b$ is assumed to be 0.

The stocks of resources and their associated shadow prices (along with other parameters) will determine which of the resources are used for energy purposes
initially. In the numeric analysis, as in practice, the initial price of fossil fuels is
estimated to be below that of obtaining energy from biomass, implying that fossil
fuels will be used for energy until their price rises due to scarcity (or the price of
energy derived from biomass declines\(^1\)). Equation (4.20) can be integrated directly to
show the price path for fossil fuels as

\[
r(t) = r_0 e^{i\phi}
\]

(4.20a)

Meanwhile, biomass stock grows until time \(t\), when its price decreases to the point
where a switch in energy fuel is made from fossil fuel to its perfect substitute –
biomass. Thus, fossil fuel use for energy vanishes and hence \(y_1(t > t) = 0\) after this
switching point.

To analyze the impact of this simplifying assumption, recall the Kuhn-Tucker
conditions used to develop (4.13):

\[
x_1 v = 0, \quad x_2 \geq 0, \quad v \geq 0
\]

Since \(\ln(\phi x_2 + y_2) \to -\infty\) as \(\phi x_2 + y_2 \to 0\), one may reason that if \(y_1(t > t) = 0\) after
the switching point, then \(x_1 > 0\); and \(x_2 > 0 \Rightarrow v = 0\). This reasoning leads to the
modified FOC:

\[
x_2 = \frac{1}{q - p}
\]

(4.13a)

The steady state of a system is typically characterized by unchanging time
derivatives. The variables \(\dot{R}, \dot{V}, \dot{p}, \dot{q}\) can achieve such a state, but fossil fuels are

\(^1\) The model could be extended to allow for technological progress in producing energy from
renewable resources.
not renewable and the structure of the problem (4.8) requires positive fossil fuel consumption for non-energy purposes, that is, \( \gamma_1(t) > 0 \ \forall t \). This path of consumption will exponentially deplete fossil fuel reserves in a manner analogous to the famous "cake-eating" problem\(^6\), where no steady state is attainable in an infinitely lived system.

With \( b = 0 \) (so carbon in petrochemical products remains locked up away from the atmosphere/biomass carbon cycle) and \( \gamma_2(t) = 0 \) (so fossil fuels are no longer being used as a source of energy), fossil fuels are effectively separated from the CO\(_2\)-biomass interaction. This separation enables a reduced system of applicable state and co-state variables to reach a steady state:

\[
\dot{R} = 0 \Rightarrow gR - \frac{g}{A} R^2 + \gamma_1 R (V - \dot{V}) = x_1 + x_2 \tag{4.24}
\]

\[
\dot{V} = 0 \Rightarrow \gamma_2 R (V - \dot{V}) + \mu (V - \dot{V}) = \alpha x_1 + x_2 \tag{4.25}
\]

\[
\dot{p} = 0 \Rightarrow p[\delta + \gamma_1 R + \mu] + q[-\gamma_1 R] = -\tau \tag{4.26}
\]

\[
\dot{q} = 0 \Rightarrow p[\gamma_2 (V - \dot{V})] + q\left[\delta - \gamma_1 (V - \dot{V}) - g\left(1 - \frac{2R}{A}\right)\right] = 0 \tag{4.27}
\]

Steady-state equations (4.24)-(4.27), and first order conditions (4.12)-(4.13) represent six equations in the six unknowns \( R, V, p, q, x_1, x_2 \). Thus, a steady state is feasible. The solution of this simultaneous system uses a Gaussian elimination method, and Cramer's rule. Thus, a square matrix is made of equations (4.26)-(4.27):

---

\(^6\) The cake-eating problem was applied to Hotelling's (1931) analysis for non-renewable resource depletion.
\[
\begin{bmatrix}
\delta + \gamma_2 R + \mu & -\gamma_1 R \\
\gamma_2 (V - \hat{V}) & \delta - \gamma_1 (V - \hat{V}) - g \left( 1 - \frac{2R}{A} \right)
\end{bmatrix}
\begin{bmatrix}
p \\ q
\end{bmatrix} = \begin{bmatrix}
-\tau \\
0
\end{bmatrix}
\] (4.28)

The determinant, \( \Delta \), is then
\[
\Delta = \delta^2 + \delta \gamma_2 R + \delta \mu - \delta \gamma_1 (V - \hat{V}) - \mu \gamma_1 (V - \hat{V}) - \delta g - g \gamma_2 R
\]
\[-\mu g + 2 \delta \frac{g}{A} R + 2 \gamma_2 \frac{g}{A} R^2 + 2 \mu \frac{g}{A} R \] (4.29)

Applying Cramer’s rule to isolate the shadow prices gives
\[
p = \frac{\begin{bmatrix}
-\tau & -\gamma_1 R \\
0 & \delta - \gamma_1 (V - \hat{V}) - g \left( 1 - \frac{2R}{A} \right)
\end{bmatrix}}{\Delta}
\]
\[= \frac{-\tau \delta + \tau \gamma_1 (V - \hat{V}) + \tau g \left( 1 - \frac{2R}{A} \right)}{\Delta} \] (4.30)
\[q = \frac{\begin{bmatrix}
\delta + \mu + \gamma_2 R & -\tau \\
\gamma_2 (V - \hat{V}) & 0
\end{bmatrix}}{\Delta} = \frac{\tau \gamma_2 (V - \hat{V})}{\Delta} \] (4.31)

Recall \( x_1 > 0 \) and \( y_2 = 0 \) in this region of phase space, which resulted in (4.13a).
while (4.12) gives \( x_1 \) in terms of \( q \) and \( p \). These results for \( x_1 \) and \( x_2 \) are substituted into (4.24) and (4.25), giving the following results:
\[
gR - \frac{g}{A} R^2 + \gamma_1 R(V - \hat{V}) = \frac{\alpha_1}{q - ap} + \frac{1}{q - p} \] (4.32)
\[
\mu(V - \hat{V}) + \gamma_2 R(V - \hat{V}) = \frac{a \alpha_1}{q - ap} + \frac{1}{q - p} \] (4.33)

The next step is to substitute the equivalencies of (4.30) and (4.31) for the appropriate variables in each of (4.32) and (4.33), eliminating \( p \) and \( q \) and giving two equations in the two state variable unknowns, \( R \) and \( V \). The results can be solved
simultaneously to yield an expression for the steady-state stock of carbon dioxide \( V \) in terms of only parameters. The other steady-state values for the remaining variables are then determined. The numerical analysis for steady-state values was performed using Maple\textsuperscript{®}V Release 4 (Appendix B), and the results are presented in Section VB.

3. Optimal Time Path

The goal of the central planner is to find the optimal initial imputed prices for the state variables. The optimal shadow prices for the stocks of biomass and CO\(_2\) will induce optimal consumption to maximize utility, and guide the paths (trajectories) of these variables to a steady state. From a policy perspective, the optimal shadow prices may indicate the taxes (or subsidies) that ought to apply to the use of fossil fuel today. Determining the shadow prices is one of the major problems in control theory. In an infinite-lived model in which a steady state exists, the only values known prior to attempting to solve the problem are the initial values for the state variables. The “terminal” (steady-state) values for the control and state variables then become known only if a steady state can be determined. Finally, the initial values for the co-state variables (shadow, or imputed prices) are determined by starting at a slight perturbation from their steady-state values, and re-computing these values in reverse time increments, based on the relations dictated by the Maximum Principle. Since the relations are continuous, this process will establish policy by identifying the “proper” imputed price of each state variable at any given time from steady state.
This policy decision has implications affecting all sectors involved in the economy described by the system. Consumption of biomass for energy, food and other products can be maintained indefinitely (or at least for a long period of time) without threat of exhausting renewable biomass. The quantity of atmospheric CO₂ is adjusted ultimately to a constant level, consistent with its costs and benefits. Fossil fuel carbon stocks used for energy, given their many-fold quantities relative to those consumed for biomass energy, will start with a lower relative imputed price. As this fossil fuel stock becomes depleted, its imputed price increases until it equals the imputed price of its substitute, i.e., biomass, whose imputed price has been decreasing due to growth of its stock. At this point, continued consumption of the non-renewable fossil fuels for energy is not optimal, since its price [per equation (4.20a)] will continue to rise beyond that of biomass. Biomass then is substituted for fossil fuels for energy production as the fuels are then “switched”. 
V. Planetary Application

A. Bases for Parameter Selections

1. Estimate of the CO₂ Fertilization Factors: \( \gamma_1, \gamma_2 \)

a. Biological Considerations

In the attempt to determine the potential for planetary equilibrium of CO₂ levels, it is necessary to discuss, at a relatively high level of abstraction, the underlying biochemical processes that form the basis of plant photosynthesis. Plants use photosynthesis to mediate the synthesis of organic (carbon-based) compounds from gaseous carbon dioxide derived from the earth's atmosphere. All living organisms in terrestrial (but not oceanic\(^1\)) ecosystems ultimately depend directly or indirectly upon photosynthesis for sustenance.

Plant biomass is usually divided into three categories based on their respective photosynthetic properties: C₃, C₄ and CAM type plants. The third category, CAM\(^2\) plants, are comprised mostly of succulent cacti and plants such as pineapple. The differences of CAM plants and their relatively low occurrence (4%) in the biosphere do not merit material consideration in this study. The numeric subscript for the remaining two major categories indicates the number of carbon atoms in the first chemical compound produced by the biomass in the photochemical process. The plants differ substantially, however, in their mechanisms of absorbing CO₂. C₃ is the most basic photosynthetic mechanism. The C₄ plants have the property that, in addition to normal

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\(^1\) The systems based on ocean thermal events are an exception.

\(^2\) Crassulacean acid metabolism is a variant of C₄ photosynthesis in which CO₂ is not only concentrated, but also stored for half a day. Their CO₂ uptake is at night and stored until light activates their photosynthesis.
photosynthesis, oxygen will compete with carbon dioxide for the receptor molecule at the leaf surface, effectively reversing the photosynthetic process by producing carbon dioxide. This reverse process, i.e., when oxygen competes successfully, is called photorespiration, or in more precise words, the light-dependent oxidation of organic metabolites to carbon dioxide, which is then respired. This latter process takes place mainly at night. C₃ plants make up the majority (95%) of species, including all important tree and most crop species, such as wheat, rice, barley, potato and cassava. The C₄ plant species comprise only 1% of terrestrial plant species, and by virtue of separate locations in the plant stomates (absorption sites at the leaf surface), do not absorb CO₂ in the same manner. Their absorption process has an intermediate step prior to their C₄ production process. This additional step makes the absorption of CO₂ effectively independent of oxygen concentration fluctuations (and therefore carbon dioxide concentration fluctuations). Examples of C₄ plants are the tropical grasses such as sugarcane, maize, sorghum, and savanna grasses.

Aquatic plants are also divided into the categories. Emerged species photosynthesize in air with roots submersed, and thus can be categorized as C₃ or C₄ plants with respect to CO₂ enrichment. The submersed species, however, can photosynthesize with dissolved CO₂, or with the bicarbonate (HCO₃⁻) or carbonate (CO₃²⁻) ions. The submersed species are also considered outside the scope of this study. For an in-depth survey of literature on plants and CO₂, see Bowes (1993).
An important property of the C$_4$ plants is their ability to absorb CO$_2$ more rapidly when its concentration is increased, a phenomenon known as CO$_2$ fertilization. In technical language, this phenomenon can be stated more scientifically: higher CO$_2$ concentration generally leads to lower stomatal conductance and higher leaf photosynthetic rates, which are directly related to growth rates$^3$. The C$_4$ plants do not share this property of CO$_2$ fertilization and therefore respond only slightly to increased carbon dioxide levels. A comprehensive model of plant CO$_2$ absorption (photosynthesis) would thus consider both types of biota. If only C$_4$ plants were present, the model for CO$_2$ sequestration would thus be represented by

$$
\dot{v} = ax - \gamma_2 R
$$

(5.1)

indicating an only coincidental possibility for a steady state, since, in the long-term $\dot{v}$ will exhibit a monotonic trend determined by $\gamma_2$. Thus, the required relationship for steady state, $x = \gamma_2 R/a$, may not occur, and the level of CO$_2$ is then not bounded.

As mentioned above, C$_3$ plants represent roughly 95% of living, terrestrial biomass. Thus, if the model considers only C$_3$ plants, then CO$_2$ fertilization is represented by

$$
\dot{v} = ax - \gamma_2 RV
$$

(5.2)

allowing the presence of CO$_2$ to enhance its absorption, ceteris paribus. This behavior has been proved in laboratory tests (Lemon, 1983; Kimball and Idso, 1983), and has

---

$^3$ Lloyd and Farquhar (1996) argue that the statement "The CO$_2$ dependence of plant growth is a simple quantitative reflection of the CO$_2$ dependence of photosynthesis" is "often taken as axiomatic" and that there is "little theoretical justification or experimental evidence supporting [this statement]" (brackets added). Despite the caveat, they estimate $\beta$ in simulations as ranging from 1 to 2, compared to laboratory experiments (vide infra).
positive, (Eamus and Jarvis, 1989), and mixed (Luo and Mooney, 1996) results, in open natural and commercial tree farm experiments. Certain trees responded with increased CO₂ absorption rates to increased CO₂ levels, while other types did not. These studies also show that water efficiency and some disease resistance are enhanced at elevated CO₂ levels. The increasing amplitude of the annual atmospheric CO₂ fluctuation has also been suggested as evidence for increasing CO₂ absorption by plants (Idso and Idso, 2000). For the present application, however, consideration must be given to the carbon storage potential of different types of plants. Trees are the largest and densest carbon storage medium of the biota flora, and as trees are almost entirely C₃ plants, their storage mechanism and abilities will be the focus and paradigm for CO₂ sequestration.

b. Comparative Analyses

In order to quantify the parameter γ₁, it is necessary to review some previous work to characterize similar parameters, and translate those analyses into a reasonable estimate. Luo and Mooney (1996, p 382) employ a similar analytical approach “in the attempt to overcome difficulties associated with environmental heterogeneity and species characteristics...”. They briefly review the Bacastow and Keeling (1973) model, which used a single parameter called the biota growth factor, β, to account for terrestrial carbon content changes with atmospheric CO₂ concentration (C₄, in Luo and Mooney notation). Gates (1985) then defined β as a fractional change in net primary productivity (NPP) with a fractional change in C₄, similar to an economics measure of elasticity:
\[ \beta = \left( \frac{\Delta NPP}{NPP} \right) \left( \frac{\Delta C_u}{C_u} \right) \]  

(5.3)

Luo and Mooney (1996) then manipulated a mechanistic model of leaf photosynthesis (Farquhar et al. 1980) to define a factor, \( \gamma \), (leaf-level factor, in units of ppm\(^{-1}\)) as

\[ \gamma = \frac{1}{P} \frac{dP}{dC_u} \]  

(5.4)

where \( P \) is the annual global photosynthetic carbon influx, i.e., gross primary productivity. This GPP is the sum of carbon influx from total leaf area within canopies over the global surface over a period of a year, and \( C_u \) is atmospheric CO\(_2\) concentration. Thus, the \( \gamma \) factor denotes the relative leaf photosynthetic response to a unit ppm \( C_u \) change.

