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Carrots and Sticks: Enforceable Effort and the Minimum Wage

by

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A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

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ABSTRACT

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Rent controls lead not only to a smaller market quantity of rental housing; they also result in a deterioration in the quality of available rental housing. A model of the labor market is presented in which the same kind of adjustment can take place when a minimum wage is imposed. The quality of a job is represented by the effort level which workers expend on the job. We assume that firms can observe and control this level of effort. This assumption may be particularly true of the lower paying jobs which would be most affected by the imposition of a minimum wage.

Although effort can be controlled, there are limits to how much effort the firm would enforce. To begin with, enforcing effort is costly. In addition, workers have the option of working elsewhere or engaging in non-market activity.

We examine the Pareto optimum and short-run perfectly competitive versions of this problem. The two coincide when the on-the-job utility takes a form which eliminates income effects.

When a minimum wage is imposed, the short-run perfectly competitive system is no longer Pareto optimal, even in the “second best” sense. For commonly used functional
forms, we find that required effort increases enough to make utility on the job decline when a minimum wage is imposed. This contradicts the classical notion that the workers who keep their jobs under a minimum wage are better off. The overall welfare generated by the system also declines.

This type of adjustment in the quality of jobs is not accounted for in the standard analysis. When the adverse effects of a minimum wage are measured only in terms of lost employment, therefore, they may be underestimated.
Acknowledgements

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How many have along the way
Encouraged me in word and deed!
How many friends did fervent pray
For grace to help in time of need!

And I, unequal to the task,
Delayed so long, made waste of time,
But now, dear reader, this I ask,
To tell my thanks in form of rhyme

To Pastor and to Mrs. B.,
Who tender love did rich bestow,
With counsel wise, with comfort sweet,
A boon their fellowship to know.

To kin but not by fleshly birth
At Westside and away at home,
A ring which stretches round the earth,
A wealth of care which I have known.

To A. K. A., dear friend of late
Who sparkles bright with godly traits,
May God through you His Son reveal
And you from every ill conceal.

To Duen and Nancy, made of gold,
Whose worth and hospitality,
Whose children fair, both young and old,
Have made me doubt reality.
To Dr. H., advisor fine,
Whose patience far surpasses mine,
A scholar of keen mind possessed,
A gentleman with kindness blessed.

To mom and dad, what can I say
For all that you have done for me?
In lifetimes I could not repay
Your sacrifices, given free.

To Joyce, my sister, firm and true,
Where would I be, if not for you?
Who like you, gentle, like you, wise,
Who always looks through loving eyes?

To God, who worked through all these means,
Alone is all the glory due;
Sometimes too high a mountain seems,
Yet in His power I all can do.
And though my work in ways is small,
And for grand praise it does not call,
Such praise is His- not part, but all.

"He giveth power to the faint, and to those who have no might he increaseth strength." (Isaiah 40:29)

"I can do all things through Christ, who strengtheneth me." (Philippians 4:13)
Table of Contents

1 Introduction

2 "Effort" Models
   2.1 Efficiency Wages
   2.2 The Principal-Agent Problem
   2.3 Enforceable Effort
   2.4 Preview

3 The Basic Model
   3.1 Introduction
   3.2 Pareto Optimum
      3.2.1 The Problem
      3.2.2 Corner Solutions
      3.2.3 The Lagrangian Formulation
      3.2.4 Second Order Conditions
   3.3 Price-Taking Equilibrium
      3.3.1 The Problem
      3.3.2 Lagrangian Formulation
      3.3.3 Second Order Conditions
   3.4 Pareto Optimum - Price-Taking Equilibrium Equivalence
      3.4.1 The Utility Function and the Participation Constraint
      3.4.2 Pareto Optimum Revisited
      3.4.3 Price-Taking Equilibrium Revisited
3.5 Specific Formulations

3.5.1 Separable, Linear-in-Income Utility

3.5.2 Separable, Nonlinear-in-Income Utility

3.5.2.1 Pareto Optimum

3.5.2.2 Price-Taking Equilibrium

4 The Minimum Wage

4.1 The Price-Taking Equilibrium

4.2 The Pareto Optimum

4.3 Specific Formulations

4.3.1 The Price-Taking Equilibrium

4.3.2 The Pareto Optimum

5 Conclusions

Bibliography
1 Introduction

The current U.S. Administration has proposed several increases in the Federal minimum wage over the last few years. These proposals were opposed by many in Congress, making the policy a very controversial one.

The effects of the minimum wage on employment have also long been the subject of much discussion among economists. In a paper that was widely cited by supporters of minimum wages, Card and Krueger\textsuperscript{1} maintained that a minimum wage does not have appreciable effects on employment and, surprisingly, that the number of jobs may well rise when a minimum wage is introduced. Upon closer examination, it was found that their study of data from fast food franchises in Pennsylvania and New Jersey was seriously flawed.\textsuperscript{2} Accurate payroll data for their sample yielded the standard result that employment falls with the imposition of a minimum wage. However, amidst the stir created by Card and Krueger’s work and the testimonies before Congress, an impression emerged that the effect of a minimum wage on employment may not be as large as standard economic theory predicts.

The classical economic framework which describes the effects of a minimum wage is shown in the diagram below. Point E, the intersection of the supply of labor and the demand for labor, represents the equilibrium combination of wage $w$ and employment $N$ which prevails in a free market. With a minimum wage of $w_m$, the amount of labor employed by firms decreases to $N_m$. Point B represents the number of people who would like to work at the new higher wage. At least until these people become discouraged and

\textsuperscript{1} Card and Krueger (1994)
\textsuperscript{2} Employment Policies Institute (1995)
cease searching for a job, the excess of workers \( N_1 - N_m \) would produce the involuntary unemployment created by the minimum wage \( w_m \).

Note that the minimum wage results in a job loss of \( N - N_m \). This job loss is the most common way to describe the adverse effects of a minimum wage.

The "lucky" workers who find employment under the minimum wage now receive \( w_m \) instead of \( w \). These workers are better off as a result of the minimum wage. Thus some workers gain from the minimum wage, while other workers lose their jobs. At the same time, firms lose profit. Overall, the loss in total surplus due to the minimum wage is given by the area AEC in the diagram above.

With the imposition of a minimum wage, the total wage bill goes from \( w \times N \) to \( w_m \times N_m \). If the demand for labor is inelastic enough, the wage bill will increase with the imposition of a minimum wage.
It should be noted that the diagram above represents the effects of a minimum wage on a competitive labor market. If the employer is a monopsonist, the effects of a minimum wage are likely to be quite different. In particular, as long as the minimum wage remains below the marginal product of labor, employment and total surplus would increase rather than decrease as they do above.

The departure point for our analysis is the implicit assumption in the standard framework that all the “quality” aspects of a job (“working conditions”) are constant. The only variable aspect of employment is the wage and the number of employees; that is why no other variable is used to describe a job. In this context, a higher wage always means a happier worker, as well as a desire on the part of the firm to economize on a relatively more expensive input into production.

It is common, though, to hear that job satisfaction is linked as much to “working conditions” as to wages.

More importantly, it is widely recognized that price controls in product markets lead to changes in the quality of the product. The imposition of rent controls, for instance, has consistently resulted in a deterioration of the quality of rental housing, as well as a decrease in the amount of rental housing supplied.

These changes are described by Albion and Stafford as follows¹:

The effects of rent control have been analysed in an enormous number of textbooks and academic articles over very many years. In most cases the expositor draws attention to the adverse effects of legislative rent ceilings on the

¹ Albion and Stafford (1990)
quality of affected dwellings, although this predicted effect does not flow directly from the analysis undertaken. In particular, the standard neoclassical supply and demand approach assumes that dwellings are homogeneous and invariant in quality.

The claim that a price ceiling in the rental housing market has a definite adverse effect on the quality of rental housing is largely an empirical claim. Such an effect is not accounted for in the usual supply and demand framework.

But a minimum wage is also a form of price control. A minimum wage is a price floor in the labor market. Is it not possible that such a price restriction will lead to a change in the quality of the product in question, namely jobs? And is it not possible that this effect, like the one described above in rental housing, is unaccounted for in the standard supply and demand framework?

Such an adjustment can take the form of the employees being asked to work harder, to put in greater effort than before in return for the higher wage which they will receive.

It is not commonly assumed that a firm has the power to make its workers put forth greater effort. However, mention has been made of such a workplace “arrangement”. Stiglitz refers briefly to a set-up in which random shocks to production can be observed, thus allowing effort to be inferred. He describes this workplace environment as follows¹:

¹ Stiglitz (1975)
Workers voluntarily undertake to be supervised; a certain amount of compulsion will be characteristic of competitive equilibrium. They are each working harder than the incentive system itself provides an inducement to work. They submit to being compelled to work harder than direct incentives provide for, because the consequence is higher expected utility. Although each worker may resent this compulsion and feel it is unnecessary on his own part, he prefers to work for firms which use this compulsion, recognizing that without it, some of his colleagues will slough on the job, and the firms which employ some degree of compulsion are able to pay higher wages.

In a survey article, Sah also makes mention of the ability which a firm might have to control how hard its employees work¹:

The preceding discussion suggests the importance of another organizational device, namely the voluntarily agreed-upon supervision of one set of individuals by another, for modifying the nature and the intensity of the activities of employees who have different attributes and motivations. Though this device is extensively used in practice, it has not received as much economic study as it deserves.

¹ Sah (1991)
Hartley describes a firm's ability to influence the level of effort expended by its workers as follows.\(^1\)

Contrary to efficiency wage theory, we assume that the employer can to some extent observe and control the effort (net output per hours of labor input) of his employees. For example, the employer can increase supervision, decrease the number and length of work breaks, reduce the amount of socializing on the job, cut "fringe benefits," control losses from breakage or pilfering, or change the assembly line to force workers to produce more output per hour, or produce fewer defective items.

We shall refer to the firm's ability to influence the effort put forth by workers as "the stick". The "carrot" of higher wages and the "stick" of supervision (or effort enforcement) are the instruments which the firm uses to attain the "quality" of labor which it desires.

What would stop a firm from whipping its workers to death in such a setting? First, supervision is costly, and we assumed that the marginal cost of such supervision increases with the level of effort. Secondly, a firm cannot make its workers "less happy" than they would be if working for a competitor. Finally, we assume that individuals have reservation utilities from non-market activities, and they would turn to these non-market

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\(^1\) Hartley (1992)
activities if that course of action provided them with greater utility than working\textsuperscript{1}.

While the assumption of enforceable effort may not be suitable for every level of employment, it may be particularly appropriate for jobs at the lower end of the wage scale. These are the types of jobs which would probably be influenced by a minimum wage. Such low-wage employment is more likely to come with the kind of supervision which has been described above.

What we assume, therefore, is that a job should not be described only by a wage level, its other features being invariant. Rather, a job should be characterized by a wage level and an effort level. It is within the firm’s power not only to observe, but also to control the effort level which its workers will put forth.

When a minimum wage is imposed, an adjustment in the “quality” of jobs may take place. Workers could well be asked to work harder under such circumstances. Higher effort may more than offset the effects of a higher wage, so that the workers who retain their jobs could be worse off than they were before the minimum wage was imposed. This kind of adjustment is not allowed for in the usual classical framework.

The traditional ways of assessing the effects of a minimum wage ignore the adjustment described above. Thus, the adverse effects of a minimum wage could well be underestimated.

In addition to raising the required level of effort, a firm may respond to a minimum wage by cost-saving measures which would reduce the level of cleanliness or safety in the workplace. These measures would also reduce workers’ on-the-job utility

\textsuperscript{1}It is possible to think of these non-market activities as entrepreneurial ventures. However, the important point as far as our model is concerned is that these activities are an alternative source of utility for the worker.
and raise the firm’s profit. Requiring higher effort has the same effects. We will assume that the firm’s response to a minimum wage takes the form of requiring higher effort. Other features of the workplace could readily be subsumed in our “effort” variable.

2 “Effort” Models

2.1 Efficiency Wages

In most of the literature which relates to effort, it is assumed that the firm carries a much smaller stick than the one which we have outlined above. Most of the time, it is supposed that the firm cannot observe effort at all, let alone control it. The prevalent assumption is that the firm checks on a worker with a certain probability. If found shirking, a worker is fired.

Efficiency wage models, which are quite common in the literature, make use of this concept of supervision. The workings of such models are explained by Shapiro and Stiglitz¹

… we show how the information structure of the employer-employee relationships, in particular the inability of employers to costlessly observe workers’ on-the-job-effort, can explain involuntary unemployment as an equilibrium phenomenon…

¹ Shapiro and Stiglitz (1984)
The intuition behind our result is simple. Under the conventional competitive paradigm, in which all workers receive the market wage and there is no unemployment, the worst that can happen to a worker who shirks on the job is that he is fired. Since he can immediately be rehired, however, he pays no penalty for his misdemeanor. With imperfect monitoring and full employment, therefore, workers will choose to shirk.

To induce its workers not to shirk, the firm attempts to pay more than the "going wage"; then, if a worker is caught shirking and is fired, he will pay a penalty. If it pays one firm to raise the wage, however, it will pay all firms to raise their wages. When they all raise their wages, the incentive not to shirk again disappears. But as all firms raise their wages, their demand for labor decreases, and unemployment results. With unemployment, even if all firms pay the same wages, a worker has an incentive not to shirk. For, if he is fired, an individual will not immediately obtain another job. The equilibrium unemployment rate must be sufficiently large that it pays workers to work rather than to take the risk of being caught shirking.

Such models of the relationship between workers and employers seem to arise in part from the "sharecropping" framework used by Stiglitz. The employer is pictured as an absentee landlord, far removed from the scene of production. Such a metaphor may exaggerate the distance between the employer and his workers. In particular, lower paid workers are the objects of considerably closer scrutiny than the "sharecropping" framework.

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1 Stiglitz (1974)
analogy suggests.

2.2 The Principal-Agent Problem

Efficiency wage models can be thought of as part of the principal-agent framework. In this set-up, a "principal" hires an agent to perform a certain task on his behalf. Since the principal does not have perfect information regarding his agent's activities, moral hazard results. The agent has the incentive to shirk and to lay the responsibility for poor performance on factors which the principal cannot observe. In the sharecropping setting, an agent may claim that low output was the result of poor soil conditions or low levels of rainfall.

The principal must design a contract which will motivate the agent to provide a particular level of effort. In formal terms, the principal maximizes his profit subject to two constraints. The incentive compatibility constraint requires the agent's utility to be greater when providing the required effort than when shirking. The participation constraint stipulates that the agent's "no-shirking" utility must also be greater than his reservation utility.

Under our assumptions, the incentive compatibility constraint is done away with. The employer has the ability (with the limitations we have explained above) to make his employees work as hard as he wants them to. This also removes from the picture all the "signaling structures" which are commonly used to describe such a setting. As the firm can observe effort perfectly, no signaling or signal extraction is necessary.
2.3 Enforceable Effort

Hartley\(^1\) defines effective labor services as net output per hour of labor input. Effective labor services are an increasing function of effort and on-the-job utility.

In Hartley's model, effort can be observed and controlled by the firm. Effort enforcement measures such as those he mentions in the quotation above are assumed to have a direct and an indirect effect. The direct effect of enforcement is an increase in worker effort. However, enforcement decreases workers' on-the-job utility, which makes them work less effectively. Through this indirect, "negative feedback" effect, enforcement lowers "effective labor services". This effect is assumed to be smaller than the direct effect for small values of effort. In addition, enforcement is not costless.

In more technical terms, the function \(G(e, U(wH, e))\) represents the worker's effective labor services per hour worked. The variables involved are effort \(e\), wage \(w\), hours worked \(H\), and on-the-job utility \(U(wH, e)\).

