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Dynamics of Bose-Einstein Condensation in $^7$Li

by

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A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree Doctor of Philosophy

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ABSTRACT

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Bose-Einstein condensation (BEC) of $^7$Li has been investigated. Because the effective interaction between $^7$Li atoms is attractive, the condensate occupation number $N_0$ is limited to $\sim 1250$ atoms, and when this limit is exceeded, the condensate becomes unstable with respect to mechanical collapse. The interplay of this limit and the natural growth of the condensate during BEC leads to complicated dynamical behavior, which has been studied both theoretically and experimentally. It has been modeled by solving the quantum Boltzmann equation, in conjunction with results from the nonlinear Schrödinger equation. It is found that $N_0$ oscillates rapidly as the condensate alternately fills and collapses, and that the oscillations can persist for many cycles before the gas comes to equilibrium. Experimental evidence for these oscillations was obtained by repeatedly producing a condensate and measuring $N_0$. The results were seen to vary randomly from one measurement to the next, which is to be expected as the timing of the oscillations is intrinsically stochastic. The distribution of $N_0$ values occurring was measured, and provides quantitative information on the condensate growth and collapse. The equilibration process itself was also observed, by quenching the gas into degeneracy and observing its relaxation.

In order to carry out these experiments, sensitive measurement and analysis techniques were developed which enabled $N_0$ to be determined in situ with an accu-
racy of ±20% and a precision of ±60 atoms. The theoretical tools used to study quantum degenerate gases were also applied to the important experimental technique of evaporative cooling, which led to substantial optimization and improvements. As part of this study, the rate constant for dipolar relaxation was measured to be $1.05 \pm 0.1 \times 10^{-14}$ cm$^3$/s, in agreement with theoretical predictions.
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Chapter 1
Introduction

This thesis is a full report on our recent investigations of Bose-Einstein condensation (BEC) in a magnetically confined gas of $^7$Li. Such a document must serve many different purposes. For example, it should provide a readable introduction to the subject for new students and other interested persons; it should document the results obtained and the methods used with appropriate detail and clarity for other scientists to understand and, if desired, to substantively repeat our work; it should serve as a comprehensive reference and instruction manual for future students in our lab. In order to meet these requirements, the thesis is conceptually separated into three parts.

The first is this introductory chapter, which will attempt to explain and motivate our studies in a general way and with minimal distractions from the experimental and technical details. The goal is to convey to the reader the ways we have developed to think about Bose condensates, so that the interest and significance of the subject can be understood.

The second part is the body of the text, Chapters 2 through 6, in which the chief technical results are presented. Chapter 2 describes the theoretical tools used to analyze the behavior of a Bose condensate, and in particular the results obtained in the case of atoms with attractive interactions. Chapter 3 describes the kinetic model we developed to understand the dynamical response of a trapped gas as it is cooled and BEC occurs. In Chapter 4, our experimental procedure is explained and data are presented to support the theoretical models previously discussed. Chapter 5 describes additional experiments which we have attempted or are developing in order to more fully characterize the dynamics of the condensate. Finally, Chapter 6 offers the conclusions derived from this work, along with questions and suggestions for
future research.

The last part of the manuscript is the Appendices, which document the details of many of the experimental systems and analysis procedures developed for this work. This section is intended primarily as an internal reference for our lab, and is more technical than the preceding.

It is hoped that by making these divisions, readers from a variety of backgrounds will find this document useful and informative.

1.1 What is BEC?

Bose-Einstein condensation is a phase transition that occurs when a collection of identical bosons is cooled to the point that their quantum mechanical deBroglie waves overlap. What exactly this means will be discussed at some length in the following pages, but the most important point is that BEC occurs only at quite low temperatures. The deBroglie wavelength $\Lambda$ is equal to $\hbar/p$, where $\hbar$ is Planck's constant and $p$ is the momentum of the particle. At temperature $T$, typical values of $p$ will be $(2mk_BT)^{1/2}$, where $m$ is the mass of the particle and $k_B$ is Boltzmann's constant. The precise condition for BEC to occur is that

$$n\Lambda^3 = n \left( \frac{2\pi \hbar^2}{mk_BT} \right)^{3/2} = 2.612,$$

where $n$ is the number density of the particles. The lower the density, the colder the required temperature is. BEC-like behavior can occur in solid, liquid, or gas phases, with observed transition temperatures ranging from a few degrees Kelvin for the densest systems to well below one millionth of a degree for the most dilute gases. In addition, BEC is thought to play a significant role in several fundamental processes in particle physics, with transition temperatures of many billions of degrees.

BEC was predicted by Einstein in 1925 [1], in an extension to Bose's derivation of the Planck black-body radiation spectrum [2]. The theory of BEC was first applied to superfluid liquid helium in the 1930s and 40s [3,4], and shortly thereafter was used to
explain superconductivity in metals at low temperatures [5]. These two branches of condensed matter physics have been enormously important research areas since their inception. More recently, BEC and related phenomena have been studied in several gas-like systems, such as excitons in a semiconductor crystal [6, 7], liquid helium wetting a porous glass [8, 9] and spin-polarized atomic hydrogen on a liquid helium surface [10]. However, the experimental system closest to Einstein’s original vision is probably the type described here, using a spin-polarized dilute atomic gas confined in a magnetic trap.

These experiments have been performed on a range of gases, namely $^{87}$Rb, $^7$Li, $^{23}$Na and $^1$H [11–14, respectively]. Other species in which BEC is currently being pursued include the remaining alkali elements and metastable $^4$He. A variety of phenomena have been investigated, such as excitation spectra [15, 16], coherence properties [17, 18], interference [19, 20], and multiple-spin-component gases [21, 22]. However, the field is still rapidly expanding, with many new experiments recently coming online or under development.

Atoms in these experiments are confined by magnetic fields in an ultra-high vacuum chamber, so that they have no contact with their environment. This allows the gas to be kept very cold, while the remainder of the apparatus is at room temperature. In our experiment, the fields provide a nearly spherical harmonic potential, in which the $^7$Li atoms oscillate with a frequency of $\omega \approx 2\pi \times 145$ Hz. The density of the gas is highest in the center of the trap, and BEC occurs when this peak density satisfies Eq. (1.1). An equivalent formulation, in terms of total number of trapped atoms $N$, is

$$
N \left( \frac{\hbar \omega}{k_B T} \right)^3 = 1.202.
$$

(1.2)

We observe BEC to occur at a temperature of 700 nK, with roughly 1 million atoms in the trap. At this temperature, the gas forms a cloud about 0.1 mm in diameter. When BEC occurs, a significant number of atoms “condense” into the quantum mechanical
ground state of the trap. Spatially, this ground state consists of a narrow spike, about 6 μm in diameter, at the very center of the cloud. The atoms in this state have many unusual properties.

Any standard textbook on statistical mechanics gives a derivation of Eq. (1.1), and a description of BEC [23, 24]; a concise treatment is also available in C. Bradley's Ph.D. thesis [25]. Fundamentally, understanding BEC means understanding such an approach, but it is also useful to try to develop a certain physical insight and intuition about the subject. Because BEC is an explicitly quantum mechanical phenomenon, it is outside the realm of direct human experience, and is difficult to explain in ordinary terms. However, many of its characteristics are shared with at least somewhat more familiar processes, and an understanding of it can to a degree be achieved through analogy with these processes. In combination with more precise mathematical descriptions, such analogies can be very fruitful.

1.1.1 Phase Transitions

For instance, BEC is a phase transition. That is to say, it is a process in which a macroscopic collection of particles changes physical state as its temperature is reduced.* An apt example from daily life is the condensation of water from the air as dew. In the case of dew, the two phases are readily distinguished: water droplets are dense and almost incompressible, and have a well defined boundary separating them from the surrounding vapor. In general, a Bose condensate cannot be so easily identified. A better way to think of the change of state occurring in BEC is by analogy with the Curie transition in a magnetic material [26]. It well known that if a magnet is heated sufficiently, it will be demagnetized as the atomic magnetic moments composing it are randomized by thermal motion. As the material is gradually allowed to

---

*Other phenomena are also sometimes called phase transitions, when the structure of a substance changes abruptly as some parameter other than temperature is varied. The present discussion is confined to “classical” phase transitions of the usual sort.
cool, the magnetization abruptly reappears at a critical temperature known as the Curie point. The transition is abrupt because, once a few neighboring atoms find themselves aligned, they tend to force other nearby atoms to align as well, spreading the order rapidly through the material.

The Curie transition is easy to measure using a magnetometer, but lacking such a device, it would be very difficult. It could be seen through subtle changes in the heat capacity of the material, or by direct probes of the atomic moments such as neutron scattering. Nonetheless, on the microscopic level the transition marks a substantial change in the global structure and order.

When Bose-Einstein condensation occurs in a gas, the atoms develop a uniform quantum mechanical phase in much the same way that atomic moments in a magnet develop a uniform orientation. Although this phase itself is fundamentally unobservable by direct means, it has a variety of effects which can and have been observed. The analogy is, then, that BEC bears a relation to a gas which is similar to the relation between the Curie transition and a solid. In both cases, a type of global order develops which is not necessarily apparent from an external viewpoint. The analogy serves well to illustrate the fallacy of considering a Bose condensate as a “fourth phase of matter”, after solids, liquids, and gases. Just as a magnetized chunk of iron is certainly a solid, a Bose-condensed gas is still a gas.

1.1.2 The Atom Field

The above analogy is of little help, however, in interpreting the meaning of a collection of atoms having an identical quantum phase. Fortunately, another more familiar example of a similar phenomenon exists, which is laser light [27]. According to the wave/particle duality principle of quantum mechanics, a beam of light must in some respects be considered as an electromagnetic wave propagating through space and in some respects as a stream of particles, termed photons. When considered
as a wave, the notion that a laser has a definite phase is easy to comprehend: at a particular point in space, there is an oscillating electric field whose phase can be defined relative to some arbitrary reference oscillator. In ordinary incoherent light, such as that from an incandescent bulb, this phase is absent. Although an electric field is present in both cases, in the latter it is randomly fluctuating rather than smoothly oscillating in time, so a value for the phase cannot be defined.

The electric field of a beam of photons is very much analogous to the wave function of a beam of atoms, and physical intuition developed regarding the one can be fruitfully applied to the other. The fact that an electric field is ordinarily thought of as a classically defined and measurable quantity reflects the fact that it ordinarily describes a macroscopic number of photons. It is then possible to measure its amplitude and phase to high precision without appreciably disturbing its overall state. Matter wave functions more typically describe the state of only a single particle, so that the familiar uncertainty restrictions apply and the wave function cannot be directly observed. This dichotomy does not necessarily hold, however, as in the case of incoherent light described above, where the electric field of each photon is different.* Similarly, the wave function of a Bose condensate describes a large number of atoms, allowing the "atom field" to be defined and observed to an unusual degree of precision.

The analogy between a Bose condensate and an electric field does beg the question of what, exactly, an electric field is. Nonetheless, given some familiarity of how laser beams behave, the comparison is useful. It has been observed that a Bose condensate forms a standing wave when confined in a cavity such as an atom trap [28], that two condensates interfere both in space [19] and in time [20] when overlapped, and that a condensate expanding from a small source behaves much like light diffracting from a small aperture [11]. Unlike a laser, a Bose condensate is generally not monochromatic, which is to say that the energy of the atoms and thus the oscillation frequency of the

*More precisely, the probability of finding two photons in the same field mode is small.
atom field is not uniform in space. At any time, however, it is possible to define and in principle measure a definite phase relationship between any two points of a condensate. This is the reason that the order parameter associated with BEC is often described as the quantum phase, though a more precise definition would be as the condensate wave function itself.

1.1.3 Statistics of Indistinguishable Objects

The production of laser light relies on the ease of creating photons from other forms of energy. In a typical laser, a gain medium consisting of molecules in an excited state is surrounded by an optical cavity. As the molecules relax to their ground state they emit photons, a fraction of which are reflected by the cavity mirrors back to the gain medium. The electric field of this reflected light induces other molecules to radiate their energy into the cavity as well, thus amplifying the reflected light. As this continues, the amplitude of radiation in the cavity mode builds up and lasing occurs.

Bose-Einstein condensation is in many respects a quite different process, even though the end results are closely related. Since transforming other forms of energy into matter is rather difficult, atoms cannot be simply emitted into a cavity mode, but must be transferred to it from elsewhere. Further, BEC is an explicitly thermodynamical process, and describes the equilibrium state of the gas. A laser is in an explicitly nonequilibrium state, and is only maintained by continually adding energy to the system through the replenishing of the gain medium. The analogy of a Bose condensate to a laser beam does not, therefore, help answer the question of why a Bose condensate forms.

Surprisingly, the analogy to the Curie transition also fails to address this issue. The Curie transition, like all phase transitions except BEC, occurs because of interactions between the constituent particles. In the case of a magnet, neighboring atoms
are coupled by electronic forces which, in the right circumstances, creates a tendency for them to align. BEC, in contrast, is predicted to occur even in a perfectly ideal gas, and has been observed in very nearly ideal gases.

The answer lies in quantum statistical mechanics. A Bose condensate forms for no other reason than that it is the most likely configuration of a sufficiently cold collection of bosonic atoms. The standard calculation is shown in many textbooks, but involves abstractions such as the calculation of the partition function and the choice of a particular ensemble. A simple example serves better to illuminate the underlying idea. Rather than considering atoms, which have an infinite number of quantum states available, a finite system can be used. In particular, imagine a set of \( N \) “quantum coins,” identical particles which can each be in either of two states, H or T, with equal probability. Consider the probability \( P_H \) that all of the coins are in state H. If the coins were classical, distinguishable objects, each possible state of the collection could be labeled by an enumeration of the states of each coin. There are \( 2^N \) distinct enumerations, so the probability of any particular configuration, such as “HHH...H”, would be \( 2^{-N} \).

If the coins are identical bosons, however, then it is fundamentally impossible to determine whether a particular coin is in a particular state. The configurations “coin 1 in H and coin 2 in T” cannot be counted as distinct from “coin 1 in T and coin 2 in H.” In both cases all that can be said is that there is one coin in H and one in T, so that only one configuration can be counted.* Each possible configuration of \( N \) coins can thus be labeled simply by the number of coins in state H. Since this occupation number can range from 0 to \( N \), \( P_H \) is \( 1/(N+1) \). For large \( N \), this is tremendously greater than \( 2^{-N} \).

This example shows that, because of their symmetric nature, a group of identi-

---

*The quantum state of the pair can be written \( \Psi(1,2) = [\psi_H(1)\psi_T(2) \pm \psi_T(1)\psi_H(2)]/\sqrt{2} \) if \( \psi_i(k) \) represents coin \( k \) being in state \( i \). It is a fundamental property of identical bosons that their wave function must be symmetric under the exchange \( 1 \rightarrow 2 \), so the “+” sign must be used and \( \Psi \) is unique.
cal bosons is more likely to be found all in the same state than a similar group of distinguishable particles. In order to demonstrate how this brings about an abrupt transition such as BEC, it is only necessary to allow the probability of observing state $H$ to vary. In a physical system, this occurs because at a finite temperature $T$, the probability of finding an atom in a state with energy $E$ decreases as $\exp(-E/k_B T)$, a fundamental result of classical statistical mechanics [24]. So, if the probability of a single coin being in $H$ is $p_H$, then $p_H$ is analogous to the temperature of the real gas. The relative probability of observing $k$ coins in $H$ becomes $p_H^k(1 - p_H)^{N-k}$, so that the probability of observing all $N$ coins in $H$ is

$$P_H = \frac{p_H^N}{\sum_k p_H^k(1 - p_H)^{N-k}}. \quad (1.3)$$

The denominator is the normalizing sum of the relative probabilities of each possible configuration. Recognizing that the sum can be expressed as a geometric series* allows $P_H$ to be simplified to

$$P_H = \frac{p_H^N(2p_H - 1)}{p_H^{N+1} - (1 - p_H)^{N+1}}, \quad (1.4)$$

which in the limit of large $N$ becomes

$$P_H = \begin{cases} \frac{2p_H - 1}{p_H} & (p_H > .5) \\ 0 & (p_H < .5) \end{cases}. \quad (1.5)$$

The function (1.4) is plotted in Fig. 1.1 for $N = 1000$. The discontinuity at $p_H = .5$ is analogous to the phase transition of BEC, and arises for the same reason. The counting argument described, peculiar as it is compared to the methods of normal statistics, is precisely that used in the standard derivation of BEC. It is instructive that it yields similar results even in so simple a system.

The three models discussed in this section cover, it is hoped, most of the more counterintuitive aspects of BEC. However, they are analogies only, and have their

---

*The geometric series $G = \sum_{k=0}^{N} x^k$ can be summed by noting that $xG - G = x^{N+1} - 1$. 
Figure 1.1  Phase transition in a set of quantum coins. The probability $P_H$ that 1000 identical coins will be found all in state H is plotted versus the probability $p_H$ that a single coin will be found in state H.

limits. Like the Curie transition, BEC is a phase transition marked by the development of a global order that does not necessarily correspond to a visible change in physical state. In the particular case of a magnetically confined gas, there in fact is an observable change of state, since the Bose condensate is localized in the center of the trap. But the point of the comparison remains true, because the development of a coherent quantum phase is certainly the fundamental characteristic of BEC. The meaning of the quantum phase can be understood by analogy with a laser beam, but the phenomena are not the same. Atoms obey the Schrödinger equation, while photons obey Maxwell’s equations; the detailed behavior of the two systems will be different. And although a set of quantum coins illustrates how the indistinguishability of quantum particles can lead to a phase transition, the transition obtained is not really BEC. For instance, the number of atoms in the lowest-energy state increases abruptly when BEC occurs, but when there are an infinite number of states available, the probability for all the atoms to be in the ground state remains negligibly small at any nonzero temperature.
1.2 Interactions

As mentioned above, Bose-Einstein condensation occurs even in an ideal gas of non-interacting point-like particles. However, all real gases are composed of atoms which do interact. As the density of the gas is lowered, the importance of interactions is reduced as the atoms spend less and less time near each other. However, the interactions never vanish completely, and necessarily have a significant impact on BEC. Because of this significance, the different effects of interactions are discussed at some length here.

The vast majority of work on BEC has involved liquid helium or superconductivity, where interactions are complicated and have as important an effect on the behavior of the system as have the quantum statistical effects. However, the benefit of studying BEC in a dilute gas is that interactions cause only a small perturbation to ideal-gas behavior. In particular, the density is low enough that there is a vanishingly small probability for three atoms to be near each other at the same time, so that three-body processes can usually be neglected compared to two-body binary interactions.

1.2.1 Elastic Collisions

An obvious binary process is the elastic collision, where two atoms approach one another, interact as they pass, and then fly apart on altered trajectories. The total kinetic energy of the two atoms is unchanged, but energy can be exchanged between them. It is this exchange of energy which allows the gas to come to thermal equilibrium, so that elastic collisions are directly responsible for the actual occurrence of BEC. In fact, consideration of these collisions alone provides a means of analyzing the equilibrium state of the gas. This approach is explained here because it is relatively elementary but is not so familiar as the usual development.*

Suppose that the number of atoms in a gas with energy $E$ is $f(E)$, and define

---

*The following application of the method of detailed balance was first brought to our attention by H. Stoof.
\(R_c(E_1, E_2; E_3, E_4)\) to be the rate at which atoms with energies \(E_1\) and \(E_2\) collide and acquire new energies \(E_3\) and \(E_4\). Clearly, \(R_c\) must be proportional to \(f(E_1)\) and \(f(E_2)\), since the rate for a collision to occur is proportional to the number of atoms available to collide. Less clearly, \(R_c\) depends on \(f(E_3)\) and \(f(E_4)\). This is a consequence of the indistinguishability of the atoms. If it were possible to label two atoms A and B, then the rate for A and B to collide would be independent of the presence or absence in the gas of some other atom C with the same energy as A happens to acquire. However, such labeling is impossible. One result is that \(R_c\) for a Bose gas is twice as large as for an otherwise identical gas of distinguishable particles, since it must include both collisions where the atom with energy \(E_1\) acquires energy \(E_3\) and where it acquires energy \(E_4\).\(^\dagger\) In addition, if there were already atoms with energy \(E_3\) present, then it is impossible to determine, after the collision, which atom had collided. Since all possible ways of getting to the same final configuration contribute to the rate, \(R_c\) must be proportional to the number of atoms with energy \(E_3\) after the collision, which is \(1 + f(E_3)\). An identical argument applies if atoms with energy \(E_4\) were present, so \(R_c(E_1, E_2; E_3, E_4) \propto f(E_1)f(E_2)[1 + f(E_3)][1 + f(E_4)].\)

In equilibrium, the distribution of energy is, by definition, constant. Normally, this occurs because the forward and backward rates for each possible collision become equal. Setting \(R_c(E_1, E_2; E_3, E_4) = R_c(E_3, E_4; E_1, E_2)\) yields a condition on the function \(f:\)

\[
f(E_1)f(E_2)[1 + f(E_3)][1 + f(E_4)] = f(E_3)f(E_4)[1 + f(E_1)][1 + f(E_2)].\tag{1.6}
\]

This can be simplified by defining \(\mathcal{F}(E) = f(E)/[1 + f(E)],\) which then satisfies

\(^\dagger\)This enhancement occurs for all possible collision channels, and in particular for all values of the orbital angular momentum of the colliding pair. Because of their symmetry requirements, however, identical bosons can only collide via channels with even numbers of angular momentum quanta. So at high energy, where many channels are involved, only half are available, and the total collision rate is reduced by a factor of two. These effects cancel, yielding at rate identical to the "normal" value for distinguishable particles. However, our experiments take place at low enough temperatures that only a single channel is open, and the enhancement is observable.
\( \mathcal{F}(E_1)\mathcal{F}(E_2) = \mathcal{F}(E_3)\mathcal{F}(E_4) \). A further constraint arises because the collisions are elastic, so that \( E_1 + E_2 = E_3 + E_4 \). Since Eq. (1.6) holds for all energies, it is true when \( E_3 = 0 \), which yields

\[
\mathcal{F}(E_1)\mathcal{F}(E_2) = Z\mathcal{F}(E_1 + E_2)
\]

(1.7)

with \( Z = \mathcal{F}(0) \). The only function satisfying this relation is \( \mathcal{F}(E) = Z \exp(-\beta E) \), with \( \beta \) arbitrary. Solving in turn for \( f \) yields

\[
f(E) = \frac{1}{Z^{-1} \exp(\beta E) - 1},
\]

(1.8)

the usual Bose-Einstein distribution function if \( \beta^{-1} \) is identified with the temperature and \( Z \) with the fugacity. From \( f \), all other equilibrium properties of the gas are readily determined. The power and simplicity of this argument illustrate the importance of elastic collisions.

The actual rate for elastic collisions—the prefactor multiplying \( R_e \)—must be determined numerically from knowledge of the molecular potential \( U(r) \), which specifies the interaction energy between two atoms separated by a distance \( r \). The way this calculation is performed is discussed in Chapter 2. The result is that, at low temperatures, all of the molecular physics can be expressed by a single number, the scattering length \( a \). For \(^7\text{Li} \), \( a \) has the value -1.46 nm [29].

It is possible to understand why the scattering process can be characterized by a single parameter without working through the detailed calculation. As mentioned above, BEC occurs when the deBroglie wavelength of the atoms in a gas becomes comparable to the average interparticle spacing. If the system is to be considered a dilute gas, however, the interparticle spacing must be much larger than the range of \( U(r) \). Otherwise, the particles would be interacting continuously, and the system would be better described as a liquid. It follows, then, that \( \Lambda \) must be much larger than the range of \( U \). Typically, the molecular potential is effective across a distance on the order of 1 nm, and the deBroglie wavelength is 1000 times greater. This
means that it is incorrect to think of the atoms bouncing off each other like marbles or billiard balls. Rather, the collisions should be considered as waves diffracting off of very small obstacles. The obstacles are so small that they can be considered as point-like, in that it is impossible for the scattered wave to carry away any details of their structure. This is entirely analogous to the inability of a microscope to resolve details much smaller than the wavelength of light. The scattered field therefore must consist of a spherical wave, which is characterized only by its amplitude. This amplitude is the scattering length. The units of $a$ can also be understood from this argument, since a spherical wave is expressed as

$$\frac{e^{ikr}}{r}, \quad (1.9)$$

which requires an amplitude with units of length to cancel the factor of $r$ in the denominator.*

The probability for a collision to occur is proportional to the intensity of the scattered wave, and thus to $a^2$. Indeed, the cross section, which is the standard parameter used to characterize a collision process, is given by $\sigma = 8\pi a^2$ for identical bosons. The cross section gives the "effective size" of an atom, in the sense that if an atomic beam with a flux of $F$ atoms per unit time per unit area is incident on a single target atom, the collision rate will be $\sigma F$.

1.2.2 Mean Field Energy

The previous discussion pointed out that an atom in an ultracold gas is in one sense very small, since another atom must be within a range of 1 nm or so in order to interact with it, but in another sense very large, since its wave function extends across 1 $\mu m$ or more. As a consequence, in a Bose condensate, where the interparticle spacing is small compared to $\Lambda$, an atom finds itself in a sense interacting with many other

*Note that this argument only applies to three-dimensional scattering. In other cases, the scattering length cannot be identified with the amplitude of the scattered wave.
atoms at once, but colliding only infrequently. This seemingly paradoxical result is, of course, an expression of the particle/wave duality fundamental to quantum mechanics discussed earlier.†

One observable consequence of this situation arises because, when two atoms have overlapping wave functions, the molecular interaction shifts the energy of the pair. This occurs because the interaction energy is very large when two atoms are close, so that the average interaction energy can be appreciable even if the probability of the atoms being close to one another is small. The size of the energy shift is proportional to the scattering length, since the amplitude of the scattered wave is a measure of the effective strength of the interaction potential. The magnitude of the shift can be computed from this fact, using dimensionality arguments: if a pair of particles has an unperturbed energy $E$ which is shifted by $\Delta E$, then the dimensionless ratio $\Delta E/E$ must be proportional to $a/\Lambda$, since $\Lambda$ is the only other physical length scale characterizing the particles. If there are more than two atoms within a distance $\Lambda$, as is usually the case, then the total shift will be

$$\frac{\Delta E}{E} \sim n \Lambda^3 \frac{a}{\Lambda},$$

where $n$ is the density of atoms. The unperturbed energy $E$ is, however, related to $\Lambda$ through $E \sim \hbar^2 / m \Lambda^2$, so that the shift itself is approximately

$$\Delta E \sim \frac{\hbar^2}{m \Lambda^2} n \Lambda^3 \frac{a}{\Lambda} = \frac{\hbar^2 n a}{m}.$$  \hspace{1cm} (1.11)

A precise derivation is provided in Chapter 2, and yields $\Delta E = 4\pi \hbar^2 n a / m$. This interaction energy is a mean-field effect, since it is the result of averaging over the possible locations of the atoms in the gas.

At the relatively low densities of a dilute gas, $\Delta E$ is small compared to the tran-

†Mean-field effects do also occur in classical systems, such as in the Van der Waals equation of state.
sition temperature for BEC:

\[
\frac{k_B T_c}{\Delta E} = \frac{(2\pi \hbar^2/m)(n/2.6)^{2/3}}{(4\pi \hbar^2 an)/m} \approx \frac{1}{an^{1/3}}. \tag{1.12}
\]

The condition that the gas be dilute is just that \(na^3 \ll 1\), so \(\Delta E \ll k_B T_c\). However, \(\Delta E\) can easily be large compared to the unperturbed energy of the Bose condensate. If the condensate is confined to a volume of size \(\ell^3\), its energy will be \(E_0 \approx \hbar^2/m\ell^2\).\(^*\)

Therefore,

\[
\frac{\Delta E}{E_0} \approx \frac{\hbar^2 an/m}{\hbar^2/m\ell^2} = na\ell^2 \approx N_0 \frac{a}{\ell}, \tag{1.13}
\]

where \(N_0\) is the number of atoms in the condensate and in the last step the relation \(n \approx N_0/\ell^3\) was used. The dilute-gas condition, expressed in terms of \(N_0\), \(a\), and \(\ell\), can be written \(N_0^{1/3} a/\ell \ll 1\). Applying this constraint to Eq. (1.13) shows that \(\Delta E/E_0\) must be small compared to \(N_0^{2/3}\). However, \(N_0\) can be very large, so that \(\Delta E/E_0\) can still be much larger than 1.

The scattering length \(a\) can have either sign. It is negative for \(^7\)Li, and positive for the other gases in which BEC has been observed. When \(a < 0\), \(\Delta E\) is also negative, meaning that the interaction between atoms is effectively attractive. This presents a problem for large \(N_0\), because the derivative of the condensate energy with respect to density will be negative. To understand the difficulty, suppose that the condensate would have size \(\ell\) in the absence of interactions. Then in the presence of interactions, the condensate will be smaller as the gas attracts itself together. But when \(\ell\) decreases, the density rises, making the interactions even stronger, which in turn further decreases \(\ell\). There is no end to the process: the condensate can always reduce its energy by decreasing \(\ell\) and increasing \(n\). Eventually, \(n\) reaches \(a^{-3}\), and the condensate can no longer be considered a dilute gas. Rather than BEC, some other type of phase transition occurs, from a gas to a denser liquid or solid phase.

\(^*\)The zero-point energy \(E_0\) is just the kinetic energy required to satisfy Heisenberg's uncertainty principle \(\Delta x \Delta p \geq \hbar/2\). Since the condensate is confined to a volume \(\ell^3\), \(\Delta x \sim \ell/2\) and \(p\) can be no smaller than \(\hbar/\ell\). The corresponding energy \(p^2/2m\) is \(E_0\).
Because of this instability, it was long thought that BEC could not occur in a gas with \( a < 0 \) \cite{30,31}. However, the problem only arises when \( N_0 \) is so large that \( E_0 \) is negligible compared to \( \Delta E \). For smaller \( N_0 \), the condensate energy will be

\[
E = E_0 + \Delta E \approx \frac{\hbar^2}{2ml^2} + \frac{4\pi\hbar^2N_0a}{m\ell^3}.
\]  

(1.14)

Since \( E_0 \) increases as \( \ell \) decreases, it provides a positive pressure offsetting the attractive interactions, as long as \( N_0 \) is small enough. The maximum possible \( N_0 \) value which can be supported is that at which the compressibility \( d^2E/d\ell^2 = 0 \). Roughly, this occurs at \( N_0 = N_m \approx \ell/|a| \). A more precise calculation is discussed in Chapter 2, and yields \( N_m = 0.57\ell/|a| \) for atoms confined in a spherically symmetric harmonic trap \cite{32}.

In our experiments \( \ell \) is approximately 3 \( \mu \)m, so \( N_m \sim 1250 \) atoms, substantially larger than 1. With this many atoms, the quantum phase of the wave function is certainly well-defined and measurable, so the atoms do form a genuine condensate. Furthermore, \( N_m \) increases as \( \ell \) increases, suggesting that having negative \( a \) is not a significant limitation. However, as \( \ell \) increases, the maximum condensate density \( N_m/\ell^3 \) decreases rapidly. In comparison, the critical density for the BEC transition depends only on temperature, so that the fractional density of the condensate vanishes for large \( \ell \). If the condensate is to be distinguished, its density must be appreciable compared to the density of uncondensed gas, so that in a large container, BEC can only be observed at vanishingly small temperatures when \( a < 0 \).

1.2.3 Inelastic Collisions

The third and last significant type of interactions are inelastic collisions which lead to heating and a loss of atoms from the trap. Inelastic processes occur because the atoms are in a highly excited, metastable state which can decay and release energy. The atoms are excited in two different ways.
First, each individual atom is in an excited spin state. The energy of an atom with magnetic moment \( \mathbf{m} \) in a magnetic field \( \mathbf{B} \) is given by \( -\mathbf{m} \cdot \mathbf{B} \). The lowest energy state is, therefore, the one with \( \mathbf{m} \) and \( \mathbf{B} \) maximally aligned. The energy of this state will always decrease as the field strength increases, so an atom in this state will be attracted to a maximum in \( \mathcal{E} \). However, Maxwell's equations prohibit the existence of a maximum of \( B \) in free space, so a magnetic trap is ordinarily constructed by establishing a minimum in \( B \), and trapping an atomic state with \( \mathbf{m} \) antiparallel to \( \mathbf{B} \). It is therefore possible for the spin, which determines \( \mathbf{m} \), to relax to a lower energy state.

In principle, an excited atom will eventually decay by spontaneously emitting a photon, but the rate for this to occur is very slow, on the order of once every 10 million years. Instead, relaxation usually occurs during a collision, where the internal energy can be transferred to the motion of the atoms. Linear momentum is conserved by splitting the energy evenly between the two atoms. However, since the spin state is changing, it is necessary to conserve angular momentum too. If the atomic moments are not already perfectly anti-aligned with \( \mathbf{B} \), it is possible for one atom to flip and transfer its spin to the other; this is called spin exchange. This process happens relatively easily, but by storing the atoms with \( \mathbf{m} \) and \( \mathbf{B} \) perfectly anti-parallel, we suppress it. Since both atoms start out with the maximal amount of angular momentum, it is impossible to transfer any from one to the other.

The second way to conserve angular momentum is to transfer it from the spin of the atom to the orbital motion of the two atoms about their center of mass. The spin and motional degrees of freedom are coupled by the interaction of the two magnetic dipole moments, so this process is termed dipolar relaxation. The dipole-dipole interaction is very weak, so collisions of this type occur only infrequently, but they are nonetheless the dominant loss mechanism in our experiment. They proceed at a rate proportional to the density, since the rate for a given atom to collide depends
linearly on the number of other atoms available for it to collide with.