Luo and Mooney then establish limits for the range of values for the \( \gamma \) factor of terrestrial \( C_1 \) plants by citing the "well-established" Farquhar et al (1980)\(^4\) study and justifying values for the parameters. In their words

```
We find that the \( \gamma \) factor is an approximate constant for any \( C_1 \) plant, regardless of the geographical location and canopy position.
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Much of the remainder of their article cites supporting research, and justifies the extension of the model to global proportions. The lower limit of \( C_1 \) photosynthesis change is characterized by a constrained electron transport environment (limited sunlight) and is given by

\[ L_u = \frac{15\alpha}{(\alpha C_u - \Gamma)(4.5\alpha C_u + 10.5\Gamma)} \quad (5.5) \]

whereas the upper limit is characterized by a constrained ribulose-1,5-biphosphate carboxylase/oxygenase environment (limited converter enzyme) and is given by

\[ L_u = \frac{\alpha(K + \Gamma)}{(\alpha C_u - \Gamma)(\alpha C_u + K)} \quad (5.6) \]

The constant \( \alpha \) is the ratio of intercellular CO\textsubscript{2} concentration to atmospheric CO\textsubscript{2} concentration, as the stomatal pores will maintain a pressure gradient to the ambient surroundings. The parameter \( \Gamma \) is the compensation point without dark respiration, i.e., the minimum CO\textsubscript{2} concentration required to prevent oxygen domination at the stomatal site, and hence, leaf destruction. The factor \( K \) is an enzyme kinetic parameter.

According to the Luo and Mooney references: \( \alpha \) is sensitive to water status and CO\textsubscript{2} concentration, but fairly constant among species when plants grow in their natural

Reproduction of Results of Luo and Mooney

\[
\begin{align*}
\alpha &= 0.7 \\
\Gamma &= 35 \\
K &= 650 \\
L_1(C_a) &= \frac{15\cdot\Gamma}{\alpha \cdot C_a - \Gamma \cdot 4.5\alpha C_a + 10.5\Gamma} \\
L_2(C_a) &= \frac{a \cdot K + \Gamma}{\alpha \cdot C_a - \Gamma \cdot \alpha C_a + K}
\end{align*}
\]

![Figure 8](image-url)
environments; $\Gamma$ varies little among species, but depends strongly on temperature (Luo and Mooney thus choose $\Gamma$ for a temperature of 20°C, which is slightly higher than the earth's average temperature of 16°C); and $K$ is variable among species, but only slightly affects the upper limit of the $\nu$ factor. The results of Luo and Mooney are reproduced in \textbf{Figure 8} (page 92).

To apply this analysis to our context, recall the biomass change equation

$$\dot{R} = gR - \frac{g}{A} R^2 + \gamma_1 RV - h$$

(5.6)

Assume for this derivation that changes in biomass occur only through changes in CO$_2$, which implies $g = 0$, and there is no consumption ($h = 0$). If a small change in CO$_2$ concentration, $\Delta V$, occurs, then a small change in CO$_2$ absorption will result and equation (5.6) can be rewritten for a unit time period, say one year ($\Delta t = 1$) as

$$\Delta \left( \frac{\Delta R}{R} \right) = \gamma_1 \Delta V$$

(5.8)

Equation (5.8) can be seen to equate to (5.4) above

$$\frac{\Delta \left( \frac{\Delta R}{R} \right)}{\Delta V} = \gamma_1, V' = V' = \frac{1}{P} \frac{\Delta P}{\Delta C_0}$$

(5.8)

Thus, when adjusted by the ambient CO$_2$ concentration, any value within the range of the curves shown in \textbf{Figure 8} could be used to approximate the marginal amount of CO$_2$ that would be absorbed with a small concentration increase.
It is more likely that the limiting factor in the fertilization would be enzyme-based rather that light-based. To determine the estimate for the value of the parameter $\gamma_1$, the values for the upper and lower limits at the 1998 CO$_2$ concentration of 366.7 ppm$^*$ for the Luo and Mooney $\gamma$ factors can then be weight-averaged. This weighting is estimated at 0.67 for enzyme-limited photosynthesis and 0.33 for light-limited photosynthesis. The weighted-average value for $\gamma$ is then adjusted for CO$_2$ concentration and is calculated as $\gamma_1 = 0.0000053$.

Further support for this method of determination for a value of $\gamma_1$ comes from the analysis of Luo and Mooney (1996) and Lloyd and Farquhar (1996). Luo and Mooney establish a relation between the biotic factor $\beta$ and the $\gamma$ factor by defining a photosynthetic factor $\beta_p$ as

$$\beta_p = C_a \gamma = \left( \frac{dP}{P} \right) \left( \frac{C_a}{dC_a} \right)$$ (5.10)

It is evident that equation (5.10) is similar to (5.3), since the factor $\beta_p$ describes photosynthetic changes for $C_a$ changes, whereas $\beta$ represents biomass production changes for $C_a$ changes. Thus, a simple conversion is obtained by multiplying by the appropriate CO$_2$ concentration. Lloyd and Farquhar use the original Bacastow and Keeling (1973) model

$$\frac{(N_p)}{(N_0)} = 1 + \beta \ln \left( \frac{[C_{O_2}]}{[C_{O_2}]_0} \right)$$ (5.10)

---

$^*$ From the CDIAC Web site http://cdiac.esd.ornl.gov/trends/co2/sio-mlo.htm
to estimate $\beta$ in a simulation of a forest ecosystem. The values ranged from about 1 for Southern Hemisphere coniferous forests to greater than 2 for forests with positive feedback. These values are substantially larger than values (average 0.3) observed in laboratory experiments (Kimball, 1983; Cure and Accock, 1986; Poorter, 1993).

2. Estimate of Sink Absorption of CO$_2$, $\mu$

a. Atmospheric Considerations

The benchmark for the "equilibrium" atmospheric CO$_2$ concentration is derived from estimates of the pre-industrial (before 1850) concentration. These estimates assume no net atmospheric additions of CO$_2$ from fossil fuel combustion or deforestation, and range from 260 to 290 ppm. Most of the literature cites this value at 280 ppm, which corresponds to roughly 600 gigatonnes of carbon in the atmosphere. The estimates made by 19th century scientists have been verified by modern techniques of $^{13}$C/$^{12}$C ratio measurements in both ice cores (Greenland and South Pole, e.g., Siple, Vostok) and tree rings, and $^{14}$C dating. See Houghton and Skole (1990), and Edmonds (1993). Boden, et al. (1990) for more references. Also see Crowley (1996) and Moore (1995) for brief explanations of how climate histories are inferred from geologic and other evidence.

The current atmospheric CO$_2$ concentration has been monitored by modern measurement techniques that began with the Mauna Loa and South Pole studies in the 1950s. These methods and studies, including the ice core samples used as a trend indicator, have confirmed the increase of CO$_2$ levels to the current and increasing concentration of 366.7
ppm, or about 0.037% by volume of the atmosphere. This value corresponds to about 750 gigatonnes of carbon (1 gigatonne = 1 billion metric tons, and 1 metric ton = 2204 lb.). Thus, \( \nu(0) = \nu_0 = 750 \) is used in the model without optimization to test these parameters in a calibration. Similar to Uzawa (1991, 1993), the difference between current and pre-industrial CO\(_2\) levels is used as the volume of CO\(_2\) to observe in the model. The pre-industrial CO\(_2\) level, termed \( \nu^* \), corresponds to about 600 GtC CO\(_2\) is similar to the relatively inert, predominant atmospheric gas, nitrogen (N\(_2\)), in that neither reacts with other atmospheric gasses, as do nitrogen oxides (NO\(_x\)), and the photochemical greenhouse gasses such as the halocarbons (freon, etc.). Thus, there is no atmospheric process by which the CO\(_2\) concentration is changed; consequently, the terrestrial processes of plant decay and respiration and fossil fuel combustion are the major sources of CO\(_2\), whereas the oceans and plant biomass are the only net sinks.

The Uzawa (1991) paper formulates [eqn (3.1), page 51]. the absorption as a constant fraction of the amount of CO\(_2\) present in the atmosphere. He relies on Ramanathan (1985) for the value of \( \mu \), who claims that residence time for any particular CO\(_2\) molecule in the atmosphere ranges from 7 to 14 years, depending on latitude and mixing. Uzawa claims, "The estimate is rather difficult to obtain", and asserts a value \( \mu = 0.02 \).

An alternative estimate used here revises the Uzawa value based on a simple arithmetic calculation. Referring to Figure 3 in Section 1B (page 10), the average amount of CO\(_2\)

\(^*\) The most recent CO\(_2\) data is for 1997, and (Colombo, T., and R. Santaguida; 1998) comes from a station
present in the atmosphere over the years 1980-1989 is 750 GtC. Using these values, if an average of 2 GtC/yr were absorbed into the ocean sink, then that fraction is 0.00266. This quantity is an order of magnitude smaller than the quantity used by Uzawa.

b. Oceanic Considerations

Since the earth is roughly 70% covered by water, and water both stores heat and dissolves CO₂, its role in global warming may be simultaneously the most important and the least controllable. Controversy surrounds the issue of the time frame for CO₂ and heat transfer between the ocean and the atmosphere. Scientists (see Daly, 1997) have advanced the assertion that global warming and the subsequent melting of polar ice caps could disrupt the oceanic thermocline “conveyor”, causing erratic gaps or even eliminating the earth’s general atmospheric circulation. Others, e.g., Daly (1997) and Dietze (1997) present arguments that claim the ocean can and will absorb “excess” CO₂.

No simple models of seawater absorption of CO₂ were found in the research process. The few papers that were found, e.g., Siegenthaler and Sarmiento (1993), explain the absorption process in detail but offer no general model. Certain thermodynamic properties, such as ionic mixing, are explained in water chemistry textbooks. A plausible explanation for the lack of models is that many variables influence seawater absorption so that any specific model would apply only to those measured conditions.

at Mt Cimone, Italy, measured by CDIAC of USGCRIO, is 364.34 ppm. See Appendix A for organizational references.
c. Biospheric Considerations

Estimates of the carbon in the biomass of the earth distinguish between soil level and above-ground carbon. Olson (1983) provides a comprehensive assessment of the global distribution of large-scale vegetation groups (biomes). The corresponding carbon content in above-ground photosynthetic biomass is estimated to be 562 Gt C. In a model that intended to encompass the entire biota carbon exchange, adjustments need to be made for reducing this amount to account for "unsuitable" landmasses such as deserts. As explained later (Sections VA3-4, below), these adjustments reduce the starting value for the biomass in the unoptimized model to $R(0) = R_0 = 500$ GtC.

3. Logistical Biomass Growth Factor, $g$

The parameter $g$ represents the average natural proportional growth rate of terrestrial biomass. Scant literature has been published quoting weight-based growth rates for large plant life. The original use of this measure was in small animal organisms, such as daphnia, microbes, or small rodents, and was later extended to plant matter. Since the parameter is usually measured as a weight percentage, it presents obvious difficulties for large biomass structures such as trees. To further complicate the measure, dry weight is preferred, since water weight can vary significantly depending on a number of environmental factors. Since most of the terrestrial biomass is contained in trees\(^7\), and any attempts at augmenting carbon sequestration would use this medium for implementation, a parameter for the growth rate of trees will be used. The growth rates

\(^7\) Forests comprise 90% of the carbon in terrestrial vegetation. (Marland, 1988)
of other plant matter, such as grasses and edible crops is too widely varying to yield plausible results in such a long-term study.

Growth rates for trees in particular are typically measured by a 19th century standard called Diameter Breast Height. This standard is still used today by most tree farms and the US Forest Service. Obviously, breast height is relative across human beings, but once a typical baseline for height is established, for example 5' from ground level, relative growth will compensate for any initial variations in measurements. Even measuring a diameter is not completely accurate, of course, since trees are rarely uniformly circular at any point in their boles (trunks)⁸. Growth rates for trees are well catalogued and documented by this measure. The variations can be linear multiples of one another: the fastest growing tropical trees can fix carbon three to ten times faster than other tropical trees (Schroeder. 1992) or their boreal (northern) relatives in the plant kingdom. Zavitovski and Stevens (1972) applied a biomass adaptation by Ricklefs (1967) of the logistical growth function to determine this growth parameter for a grove of red alder trees. The study took place over several not-unusual seasons in the Pacific Northwest. Another study yielding similar parameters was made in a laboratory environment for smaller grasses and flowering plants by Fresco (1973). As discussed in Section IIA. Zavitovski and Stevens calculated the value for the parameter  at 0.09.

A more conservative value was chosen for this parameter. It is based on an estimate for the amount of carbon that could be collected on average for living biomass on suitable

⁸ The ratio of tree biomass weight to diameter will of course vary by species.
land for the earth as a whole. Houghton and Skole (1990), using previous studies, estimate landmass for categories other than desert, tundra, rock and ice, at roughly 10.18 billion hectares. This land holds approximately 500 GtC, for an average of about 49 tC/ha. In another study, Marland (1988) estimated that “potential net annual growth” of US forests could collect 1.35 tC/ha/yr. Values for grazing and non-timber uses are estimated to accumulate 0.36 to 0.91 tC/ha/yr. Using a conservative value for $g$ of 0.04 or 4%, for the suitable growing biomass yields roughly 8 GtC/yr for the first 20 years of growth in an undisturbed model. This rate averages 0.786 tC per hectare, at the high end of the range for grazing and non-timber land. In contrast, some tropical trees have been estimated to grow in the 12-20% range. Thus, at an initial value of 500 GtC (where growth is slower than average in the logistical function), terrestrial biomass is projected to accumulate carbon over the first twenty years at a rate of 1.5% of its mass annually.