The functions \(G\) and \(U\) are assumed to satisfy \(G_1 > 0, G_2 > 0, U_1 > 0, U_2 < 0, U_{11} < 0, \) and \(U_{22} < 0\). In addition, the function \(G\) and its first partial derivatives are assumed to have the following limiting properties

\[\begin{align*}
\text{i) } & G \to 0 \text{ if both } e \to 0 \text{ and } U(wH, e) \to U_0, \text{ where } U_0 \text{ is the utility which the worker can obtain in his best alternative.} \\
\text{ii) } & G_1 \to 0 \text{ as } e \to \infty \\
\text{iii) } & G_2 \to 0 \text{ as } U(wH, e) \to U_0 \\
\text{iv) } & G_2 \to 0 \text{ as } U(wH, e) \to \infty \\
\end{align*}\]

---

\(^1\) Hartley (1992)
The profit of the firm per worker is given by

\[ \pi(e, w, H) = f(G(e, U(wH, e))H) - wH - c(e) \]

where \( f(GH) \) is output as a function of effective labor services (\( f' > 0, f'' < 0 \)) and \( c(e) \) is the cost of enforcing effort level \( e \) (\( c' > 0, c'' > 0 \)). With the number of workers supposed fixed, the above function can be maximized with respect to effort, wage, and hours worked. If one considers hours worked to be fixed, then profit can be maximized with effort and wage alone as the choice variables.

On the other hand, with both hours \( H \) and the number of workers \( N \) considered to be variables, the firm's total profit function is given by:

\[ \pi(N, e, w, H) = f(G(e, U(wH, e, H))HN) - wHN - c(e)N - vN \]

where \( f(GHN) \) represents output, \( wHN \) is the total wage bill, \( c(e)N \) is the total cost of effort enforcement, and \( vN \) is a training cost. The parameter \( v \) represents a fixed cost of employment, a cost having to do with hiring and training workers. The introduction of this cost is necessary to bring about a "separation" between hours \( H \) and number of workers \( N \).

Specific functional forms are then chosen for the functions above. It is shown numerically that, under reasonable assumptions, the imposition of a minimum wage lowers the firm's profit and the worker's utility.
The main drawback of this particular model is that it is monopsonistic in nature. The model does not include a participation constraint. In a competitive framework, the firm would need to provide the worker with a certain amount of utility (just as a firm in standard perfect competition must be willing to pay the prevailing market wage in order to attract workers). In such a set-up, the firm takes the market-determined on-the-job utility as given. In particular, the firm does not take account of the effect of its actions upon the prevailing market utility.

In contrast, the worker's on-the-job utility function is embedded in the firm's objective function in Hartley's model. In addition, this model leads to a worker having higher utility at his job than he would at another firm. This could be true if firm-specific human capital or imperfect information were part of the picture. However, these types of elements are not part of the model.

Our model removes utility from the production function. This pares the factors of production down, making that part of the problem slightly simpler. More importantly, on-the-job utility is determined in a market context. The firm's maximization problem leads to a demand for labor as a function of a given level of on-the-job utility.

The population is assumed to have a distribution of reservation utilities. This distribution, along with the participation constraint, leads to a supply of labor as a function of a given level of on-the-job utility. Supply and demand\(^1\) then interact to determine on-the-job utility. This is the same framework which determines the wage in a simple perfect competition model.

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1 It should be noted that supply and demand are multivariable functions in this framework. Each depends on the prevailing on-the-job utility and the dividend payment which each individual receives. This will be explained below.
We also assume that the combined profits of all firms are redistributed evenly to all members of the population. This allows us to examine the welfare properties of our model. In the absence of this assumption, we would have firms making profits and individuals enjoying utility. Trying to say something about the combination of profit and utility in that context would be like trying to add apples and oranges. Putting all the "benefits" generated by the system into individuals' hands is, roughly speaking, the way to transform the apples into oranges.

Finally, we choose to have the number of workers to be variable but hours of labor to be fixed. Having both as variables requires the introduction of a fixed cost of employment; this, or some similar assumption, is necessary to "split" the two variables apart. In such formulations, we discovered that the number of hours was determined by the parameters of the model, such as the powers used for each factor of production in a Cobb-Douglas production function.

Therefore, we decided to retain the number of workers as a variable, since it seemed to capture more economically meaningful aspects of the problem. With the hours of labor taken as fixed, it was no longer necessary to include a fixed cost of employment.

2.4 Preview

In chapter three, we present our model, considering first the Pareto optimum problem and then the competitive equilibrium. For each problem, we discuss the possibility of corner solutions, derive and interpret first order conditions, and examine
second order conditions. Why and how the two problems differ is also explained in some detail, as well as the conditions under which the two problems would coincide.

We then choose some simple specific functional forms and find solutions for each problem. In one case, analytical solutions can be obtained. However, numerical analysis is called for most of the time.

In chapter four, we examine how the Pareto optimum and the competitive equilibrium are affected by the imposition of a minimum wage. Again, numerical solutions are given for each problem for specific functional forms.

Chapter five offers some conclusions.
3 The Basic Model

3.1 Introduction

What sets our model apart from much of the literature is the assumption which we make concerning effort. Following Hartley (1992), we assume that effort can be observed perfectly by the firm and that the firm can enforce a particular level of effort. In other words, the firm chooses “how hard” its employees will work.

Output depends on two inputs, effort and the number of workers. We assume that diminishing marginal productivity holds for each input. Let $e$ and $N$ denote the level of effort and number of workers respectively. Our production function can therefore be represented as $F(e, N)$, with $F_e, F_N > 0$ and $F_{ee}, F_{NN} < 0$. We assume that the production function is strictly concave.

Two considerations limit the level of effort which the firm will require of its workers. In the first place, the firm incurs some cost for enforcing a particular effort level, and this cost rises as the effort level rises. We also assume that the marginal cost of effort enforcement increases as the level of effort increases. Thus, the cost of enforcing effort level $e$ per worker can be represented as $c(e)$, with $c'(e) > 0$ and $c''(e) > 0$. With $N$ workers, the total cost of enforcing effort level $e$ will be $c(e)N$.

We can now define the representative firm’s profit as

$$\pi (N, e, w) = F(e, N) - wN - c(e)N,$$

where $w$ represents the wage. The first term in the profit function represents the firm’s output, the second represents the wage bill, and the third represents the cost of enforcing effort. Note that all workers are identical in producing market output. They are also
treated identically, as they paid the same wage and are required to expend the same effort.

The second consideration limiting the effort which the firm requires of its workers is that each firm must ensure that the worker is not worse off than he would be if working for another firm. Firms are assumed to be identical, and each firm is assumed to be infinitesimal compared to the market as a whole. Under such conditions, the value of the on-the-job utility rate is determined by the market. A single firm’s actions cannot influence this prevailing utility rate. Note that this is the standard perfectly competitive framework, except that the firm has two variables it can manipulate to match the job offer of its competitors. Let $\bar{U}$ denote the going on-the-job utility rate.

On-the-job utility is assumed to depend positively on the income a worker receives and negatively on the level of effort which he must expend. An individual’s utility can then be represented as $U(I, e)$, with $U_1 > 0$, $U_e < 0$, and $U_{11}, U_{ee} < 0$. We assume that the utility function is strictly concave.

We assume that the total number of firms is equal to 1. When each “fractional” firm makes a particular profit level $\pi$, the combined profits of all firms will be $\pi \times 1$, which is equal to $\pi$. Recall that all firms are assumed to be identical.

Combined profits of $\pi$ are redistributed evenly across the population. We assume that the individual is an infinitesimal part of the population. As in the case of the firms,

---

1 These are the most general set of assumptions which characterize the individual’s utility. Later on, we will examine more closely what occurs when the utility function is separable in income and effort and $U_{11} = 0$. 
we assume that the total number of individuals is equal to one. Thus, each individual receives \( \pi \), which can be thought of as a universal dividend payment.\(^1\)

Individuals who work therefore receive income in two "forms". Those who work are paid wage \( w \) for their labor, and they also receive \( \pi \) in dividends. Since he also expends effort \( e \) on the job, a worker's utility can be represented as \( U(\pi + w, e) \). Any individual who does not work still receives \( \pi \) in dividends. However, for non-workers, wage and effort are equal to zero.

Individuals work if their on-the-job utility exceeds the level of utility they would enjoy off-the-job. All individuals have the option of engaging in some non-market activity instead of working. These non-market activities will be referred to collectively as "picking daisies".

Individuals differ in the extent to which they enjoy picking daisies. Let \( v(n) \) represent the utility which individual \( n \) would enjoy in his preferred non-market activity. We assume that individuals are arranged so that \( v'(n) > 0 \). Thus, all individuals before the \( n \)th derive less utility from picking daisies than the \( n \)th individual does.

If individual \( n \) does not work, he enjoys the utility of picking daisies in addition to the utility he derives from his dividend. The total utility which such a person would enjoy can be represented as \( U(\pi, 0) + v(n) \).

Given a prevailing market on-the-job utility of \( \bar{U} \), individual \( n \) will choose to work if \( U(\pi, 0) + v(n) \leq \bar{U} \). He will pick daisies instead if \( U(\pi, 0) + v(n) > \bar{U} \).

---

\(^1\) There is a distinction between \( \pi \) as profit and \( \pi \) as dividend. \( \pi \) as profit is measured in goods, while \( \pi \) as dividend is measured in goods per person. The two are numerically equal, since the total population is equal to 1.
Recall that \( N \) represents the number of workers. All individuals \( n \leq N \) will choose to work, since working affords them greater utility than picking daisies. All individuals \( n > N \) will choose to pick daisies. For the \( N \)th worker, it must be the case that \( U(\pi, 0) + v(N) = \bar{U} \).

### 3.2 Pareto Optimum

#### 3.2.1 The Problem

An omniscient central planner maximizes total welfare by choosing the number of workers \( N \), the effort level \( e \), the wage \( w \), and the profit level \( \pi \), subject to two constraints. The problem which the central planner deals with can be represented as

\[
\text{Max } W(N, e, w, \pi) = U(\pi + w, e)N + U(\pi, 0) (1 - N) + \int_{N}^{1} v(n)dn
\]

subject to \( F(e, N) - c(e)N \geq \pi + wN \)

and \( U(\pi + w, e) \geq U(\pi, 0) + v(N) \)

We assume that the variables \( N, e, w, \) and \( \pi \) are non-negative.

The first term in the welfare function \( W \) represents the utility per worker multiplied by the number of workers. The second term represents the utility a non-worker enjoys from receiving dividend \( \pi \) multiplied by the number of non-workers. The final term in \( W \) is the sum of the utilities which the non-workers derive from picking daisies.

The first inequality is a resource constraint. It states that the profit and the wage bill chosen by the central planner cannot exceed the difference between output and the
cost of supervision. In essence, this says that the profit and wage bill have to come from output, not out of thin air.

The second inequality is a participation constraint. For the Nth worker to be willing to work, his off-the-job utility must not exceed his on-the-job utility.¹

¹ Before moving on, we should note that \( \int_0^1 u(n) \, dn \) is measured in utils, if we take utility to be in utils/person. Welfare \( W \), however, should be denominated in welfare units ("welfares"). Therefore, our maximand \( W \) would be more properly expressed as

\[
W(N, e, w, \pi) = S \left[ \int_0^1 u(n) \, dn \right],
\]

where \( S \) is a function which transforms utils into welfares. However, since \( S' = 1 \), we will ignore \( S \) as well as its derivative in what follows. Lagrangian multipliers are measured in units of the maximand per unit of either goods or utils.
3.2.2 Corner Solutions

We can reformulate the above as a Lagrangian function, with $\lambda$ as the multiplier of the resource constraint and $\delta$ as the multiplier of the participation constraint:

$$L(N, e, w, \pi, \lambda, \delta) = U(\pi + w, e)N + U(\pi, 0)(1 - N) + \int_0^1 v(n)dn + \lambda \{ F(e, N) - wN - c(e)N - \pi \}$$

$$+ \delta \{ U(\pi + w, e) - U(\pi, 0) - v(N) \}$$

$\lambda$ and $\delta$ satisfy the Kuhn-Tucker conditions, which we list below along with the first order conditions for the variables $N$, $e$, $w$, and $\pi$.

$$N: \quad U(\pi + w, e) - U(\pi, 0) - v(N) + \lambda \left[ \frac{\partial F(e, N)}{\partial N} - w - c(e) \right] - \delta v'(N) = 0 \quad (1)$$

$$e: \quad \frac{\partial U(\pi + w, e)}{\partial e}N + \lambda \left[ \frac{\partial F(e, N)}{\partial e} - c'(e)N \right] + \delta \frac{\partial U(\pi + w, e)}{\partial e} = 0 \quad (2)$$

$$w: \quad \frac{\partial U(\pi + w, e)}{\partial w}N - \lambda N + \delta \frac{\partial U(\pi + w, e)}{\partial w} = 0 \quad (3)$$

$$\pi: \quad \frac{\partial U(\pi + w, e)}{\partial \pi}N + \frac{\partial U(\pi, 0)}{\partial \pi}(1 - N) - \lambda + \delta \left[ \frac{\partial U(\pi + w, e)}{\partial \pi} - \frac{\partial U(\pi, 0)}{\partial \pi} \right] = 0 \quad (4)$$

$$\lambda: \quad \lambda \geq 0; \quad F(e, N) - wN - c(e)N - \pi \geq 0; \quad \lambda \left[ F(e, N) - wN - c(e)N - \pi \right] = 0 \quad (5)$$

$$\delta: \quad \delta \geq 0; \quad U(\pi + w, e) - U(\pi, 0) - v(N) \geq 0; \quad \delta \left[ U(\pi + w, e) - U(\pi, 0) - v(N) \right] = 0 \quad (6)$$

Under what conditions will each of the constraints bind? This is equivalent to asking when the constraints will be satisfied with the "=" sign, not the ">" sign.
We begin with $\lambda$ and the resource constraint, with which $\lambda$ is associated. Assume that the resource constraint does not bind. Then, we will have

$$F(e, N) - wN - c(e)N - \pi > 0.$$  
This, along with the last part of (5), implies that $\lambda = 0$.

Given $\lambda = 0$, equations (3) and (4) can be rewritten as:

$$\frac{\partial U(\pi + w, e)}{\partial w} N + \delta \frac{\partial U(\pi + w, e)}{\partial w} = 0 \quad (3')$$

$$\frac{\partial U(\pi + w, e)}{\partial \pi} N + \frac{\partial U(\pi, 0)}{\partial \pi} (1 - N) + \delta \left[ \frac{\partial U(\pi + w, e)}{\partial \pi} - \frac{\partial U(\pi, 0)}{\partial \pi} \right] = 0 \quad (4')$$

Now, (3') implies that $\delta = -N$. Since $\delta \geq 0$ and $N \geq 0$, it can only be the case that $\delta = N = 0$. Given that this holds, (4') reduces to

$$\frac{\partial U(\pi, 0)}{\partial \pi} = 0,$$

which contradicts one of our basic assumptions about the utility function. $\frac{\partial U(\pi, 0)}{\partial \pi}$ represents the marginal utility of income when the individual is not working. We have assumed that the marginal utility of income is always positive for any values of income and effort.

The assumption that $F(e, N) - c(e)N - \pi - wN > 0$ leads, as we have just seen, to a contradiction. Therefore, the assumption is rejected. We are left with

$$F(e, N) - c(e)N - \pi - wN = 0,$$

and the resource constraint binds.

If the resource constraint did not bind, then we would have

$$F(e, N) - c(e)N > \pi + wN.$$  
That would mean that part of output was not being paid out in the form of wages or redistributed as dividends. This is equivalent to saying that some
part of output is "thrown away", which is incompatible with maximizing behavior as long as extra income brings extra utility.

