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Complexity of Exotic $\mathbb{R}^4$'s

by

Sanford Ganzell

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Abstract

Complexity of Exotic $\mathbb{R}^4$'s

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A smooth manifold which is homeomorphic but not diffeomorphic to $\mathbb{R}^4$ is called an exotic $\mathbb{R}^4$. In this paper we construct several examples of such spaces, and define a notion of complexity for smooth manifolds homeomorphic to $\mathbb{R}^4$. In particular, for $R$ homeomorphic to $\mathbb{R}^4$, we define the complexity, $m(R)$, to be the supremum, taken over all compact codimension zero submanifolds $K$ of $R$, of the minimal first betti number of any smooth 3-manifold $\Sigma^3$ which separates $K$ from infinity, i.e.,

$$m(R) = \sup_{K} \min_{\Sigma} (b_1(\Sigma)) \in \{0, 1, 2, \ldots, \infty\}.$$ 

Examples are given, and various upper and lower bounds for complexity are computed. We also extend a result of Biţaca and Etnyre by showing how end summing with exotic $\mathbb{R}^4$'s of increasing complexity can be used to construct infinitely many smooth structures on any open 4-manifold with at least one topologically collarable end, i.e., an end homeomorphic to $\Sigma^3 \times \mathbb{R}$ for some closed 3-manifold $\Sigma^3$. 
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Chapter 1

Introduction

1.1 Smooth structures

In dimensions $n \neq 4$, nondiffeomorphic smooth structures are detected via bundle theoretic obstructions. Since $\mathbb{R}^n$ is contractible, any associated bundle must be trivial, and so no such obstruction can exist. Hence there is a unique differentiable structure on $\mathbb{R}^n$. In contrast, $\mathbb{R}^4$ does not have a unique smooth structure: there are uncountably many nondiffeomorphic manifolds homeomorphic to $\mathbb{R}^4$, the so-called exotic $\mathbb{R}^4$'s.

A homeomorphism $f : M^n \to N^n$ is a diffeomorphism if both $f$ and $f^{-1}$ are $C^\infty$ maps. We use the term *exotic* to refer to manifolds which are homeomorphic but not diffeomorphic, i.e. a manifold $N$ is called an exotic $M$ if $N$ is homeomorphic to $M$ but not diffeomorphic to $M$. In dimensions 3 and below, every topological manifold
admits a unique smooth structure [Moi], but not so in dimension 4. There are no examples of 4-manifolds which are known to have a unique smooth structure, and several classes of manifolds are known to have infinitely many. This phenomenon is discussed in greater detail in chapter 2. However, the following remains unresolved at the time of this writing:

**Conjecture 1.1 (Smooth Poincaré Conjecture)**  There is no exotic 4-sphere.

### 1.2 Intersection forms

Our main algebraic tool for dealing with 4-manifolds will be the intersection form. There is a pairing \( H^2(M; \mathbb{Z})/T \otimes H^2(M; \mathbb{Z})/T \to \mathbb{Z} \) given by the cup product in the cohomology ring:

\[
\hat{\alpha} \otimes \hat{\beta} \to \hat{\alpha} \cup \hat{\beta} \in H^4 \cong \mathbb{Z}.
\]

(Here \( T \) is the torsion subgroup.)

If \( \alpha \) and \( \beta \) are the Poincaré duals to \( \hat{\alpha} \) and \( \hat{\beta} \), then \( \alpha \) and \( \beta \) (\( \in H_2 \)) can be represented by smoothly embedded surfaces \( F_\alpha \) and \( F_\beta \). We can assume these are transverse, so that they intersect in distinct points \( p_1, \ldots, p_n \). Count these with sign \((+1 \text{ if } T_{F_\alpha}|_{p_i} \oplus T_{F_\beta}|_{p_i} \text{ has the same orientation as } T_M|_{p_i}, -1 \text{ otherwise})\) to get the algebraic number of intersection points, \( \alpha \cdot \beta \). By Poincaré duality \( \alpha \cdot \beta = \hat{\alpha} \cup \hat{\beta} \).

The result is a symmetric bilinear form \( q_M \) associated to the manifold \( M \). If \( M \) is closed then \( q_M \) is unimodular, i.e. \( \det(q_M) = \pm1 \).
Examples: $M = S^2 \times S^2$ has two generators in second homology. By calculating the cohomology ring or by observing the geometric structure of $M$, we find that $q_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We refer to this as the hyperbolic form and denote it $H$. $\mathbb{C}P^2$ has the form $(1)$, and $\overline{\mathbb{C}P^2}$, the complex projective plane with opposite orientation, has the form $(-1)$. The 4-sphere $S^4$ has trivial second homology and so corresponds to the empty form. We define an additional important form which is denoted $E_8$.

$$E_8 = \begin{pmatrix} 2 & 1 & & & & & 0 \\ 1 & 2 & 1 & & & & \\ & 1 & 2 & 1 & & & \\ & & 1 & 2 & 1 & & \\ & & & 1 & 2 & 1 & 1 \\ & & & & 1 & 2 & 1 \\ & & & & & 1 & 2 & 0 \\ & & & & & & 1 & 0 & 0 & 2 \end{pmatrix}.$$ 

As the following theorem indicates, intersection forms classify simply connected 4-manifolds up to homotopy.

**Theorem 1.2 (Whitehead)** Let $M$ and $N$ be closed, simply connected topological 4-manifolds. Then $M \simeq N$ iff $q_M \cong q_N$. I.e. $M$ is homotopy equivalent to $N$ if and only if $q_M$ and $q_N$ are isomorphic as forms.

There are three invariants associated to intersection forms: rank, type and signature. Rank is defined as $\dim(H^2(M; \mathbb{Z})/T)$, and is equal to $b_2(M)$, the second betti number of $M$. Type is even if $\alpha \cdot \alpha \equiv 0 \mod 2$ for all $\alpha \in H_2$, and odd otherwise. The signature, $\sigma$, is the number of positive eigenvalues, $b_2^+$, minus the number of
negative eigenvalues, $b_2^-$, of the matrix associated to the form. So $b_2 = b_2^+ + b_2^-$, and 
$\sigma = b_2^+ - b_2^-$. We will often refer to the signature of an intersection form representing
a 4-manifold $M$, as the signature $\sigma(M)$ of the manifold. Additionally, $q_M$ is called
positive (negative) definite if $\alpha \cdot \alpha > 0$ ($\alpha \cdot \alpha < 0$) for all $\alpha \neq 0$.