Just as each atom is individually in an excited state, the collection of atoms as a whole is only metastable. Lithium is a solid metal at room temperature, and certainly prefers to be a solid at the ultracold temperature at which our experiments are performed. In order for a solid to form, however, atoms must first bond to form molecules, molecules must join to form clusters, and clusters must come together to form a crystal. The initial stage of this recombination process cannot occur during a binary collision. If two atoms were to collide and stick together, the molecule formed would necessarily be at rest in the center-of-mass frame. However, a tremendous amount of energy is released when the atoms bind, which then has nowhere to go. If the process is to occur, a third atom must be present in the collision to carry this excess energy away. Even then, the energy released is generally sufficient to allow all three atoms to escape the trap, so the thermodynamically favored crystallization process can never get started. Ordinarily, the density of the gas is so low that three-body collisions are very uncommon, and this loss mechanism is not observed in our experiments. However, mean-field interactions cause the density to increase greatly when a condensate is unstable, and molecular recombination does occur in that circumstance.

A final loss mechanism is due, not to interactions between trapped atoms, but between trapped atoms and untrapped, room temperature molecules in the vacuum chamber. Collisions with these background molecules knock atoms out of the trap at a steady rate. In our experiment, the vacuum pressure is low enough that this rate is small compared to that for dipolar relaxation.

1.3 Dynamics of BEC in $^7$Li

The three types of interactions discussed above combine and lead to a surprising richness of behavior in BEC. This thesis will necessarily focus on the particular case of $^7$Li, but other systems are interesting as well. To give just one example, if two
condensates of different species are created in the same trap and the mutual scattering length of the two species is positive, the condensates will repel one another [22, 33]. When the densities are high enough, the condensates are immiscible, just like oil and water. As mentioned previously, however, the condensates are not liquids, but definitely gases. The possibility of oxygen and nitrogen in the air spontaneously separating is an alarming one, but the phenomenon occurs only because the condensates are, in a sense, strongly interacting gases. Such gases represent a new domain of physics, even aside from their phase-coherent properties as condensates.

When $a < 0$, the strongly interacting regime is unattainable, since $N_0$ is limited. What happens, though, if $N_0$ is gradually raised from below $N_m$ to above it? This occurs naturally as the gas is cooled, since the condensate fraction grows as $T$ is reduced. Clearly, at some point the condensate becomes unstable and begins to collapse upon itself. But how is this collapse initiated? How does it proceed? What remains in the trap when it is completed? It is these questions which are the subject of our investigations here.

What is desired is an accurate understanding of the dynamics of the BEC process and the eventual collapse. To this end, it was necessary to develop time-dependent experimental and theoretical methods, which posed challenges beyond the production and understanding of BEC itself. Although many details remain to be clarified, the general idea of this dynamics is now known.

When a $^7$Li gas is cooled down to the critical temperature for BEC, the condensate forms and begins to grow. Atoms are transferred into the condensate through elastic collisions, but because the collision rate is finite and an atom must generally undergo several collisions before entering the condensate, the growth of the condensate lags the cooling by an appreciable amount. At first, the mean field interactions in the condensate are negligible, and the condensate is essentially like that of an ideal gas. As $N_0$ rises, however, the attractive interactions start to draw the condensate together.
In our trap, it shrinks from a ball initially about 6 \(\mu\text{m}\) in diameter to about 4 \(\mu\text{m}\) diameter before becoming unstable.

As \(N_0\) approaches \(N_m\), the stability of the condensate becomes more and more marginal. Because the gas is not at zero temperature, there is some thermal excitation of the condensate motion: it jiggles, ripples, and pulsates in the trap. At some point, this thermal motion causes the density to increase enough that the condensate begins to collapse. Even if the gas were at \(T = 0\), the collapse would be initiated for \(N_0 < N_m\) by quantum mechanical tunneling of the condensate from the marginally stable state to a denser collapsing state. As usual, an unstable equilibrium is no equilibrium at all.

Once initiated, the collapse proceeds quickly, in a time somewhat shorter than the oscillation period of atom in the trap. During the collapse, the density rises, and with it the rate for inelastic collisions. When the density becomes high enough for three-body recombination to be significant, the losses start to limit the density increase and the collapse begins to decelerate. It appears currently that the collapse halts with about two hundred atoms remaining in the condensate, which then re-expand. The atoms which are lost to inelastic collisions acquire a large amount of energy, and rapidly leave the trap. The process is thus reminiscent of a stellar nova, where a star first implodes under its gravitational attraction and then explodes, blowing off a fraction of its atmosphere and leaving a core behind.

In the case of the trapped atoms, however, a surrounding cloud of noncondensed gas remains. It is essentially unaffected by the collapse, since the noncondensed atoms have too much kinetic energy to be caught up in the collapse itself, and the ejected atoms most likely pass through the cloud without suffering further collisions. After the collapse is complete, then, the filling process continues just as before. This leads to a cycle of filling and collapse which continues until either the noncondensed cloud runs out of atoms, or the gas is allowed to come to equilibrium.
The following chapters detail the experimental and theoretical efforts that have gone into developing the preceding story, and outline the questions that remain to be answered.
Chapter 2
Collapsing Condensates

In the limiting case of an ideal gas, a Bose condensate is a reasonably simple object. It consists of many atoms, each having the same wave function $\psi_0$. This wave function obeys the single-particle Schrödinger equation, so that its characteristics and dynamical response can be readily calculated using standard techniques. Any observable quantity can be expressed as an expectation value over $\psi_0$ and evaluated.

The behavior of a real gas, however, will be affected by interactions between its constituent particles. Even when the gas is dilute and the interactions weak, they can have a profound influence. This is particularly the case for attractive interactions, which limit the occupation number of the condensate and effectively prevent BEC from occurring in a spatially homogeneous system.

This chapter develops the methods used to account for interactions, and to predict the dynamical behavior of the condensate under their influence. The presence of noncondensed atoms is mostly ignored for the moment, but will be taken up again in Chapter 3.

2.1 Theory of Interactions

In a dilute gas, the basic building block describing interactions is the scattering event: two more or less independent atoms approach each other, interact strongly but briefly, and then depart. Any theory of many-particle interaction must start by correctly describing this two-body process. The description given here is very much \textit{ad hoc}. but a comprehensive treatment can be found in [34] and a more basic one in [35, Ch. 13].
2.1.1 Binary Scattering

In the center-of-mass frame, a collision between two atoms which interact via a molecular potential \( U(|r_1 - r_2|) \) is mathematically equivalent to the scattering of a single fictitious particle with reduced mass \( \tilde{m} = m/2 \) by a fixed central potential \( U(r) \). The wave function of the particle will satisfy the Schrödinger equation,

\[
\left( -\frac{\hbar^2}{2\tilde{m}} \nabla^2 + U(r) \right) \psi(r) = E\psi(r). \tag{2.1}
\]

At large \( r \), it takes the asymptotic form

\[
\psi(r) \xrightarrow{r \to \infty} e^{ikz} + f(\theta, \varphi) \frac{e^{ikr}}{r}, \tag{2.2}
\]

where the first term represents the incoming wave propagating along an arbitrary direction \( z \), and the second term is a scattered wave expanding from the origin. The wave number \( k \) is given by \( k = (2\tilde{m}E/\hbar^2)^{1/2} \). The function \( f \) is termed the scattering amplitude, and is to be determined in terms of \( U \).

For a spherically symmetric potential, \( \psi \) can be expanded as a superposition of eigenfunctions of angular momentum, \( Y_l(m, \theta) \). Since the boundary condition (2.2) has azimuthal symmetry, \( m \) will be zero and both \( \psi \) and \( f \) are independent of \( \varphi \). If \( \psi(r) = r^{-1}u_l(r)Y_{l0}(\theta) \), the radial wave function \( u_l \) will satisfy

\[
\left( \frac{d^2}{dr^2} + k^2 - \frac{l(l+1)}{r^2} - \frac{2\tilde{m}}{\hbar^2}U(r) \right) u_l = 0. \tag{2.3}
\]

At large distances, the two atoms interact via the van der Waals force, so \( U \) behaves as \( C_6/r^6 \) for \( r \to \infty \). The potential term will therefore be negligible compared to the angular momentum barrier at large \( r \), unless \( l = 0 \). However, in the low energy limit \( k \to 0 \), the classical turning point of the motion will be at \( r \sim l/k \) and the particle will be unable to penetrate to small \( r \) and sample the potential. The solution to Eq. (2.3) will therefore be independent of \( U' \), so no scattering can occur. For this reason, the only partial wave contributing to \( f \) as \( k \to 0 \) will be \( l = 0 \).
For fixed small \( k \), \( U \) can be neglected when \( r \gg (C_s/k^2)^{1/6} \). At larger \( r \), \( u \) will be a solution to the free wave equation, which can generally be expressed

\[
u(r) \xrightarrow{r \to \infty} A \sin(kr + \delta)
\]

(2.4)

for arbitrary \( A \) and \( \delta \). The phase \( \delta \) is determined by the behavior of \( u \) at small \( r \); one way to determine it is to numerically solve Eq. (2.3) for \( u(r) \), and fit its asymptotic form to Eq. (2.4).

The total wave function \( \psi \) is therefore asymptotic to

\[
\psi(r) \to \frac{A}{\sqrt{4\pi}} \frac{\sin(kr + \delta)}{r} = A' \frac{\sin(kr + \delta)}{r},
\]

(2.5)

since \( Y_{00} = 1/\sqrt{4\pi} \). The amplitude \( A' \) is determined by the boundary condition (2.2). The \( l = 0 \) projection of the incident plane wave \( e^{ikz} \) is \( \sin(kr)/kr \), so the boundary condition can be rewritten

\[
\psi \to \frac{1}{2ikr}[(1 + 2ikf)e^{ikr} - e^{-ikr}],
\]

(2.6)

while the asymptotic solution is

\[
\psi \to \frac{A'}{2ir} (e^{i\delta}e^{ikr} - e^{-i\delta}e^{-ikr}).
\]

(2.7)

Equating the amplitudes of the incoming waves yields

\[
A' = \frac{e^{i\delta}}{k},
\]

(2.8)

and then equating the amplitudes of the outgoing waves provides an expression for the scattering amplitude

\[
f = \frac{e^{i\delta}}{k} \sin \delta.
\]

(2.9)

To see how \( \delta \) behaves as \( k \to 0 \), it is useful to divide the range of \( r \) into an internal part, where \( U \) is appreciable, and an external part, where \( U \) can be neglected. If the

*For \(^7\text{Li}, C_s \approx 1400 \text{ atomic units (a.u.)}[36]\) and a typical value for \( k \) is \( 2\pi/(3 \text{ \mu m}) \approx 1 \times 10^{-4} \text{ a.u., so the distance to which } U \text{ extends is roughly } 70 \text{ a.u. } \approx 4 \text{ nm.} \)
division point $r_p$ is small compared to $k^{-1}$, then the form of the wave function in the internal part will not depend significantly on $k$. Therefore, the logarithmic derivative of the true radial wave function at the radius $r_p$,

$$\frac{u'}{u} \bigg|_{r_p} \equiv \gamma_p^{-1},$$  \hspace{1cm} (2.10)

will be nearly independent of $k$. In the external part, $\psi$ takes on its asymptotic form, given by Eq. (2.4). The two expressions and their derivatives must agree at $r_p$, so that

$$\gamma_p^{-1} = \frac{k}{\tan(kr_p + \delta)}. \hspace{1cm} (2.11)$$

Expanding the tangent function and solving for $\tan \delta$ yields

$$\tan \delta = \frac{k\gamma_p - \tan kr_p}{1 + k\gamma_p \tan kr_p}, \hspace{1cm} (2.12)$$

which in the limit of small $k$ becomes

$$\tan \delta = k(\gamma_p - r_p). \hspace{1cm} (2.13)$$

The term on the right of (2.13) appears to depend on $r_p$, but $\delta$ itself is a genuine physical quantity independent of the arbitrary choice of division point. The dependence on $r_p$ must therefore cancel, and $\delta$ must tend to zero like $-ka$ for some constant $a$. This constant is the s-wave scattering length, known to be -1.46 nm for $^7$Li [29]. In the zero-energy limit, the scattering amplitude $f = -a$, so the scattering length completely characterizes an ultracold collision.

The scattering amplitude directly gives the partial cross section for scattering into angle $\theta$, by

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2, \hspace{1cm} (2.14)$$

which can be derived by calculating the probability current

$$j = \frac{\hbar}{2mi}(\psi^* \nabla \psi - \psi \nabla \psi^*) \hspace{1cm} (2.15)$$
for the scattered wave function (2.2). So, in the low energy limit the total cross section is \( \sigma = 4\pi a^2 \). This result is modified for identical bosons, however, since scattering by an angle \( \theta \) is indistinguishable from scattering by \( \pi - \theta \). Both processes contribute coherently to the cross section, giving

\[
\frac{d\sigma}{d\Omega} = \frac{1}{2} |f(\theta) + f(\pi - \theta)|^2.
\tag{2.16}
\]

The additional factor of \( 1/2 \) reflects the fact that only half of the total atom flux is scattered into \( d\Omega \). The net result is to increase the s-wave cross section by a factor of 2, to \( \sigma = 8\pi a^2 \).

### 2.1.2 T-Matrix and Pseudo-potential

It will be useful to express \( a \) more directly in terms of the molecular potential \( U(r) \). One way to accomplish this is by formally writing the scattered wave function as

\[
\psi(r) = e^{ikz} + \frac{2m}{\hbar^2} \int d^3r' \, G(r',r')U(r')\psi(r').
\tag{2.17}
\]

The kernel \( G \) is the Green’s function for the free space Schrödinger equation, satisfying

\[
(\nabla^2 + k^2)G(r,r') = \delta^3(r-r').
\tag{2.18}
\]

That (2.17) gives a solution to Eq. (2.1) can be verified by direct substitution. The Green’s function is

\[
G(r,r') = -\frac{e^{\pm ik|r-r'|}}{4\pi|r-r'|},
\tag{2.19}
\]

where the + sign in the exponent must clearly be used if the boundary condition (2.2) is to be satisfied.

The asymptotic form of (2.17) is obtained by expanding \( G \) using \( |r-r'| \rightarrow r - \hat{r} \cdot r' \), and yields

\[
\psi \rightarrow e^{ikz} - \frac{m}{2\pi\hbar^2} \frac{e^{ikr}}{r} \int d^3r' \, e^{-ikr'} U(r')\psi(r').
\tag{2.20}
\]
\[ \begin{align*} \hline \hline + \quad \hline \hline \quad \hline \quad \hline \hline = \quad \hline \end{align*} \]

**Figure 2.1** Summation of the ladder diagrams. The true molecular interaction between atoms is strong, and must be calculated to all orders in perturbation theory. This is equivalent to summing the series represented on the left. Knowing the sum, however, the interaction potential can be replaced with an effective potential which gives the correct result in first order, as represented on the right.

The integral in Eq. (2.20) is denoted \( T(k_1, k_2) \), and gives the amplitude for a particle originally travelling with momentum \( \hbar k \hat{z} \) to be found with momentum \( \hbar k \hat{r} \). Comparison with the results of the preceding section shows that

\[
T(k_1, k_2) \xrightarrow{k \to 0} \frac{2\pi \hbar^2 a}{m} = \frac{4\pi \hbar^2 a}{m}.
\quad (2.21)
\]

This low-energy limit will be denoted \( T^{2B} \), reflecting the fact that it describes a two-body process.

A simple approximation suggested by the form of Eq. (2.17) is to replace \( \psi \) in \( T^{2B} \) by the incident wave function \( e^{ikz} \). This is nothing but the Born approximation, and is valid when \( U \) is weak. However, the molecular potential is not weak, and the Born approximation is a poor one. It can be recovered, however, through the use of a renormalization-like procedure. If \( U \) is replaced by a new potential which yields the correct value for \( T^{2B} \) in the Born approximation, then first-order perturbation theory can be used with this modified potential, and will give results correct to all orders. In diagrammatic terms, this is equivalent to replacing an infinite sum of two-body "ladder diagrams" with a single modified diagram, as suggested in Fig. 2.1.

Since the range of \( U \) is anyway very small compared to the wavelength \( k^{-1} \), a suitable choice for a modified potential is simply

\[
\tilde{U}(r) = T^{2B} \delta^3(r).
\quad (2.22)
\]
This $\tilde{U}$ is usually called the pseudo-potential, and has found many applications in physics. It is important to remember that it is only accurate when applied as a first-order perturbation. A true delta-function potential gives no scattering at all.

### 2.1.3 Nonlinear Schrödinger Equation

With an understanding of binary interactions in hand, the problem of an interacting condensate can now be considered [37, 38, and other references noted below]. The condensate is characterized by a many-body wave function $\Phi(r_1, r_2, \ldots, r_{N_0})$, which is a solution of the full Schrödinger equation

$$\hat{H} \Phi = \sum_i \left\{ -\frac{\hbar^2}{2m} \nabla_i^2 + V(r_i) + \frac{1}{2} \sum_{j \neq i} U(|r_i - r_j|) \right\} \Phi = E \Phi.$$  \hspace{1cm} (2.23)

The trapping potential $V$ is included. Since $\Phi$ is to describe a condensate, it is expected to consist of a product of single-particle wave functions

$$\Phi_0(r_1, r_2, \ldots, r_{N_0}) = \prod_i \psi(r_i).$$  \hspace{1cm} (2.24)

The correct function for $\psi$ can be determined using the variational principle, since the best functional form will be the one which minimizes the energy $\langle \Phi_0 | \hat{H} | \Phi_0 \rangle$. However, if arbitrary variations in $\psi$ are allowed, the energy will clearly be minimized by $\psi = 0$, which is unphysical. It is necessary to maintain the normalization of $\psi$, or equivalently, the expectation value of the number operator $\hat{N}$,

$$\langle \Phi_0 | \hat{N} | \Phi_0 \rangle \equiv \langle \Phi_0 | \sum_i \mathbb{1} | \Phi_0 \rangle = N_0 \langle \psi | \psi \rangle^{N_0},$$  \hspace{1cm} (2.25)

where $\mathbb{1}$ is the identity operator. This can be achieved by using a Lagrange multiplier $\mu$, and minimizing $\langle \Phi_0 | \hat{H} - \mu \hat{N} | \Phi_0 \rangle$. Here $\mu$ plays precisely the usual role of the chemical potential: it will subsequently be seen that $\mu$ in fact is the chemical potential.

Evaluating the expectation value to be minimized yields

$$\langle \hat{H} - \mu \hat{N} \rangle = \sum_i \int d^3r_i \left\{ \psi_i^* (H_i - \mu) \psi_i + \frac{1}{2} \sum_{j \neq i} \int d^3r_j \psi_j^* \psi_i \psi_i U_{ij} \psi_j \psi_j \right\}$$

$$= N_0 \int d^3r \left\{ \psi^* (H_0 - \mu) \psi + \frac{N_0 - 1}{2} \int d^3r' \psi^* \psi U(|r - r'|) \psi \psi \right\},$$  \hspace{1cm} (2.26)
with a subscript or prime indicating the argument of a function and
\[ H_i = -\frac{\hbar^2}{2m} \nabla_i^2 + V(r_i), \quad H_0 = -\frac{\hbar^2}{2m} \nabla^2 + V(r). \] (2.27)

The interaction term will only be important when the occupation number of the condensate is large, so its coefficient can be safely simplified to \( N_0/2 \). Minimization is carried out using the variational calculus in the standard way, and results in
\[ \delta(\hat{H} - \mu \hat{N}) = N_0 \int d^3r \left\{ \delta \psi^*(H_0 - \mu)\psi + \delta \psi(H_0 - \mu)\psi^* \right\} 
+ N_0 \int d^3r' U(|r - r'|)\psi^* \psi \delta \psi^* + \psi \delta \psi \right\}, \] (2.28)

Integration by parts was used to rewrite the \( \psi^* \delta \nabla^2 \psi \) term, and the dummy integration variables were interchanged on the two terms of the form \( \psi \delta \psi' \). Since the real and imaginary parts of \( \psi \) can be varied independently, the coefficients in the integrand of \( \delta \psi \) and \( \delta \psi^* \) must independently vanish, implying that
\[ \left\{ H_0 - \mu + N_0 \int d^3r' U(|r - r'|)\psi^* \psi' \right\} \psi = 0, \] (2.29)

which is not quite right.

The problem with Eq. (2.29) is that the integral over \( U \) is huge, and diverges for the physically plausible case of particles with perfectly hard repulsive cores. What this points to is a failure of the original trial wave function \( \Phi_0 \); in reality, the condensate wave function is not just a product of single particle wave functions, but has a many-body correction part which is important whenever \( |r_i - r_j| \) is small for some \( i \) and \( j \). Since the range of \( U \) is very small, however, the product-state approximation should be adequate almost everywhere, and all that is needed is to know the effect of the many-body part on the interaction energy. Fortunately, this effect can be included nonperturbatively using the pseudo-potential approach described in the previous section. It is simply necessary to replace \( U \) in Eq. (2.29) with \( T^{2B} \delta^3(r - r') \).

The result is the nonlinear Schrödinger equation (NLSE) \([38]\),
\[ -\frac{\hbar^2}{2m} \nabla^2 \psi + V(r)\psi + N_0 T^{2B} |\psi|^2 \psi = \mu \psi, \] (2.30)
which governs the behavior of an interacting condensate. Equation (2.30) shows that 
\( \mu \) is indeed the energy of a single atom in the presence of an existing condensate,
which defines the chemical potential \( \mu \equiv \partial E / \partial N \)[24].

In the case of \( ^7 \text{Li} \), \( a \) and thus \( T^{2B} \) is negative, indicating that the interaction
energy of the condensate is negative. In Chapter 1, it was concluded from this that
the condensate occupation number must be limited, and this conclusion is born out
by numerical solution of Eq. (2.30) [32, 39, 40]. If a spherically symmetric harmonic
potential \( V(r) = m \omega^2 r^2 / 2 \) is used, it is found that solutions exist only when \( N_0 \) is
smaller than a critical value \( N_m = .57 a / \ell_0 \), where \( \ell_0 = (\hbar / m \omega)^{1/2} \) is the length scale
of the condensate.

It will also be useful to express the total energy of the condensate as a functional
of \( \psi \), or

\[
E[\psi] = \langle \Phi_0 | H | \Phi_0 \rangle = N_0 \int d^3r \psi^* \left[ -\frac{\hbar^2}{2m} \nabla^2 + V(r) + \frac{N_0 T^{2B}}{2} |\psi|^2 \right] \psi. \tag{2.31}
\]

The pseudo-potential has again been used to express the interaction term. Note that
the total energy of the condensate is not the sum of the energies of the single particle
states. Rather,

\[
E[\psi] = N_0 \mu - \frac{1}{2} N_0 T^{2B} \int d^3r |\psi|^4. \tag{2.32}
\]

This reflects the fact that the many-body part of the condensate wave function makes
a significant contribution to the interaction energy.

The preceding discussion considered a time-independent situation, but the dynamical
equation governing the condensate can be obtained with some additional
effort [38]. As might be expected from (2.30), the time-dependent single-particle
wave function \( \Psi(r, t) \) obeys

\[
i \hbar \frac{d\Psi}{dt} = \left( -\frac{\hbar^2}{2m} \nabla^2 + V + T^{2B} |\Psi|^2 \right) \Psi \tag{2.33}
\]

with the separable solution \( \Psi = \psi(r) \exp(-i \mu t) \).
2.2 Variational Approximation

The NLSE cannot be solved analytically, and even numerical solution is difficult when the interaction term is significant and negative, as will be discussed in Section 2.3. It is therefore sensible to attempt approximate solutions, in order to obtain some physical insight and a qualitative understanding of the condensate behavior. One approach described here is based on a variational solution for $\psi$. In the absence of interactions, the NLSE reduces to the normal Schrödinger equation, and for a harmonic potential the solution for the ground state is a Gaussian function. Furthermore, because of the stability limit $N_0 < N_m$, the interaction term can never come to dominate the equation. A reasonable choice for a trial solution then, is a Gaussian function with variable width. Wave functions of this form have been studied ever since it was first realized that solutions to the NLSE existed for $a < 0$ [41-43]. However, Stoof [43] has provided the most careful and complete development, and his approach is mainly followed here.

2.2.1 Limit on $N_0$

The true ground state wave function $\psi$ will minimize the condensate energy. If $V(r) = m\omega^2 r^2/2$ is used in Eq. (2.31), then the trial wave function
\begin{equation}
\psi = \left( \frac{N_0}{\pi^{3/2} \ell^3} \right)^{1/2} \exp \left( -\frac{r^2}{2\ell^2} \right)
\end{equation}
gives an energy
\begin{equation}
E(\ell) = \frac{3\hbar^2 N_0}{4m} \left[ \frac{1}{\ell^2} + \frac{\ell^2}{\ell_0^4} \frac{4}{3\sqrt{2\pi}} \frac{N_0|a|}{\ell^3} \right],
\end{equation}
with $\ell_0 = (\hbar/m\omega)^{1/2}$. Expressed in terms of the dimensionless variable $q = \ell/\ell_0$, the energy simplifies to
\begin{equation}
E(q) = \frac{3N_0\hbar\omega}{4} \left( \frac{1}{q^2} + q^2 - \frac{\beta}{q^3} \right),
\end{equation}
where
\begin{equation}
\beta = \frac{4}{3\sqrt{2\pi}} \frac{N_0|a|}{\ell_0}
\end{equation}
Figure 2.2 Variational calculation of condensate energy, using a Gaussian trial wave function. The energy $E$ of a condensate with Gaussian radius $\ell$ is shown as a function of $q = \ell/\ell_0$. The strength of the interaction is parametrized by $\beta = 0.53N_0|a|/\ell_0$, with $\beta = \beta_c = 0.36$ at the stability limit. Each curve is labeled with its value of $\beta/\beta_c$.

characterizes the strength of the interactions.

The energy $E$ is plotted in Fig. 2.2 for several values of $\beta$. A local minimum near $q = 1$ is observed when $\beta$ is small, but the absolute minimum always occurs at $q = 0$. This quantitatively confirms the argument expressed in Section 1.2.2. Note that, in reality, the energy will diverge to positive infinity as $q \to 0$, since the true molecular potential $U$ has a steep inner wall arising from exchange effects. However, long before $q$ is this small, the density will become comparable to $|a|^{-3}$ and the treatment of the condensate as a weakly interacting gas will fail. The graphs in Fig. 2.2 can only be taken to extend down to $q \gg |a|/\ell_0 \approx 5 \times 10^{-4}$ for the $^7$Li parameters. This is, however, quite far into the unstable region, so that many details of a collapsing gas can be described by the NLSE.

If a local minimum at large $q$ exists, it represents a metastable state for the condensate. The maximum value of $\beta$ at which such a state is found is determined
Figure 2.3  Condensate size in the variational approximation.

by solving simultaneously
\[
\frac{dE}{dq} = \frac{d^2E}{dq^2} = 0,
\]
which occurs at the inflection point marked with a dot in Fig. 2.2. The values thus obtained are
\[
q_c = \frac{1}{5^{1/4}},
\]
and
\[
\beta_c = \frac{8}{3} \frac{1}{5^{5/4}}.
\]

From $\beta_c$, the predicted value of $N_m$ is found to be $0.67\ell_0/|a|$, about 20% greater than the valued obtained numerically from the NLSE.* This comparison gives a measure of the accuracy of the Gaussian wave function approximation. For $N_0 < N_m$, the solution of $dE/dq = 0$ gives the size of the metastable condensate. Just before the collapse occurs, $q$ shrinks to 0.67, indicating a density increase of a factor of 3.3. The dependence of $q$ on $N_0$ is shown in Fig. 2.3.

---

*Stoof has recently extended the variational technique to give an improved estimate for $N_m$ of $0.61\ell_0/|a|$ [44].
2.2.2 Dynamics in the Gaussian Approximation

The true utility of the variational approach described above comes from its application to dynamics. Stoof has shown that the motion of the condensate can be treated as the motion of a quasiparticle with coordinate $\ell$ and mass $m^*$ moving on the potential surface $E(\ell)$ [43]. This provides a simple physical picture for the collapse process, where the quasiparticle is initially trapped in the local minimum at $q \sim 1$, but when $N_0$ is increased and the barrier falls, the condensate slides towards $\ell = 0$, like a marble rolling down a hill.

One dynamical quantity easily calculated in this framework is the frequency of the “breathing mode” oscillation of the condensate, which is determined by the curvature of $E$ at the metastable minimum. The curvature cannot be expressed analytically, but is readily determined numerically by solving

$$\frac{dE}{d\ell} \propto q - q^5 - \frac{3}{2} \beta = 0$$  \hspace{1cm} (2.41)

for $q_\beta$ and and using this value in

$$\frac{d^2E}{d\ell^2} = \frac{3N_0 \hbar \omega}{2 \ell_0^2} \left( \frac{3}{q^4} + 1 - \frac{6\beta}{q^5} \right) = \frac{3}{2} N_0 m \omega^2 \left( 5 - \frac{1}{q_\beta^2} \right).$$  \hspace{1cm} (2.42)

Eq. (2.41) was used to partially simplify the last equality. The breathing-mode frequency $\omega_B$ is then obtained from

$$\frac{1}{2} m^* \omega_B^2 \ell_0^2 (q - q_\beta)^2 = \frac{1}{2} \frac{d^2E}{d\ell^2} \ell_0^2 (q - q_\beta)^2,$$  \hspace{1cm} (2.43)

or

$$\omega_B = \omega \sqrt{\frac{3N_0 m}{2m^*} \left( 5 - q_\beta^{-4} \right)}.$$  \hspace{1cm} (2.44)

The effective mass $m^*$ is determined by ensuring that $\omega_B$ takes the correct value in the limit of a noninteracting gas $q_\beta \to 1$. Since each atom makes one half of a trap oscillation in one breathing mode oscillation, $\omega_B \to 2\omega$ and $m^* = 3N_0 m/2$. Thus $\omega_B$ is simply

$$\omega_B = \omega \sqrt{5 - q_\beta^{-4}}.$$  \hspace{1cm} (2.45)
Figure 2.4  Breathing mode oscillation frequency $\omega_B$ for $^7$Li in a symmetric trap with $\omega = 145$ Hz. The frequency approaches zero as the condensate becomes unstable with respect to compression.

which correctly approaches zero as $q_\beta \to q_c$. The dependence of $\omega_B$ on $N_0$ is shown in Fig. 2.4; a more complete treatment can be found in [45].

The dynamical picture can also be used to understand the initiation of a collapse. If $N_0$ is close to $N_m$ so that a barrier is present but sufficiently small, the condensate will be able to tunnel through it. Stoof calculates the semi-classical approximation for the tunneling rate to be

$$\Gamma_0 = \sqrt{\frac{m^* \omega_B v_B^2}{\pi \hbar}} \exp \left\{ -\frac{1}{\hbar} \int_{\ell_1}^{\ell_2} d\ell \left( 2m^* [E(\ell) - E(\ell_1)] \right) \right\}, \quad (2.46)$$

where $\ell_1$ is the location of the metastable minimum, $\ell_2$ the location on the inner side of the barrier where $E = E(\ell_1)$, and $v_B$ is a parameter characterizing the barrier, which is on the order of $\omega_B$ times the peak width. The rate is plotted as the dashed curve in Fig. 2.5. Note that the tunneling process is a coherent one involving all $N_0$ atoms in the condensate. This is one of very few situations in physics where a relatively large composite object is predicted to exhibit tunneling, and observation of the phenomenon would be a truly remarkable demonstration of quantum mechanics. Additional discussions of the tunneling rate are given in [42, 46, 47].
Figure 2.5  Collapse initiation rates $\Gamma_C$ as a function of condensate occupation number, for $^7\text{Li}$ in a symmetric trap with $\omega = 145$ Hz. The solid curves give the rates for thermal fluctuations at the temperatures shown, and the dashed curve gives the quantum tunneling rate.

For a condensate in a finite-temperature gas, however, the tunneling effect is masked by possibility for the collapse to be initiated by thermal fluctuations. In equilibrium, the breathing mode will be thermally excited, which can drive the condensate over the barrier when the barrier height becomes comparable to $k_B T$. The rate for this to occur is approximately

$$\Gamma_T = \frac{\omega_B}{2\pi} \exp \left\{ -\frac{E(\ell_m) - E(\ell_0)}{k_B T} \right\},$$

(2.47)

where $\ell_m$ is the position of the barrier peak. The rate is plotted for several temperatures in Fig. 2.5. As can be seen, the thermal excitation rate dominates even for quite low temperatures; in order for tunneling to be observed, an essentially pure condensate is required.

Finally, the Gaussian approximation can be used to investigate the dynamics of the collapse itself. Treated semi-classically, the condensate in the unstable regime will obey Newton's equations

$$m^* \ddot{\ell} = -\frac{dE}{d\ell},$$

(2.48)
which can be solved numerically. However, during the collapse, atoms are lost due to inelastic collisions, and these losses must be accounted for by allowing $N_0$ to vary in time. Atoms are lost by two mechanisms, dipolar relaxation and molecular recombination. Dipolar relaxation is a two-body process which scales as the density $n^2$, while molecular recombination is a three body process scaling as $n^3$. The loss rate from the condensate is therefore

$$\dot{N}_0 = -\int d^3r \left[ \frac{2G_2}{2}n(r)^2 - \frac{3G_3}{6}n(r)^3 \right], \quad (2.49)$$

which evaluates to

$$\dot{N}_0 = -\frac{N_0^2}{\pi^{3/2}\ell^3} \left( \frac{G_2}{2\sqrt{2}} + \frac{N_0G_3}{6\sqrt{3}\pi^{3/2}\ell^3} \right) \quad (2.50)$$

for a Gaussian density distribution. The $G_i$ are the rate constants for the inelastic collisions.* The two coupled equations for $q$ and $N_0$ can be solved together using standard techniques. The results are shown in Fig. 2.6. It is seen that the entire collapse occurs in less than a millisecond, and that virtually all of the atoms are lost.

The response of the condensate to a dynamic trapping potential $V$ has also been investigated, by allowing $\ell_0$ to vary in time. The results are discussed in Section 5.4.