Does the participation constraint bind as well? Assume that it does not. This is equivalent to assuming that \( U(\pi + w, e) - U(\pi, 0) - \nu(N) > 0 \). By virtue of (6), \( \delta \), the multiplier associated with the participation constraint, must be equal to zero.

Given \( \delta = 0 \), equations (3) and (4) can be rewritten as:

\[
\frac{\partial U(\pi + w, e)}{\partial w} N - \lambda N = 0 \tag{3'}
\]

\[
\frac{\partial U(\pi + w, e)}{\partial \pi} N + \frac{\partial U(\pi, 0)}{\partial \pi} (1 - N) - \lambda = 0 \tag{4''}
\]

(3') implies that \( \lambda = \frac{\partial U(\pi + w, e)}{\partial w} \). We know that

\[
\frac{\partial U(\pi + w, e)}{\partial w} = \frac{\partial U(\pi + w, e)}{\partial \pi}; \text{ the marginal utility of wage income and the marginal utility of dividend income are the same for a worker. Therefore, we have}
\]

\[
\lambda = \frac{\partial U(\pi + w, e)}{\partial \pi}. \text{ Substituting the latter expression for } \lambda \text{ in (4''), we obtain the following relation:}
\]

\[
\left[ \frac{\partial U(\pi, 0)}{\partial \pi} - \frac{\partial U(\pi + w, e)}{\partial \pi} \right] [1 - N] = 0
\]

The first factor in the above product will only be equal to zero if the effect of extra wage income \( w \) and the effect of the extra effort \( e \) on marginal utility are opposite
in sign and exactly equal in magnitude. We can safely assert that this will not happen in
general. Therefore, in order for the product to be zero, we must have \( N = 1 \).

But this would lead our model into a trivial case. All individuals would now be
working. Even the last individual, the person who derives the greatest utility out of daisy-
picking, chooses to work. We would have \( U(\pi + w, e) > U(\pi, 0) + v(N) \) with \( N = 1 \).
The last worker could be regarded, technically speaking, as the "marginal worker".
However, in another sense, we would have no real marginal worker, no individual who is
indifferent between working and picking daisies.

Since we do not wish our model to be confined to a trivial outcome, we will
assume that the participation constraint binds. In other words, we will assume that
\[ U(\pi + w, e) - U(\pi, 0) - v(N) = 0. \]

Thus both constraints bind. We will restate the problem before proceeding further.
3.2.3 The Lagrangian Formulation

The welfare function can be reformulated as a Lagrangian function in which both constraints bind:

\[
L(N, e, w, \pi, \lambda, \delta) = U(\pi + w, e)N + U(\pi, 0) (1 - N) + \int_N^1 v(n)dn + \lambda \left[ F(e, N) - wN - c(e)N - \pi \right] + \delta \left[ U(\pi + w, e) - U(\pi, 0) - v(N) \right]
\]

Note that all the variables are assumed to be non-negative.

This leads to the following set of first order conditions:

\[
N: \quad U(\pi + w, e) - U(\pi, 0) - v(N) + \lambda \left[ \frac{\partial F(e, N)}{\partial N} - w - c(e) \right] - \delta v'(N) = 0 \quad (1)
\]

\[
e: \quad \frac{\partial U(\pi + w, e)}{\partial e} N + \lambda \left[ \frac{\partial F(e, N)}{\partial e} - c'(e)N \right] + \delta \frac{\partial U(\pi + w, e)}{\partial e} = 0 \quad (2)
\]

\[
w: \quad \frac{\partial U(\pi + w, e)}{\partial w} N - \lambda N + \delta \frac{\partial U(\pi + w, e)}{\partial w} = 0 \quad (3)
\]

\[
\pi: \quad \frac{\partial U(\pi + w, e)}{\partial \pi} N + \frac{\partial U(\pi, 0)}{\partial \pi} (1 - N) - \lambda + \delta \left[ \frac{\partial U(\pi + w, e)}{\partial \pi} - \frac{\partial U(\pi, 0)}{\partial \pi} \right] = 0 \quad (4)
\]

\[
\lambda: \quad F(e, N) - wN - c(e)N - \pi = 0 \quad (5)
\]

\[
\delta: \quad U(\pi + w, e) - U(\pi, 0) - v(N) = 0 \quad (6)
\]

Note that the first three terms in (1) disappear by virtue of (6). Also, recall that

\[
\frac{\partial U(\pi + w, e)}{\partial w} = \frac{\partial U(\pi + w, e)}{\partial \pi}.
\]
Equations (3) and (4) can be used to solve for the multipliers $\lambda$ and $\delta$, yielding

$$\lambda = \frac{\frac{\partial U(\pi, 0)}{\partial \pi} \left( \frac{\partial U(\pi + w, e)}{\partial \pi} \right)}{\frac{\partial U(\pi, 0)}{\partial \pi} N + \left( \frac{\partial U(\pi + w, e)}{\partial \pi} \right) (1 - N)}$$

and

$$\delta = \frac{\frac{\partial U(\pi, 0)}{\partial \pi} \left( \frac{\partial U(\pi + w, e)}{\partial \pi} \right)}{\frac{\partial U(\pi, 0)}{\partial \pi} N + \left( \frac{\partial U(\pi + w, e)}{\partial \pi} \right) (1 - N)}$$

As stated above, $\lambda$ will be positive as long as the marginal utility of income is positive. This is reflected in the expression for $\lambda$ presented above.

In order for $\delta$, the multiplier on the participation constraint, to be positive, we need the inequality $\frac{\partial U(\pi, 0)}{\partial \pi} \geq \frac{\partial U(\pi + w, e)}{\partial \pi}$ to hold. Recall that we assume that $\frac{\partial^2 U(\pi + w, e)}{\partial \pi^2} < 0$ for all values of the variables involved. This means that an increase in income from $\pi$ to $(\pi + w)$ would tend to lower marginal utility, making it likely that the inequality will be satisfied.

What influence does the increase in the effort level from 0 to the value $e$ have on the marginal utility of income? If extra effort leaves the worker "tired out", so that he cannot enjoy his extra income as much as before, then the marginal utility of income would diminish with effort. Under such an assumption, the cross derivative $\frac{\partial^2 U(\pi + w, e)}{\partial e \partial \pi}$ would be negative for all values of the variables $\pi$, $e$, and $w$. This constitutes a sufficient condition for the inequality $\frac{\partial U(\pi, 0)}{\partial \pi} \geq \frac{\partial U(\pi + w, e)}{\partial \pi}$ to hold.

If, however, the cross derivative is positive and the value of extra income increases with effort, it is still possible for the inequality to hold. This would be the case
as long as the "cross effect" of the higher effort on marginal utility is smaller in magnitude than the effect of diminishing marginal utility. Cross effects are commonly assumed to be small in magnitude.

It is sufficient therefore, but not necessary for the cross derivative \( \frac{\partial^2 U(\pi, w, e)}{\partial e \partial \pi} \) to be negative for the inequality \( \frac{\partial U(\pi, 0)}{\partial \pi} \geq \frac{\partial U(\pi + w, e)}{\partial \pi} \) to hold.

Substituting the expressions for \( \lambda \) and \( \delta \) into equations (1), (2), and combining with (5), and (6), we obtain four equations in the four unknowns \( N, e, w, \) and \( \pi \).

\[
\frac{\partial F(e, N)}{\partial N} - w - c(e) - \frac{(1-N)N}{\pi} \left( \frac{\partial U(\pi, 0)}{\partial \pi} - \frac{\partial U(\pi + w, e)}{\partial \pi} \right) = 0
\]

\[
\frac{\partial F(e, N)}{\partial e} - c'(e)N + \left( \frac{\partial U(\pi + w, e)}{\partial w} \right) N = 0
\]

\[
F(e, N) - c(e)N = \pi + wN \tag{5}
\]

\[
U(\pi + w, e) = U(\pi, 0) + v(N) \tag{6}
\]

We have already spoken of the economic significance of (5) (the resource constraint) and (6) (the participation constraint).

The three terms of (2') show the various effects of a change in effort. With a rise in effort, there would be an increase in production (the first term) and an increase in the cost of enforcing effort (the second term). These are not the only effects of an increase in effort, however.
As workers are required to work harder, their on-the-job utility will decrease. One possible outcome is for the number of individuals who choose to work to go down. However, changes in N are dealt with in (1'), not (2'). (2') is concerned with changes in effort. As it contains elements drawn from (3) and (4), (2') also incorporates the effects of changes in wages and dividends. But the number of workers N should be taken to be constant in discussing the meaning of (2').

Economically, N can only remain constant if the on-the-job utility remains constant. This will be the case if the individual worker's income rises in a manner which exactly compensates him for the loss of utility brought about by a higher effort level. If the worker must put in more effort, he must be paid more to be as happy as he was before.

\[
\frac{\partial U(\pi + w, e)}{\partial e} \text{ represents the loss in utility from having to work harder.}
\]

\[
\frac{\partial U(\pi + w, e)}{\partial w} \text{ represents the change in utility from extra wage income. When the former is divided by the latter (as in the third term of (2') above), the quotient represents the amount which must be paid to restore the worker to the same level of utility he enjoyed before an increase in effort. This "compensation" is multiplied by the number of workers, and that product constitutes the third term in (2').}
\]

Note that the "compensation" takes place through an increase in the worker's wage, not an increase in the dividend payment. The latter would raise the worker's on-the-job utility. However, it would mean higher dividend payments to non-workers as well. This would be wasteful; the compensation should properly apply to workers only.
To understand the effect of a change in the number of workers, it may be easier to examine (1) than (1'). We reproduce (1) below:

$$U(\pi + w, e) - U(\pi, 0) - v(N) + \lambda \left[ \frac{\partial F(e, N)}{\partial N} - w - c(e) \right] - \delta v'(N) = 0 \quad (1)$$

For the sake of definiteness, we will discuss what happens with an increase in N.

By virtue of the participation constraint, the first three terms in the equation above disappear. The marginal worker is, by definition, indifferent between working and not working. The utility which he gains by working is $U(\pi + w, e)$. The utility he would have enjoyed as a daisy-picker is $U(\pi, 0) + v(N)$. These utilities are equal, so that there is no change in the utility of the worker himself when he stops picking daisies and starts working.

Therefore, (1) reduces to the following:

$$\lambda \left[ \frac{\partial F(e, N)}{\partial N} - w - c(e) \right] - \delta v'(N) = 0$$

Recall that profit is defined as $\pi = F(e, N) - wN - c(e)N$. The term in the square brackets can therefore be thought of as $\frac{\partial \pi}{\partial N}$, the increase in profit with an increase in the number of workers. When an extra worker is hired, profit increases, and this raises the dividend which all individuals receive. Remember that as we substitute the value of the Lagrangian multipliers into (1), we incorporate into (1) the effects of changes in wages and dividends.
\( \lambda \) tells us the value of an increase in income in utility terms. To better see this, we can rearrange the expression for \( \lambda \) as

\[
\lambda = \frac{1}{N} \frac{\partial U(\pi + w, e)}{\partial \pi} + \frac{1-N}{\partial U(\pi,0)}
\]

\( N \) is the number of workers, and \( \frac{\partial U(\pi + w, e)}{\partial \pi} \) is the increase in the utility of a worker with an increase in his income. \((1-N)\) is the number of daisy-pickers, and \( \frac{\partial U(\pi,0)}{\partial \pi} \) is the increase in the utility of a daisy picker with an increase in his income.

Roughly speaking, the denominator of the expression above can be thought of as

\[
\frac{\partial \pi}{\partial U} \cdot N + \frac{\partial \pi}{\partial U} (1-N), \text{ or as a weighted average giving an overall figure for}
\]

\[ \frac{\partial \pi}{\partial U}. \text{ Thus, } \lambda \text{ itself can be regarded as having the same units as } \frac{\partial U}{\partial \pi}. \text{ When } \lambda \text{ is multiplied by the term inside the square brackets, a term which represents } \frac{\partial \pi}{\partial N}, \text{ the result is the change in utility as a result of the extra profit which comes from hiring an extra worker.} \]
The second term in the reduced equation (1) is $-\delta v'(N)$. We can rearrange the expression for $\delta$ so that this second term is given by

$$-\delta v'(N) = \left( \frac{\partial U(\pi + w, e)}{\partial \pi} \frac{\partial U(\pi, 0)}{\partial \pi} \right) N + \left( \frac{\partial U(\pi + w, e)}{\partial \pi} \frac{\partial U(\pi, 0)}{\partial \pi} \right) 1 - N \quad \nu'(N)$$

This term has to do with the participation constraint, which can be represented as

$$U(\pi + w, e) - U(\pi, 0) = \nu(N)$$

An increase in $N$ leads to an increase in the right hand side of the above by the amount $\nu'(N)$. In order for the participation constraint to continue to hold, the left hand side must increase by $\nu'(N)$ as well. Economically speaking, the wage and dividend payments must change in order to raise on-the-job utility and cause an extra individual to abandon daisy-picking for the workforce. These changes in wages and dividends will affect not just the extra worker, but all the previous workers as well.

Unfortunately, it is not easy to arrive at a clear interpretation of the expression for $-\delta v'(N)$ given above. The changes in wages and dividends lead to income effects which
are difficult to separate, even if one maintains a distinction between $\frac{\partial U(\pi + w, e)}{\partial \pi}$ and $\frac{\partial U(\pi + w, e)}{\partial w}$.

An increase in the dividend $\pi$ will raise both $U(\pi + w, e)$ and $U(\pi, 0)$. This is possibly reflected in the numerator of the expression for $-\delta v'(N)$ above. The net effect of such an increase on $[U(\pi + w, e) - U(\pi, 0)]$ is ambiguous a priori. However, the constraint that $\delta \geq 0$ tells us that $[U(\pi + w, e) - U(\pi, 0)]$ will indeed decrease with $\pi$.

On the other hand, the effect of an increase in wage $w$ is more clear cut. Such a change would obviously raise $[U(\pi + w, e) - U(\pi, 0)]$. Thus, an increase in $w$ is very likely to be associated with an increase in $N$.

Once again, the term $-\delta v'(N)$ summarizes the effect on total welfare of the changes in $\pi$ and $w$ which are necessary to attract an extra person into the workforce.

\[ -\delta v'(N) = \frac{\partial U(\pi + w, e)}{\partial \pi} \frac{\partial U(\pi, 0)}{\partial \pi} (1 - N) - \frac{\partial U(\pi + w, e)}{\partial w} \frac{\partial U(\pi + w, e)}{\partial \pi} \frac{\partial U(\pi, 0)}{\partial \pi} \frac{\partial U(\pi + w, e)}{\partial w} \]

\footnote{With such a distinction, the following expression can be derived:}
3.2.4 Second Order Conditions

For the solution of the first order conditions of a constrained optimization problem to represent a unique global maximum, it is sufficient to have a) a strictly quasiconcave objective function and b) a convex constraint set.

Our objective function is given by

\[ W(N, e, w, \pi) = U(\pi + w, e)N + U(\pi, 0) (1 - N) + \int_0^1 v(n)dn \]

The term \( U(\pi + w, e)N \) can be regarded as an infinite sum. This can be represented mathematically as

\[ U(\pi + w, e) = \int_0^N U(\pi + w)dn = U(\pi + w, e) + U(\pi + w, e) + U(\pi + w, e) + \ldots \]

The utility function \( U \) is assumed to be strictly concave; therefore, each element in the sum \( \int_0^N U(\pi + w)dn \) is a strictly concave function. However, the sum of strictly concave functions is strictly concave. Thus, \( U(\pi + w, e)N \) is strictly concave.

By the same reasoning, \( U(\pi, 0) (1 - N) \) will also be a strictly concave function.