A complete treatment of symmetric bilinear forms can be found in [MH]. We
mention here the results we will use.

**Lemma 1.3** Let $\alpha \in H_2(M; \mathbb{Z})$. If $\alpha$ is represented by an embedded 2-sphere $F$, and $\alpha^2 = 1$, then $M = N \# \mathbb{C}P^2$ for some 4-manifold $N$.

**Proof.** There is a tubular neighborhood of $F$ in $M$ which is diffeomorphic to
the total space of the normal bundle $\nu \to F$. In this case, $\nu$ is the $B^2$ bundle over
the 2-sphere, with Euler class $e(\nu) = \alpha^2 = 1$, so is diffeomorphic to $\mathbb{C}P^2$ with the
4-handle removed. The boundary of $M - \nu$ is $S^3$, and so we cut out $\nu$ and glue in
$B^4$ to obtain $N$. ■

An element $\alpha \in H_2(M; \mathbb{Z})$ is called characteristic if $\alpha \cdot x \equiv x \cdot x \mod 2$ for all
$x \in H_2(M; \mathbb{Z})$. In the following theorem, assume $M$ is a closed, simply connected,
smooth 4-manifold.

**Theorem 1.4 ([GS])** Let $F \hookleftarrow M$ be an embedded 2-sphere which represents the
element $\alpha \in H_2(M; \mathbb{Z})$. If $\alpha$ is characteristic, then $\alpha^2 \equiv \sigma(M) \mod 16$.

The classification of integral, unimodular, symmetric bilinear forms is now given.
See [MH] for details.
In the indefinite case, if \( q_M \) is odd, then \( q_M \cong r \oplus (1) \oplus (-1) \), and is represented by \( \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \). If \( q_M \) is even, then \( q_M \cong \pm r \oplus E_8 \oplus^0 H \). \(-2E_8 \oplus 3H\) is represented by the \( K3 \) surface, defined as \( \{ [x, y, z, w] \in \mathbb{C}P^3 | x^4 + y^4 + z^4 + w^4 = 0 \} \).

Definite forms are probably impossible to classify. Any positive definite, symmetric, bilinear, unimodular form decomposes as a direct sum of indecomposable, positive definite, symmetric, bilinear, unimodular forms. There are more than \( 10^{50} \) nonisomorphic, indecomposable, positive definite, symmetric, bilinear, unimodular, even forms of rank \( \leq 40 \).

### 1.3 Classification theorems

In each of the following theorems, let \( M^4 \) be closed, smooth and oriented.

**Theorem 1.5 (Rohlin, 1952)** If \( M \) is spin, then \( \sigma(M) \equiv 0 \mod 16 \).

**Corollary 1.6** \( E_8 \) does not correspond to any smooth, closed, simply connected \( 4 \)-manifold.

\( E_8 \) is even and has signature 8. A simply connected 4-manifold \( M \) will have a spin structure if \( q_M \) is even. Spin 4-manifolds necessarily have even intersection forms.

**Theorem 1.7 (Freedman,1982)** Given an even (odd) form, there exists exactly one (two) simply connected, closed, topological 4-manifold(s) representing that form.

**Theorem 1.8 (Donaldson, 1987)** If \( q_M \) is negative definite, then \( q_M \cong \oplus (-1) \).
Corollary 1.9 \(-E_8 \oplus (-1)\) does not correspond to any smooth, closed 4-manifold.

The form \(\mathbb{R} \oplus (1)\) is represented by \(\# \mathbb{C}P^1\). \(\mathbb{R} \oplus (-1)\) is represented by \(\# \mathbb{C}P^2\).

Theorem 1.10 (Furuta, 1997) If \(q_M = 2kE_8 \oplus lH\), and \(k \neq 0\), then \(l > 2|k|\).

Corollary 1.11 \(K3\) cannot be written as a connected sum \(X\#(S^2 \times S^2)\).

Actually, this corollary was first proved by Donaldson in 1986, though that work does not extend to show, for example, that \(K3\#K3\) cannot split off two \(S^2 \times S^2\) summands. Furuta's theorem is sometimes called the \(\frac{10}{8}\)-theorem since it has an equivalent formulation as \(\frac{b_2}{|\sigma|} > \frac{10}{8}\). The \(\frac{11}{8}\)-conjecture, \(\frac{b_2}{|\sigma|} \geq \frac{11}{8}\), remains open. Equality is achieved by the \(K3\) surface: \(b_2(K3) = 22\), \(\sigma(K3) = -16\).

The above nonexistence results will be used to construct a variety of exotic smooth structures on \(\mathbb{R}^4\). Other techniques will be discussed in Chapter 4. The study of exotic \(\mathbb{R}^4\)'s will be the main subject of this paper, and so we give a characterization of spaces homeomorphic to \(\mathbb{R}^4\). An open topological space \(R\) is called simply connected at infinity if every compact subset \(C\) of \(R\) is contained in a compact subset \(D\) of \(R\), such that \(R - D\) is connected, and the inclusion \(R - D \to R - C\) induces the trivial map on \(\pi_1\). Freedman proved [Fr], any open 4-manifold which is contractible and simply connected at infinity is homeomorphic to \(\mathbb{R}^4\).
Chapter 2

Exotic constructions

2.1 Subsets of definite manifolds

We begin with the construction of a particular exotic $\mathbb{R}^4$. Let $M^4 = \mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}$, and consider the intersection form $q_M = (1) \oplus 8(-1) \oplus (-1)$ with basis $\{e_0, e_1, \ldots, e_9\}$, $e_0$ representing the $(1)$. The element $\alpha = 3e_0 + e_1 + \cdots + e_8 \in (1) \oplus 8(-1)$ is characteristic and $\alpha^2 = 1$. We claim $(\alpha) \perp \cong -E_8$. To justify this claim it suffices to demonstrate a basis whose corresponding form is $-E_8$, and whose elements are orthogonal to $\alpha$. One checks that

$$\{e_2 - e_1, e_4 - e_3, e_4 - e_3, e_6 - e_5, e_6 - e_5, e_8 - e_7, e_8 - e_7, e_0 + e_6 + e_7 + e_8\}$$

is such a basis. Thus in $q_M$, $(\alpha) \perp \cong -E_8 \oplus (-1)$. But by Donaldson’s theorem, $-E_8 \oplus (-1)$ cannot represent any closed smooth 4-manifold, so $\alpha \in H_2(\mathbb{C}P^2 \# 9\overline{\mathbb{C}P^2}; \mathbb{Z})$ cannot be represented by a smoothly embedded 2-sphere. We can, however, [Ca]
represent \( \alpha \) by a Casson handle, CH, attached to \( B^4 \) along an unknot with framing 1. Since Freedman has shown all Casson handles are homeomorphic to \( B^2 \times \text{int} \ B^2, \text{rel} \ \partial \), it follows that \( U = \text{int}(B^4 \cup CH) \) is homeomorphic to \( \mathbb{C}P^2 - \{\text{pt.}\} \). The (topological) core 2-sphere \( S \) of \( U \) in \( M \) carries the homology class \( \alpha \), and since any Casson handle can be embedded into the standard 2-handle, we have \( S \subset U \subset \mathbb{C}P^2 \). See Figure 2.1. Here \( N^4 = \mathbb{C}P^2 \).