2.2.3 Asymmetric Trap

In the forgoing, the condensate was assumed to be spherically symmetric. However, the actual trap used in the experiments is more nearly cylindrically symmetric, with an axial frequency of 131.5 Hz and radial frequencies of 150.5 Hz and 151.5 Hz. The Gaussian variational approach can be used to examine this case as well. The trapping potential is taken to be

$$V(\rho, z) = \frac{m}{2}(\omega_\rho^2\rho^2 + \omega_z z^2) \quad (2.51)$$

*The coefficients in Eq. (2.49) reflect the definition of $G_i$ as the rate coefficient for collisions in a nondegenerate gas. In each $i$-body collision, $i$ atoms are lost, which provides the numerators. The denominators arise from the coherence properties of a condensate, which reduce the rate for $i$-body collisions by $i!$ [17].
and the trial wave function

$$\psi = \left( \frac{N_0}{\pi^{3/2}\ell^2_{\rho}\ell_z} \right)^{1/2} \exp \left( -\frac{\rho^2}{2\ell^2_{\rho}} - \frac{z^2}{2\ell^2_z} \right).$$

(2.52)

The condensate energy $E$ is then

$$E(\ell_\rho, \ell_z) = \frac{N_0 \hbar^2}{4m} \left( \frac{2}{\ell^2_{\rho}} + \frac{1}{\ell^2_z} + \frac{2\ell^2_{\rho}}{\ell^2_{\rho} + \ell^2_z} + \frac{\ell^2_z}{\ell^2_{\rho} + \ell^2_z} - \frac{4}{\sqrt{2\pi}} \frac{N_0 |a|}{\ell^2_{\rho} \ell_z} \right),$$

(2.53)

with $\ell_{0i} = (\hbar/m\omega_i)^{1/2}$.

As before, the condensate will be stable if $E$ has a minimum at $\ell_i \neq 0$. To determine the stability limit, it is necessary to find the value of $N_0$ at which this minimum disappears. At an extremum, $\nabla_{\ell} E = 0$, which implies that

$$-\frac{1}{q^3_{\rho}} + q_\rho + \frac{\beta}{q^2_{\rho}q_z} = 0$$

(2.54)

and

$$-\frac{1}{q^3_z} + q_z + \frac{\beta \ell}{q^2_{\rho}q_z^2} = 0,$$

(2.55)
Figure 2.7  Stability criterion in an asymmetric trap. (a) The function $Q$ in Eq. (2.56) with $\epsilon = 2$ and values of $\beta$ as indicated. A stable condensate exists when a root $Q = 0$ is present. (b) The ratio of the maximum condensate number $N_m$ to the quantity $\ell_{\min} / |a|$ as a function of trap asymmetry $\epsilon = \omega_\rho / \omega_z$.

where $q_i = \ell_i / \ell_0$, $\epsilon = \ell_0 / \ell_\rho$, and now $\beta = (2/\pi)^{1/2} N_0 |a| / \ell_0$. Eliminating $q_\rho$ from these equations yields $q_\rho = (1 - \beta / q_z)^{1/4}$, and

$$Q(q_z) \equiv q_z^4 + \beta \epsilon \frac{q_z^{3/2}}{\sqrt{q_z - \beta}} - 1 = 0.$$  

A minimum in $E$ will exist for a given value of $\beta$ only if Eq. (2.56) has a solution. Technically, it is necessary to check whether the extremum corresponding to a solution is a minimum or a maximum, but it is clear from the general shape seen in Fig. 2.2 that if any extrema exist, they will be a pair with one minimum and one maximum except when $\beta = \beta_c$. This is indeed the case, as can be seen in Fig. 2.7(a).

The critical value $\beta_c$ is determined by the condition $Q(q_z) = dQ/dq_z = 0$. These equations cannot be solved analytically, except in the limits of large and small $\epsilon$. For $\epsilon \gg 1$, the roots of $Q$ will occur at small $\beta$ and small $q_z$, since the middle term in (2.56) must be comparable to 1. It is therefore possible to neglect the $q_z^4$ term, and the resulting equations yield $\beta_c \approx 0.62 / \epsilon^{1/2}$, or $N_m \approx 0.78 \ell_0 / |a|$. For $\epsilon \ll 1$, the critical values of $q_z$ and $\beta$ can be expanded about $q_z = 1$ and $\beta = 1$, yielding
\( \beta_c = 1 - 0.75\epsilon^{2/3} \). The resulting value of \( N_m \) is approximately \( 1.25\ell_0z/|a| \). In both limits, the stability condition can be expressed as

\[
N_{0m} \approx \frac{\ell_{\min}}{|a|},
\]

(2.57)

where \( \ell_{\min} \) is the lesser of \( \ell_{0x} \) and \( \ell_{0z} \). This result is confirmed by numerical calculation of \( \beta_c \), as shown in Fig. 2.7(b).

The shape of the condensate at the critical point is also interesting. For large \( \epsilon \), indicating a cigar-shaped trap, the critical value of \( \ell_z \) is approximately \( \ell_{0x} \), while \( \ell_\rho \) is always close to \( \ell_{0x} \). Thus, the long axis of the condensate shrinks as \( N_0 \) increases, and the collapse occurs when the condensate is roughly spherical. In contrast, for a pancake-shaped trap, \( \ell_z \) remains constant at \( \ell_{0x} \) but \( \ell_\rho \) approaches roughly \( (\ell_{0z}\ell_{0x}^2)^{1/3} \). So, the condensate becomes less oblate before collapsing, but hardly spherical.

An asymmetric variational wave function can also be used to obtain the three lowest excitation frequencies of the condensate, by allowing separate variation of \( \ell_x, \ell_y, \) and \( \ell_z \) in \( \psi \) [48]. The frequencies are obtained by diagonalizing the curvature matrix of \( E \) and using the low-\( N_0 \) limit to determine the effective mass \( m^* \). The eigenvalues for the experimental trap parameters are shown in Fig. 2.8, while the eigenvectors identify the nature of the oscillations. For small \( N_0 \), each mode corresponds to a one-dimensional breathing motion along the corresponding trap axis. At high \( N_0 \), the lowest frequency describes the three dimensional breathing mode, the middle frequency is the quadrupolar mode in which the condensate alternates axial and radial compression, and the highest frequency is the quadrupolar mode in which the two radial sizes oscillate out of phase while the axial size remains constant.

### 2.3 Numerical Solution of the NLSE

The variational method described in the previous section is a powerful technique which provides both physical insight and reasonably accurate quantitative results, at least in the stable regime. Although it makes predictions for the collapse dynamics,
Figure 2.8 Condensate oscillation frequencies in an asymmetric trap using experimental parameters.

as in Fig. 2.6, it is possible that the assumption of a Gaussian wave function becomes inaccurate as the interaction energy becomes dominant. It is therefore desirable to compare the variational predictions with numerical solutions of the NLSE.

Numerical studies of the NLSE have a substantial history, since the same equation governs several classical nonlinear wave phenomena, notably in the propagation of deep-water waves [49], the collapse of Langmuir waves in plasmas [50], and the self-focusing of light waves in nonlinear media [51]. Zhakharov, in particular, has devoted much effort to the subject [52]. Accurate modeling of the collapse is known to be difficult, because of the large dynamic range in time and length scales required. In addition, the numerical simulation of trapped condensates involves several novel issues as to the growth and loss mechanisms at work.

The most straightforward way to include collisional losses in the NLSE is to add inelastic terms to the right-hand side of Eq. (2.33),

$$-i \left( G_2 |\psi|^2 + \frac{G_3}{2} |\psi|^4 \right) \psi, \quad (2.58)$$

as in Eq. (2.49). However, a rigorous derivation showing this to be correct is not at
hand, and Stoof for one disagrees with its validity, arguing that incoherent effects such as collisions additionally require the addition of a stochastic noise term to the NLSE [44]. Kagan et al. have nonetheless used (2.58), and also include a "filling" term,

\[ +i\xi \psi, \tag{2.59} \]

representing the loading of atoms into the condensate from a non-zero temperature gas which is being cooled [53]. Again, this makes certain assumptions about the effects of elastic collisions. If, for instance, the condensate is in a highly excited state, (2.59) will represent collisions as feeding atoms directly into this excited state, whereas the thermalization process is usually thought to cause condensation into the ground state of the trap. Indeed, one might intuitively expect that elastic collisions would cause an excited state of the condensate to decay, rather than build up.

Lacking better alternatives, however, it is interesting to see the effect of the above additions to the NLSE. Kagan et al. report two main results. First, when a collapse occurs, roughly half the atoms in the condensate are lost. An example of the time dependence of \( N_0 \) is shown in Fig. 2.9. This is in contrast to the variational model, which predicts nearly all the atoms to be lost. The second result is that the condensate remaining is in a excited state. This excitation comes about because, after the collapse is halted, the atoms which were caught up in the collapse but not lost find themselves localized in a very small volume, and thus have a great deal of kinetic energy.* As this excitation evolves, the condensate wave function becomes substantially different from a Gaussian, which calls into question the approximations of the variational approach.

Unfortunately, the parameters used by Kagan et al. in their study did not correspond well to the experimental parameters of interest. As can be seen from Fig. 2.9, the filling rate used is roughly \( 3 \times 10^4 \) atoms/s, while the experimental value is in the range \( 10^2 - 10^3 \) atoms/s for the data reported here. Also, the authors neglected

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*This is seen to occur in the variational approach as well, but with only \( \sim 1 \) atom remaining to be excited, the effect is negligible.
Figure 2.9  Numerical solution of the NLSE using the parameters of Kagan et al. [53]. The calculation was performed by E. Abbondanzieri.

$G_2$ and used a value of $G_3$ which is $\sim 500$ times greater than theoretical estimates.† A collaborative effort is currently underway at Rice to repeat and further investigate these studies under more realistic conditions [55].

The results by Kagan et al. do, however, point out that many aspects of the collapse process and condensate dynamics are not well understood, and suggest a rich and complex behavior rewarding of further study. It is also to be hoped that experimental efforts will help shed light on these issues, a question which will be taken up again in Chapter 4.

†Kagan et al. use a dimensionless coefficient of $10^{-3}$ in their equation. Their time unit is $2/\omega \approx 2$ ms, and their density unit is $(8\pi \ell_0^2 |a|)^{-1} \approx 3 \times 10^{12}$ cm$^{-3}$. These yield a dimensioned loss coefficient of $5 \times 10^{-26}$ cm$^9$/s, which should be compared with $G_3/2$ for atoms in a condensate. Moerdijk et al. predict $G_3 = 2.6 \times 10^{-28}$ cm$^9$/s [54].
Chapter 3
Kinetics of BEC

The previous chapter dealt with the properties of a Bose condensate, but if a condensate is to be studied, it must first be formed. Two processes are involved in this formation. First is the essentially experimental issue of creating a gas which is cold and dense enough for BEC to occur. One of the techniques used is evaporative cooling, in which the most energetic atoms in a thermal distribution are systematically removed. As the gas re-equilibrates, it cools. We use evaporative cooling to reduce the gas temperature by a factor of a thousand; it is a crucial stage of the experiment and without it BEC would not yet have been observed in a dilute gas. The second process is the action of condensation itself, in which elastic collisions deposit atoms in to the ground state of the trap once the gas is cold enough. It is important to understand the rate at which this occurs, so that experiments can be performed on a suitable time scale. Furthermore, when $\alpha < 0$, the filling process affects the condensate behavior directly, and must be understood if the dynamics of the collapse are to be accurately modeled.

Both of these issues relate to the redistribution of atomic energies through elastic collisions, properly the study of gas kinetics. In this chapter, kinetic theory is briefly developed and our implementation described. The model is then applied first to the problem of evaporative cooling [56], and then to the behavior of a quantum degenerate gas [57].

3.1 The Quantum Boltzmann Equation (QBE)

The goal of kinetic theory is to understand the behavior of the distribution of atoms in a gas. In classical mechanics, this distribution is characterized by the func-
tion $f(r, p)$, where

$$dN = \frac{d^3r \, d^3p}{(2\pi\hbar)^3} f(r, p)$$

(3.1)

is the number of particles in the volume $d^3r$ at position $r$ having momentum $p$ in a range $d^3p$. For a quantum gas, this definition is inadequate since an atom does not have a well-defined position or momentum. The analog of $f$ is then the Wigner function, which is an expression of the density matrix in a mixed position-momentum basis [58]. For a thermal gas, however, the quantum effects are small and a semi-classical approximation is used here.

### 3.1.1 Boltzmann Transport Equation

The classical equation governing the distribution function $f$ is [23]

$$\left( \frac{p}{m} \cdot \nabla_r - (\nabla_r V) \cdot (\nabla_p) + \frac{\partial}{\partial t} \right) f(r, p) = \mathcal{I}(r, p) - G(r)f(r, p).$$

(3.2)

The left-hand side describes the motion of the atoms in the potential $V(r)$. Because atoms in a trap are not flowing as a mass, it can be simplified considerably as will be seen shortly. The right-hand side reflects collision processes. The elastic term $\mathcal{I}$ is given by

$$\mathcal{I}(r, p_1) = \frac{2\pi}{\hbar} \frac{|T^{2B}|^2}{(2\pi\hbar)^6} \int d^3p_2 \, d^3p_3 \, d^3p_4 \, \delta^3(p_1 + p_2 - p_3 - p_4)$$

$$\times \delta(E_1 + E_2 - E_3 - E_4) [f(r, p_3)f(r, p_4) - f(r, p_1)f(r, p_2)],$$

(3.3)

with $E_i = p_i^2/2m$ and $T^{2B}$ is the T-matrix defined in Chapter 2. This is essentially an expression of Fermi's Golden Rule for a transition from two particles with momentum $(p_3, p_4) \rightarrow (p_1, p_2)$ and the reverse. The population $f(r, p_1)$ increases when two atoms with momenta $p_3$ and $p_4$ collide and one acquires momentum $p_1$, and the population decreases when an atom with momentum $p_1$ collides and acquires a new momentum. The transition rates are summed over the three other momenta involved, with the delta functions conserving energy and momentum and expressing the density.
of states. The T-matrix is, as noted in the discussion following Eq. (2.20), the matrix element for an atom to make a transition between momentum states. In the low-energy limit, it is a constant and is taken outside the integral. The factors of $\hbar$ and $2\pi$ provide the correct normalization for the integral. Substituting the value of $T^{2B}$ and simplifying, the prefactor becomes

$$\frac{1}{8\pi^4} \frac{4\pi|a|^2}{m^2\hbar^3}, \quad (3.4)$$

where the expression $4\pi a^2$ is recognized as the s-wave collisional cross section $\sigma$. As discussed in Section 2.1.1, the cross section is enhanced by a factor of two for a Bose gas, and the same factor must be applied to Eq. (3.3). The correct prefactor is therefore

$$\frac{\sigma}{8\pi^4m^2\hbar^3}, \quad (3.5)$$

with $\sigma = 8\pi a^2$.

The second term in Eq. (3.2) represents losses from the trap, and is given by

$$G(r) = G_1 + \frac{2G_2}{(2\pi\hbar)^3} \int d^3p f(r, p). \quad (3.6)$$

Here $G_1$ accounts for density-independent losses due to collisions with background-gas atoms, and $G_2$ is the rate for dipolar relaxation collisions, as in (2.49).*

If $f$ is to represent a Bose gas, it is necessary to include the stimulated scattering factors discussed in Section 1.2.1. A genuine derivation of these factors and the correctness of the QBE in general is beyond the scope of this work [59]. However, the Bose factors are a simple extension to the classical equation, modifying $\mathcal{I}$ to

$$\mathcal{I}(r, \mathbf{p}_1) = \frac{\sigma}{4\pi^4m\hbar^3} \int d^3p_2 d^3p_3 d^3p_4 \delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \delta(p_1^2 + p_2^2 - p_3^2 - p_4^2)$$

$$\times [f_3f_4(1 + f_1)(1 + f_2) - f_1f_2(1 + f_3)(1 + f_4)]. \quad (3.7)$$

*In our experiment, three-body molecular recombination is only important in the case of a collapsing condensate. Loss due to background gas is generally negligible as well, but its inclusion is trivial.
3.1.2 Ergodic Approximation

The QBE is a nonlinear 7-dimensional partial integro-differential equation, and numerical solution of it is a substantial computational challenge. A convenient simplifying approach is to assume that the motion of atoms in the trap is ergodic [60, 61]. This means that an atom with a given energy is equally likely to be found at any accessible point in phase space, and that the motions of different atoms are uncorrelated. This excludes collective motions of the gas as a whole, such as pulsating, swirling, or sloshing back and forth in the trap. However, such motions are not driven in the experiment, and any motion created, perhaps when the trap is loaded, would be rapidly damped by collisions and by the trap anisotropy. The ergodic approximation has been numerically justified by Monte Carlo simulations of trapped atom distributions, both by ourselves and others [62, 63].

Mathematically, the ergodic approximation means that the distribution function \( f(r, p) \) depends only on the energy \( H(r, p) = p^2/2m + V(r) \). In general, the energy distribution function \( f(E) \) is defined by

\[
g(E)f(E) = \frac{1}{(2\pi\hbar)^3} \int d^3r \, d^3p \, \delta(E - H(r, p))f(r, p),
\]

so that the number of atoms with energy \( E \) is

\[
dN = g(E)f(E) \, dE,
\]

with density of states \( g(E) \). For a harmonic oscillator potential \( V(r) = m(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2)/2 \), the density of states is

\[
g(E) \equiv \frac{1}{(2\pi\hbar)^3} \int d^3r \, d^3p \, \delta(E - H(r, p)) = \frac{E^2}{2(\hbar \omega)^3},
\]

where \( \omega \equiv (\omega_x \omega_y \omega_z)^{1/3} \). The ergodic approximation consists of the assumption that

\[
f(r, p) = \int dE \, \delta(H(r, p) - E) \, f(E).
\]
The equation governing $f(E)$ is found by multiplying the QBE by $\delta(E - H(r, p))$ and integrating over $r$ and $p$. The result is

$$g(E) \frac{\partial f(E)}{\partial t} = \mathcal{I}(E) - G(E)g(E)f(E).$$

(3.12)

The flow terms on the left-hand side of (3.2) cancel, the collision integral becomes

$$\mathcal{I}(E_1) = \frac{m \sigma}{\pi^2 \hbar^3} \int dE_2 dE_3 dE_4 g(E_{\text{min}}) \delta(E_1 + E_2 - E_3 - E_4) \times [f_3 f_4 (1 + f_1) (1 + f_2) - f_1 f_2 (1 + f_3) (1 + f_4)],$$

(3.13)

and the loss rate is

$$G(E) = G_1 + G_2 (2m)^{3/2} \left( \frac{2}{\pi \hbar} \right)^3 \int_0^\infty dE' \frac{E_2^{5/2}}{E^2} h \left( \frac{E_>} {E_<} \right) f(E').$$

(3.14)

The energy $E_{\text{min}} = \min\{E_1, E_2, E_3, E_4\}$, and $E_<$ is the lesser of $E$ and $E'$. The function $h$ is the definite integral

$$h(x) = \int_0^1 dz z^2 \sqrt{1 - z^2} \sqrt{x - z^2},$$

(3.15)

well-defined for $x > 1$.

The derivation of (3.14) is straightforward. It is useful to note that the integral over $r = (x, y, z)$ can be transformed to an integral over $s \equiv (\omega_x x, \omega_y y, \omega_z z)$, which restores spherical symmetry to the anisotropic problem. The function $h$ cannot be reduced to closed form, but must be evaluated numerically.

In comparison, the simplicity of expression (3.13) is a truly remarkable result. Its derivation, however, is more subtle, and for this reason is presented here.* The expression obtained directly from (3.7) is

$$\mathcal{I}(E_1) = \frac{A}{(2\pi \hbar)^3} \int dE_2 dE_3 dE_4 \gamma(E_1, E_2, E_3, E_4) \times \int d^3 p_1 \ldots d^3 p_4 \delta^3(p_1 + p_2 - p_3 - p_4) \delta(p_1^2 + p_2^2 - p_3^2 - p_4^2) \times \prod_{i=1}^4 \delta \left( \frac{p_i^2}{2m} + V(r) - E_i \right).$$

(3.16)

*See also [60].
The function \( \Upsilon \) contains the factors of \( f \), all of which have been expressed in terms of energy using (3.11). The prefactor of (3.7) is abbreviated \( A \). Consider first the spatial and momentum integrals, which are completely symmetric in the \( p_i \) and \( E_i \). The variables can therefore be relabeled to make \( E_1 = \min\{E_i\} \), so that for fixed \( r \), \( p_1 \) will be the smallest of the \( p_i \). The integrals can then be rearranged as

\[
\int d^3r \, d^3p_1 \, \delta \left( \frac{p_1^2}{2m} + V(r) - E_1 \right) \frac{1}{2m} \delta(E_1 + E_2 - E_3 - E_4) \\
\times \int dp_2 \, dp_3 \, dp_4 \, \prod_{i=2}^4 \delta \left( \frac{p_i^2}{2m} + V(r) - E_i \right) \\
\times \int d^2\Omega_2 \, d^2\Omega_3 \, d^2\Omega_4 \, h(x) = \int_0^1 dz \, z^2 \sqrt{1 - z^2} \delta^3(p_1 + p_2 - p_3 - p_4)
\]

The integral over the solid angles is evaluated as

\[
\int d^2\Omega_2 \, d^2\Omega_3 \, d^2\Omega_4 \, \delta^3(p_1 + p_2 - p_3 - p_4) \\
= \int d^2\Omega_2 \, d^2\Omega_3 \, \frac{1}{p_4^2} \delta(p_4 - |p_1 + p_2 - p_3|) \\
= \frac{4\pi}{p_4} \int d^2\Omega_2 \, \int d\theta_3 \, \sin \theta_3 \, \delta(p_4^2 - p_1^2 - p_2^2 + 2Pp_3 \cos \theta_3), \quad (3.18)
\]

with \( P \) defined to be \(|p_1 + p_2|\) and \( \theta_3 \) taken as the angle between \( p_3 \) and \( P \). For \( p_1 = \min p_i \), the delta-function contributes for all angles \( \Omega_2 \), and that the final result is

\[
\frac{2\pi}{p_3 p_4} \int d^2\Omega_2 \, \frac{1}{|p_1 + p_2|} = \frac{8\pi^2}{p_2 p_3 p_4}. \quad (3.19)
\]

The last step is elementary, and again relies on \( p_1 \) being smaller than \( p_2 \). For the remaining integrals in (3.16), it is necessary only to evaluate the delta-functions in \( p \) and recognize the definition of the density of states, yielding

\[
\mathcal{I}(E_1) = 4\pi^2 m^2 A \int dE_2 \, dE_3 \, dE_4 \, g(E_{\min}) \delta(E_1 + E_2 - E_3 - E_4) \Upsilon. \quad (3.20)
\]

which expands to (3.13).
3.1.3 Numerical Solution

Equation (3.12) is solved as a set of coupled ordinary differential equations in time on a discrete energy grid, so that \( f(E) \to f_n \). For a classical gas, the grid spacing is arbitrary, but quantum effects can be included by demanding that the grid spacing \( \delta E = j\hbar\omega \), with integer \( j \). Typically, 200 grid points are used, with the highest energy corresponding to roughly \( 15k_B T \) for a gas at temperature \( T \).

Evaporative cooling is modeled by setting \( f(E) = 0 \) above a cutoff energy \( E_T \). As the gas cools, the energy grid spacing is dynamically reduced by decreasing \( j \), which is initially on the order of \( 10^3 \) for a gas at \( T = 200 \mu K \). The results of the model and its experimental realization are discussed in Section 3.2.

After BEC has occurred, it is necessary to account for the instability of the condensate. During a time step \( dt \), the rate for the condensate to collapse is calculated from \( \Gamma_C \) of Eqs. (2.46) and (2.47). The temperature of the gas is estimated from the distribution of population in the low lying levels. A random number \( x \) is then chosen, and if \( x < \Gamma_C dt \), the condensate is taken to collapse. The collapse occurs instantaneously on the time scale of the QBE, so it is necessary only to know what is left when the collapse is over. As discussed in Section 2.2.2, this issue is currently unclear. Typically, the collapse is assumed to consume the entire condensate, and \( f(0) \) is set to zero. Other models can be as easily used, however.

The correct treatment of a degenerate gas requires some care. By setting the minimum energy point of the grid to the zero-point energy \( E_0 = 3\hbar\omega/2 \), approximately the right density of states is obtained

\[
g(E) \to g_n = \frac{(E_0 + n\hbar\omega)^2}{2(\hbar\omega)^3} = \frac{n^2 + 3n + 9/4}{2\hbar\omega},
\]

where the actual degeneracy in an isotropic harmonic oscillator potential is \( (n + 1)(n + 2)/2 \). If the zero-point energy were not included, the condensate population

\*The asymmetry of the trap is ignored here.
\( f(0) \) would be undefined since \( g(0) \) would be zero. Also, since \( f \) can be large and discontinuous at \( n = 0 \), the ground state is always treated singly, and not lumped with nearby states when \( j > 1 \).

It is reasonable to question the validity of the semi-classical approximation for atoms in very low-lying states of the trap. An estimate of the accuracy of the collision rates used can be obtained by calculating the rate for inelastic collisions between condensate atoms, given by \( G(0) \). This can be compared with the actual rate determined using the Gaussian ground-state wave function. The distribution function for a gas consisting of \( N_0 \) atoms at energy \( E_0 \) is \( f(E) = N_0 \delta(E - E_0)/g(E_0) \). Using this \( f \), the semi-classical approximation to the loss rate from dipolar decay is found using (3.14) to be

\[
\frac{dN_0}{dt} \bigg|_{sc} = -\frac{128\sqrt{3}}{135\pi^3} G_2 \frac{N_0^2}{\ell_0^3} \approx 0.05G_2 \frac{N_0^2}{\ell_0^3}.
\]

(3.22)

Recall that the loss rate is reduced by a factor of 2 due to the coherence properties of the condensate. From Eq. (2.50), the loss rate in the variational quantum calculation is

\[
\frac{dN_0}{dt} \bigg|_q = -\frac{1}{(2\pi)^{3/2}} G_2 \frac{N_0}{\ell^3} \approx 0.06G_2 \frac{N_0^2}{\ell^3}.
\]

(3.23)

For small \( N_0 \), mean-field interaction effects are small and \( \ell \approx \ell_0 \). The error due to the semi-classical approximation is therefore 20\%. The collision terms for other low-lying states will be inaccurate by similar but smaller amounts.

Note, however, that for large \( N_0 \), mean-field effects become important. For example, \( \ell \) can become significantly smaller than \( \ell_0 \), as seen in Fig. 2.2. This leads to a substantial error in the collision terms involving the condensate. In addition, interactions decrease the energy of the condensate appreciably, changing the equilibrium population distribution. Correctly accounting for interactions would require at each time step a self-consistent calculation of the energy spectrum and collision rates. This would comprise a substantial theoretical effort which has yet to be fully achieved [44, 64]. A discussion of some of the effects which might be expected is given in Sec-
tion 3.3.2. Otherwise, the model used here can provide only a qualitative description of the expected behavior at large $N_0$.

Another numerical technique for solving the QBE is the Monte Carlo method, in which the gas is represented by several thousand model atoms whose trajectories and interactions are calculated by various approximation schemes [62, 63, 65, 66]. Averaging of the states of the model atoms then gives the thermodynamic properties of the gas, to some degree of precision. This allows, for example, relaxation of the ergodic approximation, and might more easily permit the incorporation of mean-field interactions. A direct comparison of calculation speeds is not available, but typically Monte Carlo methods are slow.

3.2 Evaporative Cooling

The QBE is first applied to the simulation of evaporative cooling, an essentially classical phenomenon. Evaporative cooling has been generally described in [67], and our implementation of it in [25], so it is summarized only briefly here. The QBE is then applied to the important problem of optimization of the cooling process. This optimization relies on an accurate knowledge of the loss coefficients $G_i$, so our measurements of these values are also discussed.

3.2.1 Experimental Implementation

Evaporative cooling requires a mechanism to selectively remove atoms from the trap based on their energy. As discussed in Section 1.2.3, the atoms are confined only if their magnetic moment $\mathbf{m}$ is anti-parallel to $\mathbf{B}$. Atoms can therefore be removed by driving a transition between a trapped and untrapped spin state using an oscillating transverse field. If the untrapped state experiences a repelling potential energy $V^*(r)$, then the energy difference $V(r) - V^*(r)$ has a strong spatial dependence, and increases with $r$. An applied field with frequency $\Omega_T$ will be resonant only at positions where $V(r) - V^*(r) = \hbar \Omega_T$. Atoms with energy $E < V(r)$ will never be found at such $r$. 
and thus will remain trapped, while atoms with greater energies will be lost. In this way, $\Omega_T$ defines the depth of the trap $E_T$.

The trap depth is typically set to be 2-3 times higher than the average energy of the atoms in the cloud, where $\langle E \rangle = 3k_B T$. When the oscillating field is imposed, $\langle E \rangle$ immediately decreases, and as elastic collisions attempt to repopulate the missing tail of the distribution, cooling continues. As $\langle E \rangle$ decreases, $E_T$ is lowered to keep pace. If $E_T$ is reduced too quickly, too many atoms are lost for the amount of cooling produced, but if $E_T$ changes too slowly, atoms are lost due to background-gas collisions and dipolar relaxation.

We perform evaporative cooling on a sample of $\sim 2 \times 10^8$ $^7$Li atoms loaded into the magnetic trap and laser cooled to a temperature of $\sim 250 \ \mu$K. The atoms are held in the $(F = 2; m_F = 2)$ doubly spin-polarized hyperfine state, and are driven to the $(F = 1; m_F = 1)$ untrapped state. The magnetic field at the center of the trap is $10^3 \ \text{G}$, and the frequency of the spin-flip transition is approximately 3.4 GHz. The microwave field is obtained from a digital frequency synthesizer, and applied to a 2-cm-diameter loop antenna placed $\sim 2.5 \ \text{cm}$ from the trap center. Roughly 100 mW of microwave power is applied to the antenna, of which an estimated half is coupled into the trap chamber. The frequency $\Omega_T$ is initially 70 MHz above the trap-bottom resonance, and is gradually reduced. BEC is reached after approximately three minutes of cooling, when $\Omega_T \approx 150 \ \text{kHz}$, $N \approx 10^8$ atoms, and $T \approx 700 \ \text{nK}$. We use an optimized trajectory $\Omega_T(t)$, which is important both because it increases the temperature and number of atoms at which BEC occurs and because it decreases the cooling time required.

### 3.2.2 Optimization

The optimization problem is to determine the trajectory $E_T(t)$ which gives the largest possible decrease in temperature while the number of trapped atoms is re-
duced from $N_i$ to $N_f$. Therefore, the optimization process requires that the entire trajectory $E_T(t)$ be varied and the response of the atom distribution calculated. This is computationally difficult, but a simpler approach is to optimize the trajectory locally, by maximizing the efficiency of cooling at each time $t$. This was first suggested by Hess [68], but not actively pursued until our work in [56].

This optimization strategy would be exact if the evolution of the system depended only on $N$ and $T$. This can be understood by considering the temperature achieved by the instantaneously-optimized trajectory, $T_{\text{inst}}(N)$, and that of an alternative trajectory $T_{\text{alt}}(N)$. Since $T_{\text{inst}}$ is optimized at the beginning of the trajectory, initially $T_{\text{alt}} > T_{\text{inst}}$. If $T_{\text{alt}}(N_f)$ is to be less than $T_{\text{inst}}(N_f)$, the two trajectories must cross at some point $N'$. This can only occur if

$$\frac{dT_{\text{alt}}(N')}{dN} > \frac{dT_{\text{inst}}(N')}{dN},$$

(3.24)

which is impossible since the instantaneously optimized trajectory was chosen to be maximally efficient at $N'$.

Because the evolution of the distribution does depend on the form of $f(E)$, this optimization scheme is only approximately correct. For instance, at $N'$, $T_{\text{alt}} = T_{\text{inst}}$, but if $f_{\text{alt}}(E) \neq f_{\text{inst}}(E)$, then it is possible for Eq. (3.24) to hold, so that $T_{\text{alt}}(N_f)$ could be lower than $T_{\text{inst}}(N_f)$. Though in principle this fact can be used to construct a more efficient trajectory, it is unlikely that significant gain in phase-space density could be obtained, because the efficiency possible depends much more on $N$ and $T$ than on the details of $f$. To test this sensitivity, we modeled evaporation in our trap starting from several non-equilibrium distributions with fixed initial $N$ and $\langle E \rangle$. We found that increasing the initial number of atoms with $E > 2\langle E \rangle$ by a factor of three produces the same variation in the final temperature as is caused by a 5% increase in $N_i$.

To implement the optimization scheme, a precise definition of the instantaneous efficiency must be formulated. This is an issue because, during evaporation, the gas is
not in thermal equilibrium. However, $T$ can be defined to be the temperature which would be achieved if cooling were halted and the gas allowed to equilibrate. The temperature is therefore proportional to $\langle E \rangle$, as long as the gas is nondegenerate. Also, it is better to define the efficiency in terms of phase-space density $\varrho \propto N/T^3$, since $T$ can be changed adiabatically by relaxing or compressing the trap. This should not be considered true cooling.

The instantaneous efficiency is, then, defined as

$$\varepsilon = \frac{N}{\varrho} \frac{d\varrho}{dN}. \quad (3.25)$$

It can be evaluated by noting that $\varrho \propto N^4/E_\text{tot}^3$, where $E_\text{tot} = N\langle E \rangle$ is the total energy of the atoms in the trap. The efficiency can therefore be expressed as

$$\varepsilon = \frac{3}{E_\text{tot}} \frac{dE_\text{tot}}{dN} - 4. \quad (3.26)$$

The energy is given by $E_\text{tot} = \int dE \, E \, g(E) f(E)$, and its derivative is expressed as

$$\frac{dE_\text{tot}}{dN} = \frac{dE_\text{tot}}{dt} - \frac{\int dE \, E \, g(E) \, df/df}{\int dE \, g(E) \, df/df}. \quad (3.27)$$

Equation (3.12) gives the derivative $df/df$, so the function $\varepsilon(E_T)$ is well defined and a single-variable optimization algorithm may be used to determine the best $E_T$.