4. Estimate of the Biomass Capacity, $A$

Several studies around 1980 used recently launched satellites to determine the area of landmass and associated ecosystems that covered the earth. Matthews (1983) presents a summary of these studies and compares them to her own work. One of the works (Olson, et al. 1983), included numerous vegetation types and identification of their geographic locations. Another work, Rodin, et al. (work done in 1971. published in 1975) was not included in Matthews (1983), but was used as the basis for estimating pre-agricultural terrestrial phytomass in a later study (Olson, 1974). This latter estimate may be used as a proxy for the biomass capacity, $A$. 
Estimates of Global Land Mass

Area (10^6 km^2)

<table>
<thead>
<tr>
<th>Source</th>
<th>Area (10^6 km^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lieth (1975) [Matthews, 1983]</td>
<td>131.5</td>
</tr>
<tr>
<td>Olson, Watts, Allison (1983)</td>
<td>133.4</td>
</tr>
<tr>
<td>Matthews (1983)</td>
<td>132.4</td>
</tr>
<tr>
<td>Rodin, Bazilevich, Rozov (1975)</td>
<td>133.4</td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td><strong>133.1</strong></td>
</tr>
</tbody>
</table>

Table 3

Using the average of five estimates of total land area tabulated in Table 3 above, the capacity estimate of Olson (1985) may be verified. The table shows the maximum global surface area which could hold biomass as roughly 133 million square kilometers, or 13.3 billion hectares. Glaciers of the Polar Regions are not included. Deserts must also be removed. Using Houghton and Skole (1990), the global surface area, which could hold productive biomass, is then estimated at 10.18 billion hectares, as mentioned in the previous section. This estimate may be used for carbon density comparisons.

To achieve this end, estimates for biomass quantities on land areas of square kilometers or hectares can be made using data measurements of cubic meters of wood per hectare of forestland. These data are then converted to weight measure using wood densities for the particular species of tree. Schroeder (1992) determined mean carbon storage values for selected tropical tree plantations as ranging from 8 to 72 tC/ha. The well-managed sites spanned the upper half of the range, 42-72 tC/ha. Sedjo (1989) makes the conservative adjustment for growth on managed tree plantations using only the stemwood of a tree to
calculate carbon storage per hectare, even though other biomass such as roots, branches, and leaves also store carbon, but with lower densities. His example uses plantation forests in the Pacific Northwest and southern US of 15 m³/ha/yr. After making such an adjustment, the estimate of 15 m³/ha/yr translates into carbon fixed at 6.24 tC/ha/yr. Finally, 6.24 tC/ha/yr over 10.18 x 10⁸ ha calculates to 63.5 GtC/yr of possible growth. This value is high compared to other estimates. The Marland (1988) study mentioned in the previous section suggests a figure based on more sites as closer to 1.35 tC/ha/yr. The calculations are based on trees, since woody trees contain higher carbon ratios than other nutrient plants, and since 90% of terrestrial vegetation is contained in forests. Some tropical species can sequester on the average 20-25 m³/ha annually, while tundra species such as Siberian larch will store only 3-5 m³/ha. Other studies have determined similar values (Schroeder, 1992) or used similar estimates between these ranges (Marland, 1988). The use of such an average can be justified in light of a non-linear, logistical growth function since the value is used only to calculate a maximum quantity for terrestrial biomass. Certainly one may argue that deserts will not produce such an output, but this parameter can be viewed as a choice variable: the planner can decide the maximum allowable area for terrestrial carbon storage.

To verify these estimates further, areal phytomass densities can be compared by starting with a total weight value, then dividing by area to obtain a density measure. The value for the total carbon present in terrestrial vegetation prior to the inception of agriculture was developed by Olson (1974) using the calculations of Bazilevich (1971) and was
estimated at 1070 GtC. This number was later revised downward in Olson, et al (1985) to 900 GtC (756GtC in woody plants). The difficulty in such an undertaking is noted:

The amount of carbon per unit area has also diminished in most areas because of human disturbances. Therefore, we cannot multiply representative modern carbon densities by the areas of modern (or even original) cover types and obtain valid estimates of primordial plant carbon; the product would be too low. Nor would it be valid to use data on mature stands and multiply by forest area; the estimate would be too high for modern and even for virgin landscape complexes; the latter had significant areas still undergoing recovery from fires and other disturbances at any point of time.

Despite the caveat, a undisturbed phytomass weight estimate of 900 GtC divided by 10.18 G ha corresponds to a density of about 88 tC/ha, which falls less than the Houghton and Skole (1990) value of 95tC/ha for temperate forests. and between tropical forests and northern, boreal forests. The plausibility of this estimate can be visualized by contemplating the lush forests and dense jungles covering large expanses described by European authors and explorers as new land discoveries were being documented. 9

5. Discount Rate and Social Rate of Time Preference-SRTP, δ

The rate at which future values are exchanged for present values, i.e., the discount rate, or, in the case of a social planner, the social rate of time preference (SRTP) is a key parameter in the analysis. If the SRTP were greater than the biological growth rate of biomass, no tree stocks would be maintained. The deforestation of the lands surrounding

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9 For an interesting historical account of the pre-industrial world social and political impetus for deforestation, see Perlin, John: A Forest Journey, WW Norton, NY
the Mediterranean Sea in pre-fossil fuel history is consistent with this reasoning\(^{10}\). Wood was used both as a primary fuel and also converted to charcoal for heating metal to its malleable state (forging), for arms manufacture. This myopic tendency may still exist in developing countries where wood is a primary fuel and building material.

The social rate of discount has been a difficult issue to resolve. The issue is usually divided into the questions of whether to discount at all \(r = 0\) or, if discounting is necessary, at what rate should future benefits be discounted. From a purely mathematical standpoint, the objective function in the Ramsey problem will not converge if future benefits are not discounted. However, a very small \((-10^4)\) discount rate would permit convergence while implying optimal behavior in the short run that is close to what one would obtain without discounting. The U.S. Forest Service has traditionally not discounted their valuation models, and has based their forest rotations on Maximum Sustained Yield (MSY). Several authors (e.g., Plourde, 1970) have shown that an MSY program will not be optimal with discounting.

Arrow and Kurz (1970b) bifurcate the decision of discounting into polar extremes depending upon the degree of centralization in the applicable society. For a totally centralized society, a planning board establishes objectives and allocates resources to maximize these objectives. Social benefits are determined by the planner, unconstrained by individual desires. At the opposite extreme, in a totally decentralized society.

---

\(^{10}\) See footnote 9 above. Some of the deforestation could be attributed to property rights that were difficult to enforce. Most of the forestlands in Europe, however, were part of landed estates.
discounting is determined as individual agents maximize their own objective functions in the context of the social opportunities for transferring resources intertemporally through investments. If private capital markets were perfect, and if social and private benefits were non-divergent, then the discount rates would be identical also. Similar to the case of differing marginal utilities for an individual, differing discount rates between the public and private sectors (if they could be established) may provide an impetus for increasing overall welfare by a reallocation of resources. Uncertainty also becomes an issue (not included in our analysis), which is discussed by Arrow-Lind (1970).

One may take the position that any project for which resources are allocated must compete with other projects in the relevant market. Nordhaus (1991c) presents both sides of the argument, and includes expected returns on investment for many world "projects", e.g., third world education, which could compete favorably with his estimate for the return expected for resources allocated to counteract global warming. The opportunity cost becomes the benchmark against which the value is measured.

In another view, Broome (1992), distinguishes between the pure discount rate method and the consumer-price method. Then goes further by stating that the consumer-price method is a marginal analysis applying only to small projects. These small projects are not expected to affect prices. Global warming, in its massive scale, is neither marginal nor small, and will consequently affect prices.
Cline (1991, Chapter 6) surveys and critiques several analyses on both the "demand" (consumer-price) and "supply" (competing investments) views of the discount rate issue. He argues for a discount rate of 1.5%, and suggests it should even be as low as 0.5% on the grounds that higher rates for a project of such dimension, where the benefits will tend be realized only long after the costs, will bias policy toward inaction. Lewis and Seidman (1995) use "out of conciseness, not out of conviction" the same 1.5% value in a discrete time, 1000-year model.

To accommodate the foregoing arguments, the base discount rate will be chosen as the midpoint of a range using a "low" discount rate of 1% and a "high" discount rate of 5%. Thus, the base case uses 3%. This rate is consistent with other studies. (the Uzawa, 1991, study used 5%), but leans to the socially conservative side. The steady-state biomass and atmospheric CO₂ levels will be tested for their sensitivity to this parameter. Previous studies and their comments have indicated models are highly sensitive to the discount rate.

6. Estimate of the Utility Effect of CO₂. \( \tau \)

As previously mentioned, \( \tau \) is a coefficient which represents the presumed perceived net utility effect of a quantity of CO₂ in the atmosphere. The net effect could be positive or negative depending on a variety of factors, most important of which would be a combination of global location and regional industries. As a result, potential warming may have positive or negative effects. This variable is intended to be a policy variable, with suggested limits. As discussed in Section IA, organizations, especially certain
groups of the United Nations such as the UN Environmental Programme (UNEP; see Appendix A for a description), have held conferences and funded various studies to assess the impact of modeled warming rates. Some of these studies were identified in Section IC.

a. Kyoto COP3, December 1997

Probably the most significant of the recent Conference of the Parties (COP3) was sponsored by the IPCC in 1997 at Kyoto, Japan. This conference was held with the intent of forcing countries to commit to specific greenhouse gas reduction goals. The commitment was documented as the Kyoto Protocol, and the range of reductions for most countries was 5-8% of 1990 levels by 2012. The overall goal for the developed world is 5.2% reduction from 1990 levels by 2008. This number will be used as a basis for estimating \( r \) in the base case of the simulation. Most countries agreed in principle to their goals, including the US.\(^\text{11}\) Within a year, further negotiations led to the decision (again, by the same parties described below, and under pressure from European countries) to reduce emissions by 7% over the ten years to 2008. The agreement was not officially signed by the acting US ambassador to the UN until November 12, 1998, at the Buenos Aires COP4, and has yet to be ratified by the US Senate. The likelihood of the agreement being ratified by the US Senate remains low, since large, developing country emitters such as China and India remain opposed to the treaty. claiming that developed countries

\(^{11}\) Vice-president Albert Gore became a champion of this cause, and along with President Bill Clinton, his administration, and certain environmental groups were among the consenters to the Protocol. However, several months prior to the conference, the Byrd-Hagel resolution was passed in the US Senate resolving that “the United States should not be a signatory to any protocol...” without ratification of the Senate.
are responsible for the current CO₂ levels and its history of accumulation. Moreover, the
US Senate has insisted that any agreement include a system for trading pollution rights
among countries. Citations for papers on such systems were in Section 1C.

b. Buenos Aires COP4, November 1998

Negotiations continued at the conference in Buenos Aires, with the United States
formally signing the protocol, three days before the deadline. The signing is largely
symbolic, since US Senate ratification is unlikely. However, a press release by the
UNFCCC claimed that “The developed country signatories to date account for 78.7% of
the group's emissions.”¹² The main effort of this meeting was to develop strategies to
carry out the treaty specifications. A self-imposed deadline for the development was set
at January 1, 2000.

7. Utility Coefficients for Non-Energy Consumption Elements α₁, α₂

These parameters represent multipliers to establish the relative importance of
consumption sectors in the utility function. The objective function is divided primarily
into biomass and fossil fuel segments, then secondarily into energy and non-energy uses
for these resource consumptions. α₁ and α₂ are the coefficients of x₁ and x₂,
respectively, which are the control variables for non-energy biomass consumption and