Let \( X = M - S \). Then \( q_X \cong -E_8 \oplus (-1) \). Let \( C \) be any smooth, compact submanifold of \( U \) so that \( S \subset \text{int} \ C \), and define \( Y = X - \text{int} \ C \). Finally, let \( R = \mathbb{C}P^2 - S \) and \( D = \mathbb{C}P^2 - \text{int} \ C \).

\( R \) is contractible and simply connected at infinity, so by Freedman [Fr], \( R \) is homeomorphic to \( \mathbb{R}^4 \). \( R \) is an open subset of \( \mathbb{C}P^2 \) and so inherits a smooth structure which we show to be exotic.

Suppose \( R \) is standard, i.e. diffeomorphic to \( \mathbb{R}^4 \). Then there would be a smoothly
embedded 3-sphere in $D$ which separates the compact set $K = CP^2 - U$ from $C$. This 3-sphere also embeds smoothly in $M$ with $\alpha$ (represented by $S$) on one side and $-E_8 \oplus (-1)$ on the other. We now cut $M^4$ along this 3-sphere and glue in a 4-ball. The result is a smooth, simply connected, closed 4-manifold with intersection form $-E_8 \oplus (-1)$, which contradicts Donaldson's theorem.

2.2 Subsets of spin manifolds

Another (presumably different) exotic $\mathbb{R}^4$ can be constructed similarly, with $M^4$ being the $K3$-surface, taken with reversed orientation, so that $q_M = 2E_8 \oplus 3H$, and $N^4 = \# (S^2 \times S^2)$. In this case [Ca], the six elements of $H_2(M; \mathbb{Z})$ which span the $3H$ can be represented by six Casson handles, $CH_i$, $1 \leq i \leq 6$. Set $U = \text{int}(B^4 \cup CH_i)$ and we find $U$ is homeomorphic to $\# (S^2 \times S^2) - \{\text{pt.}\}$. See Figure 2.1. Here, $S$ is the union of cores of the Casson handles, and $R = N^4 - S$ is readily seen to be homeomorphic to $\mathbb{R}^4$.

Suppose as before that $R$ is standard. Then we could find a smoothly embedded 3-sphere in $D = \# (S^2 \times S^2) - \text{int} \ C$ which separates the compact set $K = \# (S^2 \times S^2) - U$ from $C$. Find this 3-sphere in $M$ and cut it out. Glue in a 4-ball to produce a smooth (spin), simply connected 4-manifold with intersection form $2E_8$, which contradicts Furuta's theorem. Thus $R$ is an exotic $\mathbb{R}^4$ which we will denote $R_1$. We will see that $R_1$ is the first in a sequence of exotic $\mathbb{R}^4$'s which can be used to "corrupt" the ends of open 4-manifolds, producing many nondiffeomorphic smooth structures on a
given manifold.

We have seen that $K$ cannot be embedded into $\mathbb{R}^4$. $K$ also cannot be embedded into $S^4$ since $K$ is a compact manifold with boundary, so $S^4 - K$ would be non-empty. Thus we could delete a point from $S^4 - K$ to obtain an embedding of $K$ in $\mathbb{R}^4$. Moreover, we have the following

**Lemma 2.1** $R_1$ contains a compact subset $K_1$ which cannot be embedded into $\# (S^2 \times S^2)$.

**Proof.** (Compare [GS] 9.4.3) Fix a homeomorphism $h : \mathbb{R}^4 \to R_1$. Let $B = h(B^4_r)$ be the image of a ball of sufficiently large radius so that $K \subset \text{int } B$. $R_1 - B$ is then homeomorphic to $S^3 \times \mathbb{R}$. Perturb the projection map $f : S^3 \times \mathbb{R} \to \mathbb{R}$ so that it is smooth and proper, and define $K_1 = B \cup f^{-1}(-\infty, a]$ for any regular value $a$.

Let $A$ be a neighborhood of the end of $\text{int } K_1$, and suppose $K_1$ was embedded into $\# (S^2 \times S^2)$. From the image of this embedding remove the complement of $A$ in $K_1$

![Diagram](image-url)

**Figure 2.2: Construction of $K_1$**
and denote the resulting open manifold by $X_0$. The end of $X_0$ has a neighborhood diffeomorphic to $A$. But $U \subset K3$ also contains an embedding of $A$. So we delete from $K3$ the portion of $U - A$ which contains the Casson handles, and attach $X_0$ by identifying the (diffeomorphic) ends. The result is a manifold whose intersection form is $2E_8 \oplus 2H$, which is impossible by Furuta’s theorem. ■

**Remark 2.2** Since the only information about $\# (S^2 \times S^2)$ used in the proof was the intersection form, $K_1$ cannot be embedded into any smooth 4-manifold (regardless of fundamental group) whose intersection form is $2H$.

**Definition 2.3** (Gompf) Let $X_1$, $X_2$ be noncompact oriented 4-manifolds that are simply connected at infinity (e.g. any exotic $\mathbb{R}^4$). Choose proper embeddings $\gamma_i : [0, \infty) \hookrightarrow X_i$, remove a tubular neighborhood of $\gamma_i(0, \infty)$ from $X_i$ and glue the resulting $\mathbb{R}^3$ boundaries together by an orientation reversing diffeomorphism to obtain an oriented manifold called the end sum $X_1 \sharp X_2$.

Let $R_n = \sharp^n R_1$, i.e. the end sum of $n$ copies of $R_1$. Each copy of $R_1$ inside $R_n$ contains a copy of $K_1$, and by connecting these $K_1$'s with neighborhoods of arcs we can form a boundary connected sum $K_n = \sharp^n K_1$. Define $R_\infty = \infty \sharp R_1$.