An optimized trajectory is constructed by calculating $E_T$ for the initial distribution, and applying the cutoff to $f$. Equation (3.12) is then integrated forward in time for a period $\Delta t$, and then $E_T$ is re-optimized. This continues until the specified final number of atoms is reached. Rather than adjusting $E_T$ directly, it is more efficient to optimize the parameter $\eta_T = E_T/\langle E \rangle$, because the best value of $\eta_T$ changes only slowly as evaporation proceeds, permitting a larger $\Delta t$.

The optimum trajectories in our $^7\text{Li}$ trap for a number of initial conditions are shown in Fig. 3.1. The values of $E_T(t)$ are given in (a), and the corresponding values of the phase-space density are shown in (b). In reality, the optimization cannot be continued past the point of degeneracy, since the condensate rapidly fills to its
maximum and starts to collapse. This causes the suitability of the definition of $\varepsilon$ to break down: on the one hand, $N$ and $\langle E \rangle$ no longer evolve smoothly, so that the derivatives in (3.27) become indefinite. This could presumably be handled for $N \gg N_m$ by using a time-averaged derivative. On the other hand, however, it is not clear what the goal of the optimization should be. Maximization of $\varepsilon$ will drive the gas as far from equilibrium as possible, but it might be more desirable to, for example, maintain the degeneracy of the gas as long as possible instead. The choice will depend on the particular experiment in mind. For the trajectories shown in Fig. 3.1, the Bose stimulation factors $(1 + f_i)$ were artificially suppressed to demonstrate the predicted behavior of a classical gas.

To illustrate the benefit obtained by optimization, the response of the gas to a non-optimized trajectory was calculated. An exponentially decreasing $E_T(t)$ was used to attempt to maintain a constant $\eta_T$, and the time constant of 100 s was chosen to make $\eta_T = 2.5$ initially. During cooling, $\eta_T$ varied between 2 and 2.5. These are typical values which are expected to work reasonably well [67]. The trajectory and result are shown as the long-dashed curves in Fig. 3.1(a) and (b). The optimized trajectory which begins with the same conditions reaches the BEC transition in 80% the time and with 4 times more atoms than the non-optimized case, both of which are significant gains. In addition to aiding in the execution of evaporative cooling experiments, a set of trajectories such as shown in Fig. 3.1(b) is useful when designing an experiment, since it shows precisely what initial conditions are required in order to obtain BEC.

All of the trajectories in Fig. 3.1(b) roll over at low $N$, indicating that evaporative cooling becomes inefficient. This occurs because of the presence of two-body losses. Evaporative cooling relies on having a large ratio of elastic collisions to losses, so that energetic atoms can be produced and evaporated before the gas is gone. The elastic collision rate is roughly $n \sigma v$, where $n$ is the average density and $v$ the average speed
Figure 3.1  Optimized evaporative cooling trajectories. (a) Solid lines are optimized values of $E_T(t)$ for the initial values of $N$ shown. The initial temperature is 500 μK in each case. Dots show the time at which BEC would be reached, if the Bose stimulation factors were included in the calculation. (b) Response of the gas to the cooling trajectories in (a). The phase-space density is shown in terms of the critical point for BEC, $\varrho = 1$. The long dashed curves show a typical unoptimized trajectory for comparison. For these calculations, $G_1 = 10^{-4}$ s$^{-1}$ and $G_2 = 10^{-14}$ cm$^3$/s, consistent with the results of Section 3.2.3. The dotted curve shows the cooling achieved when $G_2 = 10^{-15}$ cm$^3$/s.
of the atoms. The loss rate, however, is \( G = G_1 + G_2 n \). At large \( n \), the \( G_1 \) term can be neglected, so that the ratio of good to bad collisions is \( \sigma v / G_2 \). Since \( v \) decreases with the temperature of the gas, the efficiency of evaporative cooling declines as it proceeds. The dotted curve in Fig. 3.1(b) shows the results obtained with \( G_2 \) reduced by a factor of 10.*

Although evaporative cooling always becomes inefficient eventually, the cooling shown in Fig. 3.1 is efficient at first, even for relatively low \( N_i \). This is because the loss rate \( G_1 \) is nearly negligible, so that the ratio of elastic to inelastic collisions is independent of density regardless of \( n \). When this is not the case, a threshold behavior is observed, in which \( N_i \) must be sufficiently high for evaporative cooling to be successfully initiated. The threshold occurs at approximately \( n \sigma v / G_1 = 100 \). An example of this situation is shown in Fig. 3.2, where \( G_1 \) has been increased by a factor of 100.

Determination of optimized trajectories using the method described is computationally slow. The calculation can be simplified by approximating the distribution

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*See also the discussion of evaporative cooling in the \((F = 1, m_F = -1)\) state in Chapter 6.
at all times by a Boltzmann distribution \( f(E) = z \exp(-E/k_B T) \), truncated at the evaporation cutoff \( E_T \). This allows the integrals in Eq. (3.12) to be calculated analytically [60]. We tested this approximation and found it to be reasonably accurate, having approximately the same effect on the final value of \( \varrho \) as a 5% to 10% variation in \( N_i \). Under most circumstances, this is adequate. However, near the cooling threshold described above, the final result of evaporation becomes very sensitive to the initial conditions and the details of the trajectory. Near this threshold, the truncated-Boltzmann approximation cannot be used. None of the results reported here were calculated with the truncated Boltzmann approximation.

### 3.2.3 Measuring Loss Coefficients

As is evident from the preceding discussion, the results obtained from evaporative cooling are sensitive to the loss coefficients \( G_1 \) and \( G_2 \). The dipolar-relaxation rate constant \( G_2 \) is also intrinsically interesting, since it can be calculated theoretically from the molecular potential \( U \). Experimental measurements test these theories, or if the theories are accepted, yield information about \( U \).

Theoretical methods for dipolar-relaxation collisions at low temperatures were initially developed for spin-polarized hydrogen [69, 70], and the predicted rate constants were consistent with experimental measurements [71, 72]. However, the recently measured value of \( G_2 \) for the doubly spin-polarized state of Cs was found to depend strongly on temperature, suggesting the presence of a zero-energy resonance [73]. The value at 8 \( \mu K \) was nearly 3 orders of magnitude larger than predicted [74]. Furthermore, measurements of the spin-relaxation rate of a near-room-temperature spin-polarized Rb vapor have revealed a magnetic field dependence which is not understood [75]. Lithium does not appear to share these difficulties with the heavier alkalies, as the measurements described here agree with theory.

The loss rate is measured by monitoring the evolution of nondegenerate gas cloud
in time. The gas is near equilibrium, so it is not necessary to use the full QBE model and the rate of change of the number of trapped atoms \( N \) is simply

\[
\frac{dN}{dt} = -\int d^3 r \ G(r) n(r),
\]

(3.28)

where

\[
G(r) = G_1 + 2 G_2 n(r) + 3 G_3 n^2(r).
\]

(3.29)

The possibility for three-body recombination is included, but no evidence for this process was found, supporting its omission in Eq. (3.6). For a Boltzmann energy distribution, the density \( n \) is

\[
n(r) = N \left( \frac{m \omega^2}{2 \pi k_B T} \right)^{3/2} \exp \left( -\frac{V(r)}{k_B T} \right),
\]

(3.30)

which allows (3.29) to be integrated and gives

\[
\frac{1}{N} \frac{dN}{dt} = -G_1 - 2 G_2 \bar{n} - \frac{8}{\sqrt{3}} G_3 \bar{n}^2
\]

(3.31)

in terms of the average density \( \bar{n} = N(m \omega^2/4 \pi k_B T)^{3/2} \). The rate constants are determined by comparing the time evolution of \( \dot{N}/N \) and \( \bar{n} \).

A sample is prepared by loading the trap and evaporatively cooling so \( N \approx 4 \times 10^6 \) atoms at \( T \approx 2 \ \mu K \). This provides a relatively high density of \( \bar{n} = 5 \times 10^{11} \) cm\(^{-3} \), which allows the collision rates to measured on a reasonable time scale. The total number is 15% of that required for BEC, so effects of quantum degeneracy can be safely neglected. After evaporative cooling is halted, the microwave field is raised to a high value and the atoms allowed to freely evolve for a time \( \tau \).

After the delay, the cloud is imaged using a phase contrast technique which will be described further in Chapter 4. The image produced consists of a signal proportional to the optical density of the cloud. This signal is fit to a Gaussian function

*Compare Eq. (2.49).
to determine \( N \) and \( \bar{n} \). In order to obtain low-noise and high-resolution images, destructive probing is used and only one image is obtained per sample. The experiment is repeated many times and the results of several cycles are averaged at each value of \( \tau \).

The dependence of \( N \) on \( \tau \) is shown in Fig. 3.3. The error bars represent the statistical uncertainty \( \sigma_i \) in the mean of several measurements at time \( \tau_i \). Systematic uncertainties are discussed in Appendix B, and are estimated to be \( \sim 5\% \). In order to determine \( dN/dt \), the number is fit to an empirical form

\[
N(\tau) = N_1(1 + \kappa_N \tau)^\gamma_N.
\]  

(3.32)

The result is shown as the dotted curve through the points. Uncertainties in the fit parameters are determined as the changes required to increase by one the \( \chi^2 \) parameter of the fit [76].

\[
\chi^2 \equiv \sum_i \left( \frac{N_{\text{fit}}(\tau_i) - N_{\text{data}}(\tau_i)}{\sigma_i} \right)^2.
\]

(3.33)

The values obtained are given in the figure caption. Data with \( \tau < 10 \text{ s} \) are neglected, since these distributions have not necessarily equilibrated after evaporative cooling, which can introduce systematic errors in the determination of \( \bar{n} \). The elastic collision rate is \( \sim 1 \text{ Hz} \), and after each atom has undergone several collisions, the gas should be well equilibrated.

The ratio \( \dot{N}/N = \kappa_N \gamma_N/(1 + \kappa_N \tau) \) is plotted as a function of \( \bar{n} \) in Fig. 3.4. The horizontal error bars are the statistical uncertainties in the mean of \( \bar{n}(\tau) \), and the vertical error bars are derived directly from the uncertainties in \( \kappa_N \) and \( \gamma_N \).* The

---

*In fitting \( N(\tau) \) the errors in \( \kappa_N \) and \( \gamma_N \) are correlated. This can be understood to mean that contours of constant \( \chi^2 \) in the \( \kappa_N - \gamma_N \) plane are eccentric ellipses which are tilted at some angle with respect to the coordinate axes. The simplest way to remove this correlation is to scale \( \kappa_N \) to \( \kappa'_N \), so that the uncertainty in \( \kappa'_N \) is numerically equal to that of \( \gamma_N \). The \( \chi^2 \)-ellipses in the \( \kappa'_N - \gamma_N \) plane must then lie at 45° to the axes, so that the variables \( u_1 = \kappa'_N + \gamma_N \) and \( u_2 = \kappa'_N - \gamma_N \) will have uncorrelated uncertainties. Using this procedure in the analysis provides the most accurate error estimate.
Figure 3.3  Decay of total number of trapped atoms $N$. The dotted line is a fit to the form (3.32) yielding $N_i = (3.88 \pm 0.09) \times 10^6$, $\kappa_N = (3.5 \pm 0.8) \times 10^{-2}$ s$^{-1}$, and $\gamma_N = -0.50 \pm 0.04$.

Figure 3.4  Determination of $G_2$. Black squares correspond to the data shown in Fig. 3.3, and open circles are from data taken on a different day. The dotted line is a quadratic fit.
dotted line is a quadratic function fit to the data giving

\[ G_1 = (-0.3 \pm 1.0) \times 10^{-4} \text{ s}^{-1}, \]
\[ G_2 = (1.05 \pm 0.08) \times 10^{-14} \text{ cm}^3/\text{s} \text{ and} \]
\[ G_3 = (0.5 \pm 8.1) \times 10^{-28} \text{ cm}^6/\text{s}. \]

Systematic errors in the determination of \( N \) and \( \bar{n} \) are discussed in Appendix B, and contribute an additional 5% uncertainty to \( G_2 \), giving a total uncertainty estimate of \( \pm 0.1 \times 10^{-14} \text{ cm}^3/\text{s} \) when combined in quadrature.

The value of \( G_2 \) predicted by Moerdijk \textit{et al.} is \((9.4 \pm 0.2) \times 10^{-15} \) cm\(^3\)/s at a magnetic field strength of \( 10^3 \) G \cite{54}, in agreement with our result. The measured upper bound on \( G_3 \) of \( 10^{-27} \) cm\(^6\)/s is too large to provide a stringent test of theory, but is consistent with calculations also performed by Moerdijk \textit{et al.}, predicting \( G_3 = 2.6 \times 10^{-28} \) cm\(^6\)/s at zero magnetic field \cite{54}. A strong dependence on magnetic field strength is not expected.

The measured value of \( G_1 \) is very low compared to other similar experiments, and extrapolation from previous measurements at higher background-gas pressures indicates a current pressure of \( \sim 10^{-12} \) torr. Our vacuum apparatus and procedures are described in Ref. \cite{77}.

In addition to a loss of atoms, a significant heating effect is observed in the trap as seen in Fig. 3.5. This heating can partially be explained by dipolar relaxation, since atoms are preferentially lost from the center of the trap where their energy is lower than average. This effect can be calculated by noting that the rate of change of the total energy in the trap is

\[
\frac{dE_{\text{tot}}}{dt} = -\int d^3r \ n(r) G(r) \langle E \rangle(r) = -2G_2 \int d^3r \ n(r)^2 \left[ V(r) + \frac{3}{2}k_B T \right] \tag{3.34}
\]

for a gas in equilibrium. Here \( \langle E \rangle(r) \) is the average energy of the atoms located at position \( r \), and is given by the sum of their kinetic and potential energies. Evaluation
Figure 3.5  Heating of trapped atoms. The temperature $T$ is obtained from the same data used in Fig. 3.3. The solid curve is a fit to the form (3.38), yielding $T_0 = 1.55 \, \mu K$ and $\dot{R}_H = 4.1 \, nK/s$. The dotted curve shows the predicted behavior for $\dot{R}_H = 0$.

Figure 3.6  Decay of mean density $\bar{n}$, from the data set of Fig. 3.3. The curve is a fit to the form (3.37) yielding $n_1 = 7.3 \times 10^{11}$, $\kappa_n = 1.6 \times 10^{-2} \, s^{-1}$, and $\gamma_n = -1.44$. 
the integral using (3.30) gives
\[
\frac{dE_{\text{tot}}}{dt} = -\frac{3}{2} G_2 \bar{n} E_{\text{tot}},
\]
(3.35)
which in turn implies
\[
k_B \frac{dT}{dt} = \frac{1}{3N} \frac{dE_{\text{tot}}}{dt} - \frac{E_{\text{tot}}}{3N^2} \frac{dN}{dt}
= -\frac{3}{2} G_2 \bar{n} \frac{E_{\text{tot}}}{3N} + 2G_2 \bar{n} \frac{E_{\text{tot}}}{3N} = \frac{1}{2} G_2 \bar{n} k_B T
\]
(3.36)
since \(E_{\text{tot}} = 3N k_B T\). The average density is shown as a function of time in Fig. 3.6, along with a fit
\[
\bar{n}(t) = n_1 (1 + \kappa_n t)^{\gamma_n}.
\]
(3.37)
Using this form for \(\bar{n}\) and including a constant heating rate \(R_H\), Eq. (3.36) can be solved, yielding
\[
T(t) = T(0) \exp \left\{ -\frac{G_2 n_1}{2\kappa_n (\gamma_n + 1)} \left[ (1 + \kappa_n t)^{\gamma_n+1} - 1 \right] \right\} + R_H t.
\]
(3.38)
Fitting the measured temperatures to this form yields \(T(0) = 1.6 \, \mu\text{K} \) and \(R_H = 5.5 \, \text{nK/s}\). The solid curve in Fig. 3.5 shows the results, and the dashed curve indicates the significance of \(R_H\).

One possible explanation for such a constant heating rate is that it arises from glancing collisions with background-gas atoms which do not transfer enough energy to eject a trapped atom. During the delay time \(\tau\), the evaporation frequency \(\Omega_T\) was set to be 10 MHz above the trap bottom resonance, corresponding to \(E_T = \hbar \Omega_T / 2 \approx 250 \, \mu\text{K}\). The heating effect comes from collisions which impart less than \(E_T\) of energy. In a glancing collision with scattering angle \(\theta \ll 1\) in the center-of-mass frame, the energy transferred is
\[
\Delta E = \frac{\overline{m}}{m_{\text{tot}}} E_i \theta^2,
\]
(3.39)
where \(E_i\) is the energy of the incident molecule, \(\overline{m}\) is the reduced mass and \(m_{\text{tot}}\) is the total mass of the pair. Setting \(\Delta E = E_T\) gives the minimum angle \(\theta_{\text{min}}\) required
to eject the trapped atom. From diffraction theory, the maximum scattering angle likely to occur is

$$\theta_{\text{max}} \sim \frac{1}{kR_U},$$

(3.40)

where is $R_U$ is the range of the interaction potential and $k$ is the wave number for the relative motion of the particles. Typical length scales for molecular potentials are $R_U \sim 1$ nm, while $k = \bar{m}v_i/\hbar$ if $v_i = (2E_i/m_i)^{1/2}$ is the speed of the incident molecule with mass $m_i$. The fraction of collisions which leave the target atom trapped is therefore

$$\frac{\theta_{\text{min}}^2}{\theta_{\text{max}}^2} = \frac{2m_i R_U^2}{E_T \hbar} \approx 0.007.$$

(3.41)

If the total collision rate is $10^{-4}$ s$^{-1}$ and the average energy deposited per glancing collision is $\sim 200 \mu K$, the resulting heating rate is 0.15 nK/s, considerably smaller than measured. A more accurate calculation could be performed, but would require greater knowledge of the molecular species and interaction potentials involved. It seems unlikely, however, that the simple estimate here is wrong by the observed factor of 35.

Other possible sources of heating include scattering of stray photons and mechanical vibration of the trap. Estimating the mechanical effect is difficult, since the amplitude and spectrum of the vibrations are unknown. Heating due to light scattering can be more easily considered. Each photon scattered adds $\hbar^2 k^2/2m \sim 6 \mu K$ of energy to the trap, so a heating rate of 6 nK/s implies a scattering rate of roughly $10^{-3}$ s$^{-1}$ per atom. The scattering rate is related to the incident intensity $I$ and detuning $\Delta$ by

$$R_s = \frac{I/I_{\text{sat}}}{4\Delta^2 + \Gamma^2} \Gamma,$$

(3.42)

where the linewidth $\Gamma = 3.7 \times 10^8$ s$^{-1}$ and the saturation intensity $I_{\text{sat}} \approx 7$ mW/cm$^2$. During evaporation, the nearest-resonant laser frequency in the room is the probe beam at $\Delta \approx 40\Gamma$, so the heating rate could be explained by an intensity of $\sim 1$ nW/cm$^2$. Although the beams themselves were blocked with shutters, no special
measures were taken to shield the trap from stray light, so this explanation is plausible. Alternatively, ambient light having a bandwidth of $\sim 10^{14}$ Hz would require an intensity of a few $\mu$W/cm$^2$. The ambient intensity in the trap is estimated to be less than 100 nW/cm$^2$ from measurements in similarly shaded conditions.

The different possible mechanisms could be investigated by measuring the dependence of the heating rate on $E_T$, if the heating effect is a concern. Neither photon scattering nor background-gas collisions will cause significant heating during the evaporative cooling process, since then $E_T$ is always on the order of the average energy of the atoms. If the heating is entirely due to stray laser light, then the effective loss rate $G_1$ would rise to the photon scattering rate of of $10^{-3}$ s$^{-1}$ when $E_T < 6$ $\mu$K. Although this is a relatively large increase from $G_1 \leq 10^{-4}$ s$^{-1}$, the dominant loss mechanism would still be dipolar relaxation.

### 3.3 Modeling BEC

Having addressed the technical issue of producing a quantum degenerate gas using evaporative cooling, we now apply the quantum Boltzmann model to BEC itself. Of particular interest are the effects of having the condensate collapse, and the kinetics of condensate formation. It is also useful to characterize the nonequilibrium distributions observed in the model.

#### 3.3.1 Effects of the Collapse

If the condensate did not collapse, then evaporative cooling would produce results such as those shown in Fig. 3.7(a). The BEC phase transition is clearly seen as a rapid jump in condensate number. The subsequent decline in $N_0$ is caused by dipolar-relaxation collisions in the relatively dense condensate. This is not a quantitatively accurate model of the behavior of a gas with $a > 0$, because it neglects mean-field interactions which become important for large $N_0$ [64]. Qualitatively, however, the picture is correct. In contrast, when collapses are included they yield the behavior
Figure 3.7  Modeling BEC with the quantum Boltzmann equation. (a) Behavior obtained when the instability of the condensate is disregarded. (b) Behavior when collapses are included and assumed to reduce \( N_0 \) to zero. For both plots, the gas is initially in a non-degenerate equilibrium distribution with \( N = 4.5 \times 10^6 \) atoms at \( T = 2.1 \mu \text{K} \). Evaporative cooling is applied with a trap depth of

\[
E_T(t) = 2 \text{ mK} \times \exp[-(2.53 \times 10^{-2} \text{ s}^{-1})(t + t_0) - (1.31 \times 10^{-4} \text{ s}^{-2})(t + t_0)^2]
\]

for \( t_0 = 132 \text{ s} \). These parameters simulate those used in the experiments described in Chapter 4.
Figure 3.8  Loss of atoms during evaporative cooling of a degenerate gas. The total number of trapped atoms $N$ corresponding to the curve of Fig. 3.7(b) is shown. Until the very final stages of cooling, $N \gg N_0$.

shown in Fig. 3.7(b).  Roughly 150 cycles of condensate growth and collapse occur, finally ceasing when the total number of trapped atoms $N$ drops too low to allow further cooling. A plot of $N(t)$ is shown in Fig. 3.8. In the experiments to be described, condensates can be observed most accurately when $N < 5 \times 10^4$, so the final portion of the time range in Fig. 3.7 is usually the focus of attention.

The details of the behavior seen in Fig. 3.7(b) depend on the collapse model used. As discussed in Section 2.3, there is no generally accepted theoretical prediction for the number of atoms left in the condensate after a collapse. It was assumed for the Figure that the entire condensate is eliminated, but other models can also be applied. Fig. 3.9 shows the results obtained when it is assumed (a) that the collapse leaves half the atoms behind, and (b) that the collapse leaves a variable number of atoms,

*Since collapse initiation rates from the variational approximation are used in the QBE, the effective value of $N_m$ is 1470 atoms. In all the plots shown here, $N_0$ has been scaled to give the correct $N_m$ of 1250 atoms.
Figure 3.9  Alternative collapse models. The simulations of Fig. 3.7 are repeated assuming that (a) each collapse reduces $N_0$ to $N_m/2$, as proposed by Kagan et al., and (b) that a collapse leaves behind a random number of atoms according to a Gaussian distribution (3.43).
with a probability distribution

\[ P_R(N_0) = A \exp \left[ -\frac{(N_0 - N_R)^2}{w^2} \right], \]

with \( N_R = 200 \) atoms and \( w = 100 \) atoms. Of these possibilities, Fig. 3.9(b) agrees best with the experimental evidence, as will be discussed in Chapter 4. However, all three predict qualitatively similar behavior, in that \( N_0 \) oscillates between some minimum and \( N_m \) as the condensate alternately fills and collapses.

It is interesting to note that the last collapse in the trajectories of Figs. 3.7(b), 3.9(a), and 3.9(b) all occur at nearly the same time. In part, this can be explained because the atoms lost in the collapses make up only about 20% of the total loss in \( N \), so that the kinetics are mainly determined by losses from evaporative cooling and dipolar relaxation. Furthermore, the total number of atoms lost in collapses is very similar in all three cases. The number of collapses is 161 for Fig. 3.7(b), 335 for Fig. 3.9(a), and 190 for Fig. 3.9(b), but the sum of the losses for each is \( 1.95 \pm 0.01 \times 10^8 \) atoms. Clearly, atoms are driven into the condensate at an average rate which is fundamentally fixed by the evaporative cooling process.

The dynamics of \( N_0 \) also depend on the initiation of the collapse by thermal fluctuations. Because of this stochastic element, there is a dephasing of the collapse/fill cycles. If an identical trajectory is simulated twice with different random number generation sequences, the results eventually become uncorrelated. Typically this takes about 40 collapses in the QBE model, corresponding to roughly 5 s in Fig. 3.7. In principle, a measurement of this decorrelation time under various conditions could provide information about the initiation process. However, \( N_0(t) \) is also quite sensitive to experimental variations in the temperature and number of atoms at the beginning of evaporative cooling. A 10% variation in the initial number of atoms shifts the time of the first collapse by \( \sim 2 \) s. So, if the dephasing due to thermal fluctuations is to be observed, the trap loading process must be well controlled. In any case, it is clear that once many collapses have occurred, the value of \( N_0 \) at a particular time will be
Figure 3.10  Modeling equilibration. Evaporative cooling is simulated as in Fig. 3.7(b), but at $t = 17.6$ s, $E_T$ is abruptly lowered to 12 kHz = 500 nK. Cooling is then halted and the gas allowed to equilibrate. This corresponds to the experimental procedure used in Chapter 4.

unpredictable.

In Fig. 3.7, the gas is never in equilibrium, since evaporative cooling is taking place. To exhibit the equilibration process itself, cooling can be halted and the gas allowed to evolve freely. An example of the response is shown in Fig. 3.10. Inelastic collisions in the gas and in the condensate cause $N$ to decrease and $T$ to rise, lowering the phase-space density $\varrho$ and slowing down the rate at which atoms enter the condensate. Eventually, the gas comes to equilibrium with some $N_0 < N_m$, after which time $N_0$ decreases as $\varrho$ continues to drop. In Fig. 3.10, equilibration occurs 40 s after the cooling perturbation.

This equilibration time is surprisingly long. The mean collision rate for the conditions shown is approximately 0.5 Hz, so that each atom collides 20 times on average before $N_0$ stops increasing. This indicates that it is unlikely for an average collision to produce a condensate atom, even though the rate for this process is enhanced by a factor of $N_0$. If the condensate number were not limited so that $N_0$ could be larger, equilibration would be much faster, although the higher condensate density in this
case also increases the loss rate and makes direct comparison difficult. Equilibration is also faster in a nondegenerate gas, requiring only $\sim 5$ collisions per atom.

### 3.3.2 Condensate Formation

The relatively slow equilibration of a degenerate gas indicates that an average collision is unlikely to produce a condensate atom. This can be understood from the fact that the volume of the condensate in phase space is very small. A collision giving an increase in $N_0$ must take place at the very center of the cloud, and between two relatively cold atoms. This statement can be quantified using Eq. (3.13), which gives the rate for any particular collision process. The rate of increase of the condensate population is

$$g(E_0) \frac{df(E_0)}{dt} = A g(E_0) \int dE_3 dE_4 f(E_3) f(E_4) [1 + f(E_0)][1 + f(E_3 + E_4 - E_0)],$$

(3.44)

which for $N_0 \gg 1$ can be expressed as

$$\frac{dN_0}{dt} = N_0 \int dE g(E) f(E) R_0(E),$$

(3.45)

where $R_0(E)$ is the rate at which an atom with energy $E$ is likely to enter the condensate. It is given by

$$R_0(E) = \frac{A}{g(E)} \int dE' f(E')[1 + f(E + E')],$$

(3.46)

and is plotted in Fig. 3.11 for an equilibrium gas. It does indeed decrease quickly as a function of $E$, indicating that most of the atoms entering the condensate come from a relatively small number of low-lying energy levels.

This argument suggests the three-state picture shown in Fig. 3.12. The upper level $|b\rangle$ represents the majority of the trapped states, which are not well-connected to the condensate state $|0\rangle$. Instead, they are weakly connected to a group of low-lying states represented by $|a\rangle$. The $\gamma_{ij}$ parameters describe the coupling between levels, with

$$\gamma_{ab} \ll \gamma_{0a}.$$

(3.47)
Figure 3.11  Rate $R_0$ for atoms to enter condensate, as a function of level energy $E$.
From Eq. (3.46), for $N_0 = 1000$ atoms and $T = 150$ nK.

Figure 3.12  Three-level model of condensation process. Level $|0\rangle$ represents the condensate, level $|a\rangle$ a group of low-lying states, and $|b\rangle$ the majority of the occupied states of the trap. Population is transferred between levels at the rates $\gamma_{ij}$. 
Figure 3.13  Initiation of BEC. A nondegenerate gas initially at $T = 500 \text{ nK}$ with $N = 2.2 \times 10^5$ atoms is cooled at $t = 0$ by removing all atoms having $E > T$. The gas then freely evolves, undergoing BEC at $t \approx 10 \text{ s}$.

Because of their strong coupling, the populations of states $|0\rangle$ and $|a\rangle$ are always close to equilibrium. The population $N_a$ therefore saturates at some level given by the Bose-Einstein distribution, with any excess atoms rapidly entering the condensate. Because of this saturation, the Bose enhancement of $\gamma_{ab}$ never gets very large.

This picture of the filling process explains many aspects of the condensate dynamics. For instance, Fig. 3.13 shows the response of an initially nondegenerate gas when evaporative cooling is suddenly applied. At $t = 0$, all atoms with energy greater than $T$ are removed, and the gas is then allowed to equilibrate. As can be seen, there is an appreciable delay before condensation occurs. This delay can be understood as the time required for enough population to be transferred from the high-lying energies $|b\rangle$ to saturate the states $|a\rangle$. Once this occurs, $N_0$ starts to grow.

The picture also explains the shape of $N_0(t)$ following a collapse, as seen in Figs. 3.7 and 3.9. An expanded view is shown in Fig. 3.14. There are two phases to this filling process. The condensate first must come to equilibrium with the low-lying states $|a\rangle$. The transfer rate $\gamma_{ab}$ depends strongly on $N_0$ through the Bose stimulation factor, so
Figure 3.14  Filling curve. An expanded view of a portion of Fig. 3.10, showing the shape of $N_0(t)$ after a collapse.

This process accelerates as the condensate fills. However, once equilibrium between $|0\rangle$ and $|\alpha\rangle$ is established, the growth in $N_0$ levels out. In Fig. 3.14, this occurs at $N_0 \approx 200$ atoms. After this point, $N_0$ grows linearly, reflecting the constant addition of atoms to the $|0\rangle-|\alpha\rangle$ subsystem from $|b\rangle$. If the gas as a whole is approaching equilibrium, $\gamma_{ab}$ itself decreases, causing the condensate growth to slow and correspondingly slowing down the collapse/fill cycle. This can be seen in the long-time behavior of Fig 3.10.

Recall, however, the discussion at the end of Section 3.1.3, noting that the QBE model neglects important mean-field interaction effects when $N_0$ is large. In particular, as the condensate fills, $E_0$ decreases, which shifts the equilibrium between the $|0\rangle$ and $|\alpha\rangle$ states. This should cause the condensate to fill faster than the QBE predicts. An estimate of the size of the effect can be made by considering the shift in the equilibrium population of the condensate due the interaction energy. In equilibrium, the population of each quantum state is given by the Bose-Einstein distribution

$$f(E) = \frac{1}{\exp[(E - \mu)/k_B T] - 1}, \quad (3.48)$$
where the chemical potential $\mu$ satisfies
\[ e^{-\beta \mu} = 1 + \frac{1}{N_0} \] (3.49)
when the ground state energy $E_0$ is taken to be zero. For specified values of $N$ and $T$, $N_0$ is found by adjusting $\mu$ to give
\[ N = \int dE g(E) f(E) \to \sum_n \frac{(n + 1)(n + 2)}{2} f(E_n). \] (3.50)
Ordinarily, the energy of the $n^{\text{th}}$ excited state is $E_n = n\hbar \omega$, but if interactions lower $E_0$ by $\delta \hbar \omega$ then the excited-state energies are effectively raised, to
\[ E_n = (n + \delta)\hbar \omega. \] (3.51)
Just before the condensate collapses, the energy shift obtained from Eq. (2.35) is $0.4\hbar \omega$. Comparing the determinations of $N_0$ with and without this $\delta$ indicates that the interactions increase $N_0$ by $\sim 350$ atoms at $T = 150$ nK and $N = 1.44 \times 10^4$. This is a significant effect, since it is a sizable fraction of $N_m$. It is, however, only an upper bound since interactions will lower the energy of the excited states as well. The equilibrium effect could be calculated accurately using the methods of [39] or [40].

An accurate calculation of the dynamical effect of this interaction energy is more difficult, for the reasons previously discussed. An attempt was made to use the three-level system as an approximate model, and indicated that the interaction shift causes roughly a 20% increase in the condensate fill rate for large $N_0$. However, the quantitative accuracy of the three-level model is questionable. In particular, the choice of energy level splittings required to reproduce the filling curve of Fig. 3.14 gives a substantially smaller equilibrium population shift than that calculated above. This suggests that the dynamical effect may in reality be somewhat larger.

### 3.3.3 Nonequilibrium Distributions

For the experimental work described in Chapter 4, it is necessary to determine $N_0$ from a spatial image of the gas cloud. This is facilitated by having a simple model
Figure 3.15  Characterizing nonequilibrium distributions. Points show a typical energy distribution function predicted from the QBE model. The solid curve is a fit to the nonequilibrium distribution function of Eq. (3.52), with \( N_0 = 1006 \) atoms, \( \beta^{-1} = k_B \times 106 \) nK, and \( A = 4.7 \). The actual condensate population is 1014 atoms. The dotted curves show equilibrium Bose-Einstein distributions with the same \( N_0 \) values. For the lower curve, \( T = 106 \) nK, and for the upper \( T = 334 \) nK.  

for the expected energy distribution of the atoms. Since the gas is out of equilibrium while the collapses are occurring, the usual Bose-Einstein distribution is inadequate. However, the distributions \( f(E) \) observed in the QBE model can serve as a guide. They suggest that, after a perturbation, the gas quickly reaches a state of “local equilibrium” in the sense that the populations of nearby energy levels bear nearly their equilibrium relationships. As a consequence, the nonequilibrium distributions observed can be described relatively easily.