The last part of our objective function is the term \( \int_0^1 v(n)dn \), which represents the utility derived from daisy picking. The first and second derivatives of this term with respect to the number of workers \( N \) are given by:

\[ \frac{d}{dN} \int_0^1 v(n)dn = -v(N) \]
\[ \frac{d^2}{dN^2} \int_0^1 v(n)dn = -v'(N) \]
Recall that $v'(N) > 0$. Thus, the second derivative of $\int_0^1 v(n)dn$ with respect to $N$ is strictly negative. This implies that the term $\int_0^1 v(n)dn$ is strictly concave as well.

The three terms $U(\pi + w, e)N$, $U(\pi, 0) (1 - N)$, and $\int_0^1 v(n)dn$ are each strictly concave. Therefore, the objective function, which is the sum of these three terms, is strictly concave as well. This means that our objective function meets the criterion of strict quasiconcavity, since a strictly concave function is also strictly quasiconcave.

The objective function $W(N, e, w, \pi)$ is maximized subject to a resource constraint $(F(e, N) - wN - c(e)N - \pi \geq 0)$ and a participation constraint $(U(\pi + w, e) - U(\pi, 0) - v(N) \geq 0)$. Suppose that two distinct points $A(N_1, e_1, w_1, \pi_1)$ and $B(N_2, e_2, w_2, \pi_2)$ satisfy the resource constraint. Does the convex combination $C(\rho N_1 + (1-\rho)N_1, \rho e_1 + (1-\rho)e_2, \rho w_1 + (1-\rho)w_2, \rho \pi_1 + (1-\rho)\pi_2)$ also satisfy the resource constraint for any value $\rho$ such that $0 < \rho < 1$?

Unfortunately, it cannot be established that the convex combination $C$ satisfies the resource constraint. The same is true for the participation constraint. Therefore, we cannot prove that our constraint set is convex. As a result, our problem does not meet the sufficient conditions outlined above. It should be emphasized, though, that these conditions are sufficient, but not necessary.

It should also be noted that problems with self-selection and/or participation constraints often do not have convex constraint sets. With non-linear constraints, one is less likely to have a convex constraint set.
3.3 Price-Taking Equilibrium

3.3.1 The Problem

Before proceeding, we should explain briefly what we mean by "price-taking" equilibrium. Our "price-taking" framework has all the features of perfect competition except for free entry and exit. In price-taking equilibrium, we consider the number of firms to be fixed. This framework can also be thought of, therefore, as short-run perfect competition.

The price-taking firm maximizes profit by choosing the number of workers $N$, and effort level $e$, and the wage $w$, subject to one constraint. The problem which the firm faces can be represented as

$$\text{Max } \pi (N, e, w) = F(e, N) - wN - c(e)N$$

subject to $U(\bar{\pi} + w, e) \geq \bar{U}$ where $\bar{U} = U(\bar{\pi}, 0) + \bar{v}$

Equilibrium conditions: $\bar{v} = v(N)$

$$\bar{\pi} = \pi = F(e, N) - wN - c(e)N$$

The price-taking firm assumes that its actions cannot affect the prevailing market utility $\bar{U}$ or the dividend $\bar{\pi}$. A single firm is infinitesimal compared to the market as a whole, and $\bar{U}$ and $\bar{\pi}$ are market quantities which are determined jointly by the actions of all firms. Therefore, the firm assumes that these quantities are independent of its actions.

The constraint which the firm faces is a "competitiveness" constraint. The firm must make sure that each worker receives at least the prevailing market on-the-job utility. If not, then the workers will find employment elsewhere instead.
This competitiveness constraint will be met with equality. If not, then the firm would be providing on-the-job utility which exceeds the prevailing market value. All individuals who wish to work would then want to work for that particular firm. As firms are infinitesimal, this is not possible. Again, this is the same reasoning by which a firm in standard perfect competition does not offer a wage higher than the market wage.

The on-the-job utility will be equal to the utility of the marginal worker. Note that by “marginal worker”, we do not mean the last person hired by the individual firm, but rather the last person who forsakes daisy picking for the workforce. This individual will be “the marginal worker” as far as the market as a whole is concerned. His utility can be expressed as $U(\bar{\pi}, 0) + v(N)$.

As explained above, the firm regards the dividend $\bar{\pi}$ as a constant. In addition, the individual firm regards total market employment as a quantity which is independent of its actions. The firm is infinitesimal, and thus its hiring decisions are only a “drop in the bucket” compared to the market as a whole.

Recall that $v(N)$ represents the utility which the market “marginal worker” derives from daisy-picking. Since the firm considers the number of workers in the market to be fixed, it believes it cannot influence $v(N)$. Thus, this quantity is represented as $\bar{v}$. 
It follows that the on-the-job utility \( \bar{U} \) can be expressed, as far as the firm is concerned, as\(^1\)

\[
\bar{U} = U(\bar{\pi}, 0) + \bar{v}
\]

Given a particular level of the market on-the-job utility rate, as well as dividends, the firm chooses the number of workers, effort level, and wage which maximize its profits.

The first equilibrium condition states that the constant \( \bar{v} \) which the firm takes to be the marginal worker’s daisy-picking utility is, indeed, equal to the marginal worker’s daisy picking utility.

The second equilibrium condition just affirms that the maximized profit which the firm achieves will indeed be the profit which is realized by each firm. Thus, this level of profit will also be the dividend which is redistributed across all individuals.

\(^1\)The relationship \( \bar{U} = U(\bar{\pi}, 0) + \bar{v} \) can be thought of as an equilibrium condition instead of a definition of \( \bar{U} \). However, since this relationship only includes fixed quantities, either view will lead to the same outcome.
3.3.2 The Lagrangian Formulation

As explained above, the competitiveness constraint will bind. The firm's problem, therefore, can be reformulated as a Lagrangian function with \( \theta \) as the multiplier on the competitiveness constraint:

\[
L(N, e, w, \theta) = F(e, N) - wN - c(e)N + \theta [ U(\bar{\pi} + w, e) - U(\bar{\pi}, 0) - \bar{\nu} ]
\]

This leads to the following first order conditions:

\[
N: \quad \frac{\partial F(e, N)}{\partial N} - w - c(e) = 0
\]  

(7)

\[
e: \quad \frac{\partial F(e, N)}{\partial e} - c'(e)N + \theta \frac{\partial U(\bar{\pi} + w, e)}{\partial e} = 0
\]  

(8)

\[
w: \quad -N + \theta \frac{\partial U(\bar{\pi} + w, e)}{\partial w} = 0
\]  

(9)

\[
\theta: \quad U(\bar{\pi} + w, e) - U(\bar{\pi}, 0) - \bar{\nu} = 0
\]  

(10)

EC1: \quad \bar{\nu} = \nu(N)

(11)

EC2: \quad \bar{\pi} = F(e, N) - wN - c(e)N

(12)

where EC1 and EC2 are equilibrium condition 1 and 2 respectively.
Equation (9) can be used to find the following expression for the multiplier $\theta$:

$$\theta = \frac{N}{\frac{\partial U(\pi + w, e)}{\partial w}}$$

The denominator of the above expression represents the marginal utility of wage income, which is always positive. Therefore, $\theta$ will be positive.

By making use of the Lagrangian function above, we can see that $\theta = -\frac{\partial L}{\partial U}$. The value of $\theta$ can therefore be interpreted as the increase in the firm's profit from a fall in the market on-the-job utility $\bar{U}$. Roughly speaking, $\theta$ can be thought of as $N \frac{\partial w}{\partial U}$.

With a decrease in market on-the-job utility, the firm can pay its workers lower wages and still meet the competitiveness constraint. The term $\frac{\partial w}{\partial U}$ expresses how much the firm can save, with a lower $\bar{U}$, in wages per worker. When this is multiplied by the number of workers $N$, the result is the total savings in the wage bill with a fall in market on-the-job utility.

The expression given above can be used to eliminate $\theta$ from equation (8). This gives rise to the following set of four equations in four unknowns, $N, e, w,$ and $\pi$: 
\[
\frac{\partial F(e, N)}{\partial N} - w - c(e) = 0 \quad (7)
\]
\[
\frac{\partial F(e, N)}{\partial e} - c'(e)N + \left( \frac{\partial U(\pi + w, e)}{\partial e} \right) N = 0 \quad (8')
\]
\[
U(\pi + w, e) - U(\pi, 0) - v(N) = 0 \quad (10')
\]
\[
F(e, N) - wN - c(e)N - \pi = 0 \quad (12')
\]

We have used (11) to eliminate \( v \) from (10). Also, in equilibrium, the profit \( \pi \) of the representative firm will be equal to the profit per firm \( \bar{\pi} \); therefore, we can replace \( \bar{\pi} \) by \( \pi \).

The system above \( (7, 8', 10', 12') \) differs from the corresponding system for the Pareto optimum problem \( (1', 2', 5, 6) \) in one respect. The resource constraint \( (5, 12') \) and the participation constraint \( (6, 10') \) are the same. Equations \( (2') \) and \( (8') \), which deal primarily which the effects of changes in the effort level, are also identical.

However, equation \( (1') \) incorporates a term which equation \( (7) \) does not.

This extra term in equation \( (1') \) deals with the effects which hiring more workers have on the participation constraint. Recall that the Pareto optimum participation constraint can be expressed as

\[
U(\pi + w, e) - U(\pi, 0) = v(N).
\]
A rise in N results in an increase in the right hand side of magnitude \( v' \) (N); the left hand side must then increase as well through appropriate changes in \( \pi \) and \( w \), changes which have an influence on the firm's profit as well as overall welfare.

However, the individual firm in price-taking equilibrium faces a different competitiveness constraint. The firm assumes, as mentioned above, that its actions will have no influence on the prevailing market on-the-job utility rate. Its profit, as well, is an infinitesimal part of the total profit which will be redistributed to the population. Therefore, the firm views on-the-job utility and the dividend payment as quantities which are fixed and independent of its own actions. This results in the following competitiveness constraint:

\[
U(\tilde{\pi} + w, e) - U(\tilde{\pi}, 0) - \tilde{v} = 0
\]

Note that this constraint is not affected by how many workers the individual firm hires. In particular, the magnitude of \( \bar{U} = U(\tilde{\pi}, 0) + \tilde{v} \) is determined in the market for labor, in which the individual firm's actions are only a "drop in a bucket."

This line of reasoning is only partly true, however. The decisions which a single, infinitesimal firm makes will not have an influence on market variables. However, the firm in question is a representative firm. In other words, when this firm makes a particular decision, all the other identical firms make the same decision too. And when each firm makes the same decision, the sum of those independent decisions is no longer infinitesimal compared to the market as a whole.
We can rewrite the competitiveness constraint which the price-taking firm faces as

$$U(\bar{\pi} + w, e) - U(\bar{\pi}, 0) = \bar{v}$$

When the firms hire more workers, we expect profits to grow. An increase in profits alone will lead to a decrease in the left hand side of the constraint above, since

$$\frac{\partial U(\pi, 0)}{\partial \pi} \geq \frac{\partial U(\pi + w, e)}{\partial \pi}.$$  

At the same time, the right hand side would increase, since

$$\bar{v} = v(N) \text{ and } v'(N) > 0.$$  

The equality is maintained, however, by an increasing $w$ and a decreasing $e$. These latter changes cut into the growth of profits.

Thus, the firm's assumptions that $\bar{\pi}$ and $\bar{U}$ are unchanged will not actually hold. Furthermore, the firm, through its assumption of a fixed $\bar{v}$, does not take account of the fact that as it hires more workers, an increase in the wage and/or a decrease in the effort level is called for. This leads the firm to hire "too many" workers.

An omniscient planner does take this "negative feedback" employment effect into account. Therefore, we would expect the Pareto optimum version of the model to result in a lower employment level than the price-taking equilibrium.

How else will the Pareto optimum differ from the price-taking equilibrium? Unfortunately, comparing the Pareto optimum set of equations (1', 2', 5', 6) to the corresponding set for a price-taking equilibrium (7, 8', 10', 12') is not likely to be of much help. It is true that only one equation differs in one term, but that term is quite complicated. More importantly, the difference between (1') and (7) will "interact" with
the other equations, making it extremely difficult to find out where effort, wage, and profit will be higher.

We will discuss shortly conditions under which the Pareto optimum and price-taking equilibrium coincide. However, before doing so, we need to deal with the second order conditions associated with the price-taking equilibrium problem.

3.3.3 Second Order Conditions

Once, again, for the solution of the first order conditions of a constrained optimization problem to represent a unique global maximum, it is sufficient to have a) a strictly quasiconcave objective function and b) a convex constraint set.

The objective function in the price-taking equilibrium problem is the firm's profit function, which is given by

\[ \pi (N, e, w) = F(e, N) - wN - c(e)N. \]

\( F(e, N) \) is assumed to be strictly concave. Increasing marginal cost of effort is also assumed, making \( c'(e) > 0 \). But this implies \( -c''(e) < 0 \), which means that \( -c(e) \) will be a strictly concave function. \( -c(e)N \) can then be regarded as \( \int_{0}^{N} (-c(e))\,dn \). Since the sum of strictly concave functions is strictly concave, \( -c(e)N \) will be strictly concave.

Unfortunately, the wage bill \( wN \) muddies the picture. \( wN \) is strictly quasiconcave, which makes \( -wN \) strictly quasiconvex. Thus, we cannot assert that the objective
function as a whole is strictly quasiconcave. The objective function may turn out to be strictly quasiconcave in a number of particular cases; however, we are not able to establish that this will be generally true.

The price-taking firm faces one constraint, which we can express as:

\[ U(\pi + w, e) \geq U(\pi, 0) + \nu \]

Assume that points \( A (N_1, e_1, w_1) \) and \( B (N_2, e_2, w_2) \) each satisfy the competitiveness constraint above. We can make the following argument regarding the convexity of the constraint set:

\[ U(\pi + w_1, e_1) \geq U(\pi, 0) + \nu \Rightarrow \rho \left[ U(\pi + w_1, e_1) \right] \geq \rho \left[ U(\pi, 0) + \nu \right] \]
\[ U(\pi + w_2, e_2) \geq U(\pi, 0) + \nu \Rightarrow [1 - \rho] \left[ U(\pi + w_1, e_1) \right] \geq [1 - \rho] \left[ U(\pi, 0) + \nu \right] \]

By adding the two inequalities on the right, we obtain

\[ \rho \left[ U(\pi + w_1, e_1) \right] + [1 - \rho] \left[ U(\pi + w_1, e_1) \right] \geq U(\pi, 0) + \nu \]

By the strict concavity of the utility function, we know that

\[ U(\rho(\pi + w_1) + (1 - \rho)(\pi + w_2), \rho e_1 + (1 - \rho)e_2) > \rho \left[ U(\pi + w_1, e_1) \right] \]
\[ + [1 - \rho] \left[ U(\pi + w_1, e_1) \right] \]
Therefore, we have

\[ U(\rho(\bar{\pi} + w_1) + (1 - \rho)(\bar{\pi} + w_2), p_{e_1} + (1 - p)e_2) > U(\bar{\pi}, 0) + \bar{v} \]

and the constraint set is strictly convex by virtue of the strict concavity of the utility function.

Thus, the for the price-taking firm's problem, the objective function is not in general strictly quasiconcave, but the constraint set is strictly convex.
3.4 Pareto Optimum – Price-Taking Equilibrium Equivalence

3.4.1 The Utility Function and the Participation Constraint

We have already seen that the set of equations \((1', 2', 5, 6)\) which describe the Pareto optimum differ from the set of equations which describe the price-taking equilibrium \((7, 8', 10', 12')\) by one term. This term, which is found in \((1')\), but not in \((7)\), is reproduced below:

\[
(1-N)^N \left( \frac{\partial U(\pi,0)}{\partial \pi} - \frac{\partial U(\pi+w,e)}{\partial \pi} \right)^{-1} \left( \frac{\partial U(\pi,0)}{\partial \pi} \frac{\partial U(\pi+w,e)}{\partial \pi} \right) v'(N)
\]

If this term were to equal zero, the four equations which describe the Pareto optimum and the four equations which describe the price-taking equilibrium would be identical.