**Theorem 2.4** ([BE]) For $1 \leq n < m \leq \infty$, $R_n$ and $R_m$ are not diffeomorphic.

**Proof.** It follows from the proof of Lemma 2.1 that $K_n$, and hence $R_n$ cannot be embedded smoothly into $\# (S^2 \times S^2)$, but embeds in $\# (S^2 \times S^2)$ by an analogous
construction to that of $R_1$, using $K^3$ for $M$, and $S^2 \times S^2$ for $N$. So for any integer $k > \frac{3}{2}n$, there is no embedding of $R_k$ into $S^2 \times S^2$, which implies $R_k$ is not diffeomorphic to $R_n$. Thus there are infinitely many nondiffeomorphic exotic $\mathbb{R}^4$'s in the collection $\{R_i\}$. Now suppose $R_n$ were diffeomorphic to $R_m$ for some $n < m < \infty$. Since for any $i \geq 1$, $R_{i+1} = R_i \times R_1$, we have $R_n \times R_{m-n} = R_m \times R_{m-n}$, and so $R_m = R_{2m-n}$. It follows that there are at most $m - n$ diffeomorphism classes among the $R_i$, contradicting the fact that there are infinitely many. Now for $n < \infty$, $R_n$ embeds in some (finite) connected sum of $S^2 \times S^2$'s, but by construction, $R_\infty$ does not. So $R_\infty$ is not diffeomorphic to $R_n$ for any $n < \infty$. ■

We now extend a theorem of Bižaca and Etnyre [BE] by showing how end summing can be used to create infinitely many smooth structures on certain open 4-manifolds.

**Theorem 2.5** Let $X$ be an open topological 4-manifold with at least one topologically collarable end, i.e. an end homeomorphic to $\Sigma^3 \times \mathbb{R}$ for some closed 3-manifold $\Sigma^3$. Then $X$ has infinitely many smooth structures.

**Proof.** We first prove the case where $X$ has exactly one end. Quinn [FQ] proved that any open 4-manifold is smoothable, so we may assume $X$ has a smooth structure. End summing with $\mathbb{R}^4$ (exotic or not) does not change the homeomorphism type of an open 4-manifold, so $X \times R_\infty$ is homeomorphic to $X$. Let $U$ be a neighborhood of the end of $X \times R_\infty$. Fix a homeomorphism $h : \Sigma^3 \times \mathbb{R} \to U$ such that the $n$th
copy of $K_1$ embeds into $h(\Sigma^3 \times (n, n + 1))$. Define $X_t = (X_2 \mathbin{\mathop{\oplus}} R_\infty) - h(\Sigma^3 \times [t, \infty))$, and observe that for any $t \in \mathbb{R}$, $X_t$ is homeomorphic to $X$. We show that for any positive integers $i, j$, $1 \leq i < j$, $X_i$ is not diffeomorphic to $X_j$. Thus the collection \{X_i\}, $i \in \mathbb{Z}^+$ provides infinitely many distinct smooth structures for $X$. Suppose to the contrary that $f : X_j \rightarrow X_i$ is a diffeomorphism for some integers $1 \leq i < j$.

Let $V$ be a neighborhood of the end of $X_j$ which does not intersect the copy of $K_1$ contained in $\Sigma^3 \times (i, i + 1)$. Then $f(V)$ identifies another copy of $V$ in $X_i$. Let $W = f(V) \cup (\Sigma^3 \times [i, j))$, and construct a periodic end by gluing copies of $W$ to the end of $X_j$, using $f$ to perform the gluing.

Figure 2.4: Periodic end
Choose smoothly embedded, separating 3-manifolds $M_1 \hookrightarrow \Sigma^3 \times (-\infty, 0)$ and $M_2 \hookrightarrow f(V) \subset \mathcal{X}_i$, and let $Z_0$ be the manifold bounded by $M_1$ and $M_2$ in $\mathcal{X}_i$. Since all 3-manifolds spin bound, we may cap off $Z_0$ with smooth, simply connected, spin 4-manifolds to form a closed, smooth, spin 4-manifold $Z$. Rohlin's theorem guarantees that the signature of $Z$ is a multiple of 16, so by taking a connected sum with $K3$-surfaces of the correct orientation, we may assume $Z$ has signature zero. Since $Z$ is spin, its intersection form must be even, and hence (since $\sigma(Z) = 0$) a sum of hyperbolics. Say $q_Z = kH$. Then, as in Remark 2.2, $K_n$ cannot embed into $Z$ (or any 4-manifold with the same intersection form as $Z$) when $n \geq \frac{k}{2}$. So choose $n \geq \frac{k}{2}$, and observe that $K_n$ does embed smoothly into the periodic end constructed above. Find a copy of $M_2 \subset V$ past the embedded $K_n$, and let $Z_1$ be the manifold bounded by $M_1$ and this $M_2$. Cap off $Z_1$ with the same smooth, simply connected, spin 4-manifolds as above to produce $Z'$, a manifold homotopy equivalent to $Z$ (hence same intersection form), but into which $K_n$ embeds smoothly. This contradiction establishes that $\mathcal{X}_i$ is not diffeomorphic to $\mathcal{X}_j$.

If $\mathcal{X}$ has more than one, but still finitely many ends, we use the above construction to obtain a collection $\{\mathcal{X}_i\}, i \in \mathbb{Z}^+$ of (not necessarily distinct) smooth structures for $\mathcal{X}$. In this case, $\mathcal{X}_i$ and $\mathcal{X}_j$ ($i \neq j$) can be diffeomorphic if the diffeomorphism permutes the homeomorphic ends. But there are only finitely many such permutations, and so our infinite collection provides infinitely many distinct smooth structures.

In the case where $\mathcal{X}$ has infinitely many ends, we proceed as follows: $\mathcal{X}$ is
open and hence smoothable. Group together any ends not diffeomorphic to $\Sigma^3 \times \mathbb{R}$ as one. (Note there must be at least one additional end and this “last” one is horribly non-collarable.) Now we may choose our smooth structure to respect the topological product structure on the collarable ends, so that each of our topological product ends becomes a smooth product end. With each (perhaps infinitely many) end diffeomorphic to $\Sigma^3 \times \mathbb{R}$, end sum a copy of $R_\infty$ and call the result $X^*$. Let $U^*$ be a (non-connected) neighborhood of the diffeomorphic collarable ends. Fix a diffeomorphism $h^*$ from the disjoint union of these ends to $U^*$ so that the copies of $K_1$ are distributed as in the one end case. Since these ends are all diffeomorphic, it is of no concern that $h^*$ may permute them, even if there are infinitely many. Define $X^*_i = X^* - h^*(\coprod (\Sigma^3 \times [t, \infty)))$. We claim that for integers $1 \leq i < j$, $X^*_i$ is not diffeomorphic to $X^*_j$. The construction follows exactly as in the one end case, working simultaneously on all the diffeomorphic collarable ends. The construction of $Z$ and $Z'$ establishes the contradiction as before. ■
Chapter 3

Complexity

3.1 Definition and examples

In each of our constructions, the key property of the exotic $\mathbb{R}^4$ is the compact set $K$ which no smoothly embedded 3-sphere separates from infinity. It is natural to ask what other smoothly embedded 3-manifolds cannot separate $K$ from infinity. This question is relevant, for example, to the study of topological quantum field theories in $3+1$ dimensions, in which one is concerned about the smooth embeddings of 3-manifolds into 4-manifolds, in particular, the splittings of 4-manifolds along 3-manifolds.