Fig. 3.15 shows a plot of \( f(E) \) taken from the simulation of Fig. 3.10 at \( t = 22.3 \) s. At high energies, \( f \) falls off like \( \exp(-\beta E) \), suggesting a thermal distribution at temperature \( T = 1/k_B \beta \). Also, at low energies the distribution rises like \( 1/E \), which is characteristic of a degenerate Bose-Einstein distribution. However, the chemical potential \( \mu \) suggested by the high-energy populations is considerably greater than zero, as seen from the \( y \)-intercept of \( \log f \). This is impossible for the equilibrium Bose distribution (3.48), since if \( \mu > 0 \), the condensate population \( f(0) \) would be negative,
an absurdity.

It is physically reasonable that the population of the high-lying states should be described by a chemical potential which is larger than its equilibrium value, since atoms must be transferred from these states to the condensate in order to achieve equilibrium. At the same time, the low-energy states are in equilibrium with the condensate because of the high rates for population transfer, as suggested in the three-level model. What is needed, then, is a way to patch these high- and low-energy behaviors together. A function which suffices is

$$f(E) = \frac{\exp \beta(\mu_1 - \mu_2)}{\exp \beta(E - \mu_2) - 1} \equiv \frac{\mathcal{A}}{\exp \beta(E - \mu_2) - 1}. \quad (3.52)$$

For large $E$, this function approaches $\exp \beta(\mu_1 - E)$, the form of an equilibrium distribution with chemical potential $\mu_1$ and temperature $\beta^{-1}$. However, when $\mu_1 > 0$, the condensate population remains well defined as

$$N_0 = \frac{\mathcal{A}}{\exp(-\beta \mu_2) - 1}, \quad (3.53)$$

so long as $\mu_2 < 0$. Thus $N_0$ is predominantly determined by $\mu_2$, which plays the role of the chemical potential of the condensate. The low-lying states have populations

$$f(E) \to \frac{\mathcal{A}}{\beta E} \quad (3.54)$$

when $N_0 \gg 1$. This again appears as an equilibrium distribution, but with effective temperature $\mathcal{A}/\beta$. In principle, the parameter $\beta$ is energy-dependent, as seen in the deviation of the tail of log $f$ from linearity in Fig. 3.15. However, the slight variation observed has a negligible effect.

The solid curve in Fig. 3.15 is a fit to the function (3.52); it is seen to accurately characterize the model data. Also shown as dotted curves are two equilibrium Bose-Einstein distributions for comparison. The expected accuracy to which $N_0$ can be experimentally determined was estimated by using distribution data from the QBE model to generate artificial images. These images were then analyzed using the same
routines as used for actual data, following the procedure detailed in Appendix A. Typical results are shown in Fig. 3.16. The $N_0$ values obtained from the fit generally agree well with the known values from the model distribution, differing by at most about 75 atoms. For larger $N$, however, the model performs worse and consistently overestimates $N_0$. To obtain an accuracy of ±100 atoms, $N$ must be less than about $4 \times 10^4$. In addition, at very low temperatures, the model cannot easily distinguish condensate atoms from the noncondensed cloud. This problem arises for $T$ less than about 50 nK, typically corresponding to $N \sim 10^3$. 

Figure 3.16 Accuracy of nonequilibrium model. The solid curve shows the $N_0$ values predicted by the QBE model. The points are values extracted using the procedure described in Appendix A.
Chapter 4
Measuring Condensate Distributions

The theoretical work described in the previous chapters has depicted a condensate having $a < 0$ as a highly dynamical object, with violent collapses interspersed among periods of growth. We wish to experimentally observe as many aspects of this behavior as possible. Previous work in our lab has demonstrated that the condensate occupation number is limited to a value in agreement with predictions [78], and in this chapter the first observations relating to the dynamics of the condensate are described. Owing to the difficulty of making precise measurements on samples of a thousand atoms or less, these observations are perforce indirect. Nonetheless, a wealth of information is obtained, which both qualitatively confirms the predictions of the theory and points out limitations in its quantitative accuracy. In the following chapter, several alternative experimental approaches and their possible implementation are discussed.

The experimental apparatus and procedures are only summarized here, since they are more thoroughly discussed in other works [25, 77–80]. The general techniques of laser slowing and cooling are described at a basic level in [81]. Data and analysis pertaining to our measurements of the distribution of $N_0$ values are described in detail, as well as a related series of experiments illustrating the equilibration process.

4.1 Observing BEC

The experiments reported here were carried out using the same apparatus with which BEC was originally observed in $^7$Li in 1995. A few refinements have been made, providing increased repeatability and robustness. More extensive improvements have been made in the imaging system used to detect BEC and measure $N_0$, while at the same time, image analysis techniques have been extended and better characterized.
These improvements allow the detection of as few as 100 condensate atoms, with an estimated accuracy of 20%.

4.1.1 Apparatus

A schematic of the experiment is shown in Fig. 4.1. It consists of two main sections, a slow-atom source and a magnetic trap. The source is based on a laser-slowed atomic beam. A recirculating oven [25] containing a few grams of lithium metal is heated to 600°C, producing a beam of about $10^{14}$ atoms/s. Under normal conditions, the oven is loaded about once per year, along with other routine vacuum pump maintenance. A gate valve separates the oven and pumps from the rest of the chamber to facilitate these tasks.

Atoms in the beam with velocities of about 500 m/s or less are laser cooled to a velocity of 50 m/s by a Zeeman slower. Slowing occurs as atoms scatter photons out of a counter-propagating laser beam, and an inhomogeneous magnetic field is used to compensate for the changing Doppler shift of the transition. Approximately 1% of the atoms exiting the oven are slow enough to be cooled in this way, so the beam emitted from the Zeeman slower is composed of a small low-velocity fraction together with a fast-atom background. These components are separated by a transverse cylindrically focussed laser beam. The fast atoms scatter few photons while passing through this

![Figure 4.1](image-url) Schematic of experimental apparatus.
beam, while the slow ones are deflected by an angle of 30°. The cylindrically focussed laser compensates for the Doppler shift of the atoms as they are deflected, and thus provides a more collimated slow-atom beam. The atoms are further collimated and guided by a set of four laser beams configured as a two-dimensional optical molasses. The atoms enter the trap chamber through a narrow tube which allows for good differential pumping. En route, the atoms are optically pumped into the \((F = 2, m_F = 2)\) hyperfine state.

Meanwhile, the fast atoms continue down the original beamline, and are eventually deposited onto the window through which the slowing laser enters the chamber. To prevent a coating from building up on this window, it is heated to approximately 350°C. A sapphire window is used to inhibit the chemical damage which lithium vapor inflicts on glass, and two wave plates in the slowing laser beam compensate for the birefringence of the sapphire.

The trap itself is constructed from six NdFeB-alloy permanent magnets, using the geometry shown in Fig. 4.2. The magnets provide a confining field with a minimum of 1000 G and a trap depth of 140 G, or about 10 mK. The trap itself is almost cylindrically symmetric, with atoms oscillating at frequencies of 135.5 Hz, 150.6 Hz, and 152.6 Hz along its three principle axes. Confinement is loosest in the direction
along the magnetic field, which is labeled as \( z \) in Fig. 4.1. Atoms enter the trapping region and are slowed down to \( \sim 1 \) m/s by a set of six laser beams, and the atoms are then confined. Loading saturates at an estimated \( 2 \times 10^8 \) atoms in a few seconds; this number is probably limited by losses due to optical pumping into untrapped spin states. The atoms are loaded at a temperature of \( \sim 0.5 \) mK using relatively intense laser beams, with \( I \sim 40 \) mW/cm\(^2\) per beam. After loading the intensity is reduced by a factor of \( \sim 100 \) for 10 ms, and then ramped off. This cools the atoms to 250 \( \mu \)K, near the Doppler limit of 140 \( \mu \)K. The resonant optical density of the cloud is then 10 or greater, and this high opacity is probably responsible for the inability to achieve the actual Doppler temperature limit.

The trap-loading process requires laser beams at three different frequencies, which are derived from two ring dye lasers. One laser provides the molasses beams used for trapping, while another gives the Zeeman slower and deflection/collimation beams. The frequency of the lasers is derived from a lithium heat pipe using Doppler-free saturated-absorption spectroscopy. One of the lasers is locked to the heat pipe by a Pound-Drever technique, while the second is referenced to the first by a heterodyne measurement. These systems are described in Appendix D. Together, they give both lasers a frequency accuracy of better than 2 MHz, which is smaller than \( \Gamma = 6 \) MHz, the natural linewidth of the \( ^7 \text{Li} \) 2S \( \leftrightarrow \) 2P transition.

After the atoms are loaded and cooled, all the laser beams are shut off and the atoms are held purely by magnetic forces. Evaporative cooling is then applied, as described in Section 3.2. By continually removing the high-energy tail of the thermal distribution, the atoms are gradually cooled, and after roughly three minutes BEC occurs with \( N \approx 10^6 \) and \( T \approx 700 \) nK. In the experiments described here, evaporative cooling is continued until \( N \approx 4 \times 10^4 \) atoms and \( T \approx 150 \) nK, which occurs when the evaporative cooling microwave frequency is about 10 kHz above the trap-center resonance. The 3.4 GHz microwave frequency must therefore be stable to better than
1 kHz, which is easily achieved using a digital frequency synthesizer. However, the trap magnetic field itself must similarly be stable to better than one part in $10^6$, which is more difficult. Since the magnetization of the permanent magnets depends on their temperature, the trap is regulated with heaters and the microwave power used for evaporation is adjusted during cooling to provide a constant heat load. They reduce field fluctuations to $\sim 1$ mG, corresponding to a variation in $N$ of $\sim 10^4$ atoms from one repetition of the experiment to the next. Although higher than desired, this level of variation is acceptable for the current experiments. Possible sources of these fluctuations and means of improvement are discussed in Appendix C, along with the implementations of the current stabilization schemes.

4.1.2 Imaging

Once a condensate is produced, it is observed using the phase-contrast polarization imaging technique described in Ref. [25,80]. A schematic of the imaging system is shown in Fig. 4.3. A probe laser is directed through the cloud and a lens system, onto a CCD camera. The lens system consists of two off-the-shelf achromatic doublets which relay an image of the cloud outside the vacuum chamber, a custom corrective lens to reduce spherical aberrations, and a microscope which magnifies the image onto the CCD. A vacuum viewport is in front of all the lenses, and a polarizer between the doublets is used for phase contrast imaging. The specifications and performance of the imaging system are detainted in Appendix A.

The intensity of the probe beam is reduced as atoms scatter photons out of it, and this absorption can, in principle, be imaged to provide information about the cloud. In addition, the gas as a whole has a substantial index of refraction, so that the light passing through the cloud acquires a phase shift. If the imaging system were perfect, this phase could be ignored, since it would not change the measured intensity. All real lenses, however, will make errors in reconstructing the phase, which cause distortions
Figure 4.3  Schematic of imaging system, showing: the atom cloud A, vacuum viewport B, achromatic doublets C and E, phase-contrast imaging polarizer D, correction lens F, primary image G, microscope objective H, and CCD camera J. The dark grey area represents the probe beam and the light grey the coherently scattered light.

and reduce the fidelity of the image. To avoid this problem, the probe laser is detuned significantly from resonance, lowering the index of refraction and the corresponding phase shift. Typically, the detuning $\Delta = \pm 40\Gamma$. Because the absorption decreases as $1/\Delta^2$ and the index as $1/\Delta$, making the phase shift small enough to avoid imaging distortions makes the absorption too small to observe.

This problem cannot be overcome by simply increasing the probe intensity or the exposure time, since as the atoms scatter photons they are significantly perturbed. For the experiments here, the probe is pulsed on for $\tau_p = 6 \mu s$ with an intensity of $3 \text{ W/cm}^2$. During this time each atom scatters about $n_s = 10$ photons, which heats the gas to roughly $20 \text{ } \mu K$ and destroys the condensate. However, during the pulse the atoms on average move only a distance

$$\delta x = \frac{2}{3} v_R \tau_p \sqrt{n_s}$$  \hspace{1cm} (4.1)

transverse to the probe beam, where $v_R = 8.5 \text{ cm/s}$ is the recoil velocity of a lithium atom. In our case, $\delta x \approx 1 \mu m$, which is small enough to provide an accurate image.

Because of this limitation, the phase shift caused by the cloud must be imaged directly. This can be achieved in a variety of ways [82, 83]. The technique used here relies on the fact that atoms in a large magnetic field are birefringent. The probe
laser is tuned near the \((m_J = 1/2, m_I = 3/2) \leftrightarrow (m_J = 3/2, m_I = 3/2)\) transition, so \(\Delta m = +1\). The strongest alternative transition is \((m_J = 1/2, m_I = 3/2) \leftrightarrow (m_J = 1/2, m_I = 3/2)\), which is detuned by approximately \(1.5 \text{ GHz} \approx 250\Gamma\). Therefore, only the \(\sigma^+\) polarized component of the probe interacts with the atoms, and the scattered and transmitted fields will generally have different polarizations. In our case, the probe polarization is linear with the maximum possible \(\sigma^+\) projection and the scattered field is elliptically polarized. A polarizer in the lens system projects both fields onto a common axis, so that when the fields recombine in the image plane they interfere, providing an intensity modulation sensitive to the initial phase shift.

The polarization phase contrast system is convenient in several ways. The angle \(\vartheta\) between the polarizer axis and the probe polarization is readily adjusted, so it can be optimized as experimental conditions vary. The system is also easy to set up, as it is necessary only to add a polarizer to a normal phase-insensitive imaging system. However, it gives a smaller signal than some other imaging techniques, which can be a drawback. This issue is further discussed in Section 5.1.

As mentioned, the phase shift \(\phi\) must be kept small to avoid image distortions. In terms of the resonant optical density of the cloud \(\alpha [27]\),

\[
\phi = \frac{\alpha \Delta / \Gamma}{2I_0/I_{\text{sat}} + 4(\Delta / \Gamma)^2 + 1},
\]

where \(I_0\) is the intensity of the probe and the saturation intensity \(I_{\text{sat}} = 10 \text{ mW/cm}^2\) for the transition and geometry used [25]. Typical degenerate clouds of \(4 \times 10^4\) atoms have \(\alpha \sim 5 - 10\), giving \(\phi \sim 0.05 \text{ rad}\) for a probe with \(\Delta = 40\Gamma\) and \(I_0 = 300I_{\text{sat}}\). In the limit of small \(\phi\), the signal intensity in the image plane is

\[
I_s(r) = I_0 \left[ \cos^2 \vartheta + \frac{\sqrt{3}}{4} \phi(r) \sin 2\vartheta - \frac{3}{16} \phi(r)^2 \cos 2\vartheta \right],
\]

where \(I_0\) is the incident intensity. For the experiments here, \(\vartheta = 82.5^\circ\), close to \(90^\circ\) so that the signal-to-background ratio is large, but not so close that the quadratic term
in Eq. (4.3) becomes significant. Generally, the reduced signal
\[ S \equiv \frac{I_s}{I_0 \cos^2 \vartheta} - 1 = \frac{\sqrt{3}}{2} \phi \tan \vartheta \] (4.4)
is considered, which gives an observable linearly related to \( \phi \). The background \( I_0 \cos^2 \vartheta \) is measured separately for each experimental run by taking an additional image after emptying the trap by exposing it to the molasses beams for 5 s.* Since \( \alpha \) is related to the density profile of the atoms by
\[ \alpha = \sigma_L \int dz' n(r), \] (4.5)
the signals obtained are direct images of the column density of the cloud. The light-scattering cross section \( \sigma_L = 4\pi/k^2 \), and the integral is along the propagation direction of the probe beam.

4.1.3 Image Analysis

Images are analyzed by assuming that they are derived from the nonequilibrium energy distribution function Eq. (3.52). The density \( n \) and energy distribution \( f(E) \) are related through the ergodic approximation, Eq. (3.11), using
\[ n(r) = \frac{1}{(2\pi\hbar)^3} \int d^3p f(r,p). \] (4.6)
Combining this and Eq. (4.5) gives the relation between \( \alpha \) and \( f(E) \),
\[ \alpha(x',y') = \frac{\sigma_L m}{2\pi\hbar^3 \omega_p} \epsilon \zeta \int_{E_m}^{\infty} dE f(E)(E - E_m), \] (4.7)
where \( x' \) and \( y' \) are coordinates in the image plane, \( \epsilon = \omega_p/\omega_z \approx 1.15 \) is the trap asymmetry, \( \zeta^2 = 3/(1+2\epsilon^2) \approx 0.8 \), and
\[ E_m \equiv \frac{1}{2} m \omega_z^2 (x'^2 + \zeta^2 y'^2). \] (4.8)

*This procedure has the beneficial effect of normalizing out spatial variations in the probe laser intensity \( I_0 \). To achieve good cancellation, it is important that the signal and background images be taken as close in time as possible to minimize drifts in beam pointing, and at the same laser detuning to avoid changes in interference patterns introduced by the imaging lenses.
Details of this calculation are carried out in Appendix A. For \( f(E) = A[\exp \beta(E - \mu) - 1]^{-1} \), the integral in (4.7) can be evaluated by expanding
\[
\frac{1}{e^x - 1} = \sum_{j=1}^{\infty} e^{-jx}
\] (4.9)
to give
\[
\alpha = Ae^\zeta \frac{k_BT}{\hbar \omega} \frac{\sigma_L}{\Lambda^2} \sum_{j=1}^{\infty} \frac{1}{j^2} \exp \left\{ \frac{j}{k_BT} \left[ \mu - \frac{1}{2} m\omega^2 (x'^2 + \zeta^2 y'^2) \right] \right\} + \alpha_0,
\] (4.10)
where \( \Lambda \) is the thermal deBroglie wavelength \( (2\pi \hbar^2 / mk_BT)^{1/2} \). As usual for a semiclassical calculation in a degenerate Bose gas, the contribution of the condensate itself has been lost and must be included post hoc, as
\[
\alpha_0 = \frac{N_0 \sigma}{\pi \ell_{\rho}^2} \zeta_0 \exp \left\{ -\frac{1}{\ell_{\rho}^2} \left( x'^2 + \zeta_0^2 y'^2 \right) \right\},
\] (4.11)
for \( \ell_{\rho}^2 = \hbar / m\omega \) and \( \zeta_0^2 = 3/(1 + 2\epsilon) \). The condensate size \( \ell_\rho \) can be adjusted as a function of \( N_0 \) to reflect the contraction caused by interactions, as in Fig. 2.3.

Signals are processed by averaging the two-dimensional images around ellipses of asymmetry \( \zeta \) to generate a radial signal profile. The center of the signal \( (x_c, y_c) \) is first found by determining the centroid of the brightest pixels in a manually identified central region of the image. Each pixel in the image is then subdivided into \( 3 \times 3 \) subpixels, and each subpixel assigned to a radial bin according to its mean distance from the center under the ellipsoidal metric \( r^2 = (x' - x_c)^2 + \zeta^2 (y' - y_c) \). Radial bins one pixel in width are used, and the mean radius of the bin is calculated as the average distance \( r \) of the contributing subpixels. The distinct asymmetry \( \zeta_0 \) of the condensate is ignored, since it is a small effect and cannot be resolved by our imaging system. The distance metric used scales the resulting profiles to the \( x' \) direction, so they are then fit to Eq. (4.10) with \( y' = 0 \). Examples are shown in Fig. 4.4. Agreement between data and fit is generally good.

The ripples apparent in the upper plot of Fig 4.4 are an artifact of the limited resolution of the imaging system. Neglecting interaction effects, the condensate has a
Figure 4.4  Signal profiles of degenerate clouds, taken 10 s after evaporative cooling was halted. The points represent angular averages of the reduced signal $S$ taken from three different images. The error bars on the upper trace show the estimated uncertainty due to shot noise. The solid curves are fits to the nonequilibrium distribution function Eq. (3.52) assuming the average lens aberration parameters of (4.20). For the three traces, the fits give

(a) $N = 1.17 \times 10^4$, $T = 124$ nK, $N_0 = 916$, and $A = 1.6$

(b) $N = 1.03 \times 10^4$, $T = 106$ nK, $N_0 = 405$, and $A = 2.3$

(c) $N = 1.04 \times 10^4$, $T = 108$ nK, $N_0 = 95$, and $A = 2.3$.

For clarity, the signal in trace (a) is offset by 0.02 and in trace (b) by 0.01.
Gaussian profile with a 1/e-radius of 3.1 µm in the radial direction. The imaging aperture is limited to f/5.5 by the vacuum viewport through which the cloud is observed, so a diffraction-limited imaging system would have an Airy spot of radius 4.1 µm. Since the object size is comparable to the resolution, imaging limitations are important, and it is desirable that the lens system be both as ideal and as well-characterized as possible. The imaging system used here is similar to that described in [25], but with a different microscope objective and the addition of the corrective lens. The system is nearly diffraction limited, but measurably different as discussed in Appendix A.

Aberrations are characterized by the phase error $W(\rho)$ as a function of location on the lens [82, 84]. This is the difference in optical path length experienced by a ray passing through the actual lens at position $\rho$ and an ideal lens. The effect of the aberrations on the image is found by convolving the modeled object electric field $E_{\text{obj}}$ with the point transfer function, $G(r)$:

$$E_{\text{img}}(M \mathbf{r}) = \int d^2 r' G(|\mathbf{r} - \mathbf{r}'|) E_{\text{obj}}(\mathbf{r}').$$

(4.12)

Here $M$ is the magnification. The point transfer function is simply the Fourier transform of the phase error,

$$G(r) = \frac{1}{(2\pi)^2} \int_{\Sigma} d^2 k e^{i\mathbf{k} \cdot \mathbf{r}} e^{ikW(k)},$$

(4.13)

where the wave vector $k$ corresponds to a point on the lens as shown in Fig. 4.5. The integral is over the lens aperture $\Sigma$, and yields the familiar Airy pattern when $W = 0$. For a cylindrically symmetric system, $W$ can be expanded as

$$W(\rho) = C_2 \rho^2 + C_4 \rho^4 + C_6 \rho^6;$$

(4.14)

where $\rho$ is the distance from a point in the aperture plane to the imaging axis. The phase error can be calculated from knowledge of the lens design, or measured by analyzing images of a point-like object. We determined $W$ on the bench by imaging
Figure 4.5 Calculation of point-transfer function. The phase error $W(k)$ in (4.13) is related to the position-dependent phase error $W(\rho)$ as seen above. The lens aperture $\Sigma$ is located a distance $L$ from the object. The vector labeled $k$ shows the direction of $k$; its magnitude is fixed at $2\pi/\lambda$. The lens position $\rho$ is related to the radial component of $k$ by $\rho = Lk_p(k^2 - k_p^2)^{-1/2}$.

Laser light emitted from a single-mode optical fiber with a $1/e^2$-intensity radius of 1.7 $\mu$m. Analysis of the images

$$C_2 = 0.26\lambda/cm^2 \quad C_4 = -0.36\lambda/cm^4 \quad C_6 = 0.17\lambda/cm^6$$  \hspace{1cm} (4.15)

at best focus. An alternative measurement was made in situ by adjusting $W$ to reproduce the observed ripples in images of clouds with large condensates. The resulting values are

$$C_2 = 0.34\lambda/cm^2 \quad C_4 = -0.30\lambda/cm^4 \quad C_6 = 0.094\lambda/cm^6.$$  \hspace{1cm} (4.16)

The difference might be due to focusing errors in the experimental configuration, or to additional aberrations caused by the vacuum viewport. Details of both of these measurements are presented in Appendix A.

4.1.4 Uncertainty

The goal of the above procedure is to determine $N_0$ as accurately as possible. Fundamentally, the measurement is limited by shot noise in the CCD camera. For a probe pulse which generates $s$ photoelectrons in a pixel, the shot noise will be $\sigma = \sqrt{s}$. 
If the signal on a pixel is $s_a$ against a no-atom background of $s_b$, the noise on the reduced signal $S = s_a/s_b - 1$ will be

$$
\sigma_S^2 = \left( \frac{\sigma_a}{s_b} \right)^2 + \left( \frac{s_a \sigma_b}{s_b^2} \right)^2 = \frac{(1 + S)(2 + S)}{s_b}.
$$

(4.17)

In radial profiles such as those of Fig. 4.4, more pixels contribute to points at larger radius, with $\zeta \pi(2j + 1)$ pixels contributing to the $j^{th}$ annular bin. The statistical uncertainty in bin $j$ is therefore

$$
\sigma_j^2 = \frac{(1 + S_j)(2 + S_j)}{s_b \zeta \pi(2j + 1)}.
$$

(4.18)

Using this error estimate, the value of $\chi^2$ for the fit can be calculated by

$$
\chi^2 = \sum_j \frac{|S_j - F_j|^2}{\sigma_j^2},
$$

(4.19)

where $F_j$ is the value of the fitting function for bin $j$. The ratio of $\chi^2$ to the number of degrees of freedom $\nu$ should be near unity for an accurate fitting function. Four fitting parameters are used: $N_0$, $\beta$, $A$, and an offset allowing for variation in probe intensity between the signal and background images. So, for $n_b$ radial bins, $\nu = n_b - 4$. A histogram of values of reduced $\chi^2$ is shown in Fig. 4.6, and demonstrates that they are indeed typically close to one.

The statistical uncertainty in the condensate number is determined by fixing $N_0$, fitting the remaining parameters, and finding the variation in $N_0$ required to increase $\chi^2$ by one [76]. Figure 4.7 shows a plot of the measured uncertainty versus $N_0$ value for the data set used in Fig. 4.6. For $N_0 > 100$, the uncertainty is $\pm 60$ atoms. This is the only source of statistical error in the experiment, in the sense that if two identical atom clouds were imaged and analyzed, the only difference in the results would be that caused by variation in the shot noise.

The dominant sources of error in the experiment, however, are not statistical but systematic. Chief among these is the uncertainty in the lens aberrations discussed
Figure 4.6  Observed distribution of $\chi^2_v = \chi^2/(n_b - 4)$. The distribution is peaked around one, indicating that the data are consistent with the fitting function. A total of 201 images were used, all taken 10 s after evaporative cooling was halted.

Figure 4.7  Statistical uncertainty in $N_0$ due to shot noise in camera.
Figure 4.8  Effect of imaging model on $N_0$. The fitted $N_0$ values obtained using the lens parameters derived from images of an optical fiber (4.15) are compared with those obtained using the parameters derived from distortions of data images (4.16). The line is a linear fit with a slope of 1.35.

in the previous section. The two different sets of parameters (4.15) and (4.16) give $N_0$ values which differ by about 40%, as seen in Fig. 4.8. We assume that the actual aberrations lie somewhere in this range, and report $N_0$ values based on the average parameters

$$C_2 = 0.30\lambda/cm^2 \quad C_4 = -0.33\lambda/cm^4 \quad C_6 = 0.13\lambda/cm^6,$$

(4.20)

with an uncertainty of ±20%.

Other sources of uncertainty contribute to a lesser degree. As discussed in Section 3.3.3, inaccuracy of the nonequilibrium fitting function is expected to lead to an error of ±75 atoms. This is partially confirmed experimentally by analyzing images of clouds which have been allowed to equilibrate, fitting to both nonequilibrium and equilibrium energy distributions. The difference in $N_0$ values is close to zero on average. When images of clouds with small $N_0$ and large $N_0$ are considered separately, the nonequilibrium fit gives values larger by about 50 atoms in the first case and smaller by 50 atoms in the second. Although this cannot be directly compared with
the results in Fig. 3.16, it suggests that the error introduced by the nonequilibrium function is not much larger than predicted.

Additional uncertainty in $N_0$ arises from the effect of interactions on the size and shape of the condensate, since this has not been observed experimentally and is not necessarily well-understood theoretically. The effect of including the compression predicted by the variational model, seen in Fig. 2.3, is to lower the highest values of $N_0$ by about 10%. This provides an estimate of the uncertainty introduced by this variability. Note, however, that the error could potentially be larger: Kagan et al. predict that a fraction of the condensate acquires very high energy during the collapse, and the experiment would likely be completely insensitive to this fraction. This possibility indicates that $N_0$ could be considerably higher than measured, but there is no similar mechanism which could cause $N_0$ to be grossly overestimated.

There are several other sources of systematic error which contribute little to the uncertainty in $N_0$, but more to the measurement of $N$ and $T$. These are outlined in Appendix B. The dominant uncertainties are those of the polarizer angle $\theta$ and of the imaging magnification. For a nondegenerate cloud in equilibrium, these allow measurement of $T$ to $\sim$3% and $N$ to $\sim$5%. For a gas undergoing collapses, the accuracy of the nonequilibrium model must also be included. From the mock image analysis procedure described in Appendix A, the accuracy of the model for $N$ is expected to be within that of the experimental systematics described in Appendix B. Temperature determination is more problematic since $T$ is indefinite for a nonequilibrium gas, but the analysis correctly determines the average energy $\langle E \rangle$ of the atoms to an accuracy of about 10%.

4.2 Distribution of $N_0$ Values

The preceding section has illustrated that we can measure $N_0$ with respectable precision. In order to achieve this, however, it is necessary to destroy the atom
cloud as it is being observed, which hinders the study of dynamical properties. As noted in Section 3.3.2, a given trajectory $N_0(t)$ is made unrepeatable by intrinsic quantum and thermal fluctuations, as well as by variations in experimental conditions. This prevents the dynamics from being mapped out by repeating an experiment with a variable delay before probing. Several possible methods to circumvent these limitations are discussed in Chapter 5. The simplest approach, however, is to repeat the experiment many times with a fixed probe delay, and observe the distribution of $N_0$ values obtained. Since the effect of the experimental and intrinsic fluctuations is to shift the phase of the collapses, this approach effectively averages the probability for a particular $N_0$ value to occur over the oscillation cycles.

4.2.1 Data

Data are obtained by loading atoms into the trap and evaporatively cooling until the microwave frequency $\Omega_T \approx 100$ kHz above the trap-center resonance. Approximately $4 \times 10^8$ atoms then remain, at a temperature of $\sim 400$ nK. This is well below the point at which BEC typically occurs, to ensure that the gas is well into the degenerate regime. In order to accurately determine $N_0$, $N$ must be further reduced. This is achieved by rapidly sweeping $\Omega_T$ down to $\sim 10$ kHz and then raising it again. This technique used because it is fast and because it puts the gas into a definite state at $t = 0$, from which the relaxation to equilibrium can be observed. The microwaves are swept in $\sim 100$ ms, which is fast compared to the collision rate of a few Hertz, but slow compared to the trap oscillation period so that time-of-flight broadening and non-ergodic effects are not important.

After the sweep, roughly $4 \times 10^4$ atoms remain trapped, at a temperature of 150 nK. The precise values vary from one repetition to the next because of the previously mentioned fluctuations in the trap bias field. The response predicted by the QBE is shown in Fig. 3.10, and the variations caused by fluctuations can be seen in Fig. 4.9.
Figure 4.9 Variations in QBE trajectories. Simulations of the equilibration process such as in Fig. 3.10 are repeated with varying random number sequences. The microwave sweep frequencies are also varied to simulate the effect of trap bias field fluctuations. The legend shows the energy to which the trap depth is reduced during the sweep. The time origin is taken to be the time of the sweep.

Collapses continue for 10 to 20 s after the sweep, equilibrium is reached at $t = 20-30$ s, and then the condensate slowly decays because of inelastic collisions.

The gas is allowed to thermalize for a time $\tau$, and is then probed. The results are shown in Fig. 4.10. For small $\tau$, $N_0$ varies from near zero to about 1200 atoms, as expected if the condensate is alternately filling to its maximum and collapsing. At longer time delays, the histograms change shape, narrowing somewhat at $\tau = 30$ s and exhibiting only small $N_0$ values at $\tau = 60$ s. At each time, the spread in values is much larger than the statistical uncertainty in $N_0$, so the variations are experimentally significant. This conclusion is confirmed by constructing a histogram at $\tau = 90$ s, when only small $N_0$ values should occur. As can be seen in Fig. 4.11, the results of the measurement are indeed small, and consistent with our expected statistical uncertainty.

This behavior shown in Figs. 4.10 and 4.11 is qualitatively the same as that seen in Fig. 4.9. To our knowledge, no other explanation for variations in $N_0$ of this
Figure 4.10  Frequency of occurrence of $N_0$ values, and dependence on equilibration time $\tau$. The total number of atoms $N$ ranged from $5 \times 10^3$ to $3.5 \times 10^4$ in each case. The total number of runs is 106 for $\tau = 5$ s, 185 for $\tau = 10$ s, 81 for $\tau = 30$ s, and 86 for $\tau = 60$ s. The width of the bins approximates the statistical uncertainty in $N_0$. 
Figure 4.11 Values of $N_0$ obtained at $\tau = 90$ s are always small, and consistent with the expected statistical uncertainty in the measurement. A total of 30 runs are shown.

magnitude has been proposed, and we consider their observation to strongly support the dynamical picture developed in the previous chapters.

4.2.2 Comparison with Theory

At short delays, while collapses are still occurring regularly, the shape of $N_0(t)$ during the condensate growth phase is fairly constant, as seen in Fig 3.14. The expected distribution of $N_0$ values can be expressed in terms of this shape, the probability $P_R(N_1)$ for $N_1$ atoms to remain in the condensate after a collapse, and the probability $P_S(N_1, N_2)$ for a condensate to survive without collapse while growing from $N_1$ to $N_2$:

$$P(x) = C \int_0^x dN_1 P_R(N_1) P_S(N_1, x) \left( \frac{dN_0}{dt} \bigg|_{x=0} \right)^{-1}, \quad (4.21)$$

with the constant $C$ chosen to make $\int dx P(x) = 1$. The derivative $dN_0/dt$ is evaluated at the point $N_0(t) = x$, and depends in principle on $N_1$, the value from which condensate growth began. However, it is observed in the QBE solutions that this dependence is insignificant: the growth rate of the condensate at $N_0 = x$ is nearly the same regardless of whether it reached that point by collapsing to it or by collapsing to
Figure 4.12 Distribution of $N_0$ values predicted by QBE, assuming collapses lead to complete loss of the condensate. (a) Probability function calculated from Eq. (4.22). (b) Histogram derived from the probability function.

a lower value and filling for a time. This allows the derivative to be brought outside the integral and simplifies the calculation.