The expression above will disappear if

\[
\frac{\partial U(\pi,0)}{\partial \pi} = \frac{\partial U(\pi+w,e)}{\partial \pi}
\]

Recall that \(\frac{\partial U(\pi,0)}{\partial \pi}\) represents the non-worker's marginal utility of income, while \(\frac{\partial U(\pi+w,e)}{\partial \pi}\) represents the worker's marginal utility of income.

In order for the two quantities to be equal in general, it must be the case that:

a) the utility function is separable in income and effort. If there is any interrelation of income and effort, the marginal utility of income will be a function of the effort level.
Since workers put forth effort level $e$ while non-workers expend no effort, their marginal utilities would not be equal in general.

b) the utility function is linear in income. The marginal utilities must be equal for two different levels of income $\pi$ and $(\pi + w)$. This would only take place if marginal utility were constant and thus independent of the level of income, which implies that utility itself would be linear in income.

If these conditions are met, the Pareto optimum and the price-taking equilibrium will coincide.

A utility function having the above properties would take the form

$$U(a, b) = a + h(b)$$

where "a" represents the individual's income and "b" the level of effort which he puts forth. In accord with previous assumptions with regard to the utility function and effort, we assume that $h(e) < 0$, $h'(e) < 0$, and $h''(e) < 0$ for all $e > 0$. For the sake of simplicity, we assume that $h(0) = 0$.

With such a utility function, we would have $U(\pi, 0) = \pi$ and

$$U(\pi + w, e) = \pi + w + h(e)$$

This implies that

$$\frac{\partial U(\pi, 0)}{\partial \pi} = \frac{\partial U(\pi + w, e)}{\partial \pi} = 1$$

Once again, such a utility function would make the equations describing the Pareto optimum identical with those describing the price-taking equilibrium.

What is the economic rationale behind this fact? Why is it that such a utility function makes the Pareto optimum and price-taking equilibrium problems coincide?
As we have explained above, the individual firm in price-taking equilibrium does not take account of the effects which its actions will have on the competitiveness constraint. The omniscient planner does take such effects into consideration. This leads to the divergence between the two problems.

How would a separable, linear-in-income utility function affect the constraint which the firm faces in price-taking equilibrium? The competitiveness constraint is reproduced below in general form. We then show how that constraint would be altered when we assume that the utility function is separable and linear in income:

\[
U(\pi + w, e) - U(\pi, 0) = v
\]

\[
\Rightarrow \pi + w + h(e) - \pi = v
\]

\[
\Rightarrow w + h(e) = v
\]

Now, the competitiveness constraint is not affected any more by the firm's profit. With the separable, linear-in-income utility function, extra profit benefits the worker and non-worker to the very same extent. Thus, the "position" of the marginal worker would be unchanged.

Recall that the position of this marginal worker- the "last" individual who leaves daisy-picking for the workplace- is determined by the relation

\[
U(\pi + w, e) = U(\pi, 0) + v(N)
\]

where the left hand side represents on-the-job utility while the right hand side represents the utility this individual would have outside the workforce. The marginal worker can be
thought of as the “point” in our population where the two sides of the above equation “balance” exactly.

What happens to our balance if equal weights are added to either side? If the weights are added at the same distance from the balance point, the balance point will not change. This is precisely what takes place when utility is separable and linear in income. Extra profit is added algebraically to either side of the equation above. The result is that the equation continues to hold with the same value of \( v(N) \), and hence for the same value of \( N \). In this case, the individual firm’s actions and the individual firm’s profit will not have any influence on the position of the market’s marginal worker.

On the other hand, if equal weights are added at unequal distances from the balance point, the balance point itself will shift. This is what takes place when the utility function does not satisfy the two properties mentioned above. Increasing \( \pi \) will then have a different effects on \( U(\pi + w, e) \) and \( U(\pi, 0) \), with the result that \( v(N) \), and hence \( N \), will change. In general, any action taken with the intent of increasing profit will have a partial “negative feedback” effect on profit by requiring an increase in \( w \) and a decrease in \( e \).

This, however, does not take place with a utility function which is separable and linear in income. It is not that the price-taking firm starts to take into account the effect above, an effect which an omniscient planner always takes into consideration. It is rather that the effect above ceases to exist. Therefore, the price-taking firm’s decisions and the planner’s decision now coincide.
3.4.2 Pareto Optimum Revisited

How will the Pareto optimum problem change if we assume that utility is separable and linear in income? Recall first that the welfare function in this problem is given by

\[ W(N, e, w, \pi) = U(\pi + w, e)N + U(\pi, 0)(1 - N) + \int_0^1 v(n)dn \]

Now, suppose we assume that utility takes the form \( U(a, b) = a + h(b) \), with \( h(0) = 0 \) as mentioned above. Then our welfare function becomes:

\[ W(N, e, w, \pi) = [\pi + w + h(e)]N + \pi(1 - N) + \int_0^1 v(n)dn \]

\[ = \pi N + w N + h(e)N + \pi - \pi N + \int_0^1 v(n)dn \]

\[ = \pi + wN + h(e)N + \int_0^1 v(n)dn \]

The participation constraint is also altered by the assumption of separable and linear in income utility. We give the constraint in its more general form and then show the changes:
\[ U(\pi + w, e) - U(\pi, 0) - \nu(N) = 0 \]
\[ \Rightarrow \pi + w + h(e) - \pi - \nu(N) = 0 \]
\[ \Rightarrow w + h(e) - \nu(N) = 0 \]

The resource constraint is unchanged. It continues to take the form

\[ F(e, N) - wN - c(e)N - \pi = 0. \]

Therefore, we can formulate the problem as follows:

\[ L(N, e, w, \pi) = \pi + wN + h(e)N + \int_0^1 \nu(n)dn + \lambda \left[ F(e, N) - wN - c(e)N - \pi \right] \]
\[ + \delta \left[ w + h(e) - \nu(N) \right] \]

This leads to the following set of first order conditions:

**N:** \[ w + h(e) - \nu(N) + \lambda \left[ \frac{\partial F(e, N)}{\partial N} - w - c(e) \right] - \delta v'(N) = 0 \] (13)

**e:** \[ h'(e)N + \lambda \left[ \frac{\partial F(e, N)}{\partial e} - c'(e)N \right] + \delta h'(e) = 0 \] (14)

**w:** \[ N - \lambda N + \delta = 0 \] (15)

**\pi:** \[ 1 - \lambda = 0 \] (16)

**\lambda:** \[ F(e, N) - wN - c(e)N - \pi = 0 \] (17)

**\delta:** \[ w + h(e) - \nu(N) = 0 \] (18)

Note that the first three terms of (13) disappear by virtue of (18).
Equations (15) and (16) imply that $\lambda = 1$ and $\delta = 0$. The implication of $\delta = 0$ is that the participation constraint now has no effect on the optimal $e$ and $N$. Effectively, $e$ and $N$ are chosen to maximize welfare and then $w$ is set to ensure that the "right" person $N$ is the marginal worker.

When $\lambda = 1$ and $\delta = 0$, the equations above can be recast as four equations in the four unknowns $N$, $e$, $w$, and $\pi$:

$$\frac{\partial F(e, N)}{\partial N} - w - c(e) = 0 \quad (13')$$

$$\frac{\partial F(e, N)}{\partial e} - [c'(e) - h'(e)] N = 0 \quad (14')$$

$$F(e, N) - wN - c(e)N - \pi = 0 \quad (17)$$

$$w + h(e) - v(N) = 0 \quad (18)$$

This set of equations is relatively straightforward. An increase in $N$, according to $(13')$, results in an increase in output $F(e, N)$, an increase in the wage bill $wN$, and an increase in the cost of supervision $c(e)N$. An increase in $e$, according to $(14')$, raises output $F(e, N)$, the cost of supervision $c(e)N$ and the disutility $h(e)N$ associated with effort.

In the system above, equation (17) serves to define profit $\pi$. Equation (18) implies that $w = v(N) - h(e)$. If we use the later expression for $w$ in $(13')$, we are left with a system of two equations in the two unknowns $N$ and $e$: 
\[
\frac{\partial F(e,N)}{\partial N} - [c(e) - h(e)] - v(N) = 0 \tag{13^*}
\]

\[
\frac{\partial F(e,N)}{\partial e} - [c'(e) - h'(e)] N = 0 \tag{14^*}
\]

The resource constraint defines profit \( \pi \) as \([ F(e,N) - wN - c(e)N ]. If we replace \( \pi \) in the objective function by the expression in the square brackets, we can eliminate \( \pi \) from our problem. In that case, the objective function would become

\[
W(N,e,w) = F(e,N) - c(e)N + h(e)N + \int_{N}^{1} v(n)dn
\]

which would be maximized subject to the participation constraint \( w + h(e) - v(N) = 0 \).

Notice here that \( w \) has also disappeared from the objective. That is why \( w \) can be set at whatever value is required to make worker \( N \) marginal. Changes in \( w \) do not have a direct effect on welfare.

This formulation makes the concavity of the objective function more apparent, as \( F(e,N), c(e), h(e), \) and \( \int_{N}^{1} v(n)dn \) are all strictly concave. The convexity of the constraint set cannot be established, as we have made no assumptions regarding \( v'(N) \).
3.4.3 Price-Taking Equilibrium Revisited

How will the price-taking equilibrium problem change if we assume that utility is separable and linear in income? Recall that the profit function is given by

$$\pi (N, e, w) = F(e, N) - wN - c(e)N. $$

Since the utility function does not play a role in the expression for profit, the objective function is unchanged.

In general, the constraint which the firm faces is given by

$$ U(\bar{\pi} + w, e) - \bar{U} = 0, $$

where $\bar{U} = U(\bar{\pi}, 0) + \bar{v}$. The assumption of separable, linear-in-income utility alters this constraint as follows:

$$ U(\bar{\pi} + w, e) - U(\bar{\pi}, 0) - \bar{v} = 0 $$

$$ \Rightarrow \bar{\pi} + w + h(e) - \bar{\pi} - \bar{v} = 0 $$

$$ \Rightarrow w + h(e) - \bar{v} = 0 $$

The equilibrium conditions are still $\bar{v} = v(N)$ and

$$ \bar{\pi} = \pi = F(e, N) - wN - c(e)N. $$

The quantity $\bar{\pi}$ is now not meaningful to the problem, since it no longer appears in the constraint.

Therefore, the problem may be formulated as

$$ L(N, e, w, \theta) = F(e, N) - wN - c(e)N + \theta [ w + h(e) - \bar{v} ] $$

This leads to the following first order conditions:
\( N: \quad \frac{\partial F(e, N)}{\partial N} - w - c(e) = 0 \) \hspace{1cm} (19)

\( e: \quad \frac{\partial F(e, N)}{\partial e} - c'(e)N + \theta h'(e) = 0 \) \hspace{1cm} (20)

\( w: \quad -N + \theta = 0 \) \hspace{1cm} (21)

\( \theta: \quad w + h(e) - \bar{v} = 0 \) \hspace{1cm} (22)

\( \text{EC1:} \quad \bar{v} = v(N) \) \hspace{1cm} (23)

\( \text{EC2:} \quad \bar{\pi} = \pi = F(e, N) - wN - c(e)N \) \hspace{1cm} (24)

where EC1 and EC2 are equilibrium conditions 1 and 2.

Equation (21) tells us that \( \theta = N \). Thus, we can eliminate \( \theta \) from (20). We can also use (23) to eliminate \( \bar{v} \) from (22). This leads to a system of four equations in the four unknowns \( N, e, w, \) and \( \pi \).

\[ \frac{\partial F(e, N)}{\partial N} - w - c(e) = 0 \] \hspace{1cm} (19)

\[ \frac{\partial F(e, N)}{\partial e} - c'(e)N + h'(e)N = 0 \] \hspace{1cm} (20')

\[ w + h(e) - v(N) = 0 \] \hspace{1cm} (22')

\[ \pi = F(e, N) - wN - c(e)N \] \hspace{1cm} (24)

These equations are identical to the set \((13', 14', 17, 18)\), which define the solution to the Pareto optimum problem.
3.5 Specific Formulations

3.5.1. Separable, Linear-in-Income Utility

To calculate specific solutions for our model, we need to make use of specific functional forms. A listing of these functional forms follows:

\[ F(e, N) = e^{1/2} N^{1/2} \]

\[ c(e) = ae^2 \]

\[ h(e) = -e^2 \]

\[ v(n) = n \]

Little comment is needed regarding the Cobb-Douglas-type production function. The disutility of effort function \( h(e) \) is also unremarkable. In either case, the functions selected are about the simplest which could possibly be picked. As a reminder, the properties these functions must satisfy are

\[ F_e, F_N > 0, \ F_{ee}, F_{NN} < 0, \ \text{and} \ F \ \text{strictly concave} \]

\[ h(e) < 0, \ h'(e) < 0, \ h''(e) < 0, \ \text{and} \ h(0) = 0. \]

The cost of effort enforcement \( c(e) \) is defined with a parameter "a". Higher values of the parameter a reflect higher costs of effort enforcement. By changing the value of a, we can explore the effect that such higher costs have on the model’s variables. Recall that \( c(e) \) is assumed to satisfy \( c'(e) > 0 \) and \( c''(e) > 0 \).
The variable \( n \) can be thought of a number assigned to each member of the population ("One, two, three, . . ."). With the requirement that \( v'(n) > 0 \), \( v(n) = n \) is also the simplest possible function which could be chosen.

The Lagrangian function for Pareto optimum problem will take the following form:

\[
L(N, e, w, \pi) = \pi + wN - e^2N + \int\limits_0^1 n \, dn + \lambda \left[ e^{1/2}N^{1/2} - wN - ae^2N - \pi \right] \\
+ \delta \left[ w - e^2 - N \right]
\]

\[
L(N, e, w, \pi) = \pi + wN - e^2N + \frac{1}{2} - \frac{1}{2}N^2 + \lambda \left[ e^{1/2}N^{1/2} - wN - ae^2N - \pi \right] \\
+ \delta \left[ w - e^2 - N \right]
\]

We will dispense with a listing of the first order conditions.

This formulation of our problem yields an analytical solution. Our endogenous variables are given by:

\[
N = (2)^{-b/5}(1 + a)^{-1/5}
\]

\[
e = (2)^{-b/5}(1 + a)^{-3/5}
\]

\[
w = (2 + a)(2)^{-b/5}(1 + a)^{-6/5}
\]

\[
\pi = (2)^{-11/5}(1 + a)^{-2/5}
\]

With regard to the interrelationship of the first three variables, we have
\[ w = (2 + a) e^3, \quad N = (1 + a) e^3, \quad \text{and} \quad w = \left( \frac{2+a}{1+a} \right) N. \]

The expressions above imply that welfare \( W \) will be given by:

\[
W = (5)(2)^{-3/5}(1 + a)^{-2/5} + \frac{1}{2}
\]

Note that \( N, e, w, \pi, \) and \( W \) all decrease with an increase in the cost of effort enforcement.

The following table gives the values of the variables for different values of the parameter \( a \), as well as the elasticities of the variables with respect to changes in the value of \( a \). The variables are then plotted as functions of the parameter \( a \).
### Separable and Linear-in-Income Utility, Pareto Optimum and Price-Taking Equilibrium

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### Elasticities

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<td>-0.1639</td>
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### Cost of effort enforcement

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As the parameter a increases, the greatest changes take place in the level of effort which is required of workers. This is not surprising.