We define complexity in terms of the first homology of embedded 3-manifolds, but this is not the only option. One could examine the number of generators of $\pi_1$, or the number of normal generators of $\pi_1$, for example.
**Definition 3.1** Let $R$ be homeomorphic to $\mathbb{R}^4$. Define $m(R)$ to be the supremum, taken over all compact codimension zero submanifolds $K$ of $R$, of the minimal first betti number of any smooth 3-manifold $\Sigma^3$ which separates $K$ from infinity, i.e.,

$$m(R) = \sup_K \left( \min_{\Sigma} (b_1(\Sigma)) \right) \in \{0, 1, 2, \ldots, \infty\}.$$ 

For example, if $R = \mathbb{R}^4$ then $m(R) = 0$. In particular, for any compact $K$, there is a smoothly embedded 3-sphere which separates $K$ from infinity. Note that $m$ is subadditive under end sum, i.e., for $R_1, R_2$ homeomorphic to $\mathbb{R}^4$, $m(R_1 \uplus R_2) \leq m(R_1) + m(R_2)$. We may think of $m(R)$ as a measurement of complexity of the exotic $\mathbb{R}^4$, though there is a natural extension to the measurement of complexity of any open 4-manifold, i.e., consider the supremum of complexities of each of the ends.

Our first task is to show that it is nontrivial. Let $R$ be the exotic $\mathbb{R}^4$ created in Section 2.1 as a subset of $\mathbb{C}P^2$. We have already seen that the compact subset $K$ cannot be separated from infinity by $S^3$. We now show that $K$ cannot be separated from infinity by any rational homology sphere. Let $\Sigma^3 \to R \subset \mathbb{C}P^2$ be a smoothly embedded 3-manifold which separates $K$ from infinity, and suppose $\Sigma^3$ has the same rational homology as $S^3$. By construction, $K$ can be taken to contain a topological 3-sphere which is also a subset of $U = \text{int}(B^4 \cup S)$ (recall, $S$ is the topological core of the Casson handle which carries the generator for homology in $\mathbb{C}P^2$). Find this 3-sphere in $\mathbb{C}P^2 \# 9\mathbb{C}P^2$, as per the original construction of $R$. $\Sigma^3$ is then found smoothly embedded into $\mathbb{C}P^2 \# 9\mathbb{C}P^2$, and we define $M^4 \subset \mathbb{C}P^2 \# 9\mathbb{C}P^2$ to be the 4-manifold bounded by $\Sigma^3$ which does not contain $S$. Let $N^4$ be the 4-manifold
contained in $R$ which is bounded by $\Sigma^3$.

Define $X^4 = M \cup_\Sigma N$, so that $X^4$ is closed and smooth. We examine the Mayer-Vietoris sequence of $M$ and $N$ (coefficients to be taken in $\mathbb{Q}$).

$$\cdots \longrightarrow H_2(\Sigma) \xrightarrow{\psi_*} H_2(M) \oplus H_2(N) \xrightarrow{\varphi_*} H_2(X) \xrightarrow{\eta_*} H_1(\Sigma) \longrightarrow \cdots$$

$H_2(\Sigma) \cong H_1(\Sigma) \cong 0$, so $\psi_*$ is an isomorphism. The image of $H_2(\Sigma)$ in $M$ is zero, so the intersection form $q_M \cong -E_8 \oplus (-1)$. Now since $\Sigma \subset R$, and $H_2(N) = \text{im}(H_2(\Sigma)) = 0$, $N$ has the homology type of the 4-ball. So $X$ has intersection form $-E_8 \oplus (-1)$, which is impossible by Donaldson's theorem. Thus $m(R) > 0$.

### 3.2 Upper bounds

In a similar fashion we may show that $m(R_1) > 0$, where $R_1$ is the exotic $\mathbb{R}^4$ created in Section 2.2 as a subset of $\#^3 (S^2 \times S^2)$, but since in this case our manifolds
carry spin structures, we will be able to improve our bound for $m(R)$. Suppose 
$\Sigma \hookrightarrow R_1 \subset \#^3 (S^2 \times S^2)$ is a smoothly embedded rational homology 3-sphere which
separates $K_1$ from infinity. $K_1$ contains a topological 3-sphere which is also a subset of $U = \operatorname{int}(B^4 \cup S)$ (here, $S$ is the union of the cores of Casson handles). Find this 3-sphere in the $K3$ surface so that $\Sigma$ can be found smoothly embedded into $K3$, and
define $M \subset K3$ to be the manifold bounded by $\Sigma$ which does not contain $S$. Let $N$
be the manifold contained in $R_1$ which is bounded by $\Sigma$.

Define $X^4 = M \cup_\Sigma N$, so that $X^4$ is closed, smooth and spin. We again examine
the Mayer-Vietoris sequence of $M$ and $N$.

$$\cdots \longrightarrow H_2(\Sigma) \xrightarrow{r_*} H_2(M) \oplus H_2(N) \xrightarrow{\psi_*} H_2(X) \xrightarrow{n_*} H_1(\Sigma) \longrightarrow \cdots$$

$H_2(\Sigma) \cong H_1(\Sigma) \cong 0$, so $\psi_*$ is an isomorphism. The image of $H_2(\Sigma)$ in $M$ is zero, so
the intersection form $q_M \cong 2E_8$. Thus if $q_N$ contains fewer than three hyperbolics, $X$
would violate Furuta's theorem. In fact, since $\Sigma \subset R_1$, and $H_2(N) = \operatorname{im}(H_2(\Sigma)) = 0$,$\n$N has the homology type of the 4-ball. So $X$ has intersection form $2E_8$. The contradiction establishes that $m(R_1) > 0$.