non-energy fossil fuel consumption, respectively. One might use the analogy of food and
building materials as a proxy for x₁, and the chemical industry as a proxy for x₂. The

¹² COP4 Press Releases at http://www.unfccc.de/cop4/
relative values for these coefficients will determine a scale for the consumption quantities of their respective resource inputs. A reasonable estimate for these parameters is derived from these proxies, which indicate that roughly >90% of non-energy biomass is used for food and durable goods and <10% for chemical production. Further, the estimate is made that 90+% of chemicals made from biomass end up as CO₂ rapidly, while 100% of food and <5% of building materials end up as CO₂. The rough estimate for a ratio of \( \alpha_1 : \alpha_2 \) is somewhere in the range 5:1 to 10:1. As a base case, \( \alpha_1 = 10.0 \) and \( \alpha_2 = 1.0 \) will be used.

8. Utility Coefficient for Energy Consumption Elements, \( \phi \)

This ratio could also be called the marginal substitution coefficient for the energy consumption elements in the utility function. A recent study by the International Energy Agency (IEA, also linked to OECD and funded by the Industry and Environment Division of the UNEP) estimated that biomass energy currently accounts for 14% of world energy consumption. “This IEA workshop, held in Paris on 23-24 March 1998, focused on the work done by the IEA and other organizations to understand better the transition from non-commercial to commercial energy forms.”\(^{13}\) This estimate will be used as a proxy for the utility substitution coefficient for energy consumption elements. This ratio then becomes 14 / 86 = 0.163, or in words, roughly 6 times as much utility is gained from the consumption of fossil fuels for energy as is gained from the consumption of biomass for energy.

\(^{13}\) http://www.iea.org/energy.htm
Biomass energy can be obtained directly from sources such as wood or peat. This process primarily occurs in developing countries. For developed countries, an alternative measure could be based on energy fuels derived from biomass, such as methanol. The alternative measure for $\phi$ could be the ratio of the price of methanol to the price of gasoline. Based on current prices, this ratio would be roughly 0.45 - 0.625.

9. Non-Energy Biomass Fraction Becoming CO$_2$, $a$

As specified previously, $a$ and $b$ are coefficients representing the fraction of non-energy consumption of biomass and fossil fuels, respectively, which rapidly become CO$_2$. We shall assume that 80% of the non-energy biomass is consumed as durable building materials and 20% for food. The assumptions are also made that ~100% of food rapidly ends up as CO$_2$ in the normal, animal metabolic functions, and ~0% of the durable products quickly decompose to CO$_2$. Thus, 0.2 is the estimate determined for $a$, and $b$ is estimated at zero, since fossil fuels transformed into chemicals do not rapidly decompose.

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$^{14}$ A recent wholesale price for methanol, obtained from [http://www.methanex.com/glance/price.html](http://www.methanex.com/glance/price.html), is $0.30-0.35$ per US gallon, while wholesale gasoline prices range from $0.56-0.66$ per US gallon. Gasoline prices obtained from [http://www.petroleumplace.com/Articles/TheOilPatch/4-14.html](http://www.petroleumplace.com/Articles/TheOilPatch/4-14.html).
B. Simulation

1. Biomass-CO$_2$ Interaction: Undisturbed, No Optimization

As a preliminary test of the model viability and parameter selection, a simple Microsoft Excel® spreadsheet (Appendix E) was created to simulate the differential equations involving biomass and carbon dioxide in an “undisturbed” environment. This type of environment is devoid of any optimization activities such as consumption decisions. Thus, biomass is not harvested, and is allowed to grow “naturally,” according to its logistic function, and no anthropogenic contribution to CO$_2$ stock is made via consumption.

The equations which represent this system were presented in Section IV; the state equations for biomass and carbon dioxide, respectively:

\[
\dot{R} = gR - \frac{K}{A} R^2 + \gamma_1 R(V - \hat{V})
\]

\[
\dot{V} = \mu(V - \hat{V}) + \gamma_2 R(V - \hat{V})
\]

The parameters used in this system have been selected and discussed in Section VA.

Parameters for CO$_2$-Biomass Interaction without Optimization

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
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</thead>
<tbody>
<tr>
<td><strong>Biomass Growth. $g$</strong></td>
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<td>fraction</td>
</tr>
<tr>
<td><strong>Biomass Capacity. $A$</strong></td>
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<td>GtC</td>
</tr>
<tr>
<td><strong>Fertilization Factor. $\gamma_1$</strong></td>
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<td>GtC$^{-1}$</td>
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<tr>
<td><strong>“Natural” CO$_2$ Level. $\hat{V}$</strong></td>
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<td>GtC</td>
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<tr>
<td><strong>CO$_2$ Decay Factor.</strong></td>
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<td>fraction</td>
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<tr>
<td><strong>Fertilization Factor. $\gamma_2$</strong></td>
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<td>GtC$^{-1}$</td>
</tr>
</tbody>
</table>

Table 4
The initial values for the biomass, $R_0 = 550$ GtC, and carbon dioxide, $V_0 = 750$ GtC stocks are also discussed in Section VA.

The "real world" results of the spreadsheet construction indicate that reasonable values for the parameters were selected and that the model is viable. The amount of carbon dioxide decay from its stock, roughly $3$ GtC/yr, is near the ocean absorption and "other" sinks produce values approximating the numbers presented in the Carbon Cycle (Table 1) of Section I (page 9). The annual amounts of CO$_2$ fertilization are expectedly closer to the values of $0.2$–$0.4$ GtC calculated by extrapolation in Luo and Mooney (1996), than to the values $0.5$–$2.0$ GtC estimated by various models summarized in IPCC WGI 1995 (Melillo, et al. 1996). Note that in this "undisturbed" model, the growth of terrestrial biomass will continue asymptotically to its capacity level, $\dot{A}$, and the reduction of atmospheric CO$_2$ will continue asymptotically to its pre-industrial level, $\dot{V}$. With these parameters tested in an unoptimized model, they can be now used in the optimized model. In the optimized model, as mentioned previously, a decision is made by a planner as to what prices will yield an efficient approach to an optimal steady state level of consumption, given the external costs of CO$_2$. It will also determine the optimal steady states for the state variables of biomass and CO$_2$. 
2. Determination of Steady State for the Optimization Problem

A base case solution for the steady state was sought in the 4-dimensional system (5.12)-(5.16) using Maple V Release 4 on a desktop personal computer. This process involved asking Maple to solve the following system of two equations in two unknowns, $R$ and $V$. After substitutions for the shadow prices and the determinant are made:

$$gR - \frac{g}{A} R^2 + \gamma_1 R (V - \hat{V}) = \frac{\alpha_1}{q - ap} + \frac{1}{q - p}$$  \hspace{1cm} (5.12)

$$\mu (V - \hat{V}) + \gamma_2 R (V - \hat{V}) = \frac{a \alpha_1}{q - ap} + \frac{1}{q - p}$$  \hspace{1cm} (5.13)

where

$$p = \frac{-\tau \delta + \tau \gamma_1 (V - \hat{V}) + \tau \gamma_2 \left(1 - \frac{2R}{A}\right)}{\Delta}$$  \hspace{1cm} (5.14)

$$q = \frac{\tau \gamma_1 (V - \hat{V})}{\Delta}$$  \hspace{1cm} (5.15)

and

$$\Delta = \delta^2 + \delta \gamma_2 R + \delta \mu - \delta \gamma_1 (V - \hat{V}) - \mu \gamma_1 (V - \hat{V}) - \delta g - g \gamma_2 R$$

$$-\mu g + 2 \delta \frac{g}{A} R + 2 \gamma_2 \frac{g}{A} R^2 + 2 \mu \frac{g}{A} R$$  \hspace{1cm} (5.16)

Expressions (5.12)-(5.16) combine to form a 7th-power system in $(V, R)$ space. Thus, seven roots are found as candidates for the steady state. Obviously, any solutions with negative values for the state variables of biomass, $R$ and CO$_2$, $V$ are not feasible and can be excluded immediately. Other roots can be excluded if they result in a
negative value for any of the consumption controls, \(x_1, x_2, y_1, y_2\). Any steady state which obtained a negative shadow price for a good (biomass) would not necessarily be excludable, but would be undesirable, since such a condition would indicate biomass is a "free good", which does not accord with its positive market price. To this end, Appendix F contains the results of the solutions to the system (5.12)-(5.16) for the base case parameters in two extreme cases for CO\(_2\) intolerance. These roots of the seven-power system are narrowed down to one possible solution for each suggested CO\(_2\) tolerance extreme, with explanations of the reason for any elimination.

3. Results Discussion and Comparisons

The following table presents the steady state solution values (Steady State Results Summary and Comparative Statics. Tables 6-7, pages 118-119) of our model. These values can be maintained on the earth forever for the infinite-lived optimal control problem with parameter assumptions discussed in Section VA.

Summary of Steady State Values

<table>
<thead>
<tr>
<th>Variable</th>
<th>Steady State Value ( r = 0.1 )</th>
<th>Steady State Value ( r = 0.001 )</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Biomass, ( R )</td>
<td>703</td>
<td>123</td>
<td>GtC</td>
</tr>
<tr>
<td>( \text{CO}_2, V )</td>
<td>842</td>
<td>772</td>
<td>GtC</td>
</tr>
<tr>
<td>( \text{CO}_2 ) Price, ( p )</td>
<td>-1.86</td>
<td>0.01</td>
<td>util/GtC</td>
</tr>
<tr>
<td>Biomass Price, ( q )</td>
<td>0.47</td>
<td>2.54</td>
<td>util/GtC</td>
</tr>
<tr>
<td>Non-Energy Biomass Consumption, ( x_1 )</td>
<td>6.5</td>
<td>4.0</td>
<td>GtC</td>
</tr>
<tr>
<td>Energy Biomass Consumption, ( x_2 )</td>
<td>0.5</td>
<td>0.4</td>
<td>GtC</td>
</tr>
</tbody>
</table>

Table 5
a. Biomass and CO₂ Stocks

The results show that under assumptions of high- and low-CO₂ intolerance, biomass quantities (703 GtC and 123 GtC, respectively) could sustain food and energy consumption (7.0 GtC/yr and 4.4 GtC/yr, respectively) indefinitely (given no other cataclysmic intervening forces, such as nuclear war, large asteroid collision, etc.). These biomass levels will also maintain carbon dioxide levels at near today’s level (772 GtC) to roughly 10% greater than today’s level. The case of low tolerance or high cost for excess CO₂ results in a biomass expansion of about 90 GtC, which translates into expanding global biomass by roughly 2 billion hectares\(^1\). The case of high tolerance of excess CO₂ (small \(r\)) results in significantly less steady state biomass, roughly one-fourth to one-fifth of today’s level.

Sedjo (1989), Sedjo and Solomon (1988), Marland (1988), Sedjo (1983), Dyson and Marland (1979) and others have published reports estimating the amount of reforestation necessary to reduce or postpone the buildup of atmospheric CO₂. The typical construct is one or more of the following potential solutions: 1) to afforest areas conducive to rapid growth forests. 2) reforest areas that have been cleared. 3) decrease the harvest rate of existing forest. or 4) increase the amount of current forest biomass. Sedjo estimated that 465 million hectares of new plantations, or about 3.5% of global arable landmass, would be required to sequester the then 2.9 GtC/yr of

---

\(^1\) Some countries (U.S., New Zealand, Australia) claimed forestation should be considered in the Kyoto agreements, arguing that taking CO₂ out of the atmosphere is an alternative to reducing emissions. European countries argued against this inclusion, however, and won.
contribution to atmospheric CO$_2$ (net of oceanic absorption). Thus, his estimate would maintain CO$_2$ at the then-current level. Dyson and Marland estimated that 700 million hectares of new plantings would be required to sequester all of the then 5.0 GtC/yr. Using the Sedjo number to scale this figure by 700/465, it corresponds to a 5.3% expansion in forestland.

b. Biomass Consumption Level

More biomass carbon is “consumed” in the high-CO$_2$ intolerance case than for low-CO$_2$ intolerance. The larger biomass level enables higher consumption levels, which are necessary to compensate for the “disutility” from additional levels of CO$_2$. The rates 4.35 GtC to 7.05 GtC “consumed” per year correspond to biomass harvested directly for both energy and non-energy purposes, much of which ends up in the atmosphere, or that which transpires between biosphere and atmosphere. This amount is then recovered to surface biomass via growth. Using the current world population figure of 6.1 billion, these quantities give a range of consumption level between 0.71-1.16 tons of biomass carbon per person per year (4.3-7.0 lb/day).

c. Imputed Prices of Biomass and CO$_2$

Since no currency units have been introduced to the model, no determination is made for imputed monetary values of CO$_2$ and biomass. The prices are in utility units, or utils, as some textbooks have labeled them. However, one may glean a certain aspect of the relative value of atmospheric carbon dioxide. If biomass is viewed as a good, while CO$_2$, because of its alleged potential to induce global warming, is typically
viewed as a *bad*, then logic would indicate a positive imputed price for biomass and a negative imputed price for CO$_2$. The prices could be interpreted as an incentive subsidy for growing more biomass, and a tax on the production of CO$_2$, respectively. Production of biomass, however, depends on CO$_2$, and thus it becomes necessary to include that value into the analysis. As may be seen from the Summary Table above, additional CO$_2$ may have a net positive value in some cases. Such cases include those when the intolerance for CO$_2$ is low (see Table 7, page 119).

**d. CO$_2$ Fertilization**

Tables 6-7 also show the results of the steady state values when the carbon dioxide fertilization factors, $\gamma_1$ and $\gamma_2$, are reduced substantially. In the case where the CO$_2$ tolerance is high, the stocks of biomass and CO$_2$ decrease slightly, while the imputed price for biomass increases and the imputed for CO$_2$ changes from slightly positive to slightly negative. This result would indicate that CO$_2$ fertilization (given that empirical studies and field experiments continue to yield positive results) will reduce the imputed cost of CO$_2$ in the atmosphere and thus an inherent value to additional CO$_2$ exists.