Recall that the Pareto optimum can be expressed in a slightly different form, a form in which we substitute the resource constraint into the objective function, leading to the following expression

\[ W(N, e, w) = F(e, N) - c(e)N + h(e)N + \int_0^1 v(n)dn \]

which is maximized subject to the participation constraint \( w + h(e) - v(N) = 0 \). In this context, we said that the convexity of the constraint set could not be established, as we had made no assumptions regarding \( v^*(N) \).

However, when we take \( v(n) = n \), we can prove that the constraint set is convex. Whatever its specific form, \( h(e) \) will be concave. With \( v(N) = N \), the other two terms in the participation constraint will be linear. It is relatively easy to show that the set \( w + h(e) - N \geq 0 \) will be a convex set.

With the specific functional forms mentioned above, the price-taking equilibrium problem can be expressed as:

\[ L(N, e, w, \theta) = e^{1/2}N^{1/2} - wN - ae^2N + \theta \{ w - e^2 - \bar{v} \} \]

with the equilibrium condition \( \bar{v} = N \). This leads to the same expressions for \( N, e, w, \) and \( \pi \) which are given above. Therefore, the table on the previous page applies to both the Pareto optimum and the price-taking equilibrium problems.
3.5.2 Separable, Nonlinear-in-Income Utility

3.5.2.1 Pareto Optimum

We now assume that utility is separable, but nonlinear in income. Specifically, we will use the functional form

\[ U(j, k) = j^{1/2} - k^2 \]

where “j” represents the individual’s income, and “k” represents his effort level. We will continue to use \( F(e, N) = e^{1/2} N^{1/2} \), \( c(e) = ae^2 \), and \( v(n) = n \).

With these functional forms, the Pareto optimum problem is given by:

\[
\begin{align*}
\text{Max } W(N, e, w, \pi) &= (\pi + w)^{1/2} N - e^2 N + (\pi)^{1/2} (1 - N) + \int_1^N \frac{n}{N} \, dn \\
\text{subject to } &e^{1/2} N^{1/2} - wN - ae^2 N - \pi \geq 0 \\
\text{and } & (\pi + w)^{1/2} - c^2 - (\pi)^{1/2} - N \geq 0
\end{align*}
\]

which can be expressed in Lagrangian form as:

\[
L(N, e, w, \pi, \lambda, \delta) = (\pi + w)^{1/2} N + (\pi)^{1/2} (1 - N) + \frac{1}{2} - \frac{1}{2} N^2 \\
+ \lambda \left( e^{1/2} N^{1/2} - wN - ae^2 N - \pi \right) + \delta \left( (\pi + w)^{1/2} - c^2 - (\pi)^{1/2} - N \right)
\]
The first order conditions for this problem are given by:

\[ \text{N: } (\pi + w)^{1/2} N - \varepsilon^2 - (\pi)^{1/2} - N + \lambda \left[ \frac{1}{2} e^{1/2} N^{-1/2} - w - a\varepsilon^2 \right] - \delta = 0 \]  
\[ (25) \]

\[ \text{e: } -2\varepsilon N + \lambda \left[ \frac{1}{2} e^{-1/2} N^{1/2} - 2a\varepsilon N \right] - 2\delta e = 0 \]  
\[ (26) \]

\[ \text{w: } \frac{1}{2} (\pi + w)^{-1/2} N - \lambda N + \frac{1}{2} \delta (\pi + w)^{-1/2} = 0 \]  
\[ (27) \]

\[ \text{p: } \frac{1}{2} (\pi + w)^{-1/2} N - \frac{1}{2} (\pi)^{-1/2} (1 - N) - \lambda + \delta \left[ \frac{1}{2} (\pi + w)^{-1/2} - \frac{1}{2} (\pi)^{-1/2} \right] = 0 \]  
\[ (28) \]

\[ \text{\lambda: } e^{1/2} N^{1/2} - wN - a\varepsilon^2 N - \pi = 0 \]  
\[ (29) \]

\[ \text{\delta: } (\pi + w)^{1/2} - \varepsilon^2 - (\pi)^{1/2} - N = 0 \]  
\[ (30) \]

(27) and (28) can be used to obtain the following expressions for the multipliers:

\[ \lambda = \frac{1}{2} (\pi + w)^{-1/2} (\pi)^{-1/2} \]  
\[ \text{and } \delta = \frac{N(1 - N)((\pi)^{-1/2} - (\pi + w)^{-1/2})}{(\pi + w)^{-1/2} (1 - N) + (\pi)^{-1/2} N} \]

These expressions can be used to eliminate \( \lambda \) and \( \delta \) from (25) and (26), leading to the following four equations in the four unknowns \( N \), \( e \), \( w \), and \( \pi \):
\[
\frac{1}{2} e^{1/2} N^{-1/2} - w - ae^2 - \frac{N(N-N)\left[(\pi e^{-1/2} - (\pi + w)^{-1/2}\right]}{2 (\pi + w)^{-1/2} (\pi)^{-1/2}} = 0 \tag{25'}
\]

\[
\frac{1}{2} e^{-1/2} N^{1/2} - 2aeN - 4(\pi + w)^{1/2} eN = 0 \tag{26'}
\]

\[
e^{1/2} N^{1/2} - wN - ae^3 N - \pi = 0 \tag{29}
\]

\[(\pi + w)^{1/2} - e^2 - (\pi)^{1/2} - N = 0 \tag{30}
\]

The equations above yield no analytical solution; however, we obtain the following numerically:
Separable, Nonlinear-in-Income Utility, Pareto Optimum

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<th>0.75%</th>
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Elasticities

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<tr>
<td>Welfare</td>
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Note that all the endogenous variables decrease with an increase in the cost of effort enforcement.

3.5.2.2 Price-Taking Equilibrium

With the utility function \( U(j, k) = j^{1/2} - k^2 \), the price-taking equilibrium problem is given by

\[
\text{Max } \pi(N, e, w) = e^{1/2} N^{1/2} - wN - ae^2.N \\
\text{subject to } (\pi + w)^{1/2} - \bar{e}^2 - (\bar{\pi})^{1/2} - \bar{v} \geq 0 \\
\text{with equilibrium conditions: } \bar{v} = N \\
\bar{\pi} = \pi = e^{1/2} N^{1/2} - wN - ae^2N
\]

The Lagrangian function representing this problem is given by:

\[
L(N, e, w, \theta) = e^{1/2} N^{1/2} - wN - ae^2N + \theta [(\pi + w)^{1/2} - \bar{e}^2 - (\bar{\pi})^{1/2} - \bar{v}]
\]

which leads to the following first order conditions:
\[ N: \frac{1}{2} e^{1/2} N^{-1/2} - w - ae^2 = 0 \quad (31) \]

\[ e: \frac{1}{2} e^{-1/2} N^{1/2} - 2aeN - 2\theta e = 0 \quad (32) \]

\[ w: -N + \frac{1}{2} \theta (\bar{\pi} + w)^{-1/2} = 0 \quad (33) \]

\[ \theta: (\bar{\pi} + w)^{1/2} - e^2 - (\bar{\pi})^{1/2} - \bar{v} = 0 \quad (34) \]

EC1: \[ \bar{v} = N \quad (35) \]

EC2: \[ \bar{\pi} = \pi = e^{1/2} N^{1/2} - wN - ae^2 N \quad (36) \]

(33) implies that \( \theta = 2(\pi + w)^{1/2} N \). By making use of that expression, as well as (35) and (36), we obtain four equations in the four unknowns \( N, e, w, \) and \( \pi \):

\[ \frac{1}{2} e^{1/2} N^{-1/2} - w - ae^2 = 0 \quad (31) \]

\[ \frac{1}{2} e^{-1/2} N^{1/2} - 2aeN - 4(\pi + w)^{1/2} eN = 0 \quad (32') \]

\[ (\pi + w)^{1/2} - e^2 - (\pi)^{1/2} - N = 0 \quad (34') \]

\[ e^{1/2} N^{1/2} - wN - ae^2 N - \pi = 0 \quad (36) \]

As before, equations (32', 34', 36) and equations (26', 29, 30), which describe the corresponding Pareto optimal problem, are identical. Equation (31) differs from (25') in that the latter includes the extra term we have already discussed.

Once again, the equations above do not yield an analytical solution. We obtain the following numerically:
### Separable, Nonlinear-in-Income Utility, Price-Taking Equilibrium

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### Elasticities

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<tr>
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<td>-0.1577</td>
<td>-0.1754</td>
<td>-0.1908</td>
</tr>
<tr>
<td>Profit</td>
<td>-0.1577</td>
<td>-0.1754</td>
<td>-0.1908</td>
</tr>
<tr>
<td>Welfare</td>
<td>-0.0371</td>
<td>-0.0408</td>
<td>-0.0439</td>
</tr>
</tbody>
</table>
First of all, we should note that compared to the Pareto optimum, the price-taking equilibrium problem results in lower effort, higher wages, greater employment, lower profit, and lower welfare.

Since the individual firm does not take account of the effect of its actions on the competitiveness constraint, the price-taking equilibrium ends up with a greater number of workers than the Pareto optimum.

Higher employment can only come with greater on-the-job utility. Therefore, if the number of workers is greater (as it is in price-taking equilibrium), it should be expected that effort would be lower and wages higher.

The omniscient planner maximizes welfare, while the price-taking firm maximizes profit. This objective, coupled with the fact that the omniscient planner is not “ignorant” of the effects of his actions, means that the Pareto optimum problem necessarily yields a higher level of welfare.

One might expect the price-taking equilibrium profit to exceed the Pareto optimum profit. However, the fact that the omniscient planner takes all the consequences of his decisions into consideration seems to outweigh the fact that his objective is welfare, not profit. By narrowly maximizing their own profit while ignoring the effects on the competitiveness constraint, the price-taking firms end up “fouling their own nest,” so to speak.

Note that increases in the cost of effort enforcement lower effort, wages, number of workers, profit, and welfare. The same changes result when effort becomes more costly in the Pareto optimum problem.
The elasticities of effort and wage with respect to $a$ are lower in the price-taking equilibrium, while the elasticities of the number of workers and profit are higher.

The differences between the three specific formulations we have examined so far are presented in the following table. "PTE" stands for price-taking equilibrium, while "PO" stands for Pareto optimum.

<table>
<thead>
<tr>
<th>$a = 0.5$</th>
<th>Linear Utility, PTE and PO</th>
<th>Nonlinear Utility, PTE</th>
<th>Nonlinear Utility, PO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effort</td>
<td>0.4503</td>
<td>0.3661</td>
<td>0.4008</td>
</tr>
<tr>
<td>Wage</td>
<td>0.5070</td>
<td>0.5040</td>
<td>0.4647</td>
</tr>
<tr>
<td>Number of workers</td>
<td>0.3042</td>
<td>0.2807</td>
<td>0.2197</td>
</tr>
<tr>
<td>On-the-Job Utility</td>
<td>0.4892</td>
<td>0.6810</td>
<td>0.6404</td>
</tr>
<tr>
<td>Output</td>
<td>0.3701</td>
<td>0.3206</td>
<td>0.2967</td>
</tr>
<tr>
<td>Profit</td>
<td>0.1851</td>
<td>0.1603</td>
<td>0.1770</td>
</tr>
<tr>
<td>Welfare</td>
<td>0.7313</td>
<td>0.9397</td>
<td>0.9449</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$a = 1$</th>
<th>Linear Utility, PTE and PO</th>
<th>Nonlinear Utility, PTE</th>
<th>Nonlinear Utility, PO</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effort</td>
<td>0.3789</td>
<td>0.3289</td>
<td>0.3589</td>
</tr>
<tr>
<td>Wage</td>
<td>0.4308</td>
<td>0.4405</td>
<td>0.3987</td>
</tr>
<tr>
<td>Number of workers</td>
<td>0.2872</td>
<td>0.2731</td>
<td>0.2161</td>
</tr>
<tr>
<td>On-the-Job Utility</td>
<td>0.4521</td>
<td>0.6602</td>
<td>0.6217</td>
</tr>
<tr>
<td>Output</td>
<td>0.3299</td>
<td>0.2997</td>
<td>0.2785</td>
</tr>
<tr>
<td>Profit</td>
<td>0.1649</td>
<td>0.1498</td>
<td>0.1645</td>
</tr>
<tr>
<td>Welfare</td>
<td>0.7062</td>
<td>0.9244</td>
<td>0.9289</td>
</tr>
</tbody>
</table>
4 The Minimum Wage

4.1 Price-Taking Equilibrium

Assume now that an effective minimum wage is imposed on the firm in price-taking equilibrium. We could represent this mathematically by adding another term to the firm’s Lagrangian function. This would take the form $\mu \left[ w - w_m \right]$, where $\mu$ would be a new Lagrangian multiplier and $w_m$ would represent the minimum wage. However, since we assume that the minimum wage will be effective, we know that the constraint will bind, so that $w = w_m$. Therefore, instead of adding $\mu \left[ w - w_m \right]$ to the Lagrangian function, we can change the variable $w$ into the fixed value $w_m$.

We will use the separable, linear-in-income utility function $U(j, k) = j + h(k)$. Recall that the price-taking equilibrium and the Pareto optimum coincide when the utility function takes this form. The rationale for doing this is that we want to see the effect of a minimum wage starting from an undistorted equilibrium. We shall later consider the consequences of imposing a minimum wage when the initial equilibrium is not Pareto optimal.

The firm’s problem is then given by:

$$
\text{Max } \pi (N, e) = F(e, N) - w_m N - c(c)N
$$

subject to $w_m + h(e) \geq \bar{v}$

Equilibrium condition: $\bar{v} = v(N)$

This can be represented by the following Lagrangian function:
\[ L(N, e, \theta) = F(e, N) - w_m N - c(e) N + \theta [w_m + h(e) - \bar{v}] \]

which leads to the following first order conditions:

1. N: \[ \frac{\partial F(e, N)}{\partial N} - w_m - c(e) = 0 \] (37)

2. e: \[ \frac{\partial F(e, N)}{\partial e} - c'(e) N + \theta h'(e) = 0 \] (38)

3. \[ w_m + h(e) - \bar{v} = 0 \] (39)

4. EC: \[ \bar{v} = \nu(N) \] (40)

The first order condition for e is the only equation where the Lagrangian multiplier \( \theta \) appears. Mathematically, therefore, equation (38) no longer plays a role in determining the level of effort \( e \). Instead, it ends up serving as a definition of \( \theta \).

In essence, the firm no longer chooses \( e \) based on considerations of the cost and benefit of extra effort. Instead, equation (39), the constraint on on-the-job utility, ends up determining the effort level \( e \). Recall that \( w_m \) and \( \bar{v} \) are both fixed quantities as far as the firm is concerned.

We can substitute \( \bar{v} = \nu(N) \) in equation (39) and obtain the following system of two equations in the two unknowns \( N \) and \( e \):

\[ \frac{\partial F(e, N)}{\partial N} - w_m - c(e) = 0 \] (37)

\[ w_m + h(e) - \nu(N) = 0 \] (39')
Equations \((13^\prime)\) and \((14^\prime)\), which we repeat below, describe the Pareto optimum in this problem.

\[
\frac{\partial F(e, N)}{\partial N} - \left[ c(e) - h(e) \right] - v(N) = 0 \quad (13^\prime)
\]

\[
\frac{\partial F(e, N)}{\partial e} - \left[ c'(e) - h'(e) \right] N = 0 \quad (14^\prime)
\]

Since \((37, 39')\) differ from \((13^\prime, 14^\prime)\), the introduction of a minimum wage means that the price-taking equilibrium no longer coincides with the Pareto optimum.

If the price-taking equilibrium is not "first best" anymore, is it second best? In other words, does it coincide with what an omniscient planner would choose to do when his "hands are tied" by the requirement that he meet the minimum wage constraint?