With a little more analysis we can compute a better lower bound for $m(R_1)$. Instead of assuming $\Sigma$ is a homology sphere, we only suppose $b_1(\Sigma) \leq 2$. By duality
$H_2(\Sigma) \cong H_1(\Sigma)$. The idea is that $H_2(\Sigma)$ can only contribute at most a rank two
subspace to each of $H_2(M)$ and $H_2(N)$. And so $H_2(X)$ can inherit at largest a rank
four subspace from $H_*(\Sigma)$: roughly speaking, half from $H_2(\Sigma)$ (via $\psi_*$) and half from
$H_1(\Sigma)$. This subspace has signature zero, and since $X$ is spin, must be a sum of
hyperbolics. Then $H_2(X)$ would equal $2E_8 \oplus 2H$ (at best), which would contradict Furuta's theorem, and establish $m(R_1) \geq 3$. This will be the method to prove the following

**Theorem 3.2** For $R_n$, $R_\infty$ defined as in Section 2.2, $m(R_n) \geq 2n + 1$, $m(R_\infty) = \infty$.

**Proof.** Let $\Sigma$ be a 3-manifold which separates $K_n$ from infinity. For $n < \infty$, define $M \subset K3$ to be the 4-manifold bounded by $\Sigma$, which does not contain the cores of Casson handles which carry the $3nH$ in homology. Define $N \subset (S^2 \times S^2)$ to be the 4-manifold bounded by $\Sigma$ which is contained in $R_n$, and let $X = M \cup \Sigma N$. We note that all 4-manifolds in our construction are spin, and so have even intersection form and signature $0 \mod 16$. Suppose $H_1(\Sigma)$ has rank $\leq 2n$. We make use of the above Mayer-Vietoris sequence, and of the exact sequences of the pairs $(M, \Sigma)$ and $(N, \Sigma)$ (again, coefficients in $\mathbb{Q}$).

$$
\cdots \to H_2(\Sigma) \xrightarrow{i_*} H_2(M) \xrightarrow{\alpha_*} H_2(M, \Sigma) \xrightarrow{\partial_*} H_1(\Sigma) \to \cdots 
$$

$$
\cdots \to H_2(\Sigma) \xrightarrow{j_*} H_2(N) \xrightarrow{\lambda_*} H_2(N, \Sigma) \xrightarrow{\partial_*} H_1(\Sigma) \to \cdots 
$$

Duality implies $H_2(\Sigma) \cong H_1(\Sigma)$. The end of $\text{int}(M)$ (similarly for $N$) contains a topological 3-sphere which, along with $\Sigma$, bounds a 4-manifold $A$, which is a subset (topologically) of $\mathbb{R}^4$. See Figure 3.2. So any $H_2$ in $A$ can be pushed into $\Sigma$. Thus $H_2(M, \Sigma) = 2nE_8$, $H_2(N, \Sigma) = 0$, and our sequences reveal that $H_2(M) = 2nE_8 \oplus i_*(H_2(\Sigma))$, and $H_2(N) = j_*(H_2(\Sigma))$. Now, rank($i_*$), rank($j_*$) $\leq 2n$, so rank($\text{im} \psi_*$) $\leq 2n$. By exactness, $\text{im} \psi_* = \ker \eta_*$. And since $b_1(\Sigma) \leq 2n$, $H_2(X) \subseteq$
Figure 3.2: Computation of $m(R_n)$

$2nE_8 \oplus \Gamma$, for some $\Gamma$ of (even) rank $\leq 4n$. $\Gamma$ has signature 0, and since $X$ is spin, $\Gamma$ is a sum of (at most $2n$) hyperbolics, i.e., $H_2(X) \subseteq 2nE_8 \oplus 2nH$. But $H_2(X)$ necessarily contains $2n E_8$ summands, which contradicts Furuta's theorem. Note that we have actually proved any exotic $\mathbb{R}^4$ which contains $K_n$ has complexity at least $2n + 1$, even if it is not diffeomorphic to $R_n$.

Now suppose $m(R_\infty) = s < \infty$. Then for any compact set $K \subset R_\infty$, $K$ can be separated from infinity by a 3-manifold $\Sigma$, with $b_1(\Sigma) \leq s$. In particular, $K_s \subset K_\infty$ is compact, but as seen above, $K_s$ cannot be separated from infinity by a 3-manifold whose first betti number is smaller than $2s + 1$, implying $m(R_\infty) > 2s$. The contradiction establishes $m(R_\infty) = \infty$. \qed
Chapter 4

Small exotic constructions

An exotic $\mathbb{R}^4$ is called large if it does not smoothly embed into $S^4$. All of our exotic constructions so far have resulted in large exotic $\mathbb{R}^4$'s. It is a remarkable fact that there are (in fact, uncountably many) small exotic $\mathbb{R}^4$'s, i.e., manifolds homeomorphic but not diffeomorphic to $\mathbb{R}^4$, which embed smoothly into $S^4$. These will be the subject of the present chapter. An important part of the construction will rely on Kirby's calculus for framed links, and so a summary of this technique is presented here. Details can be found in [Ki] or [GS].

4.1 Kirby calculus

By a $k$-handle in dimension $n$, we mean a pair $(B^k \times B^{n-k}, \partial B^k \times B^{n-k})$, where $B^k \times B^{n-k}$ is the handle and $\partial B^k \times B^{n-k} = S^{k-1} \times B^{n-k}$ is the attaching region. A handle is added to an $n$-manifold $M^n$ via an attaching map $f : S^{k-1} \times B^{n-k} \hookrightarrow \partial M^n$, where
$f$ is taken to be a smooth embedding of the attaching region into the boundary of the manifold. A smooth handlebody decomposition of a smooth, compact $n$-manifold $M^n$ is a sequence $B^n = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_d = M^n$, where $M_i$ is obtained from $M_{i-1}$ by adding a handle. It is a fact that every smooth manifold admits a smooth handlebody decomposition. Topological handlebody decompositions exist for manifolds in all dimensions except dimension four, where manifolds admit handlebody decompositions if and only if they are smoothable. Figure 4.1 shows a 1-handle and a 2-handle attached to $B^3$. Closed, connected manifolds can be taken to have exactly one 0-handle and one $n$-handle.