<table>
<thead>
<tr>
<th>Biomass Growth</th>
<th>6.15</th>
<th>6.15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gameplan, R(γ)-VVAR</td>
<td>0.90</td>
<td>1.00</td>
</tr>
<tr>
<td>Min(VAR)</td>
<td>4.84</td>
<td>4.84</td>
</tr>
</tbody>
</table>

Other Cells @ Steady State

<table>
<thead>
<tr>
<th>X2 (energy bio cons)</th>
<th>0.53 GIC</th>
<th>0.53 GIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>X1 (non-energy bio cons)</td>
<td>6.25 GIC</td>
<td>6.25 GIC</td>
</tr>
<tr>
<td>G (implied biomass price)</td>
<td>0.05 utility GIC</td>
<td>0.05 utility GIC</td>
</tr>
<tr>
<td>D (implied CO2 price)</td>
<td>-1.86 utility GIC</td>
<td>-1.86 utility GIC</td>
</tr>
<tr>
<td>Λ (atmosphere)</td>
<td>0.027 GIC</td>
<td>0.027 GIC</td>
</tr>
<tr>
<td>R (biomass)</td>
<td>0.40 GIC</td>
<td>0.40 GIC</td>
</tr>
</tbody>
</table>

Steady State Values

| Steady State Values | 0.00 GIC | 0.00 GIC |

Note...

Note...

<table>
<thead>
<tr>
<th>Parameters</th>
<th>BASE CASE Units</th>
<th>New Parameters and Resulting Values...</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gameplan</td>
<td>0.0000000000</td>
<td>0.0000000000</td>
</tr>
<tr>
<td>Gameplan</td>
<td>0.0000000000</td>
<td>0.0000000000</td>
</tr>
</tbody>
</table>

Summary

Steady State Results Summary
### Table 7

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value 1</th>
<th>Value 2</th>
<th>Value 3</th>
<th>Value 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Biomass Growth</td>
<td>10.61</td>
<td>4.29</td>
<td>0.01</td>
<td>0.82</td>
</tr>
<tr>
<td>g (non-biomass price)</td>
<td>3.94</td>
<td>3.96</td>
<td>3.97</td>
<td>3.98</td>
</tr>
<tr>
<td>X (non-biomass)</td>
<td>7.22</td>
<td>7.23</td>
<td>7.24</td>
<td>7.25</td>
</tr>
<tr>
<td>Y (atmosphere)</td>
<td>12.34</td>
<td>12.35</td>
<td>12.36</td>
<td>12.37</td>
</tr>
</tbody>
</table>

#### Energy Costs

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value 1</th>
<th>Value 2</th>
<th>Value 3</th>
<th>Value 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>DC</td>
<td>1.01</td>
<td>1.02</td>
<td>1.03</td>
<td>1.04</td>
</tr>
<tr>
<td>Price</td>
<td>2.01</td>
<td>2.02</td>
<td>2.03</td>
<td>2.04</td>
</tr>
<tr>
<td>LC</td>
<td>3.01</td>
<td>3.02</td>
<td>3.03</td>
<td>3.04</td>
</tr>
</tbody>
</table>

#### Steady State Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value 1</th>
<th>Value 2</th>
<th>Value 3</th>
<th>Value 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>V</td>
<td>6.00</td>
<td>6.01</td>
<td>6.02</td>
<td>6.03</td>
</tr>
<tr>
<td>U</td>
<td>3.00</td>
<td>3.01</td>
<td>3.02</td>
<td>3.03</td>
</tr>
<tr>
<td>G</td>
<td>4.00</td>
<td>4.01</td>
<td>4.02</td>
<td>4.03</td>
</tr>
</tbody>
</table>

#### New Parameters and Resulting Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value 1</th>
<th>Value 2</th>
<th>Value 3</th>
<th>Value 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>g</td>
<td>0.01</td>
<td>0.02</td>
<td>0.03</td>
<td>0.04</td>
</tr>
<tr>
<td>Y</td>
<td>0.05</td>
<td>0.06</td>
<td>0.07</td>
<td>0.08</td>
</tr>
<tr>
<td>X</td>
<td>0.09</td>
<td>0.10</td>
<td>0.11</td>
<td>0.12</td>
</tr>
</tbody>
</table>

#### Summary

Steady State Results
e. Optimal Path

The optimal path is inferred from a perturbation of the feasible steady state. The feasible steady state solutions were culled from the candidate solutions calculated by the Maple\textsuperscript{\textregistered} analyses. The perturbed steady states were used as a starting point to “run the system backwards”. Trajectories were graphed in different spaces for a more varied visual perception. As expected from the model structure, monotonic paths are obtained. The bias of the perturbation from steady state does not appear to determine the direction from which that path would lead to steady state. For example, a starting point that was slightly less in biomass (with a more positive associated imputed price for biomass) and slightly higher in CO\textsubscript{2} (with a more negative associated imputed price for CO\textsubscript{2}) will approach the steady state from different directions, depending the case being examined.

The graphs in Figures 9-12 (pages 122-123) show these trajectories of the state variables over a 100-year time horizon. Starting with the following combinations of perturbations.

**Perturbations from Steady State**

<table>
<thead>
<tr>
<th>Variable</th>
<th>Perturbation 1</th>
<th>Perturbation 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Biomass. ( R )</td>
<td>+0.01%</td>
<td>-0.01%</td>
</tr>
<tr>
<td>CO\textsubscript{2}. ( V )</td>
<td>-0.01%</td>
<td>+0.01%</td>
</tr>
<tr>
<td>Biomass price. ( p )</td>
<td>-1.0%</td>
<td>+1.0%</td>
</tr>
<tr>
<td>CO\textsubscript{2} price. ( q )</td>
<td>+1.0%</td>
<td>-1.0%</td>
</tr>
</tbody>
</table>

*Table 8*
the trajectory evolves over a continuous path through possible points where the system could have begun. One may determine from Figure 9 that the Perturbation 1 for low CO$_2$-intolerance leads to steady state via decreasing biomass stock (and hence decreasing consumption). Consequently, CO$_2$ stock will increases to its steady state value. The opposite behavior holds for Perturbation 2 (Figure 10). These phenomena are reversed for the high CO$_2$-intolerance case (Figures 11-12). The graph titled Biomass vs. CO$_2$ in State Space (along with others associated graphs) in Appendix G also supports these observations. Graphs of consumption, imputed prices, and state variable paths over time are also contained in Appendix G. From these graphs, one may also observe the slowly changing, monotonic behavior of these variables over the 100-year time period, as the variables approach steady state.

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$^2$ For clarity, the axes are inverted in Figures 9-10 and again in Figures 10-11.
State Variable Evolution over Time, $\tau = 0.001$, Perturbation 1

Figure 9
State Variable Evolution over Time, \( \tau = 0.001 \), Perturbation 2

Figure 10
State Variable Evolution over Time, \( \tau = 0.1 \), Perturbation 1

Figure 11
State Variable Evolution over Time, \( \tau = 0.1 \), Perturbation 2

Figure 12
4. Sensitivities

The process of solving the model established the existence of acceptable steady states under two extreme parametric assumptions. Changes were made in some of the parameters to examine sensitivities. Two objectives were sought in this process. The first goal was to exclude from further sensitivity testing those parameters which had insignificant (a somewhat arbitrary notion) elasticities. The second goal was to test whether "reasonable" variations in parameters which may be controversial in nature, e.g., the discount rate or SRTP $\delta$, the perception of CO$_2$ utility, $\tau$ (CO$_2$ intolerance), and the marginal utility of biomass $\alpha_1$, could also yield "reasonable" results. Our paper has focussed on CO$_2$ intolerance as a potentially important parameter that could become a policy variable. This determination might also establish base parameters for other studies, or provide a default scenario for mitigation activity in such a policy.

Following this scheme, the results in Tables 6-7 leads one to conclude that none of the parameters plays an insignificant role in all cases. The marginal utility of biomass, $\alpha_1$, appears to play a minor role in the low CO$_2$-intolerance case, but only in the levels of the state variables. Note that the consumption ratio $x_1/x_2$ is the same (within rounding error) in the low CO$_2$-intolerance case, but not for the other extreme. The social rate of time preference, $\delta$, critical in some studies (as mentioned in Section VA) does significantly affect most variables, but not to catastrophic degrees. An increase in the growth rate, $g$, results in much higher levels of biomass and CO$_2$ in the low-CO$_2$ intolerance case than in the high intolerance case.
Another noteworthy phenomenon is the increase in the number of real-numbered solutions found in many cases when CO$_2$ intolerance is high. In many of the cases all seven roots are real, but many can be excluded because they result in negative consumption. Those subcases are noted on the spreadsheet. However, in special subcases, some of these roots yield viable solutions, and even very close utility levels. This result would lead one to examine stability of the steady state via second order testing. Given the degree of non-linearity in the problem and the secondary importance of this level of sensitivity, it was deemed beyond the scope of the current investigation.

As will be mentioned later as a possible model modification, the fertilization parameters $\gamma_1$ and $\gamma_2$ should probably be treated as variables instead of parameters. Research discussed in Section VA showed laboratory results of a monotonic functional relationship between CO$_2$ absorption by plants and CO$_2$ concentration. Assuming this relationship as constant was purely for convenience sake. At a minimum, however, an extreme value for these parameters should be tested. This extreme was to reduce to a minimum all aspects of CO$_2$ fertilization. The steady state values were not particularly sensitive to this change, except for one of the possible solutions in the low CO$_2$ intolerance case, where the imputed price of CO$_2$ changes from positive to negative. Thus, as one would expect, the lack benefits of CO$_2$ fertilization are not seen to reduce the imputed price. A similar result is seen in the high CO$_2$ intolerance case. In general for this model, the marginal benefit of biomass
is substantially reduced when fertilization effects are reduced. Overall, it is unlikely
that no carbon dioxide fertilization occurs, although extreme levels of \( \text{CO}_2 \) will
undoubtedly threaten plant life.

C. Summary and Conclusions

Although the carbon cycle model presented and simulated in this study is abstract and
highly simplified, the results can be quickly and seamlessly interpreted, with the help
of some key assumptions. This approach can be contrasted to other research, which
typically uses hundreds of variables to account for the complex interactions of ocean,
land biomass, and atmosphere.

The effort undertaken in this study operated under a critical chain of assumptions:

- atmospheric carbon dioxide concentrations will increase
- increased atmospheric carbon dioxide concentrations will cause atmospheric
  warming
- atmospheric warming will threaten living conditions
- threatened living conditions will require measures to mitigate the threat.

No significant attempts were made to determine a conclusion to the controversy
present in the debate over global warming. Research material was presented to
introduce the reader to the concept of atmospheric warming, how it may become
manifest, and some results of previous studies on both sides, including the results
achieved in this work.
With managed deforestation, it has been shown that a steady state can be achieved in an infinite-lived system for the interaction between atmospheric carbon dioxide and terrestrial biomass after a price-based switch from fossil fuels has been made. A steady-state biomass is feasible, measurable in practical units, and is not inconsistent with other studies discussing world-scale forestation. The carbon dioxide level, also measurable in practical units, is feasible at steady state. Only in a few extreme cases do biomass and CO₂ approach critical levels. Biomass and CO₂ levels both increase under a CO₂-intolerant scenario, while biomass can be substantially reduced without increasing or decreasing CO₂ present levels.

In considering biomass expansion, to even conceive that all suitable growing areas of the earth can be covered with CO₂-absorbing biomass is unrealistic. It is, nevertheless, a possible maximum. Several other authors have shown that at a minimum, substantial reductions in carbon dioxide are possible with major expansions in biomass, especially forests.

Imputed prices of biomass and CO₂, while not in practical units, have interpretations relative to other parameters. In particular, it has been shown that under certain parametric assumptions, CO₂ is seen to be a benefit to the ecosystem. Biomass is consistently seen as a benefit in all parametric variations, except when little CO₂ fertilization can occur in a CO₂-intolerant world, and even then it is not a cost.
The model presented changed the biomass accumulation function from the widely used linear version of biotic growth to one believed to be more characteristic of living organisms. Also, while plant growth under enhanced carbon dioxide concentrations has been proven in carefully controlled laboratory experiments, its effects in field studies are not as conclusive, especially in harsher environments where other factors may limit growth. Moderate temperature increases, as would be expected in a warmed environment, enhance the fertilization effect. Despite some cautions, some CO₂ fertilization most likely will take place, and to some degree, already has taken place. We used a moderate value for this parameter, but in reality it should probably be treated as a variable. Further, if one believes that the deep ocean absorption lag is as long as several hundred years, then CO₂ sequestration via biomass expansion may be either an adaptation or a mitigation measure, which may be enhanced by CO₂ fertilization.

**D. Suggestions for Further Research and Analysis**

A model of the competitive equilibrium would be an alternative possibility for analysis. This model would need to allow for open loop feedback, starting when the initial imputed prices are set and the subsequent path observed. The equilibrium outcome could be contrasted with the optimum calculated in this thesis. A correction mechanism could be interposed at selected intervals to "guide" the system to an optimal steady state. This method has the convenience of allowing changes to the parameters and the model, which may be revised as technology evolves to better characterize variable relationships.