### 4.2 Pareto Optimum

Once again, the imposition of a minimum wage means that the wage is no longer a choice variable. Under such conditions, the problem which an omniscient planner faces can be represented as:

\[
\text{Max } W(N, e, \pi) = \pi + w_mN + h(e)N + \int_{N}^{1} v(n)dn
\]

subject to

\[
F(e, N) - w_mN - c(e)N - \pi \geq 0
\]

\[
w_m + h(e) - v(N) \geq 0
\]

The Lagrangian function for this problem is given by:
\[ L(N, e, \pi, \lambda, \delta) = \pi + w_m N + h(e)N + \int_0^N v(n)dn + \lambda \left[ \frac{\partial F(e, N)}{\partial N} - w_m - c(e)N - \pi \right] \]

\[ + \delta \left[ w_m + h(e) - v(N) \right] \]

which leads to the following first order conditions:

\[ N: \quad w_m + h(e) - v(N) + \lambda \left[ \frac{\partial F(e, N)}{\partial N} - w_m - c(e) \right] - \delta v'(N) = 0 \quad (41) \]

\[ e: \quad h'(e)N + \lambda \left[ \frac{\partial F(e, N)}{\partial e} - c'(e)N \right] + \delta h'(e) = 0 \quad (42) \]

\[ \pi: \quad 1 - \lambda = 0 \quad (43) \]

\[ \lambda: \quad F(e, N) - w_m N - c(e)N - \pi = 0 \quad (44) \]

\[ \delta: \quad w_m + h(e) - v(N) = 0 \quad (45) \]

Note that the first three terms of (41) disappear by virtue of (45). Also, equation (44) is definitional for \( \pi \).

Equation (43) implies that \( \lambda = 1 \), and (42) yields the following expression for the Lagrangian multiplier \( \delta \):

\[ \delta = \frac{\left[ c'(e) - h'(e) \right]N - \frac{\partial F(e, N)}{\partial e}}{h'(e)} \]

By substituting \( \lambda = 1 \) and the above expression for \( \delta \) in equation (41), we can obtain the following system of two equations in the two unknowns \( N \) and \( e \):
\[
\frac{\partial F(e, N)}{\partial N} - w_m - c(e) - \frac{[c'(e) - h'(e)]N - \frac{\partial F(e, N)}{\partial e}}{h'(e)} v'(N) = 0 \quad (41')
\]
\[
w_m + h(e) - v(N) = 0 \quad (45)
\]

An increase in the number of workers must be associated with an increase in on-the-job utility. Since the wage is no longer a choice variable, if the planner wishes to attract an extra worker, he must lower effort to achieve higher-on-the-job utility. This change in effort affects overall welfare, and this effect is captured by the last term of (41').

A price-taking firm does not take account of such changes, since it believes that its actions will have no effect on market employment N.

The two equations above define the values of N and e which are optimal given the minimum wage. The corresponding equations for a price-taking equilibrium are (37, 39'), which are reprinted below:

\[
\frac{\partial F(e, N)}{\partial N} - w_m - c(e) = 0 \quad (37)
\]
\[
w_m + h(e) - v(N) = 0 \quad (39')
\]

Since (37) and (41') are different, the price-taking equilibrium solution under a minimum wage is not “second best”. It does not represent what a planner would do under the same constraint, even though the undistorted equilibrium was Pareto optimal under our present assumptions.
4.3 Specific Formulations

4.3.1 The Price-Taking Equilibrium

By making use of the specific functional forms which we used earlier, we can formulate the price-taking firm's problem as follows:

\[
\text{Max } \pi (N, e) = e^{1/2}N^{1/2} - w_m N - ae^2 N \\
\text{subject to: } w_m - e^3 - \bar{v} \geq 0 \\
\text{Equilibrium condition: } \bar{v} = N
\]

which leads to the following Lagrangian function:

\[
L (N, e, w, \theta) = e^{1/2}N^{1/2} - w_m N - ae^2 N + \theta [ w - e^3 - \bar{v} ]
\]

and the set of first order conditions listed below:

\[
N: \quad \frac{1}{2} e^{1/2}N^{-1/2} - w_m - ae^2 = 0 \quad (46)
\]

\[
e: \quad \frac{1}{2} e^{-1/2}N^{1/2} - 2aeN - 2\theta e = 0 \quad (47)
\]

\[
\theta: \quad w_m - e^3 - \bar{v} = 0 \quad (48)
\]

\[
\text{EC: } \bar{v} = N \quad (49)
\]

Equation (47) just gives us a value for the multiplier \(\theta\). The remaining three equations can be reduced to the following two equations in the two unknowns \(N\) and \(e\):
\[
\frac{1}{2} e^{1/2} N^{1/2} - w_m - ae^2 = 0 \quad (46)
\]

\[
w_m - e^2 - N = 0 \quad (48')
\]

The equations above can be used to solve for the variables numerically, for different values of the cost parameter \( a \) and the minimum wage \( w_m \):
### Separable, Linear-in-Income Utility, Price-Taking Equilibrium ($a = 0.25$)

<table>
<thead>
<tr>
<th>Minimum Wage</th>
<th>0.57</th>
<th>0.575</th>
<th>0.58</th>
<th>0.585</th>
<th>0.59</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effort</td>
<td>0.50238</td>
<td>0.50547</td>
<td>0.51264</td>
<td>0.51973</td>
<td>0.52675</td>
</tr>
<tr>
<td>Wage</td>
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<td>0.57000</td>
<td>0.57500</td>
<td>0.58000</td>
<td>0.58500</td>
</tr>
<tr>
<td>Number of workers</td>
<td>0.31548</td>
<td>0.31450</td>
<td>0.31221</td>
<td>0.30988</td>
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</tr>
<tr>
<td>On-the-Job Utility</td>
<td>0.51453</td>
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<td>0.51223</td>
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</tr>
<tr>
<td>Output Utility</td>
<td>0.39811</td>
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<tr>
<td>Profit</td>
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<td>0.19936</td>
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<tr>
<td>Welfare</td>
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<td>0.74863</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Minimum Wage</th>
<th>0.57</th>
<th>0.575</th>
<th>0.58</th>
<th>0.585</th>
<th>0.59</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effort</td>
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<tr>
<td>Wage</td>
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<td>0.60000</td>
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<td>0.61500</td>
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<tr>
<td>Number of workers</td>
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<tr>
<td>On-the-Job Utility</td>
<td>0.50509</td>
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</tr>
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<td>0.40458</td>
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<tr>
<td>Profit</td>
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</table>

### Elasticities

<table>
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<tr>
<th>Minimum Wage</th>
<th>0.57</th>
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<th>0.585</th>
<th>0.59</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effort</td>
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<td>1.6125</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<tr>
<td>Number of workers</td>
<td>-0.8232</td>
<td>-0.8400</td>
<td>-0.8624</td>
<td>-0.8844</td>
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<td>-0.3479</td>
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<td>Output Utility</td>
<td>0.4038</td>
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<tr>
<td>Profit</td>
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<td>Welfare</td>
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</table>

<table>
<thead>
<tr>
<th>Minimum Wage</th>
<th>0.57</th>
<th>0.575</th>
<th>0.58</th>
<th>0.585</th>
<th>0.59</th>
</tr>
</thead>
<tbody>
<tr>
<td>Effort</td>
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<td>1.4916</td>
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<td>1</td>
<td>1</td>
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<tr>
<td>Number of workers</td>
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</tr>
<tr>
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<td>0.2524</td>
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</tbody>
</table>
Separable, Linear-in-Income Utility, Price-Taking Equilibrium (a = 0.75)

<table>
<thead>
<tr>
<th>Minimum Wage</th>
<th>0.410538</th>
<th>0.413804</th>
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<th>0.435036</th>
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<tbody>
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<td>0.48</td>
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<td>0.293767</td>
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<tr>
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<tr>
<td>On-the-Job Utility</td>
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<tr>
<td>Output</td>
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<td>0.175371</td>
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<tr>
<td>Profit</td>
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<table>
<thead>
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<tbody>
<tr>
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<td>0.495</td>
<td>0.50</td>
<td>0.505</td>
<td>0.51</td>
<td>0.515</td>
</tr>
<tr>
<td>Wage</td>
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<tr>
<td>Number of Workers</td>
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<td>0.444219</td>
<td>0.440999</td>
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<tr>
<td>On-the-Job Utility</td>
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<td>0.357647</td>
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<tr>
<td>Output</td>
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<td>0.179210</td>
<td>0.179512</td>
<td>0.179736</td>
<td>0.179888</td>
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<tr>
<td>Profit</td>
<td>0.715698</td>
<td>0.715047</td>
<td>0.714325</td>
<td>0.713543</td>
<td>0.712710</td>
<td>0.711834</td>
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Elasticities

<table>
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<tr>
<th>Minimum Wage</th>
<th>2.4337</th>
<th>2.3871</th>
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<tr>
<td>Effort</td>
<td>-1.2320</td>
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<td>Wage</td>
<td>-0.5507</td>
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<td>-0.7313</td>
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<tr>
<td>Number of Workers</td>
<td>0.6008</td>
<td>0.5574</td>
<td>0.4925</td>
<td>0.4306</td>
<td>0.3718</td>
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<tr>
<td>On-the-Job Utility</td>
<td>0.6090</td>
<td>0.5574</td>
<td>0.4925</td>
<td>0.4306</td>
<td>0.3718</td>
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<tr>
<td>Output</td>
<td>-0.0029</td>
<td>-0.0152</td>
<td>-0.0329</td>
<td>-0.0492</td>
<td>-0.0641</td>
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<tr>
<td>Profit</td>
<td></td>
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<thead>
<tr>
<th>Minimum Wage</th>
<th>2.1067</th>
<th>2.0400</th>
<th>1.9757</th>
<th>1.9137</th>
<th>1.8541</th>
<th>1.7971</th>
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<tbody>
<tr>
<td>Effort</td>
<td>-1.4743</td>
<td>-1.5121</td>
<td>-1.5459</td>
<td>-1.5754</td>
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<td>Wage</td>
<td>-0.7712</td>
<td>-0.8070</td>
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<td>-0.9160</td>
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<tr>
<td>Number of Workers</td>
<td>0.3162</td>
<td>0.2640</td>
<td>0.2149</td>
<td>0.1691</td>
<td>0.1264</td>
<td>0.0867</td>
</tr>
<tr>
<td>On-the-Job Utility</td>
<td>0.3163</td>
<td>0.2639</td>
<td>0.2149</td>
<td>0.1691</td>
<td>0.1264</td>
<td>0.0867</td>
</tr>
<tr>
<td>Output</td>
<td>-0.0775</td>
<td>-0.0896</td>
<td>-0.1005</td>
<td>-0.1101</td>
<td>-0.1186</td>
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Separable, Linear-in-Income Utility, Price-Taking Equilibrium (a = 1.25)

<table>
<thead>
<tr>
<th>Minimum Wages</th>
<th>0.3531</th>
<th>0.3687</th>
<th>0.3844</th>
<th>0.3998</th>
<th>0.4145</th>
<th>0.4297</th>
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<tr>
<td>Effort</td>
<td>0.4062</td>
<td>0.410</td>
<td>0.415</td>
<td>0.420</td>
<td>0.425</td>
<td>0.430</td>
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<td>Wage</td>
<td>0.2805</td>
<td>0.2741</td>
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<td>0.4214</td>
<td>0.4152</td>
<td>0.4087</td>
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<td>On-the-job Utility</td>
<td>0.3147</td>
<td>0.3179</td>
<td>0.3205</td>
<td>0.3225</td>
<td>0.3240</td>
<td>0.3249</td>
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<td>Output</td>
<td>0.1573</td>
<td>0.1589</td>
<td>0.1603</td>
<td>0.1613</td>
<td>0.1620</td>
<td>0.1624</td>
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<tr>
<td>Profit</td>
<td>0.6967</td>
<td>0.6965</td>
<td>0.6960</td>
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<td>0.6940</td>
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<table>
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<tr>
<th>Minimum Wages</th>
<th>0.4422</th>
<th>0.4550</th>
<th>0.4673</th>
<th>0.4790</th>
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<tr>
<td>Effort</td>
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<td>0.440</td>
<td>0.445</td>
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<td>0.455</td>
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<tr>
<td>Wage</td>
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<td>0.2093</td>
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<td>Number of Workers</td>
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<td>0.3894</td>
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<td>On-the-job Utility</td>
<td>0.3254</td>
<td>0.3256</td>
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<td>Output</td>
<td>0.1627</td>
<td>0.1628</td>
<td>0.1627</td>
<td>0.1625</td>
<td>0.1622</td>
<td>0.1619</td>
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<tr>
<td>Profit</td>
<td>0.6914</td>
<td>0.6899</td>
<td>0.6884</td>
<td>0.6869</td>
<td>0.6853</td>
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Elasticities

<table>
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<tr>
<th>Minimum Wages</th>
<th>3.6332</th>
<th>3.4555</th>
<th>3.2613</th>
<th>3.0627</th>
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<td>Effort</td>
<td>1.1915</td>
<td>1.1558</td>
<td>1.1454</td>
<td>1.1399</td>
<td>1.1362</td>
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<tr>
<td>Wage</td>
<td>-1.9429</td>
<td>-2.0968</td>
<td>-2.2197</td>
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<td>Number of Workers</td>
<td>-0.9320</td>
<td>-1.0669</td>
<td>-1.1817</td>
<td>-1.2718</td>
<td>-1.3392</td>
</tr>
<tr>
<td>On-the-job Utility</td>
<td>0.8454</td>
<td>0.6796</td>
<td>0.5209</td>
<td>0.3766</td>
<td>0.2487</td>
</tr>
<tr>
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<td>0.8454</td>
<td>0.6796</td>
<td>0.5210</td>
<td>0.3766</td>
<td>0.2487</td>
</tr>
<tr>
<td>Profit</td>
<td>-0.0225</td>
<td>-0.0648</td>
<td>-0.1015</td>
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<table>
<thead>
<tr>
<th>Minimum Wages</th>
<th>2.6827</th>
<th>2.5102</th>
<th>2.3521</th>
<th>2.2085</th>
<th>2.0790</th>
<th>1.9626</th>
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<tbody>
<tr>
<td>Effort</td>
<td>1.1915</td>
<td>1.1558</td>
<td>1.1454</td>
<td>1.1399</td>
<td>1.1362</td>
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</tr>
<tr>
<td>Wage</td>
<td>-2.4074</td>
<td>-2.4248</td>
<td>-2.4279</td>
<td>-2.4204</td>
<td>-2.4055</td>
<td>-2.3855</td>
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<tr>
<td>Number of Workers</td>
<td>-1.3968</td>
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<td>-1.4476</td>
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<tr>
<td>On-the-job Utility</td>
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<td>0.0427</td>
<td>-0.0379</td>
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<td>-0.1633</td>
<td>-0.2115</td>
</tr>
<tr>
<td>Output</td>
<td>0.1377</td>
<td>0.0427</td>
<td>-0.0379</td>
<td>-0.1060</td>
<td>-0.1633</td>
<td>-0.2115</td>
</tr>
<tr>
<td>Profit</td>
<td>-0.1728</td>
<td>-0.1859</td>
<td>-0.1950</td>
<td>-0.2011</td>
<td>-0.2048</td>
<td>-0.2068</td>
</tr>
</tbody>
</table>
The figures listed under "none" are the values which the variables take on before the imposition of a minimum wage. The elasticities are all taken with respect to the minimum wage and are calculated through a simple point elasticity formula.

Note first that the number of workers declines as the minimum wage rises. This decline grows sharper as the minimum wage goes up. Also, the decline is significantly larger, in percentage terms, when the cost of effort enforcement is greater. The imposition of a minimum wage reduces the equilibrium level of employment just as in the classical model.