![3-dimensional handles](image)

Figure 4.1: 3-dimensional handles

In dimension 4 this is harder to draw, so we visualize handlebodies by drawing their attaching maps. Start with the 0-handle = $B^4$. Its boundary is $S^3$, which we take to be the paper. 1-handles $B^1 \times B^3$, are attached along $S^0 \times B^3$, so we draw a pair of 3-balls. It is often more convenient to represent a 1-handle by an unknotted circle with a dot. One can visualize the dotted circle as an equator of a linking sphere to the 1-handle, so that arcs which go across the 1-handle are drawn to link the dotted circle. Equivalently, one can interpret the dotted circle as the boundary in
$S^3$ of a disc whose interior has been pushed into the interior of the 4-ball. Removing
a tubular neighborhood of such a disc is equivalent to adding a 1-handle.

2-handles $B^2 \times B^2$, are attached along $S^1 \times B^2$. We draw $S^1 \times \{0\}$, a knot in
$S^3$, and label it with an integer, its framing, to indicate how the remainder of the
attaching region embeds in $S^3$. 3 and 4-handles are not drawn. The resulting picture
is called a Kirby diagram. When a Kirby diagram has no 1-handles, we can associate
to it a linking/framing matrix $A = [a_{ij}]$. In this case, if $i \neq j$, $a_{ij}$ is the algebraic
linking number of the $i$th attaching circle with the $j$th. The diagonal entries are the
framings of the corresponding circles. $A$ is readily seen to be the intersection form
of the 4-manifold represented by the Kirby diagram.

A 1-handle and 2-handle taken together will collapse into the 4-ball if the attaching
region of the 2-handle passes across the 1-handle geometrically once. Similarly, a 2-3
handle pair will cancel (see Figure 4.2). One should also note that isotopy of the
attaching maps of handles does not change the diffeomorphism type of the underlying
manifold. Conversely we have the following

**Theorem 4.1 (Cerf)** Two smooth 4-manifolds $M$ and $N$ are diffeomorphic if and
only if their handlebody decompositions are equivalent by isotopy of the attaching maps
and by creation or cancellation of handle pairs.

Kirby calculus is a method for applying Cerf’s theorem directly to Kirby diagrams.
A creation (cancellation) of a handle pair corresponds to the addition (removal) of
one of the following, disjoint from the remainder of the diagram.
Isotopy of an attaching region corresponds to the passing of one handle over another: 1-handles may be passed over 1-handles, 2-handles may be passed over 2-handles, and 2-handles may be passed over 1-handles if we view the dotted circle sitting in $S^3$ as above. To pass one 2-handle over another (picture a grapefruit rind passing over an orange rind), band the first attaching circle to a parallel copy of the second, noting the framing when finding the parallel. Figure 4.3 demonstrates that $S^2 \times S^2$ is diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, by passing one 2-handle over another.

Similarly, a 2-handle may be passed over a 1-handle by treating the dotted circle as a 0-framed circle.

If our handlebody decomposition does not contain a 4-handle, then the underlying manifold has nonempty boundary. In studying this boundary 3-manifold, we are
permitted two additional moves (though each changes the 4-manifold):

1. We may remove a dotted circle and replace it with a zero framed circle. This has the effect of surgering the circle defined by the 1-handle. Adding a 1-handle to $B^4$ yields $S^1 \times B^3$, with boundary $S^1 \times S^2$. Adding a 2-handle to a zero framed unknot yields $S^2 \times B^2$, which has the same boundary.

2. We may add (delete) a $\pm 1$ framed unknot, disjoint from the remainder of the diagram. This process is called blowing up (blowing down). From the perspective of the 3-manifold, we are just taking a connected sum with $S^3$ in an interesting way. The 4-manifold, however, sees a connected sum with $CP^2$ or $\overline{CP^2}$, depending on the sign of the framing.

## 4.2 The Seiberg-Witten invariant

The last tool we will need in our construction is an invariant of smooth, closed, oriented, simply connected 4-manifolds known as the Seiberg-Witten invariant. We will not give an explicit definition here, rather we will describe the invariant and state without proof the main theorems which we will use.

Let $X$ be a smooth, closed, oriented, simply connected 4-manifold with $b_2^+ > 1$ and odd. Let $C_X$ be the set of characteristic elements in $H^2(X; \mathbb{Z})$. The Seiberg-Witten function, $SW_X : C_X \to \mathbb{Z}$, is a function on $C_X$ which yields an integer (the Seiberg-Witten invariant) which is invariant under diffeomorphism of $X$. I.e., for an
orientation preserving diffeomorphism \( f : X \to X' \), we have \( SW_{X'}(\alpha) = \pm SW_X(f^*\alpha) \).

(The sign is determined by a choice of orientation on the moduli space of solutions to the monopole equation. See \([\text{Mor}]\) or \([\text{GS}]\) for details.)

We now state an important vanishing theorem for \( SW_X \).

**Theorem 4.2** Let \( X \) be a smooth, closed, oriented, simply connected 4-manifold with \( b_2^+ > 1 \) and odd. If \( X = X_1 \# X_2 \) and \( b_2^+(X_i) > 0 \) for \( i = 1, 2 \), then \( SW_X \equiv 0 \).

For complex surfaces, we have the following nonvanishing result. Note that for any complex surface, \( b_2^+ \) is odd.

**Theorem 4.3** If \( X \) is a simply connected complex surface and \( b_2^+ > 1 \), then \( SW_X \) is nontrivial. In particular, \( SW_X(\pm c_1(X)) \neq 0 \), where \( c_1(X) \) represents the first Chern class of \( X \).

In Section 4.3 we will use these results to prove certain 4-manifolds nondiffeomorphic.

### 4.3 An exotic construction

Let \( W \) be a compact, smooth manifold with two simply connected boundary components \( M \) and \( M' \), such that the inclusions \( M \to W \) and \( M' \to W \) are homotopy equivalences. Then \( W \) is said to be an \( h \)-cobordism between \( M \) and \( M' \). The following is known as the \( h \)-cobordism theorem.
Theorem 4.4 (Smale) If $W$ is an $h$-cobordism between the simply connected $n$-manifolds $M$ and $M'$, and $n \geq 5$, then $W$ is diffeomorphic to $M \times [0,1]$ and (consequently) $M$ is diffeomorphic to $M'$.