Several structural assumptions are made in the model, such as concavity, continuity, and differentiability of the objective function, production function constraints, and convexity of the control sets. Relaxation of some of these restrictions may allow more flexibility in model structures. Also, due to the complex nature of the multi-dimensional problem, no second derivative test was performed. Thus the stability and other characteristics of the steady state were not determined. However, the fact that the convergent path was easy to calculate and displayed no instabilities suggests that the model is stable. Most optimal control models of the type exhibit saddle point stability at the steady state.

Fossil fuels were introduced into the model as a resource that is ultimately exhausted, and consumed in both an energy sector and a non-energy sector. When fossil fuels become priced sufficiently high as a result of this exhaustion, a switch to biomass fuels occurs. Fossil fuels are effectively separated from the model system from that point in time forward. Since no specific initial stocks were considered, and imputed prices are not monetized, it was not practical to determine the optimal switching point. A more comprehensive model would incorporate these specifics and use additional estimates of monetary parameters, such as the "current prices" of biomass and fossil fuels to test whether these prices could guide the system to steady state. Iterations could be made to arrive at the appropriate prices.

Another variation in the model, using an analogy of classic growth models, would be to partition biomass harvesting into both consumption items and capital. A flow of
services would come from the capital in the form of building materials and other durable goods such as furniture. This feature would enhance the storage of carbon, since wood, once converted, typically lasts longer than fully-grown trees. Thus, a biomass storage unit could be thought of as a forest or a wood building, both delivering a flow of services in the form of amenity values.

Another aspect of the biomass which applies more to forests and woodlands is their amenity value. This value could be derived from observation of market consumption in outdoor recreation, which has expanded significantly in the baby-boomer generation. Hartman (1976) responded to Samuelson (1976), demonstrating the difference in forestry rotations when a standing forest has an amenity value. A term such as $W(\tilde{R})$ could be added to the value function, where presumably $W''(\tilde{R}) > 0$. In this case, $\tilde{R}$ would stand for the quantity of biomass that has an amenity value, and this value increases with the quantity.

The growth function used in this paper, the logistic function, models organic growth accurately in its tapering off of growth as biomass accumulates, but fails in one key respect: the growth is not specifically dependent upon carbon dioxide. A fertilization term was then added to compensate for this effect. Empirical studies have created growth curves for most species of trees, and a curve-fit model could be used in its place.
No stock effects of harvesting were introduced into the model. The harvesting function used is linear with no diminishing returns to scale. A non-linear harvesting function, such as a quadratic or concave, would more accurately reflect the effort required in realizing the gains from harvesting. Using such a function would also make the model more complicated.

As mentioned earlier in the paper (page 75), future energy fuels such as solar, nuclear, hydroelectric, geothermal, and wind may not be carbon based. Inclusion of such energy sources could be modeled as an additional set of state-costate variables and controls that use this energy.
VI. Bibliography


Denman, Kenneth L, Hofman, Eileen E, and Marchant H; (1996) “Marine Biotic Responses to Environmental Change and Feedbacks to Climate” in IPCC Working Group I. Chapter 10


Edmonds, Jae A; (1992) “Understanding Natural Sinks and Sources of CO\textsubscript{2}”. *Water, Air, and Soil Pollution*. v 62. no 1/2. August, pp 11-21


Forster, Bruce A; (1973) “Optimal Consumption Planning in a Polluted Environment”. Economic Record. v 49, December, pp 534-545


Hartley, Peter; (1997) “Can International Tradeable Carbon Dioxide Emission Quotas Work ?”. presented to the Tasman Institute


Koopmans, Tjalling C; (1964) “Economic Growth at a Maximal Rate”. Quarterly Journal of Economics, vol 78, no 3, August, pp 355-394


Lotka, A J; (1925) Elements of Physical Biology. Baltimore MD: Williams and Wilkins


Manawar, P B; (1945) "Size, Shape, and Age", in Thompson, D'Arcy W; *Essays on Growth and Form*. Oxford: Clarendon Press, pp 157-187


Marland, Gregg, and Marland, Scott; (1992) "Should We Store Carbon in Trees?", *Water, Air and Soil Pollution*, v 64, no 1-2, August, pp 181-195


Pethig, Rüdiger ed; (1992) Conflicts and Cooperation in Managing Environmental Resources. New York: Springer-Verlag


Pigou, Arthur C; (1920) The Economics of Welfare. London: Macmillan


Poorter, H; (1993) “Interspecific Variation in the Growth Response of Plants to an Elevated CO₂ Concentration”. Vegetatio. v 104/105. pp 77-97


Richardson, Lewis F; (1960) *Arms and Insecurity*. Chicago: Boxwood Press


Sandler, Todd; (1992b) “After the Cold War, Secure the Global Commons”, *Challenge*. July-August, pp 16-23


Verhulst, P F; (1838) “Notice sur la loi que la population suit dans son accroissement”. Correspondence Mathématique et Physique. v 10, pp113-121


Appendix A: Selected Climate and Impact Study Organizations\(^1\)

The Intergovernmental Panel on Climate Change (IPCC)

http://www.ipcc.ch

The IPCC was jointly established by the World Meteorological Organization (WMO) and the United Nations Environmental Programme (UNEP) in 1988, in order to: (i) assess available scientific information on climate change, (ii) assess the environmental and socio-economic impacts of climate change, and (iii) formulate response strategies. Three working groups were established, originally titled the Scientific Working Group (Working Group I), the Impacts Assessment Working Group (Working Group II), and the Response Strategies Working Group (Working III). In 1992-1993, the IPCC reorganized WGII and WGIII, and committed to the 1995 Second Assessment Report. WGII continued to analyze the functioning of the climate change system and its potential changes resulting from human activities. WGII was to assess potential impacts of climate change, as well as adaptation strategies and measures for reducing GHG emissions. WGIII was to focus on evaluating economic implications of climate change, including assessments of potential economic damages, costs of reducing emissions, and the applicability of cost-benefit analysis to decision making.

At the IPCC Thirteenth Session in late September 1997, WGII and WGIII were restructured again, in preparation for the Third Assessment Report, due in 2000. "WGII now assesses the scientific, technical, environmental, economic, and social aspects of the vulnerability (sensitivity and adaptability) to climate change of, and the negative and positive consequences (impacts) for, ecological systems, socio-economic sectors and human health, with an emphasis on regional sectoral and cross-sectoral issues. WGIII assesses the scientific, technical, environmental, economic, and social aspects of the mitigation of climate change, and a through a task group (multidisciplinary team), will assess the methodological aspects of cross-cutting issues (e.g., equity, discount rates and decision-making frameworks)." USGCRP (1997)

World Meteorological Organization (WMO)

http://www.wmo.ch

From weather prediction to air pollution research, climate change related activities, ozone layer depletion studies and tropical storm forecasting, the World Meteorological Organization coordinates global scientific activity to allow increasingly prompt and accurate weather information and other services for public, private and commercial use, including international airline and shipping industries. WMO's activities contribute to the safety of life and property, the socio-economic development of nations and the protection of the environment.

Within the United Nations, the Geneva-based 185-Member Organization provides the authoritative scientific voice on the state and behaviour of the Earth's atmosphere and climate.

The World Meteorological Convention, by which the World Meteorological Organization was created, was adopted at the Twelfth Conference of Directors of the International Meteorological Organization (IMO), which met in Washington in 1947. Although the Convention itself came into force in 1950, WMO commenced operations as the successor to IMO in 1951 and, later that year, was established as a specialized agency of the United Nations by agreement between the UN and WMO.

\(^1\) Except for abridging the text on IPCC, the organization texts were copied from the associated Web sites.
The purposes of WMO are to facilitate international cooperation in the establishment of networks of stations for making meteorological, hydrological and other observations; and to promote the rapid exchange of meteorological information, the standardization of meteorological observations and the uniform publication of observations and statistics. It also furthers the application of meteorology to aviation, shipping, water problems, agriculture and other human activities. promotes operational hydrology and encourages research and training in meteorology.

**United Nations Environmental Programme (UNEP)**

http://www.unep.org

The United Nations Environment Programme (UNEP) is built on a heritage of service to the environment. As one of the productive consequences of the 1972 Stockholm Conference on the Human Environment, UNEP provides an integrative and interactive mechanism through which a large number of separate efforts by intergovernmental, non-governmental, national and regional bodies in the service of the environment are reinforced and interrelated. UNEP was established as the environmental conscience of the United Nations system, and has been creating a basis for comprehensive consideration and coordinated action within the UN on the problems of the human environment.

Today, 24 years after the Stockholm Conference and four years after UNCED, the challenge before UNEP is to further catalyze, promote and implement an environmental agenda that is integrated strategically with the goals of economic development and social well-being - an agenda for sustainable development. emphasizes relationships between socio-economic driving forces, environmental changes and impacts on human well being. Equipped with stronger regional presence and marked by a process of continuous monitoring and assessment of its implementation, UNEP's programme of work for 1996-97 focuses on the following areas: sustainable management and use of natural resources sustainable production and consumption, a better environment for human health and well-being; and globalization of the economy and the environment.

**International Energy Agency (IEA)**

http://www.iea.org

The IEA, based in Paris, is an autonomous agency linked with the Organisation for Economic Co-operation and Development (OECD). The 24 member countries of the International Energy Agency (IEA) seek to create the conditions in which the energy sectors of their economies can make the fullest possible contribution to sustainable economic development and the well-being of their people and of the environment. In formulating energy policies, the establishment of free and open markets is a fundamental point of departure, though energy security and environmental protection need to be given particular emphasis by governments. IEA countries recognise the significance of increasing global interdependence in energy. They therefore seek to promote the effective operation of international energy markets and encourage dialogue with all participants.
United States Global Climate Research Program (USGCRP)

http://www.usgcrp.org

The USGCRP was created as a Presidential Initiative in 1989 and formalized in 1990 by the Global Climate Change Research Act of 1990. Since that time, global change research has remained a key science initiative. Continuing to improve scientific understanding of the Earth system is a priority of the National Science and Technology Council’s Committee on Environment and Natural Sciences.

Global change research brings significant benefits to the nation and the world by providing a well-founded scientific understanding of the Earth system to ensure the availability of future resources essential for human well-being, including water, food, fiber, ecosystems, and human health. The U.S. Global Change Research Program (USGCRP), working with research institutions in the U.S. and beyond our borders, provides the foundation for increasing the skill of predictions of seasonal-to-interannual climate fluctuations (which can bring excessively wet and dry periods) and long-term climate change. The USGCRP also sponsors research to understand the vulnerabilities to changes in important environmental factors, including changes in climate, ultraviolet (UV) radiation at the Earth's surface, and land cover. Scientific knowledge is essential for informed decision making on environmental issues and to ensure the social and economic health of future generations. Thus global change research is a critical investment for the future of this nation, its economy, and the health and safety of its citizens.

Carbon Dioxide Information Analysis Center (CDIAC)

http://www.cdiac.org

The Carbon Dioxide Information Analysis Center (CDIAC), located within the Environmental Sciences Division of Oak Ridge National Laboratory in Oak Ridge, Tennessee, provides information to help international researchers, policymakers, and educators evaluate complex environmental issues associated with elevated levels of atmospheric carbon dioxide (CO2) and other radiatively active trace gases, including potential climate change. CDIAC is sponsored by the U.S. Department of Energy's (DOE) Global Change Research Program (GCRP). Michelle Broido is the Director of DOE's Environmental Sciences Division and Program Manager of GCRP. Bobbi Apr is the GCRP program manager with responsibility for CDIAC. Robert M. Cushion is Director of CDIAC. In operation since 1982, CDIAC (1) obtains, evaluates, and archives data,(2) compiles and distributes digital numeric data packages and computer model packages,(3) distributes related reports,(4) produces the newsletter, CDIAC Communications, and(5) in general acts as the information focus for the U.S. DOE research programs. Since its inception, CDIAC has responded to thousands of requests for information, and since 1985 has distributed more than 95,000 reports and numeric data packages to over 110 countries worldwide. The center's staff members explore both general and technical aspects of issues related to carbon dioxide, methane, and other trace gas emissions; the carbon cycle; and other climate-change topics. The center's eclectic data holdings are related to the issues of concern rather than the specific scientific disciplines.