In terms of the collapse initiation rate $\Gamma_C$ of Fig. 2.5, the probability for the condensate to collapse at $N_0 = x$ is $\Gamma_C(x)(dN_0/dt)_x^{-1}$. The survival probability $P_S$ is then

$$P_S(N_1, N_2) = \exp \left[ - \int_{N_1}^{N_2} dN' \Gamma_C(N') \left( \frac{dN_0}{dt} \bigg|_{N'} \right)^{-1} \right]. \quad (4.22)$$

Figure 4.12 shows the predicted distribution at $\tau = 5$ s based on the trajectories of Fig. 4.9, where $P_R(N_0) = \delta(N_0)$. It is highly peaked at small $N_0$ because the condensate growth rate is initially slow, and levels out as the growth saturates.*

The observed histograms differ quantitatively from this prediction in two major respects. First, there is no peak seen at low $N_0$. Rather, a broad hump ranges from $N_0 = 200$–700 atoms. Second, the measured data drop smoothly to zero as $N_0 \to N_m$, rather than remaining flat. These differences must arise from inaccuracies in the predicted forms for $N_0(t)$, $P_R$ and $P_S$.

*See Section 3.3.2 for discussion.
Figure 4.13 Predicted histogram when collapsing to a Gaussian distribution centered at $N_0 = 200$ atoms and with a $1/e$ half-width of 100 atoms.

The deviation at low numbers could indicate that $N_0(t)$ rises very quickly after a collapse, with the growth subsequently slowing when $N_0$ reaches $\sim 200$ atoms. It is hard to motivate this hypothesis, however, since Bose stimulation must certainly play an important role in condensate formation and this factor only increases as $N_0$ grows. A more likely explanation is that the collapse does not lead to complete loss of the condensate. As noted in Section 2.3, Kagan et al. predict that the collapse reduces $N_0$ to $N_m/2$ [53]. This possibility is clearly ruled out by the large number of points observed with small $N_0$. There is, in fact, no fixed value to which the collapse could reduce $N_0$ which reproduces the observed data, since the probability of occurrence is seen to rise gradually as $N_0$ increases from 0 to 200 atoms. The data are described better if $P_R$ is assumed to have a nonzero width, as in the function (3.43) used for Fig. 3.9(b). The predicted histogram for this case is shown in Fig. 4.13, and agrees reasonably well with the data for low $N_0$.

The deviation at high numbers can similarly be explained either by an incorrect expectation for $N_0(t)$ or by an incorrect decay rate $\Gamma_C$. In this case, however, either explanation is reasonable. As discussed in Chapter 3, the QBE model neglects siz-
able mean-field interaction effects, which might be expected to accelerate condensate growth for large $N_0$. This would have the observed effect of reducing the probability of observing $N_0$ near $N_m$. Alternatively, if collapses are initiated at lower values than predicted, the probability of condensates surviving to large $N_0$ will be reduced. In their numerical solutions of the NLSE, Kagan et al. observe the collapse to leave the condensate in an energetically excited state. This excess energy might decrease the stability of the condensate and allow it to collapse at relatively low $N_0$ values, since when the breathing mode is significantly excited, the condensate can be carried over the barrier preventing collapse. Indeed, their solutions exhibit a number of small collapses as the condensate fills, as seen in Fig. 2.9. If this were the dominant collapse mechanism, however, the histograms at $\tau = 5$ and 10 s should differ, since in the latter case the condensate spends more time with large $N_0$ and thus should tend to collapse earlier.

At longer delay times, quantitative comparison with theory is difficult because the $N_0$ values predicted by the QBE model depend sensitively on the time at which the last collapse occurs. However, from the simulation data shown in Fig. 4.9, the model appears to be reasonably consistent with the measured distribution at $\tau = 30$ s, but to predict slightly higher values than typically observed at $\tau = 60$ s. The discrepancy at long times could be explained by technical sources of heating at the 1 nK/s level or lower.* Heating of this magnitude cannot be measured experimentally, because of the run-to-run fluctuations in $N$ and $T$, but it can be compared to the heating observed at higher temperatures, discussed in Section 3.2.3.

In the low temperature experiments, the trap depth $E_T$ is set to 2.5 $\mu$K after the sweep, so heating from collisions with background-gas atoms will be virtually eliminated, and that from stray laser light will be reduced. Since $E_T$ is less than the

---

*This is in addition to the heating caused by dipolar decay and the collapses, which are included in the model.
photon recoil energy of $E_R = 6 \mu K$, only atoms scattering photons by an angle of

$$\cos^{-1}\left(1 - \frac{E_T}{2E_R}\right) \approx 37^\circ$$

or less will remain trapped and cause heating. The probability for this to occur is $E_T/4E_R \approx 0.1$. The implied scattering rate at large $E_T$ was $10^{-3}$ s$^{-1}$, which gives a heating rate for the current experiment of roughly $0.3$ nK/s. For $\tau = 60$ s, this would give a $20$ nK increase in temperature, corresponding to a $30\%$ decrease in phase-space density for a cloud with $T \approx 150$ nK. It is reasonable that this could significantly reduce the observed $N_0$ values.

### 4.2.3 Correlations

If the observed variations in $N_0$ truly reflect the underlying dynamics of the gas being sampled, they will be uncorrelated with other experimental variables. Searching for such correlations is, therefore, an important check on the results presented above.

Figure 4.14 shows typical results for $N_0$ as a function of total number $N$, the goodness of fit $\chi^2$, and the value of $N_0$ obtained on the previous experimental run. In each case, the two variables are clearly independent. Other variables examined include the temperature $T$, probe detuning $\Delta$, imaging polarizer angle $\vartheta$, lens aberration parameters $C_i$, and time of day.

A quantitative measure of correlation significance can be obtained through contingency table analysis [85, Section 14.4]. In this method, a set of data such as in Fig. 4.14(a) is used to generate a joint probability distribution $P(N_0, N)$, which is the probability of observing a condensate number of $N_0$ at the same time as a total number of $N$. The same data also provides the single-variable probabilities $P(N_0)$ and $P(N)$. If $N_0$ and $N$ are independent, then $P(N_0, N) = P(N_0)P(N)$, a simple prediction which is readily tested. Because a finite number of data points are available, the measured probability values are subject to shot noise, and the significance of any observed association must be parameterized by the $\chi^2$ statistic for the difference
Figure 4.14  Searching for correlations with $N_0$. Relation between condensate number $N_0$ and: (a) total number of trapped atoms $N$, (b) goodness of fit parameter $\chi^2_y$, and (c) the value of $N_0$ found in the previous experimental run, $N_0'$. No correlations are observed. The data shown have $\tau = 10$ s.
<table>
<thead>
<tr>
<th>Variable</th>
<th>Range</th>
<th># Bins</th>
<th>( \nu )</th>
<th>( \chi^2_\nu )</th>
<th>( P_\nu )</th>
<th>( V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>( 5 \times 10^3 - 3.5 \times 10^4 )</td>
<td>8</td>
<td>42</td>
<td>0.75</td>
<td>0.93</td>
<td>0.17</td>
</tr>
<tr>
<td>( \chi^2_{\text{fit}} )</td>
<td>0.4-2.5</td>
<td>8</td>
<td>49</td>
<td>1.00</td>
<td>0.45</td>
<td>0.18</td>
</tr>
<tr>
<td>( N_0' )</td>
<td>0-1200</td>
<td>8</td>
<td>49</td>
<td>1.06</td>
<td>0.35</td>
<td>0.19</td>
</tr>
<tr>
<td>( T )</td>
<td>60-230 nK</td>
<td>8</td>
<td>49</td>
<td>0.94</td>
<td>0.60</td>
<td>0.18</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>-43( \Gamma )-44( \Gamma )</td>
<td>2</td>
<td>7</td>
<td>1.26</td>
<td>0.27</td>
<td>0.21</td>
</tr>
<tr>
<td>Sequence #</td>
<td>1-210</td>
<td>8</td>
<td>49</td>
<td>0.71</td>
<td>0.95</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Table 4.1  Contingency-square analysis of correlations, for the \( \tau = 10 \) s data set of Fig. 4.10. Association of \( N_0 \) with the specified variable is examined. The observed values are lumped into the specified number of bins, using the ranges shown; \( N_0 \) itself ranges from 0 to 1200 atoms with 8 bins. Contingency-square analysis yields the indicated values of \( \chi^2_\nu \) and Cramer’s \( V \), for the specified number of degrees of freedom \( \nu \). The probability \( P_\nu \) is the likelihood that measurement of a fundamentally uncorrelated joint probability distribution would give a value of \( \chi^2_\nu \) equal to or greater than that observed. In the variables column, “Sequence #” stands for the order in which the images were obtained, and “\( \chi^2_{\text{fit}} \)” is the goodness-of-fit parameter from the image analysis.

between the predicted and observed values of \( P(N_0, N) \). From \( \chi^2 \) in turn can be found the statistical likelihood for correlations of the measured strength or greater to be observed when the underlying distributions are, in fact, independent [76, Table C.4]. Results of this analysis indicate this likelihood to be at least 25\%, and typically \( \sim 60\% \). If a correlation is assumed to be present, its strength can be measured with the Cramer’s \( V \) statistic, which ranges from 0 for no association to 1 for perfect association. The observed values of Cramer’s \( V \) are \( \sim 0.2 \), indicating that whatever correlation is present is weak.* Results of this analysis are shown in Table 4.1.

We do, however, find \( N \) and \( T \) to be correlated, as seen in Fig 4.15. This is to be expected if variations in both are caused by fluctuations in the trap bias field at the end of evaporative cooling. The number and temperature are seen to be related approximately as

\[
T \propto N^{1/2}. \tag{4.24}
\]

At low energy, the energy distribution function is approximately proportional to \( E^{-1} \),

*The distinction between Cramer’s \( V \) and \( \chi^2 \) can be understood as follows: if a very large number of data points are considered, even a weak correlation can be measured with high confidence. In this case, Cramer’s \( V \) would be small, but \( \chi^2 \) would be large.
Figure 4.15  Correlation of number $N$ and temperature $T$. The solid curve is a fit to the form $N \propto T^{1/2}$. Fluctuations in both quantities arise from variations in the trap bias field during evaporative cooling. Contingency-square analysis of this data yields $\chi^2 = 7.9$, and Cramer's $V = 0.54$.

so that the number of atoms remaining after a sweep to energy $E_T$ is

$$\int_0^{E_T} dE g(E)f(E) \propto \int_0^{E_T} dE E^{-1/2} \propto E_T^2,$$  

(4.25)

while the average energy is

$$\frac{1}{N} \int_0^{E_T} dE E g(E)f(E) \propto \frac{E_T^3}{N} \propto E_T,$$  

(4.26)

so the observed relation is the expected one. The temperature fluctuations of $\sim 100$ nK indicate that the sweep frequency varies by $\sim 2.5$ kHz, corresponding to a bias field fluctuation of about 1 mG.

4.3 Equilibration

The cycle of condensate growth and collapse is driven by an excess of noncondensed atoms compared to a thermal distribution. This excess can be examined directly. Doing so provides a check on the interpretation of the histogram data, since the changing shape of the distributions should be consistent with the approach
to equilibrium. The equilibration process is also interesting in its own right, as it demonstrates significant differences from the behavior of a nondegenerate gas.

4.3.1 Measurement

The experimental procedures and analysis are exactly those used to generate the histogram data. However, instead of considering the values of \( N_0 \) obtained, the total number \( N \) and temperature parameter \( \beta \) are used to calculate

\[
\varrho \equiv \frac{N}{N_c} = 0.832 N (\beta \hbar \omega)^{\frac{3}{2}}
\]

as a function of delay time \( \tau \). The results obtained are shown in Fig. 4.16. Because of the fluctuations in the trap bias field, \( \varrho \) varies by \( \sim 40\% \) from one run to the next, and the points shown are averages over several repetitions at each \( \tau \). Also, as noted in Section 3.3.3, systematic errors limit the accuracy of \( N \) to \( \sim 5\% \) and \( \beta \) to \( \sim 10\% \), so that the systematic error in \( \varrho \) is roughly 30\%. This is reduced for \( \varrho \leq 1 \) by fitting the images to an equilibrium Bose-Einstein distribution function. The error in \( \beta \) is then \( \sim 3\% \), giving a total error of approximately 1\%.

For each value of \( \tau \) in Fig. 4.16, the microwave sweep at the end of evaporative

![Figure 4.16](image-url)

**Figure 4.16** Equilibration of a degenerate gas. The critical parameter \( \varrho \) of Eq. (4.27) is shown as a function of delay time after the microwave sweep \( \tau \).
cooling was adjusted to maintain the average number $N$ constant at $\sim 2 \times 10^4$ atoms, to agree with the data used for the histogram plots. Because of this, the points shown do not represent the evolution of any single cloud in time, making interpretation slightly more difficult. This approach was used because the trap bias field drifts, and the sweep must be occasionally adjusted to maintain a constant average $N$. If the experiment were to be carried out with constant initial conditions, it would be necessary to continually recheck the $\tau = 0$ point in order to correct for this drift. As performed, the drift was monitored as the data was taken, which decreased the time required for the experiments, and correspondingly increased the amount of data available. Correcting for the varying initial conditions is discussed in the next section, and does not present any serious difficulties.

Because $N_m$ is small compared to $N$, equilibrium is reached when $\theta \approx 1$. As can be seen from the Fig. 4.16, this occurs at $\tau \approx 50$ s on average. Comparison with Fig. 4.10 shows that this is consistent with the time at which the shape of the histograms begins to change, helping to confirm the interpretation of the histograms as reflecting the underlying dynamical behavior. Another way to observe the approach to equilibrium is to fit the image data used in Fig. 4.16 to an equilibrium distribution, and watch $\chi^2$ decrease as the gas thermalizes. The results are shown in Fig. 4.17, and suggest an average equilibration time of $\sim 20$ s. It is reasonable that this measure should give a shorter time, both because the gas is essentially thermalized some time before $\theta = 1$, and because shot noise limits the sensitivity of $\chi^2$ to deviations from equilibrium.

4.3.2 Interpretation

The solid curve in Fig. 4.16 is a fit to a power law form

$$\theta(\tau) = \theta_1 (1 + \kappa \tau)^{\gamma_\theta},$$

(4.28)
Figure 4.17 Fitting to equilibrium distributions. The goodness-of-fit parameter $\chi^2$ provides an alternative measure of the equilibration time.

with $q_1 = 12.5$, $\kappa = 2.5$ s$^{-1}$, and $\gamma_\theta = -0.52$. This is a solution to the nonlinear differential equation

$$\frac{dq}{dt} \propto q^{1-1/\gamma_\theta} \approx -q^3.$$  \hspace{1cm} (4.29)

A nonlinear equation is to be expected, since the QBE is itself nonlinear. Of particular interest is the exponent $\gamma_\theta$, since it should be comparable with theoretical predictions.

For such a comparison to be made, it is necessary to correct for the variation in initial conditions. If the number and temperature also vary according to a power law, as

$$N(t) = N_i (1 + \kappa \tau)^{\gamma_N}$$ \hspace{1cm} (4.30)

and

$$T(t) = T_i (1 + \kappa \tau)^{\gamma_T}$$ \hspace{1cm} (4.31)

respectively, then

$$q(\tau) \sim \frac{N_i}{T_i^{3/2}} (1 + \kappa \tau)^{1/2 - 3\gamma_T}.$$ \hspace{1cm} (4.32)

In the experiment, both $N_i$ and $T_i$ are varied with $\tau$, but are related according to
Figure 4.18  Modeling total number $N$ and temperature $T$ during equilibration, from the trajectory shown in Fig. 3.10. The lines are fits to a power law form as in (4.28), but with $\kappa$ fixed at 2.5 s$^{-1}$. The fits give exponents of -0.2 for $N$ and 0.1 for $T$. In the simulation, the gas is degenerate for the entire period shown, with $N_0 = 80$ at $t = 100$ s.

Eq. (4.24). Since $N$ is held fixed, $N_1$ must vary as $\tau^{-\gamma_N}$ and $T_1$ as $\tau^{-\gamma_N/2}$. Thus,

$$\varrho(\tau) \propto \frac{N}{T_1^2(1 + \kappa \tau)^3} \propto (1 + \kappa \tau)^{3\gamma_N/2 - 3\gamma_T},$$

(4.33)

and the experimental exponent is $\gamma_\varrho = 3(\gamma_N/2 - \gamma_T)$.

Values of $N$ and $T$ from the QBE simulation of Fig. 3.10 are shown in Fig. 4.18. The temperatures were obtained using the mock imaging procedure described in Appendix A. The fit curves give $\gamma_N = -0.2$ and $\gamma_T = 0.1$, predicting $\gamma_\varrho = -0.6$ in reasonable agreement with the observed result.

The values of $\gamma_N$ and $\gamma_T$ are, however, surprising. For a nondegenerate gas decaying via two-body inelastic collisions, the results of Section 3.2.3 can be applied. From the ratio of Eqs. (3.31) and (3.36),

$$\frac{dN}{dT} = -\frac{1}{4} \frac{N}{T},$$

(4.34)

so $\gamma_T$ should equal $-\gamma_N/4$. Also, if $N$ decays according to a power law, then since $\dot{N} \propto \tilde{n} N$, the density itself must decay as $t^{-1}$ to provide the same time exponent on
both sides. However, \( \bar{n} \propto N/T^{3/2} \), and equating exponents shows that \( \gamma_N = -8/11 \)
and \( \gamma_T = 2/11 \), considerably different than observed. The exponents obtained in the
QBE model are found to be independent of the value of the loss coefficient \( G_2 \), and
of the trap depth \( E_T \) applied after the microwave sweep. The difference therefore
appears to be a signature of the nonclassical nature of the Bose-Einstein distribution
function.

An additional complication in the experiment arises because of the possibility for
extra heating caused by technical sources. In principle, the effect could be important,
since the average heating rate observed in the QBE model is only \( \sim 0.5 \) nK/s, com-
parable to the effect of photon scattering estimated above. Including heating in the
QBE model is difficult without knowing more about its nature and magnitude, but
evidently its effect is not too great, since the observed and predicted values of \( \gamma_e \) are
similar. The issue does, however, merit further investigation.

It is interesting to consider what mechanism is mainly responsible for reducing
the excess of atoms and allowing the gas to equilibrate. Two loss mechanisms are
included in the model, dipolar relaxation and the collapse. In the trajectory shown
in Fig. 3.10, fifteen collapses occur, in which \( 1.6 \times 10^4 \) atoms are ejected from the
trap. Between the microwave sweep and the final collapse, a total \( 2.4 \times 10^4 \) atoms
are lost, so the collapses and dipolar relaxation both play important roles. From the
last collapse at \( \tau = 18 \) s to \( \tau = 100 \) s, an additional \( 4 \times 10^3 \) atoms are lost. The
rise in temperature and thus \( N_c \) is also important, but the effects of the collapse and
of dipolar decay are more difficult to distinguish. Immediately after the sweep,
\( N = 4 \times 10^4 \), so from the collapses alone, the average energy increases by a factor of

\[
\frac{4 \times 10^4}{4 \times 10^4 - 1.6 \times 10^4} = 1.66,
\]

from \( \langle E \rangle / k_B = 54 \) nK to 89 nK. However, because of the additional decrease in
\( N \) due to dipolar decay, the collapses cause a relatively larger change in \( N \) than
otherwise, and the heating rate is increased. If the losses due to dipolar relaxation
occurred all at once after the sweep, the collapses would then raise $\langle E \rangle$ by a factor of 2; the true value will lie between these extremes. By the last collapse the actual average energy is 117 nK, so roughly 20 nK heating can be attributed to dipolar decay. At $\tau = 100$ s, the average energy is 133 nK. Note that the temperature as defined by the nonequilibrium model is generally higher than $\langle E \rangle / 3k_B$ by factors of 2–3, depending on the degree of degeneracy and distance from equilibrium.
Chapter 5
Other Experimental Approaches

The results presented in Chapter 4 strongly suggest that a condensate with $a < 0$ undergoes dynamical oscillations as predicted by the collapse/fill model. The observed fluctuations, however, were not positively shown to represent a dynamical effect as opposed to some other unexplained source. It is therefore desirable to pursue these experiments further, attempting to study the system in a number of different ways to ensure that a consistent picture emerges. In addition, the data available is far from complete. As expressed in Eq. (4.21), the distributions shown by the histograms depend on the condensate filling rate and the collapse probability together. Further measurements may allow these processes to be detangled and understood independently.

This chapter proposes several different methods by which BEC could be studied using the existing apparatus, and describes the preliminary investigations made to date. Section 5.1 explains the impossibility of making high-precision, nonperturbative measurements of $N_0$ in time, and discusses the information which might nonetheless be available from low-precision, minimally perturbative techniques. Section 5.2 describes the use of photoassociation to modify the interatomic interactions, and how this can provide a probe of the condensate behavior. Section 5.3 discusses observing the initial onset of BEC and the first few collapses, and finally Section 5.4 discusses mechanical manipulation of the condensate using externally applied magnetic fields.

5.1 Nonperturbative Imaging

The ideal way to observe the dynamics of the condensate would be to monitor $N_0$ in time and watch what it does. Unfortunately, the relatively low value of $N_m$ precludes this simple approach, since any accurate measurement will disturb the state of the
system. Even so, some information can be obtained from repeated measurements of the same cloud, and the clear connection to dynamics motivates investigation along these lines.

5.1.1 Optimization of Imaging Parameters

For a fixed probe perturbation, it is desirable to optimize the imaging measurement by suitable choice of the probe detuning $\Delta$, intensity $I_0$, pulse time $\tau_p$, and imaging polarizer angle $\vartheta$. For large $\Delta$, the average number of photons scattered per atom is

$$n_s = \frac{I_0 \tau_p \sigma_L}{\hbar \omega_L} \frac{\Gamma^2}{4\Delta^2},$$

where the light-scattering cross section $\sigma_L$, the natural linewidth $\Gamma$, and the transition frequency $\omega_L$ are constants.

The quantity $n_s$ can be directly related to the signal-to-noise ratio of the images. The signal detected is related to the optical phase shift induced by the atoms,

$$\phi = \frac{\Gamma}{4\Delta} \alpha,$$

through

$$S = I_0 \tau_p \left( \cos^2 \vartheta + \frac{\sqrt{3}}{2} \phi \sin \vartheta \cos \vartheta - \frac{3}{16} \phi^2 \cos 2\vartheta \right).$$

Here as in Section 4.1.2, $\alpha$ denotes the resonant optical density of the atom cloud (4.5). For now, the quadratic term is neglected as $\phi$ is typically small, but it will be reconsidered in Section 5.1.4 below. Expressing $S$ in terms of $n_s$ gives

$$S = 4n_s \frac{\hbar \omega_L}{\sigma_L} \left[ \left( \frac{\Delta}{\Gamma} \right)^2 \cos^2 \vartheta + \frac{\sqrt{3} \Delta}{8 \Gamma} \alpha \sin \vartheta \cos \vartheta \right].$$

If this signal is measured by a camera with pixel area $A_P$ and quantum efficiency $\varepsilon_q$, then the number of counts per pixel will be $s = A_P \varepsilon_q S / \hbar \omega_L$, if $\alpha$ is interpreted as the average of the optical density across the pixel. To correctly account for the magnification $M$ of the imaging system, $A_P$ must represent the effective area of the pixel as measured in the object plane, or the actual pixel area divided by $M^2$. 
If the imaging system were perfect, the noise in the measurement \( s \) would be just shot noise \( s^{1/2} \). Generally, the portion of \( s \) depending on \( \alpha \) is small compared to the background, so that the signal-to-noise ratio is approximately

\[
S/N = \frac{\sqrt{3}}{2} n_s \Delta \frac{A_P}{\sigma_L} \frac{\Delta}{\Gamma} \alpha \sin \vartheta \cos \vartheta \left[ \frac{4n_s \Delta}{\sigma_L} \left( \frac{\Delta \cos \vartheta}{\Gamma} \right) \right]^{-1/2}
\]

\[
= \frac{\sqrt{3}}{4} \alpha \sin \vartheta \left( \frac{n_s \Delta}{\sigma_L} \right)^{1/2}.
\]

(5.5)

Clearly, \( \vartheta \) should be set near \( \pi/2 \) for maximum \( S/N \), but a small offset is required: if the angle is too close the quadratic term in \( S \) cannot be neglected. Since sin(\( \pi/2 - \delta \)) \( \approx 1 - \delta^2/2 \), an offset of several degrees has little effect on \( S/N \).

Remarkably, then, \( S/N \) is independent of \( \Delta \), \( I_0 \) and \( \tau \), except through the dependence of \( n_s \). It is necessary only that \( \Delta >> \Gamma \), both to justify the approximations used and, experimentally, to alleviate the imaging difficulties encountered for large \( \phi \) which were described in Section 4.1.2. Therefore, the choice of these parameters has no bearing on the degree to which the measurement can be both accurate and nonperturbative, as long as the uncertainty is dominated by shot noise. Two significant sources of technical noise must be considered: ripples in the intensity of the probe beam can normalized out only to an accuracy of \( \sim 1\% \), and read noise in the A-to-D circuit of the camera gives \( \sim 10 \) spurious counts per pixel. Shot noise will therefore dominate if the intensity is adjusted to give between \( 10^2 \) and \( 10^4 \) counts per pixel. In the destructive imaging experiments of Chapter 4, we typically used \( \sim 3000 \) counts/pixel.

The only experimentally adjustable parameter which does bear on the question of perturbation is the pixel area \( A_P \).\(^*\) Because the condensate being observed is small, the area cannot be made arbitrarily large, for as the pixel size is increased, it becomes difficult to distinguish the condensate from the remainder of the cloud. For

\(^*\)\( A_P \) can be adjusted both via the magnification and by combining signals from adjacent physical pixels on the CCD camera.
the histogram measurements, the effective pixel size was 1.3 μm square and $n_s \approx 10$. With these values, the uncertainty in $N_0$ due to shot noise was 65 atoms, and that due to the distinguishability of the condensate was 75 atoms. The largest value of $n_s$ which might reasonably be considered nonperturbative is $n_s = 0.1$, which for constant $A_P$ would increase the shot noise by a factor of 10 to ±650 atoms, too close to $N_m$ to be useful. However, if $A_P$ is increased by a factor of 100 to compensate, the pixel becomes large compared to the 6 μm-diameter condensate. The value of $N_0$ would then be impossible to determine.

Even if $\varepsilon_q = 1$, representing an ideal detector,* it would still be impossible to measure $N_0$ with an accuracy of $N_m / 10$ and a 10% perturbation. Better results can be obtained by other imaging methods, as discussed below, but none can provide the necessary order-of-magnitude enhancement.

5.1.2 Differential Measurement of $N_0$

Although using a large pixel size prevents the accurate determination of $N_0$, it does in principle allow the number of atoms in the center of the cloud to be measured. Indeed, inspection of Eq. (4.5) shows that the number of atoms in the column imaged onto pixel $P$ is just

$$N_P = \int_P dA \int dl \ n(r) = \frac{A_P}{\sigma_L} \alpha. \quad (5.6)$$

In the linear signal regime, the uncertainty $\sigma_{N_P} = N_P (S/N)^{-1}$, or

$$\sigma_{N_P} = \frac{4}{\sqrt{3}} \left( \frac{A_P}{n_s \varepsilon_q \sigma_L} \right)^{1/2}. \quad (5.7)$$

When the condensate collapses, $N_P$ for the central pixel will change suddenly by $\sim N_m$. This change can be observed, even if $N_0$ cannot be itself measured. Such a differential measurement of the change in $N_0$ would indicate the number of atoms lost.

---

*For the current imaging system, $\varepsilon_q \approx 0.3$, limited by the 40% quantum efficiency of the CCD camera and the 85% transmission of the polarizer.
in the collapse and the frequency at which collapses occur, both useful quantities for characterizing the behavior of the gas.

To implement this approach, $\sigma_{N_P}$ must be small compared to $N_m$. As seen in Eq. (5.7), the absolute uncertainty $\sigma_{N_P}$ increases with $A_P$, suggesting that a small pixel size be used. If, however, the collapse is to be observed, the condensate must fit within the pixel $P$. The optimum pixel size is therefore $\sim 6 \mu m$, which gives $\sigma_{N_P} \approx 200$ atoms for $n_s = 0.1$. Though imperfect, this is low enough to be promising. Unfortunately, $A_P$ must be increased slightly owing to vibrations of the imaging system and the trap. This motion causes the observed location of the cloud center on the camera to vary by 1–2 $\mu m$ from one image to the next. In order for the differential measurement to be accurate, the condensate must be contained in the central pixel with high probability, so a pixel size of 7–8 $\mu m$ should be used, which raises $\sigma_{N_P}$ to $\sim 300$ atoms.

5.1.3 Simulation and Experiment

The time interval $\Delta t$ between images is a critical parameter for the differential measurement. It should be small compared to the elastic collision rate and condensate filling time, so that an observed change in $N_P$ can be confidently attributed to a collapse. If it is too small, however, a collapse is unlikely to be observed and it will be impossible to obtain adequate statistics. It is also important for the cloud to be probed well before it has equilibrated, since the 10% perturbation will drive the gas closer to equilibrium and reduce the rate for the condensate to fill. If this rate is reduced too far, $N_0$ will not reach $N_m$ and no collapses will occur.

Figure 5.1 shows the results obtained when the probing perturbation is simulated with the QBE model. Evaporative cooling was applied as in Fig. 3.10, but 5 seconds after the microwave sweep, $f(E)$ was uniformly reduced by $e^{-0.1}$ to represent the probe pulse. A similar perturbation was applied every half second for a total of six
Figure 5.1  Modeling minimally perturbative imaging. The difference in the simulations reflects a sweep frequency difference of 2 kHz.

 pulses, as indicated by crosses in the Figure. In Fig. 5.1(a), a collapse occurs between the first and second pulses, while in Fig. 5.1(b), \( N_0 \) is initially too low and no collapse takes place. Clearly, for these conditions a collapse will only rarely be observed. If the cloud is probed sooner after the sweep, or if the delay between probe pulses is increased, then the probability of observing a collapse will rise, but the change in signal measured when no collapse occurs will also be larger.

The differential measurement experiment was initially attempted in November 1997, with poor results. At the time, trap loading and evaporative cooling were not well optimized, and the BEC transition occurred at \( N \sim 10^5 \) atoms and \( T \sim 300 \) nK. Cooling was continued to \( T \approx 200 \) nK, and the microwave sweep was then applied. The gas was allowed to equilibrate for 10 s before being probed by a sequence of five pulses. The probe was applied for \( \tau_p = 8 \) \( \mu \)s at a detuning of \( \Delta = 100 \) MHz and intensity \( I_0 = 3 \) mW/cm\(^2\), yielding \( n_s \approx 0.08 \) photons per atom. The pixel area was \((9 \mu m)^2\), giving \( \sigma_{Np} \approx 350 \) atoms.

The experiment was repeated many times and the measured changes \( \Delta N_P \) were collated into a histogram. It was hoped that the occurrence of collapses would result
Figure 5.2 Histogram of measured $\Delta N_P$ values. Only the first two probe pulses are included, since for the parameters used, the probability for a collapse to occur later is low.

in a bimodal distribution with $\Delta N_P$ typically near zero but sometimes near -1200. However, only a single broad distribution was observed, as seen in Fig. 5.2. No signature of the collapse is evident.

The mean of the distribution Fig. 5.2 is $\Delta N_P = -100$ atoms, and the variance is 500 atoms. This is very close to the expected $2^{1/2}\sigma_{N_P}$, and is too large to allow any definite conclusions to be drawn. In addition, the 10 second delay before the first probe was probably too long; at the time, the value of $G_2$ being used in the QBE model was low by a factor of 2, so the model was predicting longer equilibration times than were actually occurring.

The experiment could be readily improved in several ways. The pixel size could be reduced to 7.5 $\mu$m, and a back-thinned CCD camera with $\varepsilon_\gamma = 0.8$ is available to be used. These changes would decrease $\sigma_{N_P}$ to $\sim200$ atoms. Also, if only two probe pulses are applied, the second could be made with a higher intensity, since the last probe need not be nonperturbative. This would reduce the uncertainty in the difference $\Delta N_P$ by an additional factor of $2^{1/2}$. Finally, accurate modeling suggests that a 5 s delay before the first probe and a 0.5–1.0 s delay between probes would be
more appropriate.

5.1.4 Other Imaging Techniques

Although the polarization phase contrast imaging technique used in Chapter 4 is convenient, it does not provide the best possible performance in terms of the relation between \( S/N \) and \( n_s \). Some alternatives are discussed here. Interestingly, each of the methods shows the same scaling behavior

\[
S/N = \Xi \alpha \left( n_s \varepsilon_q \frac{A_P}{\sigma_L} \right)^{1/2},
\]

(5.8)

with only small differences in the constant of proportionality \( \Xi \). This suggests an underlying universal imaging principle, but how such a principle might be formulated is unclear.

The simplest alternative imaging method is to set \( \theta = \pi/2 \) in Eq. (5.4), and consider the term proportional to \( \phi^2 \). Because there is no background signal in the absence of atoms, this is termed a dark-ground imaging technique. The calculation of \( S/N \) proceeds as above, except that all of \( s \) contributes to both the signal and noise, giving \( S/N = s^{1/2} \). A relation of the form (5.8) results, with \( \Xi = 3^{1/2}/8 \), a factor of two worse than the linear imaging result. However, because the signal depends quadratically on \( N_P \),

\[
\frac{\sigma_s}{S} \equiv (S/N)^{-1} = 2 \frac{\sigma_{N_P}}{N_P},
\]

(5.9)

and

\[
\sigma_{N_P} = \frac{1}{2} N_P (S/N)^{-1}.
\]

(5.10)

The two factors of two cancel, and the net uncertainty is unchanged.