The new element in this model is that effort can vary. Effort rises with the imposition of a minimum wage. As the minimum wage increases, effort continues to increase, albeit at a diminishing rate. In percentage terms, the growth in effort is greatest when the cost of effort enforcement is high.

Output increases as long as the growth in effort is greater than the decline in the number of workers.

Welfare falls at an increasing rate with increases in the minimum wage. The decline in welfare is sharper when the cost of effort enforcement is high.

Profit actually rises initially with increases in the minimum wage. This increase takes place at a decreasing rate, and it is slower for higher values of the parameter $a$. For $a = 1.75$, profit begins to decline within the range of values we used for the minimum wage.

In mathematical terms, the increase in profit is the result of two opposing "effects". Profit $\pi (N, e; w_m)$ can be thought of as a three dimensional surface. An increase in $w_m$ lowers this surface, since it leads to a greater wage bill. At the same time,
an increase in $w_m$ leads to a decrease in $N$. In the solution to the system, $\bar{v}$ takes on the value $N$. Therefore a decrease in $N$ is roughly equivalent to having a smaller $\bar{v}$.

However, the firm’s constraint is given by $w_m - e^2 \geq \overline{v}$. A smaller value for $\overline{v}$ means a looser constraint. This means that the firm can choose combinations of $e$ and $N$ which it could not before.

On the one hand, the profit surface is falling. On the other, the constraint is growing looser, making combinations of $e$ and $N$ closer to the maximum feasible. The latter effect is the stronger one initially. As the minimum wage continues to increase, the “looser constraint” diminishes in strength relative to the “falling surface.”

The effect to be emphasized is the increase in effort, a change which exceeds in magnitude the change in any other variable in the system. The elasticity of effort with respect to the minimum wage is significantly greater than one for the parameter values which we used.

The worker does derive extra utility from the higher dividends. However, the extra utility and the higher wage do not compensate the worker for the higher level of effort. Those who keep their jobs are worse off. This is contrary to the simple classical view of the effect of a minimum wage. According to that framework, those who maintain their jobs under a minimum wage will be better off.

If we hold effort constant, as is assumed implicitly in the classical case, we might expect to get an even greater decline in employment. However, our model cannot accommodate the loss of another variable. If effort is held constant, employment would be the only remaining variable. The firm would have no choice to make, and the only economic story line left would be that of the participation constraint. Since the constraint
is similar to a supply of labor relation, a rise in the minimum wage would lead to greater employment.

4.3.2 The Pareto Optimum

Under a minimum wage, the planner’s problem with our specific functional forms becomes

$$\text{Max } W(N, e, \pi) = \pi + w_m N - e^2 N + \frac{1}{N} \int_n \text{dn}$$

subject to: $$e^{1/2} N^{1/2} - w_m N - ae^2 N - \pi \geq 0$$

and $$w_m - e^2 - N \geq 0$$

This leads to the following Lagrangian function:

$$L(N, e, \pi) = \pi + w_m N - e^2 N + \frac{1}{2} - \frac{1}{2} N^2 + \lambda \left[ e^{1/2} N^{1/2} - w_m N - ae^2 N - \pi \right]$$

$$+ \delta \left[ w_m - e^2 - N \right]$$

and following set of first order conditions:

$$N: \quad w_m - e^2 - N + \lambda \left[ \frac{1}{2} e^{1/2} N^{1/2} - w_m - ae^2 \right] - \delta = 0 \quad (50)$$

$$e: \quad -2eN + \lambda \left[ \frac{1}{2} e^{1/2} N^{1/2} - 2aeN \right] - 2\delta e = 0 \quad (51)$$

$$\pi: \quad 1 - \lambda = 0 \quad (52)$$

$$\lambda: \quad e^{1/2} N^{1/2} - w_m N - ae^2 N - \pi = 0 \quad (53)$$

$$\delta: \quad w_m - e^2 - N = 0 \quad (54)$$
Once again, equation (53) can be regarded as definitional for \( \pi \). Making use of \( \lambda = 1 \), we can derive the following expression for \( \delta \) from (51):

\[
\delta = \frac{\frac{1}{2} e^{-1/2} N^{1/2} - 2(1+a)N}{2e}
\]

The system of equations above can therefore be reduced to the following two equations in the two unknowns \( N \) and \( e \):

\[
\frac{1}{2} e^{1/2} N - w_m - ae^2 - \frac{\frac{1}{2} e^{-1/2} N^{1/2} - 2(1+a)N}{2e} = 0 \quad (50')
\]

\[
w_m - e^2 - N = 0 \quad (54)
\]

The system above can be solved numerically to yield the following:
Separable and Linear-in-Income Utility, Pareto Optimum (a = 0.25)

<table>
<thead>
<tr>
<th>Effort</th>
<th>0.50238</th>
<th>0.50390</th>
<th>0.50749</th>
<th>0.51111</th>
<th>0.51477</th>
<th>0.51846</th>
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<tbody>
<tr>
<td>Wage</td>
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<td>0.575</td>
<td>0.590</td>
<td>0.585</td>
<td>0.590</td>
</tr>
<tr>
<td>Number of workers</td>
<td>0.31546</td>
<td>0.31608</td>
<td>0.31745</td>
<td>0.31876</td>
<td>0.32001</td>
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</tr>
<tr>
<td>On-the-Job Utility</td>
<td>0.51453</td>
<td>0.51494</td>
<td>0.51586</td>
<td>0.51670</td>
<td>0.51748</td>
<td>0.51819</td>
</tr>
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<td>Output</td>
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<td>0.40138</td>
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<tr>
<td>Profit</td>
<td>0.19005</td>
<td>0.19886</td>
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<td>Welfare</td>
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<table>
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<th>0.52595</th>
<th>0.52975</th>
<th>0.53359</th>
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<tr>
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<td>0.600</td>
<td>0.605</td>
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<td>0.32338</td>
<td>0.32437</td>
<td>0.32528</td>
<td>0.32612</td>
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<tr>
<td>On-the-Job Utility</td>
<td>0.51883</td>
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<td>0.51990</td>
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</tr>
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<td>Output</td>
<td>0.41026</td>
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Elasticities

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<td>1</td>
<td>1</td>
<td>1</td>
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<td>0.6539</td>
<td>0.6487</td>
<td>0.6431</td>
<td>0.6373</td>
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<tr>
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<td>Welfare</td>
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<table>
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<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>Number of workers</td>
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<td>0.1154</td>
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<td>0.0654</td>
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<td>Output</td>
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<tr>
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<td>-0.3074</td>
<td>-0.3094</td>
</tr>
<tr>
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<td>-0.0273</td>
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### Separable and Linear-in-Income Utility, Pareto Optimum (a = 0.75)

<table>
<thead>
<tr>
<th>Minimum Wage</th>
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<th>0.416755</th>
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<tbody>
<tr>
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<td>0.475</td>
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<tr>
<td>Number of Workers</td>
<td>0.294947</td>
<td>0.295790</td>
<td>0.298564</td>
<td>0.301315</td>
<td>0.304042</td>
<td>0.306743</td>
</tr>
<tr>
<td>On-the-Job utility</td>
<td>0.468935</td>
<td>0.469527</td>
<td>0.471447</td>
<td>0.473305</td>
<td>0.475102</td>
<td>0.476836</td>
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<tr>
<td>Output</td>
<td>0.347976</td>
<td>0.348817</td>
<td>0.351956</td>
<td>0.354365</td>
<td>0.357124</td>
<td>0.359873</td>
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<tr>
<td>Profits</td>
<td>0.173988</td>
<td>0.173737</td>
<td>0.172883</td>
<td>0.171990</td>
<td>0.171060</td>
<td>0.170093</td>
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<tr>
<td>Welfare</td>
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<td>0.717483</td>
<td>0.717453</td>
<td>0.717385</td>
<td>0.717281</td>
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<table>
<thead>
<tr>
<th>Minimum Wage</th>
<th>0.424950</th>
<th>0.427711</th>
<th>0.430490</th>
<th>0.433288</th>
<th>0.436108</th>
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<tr>
<td>Wage</td>
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<tr>
<td>On-the-Job utility</td>
<td>0.475008</td>
<td>0.480115</td>
<td>0.481658</td>
<td>0.483136</td>
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<tr>
<td>Output</td>
<td>0.362611</td>
<td>0.365339</td>
<td>0.368057</td>
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<td>0.376143</td>
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<td>Profits</td>
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<td>0.169052</td>
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<td>0.165875</td>
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<td>Welfare</td>
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<td>0.715877</td>
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### Elasticities

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<tr>
<th>Minimum Wage</th>
<th>0.6080</th>
<th>0.6110</th>
<th>0.6158</th>
<th>0.6209</th>
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<td>1</td>
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<td>0.3876</td>
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<td>0.3718</td>
<td>0.3618</td>
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<tr>
<td>Output</td>
<td>0.7422</td>
<td>0.7419</td>
<td>0.7413</td>
<td>0.7406</td>
<td>0.7399</td>
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<td>Profits</td>
<td>-0.4428</td>
<td>-0.4610</td>
<td>-0.4892</td>
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<td>Welfare</td>
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<table>
<thead>
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<th>0.6379</th>
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<td>1</td>
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<tr>
<td>Number of Workers</td>
<td>0.8463</td>
<td>0.8386</td>
<td>0.8304</td>
<td>0.8216</td>
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<td>0.3412</td>
<td>0.3304</td>
<td>0.3193</td>
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<tr>
<td>Output</td>
<td>0.7391</td>
<td>0.7383</td>
<td>0.7374</td>
<td>0.7364</td>
<td>0.7352</td>
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<td>Profits</td>
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<td>-0.6368</td>
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<td>Welfare</td>
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<td>-0.0297</td>
<td>-0.0351</td>
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<td>-0.0462</td>
<td>-0.0518</td>
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## Separable and Linear-in-Income Utility, Pareto Optimum (a = 1.25)

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<thead>
<tr>
<th>Minimum Wage</th>
<th>0.435</th>
<th>0.445</th>
<th>0.455</th>
<th>0.465</th>
<th>0.472</th>
<th>0.481</th>
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<tbody>
<tr>
<td>Effort</td>
<td>0.36238</td>
<td>0.36446</td>
<td>0.36596</td>
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<td>0.440</td>
<td>0.445</td>
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<td>Number of workers</td>
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<td>0.30717</td>
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<td>On-the-job Utility</td>
<td>0.45319</td>
<td>0.45560</td>
<td>0.45797</td>
<td>0.46030</td>
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<td>Output</td>
<td>0.33177</td>
<td>0.33459</td>
<td>0.33740</td>
<td>0.34020</td>
<td>0.34299</td>
<td>0.34577</td>
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<tr>
<td>Profit</td>
<td>0.14991</td>
<td>0.14843</td>
<td>0.14690</td>
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<td>Welfare</td>
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<td>0.69660</td>
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<td>0.69614</td>
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## Elasticities

<table>
<thead>
<tr>
<th>Minimum Wage</th>
<th>0.435</th>
<th>0.445</th>
<th>0.455</th>
<th>0.465</th>
<th>0.472</th>
<th>0.481</th>
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<tr>
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<td>0.3944</td>
<td>0.3898</td>
<td>0.3849</td>
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<tr>
<td>Wage</td>
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</tr>
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<td>Number of workers</td>
<td>1.0898</td>
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<td>On-the-job Utility</td>
<td>0.4988</td>
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<td>0.4876</td>
<td>0.4819</td>
<td>0.4762</td>
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<tr>
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<td>0.7443</td>
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<td>0.7434</td>
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<tr>
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<td>-0.5651</td>
<td>-0.6091</td>
<td>-0.6557</td>
<td>-0.7042</td>
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<tr>
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<td>-0.0077</td>
<td>-0.0131</td>
<td>-0.0187</td>
<td>-0.0243</td>
<td></td>
</tr>
</tbody>
</table>

## Minimum Wage (x = 0.435)

| Effort                | 0.3745 | 0.3691 | 0.3633 | 0.3573 | 0.3510 | 0.3445 |
| Wage                  |       |       |       |       |       |       |
| Number of workers     | 1.1092 | 1.1134 | 1.1180 | 1.1226 | 1.1274 | 1.1324 |
| On-the-job Utility    | 0.4705 | 0.4647 | 0.4590 | 0.4533 | 0.4475 | 0.4418 |
| Output                | 0.7419 | 0.7413 | 0.7406 | 0.7399 | 0.7382 | 0.7384 |
| Profit                | -0.8076 | -0.8627 | -0.9205 | -0.9809 | -1.0443 | -1.1107 |
| Welfare               | -0.0300 | -0.0359 | -0.0418 | -0.0479 | -0.0541 | -0.0604 |
Separable, Linear-Income Utility, Price-Taking Equilibrium, Minimum Wage

(a = 1.25)

- Effort
- Wage
- Number of workers

Separable, Linear-Income Utility, Price-Taking Equilibrium, Minimum Wage

(a = 1.25)

- On-the-Job Utility
- Output
- Profit
- Welfare

Minimum Wage
Note that N increases with an increase in the minimum wage in the planner’s problem. Effort rises as well, while profit and welfare both fall at increasing rates with an increase in the minimum wage. The declines in profit and effort are sharper with higher values of the cost of effort enforcement parameter a.
5 Conclusion

The prevailing assumption in the literature is that effort can be neither observed nor controlled. Very little work has been done in which firms have greater powers of supervision.

Our model provides an example of what an “enforceable effort” framework looks like. In such a set-up, the quality of a job is represented by the level of effort that is required of the worker.

When price floors, such as rent controls, are imposed in product markets, a significant deterioration in quality takes place. The enforceable effort assumption allows for the same kind of quality adjustment to take place in the labor market when a minimum wage is imposed.

We find that workers who keep their jobs under a minimum wage are indeed forced to work harder than before. Of all the variables in the model, effort shows the greatest response to the imposition of a minimum wage. This effect is not usually accounted for in discussions of the impact of the minimum wage.

We also investigate the welfare properties of the model. Our price-taking set-up ends up being non-Pareto optimal, except when the worker’s on-the-job utility is separable and linear in income. In that one case, the price-taking firm’s decisions and the planner’s decisions coincide.

In that specific case, the imposition of a minimum wage makes the price-taking framework depart from the Pareto optimum. The behavior of price-taking firms does not even coincide with what a planner would do given the minimum wage constraint.
Therefore, the price-taking set-up is not even "second best" when a minimum wage is imposed.

The effect of a minimum wage on the welfare of workers is of particular interest. The usual assumption is that workers who keep their jobs are better off. We find, however, that the higher effort required of workers following the imposition of minimum wages makes them worse off. While this particular result might not obtain in more general environments, the adverse effects of a minimum wage on worker welfare will be underestimated when employment reductions are taken as the only adverse consequence. We also find that the overall welfare generated by the system (including firm profits as well as workers' and non-workers' utility) falls with an increase in the minimum wage.

With regard to further work, the first thing to be done should be to allow for the entry and exit of firms. This would make the price-taking framework perfectly competitive. Welfare properties of such a system would need to be examined, as well as the effects of a minimum wage.

It is also possible to construct a monopsonistic version of our model. In such a context, the imposition of a minimum wage may raise overall welfare.

Finally, it may be possible to investigate the effect that the enforceable effort assumption has on the size and structure of a firm. To do so, one would need to characterize the cost of effort enforcement in much greater detail. The firm would be characterized by several layers, each of which "supervises" the layer beneath it. The minimum wage would then have an effect on the number of layers, the size of each layer, and the wage and effort level required of employees of each layer.

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1 This structure is common in the literature. See, for instance, Calvo and Wellisz (1978, 1979) and Rosen (1982)
Bibliography