European Science and Environmental Foundation (ESEF)

http://www.esef.org

The European Science and Environment Forum is an independent, non-profit-making alliance of scientists whose aim is to ensure that environmental debates are properly aired, and that decisions which are taken, and action that is proposed, are founded on sound scientific principles.

The ESEF will be particularly concerned to address issues where it appears that the public and their representatives, and those in the media, are being given misleading or one-sided advice. In such instances the ESEF will seek to provide a platform for scientists whose views are not being heard, but who have a contribution to make.

Members are accepted from all walks of life and all branches of science. There is no membership fee. Members will be expected to offer their services in contributing to ESEF publications on issues where their expertise is germane.

To maintain its independence and impartiality, the ESEF does not accept outside funding from whatever source, the only income it receives is from the sale of its publications. Such publications will automatically be sent to members. Copies will be sent to selected opinion formers within the media and within government.

Science and Environmental Policy Project (SEPP)

http://www.sepp.org

The Science & Environmental Policy Project was founded in 1990 by atmospheric physicist S. Fred Singer on the premise that sound, credible science must form the basis for health and environmental decisions that affect millions of people and cost tens of billions of dollars every year. A non-profit 501(c)3 educational group, its mission was to clarify the diverse problems facing the planet and, where necessary, arrive at effective, cost-conscious solutions.

Over the years, SEPP’s authoritative critiques of UN documents used to shore up the Climate Treaty negotiated at the 1992 Rio de Janeiro "Earth Summit" have been widely quoted. Its debunking of NASA’s announcement of "record" chlorine in the Arctic stratosphere (the "ozone hole over Kennebunkport") attracted the attention of the press and Congress. The Project has been tapped by both Democrats and Republicans on Capitol Hill for objective, science-based information on global warming, ozone depletion, chemical risk, clean air standards, and other issues. The Project has been cited hundreds of times by the major news media. Articles and editorials by SEPP-affiliated scientists have been published in leading journals and newspapers, including the Wall Street Journal, Miami Herald, Detroit News, Chicago Tribune, Cleveland Plain Dealer, Memphis Commercial-Appeal, Seattle Times, Orange County Register, The Bulletin of the Atomic Scientists, New Straits Times (Malaysia), and Finanz und Wirtschaft (Switzerland), among many others.
Commonwealth Scientific and Industrial Research Organisation

http://www.csiro.au

Australia's Commonwealth Scientific and Industrial Research Organisation (CSIRO) is an independent statutory authority constituted and operating under the provisions of the Science and Industry Research Act 1949. CSIRO's current structure was established by the Science and Industry Research Amendment Act 1986. This act established a ten-member CSIRO Board responsible for determining policy and ensuring the efficient functioning of CSIRO. The Chief Executive, who is a member of the Board, is responsible for the Organisation's activities. The Chief Executive and four Deputy Chief Executives form the Executive Committee, which assists the Chief Executive in managing the activities of the Organisation.

CSIRO staff are employed under Section 32 of the Science and Industry Research Act 1949. At 30 June 1996 CSIRO had a total staff of 7497, including nearly 3000 scientists, working in laboratories and field stations around Australia. Their work covers a broad range of areas of economic or social value to the nation, including agriculture, minerals and energy, manufacturing, communications, construction, health and the environment. The emphasis is on bringing teams together from different scientific fields to find solutions to major problems facing Australia. CSIRO is an agency in the Department of Industry, Science and Tourism government portfolio under the Minister for Industry, Science and Tourism.

Competitive Enterprise Institute

http://cei.org

The Competitive Enterprise Institute (CEI) is a pro-market, public policy group committed to advancing the principles of free enterprise and limited government. Founded in 1984 by Fred L. Smith, Jr., CEI emphasizes the marketing and implementation of classical liberal ideals.

CEI utilizes a five-point management approach to affecting public policy: analysis, education, coalition building, advocacy and litigation. Its purpose is to advance the free-market agenda, believing limited government and competition best serve the public interest.

A non-profit, tax-exempt organization under Section 501(c)(3) of the Internal Revenue Code, CEI relies entirely on donations from corporations, foundations, and private individuals with an interest in restoring individual liberties and economic freedom.

Cato Institute

http://www.cato.org

Founded in 1977, the Cato Institute is a nonpartisan public policy research foundation headquartered in Washington, D.C. The Institute is named for Cato's Letters, libertarian pamphlets that helped lay the philosophical foundation for the American Revolution.

The Cato Institute seeks to broaden the parameters of public policy debate to allow consideration of more options that are consistent with the traditional American principles of limited government, individual liberty, and peace. Toward that goal, the Institute strives to achieve greater involvement of the intelligent, concerned lay public in questions of policy and the proper role of government.
Center for the Study of Carbon Dioxide and Global Change

http://www.co2science.org

The Center for the Study of Carbon Dioxide and Global Change was created to disseminate factual reports and sound commentary on new developments in the world-wide scientific quest to determine the climatic and biological consequences of the ongoing rise in the air's CO2 content. This it does through mini-reviews of recently published peer-reviewed scientific journal articles, books, and other educational materials. In this endeavor it attempts to separate reality from rhetoric in the emotionally charged debate that swirls around the subject of carbon dioxide and global change. The Center additionally produces editorials on topics of current concern and highly publicized claims that are designed to determine if they are fact or fiction. It also answers particularly good questions from the public that address pertinent scientific subjects. In addition, to help students and teachers gain greater insight into the biological aspects of potential global change, the Center maintains on-line instructions on how to conduct CO2 enrichment and depletion experiments in its Global Change Laboratory (located in its Experiments section), which allow interested parties to conduct similar studies in their own homes and classrooms.
Appendix B

Maple Analyses

- The procedures contained in this appendix were performed with Maple V® Release 4 on a Microsoft Windows 98®-based personal computer. The first case is $\tau = 0.1$, the second is $\tau = 0.001$. 
> restart:
> #CASE tau = 0.1:
> #Begin with the two steady-state state variable equations in
> biomass and CO2, i.e., (4.24, 4.25):
> Vdot := a*x1+x2=gamma2*R*(V-Vh)+mu*(V-Vh);
> Vdot := a x1 + x2 = γ2 R (V - Vh) + μ (V - Vh)
> Rdot := gamma1*(V-Vh)*R+g*R-(g/A)*R^2=x1+x2:
> Rdot := γ1 (V − Vh) R + g R − \frac{g R^2}{A} = x1 + x2
> #Recall the First Order Condition (4.12) and First Order Condition
> (4.13) which was modified by Kuhn-Tucker conditions and assumption
> y2=0 after the switching point to (4.13a):
> x1 := alphal/(q-a*p):
> x1 := \frac{α1}{q - a p}
> x2 := l/(q-p):
> x2 := \frac{l}{q - p}
> #Substitute these FOCs into the state equations to reproduce
> equations (4.32,4.33):
> Vdot:
> \frac{α1}{q - a p} + \frac{1}{q - p} = γ2 R (V - Vh) + μ (V - Vh)
> Rdot:
> γ1 (V − Vh) R + g R − \frac{g R^2}{A} = \frac{α1}{q - a p} + \frac{1}{q - p}
> #Recall the expressions for the imputed prices 4.30 4.31 solved
> by using Kramer's Rule:
> p := -delta*tau+gamma1*tau*gamma2*(V-Vh)+tau*g*(1-(2/A)*R) / dt:
> p := -δ τ + γ1 τ (V − Vh) + τ g \left(1 - 2 \frac{R}{A}\right)
> q := tau*gamma2*(V-Vh) / dt:
> q := τ γ2 (V − Vh)
> #The denominator in the preceeding two equations is the
> determinant of the matrix formed by the co-state equations and is
> equivalent to (4.29):
> dt := delta^2-gamma2*delta*R+mu*delta-g*delta-gamma1*delta*(V-Vh)-mu
> *gamma2*(V-Vh)-g*gamma2*R-mu*g+2*gamma2*(g/A)*R^2+2*mu*(g/A)*R+2*delta
> *gamma1*(g/A)*R;

Page 1
\[ \textstyle{dt := \delta^2 + \gamma_2 \delta R + \mu \delta - g \delta - \gamma_1 \delta (V - Vh) - \mu \gamma_1 (V - Vh) - g \gamma_2 R - \mu g + 2 \frac{\gamma_2 g R^2}{A}} + 2 \frac{\mu g R}{A} + 2 \frac{\delta g R}{A} \]

> # The expressions for the imputed prices are substituted into the expressions for Vdot and Rdot to obtain two equations in two unknowns, R, V, and parameters only:

> Vdot:

\[ \textstyle{a \alpha 1 \frac{\tau \gamma_2 (V - Vh)}{\alpha 1} - \frac{\alpha \left(-\delta \tau + \gamma_1 \tau (V - Vh) + \tau g \left(1 - 2 \frac{R}{A}\right)\right)}{\alpha 1}} = \gamma_2 R (V - Vh) + \mu (V - Vh) \]

\[ \textstyle{\frac{\alpha 1}{\alpha 1} \frac{\tau \gamma_2 (V - Vh)}{\alpha 1} - \frac{\alpha \left(-\delta \tau + \gamma_1 \tau (V - Vh) + \tau g \left(1 - 2 \frac{R}{A}\right)\right)}{\alpha 1}} = \gamma_2 R (V - Vh) + \mu (V - Vh) \]

\[ \alpha 1 := \delta^2 + \gamma_2 \delta R + \mu \delta - g \delta - \gamma_1 \delta (V - Vh) - \mu \gamma_1 (V - Vh) - g \gamma_2 R - \mu g + 2 \frac{\gamma_2 g R^2}{A} + 2 \frac{\mu g R}{A} + 2 \frac{\delta g R}{A} \]

> Rdot:

\[ \textstyle{\gamma_1 (V - Vh) R + g R - \frac{g R^2}{A} = \frac{\alpha 1 \frac{\tau \gamma_2 (V - Vh)}{\alpha 1} - \frac{\alpha \left(-\delta \tau + \gamma_1 \tau (V - Vh) + \tau g \left(1 - 2 \frac{R}{A}\right)\right)}{\alpha 1}}{\alpha 1}} \]

\[ \textstyle{\frac{\alpha 1}{\alpha 1} \frac{\tau \gamma_2 (V - Vh)}{\alpha 1} - \frac{\alpha \left(-\delta \tau + \gamma_1 \tau (V - Vh) + \tau g \left(1 - 2 \frac{R}{A}\right)\right)}{\alpha 1}} = \gamma_2 R (V - Vh) + \mu (V - Vh) \]

\[ \alpha 1 := \delta^2 + \gamma_2 \delta R + \mu \delta - g \delta - \gamma_1 \delta (V - Vh) - \mu \gamma_1 (V - Vh) - g \gamma_2 R - \mu g + 2 \frac{\gamma_2 g R^2}{A} + 2 \frac{\mu g R}{A} + 2 \frac{\delta g R}{A} \]

> # Computations begin after substituting parameters:

> alpha1 := 10.0:

\[ \alpha 1 := 10.0 \]

> a := 0.8:

\[ a := .8 \]

> gamma1 := 0.0000053:

\[ \gamma_1 := .53 \times 10^{-5} \]

> gamma2 := 0.0000053:
\( \gamma_2 := 0.53 \times 10^{-5} \)

\( \mu := 0.02 \)

\( g := 0.04 \cdot \)

\( \tau := 0.1 \cdot \)

\( \lambda := 900 \cdot \)

\( \delta := 0.03 \cdot \)

\( V_h := 600 \cdot \)

\( V_h := 600 \)

> The equations are now set forth in terms of numeric values and the two unknowns, \( R \) and \( V \)

\( \frac{\partial t}{\partial t} = \frac{\text{.000341000 + .4391444444} \times 10^{-5} R - .265 \times 10^{-6} V + .4711111110 \times 10^{-9} R^2}{\text{.000341000 + .4391444444} \times 10^{-5} R - .265 \times 10^{-6} V + .4711111110 \times 10^{-9} R^2} \)

\( \frac{\partial R}{\partial t} = \frac{V - 600}{\text{.000341000 + .4391444444} \times 10^{-5} R - .265 \times 10^{-6} V + .4711111110 \times 10^{-9} R^2} \)

\( \frac{\partial V}{\partial t} = \frac{8.00}{\text{.00068200 + .53} \times 10^{-6} V - .8888888888 \times 10^{-5} R} \)

\( \frac{\text{.53} \times 10^{-6} V - 600}{\text{.00068200 + .53} \times 10^{-6} V - .8888888888 \times 10^{-5} R} - \frac{1}{\text{.53} \times 10^{-6} V - 600} \cdot \frac{\text{.00068200 + .53} \times 10^{-6} V - .8888888888 \times 10^{-5} R}{\text{.53} \times 10^{-5} R (V - 600) + .02 V - 12.00} \)

\( \%1 := \text{-0.000341000 + .4391444444} \times 10^{-5} R - .265 \times 10^{-6} V + .4711111110 \times 10^{-9} R^2 \)

\( \text{Rdot} := \text{.53} \times 10^{-5} R (V - 600) + .04 R - \text{.00004444444444} \cdot R^2 = 10.0 \)

\( \frac{\text{.53} \times 10^{-6} V - 600}{\text{.00068200 + .53} \times 10^{-6} V - .8888888888 \times 10^{-5} R} - \frac{1}{\text{.53} \times 10^{-6} V - 600} \cdot \frac{\text{.00068200 + .53} \times 10^{-6} V - .8888888888 \times 10^{-5} R}{\text{.53} \times 10^{-5} R (V - 600) + .02 V - 12.00} \)

\( \%1 := \text{-0.000341000 + .4391444444} \times 10^{-5} R - .265 \times 10^{-6} V + .4711111110 \times 10^{-9} R^2 \)
> Finally, the solver is asked to determine values for the two unknowns in two equations:
> h1 := solve({Vdot,Rdot},{V,R}):
> h1 := \{R = -38520.61159, \} V' = .2592364035 10^7 \}, \{ V' = 38590.84045, R = 3773.554231 \}, \{ V' = 1067.899732, R = 112.0480728 \}, \{ V' = 600.0000000, R = 112.5000000 \},
> \{ R = 150.2609126 - 35.34031384 1, \} V' = 805.0049597 - 37.64752519 1, \} V' = 805.0049597 + 37.64752519 1, \} V' = 842.0711086, R = 703.4013343 \}
> # Note the existence of seven solutions. Any solution which contains negative numbers can be excluded due to state variable constraint violation. The solution (R=703.4, V=842.1) is analyzed later, returns negative values for a control variable, and is excluded due to control variable constraint violation:
> # The other steady-state variables can now be found and tested with the candidate solution:
> R := 703.4013343:
> R := 703.4013343
> V := 842.0711086:
> V := 842.0711086
> Rdot:
> 7.04857048 = 7.048570481
> Vdot:
> 5.743869816 = 5.743869818
> dt:
> .002757892301
> p:
> -1.857998086
> q:
> .04652019499
> x1:
> 6.523503324
> x2:
> .5250671574
>
#CASE \ tau = 0.001

#Begin with the two steady-state state variable equations in biomass and CO2, i.e., (4.24, 4.25):

\begin{align*}
V & = x_1 + x_2 = \gamma_2 R (I' - Vh) + \mu (I' - Vh) \\
R & = \gamma_1 (V' - Vh) + g (g - (g/\gamma_1)) R^2 = x_1 + x_2
\end{align*}

#Recall the First Order Condition (4.12) and First Order Condition (4.13) which was modified by Kuhn-Tucker conditions and assumption \( \gamma_2 = 0 \) after the switching point to (4.13a):

\begin{align*}
x_1 & = \alpha_1 / (q - a \cdot p) \\
x_2 & = 1 / (q - p)
\end{align*}

#Substitute these FOCs into the state equations to reproduce equations (4.32, 4.33):

\begin{align*}
\dot{V} & = \frac{a \cdot \alpha_1}{q - a \cdot p} + \frac{1}{q - p} = \gamma_2 R (I' - Vh) + \mu (I' - Vh) \\
\dot{R} & = \gamma_1 (I' - Vh) + g R - \frac{g R^2}{A} = \frac{\alpha_1}{q - a \cdot p} - \frac{1}{q - p}
\end{align*}

#Recall the expressions for the imputed prices (4.30, 4.31). Solved by using Cramer's Rule:

\begin{align*}
p & = (-\delta \tau + \gamma_1 \tau (I' - Vh) + \tau g (1 - 2/R)) / \delta \\
q & = \tau \gamma_2 (I' - Vh)
\end{align*}

#The denominator in the preceding two equations is the determinant of the matrix formed by the co-state equations and is equivalent to (4.29):

\begin{align*}
\delta t & = \delta + \gamma_1 \tau (I' - Vh) + \tau g (1 - 2/R) \\
\frac{\tau \gamma_2 (I' - Vh)}{dt}
\end{align*}
\[
\frac{\partial t}{\partial \tau} = \delta^2 + \gamma_2 \delta R + \mu \delta - g \delta - \gamma_1 \delta (V - V') - \mu \gamma_1 (V - V') - g \gamma_2 R - \mu g + 2 \frac{\gamma_2 g R^2}{A} + 2 \frac{\mu g R}{A} + 2 \frac{\delta g R}{A}.
\]

The expressions for the imputed prices are substituted into the expressions for Vdot and Rdot to obtain two equations in two unknowns, R and V, and parameters only:

\[\text{Vdot:}\]

\[
\frac{a \alpha}{\tau \gamma_2 (V - V')} = \frac{a \left(-\delta \tau + \gamma_1 \tau (V - V') + \tau g \left(1 - 2 \frac{R}{A}\right)\right)}{\gamma_2 R (V - V') + \mu (V - V')} + \frac{\gamma_2 g R^2}{A} + 2 \frac{\mu g R}{A} + 2 \frac{\delta g R}{A}.
\]

\[\text{Rdot:}\]

\[
\gamma_1 (V - V') R + g R - \frac{g R^2}{A} = \frac{\alpha}{\tau \gamma_2 (V - V')} - \frac{a \left(-\delta \tau + \gamma_1 \tau (V - V') + \tau g \left(1 - 2 \frac{R}{A}\right)\right)}{\gamma_2 R (V - V') + \mu (V - V') \gamma_2 R - \mu g + 2 \gamma_2 g R^2} + \frac{2 \mu g R}{A} + 2 \frac{\delta g R}{A}.
\]

Computations begin after substituting parameters:

\[\alpha_1 := 100.0;\]

\[\alpha := 10.0;\]

\[a := 0.3;\]

\[\gamma_1 := 0.0000053;\]

\[\gamma_1 := 53 \times 10^{-5};\]

\[\gamma_1 := 0.0000053;\]
\( \gamma_2 := 0.02 \)
\( \mu := 0.02 \)
\( g := 0.04 \)
\( \tau := 0.01 \)
\( \Delta := 900 \)
\( \delta := 0.03 \)
\( \nu h := 600 \)

The equations are now set forth in terms of numeric values and the two unknowns, \( R \) and \( V \)

\[ \frac{\text{d}t}{\text{d}t} = \frac{-0.00341000 + 0.4391444444 \times 10^5 R - 265 \times 10^{-6} \nu + 0.4711111110 \times 10^{-9} R^2}{\text{d}t} \]

\[ p = \frac{0.68200 \times 10^5 + 0.53 \times 10^8 \nu - \ldots 10^{-7} R}{-0.00341000 + 0.4391444444 \times 10^5 R - 265 \times 10^{-6} \nu + 0.4711111110 \times 10^{-9} R^2} \]

\[ q = \frac{0.53 \times 10^{-8} (\nu - 600)}{-0.00341000 + 0.4391444444 \times 10^5 R - 265 \times 10^{-6} \nu + 0.4711111110 \times 10^{-9} R^2} \]

\[ \nu \dot{v} = 8.00 \]

\[ \frac{0.53 \times 10^{-8} (\nu - 600)}{0.01} - 0.8 \frac{0.68200 \times 10^5 + 0.53 \times 10^8 \nu - \ldots 10^{-7} R}{0.01} = \]

\[ 0.53 \times 10^{-5} R (\nu - 600) + 0.02 \nu - 12.00 \]

\[ \%1 := -0.00341000 + 0.4391444444 \times 10^5 R - 265 \times 10^{-6} \nu + 0.4711111110 \times 10^{-9} R^2 \]

\[ R \dot{v} = 10.0 \]

\[ \frac{0.53 \times 10^{-8} (\nu - 600)}{0.01} - 0.8 \frac{0.68200 \times 10^5 + 0.53 \times 10^8 \nu - \ldots 10^{-7} R}{0.01} = \]

\[ 0.53 \times 10^{-5} R (\nu - 600) + 0.04 R - 0.0004444444444 \times 10^2 \]

Page 3
> Finally, the solver is asked to determine values for the two unknowns in two equations:
> h1 := solve({Vdot, Rdot}, {V, R});
> h1 := [R = -38476.23910, V = .2591875404 10^7],
>      {R = 82.13439972, V = 28240.02427},
>      {R = 122.5412576 - 3694.017184 I, V = 25346.05745 + 147.4642373 I},
>      {V = 25346.05745 - 147.4642373 I, R = 122.5412576 + 3694.017184 I},
>      {R = 122.7289009, V = 772.4270071];
> #Note the existence of seven solutions. Any solution which contains negative numbers can be excluded due to state variable constraint violation. The solution (R=122.7, V=772.4) is analyzed later, returns negative values for a control variable, and is excluded due to control variable constraint violation:
> #The other steady-state variables can now be found and tested with a the candidate solution:
> R := 122.7289009;
> V := 772.4270071;
> Rdot := 4.351874205 = 4.351870413
> Vdot := 3.560694459 = 3.560697561
> it := .3600488 10^6
> p := .01285245222
> q := 2.538164653
> x1 := 3.955879771
> x2 := .3959906421
Appendix C

MatLab Programs

- The programs contained in this appendix were performed using Matlab® version 5.0 (Alpha 5) on a Microsoft Windows 98®-based personal computer. The ccycle.m program calls the parameter file (parms.m) and the differential equation processing program (sequest.m). In all cases, the function ccycle(tfinal) used the value 100 as an argument for the number of years.
These parameters are generated from analysis discussed in Section VA, and are identical to those used in the Maple base case analysis.

\[
\begin{align*}
\alpha_1 &= 10.0; \\
a &= 0.8; \\
\gamma_1 &= 0.0000053; \\
\gamma_2 &= 0.0000053; \\
\mu &= 0.02; \\
g &= 0.04; \\
\Delta &= 900; \\
\delta &= 0.03; \\
V_h &= 600;
\end{align*}
\]

\text{Turn on the particular case}

\[
\begin{align*}
\tau &= 0.001; \\
R_{ss} &= 122.729; \\
V_{ss} &= 772.427; \\
\tau &= 0.1; \\
R_{ss} &= 703.4013; \\
V_{ss} &= 942.0711;
\end{align*}
\]

\text{// Steady-state conditions}

\[
\begin{align*}
\frac{dg}{dt} &= g + g_2 + R_{ss} + \mu \frac{dg}{dt} - \mu g_1 \frac{dg}{dt} - g g_2 R_{ss} - g g_2 R_{ss}^2 + 2 \mu R_{ss} + 2 \mu g_1 R_{ss} \\
qss &= \frac{\tau g g_2 (Vss - Vh)}{dt}; \\
pss &= \frac{-\tau g g_2 (Vss - Vh) + \tau g^2 - \tau (g/A) Rss}{dt}; \\
rss &= \frac{\tau g g_1 Rss (Vss - Vh)}{dt}; \\
xiss &= \frac{\alpha_1}{qss + pss}; \\
xiss &= \frac{1}{qss - pss};
\end{align*}
\]
%This program segment processes the differential equations
function yp = sequest(t,y);

parms

R = y(1);
q = y(2);
V = y(3);
p = y(4);

x1 = alphl/(q - a*p);
x2 = 1/(q - p);

dVdt = - mu*(V-Vh) - gam2*R*(V-Vh) + a*x1 + x2;
JRdt = gaml*(V-Vh)^2*R + q*R - (q/A)^2*R^2 - x1 - x2;

dqdt = p*gam2*(V-Vh) + q*(delt - q*(1-(2*R/A)) - gaml*(V-Vh));
dpdt = p*(delt - mu + gam2*R) - q*gaml*R + tau;

yp(1) = dRdt;
yp(2) = dqdt;
yp(3) = dVdt;
yp(4) = dpdt;

yp = yp';
This program segment controls the program flow and generates graphs.

function ccycle(tfinal)

parms

\% Define initial conditions,
\% If p and q are known initially,
\% one may run with these conditions
\% y0 = [R0, q0, V0, p0];
\% To determine path from equilibrium,
\% equilibrium must be perturbed
\% Turn on the perturbation bias and change figure labels below
\% The following corresponds to Perturbation 1
perturbation = [+.0001*Rss, -.01*qss, -.0001*Vss, -.01*pss];
\% The following corresponds to Perturbation 2
\% perturbation = [-.0001*Rss, -.01*qss, +.0001*Vss, -.01*pss];

y0 = [Rss, qss, Vss, pss] + perturbation;
tspan = t0:1:tfinal;

tspan = -tspan;
[t,y] = ode45('sequest',tspan,y0);

R = y(:,1);
q = y(:,2);
V = y(:,3);
p = y(:,4);

x1 = alph1./[q - a*p] +
x2 = 1./[q - p] +
pqRn

(The axes in Figure 1 are reversed for each case to better depict the trajectory figure1)

clf
plot(t,R,V)
title('State Variable Evolution over Time, $\tau = 0.001$, Perturbation 1')
xlabel('-t')
ylabel('V')
zlabel('R')
text(0.122,772.4,'Steady State Value')
text(0.842.1,703.4,'Steady State Value')

figure2)
clf
plot([R,V])
title('Biomass vs CO2 in State Space, $\tau = 0.001$, Perturbation 1')
xlabel('R')
ylabel('V')

figure3)
clf
plot(t,q,t,p,'-')
title('Imputed Prices over Time, $\tau = 0.001$, Perturbation 1')
xlabel('t')
ylabel('p, q')
legend('q', 'p')

figure(4)
clf
plot(t, R, t, V, '--');
title('State Variables over Time, tau = 0.001, Perturbation 1')
xlabel('t')
ylabel('R, V')
legend('R', 'V')

% diagram : x vs t
figure(5)
clf
plot(t, x1, t, x2, ':');
title('Consumption Paths over Time, tau = 0.001, Perturbation 1')
xlabel('t')
ylabel('x1, x2')
legend('x1', 'x2')
Appendix D

MathCAD Worksheets

- The worksheets contained in this appendix were performed using MathCAD®8 on a Microsoft Windows 98®-based personal computer. The graphs on the second two pages correspond to the equations for the variants of the logistic function, listed on the first page.
Variants of the Logistic Function
Comparison of Accumulation Function

\[ R_0 := .05 \quad R := 0, .05, 1 \quad g := .08 \quad A := 1 \quad t := 0, 1, 60 \]

\[ \text{Logistic} \]
\[ R_1(t) := \frac{\frac{A}{R_0}}{1 + \frac{A}{R_0} - 1 \cdot \exp(-g \cdot t)} \]

\[ \text{von Bertalanffy} \]
\[ R_3(t) := \left[ \frac{1}{A^3} - \frac{1}{A^3} - \frac{1}{R_0^3} \cdot \exp(-g \cdot t) \right]^3 \]

\[ \text{Gompertz} \]
\[ R_2(t) := R_0 \exp(-g \cdot t) \]

\[ \text{Concave} \]
\[ R_4(t) := A + (R_0 - A) \cdot \exp\left(-\frac{g}{A} \cdot t\right) \]

\[ D_1(R) := g \cdot R - \frac{g}{A} \cdot R^2 \]

\[ D_2(R) := g - g \cdot R \cdot \ln(R) \]

\[ D_3(R) := 3 \cdot g \cdot R^2 \cdot (1 - R^\frac{1}{3}) \]

\[ D_4(R) := g - \frac{g}{A} \cdot R \]
\[ g := 0.08 \quad X := 0.05 \ldots 1 \quad A := 1 \quad t := 0, 1 \ldots 100 \quad X_0 := 0.2 \]

\[ f(X) := g \cdot X - \frac{g}{A} \cdot X^2 \]

\[ X(t) := \frac{A}{1 + \frac{A}{X_0} - 1 \cdot \exp(-g \cdot t)} \]

\[ X := 0.05 \ldots 1 \]

\[ y(X) := g - g \cdot X \cdot \ln(X) \]
\[ g := 0.08 \quad X := 0.05 \ldots 1 \quad A := 1 \quad t := 0, 1 \ldots 100 \quad X_0 := 0.2 \]

\[ f(X) := g \cdot X - \frac{g}{A} X^2 \]

\[ X(t) := \frac{A}{1 + \frac{A}{X_0 - 1} \exp(-g \cdot t)} \]

\[ X := 0, 0.05 \ldots 1 \]

\[ y(X) := g - g \cdot X \cdot \ln(X) \]
Appendix E

Excel Worksheets

- This appendix contains the unoptimized problem, used to determine “natural” parameters for the optimized model. The formulas used are those set forth in Section VB (page 111). Microsoft Excel® was used to perform the calculations on a Microsoft Windows 98®-based personal computer.
Appendix F

Elimination of Infeasible Steady-State Solutions

<table>
<thead>
<tr>
<th>Solution Pair $(R,V)$ $\tau = 0.1$</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-3.9 e4, 2.6 e6)</td>
<td>negative biomass</td>
</tr>
<tr>
<td>(-3.8 e3, -3.9 e4)</td>
<td>negative biomass and CO$_2$</td>
</tr>
<tr>
<td>(112.0, 1067.9)</td>
<td>results in negative consumption for $x_1$</td>
</tr>
<tr>
<td>(112.5, 600.0)</td>
<td>zero determinant (see text)</td>
</tr>
<tr>
<td>(150-35i, 805+38i)</td>
<td>complex</td>
</tr>
<tr>
<td>(150+35i, 805-38i)</td>
<td>complex</td>
</tr>
<tr>
<td>(703.40, 842.07)</td>
<td>Feasible Solution</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Solution Pair $(R,V)$ $\tau = 0.001$</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-3.8 e4, 2.6 e6)</td>
<td>negative biomass</td>
</tr>
<tr>
<td>(-3.8 e3, -3.9 e4)</td>
<td>negative biomass and CO$_2$</td>
</tr>
<tr>
<td>(82.1, 2.8 e4)</td>
<td>results in negative consumption for $x_1$</td>
</tr>
<tr>
<td>(112.5, 600.0)</td>
<td>zero determinant (see text)</td>
</tr>
<tr>
<td>(123-3964i, 25346+147i)</td>
<td>complex</td>
</tr>
<tr>
<td>(123+3964i, 25346-147i)</td>
<td>complex</td>
</tr>
<tr>
<td>(122.73, 772.43)</td>
<td>Feasible Solution</td>
</tr>
</tbody>
</table>
Appendix G

Miscellaneous Matlab® Graphs of Variable Dynamics

The graphs contained in this appendix are associated with the Matlab® runs of the system after perturbation. The graphs were generated by the programs contained in Appendix C (pages 161-165). The associated graphs are included in the text as Figures 9-12 (pages 122-125).

The titles of the graphs contained in this appendix are:

- Consumption Paths over Time. $\tau = 0.001$, Perturbation 1
- State Variables over Time. $\tau = 0.001$, Perturbation 1
- Imputed Prices over Time. $\tau = 0.001$, Perturbation 1
- Biomass vs CO$_2$ in State Space. $\tau = 0.001$, Perturbation 1
- Consumption Paths over Time. $\tau = 0.001$, Perturbation 2
- State Variables over Time. $\tau = 0.001$, Perturbation 2
- Imputed Prices over Time. $\tau = 0.001$, Perturbation 2
- Biomass vs CO$_2$ in State Space. $\tau = 0.001$, Perturbation 2
- Consumption Paths over Time. $\tau = 0.1$, Perturbation 1
- State Variables over Time. $\tau = 0.1$, Perturbation 1
- Imputed Prices over Time. $\tau = 0.1$, Perturbation 1
- Biomass vs CO$_2$ in State Space. $\tau = 0.1$, Perturbation 1
- Consumption Paths over Time. $\tau = 0.1$, Perturbation 2
- State Variables over Time. $\tau = 0.1$, Perturbation 2
- Imputed Prices over Time. $\tau = 0.1$, Perturbation 2
- Biomass vs CO$_2$ in State Space. $\tau = 0.1$, Perturbation 2
Consumption Paths over Time, $\tau = 0.001$, Perturbation 1

$\text{x1, x2}$
State Variables over Time, $\tau = 0.001$, Perturbation 1
Imputed Prices over Time, $\tau = 0.001$, Perturbation 1
Biomass vs CO2 in State Space, \( \tau = 0.001 \), Perturbation 1
Consumption Paths over Time, $\tau = 0.001$, Perturbation 2
State Variables over Time, $\tau = 0.001$, Perturbation 2
Imputed Prices over Time, $\tau = 0.001$, Perturbation 2

\[ 3 \]

\[ 2.5 \]

\[ 2 \]

\[ 1.5 \]

\[ 1 \]

\[ 0.5 \]

\[ -100 \quad -90 \quad -80 \quad -70 \quad -60 \quad -50 \quad -40 \quad -30 \quad -20 \quad -10 \quad 0 \]

$t$
Biomass vs CO2 in State Space, \( \tau = 0.001 \), Perturbation 2

\[ R \]

\begin{tabular}{cccccccc}
120 & 120.5 & 121 & 121.5 & 122 & 122.5 & 123 \\
\end{tabular}
State Variables over Time, $\tau = 0.1$, Perturbation 1
Imputed Prices over Time, $\tau = 0.1$, Perturbation 1

$p, q$

$q$

$p$

$t$: 

-100 -90 -80 -70 -60 -50 -40 -30 -20 -10 0
Biomass vs CO2 in State Space, tau = 0.1, Perturbation 1
Consumption Paths over Time, $\tau = 0.1$, Perturbation 2
State Variables over Time, $\tau = 0.1$, Perturbation 2

850

800

750

700

-100 -90 -80 -70 -60 -50 -40 -30 -20 -10 0
Imputed Prices over Time, $\tau = 0.1$, Perturbation 2

$p, q$

$q$

$p$

$t$
<table>
<thead>
<tr>
<th>835</th>
<th>702</th>
<th>704</th>
<th>706</th>
<th>708</th>
<th>710</th>
<th>712</th>
<th>714</th>
</tr>
</thead>
</table>

Biomass vs CO2 in State Space, \( \tau = 0.1 \), Perturbation 2