A better technique is to use standard phase-contrast imaging, depicted in Fig. 5.3 [19,28,82]. At a focus of the probe laser beam, a spatially small retarding plate is used to shift the phase of the probe by \( \pi/4 \) relative to the scattered field. Typically, the plate also has transmission \( t_P < 1 \). The retarding plate changes the object electric
Figure 5.3  Phase-contrast imaging schematic. A phase plate positioned at a focus of the probe laser beam provides a retardance of $\pi/4$ relative to the scattered light.

field $\mathbf{E} = \mathbf{E}_0 \exp i\phi \approx \mathbf{E}_0 (1 + i\phi)$ to an image field $\mathbf{E} = i\mathbf{E}_0 (t_P + \phi)$. The resulting signal intensity is

$$S_{PC} = I_0 \tau_P (t_P^2 + 2t_P\phi + \phi^2).$$  \hspace{1cm} (5.11)

Calculation of $S/N$ again yields the form (5.8), both for the bright-field case where $\phi^2$ can be neglected and the dark-ground case $t_P = 0$. The two cases again give equivalent results, with $\Xi = 1$ for bright-field and $\Xi = 1/2$ for dark-ground imaging.

Ordinary absorption imaging can also be considered, wherein the reduction of the probe intensity due to light scattered by the cloud is measured directly. For this technique to be useful, the optical density of the cloud must be relatively small, either because $\alpha$ is itself low or because the probe is off-resonance or saturated: otherwise, the light would be completely absorbed and no information would be available.* In this case, $\Xi$ does depend on the probe detuning and intensity, as

$$\Xi_{abs} = \left(1 + 2 \frac{I_0}{I_s} + 4 \frac{\Delta^2}{I_s^2}\right)^{-1/2}. \hspace{1cm} (5.12)$$

If $\alpha < 1$, so that $I_0$ and $\Delta$ can be small, $\Xi \to 1$ and absorption imaging is no more perturbative than phase-contrast imaging. However, nonperturbative imaging is in any case hopeless when $\alpha$ is small.

Finally, the cloud can be illuminated by a transverse beam and the resulting fluorescence imaged. Although a poor approach, this method is interesting because

*Recall from Section 4.1.2 that measuring absorption with an off-resonant probe requires a very high quality imaging system to correctly reconstruct the phase of the scattered light.
(5.8) does not apply. For a resonant probe and isotropic spontaneous emission, the number of counts measured on a pixel will be just the number of photons emitted by the atoms which are imaged onto the pixel, times the collection efficiency,

\[ C = n_s N_F \varepsilon_q \frac{d\Omega}{4\pi} = \varepsilon_q a n_s \frac{A_F}{\sigma_L} \frac{d\Omega}{4\pi}, \]

where \( d\Omega \) is the solid angle subtended by the imaging aperture. The resulting \( S/N = C^{1/2} \), and thus depends on \( \alpha^{1/2} \) rather than on \( \alpha \) directly. Since \( \alpha \) is large and \( d\Omega \) small, the \( S/N \) values obtainable are low. Complications also arise for large \( \alpha \), since the probe light will be unable to penetrate the cloud and scattered photons will have a substantial probability to be rescattered before escaping.

The best imaging method, therefore, appears to be the standard phase contrast approach using a phase plate. In this method, the signal-to-noise is larger by a factor of \( 4/3^{1/2} \approx 2.3 \) than for the current polarization phase contrast technique. Alternatively, the current sensitivity could be maintained while scattering only 20% as many photons per atom. With the additional increase in \( \varepsilon_q \) due to the elimination of the polarizer and the use of the back-thinned CCD, \( n_s \) is reduced by yet another factor of 2. The result of all these improvements is that in the differential measurement, \( \Delta N_F \) could be measured with an accuracy of about 100 atoms while maintaining \( n_s = 0.1 \).

It is interesting to consider whether \( S/N \) would be improved in a trap with stronger or weaker confinement. The condensate size is approximately \( \ell_0 = (\hbar/m\omega)^{1/2} \), in terms of which \( \alpha \sim N_0 \sigma_L/\ell_0^2 \) and \( A_F \sim \ell_0^2 \). Thus the imaging relation (5.8) becomes

\[ S/N \sim \frac{N_0}{\ell_0^2} (n_s \varepsilon_q \sigma_L)^{1/2}. \]

However, \( N_m \) scales as \( \ell_0/|a| \), so the dependence on \( \ell_0 \) drops out. The maximal possible \( S/N \) is thus fundamentally determined only by the scattering length \( |a| \) and, through the cross section \( \sigma_L \), the transition wavelength \( \lambda \). Note, however, that this
is true only so long as $\lambda$ is smaller than $\ell_0$. For larger $\lambda$, the condensate is imaged onto a large area and $A_F$ must be increased.

5.2 Modifying $a$

In Ref. [86], Fedichev et al. calculate that it is possible to significantly change the value of the scattering length $a$, by illuminating the cloud with a laser beam tuned near a Li$_2$ molecular resonance.* This offers the possibility of having detailed experimental control over the strength and sign of the atomic interactions. Unfortunately, the technique has only limited applicability in practice, but it remains interesting both because of its intrinsic novelty and because it can serve as a probe of the collapse dynamics.

5.2.1 Photoassociation

The idea is based on the technique of photoassociation, in which two free atoms are driven to a bound excited molecular state during a collision [88]. The process is illustrated in Fig. 5.4. Photoassociation has been used for precise spectroscopy of high-lying molecular states, since the transition will only occur when the detuning of the photoassociation laser from the atomic resonance, $\Delta_P$, is close to the binding energy of a molecular state $E_v$.† Atoms making the transition have been detected either through further excitation to an autoionizing molecular state and subsequent detection of the ions produced [89], or by spontaneous emission from the molecular state leading to trap loss [90,91]. These measurements have already yielded much information about the diatomic molecular potentials of the alkali elements, notably including the value of $a$ for $^7$Li [29].

The proposal by Fedichev et al. is to tune the photoassociation laser not directly

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*See also [87].
†For the following, $\Delta_P$ is defined as a positive quantity, although the photoassociation beam is tuned red of resonance.
Figure 5.4  Photoassociation schematic. Photoassociation occurs during a collision when a laser is tuned between the ground state of the free atoms and a bound molecular state. To conserve kinetic energy, the transition preferentially occurs when the atomic separation $R$ equals the classical turning point of the excited state, $r_t$.

onto a molecular resonance, but nearby, with $|\zeta_v| \equiv |\Delta_P - E_v| \gg \Gamma$. The probability for the pair of colliding atoms to be found in the molecular state is then small, but nonvanishing. Since atoms in the excited state interact much more strongly than in the ground state, the small amplitude to be excited can have a substantial effect on the details of the collision. In this way, the effective value of $a$ can be altered.

Fedichev et al. calculate the change in $a$ to be

$$\Delta a = -\frac{\Omega_F^2 \tilde{\beta} \zeta_v}{\Delta \varepsilon_v (\zeta_v^2 + \Gamma^2)} r_t, \quad (5.15)$$

where $\Omega_F = \Omega / 2^{1/2}$, $\zeta_v$ is the detuning of the photoassociation beam from the molecular resonance, $\Delta \varepsilon_v = (E_v - E_{v+1})/\hbar$ is the molecular level spacing, and $r_t$ is the outer classical turning point of the excited state. The Rabi frequency $\Omega$ is related to the drive intensity $I_P$ by $\Omega^2 / \Gamma^2 = I_P / I_{sat}$, with $I_{sat} \approx 5 \text{ mW/cm}^2$. The dimensionless parameter $\tilde{\beta}$ expresses the overlap integral $\Theta$ between the ground and excited state wave functions,\footnote{ $\Theta \equiv I_0$ in the notation of [88]} by

$$\tilde{\beta} = |\Theta|^2 \frac{m \Delta \varepsilon_v}{4 \pi \hbar^2 r_t}. \quad (5.16)$$
This notation lends itself to an approximation scheme for $\Theta$, because $\tilde{\beta} \sim 1$. However, we have available accurate values of $\Theta$, so (5.15) can be simplified to

$$
\frac{\Delta a}{a} = A_v \frac{\zeta_v}{\Gamma} \frac{\Omega^2}{\zeta_v^2 + \Gamma^2},
$$

(5.17)

where

$$
A_v \equiv -|\Theta|^2 \frac{m\Gamma}{8\pi \hbar a}
$$

(5.18)

expresses the overlap integral as a dimensionless quantity with no other dependence on $v$.

The definition of $\Theta$ is

$$
\Theta = \xi \int d^3r \phi_0(r)\psi_v^*(r),
$$

(5.19)

where $\xi$ is the polarization overlap between the photoassociation beam and the molecular dipole moment, and $\psi_v$ and $\phi_0$ are respectively the excited and ground state wave functions. The normalization of $\phi_0$ is specified such that $\phi_0 \to 1 - a/r$ as $r \to \infty$. Values for the overlap integral were obtained by R. Côté from numerical solution of the Schrödinger equation in the molecular potential $U$ [92]. However, his wave function normalization was different, so that his values $D_v(E)$ are related to $\Theta$ by*

$$
|\Theta|^2 = \frac{\pi \hbar^3}{m^{3/2}E^{1/2}} |D_v(E)|^2.
$$

(5.20)

The dependence on the energy $E$ of the atom pair cancels for small $E$. The resulting values of $A_v$ are plotted in Fig. 5.5, together with those resulting from the approximations used in [86]. The dips in $A_v$ occur when $r_t$ falls near a node of the ground state wave function, as discussed in [94].

The factor $\xi$ depends on the orientation of the drive laser polarization $\hat{e}$ with respect to the magnetic field of the trap $\mathbf{B}$. When $E_v$ is large compared to the fine-structure splitting of the atomic states, the electronic and nuclear spins of the atoms

*See Ref. [93], Eqs. (4.26) and (4.27), with $\phi_0 = Y_{00}u_x^{K,0}/R$ and $\psi_v = Y_{00}u_x^{v,1}/R$. The relation between $K$ and $E$ is given in the discussion following Eq. (4.14), and $\mu$ is the reduced mass $m/2$. 
Figure 5.5  Photoassociation overlap integral $A_v$, between the zero-energy atomic ground state and the $1^3\Sigma^+_g$ excited molecular state. Circles show the exact calculation by Côté, and squares the approximation suggested by Fedichev et al.

decouple from the light field, and the polarization of the transition depends only on the electronic orbital angular momentum projection $L$. The most easily accessible transition is from the $L = 0$ ground state to an $L = 0$ excited state, with no change in the angular momentum of the atoms.† This transition is therefore driven by $\pi$-polarized light with $\hat{\varepsilon}$ parallel to $\mathbf{B}$. If the $z$-axis is taken to lie along the bias field, the drive laser in the experiment propagates in the direction

$$\hat{k} = \frac{\hat{x} + \hat{y} + \hat{z}}{3^{1/2}},$$  \hspace{1cm} (5.21)

and $\hat{\varepsilon} \cdot \hat{k}$ must be zero. The maximum value of $\xi = |\hat{\varepsilon} \cdot \hat{z}|^2$ is then $2/3$, occurring for

$$\hat{\varepsilon} = \frac{\hat{x} + \hat{y} - 2\hat{z}}{6^{1/2}}.$$  \hspace{1cm} (5.22)

This polarization is perpendicular to that of the probe laser used for phase-contrast imaging.*

†Further discussion of the selection rules for photoassociation transitions can be found in [95].
*The calculations described in this section were originally performed by M. Welling. His results are consistent with those presented here, except that he did not include polarization effects.
Fedichev et al. predict the molecular resonance to be shifted by the presence of the drive laser. This shift is given by

$$\delta \zeta_v = \frac{\beta_s \Omega^2}{2\Delta \varepsilon_v},$$

(5.23)

where $\beta_s$ is approximately $0.8\pi^2(1 - a/r_i) \sim 13$ for $\nu = 69$. For large $\Omega$, this shift can be significant, and must be included in the definition of $\zeta_v$ used in Eq. (5.17). The existence of this light shift is unusual, since it does not occur in a free atomic transition. Accounting for its effect experimentally is not trivial, because at high $\Omega$, power-broadening can make the photoassociation line center difficult to identify, while at low $\Omega$ the light shift is absent. This difficulty might be overcome by measuring the shift at several low-to-moderate intensities and extrapolating.

### 5.2.2 Optimization

Equation (5.17) indicates that arbitrarily large changes in $a$ can be made, by increasing the laser intensity and thus $\Omega$. However, the photoassociation laser also causes a substantial loss of atoms. If this loss rate is too large, the atoms will not survive long enough to respond to the change in $a$. Atoms are lost by two mechanisms. First, atoms in the excited molecular state can spontaneously decay, with the recoil from the emitted photon effectively ejecting the atoms from the trap. Fedichev et al. calculate this process to occur at a rate of

$$R_P = A_v \frac{8\pi \hbar n |a|}{m} \frac{\Omega^2}{\zeta_v^2 + \Gamma^2} = 2A_v \frac{|T_{2B}| n}{\hbar} \frac{\Omega^2}{\zeta_v^2 + \Gamma^2},$$

(5.24)

for atomic density $n$. Côté describes a different calculation in [93], but his result is inaccurate as $T \to 0$. The photoassociation rate can be very large on resonance $\zeta_v = 0$, but falls off more rapidly with $\zeta_v$ than does $\Delta a$. It can therefore be suppressed for a given $\Delta a$ by increasing $\zeta_v$ and $\Omega^2$ together. The second loss process is excitation of solitary atoms in the cloud, and subsequent spontaneous emission. The off-resonant
excitation rate is

\[ R_A = \Gamma \frac{\Omega^2}{4\Delta_P^2}, \]  

(5.25)

and since \( \zeta_v \ll \Delta_P \), \( R_A \) is essentially independent of \( \zeta_v \). Note that, because \( \Delta_P \) is very large, all polarization components contribute to the atomic transition, and no factor of \( \xi \) is required. Since \( R_A \) always increases with \( \Omega^2 \), the drive intensity can not be increased arbitrarily, and thus the photoassociation loss rate \( R_P \) cannot be completely suppressed. For a given \( \Delta a \), the lowest total loss rate \( R_{\text{loss}} = R_P + R_A \) is achieved by choosing \( \zeta_v \) to make \( R_P = R_A \). This optimal detuning is in fact independent of \( \Delta a \) and \( \Omega \), and depends only on the values of \( v \) and \( n \). The calculated values of \( \zeta_v \) are shown in Fig. 5.6(a).

The density \( n \) is fixed by the number of atoms in the condensate, but the molecular level \( v \) can be chosen to minimize losses. For the optimal value of \( \zeta_v \), the ratio \( \eta \equiv \Delta a / R_{\text{loss}} \) is independent of the drive intensity \( \Omega^2 \). This figure of merit is plotted in Fig. 5.6(b), and shows that lower-lying levels are generally more suitable. However, because the overlap \( \Theta \) decreases rapidly with \( v \), the required intensity becomes very large, as seen in Fig. 5.6(c). The largest conveniently attainable intensity is \( \sim 20 \text{ kW/cm}^2 \), limiting \( v \) to be greater than \( \sim 53 \) if \( \Delta a / a = 3 \) is to be imposed. The best molecular level meeting this requirement is \( v = 54 \), but the lower intensity required for \( v = 69 \) is more easily achieved and the \( v = 69 \) level has already been observed, as reported in [96]. The parameters for the three local maxima in \( \eta \) are summarized in Table 5.1.

5.2.3 Experiments

Even with optimization, the loss rate imposed by the photoassociation beam is large, with a 1/\( e \) decay time of 9 ms for the \( v = 69 \) level. This is comparable to the fastest condensate filling times observed in the QBE model, but the substantial loss of atoms during filling would likely slow or halt condensate growth. In addition, such
Figure 5.6 Photoassociation parameters for modifying $a$. (a) The optimum value of the detuning from the molecular resonance, $\zeta_v$, chosen to make $R_P = R_A$. (b) Figure of merit parameter $\eta = \Delta a/R_{\text{loss}}$, normalized to unity at $v = 69$. (c) Drive laser intensity required to make $\Delta a/a = 3$. (d) Decay time $R_{\text{loss}}^{-1}$ when $\Delta a/a = 3$. The line shows the trap response time $\omega^{-1}$. Note that the curves in (a), (b), and (d) are proportional to each other, and that $I_P$ and $R_{\text{loss}}$ are proportional to $\Delta a/a$. For all four plots, $\xi = 2/3$ and the atom density was fixed at $2.3 \times 10^{12}$ cm$^{-3}$ corresponding to $N_0 = 1000$ atoms.
Table 5.1  Photoassociation parameters for molecular levels with high $\eta$. The optimal molecular detuning $\zeta_v$, laser power $I_P$ and loss rate $R_{\text{loss}}$ are calculated assuming that $N_0 = 1000$, $\Delta a/a = 3$, and $\xi = 2/3$. The binding energy $E_{\text{gg}}$ is given relative to the $2P_{3/2}$ atomic state, but the energies of the deeper levels are less accurately known as they have not been observed.

Rapid filling occurs only when the gas is driven very far from equilibrium, and the accuracy of the fitting procedure is then poor. It is therefore not feasible to increase $a$ and measure the increased value of $N_m$, since large $N_0$ values could never be observed. It is, however, possible to make $a$ more negative. If $N_0$ is moderately large when this is done, the condensate will immediately become unstable and collapse. Since the collapse process occurs quickly, the photoassociation beam need only be applied for a short time, and losses can be avoided.

The laser must, however, be applied long enough to take effect. As seen in Fig. 2.6, the collapse typically occurs in a time on the order of $\omega^{-1} \approx 1 \text{ ms}$. Fig. 5.6(d) shows the decay time $R_{\text{loss}}^{-1}(v)$, along with $\omega^{-1}$. Another constraint arises from the uncertainty principle, in that if the condensate is to react to a change in energy $\Delta U$, it will generally require a time $\Delta t \sim h/2\Delta U$ to do so. The energy shift $\Delta U = |T_{2B}|n\Delta a/a$, so for low $n$ the response time will be long. However, the condensate will only collapse if $N_0$ is greater than the modified maximum number $N_m' = N_m/(1 + \Delta a/a)$. If $\Delta a/a = 3$, the lowest density of interest will be $n \approx 7 \times 10^{11} \text{ cm}^{-3}$, which gives $\Delta t = 1.4 \text{ ms}$. A reasonable pulse duration to attempt experimentally is therefore 3 ms, which gives a total atom loss of 30%.

Several experiments based on this technique can be envisioned. The simplest is to observe the change in $N_m$ by applying the drive laser and then immediately probing the cloud to determine $N_0$. Repeating this procedure will generate a histogram of observed values, as in Section 4.2. If the predicted effect occurs, the distribution
will be truncated at \( N'_m \), providing a very clear signal. In addition, the change in the shape of the histogram for \( N_0 < N'_m \) conveys useful information about the collapse process. Specifically, if the undisturbed distribution of \( N_0 \) values is \( P(N_0) \), the measured distribution for \( N_0 < N'_m \) will be approximately

\[
P'(N_0) = P(N_0) + P_R'(N_0) \int_{N'_m}^{N_m} dx \ P(x),
\]

where, as in Eq. (4.21), \( P_R'(N_0) \) is the probability for \( N_0 \) atoms to remain in the condensate after a collapse with the modified value of \( a \). Since \( P(N_0) \) is known, measurement of \( P' \) can be directly used to find \( P_R' \). Such a calculation requires that the effect of the photoassociation losses on \( P \) be known, but this can be measured by observing the histograms resulting when \( \Delta a > 0 \) since in that case no collapse can be induced.

A second experiment is to use the photoassociation beam to probe the filling dynamics of the condensate. If \( \Delta a / a \) is large, then immediately following the drive pulse \( N_0 \) is necessarily small, so by delaying various times before probing the subsequent growth in \( N_0 \) could be mapped out. If repeatability is adequate, it may be possible to follow the evolution \( N_0(t) \) through several collapse/fill cycles, and thereby directly observe the predicted dynamical behavior. Note however, that if \( P_R' \) has appreciable width, as is suggested by the histogram data, then the repeatability of the conditions after the drive pulse may be too low for this approach to succeed.

Experiments to observe the modification of \( a \) are currently underway. The photoassociation beam is generated by a MOPA laser diode system, which provides \( \sim 100 \) mW of laser power. The beam is focussed onto the center of the trap with a waist of 60 \( \mu \)m, so that to achieve an intensity of 580 W/cm\(^2\), a total power of 30 mW is required. By passing the drive laser through the cloud at a small angle to the probe beam, the imaging system can be used to accurately position the drive beam with respect to the condensate, by simply overlapping the images on the camera.

The \( v = 69 \) photoassociation resonance has been observed, by evaporatively cool-
Figure 5.7 Photoassociation resonance for the \( v = 69 \) level of the \( 1^3\Sigma_g^+ \) manifold, showing the relative number of atoms remaining after a drive pulse at frequency offset \( \zeta_0 \). For each \( \zeta_0 \), the experiment was repeated several times, and the error bars indicate the spread in values observed. Even on resonance, not all the atoms are lost, perhaps because of saturation effects not included in the calculations.

Placing the cloud to \( T \approx 1 \mu \text{K} \) and applying a 25 ms pulse of \( \pi \)-polarized drive light at an intensity of 70 mW/cm\(^2\). The number of atoms remaining in the trap was then probed, with the results shown in Fig. 5.7. Further progress toward modifying \( a \) will provide an exciting test of our understanding of BEC with \( a < 0 \).

5.3 Rapid Quenching of BEC

The previous section discussed the possibility of using an induced collapse as a time marker from which the evolution of \( N_0 \) could be reconstructed by repeated experiments. A similar time marker exists naturally in the initial onset of BEC. In normal evaporative cooling, small fluctuations in initial conditions cause the time at which BEC occurs to vary from run to run, but the microwave sweep cooling technique can abruptly increase the phase-space density of the gas at a specific time. Potentially, this can serve to quench the gas into the degenerate regime, and the subsequent time evolution can be followed [97].
If this method is to succeed, it is necessary for the microwave sweep to provide a large enough increase in the phase-space density $g$. In the nondegenerate limit, this increase is readily calculated. For a gas initially containing $N$ atoms at temperature $T$, the energy distribution function is

$$f(E) = N(h\omega\beta)^3 e^{-\beta E},$$

(5.27)

with $\beta = 1/k_B T$. If all atoms with energy greater than $E_c$ are removed, the number becomes

$$N' = \int_0^{E_c} dE \frac{g(E)f(E)}{\int_0^{E_c} dE g(E)f(E)}$$

$$= N \left[ 1 - e^{-\eta_c} \left( 1 + \eta_c + \frac{\eta_c^2}{2} \right) \right] \eta_c \to 0 \, N \frac{\eta_c^3}{6}. \quad (5.28)$$

for $\eta_c = \beta E_c$. Similarly, from the average energy of the gas particles

$$T' = T \frac{N}{N'} \left[ 1 - e^{-\eta_c} \left( 1 + \eta_c + \frac{\eta_c^2}{2} + \frac{\eta_c^3}{6} \right) \right]$$

$$\eta_c \to 0 \, T \frac{\eta_c}{4} = \frac{E_c}{4k_B}. \quad (5.29)$$

The maximum increase in $g = 0.83 N(h\omega\beta)^3$ is therefore a factor of $64/6$, obtained for $\eta_c \to 0$. Figure 5.8 shows the exact results as a function of $\eta_c$. For very small $E_c$, this calculation is incorrect, since the discrete nature of the density of states becomes important. For instance, if $E_c = \hbar\omega/2$, no atoms should remain trapped and $g'$ should be zero. This effect can be safely neglected while $E_c \gg \hbar\omega \approx 150$ Hz.

Although not unlimited, the increase in $g$ provided by the sweep is large enough to take a near-degenerate gas into the degenerate regime. It is also necessary, however, that $N'$ be large compared to $N_m$, so that one or more collapses can occur. For example, if $N \approx 1.5 \times 10^6$ and $T \approx 1$ $\mu$K, then $g \approx 0.4$ initially. To achieve $N' = 2 \times 10^4$, the endpoint of the sweep must be $E_c = 0.485 k_B T \approx 20$ kHz above the trap bottom, which reduces the average energy of the atoms to 0.12 of the initial value. In the nondegenerate limit, this would give $g \approx 3.4$, but because $T > \langle E \rangle / 3$
for a degenerate gas, the actual value of $\varrho$ is lower. Nonetheless, the gas is definitely quenched into degeneracy, with $\varrho' = 1.16$. A larger increase can be obtained by starting the sweep closer to the BEC transition and by reducing further the number of atoms. Several example results are shown in Table 5.2.

The quenching procedure can be simulated with the QBE model, with results as shown in Fig. 5.9. One collapse usually occurs after the quench and its dynamics could, in principle, be mapped out in time. Unfortunately the trajectories are sensitive to the precise value of $E_c$, which varies experimentally by $\sim 3$ kHz due to fluctuations in the trap bias field. As the Figure shows, these fluctuations are expected to wash out the location of the collapse. Note that although the dotted curve represents

<table>
<thead>
<tr>
<th>$N$</th>
<th>$T$</th>
<th>$\varrho$</th>
<th>$N'$</th>
<th>$\eta_c$</th>
<th>$E_c$</th>
<th>$\langle E' \rangle/3k_B$</th>
<th>$T'$</th>
<th>$\varrho'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.5 \times 10^6$</td>
<td>1.0 $\mu$K</td>
<td>0.42</td>
<td>$2.0 \times 10^4$</td>
<td>0.485</td>
<td>20.2 kHz</td>
<td>118 nK</td>
<td>169 nK</td>
<td>1.16</td>
</tr>
<tr>
<td>1.0</td>
<td>0.7</td>
<td>0.81</td>
<td>1.0</td>
<td>0.436</td>
<td>12.7</td>
<td>74</td>
<td>126</td>
<td>1.40</td>
</tr>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0.78</td>
<td>2.0</td>
<td>0.693</td>
<td>8.6</td>
<td>100</td>
<td>162</td>
<td>1.31</td>
</tr>
</tbody>
</table>

**Table 5.2** Initiating BEC with sweep cooling. Final temperatures $T'$ are calculated from the final number $N'$ and average energy $\langle E' \rangle$ using the equilibrium Bose-Einstein distribution function.
Figure 5.9  QBE simulation of quenching BEC. For all three curves, $N$ is initially $6 \times 10^5$ and $T$ initially 600 nK. The dashed curve is truncated at $E_c = 70\hbar \omega = 10$ kHz to give $N' = 2.5 \times 10^4$ atoms, the solid curve has $E_c = 60\hbar \omega = 8.7$ kHz and $N' = 1.9 \times 10^4$ atoms, and the dotted curve has $E_c = 50\hbar \omega = 7.2$ kHz and $N' = 1.3 \times 10^4$.

Figure 5.10  Observation of the onset of BEC. A microwave sweep to $E_c \approx 20$ kHz was applied to a nondegenerate gas of $N \approx 4.3 \times 10^5$ atoms at $T \approx 650$ nK. The gas was allowed to equilibrate for time $\tau$, and then probed. Points represent the average $N_0$ values obtained for several runs, and the error bars indicate the mean variations observed.
Figure 5.11  Additional quenching experiments. Experiments were performed as in Fig. 5.10, but applying the sweep when $N \approx 4.7 \times 10^5$ and $T \approx 580$ nK. Because of fluctuations in the trap bias field, the effect of the sweep varied. To reflect this, the three plots show the results obtained for different ranges of measured number $N'$: (a) $2.6 \times 10^4 < N' < 3.4 \times 10^4$ (b) $1.8 \times 10^4 < N' < 2.6 \times 10^4$ (c) $1.0 \times 10^4 < N' < 1.8 \times 10^4$. The differences in the plots shows the sensitivity of the behavior to $E_c$. 
deeper quenching than the solid curve, the collapse is seen to occur later. This occurs because with lower $N'$, the elastic collision rate is lower for the dotted curve.

Although the prognosis for success was poor, the quenching experiment was attempted, with results shown in Figs. 5.10 and 5.11. In Fig. 5.10, $g$ was initially $\sim 0.4$, well above the critical temperature. After the quench, a delay is observed before the onset of BEC as discussed in Section 3.3.2, but no evidence for the collapse itself is seen. For Fig. 5.11, the gas was initially closer to the phase transition, with $g \approx 0.7$, so that quenching had a more immediate effect. Although the decline in $N_0$ observed at $\tau \approx 10$ s in Fig. 5.11(b) is suggestive of a collapse tending to occur at that time, the statistical significance is low.

Clearly, if this approach is to be further pursued, the stability of the trap magnetic field must be improved. Possible means for such improvements are discussed in Appendix C.

### 5.4 Resonant Compression

One final way to probe the behavior of the condensate is to alter the potential in which it is confined, and observe the response. Because the trap is constructed from permanent magnets, large changes in the field cannot easily be imposed. However, by modulating the trap at a frequency near a resonance of the condensate, it may be possible to significantly alter its behavior using only a weak field. In principle, this could allow the excitation spectrum of the condensate to be explored.\(^*\)

For a spherically symmetric trap, the frequency of the breathing mode $\omega_B$ is shown in Fig. 2.4. The ideal experiment would be to apply an external field modulating the trap confinement at a frequency $\omega_M$ in a range near $2\omega$, and then measure the distribution of $N_0$ values. The modulation will have little effect except when $\omega_B = \omega_M$, but if $\omega_M$ is slightly less than $2\omega$, the condensate will at some point come into

\(^*\)For comparison with the $a > 0$ case, see Refs. [15, 16].
Figure 5.12  Effect of resonant compression in a symmetric trap. The trap oscillation frequency of $\omega = 2\pi \times 140$ Hz was modulated with an amplitude of 1%, at the frequency $\omega_M$. The condensate filled to $N'_0$ before collapsing.

resonance as it fills. If the modulation is sufficiently strong and condensate growth sufficiently slow, the additional energy coupled to the the breathing mode can induce the condensate to collapse. If this situation could be achieved, it would be possible to experimentally determine $\omega_B$, since for a modulation frequency resonant when $N_0 = N'_0$, condensates will only be observed with $N_0 < N'_0$. The dependence of the cutoff $N'_0$ on $\omega_M$ is therefore the functional inverse of the dependence of $\omega_B$ on $N_0$.

This procedure can be modeled using the variational method of Section 2.2.2. A filling term $+R_0$ is added to Eq. (2.49) to represent roughly constant condensate growth, and it is found that for $R_0 \approx 100$ atoms/s, the condensate can be induced to collapse if the trap frequency $\omega$ is modulated by $\sim 0.5\%$. The dependence on $\omega_M$ of the maximum $N_0$ value observed is shown in Fig. 5.12. The cutoff value $N'_0$ does indeed satisfy $\omega_B(N'_0) \approx \omega_M$, as desired.

The experimental implementation of this scheme is complicated because it is difficult to compress the trap in a spherically symmetric way. The trap field itself is
approximately
\[
B_{\text{trap}} = B_1 x \dot{x} - B_1 y \dot{y} + \left( B_0 + \frac{1}{2} B_2 z^2 \right) \dot{z},
\]
(5.30)
with \( B_0 = 1000 \) G, \( B_1 = 1060 \) G/cm, and \( B_2 = 855 \) G/cm². The magnitude of the field is
\[
B_{\text{trap}} = B_0 + \frac{1}{2} \frac{B_1^2}{B_0} (x^2 + y^2) + \frac{1}{2} B_2 z^2,
\]
(5.31)
and is related to the trapping potential by \( V = -\mathbf{m} \cdot \mathbf{B}_{\text{trap}} = \mu_B B_{\text{trap}} \). External coils can be used to adjust slightly the values of \( B_0 \), \( B_1 \), and \( B_2 \), which changes the trap oscillation frequencies according to
\[
\frac{\delta \omega_z}{\omega_z} = \frac{\delta B_2}{2B_2}
\]
(5.32)
and
\[
\frac{\delta \omega_\rho}{\omega_\rho} = \frac{\delta B_1}{B_1} - \frac{\delta B_0}{2B_0}.
\]
(5.33)
The problem arises because the terms \( \delta B_0/B_0 \), \( \delta B_1/B_1 \) and \( \delta B_2/B_2 \) are always dissimilar. If the modulating field \( \delta \mathbf{B} \) is generated by a single coil, or by a set of coils of similar diameter \( D \) and distance from the trap \( L \), then, roughly,
\[
\delta B_0 \sim \frac{1}{(L^2 + D^2)^{3/2}},
\]
(5.34)
\[
\delta B_1 \sim \frac{dB_0}{dL} \sim \frac{L}{(L^2 + D^2)^{5/2}},
\]
(5.35)
and
\[
\delta B_2 \sim \frac{dB_1}{dL} \sim \frac{L^2}{(L^2 + D^2)^{7/2}}.
\]
(5.36)
Thus,
\[
\frac{\delta B_1}{\delta B_0} \sim \frac{\delta B_2}{\delta B_1} \sim \frac{L}{L^2 + D^2}
\]
(5.37)
in each case. For the trap field, \( B_1/B_0 \sim B_2/B_1 \sim 1 \text{ cm}^{-1} \). So, if the values of \( \delta B_i/B_i \) are all to be comparable, coils must be used with \( L/(L^2 + D^2) \) similarly on the order of \( 1 \text{ cm}^{-1} \). Note that this is in fact the case for the trap magnets themselves, which are a few cm in diameter and located a few cm from the atoms.
This condition clearly cannot be achieved without putting the coils inside the vacuum chamber, which is technically very challenging because of the ultra-low pressures required for the experiment. For coils outside the chamber, $L$ is larger than $\sim 10 \text{ cm}$, so the effect of $\delta B_0$ will always dominate and the condensate will primarily be compressed radially. In principle, symmetry could be restored by using a relatively small coil to generate the curvature, a medium-sized one for the gradient, and a large one to cancel the bias. However, the power requirements for such a set of coils would be prohibitive.

The experiment may, of course, be performed using only radial compression. If a coil with $D = 30 \text{ cm}$ is located 10 cm from the trap center, modulation of $\omega_\rho$ by 1% requires a current of 350 A-turns, an achievable amount. However, modeling the response of the condensate in an asymmetric situation has proven difficult. Apparently chaotic behavior is observed, but it is not yet clear whether this is fundamental or an artifact of the numerical methods used.
Chapter 6
Conclusions

The purpose of this concluding chapter is two-fold. First, it serves to summarize the results reported and indicate what has been learned in the course of this research. Second, it points out what is not yet known, and offers future directions for the research to take.