Freedman [Fr] has proved the topological $h$-cobordism theorem for 4-manifolds, i.e. substitute homeomorphism for diffeomorphism in the statement of the theorem, but counterexamples are known in the smooth case. Our exotic construction involves explicitly constructing an $h$-cobordism between two nondiffeomorphic 4-manifolds. The procedure can be carried out in greater abstraction, but we provide a specific example here due to Gompf [GS]. Begin with a handlebody decomposition of a connected sum of the $K3$ surface with $\mathbb{CP}^2$. A Kirby diagram for $K3\#\mathbb{CP}^2$ with the 4-handle removed is given in Figure 4.4(a) (for others see [HKK]). Note the submanifold which is drawn darker. In this subhandlebody, pass the 0-framed 2-handle over the 1-framed 2-handle using a half-twisted band as shown in (b). This increases the framing of the first handle from 0 to 1. Passing this newly 1-framed 2-handle over the isolated $-1$-framed 2-handle brings the framing back to 0. Considering only the submanifold $Y$ consisting of this 0-framed 2-handle and the 1-handle, yields (c), which is isotopic to (d). Note that $Y$ is contractible, since without changing the homotopy type we may reverse the crossing in the 2-handle indicated by the arrow in (d). The picture is then a cancelling 1-2 handle pair.

We now consider $\partial Y$, and surger the 1-handle to become a 0-framed 2-handle. Due to the symmetry of the diagram, we may easily find a diffeomorphism $\varphi : \partial Y \to$
Figure 4.4: Construction of the $h$-cobordism

$\partial Y$ which interchanges the two components, namely, rotate 180 degrees about a vertical line in the paper. Define $X$ to be the manifold obtained by removing $Y$ from $K3\#\overline{CP^2}$, and regluing it via $\varphi$. We claim that $X$ is not diffeomorphic to $X' = K3\#\overline{CP^2}$. First observe that $K3\#\overline{CP^2}$ is a complex surface. $b^+_2(K3\#\overline{CP^2}) = b^+_2(K3) = 3$, so by theorem 4.3, $SW_{X'}$ is nontrivial. On the other hand, we show that $SW_X$ is identically zero.
To see the effect on $X$ of cutting out $Y$ and regluing by $\varphi$, we trace the two 2-handles of $Y$ back to Figure 4.4(a). The diffeomorphism of $\partial Y$ switches the dotted circle with the dark 0-framed circle. As a result, the rightmost 1-framed circle is unlinked from the dotted circle, so it can be blown down. Thus $X$ can be written as $X_0 \# \mathbb{C}P^2$ for some smooth, closed manifold $X_0$ with $b_2^+(X_0) = 2$. Thus by Theorem 4.2, $SW_X \equiv 0$. Hence, $X$ is not diffeomorphic to $K3\#\mathbb{C}P^2$. But they are homotopy equivalent (hence homeomorphic by Theorem 1.7), and so our surgery procedure above defines an $h$-cobordism between $K3\#\mathbb{C}P^2$ and $X$. The following theorem [GS] now indicates we have constructed a small exotic $\mathbb{R}^4$.

**Theorem 4.5** Let $W$ be a smooth $h$-cobordism between closed, simply connected 4-manifolds $X$ and $X'$. Then there is an open subset $U \subset W$ homeomorphic to $I \times \mathbb{R}^4$ with a compact subset $K \subset U$ such that the pair $(W - K, U - K)$ is diffeomorphic to $I \times (X - K, U \cap X - K)$. The subsets $R = U \cap X$ and $R' = U \cap X'$ (homeomorphic to $\mathbb{R}^4$) are diffeomorphic to open subsets of $\mathbb{R}^4$. If $X$ and $X'$ are not diffeomorphic, then there is no smooth 4-ball in $R$ (resp. $R'$) containing the compact set $K \cap R$ (resp. $K \cap R'$), so both $R$ and $R'$ are exotic $\mathbb{R}^4$'s.

### 4.4 Complexity

Notice that the 4-manifold in Figure 4.5(a) is diffeomorphic to $Y$ if the unmarked 2-handle is given framing $-1$. Gompf shows [GS] that if we replace this 2-handle
with a 0-framed circle, then there is a Casson handle which we may attach there to produce the small exotic $\mathbb{R}^4$ constructed in Section 4.3. Moreover, it is shown that attaching a simpler Casson handle, namely the one with a single positive self-intersection at each stage (Figure 4.5(b)), also yields a small exotic $\mathbb{R}^4$, which we denote $R$. To see directly that $R$ is homeomorphic to $\mathbb{R}^4$, replace the Casson handle in Figure 4.5(b) with a standard 2-handle. We then have Figure 4.5(a), where the unmarked 2-handle has framing 0. That 2-handle now cancels the 1-handle it goes across (the linking dotted circle), and the remaining 1-2 handle pair cancels to leave $B^4$. Thus Freedman’s result implies $R$ is homeomorphic to $\mathbb{R}^4$.

We now compute an upper bound for the complexity of $R$. Define $Y_n$ to be the compact submanifold of $R$ obtained by cutting off the Casson handle after the $n$th stage, so that $Y_2$ is pictured in Figure 4.6(a). Begin with the handlepass indicated by the arrow, and isotope the picture to obtain (b). Another handlepass and isotopy produces (c). Passing 1-handles twice then yields (d), which is the same as (a) with one fewer 1-2 handlepair, and the rightmost 1-handle replaced by its untwisted
Figure 4.6: $Y_2$
Whitehead double. Iterating this sequence of moves, but doubling the number of 1-handle passes in (c) for each iteration allows us to cancel 1-2 handlepairs of \( Y_n \), obtaining the \( n \)-fold untwisted Whitehead double of the unknot for the rightmost dotted circle. Passing that dotted circle \( 2^n \) times over the middle dotted circle as indicated in 4.6(d), allows us to cancel another 1-2 pair to give us Figure 4.7(a) which is isotopic to 4.7(b), where the \( n \)-fold double is drawn for convenience as a thickened strand. We now wish to determine the boundary of \( Y_n \). Surger the 1-handles by

Figure 4.7: Calculation of \( \partial Y_n \)
replacing the dotted circles with 0-framed circles, and pass each strand of the \( n \)-fold double over the left circle twice as indicated in (b). The right circle is now seen to be a 0-framed meridian of the left, and so the two can be cancelled to leave (c) which is isotopic to the \( n \)-fold untwisted Whitehead double of the \((-3, 3, -3)\) pretzel knot pictured in (d).

The importance of this computation is that \( \partial Y_n \) is seen to be the 3-manifold obtained by 0-surgery on a knot in \( S^3 \) (the \( n \)-fold untwisted Whitehead double of the \((-3, 3, -3)\) pretzel knot). so a Mayer-Vietoris sequence shows \( H_1(\partial Y_n; \mathbb{Z}) = \mathbb{Z} \). Since every compact subset of \( R \) is contained in some \( Y_n \), we have \( m(R) \leq 1 \).
Bibliography