From the theoretical side, our primary contribution has been the results of modeling the quantum Boltzmann equation. For nondegenerate gases, this has provided optimized evaporative cooling trajectories to support the experiments. Perhaps more interesting is the application to degenerate gases, where a cycle of condensate growth and collapse is observed. Although this behavior can be qualitatively understood in terms of simpler models, the full calculation of the QBE is useful, as it gives detailed and quantitative predictions which can be meaningfully compared to experiment. Of particular interest are the frequency of the oscillations, the time for which they continue, and the characteristic shape of condensate growth after a collapse.

Many questions remain to be answered, however. There is as yet no good theoretical model of the collapse process itself, which limits the accuracy of the QBE results. First, an accurate prediction for the state of the condensate after a collapse is needed, with both the number of atoms remaining and their excitation energy being important. In addition, the effect of excitations on the subsequent dynamics of the condensate must be better understood. This can likely be accomplished through further analysis of the NLSE, although the correct way to account for losses from inelastic collisions is not yet clear.

A more difficult question is how to treat the interaction between the condensate and the noncondensed cloud. This interaction works two ways: the addition of atoms to the condensate as the gas cools will both depend on and affect the state of the
condensate. Modeling this situation requires the inclusion of both coherent and incoherent dynamics, so neither the NLSE nor the QBE will suffice by themselves. Stoof has formulated a scheme in which these effects are combined in a single Fokker-Plank equation governing the condensate [44], but a full and realistic solution to the problem remains challenging. A simpler model accounting only for the effect of mean-field interactions on the filling rate would be to solve the QBE with a variable condensate energy and density of states.

On the experimental side, the central result of this work is the measurement of the distribution of $N_0$ values described in Chapter 4. Together with the determination of the equilibration time, this provides strong, if indirect, evidence that the theoretical predictions for the condensate dynamics are at least qualitatively correct. It is certain, for example, that $N_0$ does not simply rise to its maximum value and remain clamped there as cooling proceeds. Furthermore, the detailed shape of the distributions allows quantitative comparison with theory. From this comparison, it is clear that none of the currently available theories are completely accurate.

The data do suggest that approximately 200 atoms remain in the condensate after a collapse, but that this number fluctuates considerably. Also, it seems that either the condensate growth accelerates faster than predicted for large $N_0$, or that collapses can occur at considerably lower $N_0$ than predicted. These findings will provide benchmarks for future theoretical efforts.

In addition, several experiments were proposed to further investigate the condensate dynamics. Although a clear and direct measurement of the dynamics will never be available, by combining several approaches a complete picture can likely be worked out. The primary experimental improvement which could be made to facilitate these experiments is a reduction of the trap magnetic field fluctuations responsible for most of the variations in evaporative cooling. A less direct but similarly important improvement would be to increase the reliability and robustness of the laser sources, so that
more time could be spent collecting data and less on preparing the apparatus.

The experiments suggested in Chapter 5 all used optical imaging as the fundamental detection method. An alternative approach to studying the collapse dynamics would be to directly detect the molecules produced by three-body recombination. Only a few hundred molecules are produced in each collapse, but it might be possible resonantly photoionize them with high efficiency, and once ionized, they could be readily detected and counted. Because of the large inhomogenous trap magnetic field, the motion of the ions en route to the detector will be complicated. However, although the molecules are hot compared to the original atom cloud, they are cold on the scale of ion experiments, with energies on the order of eV. If rapidly accelerated, they can be considered to be initially at rest, and their trajectories then calculated with some accuracy. A large position-sensitive detector such as a microchannel plate would anyway allow for a range of trajectories to be observed. Some potential problems with this experiment are the effect of the photoionizing lasers on the atom cloud, the unknown distribution of molecular states produced by recombination, and the level of background counts produced by off-resonant ionization of unwanted species. If successful, however, the experiment would directly indicate the frequency at which collapses occur and the number of atoms lost to recombination in the collapse.

Other interesting directions for future research would require more substantial changes to the apparatus. If an electromagnetic, rather than permanent magnet, trap were used, the variation of $N_m$ with the trap confinement and asymmetry could be verified. Typically such traps can be made very asymmetric with $\omega_z \ll \omega_r$ [98]. This would permit the study BEC in a quasi-one dimensional system, which is interesting when $a < 0$ because soliton-like solutions to the NLSE exist [99]. This can be inferred from the results of Section 2.2.3, where it was observed that in a pencil-like trap, the condensate does not collapse until it has contracted to a nearly spherical shape. So, if $1 \ll N_0 < N_m$, the condensate will be stable, but its axial length will be
determined by its interactions rather than by the trap confinement. In the limit \( \omega_z \to 0 \), this is a soliton. For small but nonzero \( \omega_z \), the condensate would consist of a localized density peak which could be driven to oscillate in the trap without dispersion. Observation of such behavior would be interesting in itself, but it also would enable unique experiments on the coherence of the condensate if two or more solitons could be established in the same trap and made to interfere.

In a trap with a bias field smaller than 144 G, the \((F = 1, m_F = -1)\) state is trappable. The scattering length for this state is positive [29], which would permit exploration of the strongly interacting regime unavailable when \( a < 0 \). Evaporative cooling of atoms in this state would be significantly different, because with \( a_{(1, -1)} = 0.26 \text{ nm} \), the collision cross section is reduced by a factor of 30. However, because the \((1, -1)\) state is in the lower hyperfine manifold, dipolar relaxation is suppressed so that losses during cooling will be much lower. If dipolar relaxation were eliminated under the conditions of the current experiment, trap loss would be dominated by background-gas collisions and reduced by a factor of \( \sim 20 \) initially and \( \sim 200 \) at the end of evaporative cooling, assuming \( G_1 \approx 10^{-4} \text{ s}^{-1} \). Cooling could therefore be more efficient overall, but would be significantly slower given similar initial conditions.

If BEC could be obtained in the \((1, -1)\) state, it would be the most weakly interacting condensate composed of alkali atoms, making it ideally suited for studying the transition from the ideal-gas to the Thomas-Fermi regime. The scattering length for \(^1\text{H}\) is even smaller, but because Lyman-\(\alpha\) laser light is difficult to generate and manipulate, observing the behavior of \(^7\text{Li}\) condensates will likely be much easier.

On a personal level, I feel that the ultimate endpoint of investigations of Bose-Einstein condensation will be a greatly increased understanding of the nature of quantum mechanics and the meaning of the wave function. In particular, these studies are one way of addressing the question raised by Schrödinger in his famous cat paradox, as to how quantum mechanics can be meaningfully applied to a macroscopic
system. Although we are as yet far from experimentally answering this question, the observation and understanding of BEC does mark a significant step in that direction. I have found it very satisfying to participate in such an endeavor.
Appendix A
Imaging

The imaging system used to collect data here is the fourth experimental iteration. The first consisted of a single doublet-meniscus lens combination, and had substantial aberrations. It was used to obtain our original evidence for BEC [12]. The second was a commercial f/6.4 reducing lens, the 150 mm Rodenstock Rodagon. No published data was obtained with this lens. The third system consisted of a matched pair of achromatic doublets relaying an image of the cloud outside the chamber with unit magnification, together with a conventional microscope objective. This system was used for the data described in Refs. [25, 78, 80]. It was modified for the current experiments by using a special long-working-distance microscope objective from Mitutoyu, and by adding a corrective lens after the doublet pair. The Mitutoyu objective has a large aperture to avoid clipping the probe laser beam, which reduces spatial noise in the image. The corrective lens reduces the spherical aberrations of the achromat pair.

This Appendix describes several technical aspects of this Mark IV imaging system. Section A.1 explains the construction of the system and Section A.2 its characterization. Section A.3 details the image analysis procedure and simulated image tests.

A.1 Lenses

A scale drawing of the imaging system is shown in Fig. A.1. The two achromats are \( f = 160 \) mm doublets, Melles Griot P/N 01 LAO 146, with a single-layer MgF anti-reflective coatings. They are separated by approximately \( 2f \). The microscope objective is a Mitutoyu M Plan APO 10. The five lens elements making up the objective were taken from two assemblies, using in each position the element with the fewest surface imperfections. The objective is designed to work at infinite conjugate
Figure A.1 Scale drawing of Mark IV imaging system, showing orientation of various elements. The atom cloud being imaged is to the left, and the microscope objective and CCD camera are to the right. A more detailed drawing of the correction lens is shown in Fig. A.3. The magnification of the system is $M = 1.12$.

ratio, so a second "tube lens" is required to focus the image onto the camera. The focal length of the tube lens sets the magnification of the system. The tube lens is attached to the camera via a Nikon bayonet-style F-mount, so it can be mounted and demounted easily. Two different tube lens assemblies were used. High magnification was obtained with an $f = 250$ mm singlet, Newport P/N KPX109. Low magnification was obtained with an $f = 50$ mm doublet, Newport P/N PAC040. However, in order to achieve a large field of view with the low magnification system, an additional pair of identical lenses was required to relay the image to the CCD. These tube lens systems are illustrated in Fig. A.2. Note that, due to a design flaw, the focal length of the tube lens systems are not quite correct. The microscope objective is therefore not operated exactly at infinite conjugate, but focuses an estimated $3$ m from the exit aperture. This does not observably impair the system performance. The overall system magnification is $M = 14.25 \pm .2$ with the $f = 250$ mm tube lens, and $M = 3.7$ with the $f = 50$ mm tube lens. The pixel size on the CCD camera is $19 \mu m$.

The corrective lens was designed by J. Queneille using the OSLO lens design software by Sinclair Optics, and constructed by V. Bagnato and coworkers at the University of São Paulo, Brazil. Its specifications are shown in Fig. A.3. According to the OSLO calculations, the lens should virtually eliminate all spherical aberrations, and give performance indistinguishable from a diffraction limited system. As discussed in Section 4.1.2 and in Section A.2 below, the actual performance of the system is good but effects of aberrations can clearly be seen. We believe the most likely explanation for this discrepancy is that the lens was not manufactured to the
Figure A.2  Schematic of tube lens system. The thick lens represents the microscope objective, with an effective focal length of \( \sim 20 \text{ mm} \). The hatched surface at the right represents the CCD. (a) High magnification system gives \( M = 12.7 \), for a total magnification of 14.25 in combination with the relay lenses. (b) Low magnification system gives \( M = 3.3 \), for a total of 3.7. In this system, the final doublet actually serves as the tube lens, while the first two are a relay system required to maintain the field of view without moving the CCD closer to the objective.

Figure A.3  Details of correction lens. The lens is made of BK7 glass, with a refractive index of 1.51.
specified tolerance. When the lens was first made, it did not improve the imaging performance, but rather worsened it slightly.* The lens was returned to Brazil to be reground, and then performed better. However, given the significant errors made in the first attempt, it seems reasonable to suspect more subtle errors are still present. Clearly, a fault-tolerant design should be an important consideration when developing a lens system. The correction lens changes the measured magnification of the achromat pair from $0.997 \pm 0.004$ to $1.127 \pm 0.007$.

The lenses and camera are mounted on 1/2-inch thick aluminum plates on stainless steel risers. The correction lens and second achromat are jointly held in an aluminum mount, and both achromats are mounted on a single plate, with an iris aperture, the imaging polarizer, and a folding mirror between the two. The iris provides the limiting aperture of the system, and is usually set to a diameter of 2.8 cm, giving an $f/5.7$ system. It prevents light scattered by the lens holders from reaching the camera. The polarizer is a polymer dichroic sheet from Meadowlark Optics, P/N DPM-200-HN38S, with 3.05 cm clear aperture and measured transmission of 85% for polarized light at 671 nm. The fold mirror changes the beam path to be horizontal, and can be used to finely adjust the position of the image on the camera. The camera, microscope objective, and tube lens are mounted to an aluminum box on a second plate. The system as a whole is focused by adjusting the position of the second achromat and correction lens. Their mount is attached to a micrometer-driven slide translator for that purpose.

The lenses are aligned by ensuring that the probe beam passes through their centers. This is accomplished using paper cross-hair targets placed on the lens mounts, and by minimizing deflection of the reflected and transmitted beams. The achromats are centered with an estimated 1 mm accuracy in this way. The microscope/camera assembly is less sensitive, and is only roughly centered to within a few mm.

*Surprisingly, performance could be improved by using the lens in reverse orientation, but not to a diffraction-limited level.
Focusing the imaging system in situ presents some difficulty, because with $N_0$ varying, there is no fixed small object to use as a target. The best procedure we have found is to roughly focus the system by observing the cloud with absorption imaging, in which the phase-contrast polarizer is removed and a modest detuning $|\Delta| \approx 3\Gamma$ is used. When the system is off focus, images with positive and negative $\Delta$ are significantly different, so an approximate focus can be found by alternating $\Delta$ and making the images as similar as possible. This procedure is limited by the effects of aberrations, which at the end distort the images for both signs of $\Delta$. The focus can be adjusted precisely by taking a set of $\sim 30$ images at different focal positions, and maximizing the mean value of $N_0$ for the set. The maximum can be located in this way to approximately $\pm 25$ $\mu$m. Once set, the focus is not expected to drift significantly, because the location of the image center on the camera does not drift by more than a few $\mu$m. The focus has been observed to change over long periods (\~months), but this could have been caused by the experimenters bumping the imaging optics, rather than mechanical drift or sag. In either case, the location of the image center should serve as an indicator of the steadiness of the system.

The CCD camera is a Photometrics AT200 system, with a CH250 camera head. The CCD element itself is a Thompson TH7895B Grade 2 chip, and the specified A-to-D gain of the controller is 1.65 electrons/count in high-gain mode where we ordinarily operate. The sensitivity of the camera was measured to be 0.33 counts/photon, implying a quantum efficiency $\varepsilon_q$ of 0.5 electrons/photon, in reasonable accord with the specified value of 0.4. The precise value of the quantum efficiency is not needed, because the shot noise in a measurement reflects the number of electrons generated, rather than the number of photons incident.
A.2 Characterization

The lens system was tested by imaging 671 nm laser light emitted from an optical fiber, 3M P/N FS-SN-3224, purchased from Thor Labs. The fiber supports only a single Gaussian mode, with 1/e-intensity radius of 1.2 μm. The objective assembly was tested separately, and imparted no observable artifacts to the image. The relay lens system was tested using a Pulnix CCD video camera and an 60× Newport microscope objective. The 60× objective by itself also gave a true image. The relay system both broadened the image and imposed an observable set of rings around the central spot. A radial profile of the signal is shown as the data points in Fig. A.4.

Calculation of the effects of the limited aperture and of lens aberrations in described in Section 4.1.2. A Gaussian object field was convolved with the point-transfer function (4.13), and compared with the measured data. The solid curve shows the best fit obtained by adjusting the aberration parameters $C_2$, $C_4$, and $C_6$ to agree with the data. Although agreement is good, some question arises because the observed image did not change as predicted when the focal position was changed. In particular,

![Graph](image)

**Figure A.4** Image of optical fiber using Mark IV imaging system. The intensity is normalized to unity at the center of the image. The solid curve is a fit yielding $C_2 = 0.257λ/cm^2$, $C_4 = -0.355λ/cm^4$, and $C_6 = 0.165λ/cm^6$. 
the fitted values of the aberration coefficients correspond to an off-focus system, while the data shown were taken at the experimentally determined best focus. The best focus was defined primarily as the position giving the maximum central intensity, and secondarily as the position minimizing the intensity of the first diffraction ring. It could be located experimentally to ±10 μm. This discrepancy was not investigated thoroughly, but one possible explanation is that the cylindrical expansion of the phase error is inadequate. If the lens surface were not sufficiently smooth, the phase error might vary randomly from point to point. Little gross asymmetry in the image was observed, which rules out low-order asymmetric aberrations such as coma or astigmatism. If, however, the length scale for the phase fluctuations is small compared to the lens diameter, the asymmetry would presumably wash out, but the variations would nonetheless cause some blurring of the image. This problem deserves further study, but perhaps the best course of action would be to have the correction lens remade by an established manufacturer with guaranteed standards, and determine whether agreement with the predicted performance can be obtained. Our lab lacks the proper tools for detailed analysis of lens characteristics.
Figure A.6  Comparison of Mark III (crosses), Mark IV (points), and diffraction-limited (solid curve) imaging. The three plots are normalized to represent the same total power in the image.

To see the effect of the correction lens, the Mark III lens system was also characterized. The imaging and best-fit calculation are shown in Fig. A.5. For the aberration parameters, the fit gives

\[ C_2 = -0.72 \lambda / \text{cm}^2 \quad C_4 = 0.55 \lambda / \text{cm}^4 \quad C_6 = -0.015 \lambda / \text{cm}^6, \]  

(A.1)

which compare well with the values calculated by ray tracing, as reported in [25],

\[ C_2 = -0.67 \lambda / \text{cm}^2 \quad C_4 = 0.48 \lambda / \text{cm}^4 \quad C_6 = -0.023 \lambda / \text{cm}^6. \]  

(A.2)

The Mark III system did not seem to exhibit the difficulty with focal position seen in the Mark IV. This strengthens the hypothesis that the error lies in the corrective lens. For comparison, Fig. A.6 shows the best results obtained with Mark III and Mark IV, together with the predicted performance of an aberration-free lens.

The above procedure allows the imaging system to be characterized on the bench, but performance in situ may vary for two reasons. First, the system might be off-focus. Experimentally, the best focus can be determined only to ±25 \( \mu \text{m} \), across which range
the fiber images vary considerably. However, since the condensate is \( \sim 2.5 \) times larger than the fiber, it should have a correspondingly greater depth of focus, in which case the experimental tolerance is reasonable. A second source of error is aberrations in the vacuum viewport through which the condensate is viewed. A variety of viewports were tested on the bench. Most caused some degradation of the image quality, and in several cases the degradation was severe. Viewports manufactured by Huntington and MDC were examined, and the ones by MDC were found to be superior. The actual viewport used was made by Huntington, but could not be tested since it was on the vacuum chamber. Subsequently, it was removed, and qualitatively characterized by observing its effect on a collimated laser beam with a shear plate. It was found to cause less distortion than most of the other Huntington viewports examined, but still an appreciable amount. It was replaced with an MDC viewport which caused almost no observable distortion, but the improvement in image quality has not yet been ascertained. Note that when mounted, the viewport is under considerable stress from the mounting bolts and pressure differential. This may introduce aberrations not present when bench-tested.

We attempted to experimentally determine the aberration parameters by observing images of condensates. In principle, a set of images of typical clouds could be analyzed together, with the cloud distribution parameters \( N_0, T \), etc. fit to each image, and the \( C_i \) fit to the set as a whole. However, this approach would be time consuming, and somewhat abstract since shot noise obscures the effect of the aberrations in any single image. A more desirable method would be to average several images to bring out the ripples, and then fit the averaged image to determine the \( C_i \). This could be achieved by examining pictures of clouds consisting of essentially bare condensates. Naturally, \( N_0 \) is expected to fluctuate from shot to shot, but the images could be scaled linearly to a fixed \( N_0 \) value. Alternatively, only images with similar \( N_0 \) values could be selected. In any case, by averaging several files a low-noise
image of the condensate could be generated, and the aberration coefficients adjusted to reproduce the observed ripples.

Experimentally it has proven difficult to obtain images of bare condensates, because fluctuations in the trap bias field limit the precision of evaporative cooling. Instead, images of clouds with substantial non-condensed components were examined, specifically from the $\tau = 10$ s data set of Fig. 4.10. A set of ten images with large $N_0$ values were used, and were first fit to the nonequilibrium distribution using a 3-μm Gaussian blur rather than the coherent convolution described in Section 4.1.2. This approach is described in Ref. [25], and can give accurate fit values if the correct amount of blurring is used. It does not, however, reproduce the ripples in the data observed, for instance, in Fig. 4.4. The residuals of the fit therefore exhibit the ripples clearly, in isolation from the smooth part of the profile. The results are averaged and plotted in Fig. A.7.

It is expected that the oscillations in the residuals arise primarily from the conden-
Figure A.8  Effect of imaging model on residuals for fitting with a 3-μm Gaussian blur. Points are data taken from Fig. A.7. The black curve shows the residuals predicted by the mock imaging procedure (q.v.), for the aberration parameters obtained from analysis of images of the optical fiber. When the aberration parameters are adjusted to agree with the data as closely as possible, the gray curve results.

sate itself, since only the condensate scatters light at large enough angle to encounter substantial aberrations. This expectation is supported by analysis of simulated images based on the QBE results: if several distributions having the same $N_0$ are used to generate images via the coherent convolution procedure, and the images are then analyzed using a 3-μm Gaussian blur, the residuals of the fits are essentially independent of $N$. The experimental data can therefore be analyzed by adding the measured residuals to an appropriate Gaussian function, and thereby "constructing" an image of a bare condensate. This image is then fit in the same way the fiber images were, giving the results of Eq. (4.16). The difference between the parameters of (4.16) and the bench-measured values of (4.15) can be seen in Fig. A.8, where the measured residuals are compared to those predicted by the two imaging models. The effect of the models on $N_0$ values is discussed in Section 4.1.2.
A.3 Image Analysis

The image analysis procedure relies on Eq. (4.7) to relate the observed images and the energy distribution function being fit. Derivation of this relation is straightforward, but involved enough to outline here.

The optical density is defined as

\[ \alpha(x', y') = \sigma_L \int dz' n(r), \]  

(A.3)

where the \( z' \)-axis is parallel to the probe beam. It is first necessary, then, to find the density \( n \) in terms of \( f(E) \). The density is defined as

\[ n(r) = \frac{1}{(2\pi\hbar)^3} \int d^3p/\cdot f(r, p), \]  

(A.4)

and

\[ f(r, p) = \int dE \ f(E) \ \delta \left( \frac{p^2}{2m} + V(r) - E \right). \]  

(A.5)

Substituting (A.5) into (A.4) and using the \( \delta \)-function to do the integral over momentum yields

\[ n(r) = \frac{1}{(2\pi)^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_{V(r)}^\infty dE \ f(E) \sqrt{E - V(r)}. \]  

(A.6)

The potential energy \( V(r) \) is

\[ V(r) = \frac{1}{2} m(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) = \frac{1}{2} m \omega_p^2 (x^2 + y^2 + z^2), \]  

(A.7)

where the \( z \)-axis is parallel to the trap bias field. Combining Eqs. (A.3) and (A.6) gives

\[ \alpha(x', y') = \frac{\sigma_L}{(2\pi)^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_{-\infty}^{\infty} dz' \int_{V(r)}^\infty dE \ f(E) \sqrt{E - V(r)}. \]  

(A.8)

To obtain the desired expression, the \( z' \) and \( E \) integrals must be interchanged, and the resulting \( z' \) integral performed. Evaluation of these steps requires that the relation between \( (x, y, z) \) and \( (x', y', z') \) be specified.
The transverse coordinates \((x, y)\) and \((x', y')\) can be chosen freely, so long as they form an orthogonal set with \(z\) and \(z'\), respectively. We choose \(\hat{x}\) and \(\hat{y}\) to lie along the symmetry axes of the trap quadrupole field, specifically with \(\hat{x}\) vertical. With this definition, the slow-atom beam for loading travels in the \(\hat{z} - \hat{y}\) direction, and the probe laser travels in the \(\hat{x} + \hat{y} + \hat{z}\) direction. The \(x'\)-axis is taken to lie along the polarization vector of the probe, in the direction \(\hat{x} - \hat{y}\), which then fixes \(\hat{y}'\). The primed and unprimed coordinates are therefore related by

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix}
\sqrt{3} & 1 & \sqrt{2} \\
-\sqrt{3} & 1 & \sqrt{2} \\
0 & -2 & \sqrt{2}
\end{bmatrix}
\begin{bmatrix}
x' \\
y' \\
z'
\end{bmatrix},
\] (A.9)

Using this relation,

\[
x^2 + y^2 + \epsilon^{-2} z^2 = x'^2 + \frac{1 + 2 \epsilon^{-2}}{3} y'^2 + \frac{2 + \epsilon^{-2}}{3} z'^2 + (1 - \epsilon^{-2}) \frac{2 y' z'}{3},
\] (A.10)

which allows the potential energy to be expressed in terms of the primed coordinates.

Interchanging the integrals in (A.8) yields an expression of the form

\[
\alpha(x', y') = \frac{\sigma_L}{(2\pi)^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} \int_{E_m}^{\infty} dE \int z_0' \sqrt{E - V(r)},
\] (A.11)

where \(E_m(x', y')\) is the minimum value of \(V(x', y', z')\) occurring as \(z'\) is varied, and \(z_\pm'\) are the roots of \(E = V(x', y', z')\), which will be real for \(E > E_m\). Solving \(dV/dz' = 0\) to find \(z_0'\) gives

\[
z_0' = 2^{1/2} \left( \frac{1 - \epsilon^2}{1 + 2 \epsilon^2} \right) y',
\] (A.12)

and evaluating \(V(x', y', z_0')\) yields

\[
E_m(x', y') = \frac{1}{2} m \omega_p^2 (x'^2 + \zeta^2 y'^2),
\] (A.13)

with \(\zeta^2 = 3/(1 + 2 \epsilon^2)\). Similarly, the roots \(z_\pm'\) are found to be

\[
z_\pm' = z_0' \pm \zeta \left[ \frac{\zeta}{m \omega_p^2} (E - E_m) \right]^{1/2},
\] (A.14)
as is readily verified by substitution into (A.10), using the fact that \( V(z', y', z_0') = E_m \).

From

\[
E - V = \frac{m\omega_p^2}{2\epsilon^2 \zeta^2} \left( z' - z'_- \right) \left( z'_+ - z' \right),
\]

the relation (A.11) can be expressed as

\[
\alpha(x', y') = \frac{\sigma_L}{2\pi} \left( \frac{2m}{\hbar^2} \right)^{3/2} \left( \frac{m\omega_p^2}{2\epsilon^2 \zeta^2} \right)^{1/2} \int_{E_m}^{\infty} dE f(E) \int_{z_-'}^{z'_+} dz' \left[ (z' - z'_-)(z'_+ - z') \right]^{1/2}.
\]

The \( z' \) integral is elementary [100, Eq. 855.42], and yields

\[
\frac{\pi}{8} (z'_+ - z'_-) = \frac{m\epsilon^2 \zeta^2}{m\omega_p^2} (E - E_m).
\]

The optical density is therefore

\[
\alpha(x', y') = \frac{\sigma_L}{2\pi} \frac{m}{\hbar^3 \omega_p} \epsilon \zeta \int_{E_m}^{\infty} dE f(E)(E - E_m),
\]

in agreement with (4.7).

This result can be used in conjunction with the \( f(E) \) values generated by the QBE simulation to test the analysis procedure. Pixel values for a "mock" image are produced by numerically calculating the integral in (4.7) at each point in the image plane, and from the resulting values of \( \alpha \) finding the signal intensity \( I_s \) via (4.3). This signal is then convolved with the coherent point-transfer function, assuming some values for the aberration parameters. The convolved signal is converted to a number of counts \( C \) using the quantum efficiency of the CCD camera and the pixel area, and shot noise is generated by adding to each pixel a random value taken from a Gaussian distribution with width \( C^{1/2} \). If desired, additional noise of various types can be added to represent interference fringes from the lenses, or jitter in the probe beam pointing.

An identical procedure is followed to generate a background image with \( \alpha = 0 \), and the two images are then analyzed with the same program used to analyze data images. As discussed in Section 3.3.3, the \( N_0 \) values obtained by the fit are close to
the actual values from the QBE distribution. In addition to testing the validity of the nonequilibrium model, this provides an independent check for errors in the analysis program.
Appendix B
Uncertainties in $N$ and $T$

Uncertainties in the determination of $N_0$ are discussed at some length in Section 3.3.3 and Chapter 4. However, uncertainties in $N$ and $T$ are also important, particularly for the measurement of $G_2$ in Section 3.2.3 and of equilibration in Section 4.3. This Appendix discusses possible systematic errors in these measurements. For simplicity, the case of a nondegenerate cloud is assumed, and differences arising for a degenerate gas are discussed at the end of the section. Table B.1 summarizes all the measured uncertainties and their effects.

In equilibrium, a nondegenerate cloud is completely specified by two parameters. Normally $N$ and $T$ themselves fill these roles, but the experimental parameters determined directly from the images are the peak signal height $S$ and the 1/e-radius of the cloud, $r_C$. The size $r_C$ is related to the temperature according to

$$k_B T = \frac{1}{2} m \omega^2 (Mr_C)^2,$$

(B.1)

where $M$ is the magnification of the imaging system, converting the value measured in pixels to the physical size of the cloud. Uncertainty arises from $M$ and the trap frequency $\omega$. The frequencies were measured with an accuracy of $\pm 0.25$ Hz by resonantly driving the cloud during evaporative cooling, as discussed in [25]. This uncertainty is negligible compared to that of the magnification.

The magnification uncertainty has several components, since the various lens systems and the pixel size of the camera all contribute. However, most the measurements of the magnification were made with the Photometrics camera and referenced directly to the pixel size, so the actual value of the pixel size need not be known. It is specified to be 19 $\mu$m, and for convenience the magnifications listed below assume this value. The exception is the measurement of the Mark IV relay lens system, which was made with a Pulnix camera. In that measurement, however, the Mark IV magnification was
directly compared to the Mark III magnification, which had previously been referenced to the Photometrics pixel size. The measurement results are:

<table>
<thead>
<tr>
<th>System</th>
<th>Magnification</th>
<th>Uncertainty</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mark III doublets</td>
<td>0.997</td>
<td>0.0035</td>
</tr>
<tr>
<td>Mitutoyo objective w/tube lens</td>
<td>12.68</td>
<td>0.14</td>
</tr>
<tr>
<td>Ratio Mark IV : Mark III</td>
<td>1.127</td>
<td>0.007</td>
</tr>
</tbody>
</table>

The net magnification of the Mark IV system is therefore 14.25 ± 0.2. This relative uncertainty of 1.3% yields a systematic uncertainty in T of 2.6%. The measurements of the relay systems were made by translating a laser beam a known amount in the object plane, and observing the displacement on the camera, while the objective was measured using a standard imaging target, Melles Griot P/N 04TRN003.

The uncertainty in \( N \) is more involved. The measured signal \( S \) is related to the phase shift imposed by the atoms according to

\[
S = \frac{\sqrt{3}}{2} \tan \vartheta \phi, \tag{B.2}
\]

where \( \vartheta \) is the angle between the imaging polarizer axis and the probe polarization. The phase is related to the optical density \( \alpha \) through

\[
\phi = \frac{\alpha \Delta}{2I_0/I_{sat} + 4\Delta^2 + 1}, \tag{B.3}
\]

and to \( N \) by

\[
\alpha = \frac{N \sigma_L}{\pi r_G^2}. \tag{B.4}
\]

In (B.3), the detuning \( \Delta \) is normalized by the atomic transition linewidth \( \Gamma \) for brevity. So in terms of measured quantities, \( N \) can be expressed as

\[
N = \frac{2\pi}{\sqrt{3}} \frac{r_G^2}{\sigma_L \tan \vartheta} \left( 4\Delta + \frac{2I_0/I_{sat}}{\Delta} + \frac{1}{\Delta} \right) S. \tag{B.5}
\]

Uncertainty in the magnification contributes to \( r_G \) as above. Uncertainty in \( \vartheta \) contributes approximately as

\[
\frac{\sigma_N}{N} = \frac{\sigma_{\vartheta}}{\cos \vartheta \sin \vartheta} \approx \frac{\sigma_{\vartheta}}{|90^\circ - \vartheta|}. \tag{B.6}
\]
for \( \phi \) near 90°. The polarizer is mounted in a rotation stage, and can be set perpendicular to the probe polarization with an accuracy of \( \sim 0.1^\circ \). It is then offset by 7.5 \( \pm 0.25^\circ \), using the rulings on the stage.

Since the probe detuning \( \Delta \approx 40 \gg 1 \), the uncertainty \( \sigma_\Delta \) contributes only linearly through the 4\( \Delta \) term. Uncertainty in \( \Delta \) arises mainly from the indefinite location of the set point of the heat pipe with respect to the excited state hyperfine manifold of the \( 2P_{3/2} \) state, and is estimated to be 2 MHz \( \approx 0.3 \Gamma \).

The uncertainty in the probe intensity \( I_0 \) contributes as

\[
\sigma_N = \frac{2\pi}{} \frac{r_G^2}{\sqrt{3}} \frac{1}{\sigma_L \tan \phi} \frac{2}{\Delta I_{sat}} \approx N \frac{\sigma_{I_0}}{2I_{sat}\Delta^2}, \tag{B.7}
\]

so

\[
\frac{\sigma_N}{N} = \frac{\sigma_{I_0} I_{sat}}{I_0} \frac{I_0}{2\Delta^2} \approx 0.09 \frac{\sigma_{I_0}}{I_0} \tag{B.8}
\]

for \( I_0 = 300 I_{sat} \). The intensity is measured directly from the image, with an uncertainty of \( \sim 10\% \) arising from the calibration of the CCD sensitivity and the probe pulse duration.

Finally, the cross section \( \sigma_L \) is itself uncertain, since it depends on the angle between trap bias field and the probe polarization \( \theta_p \), and that between the field and the probe propagation direction \( \theta_k \). In both cases, the cross section varies by \( \sim \lambda^2/2\pi = \sigma_L/2 \) as the angles vary by 90°. The uncertainties therefore enter as

\[
\frac{\sigma_{\sigma_L}}{\sigma_L} = \frac{1}{2} \frac{\sigma_{\theta_k}}{90^\circ}. \tag{B.9}
\]

Uncertainty in \( \theta_k \) is estimated to be 2° from measurements of the probe entrance and exit spots on the vacuum viewports, and knowledge of the trap chamber dimensions. Uncertainty in \( \theta_p \) is similar.

The size and effect of all these uncertainties is listed in Table B.1. The resulting total uncertainty in \( N \) is 4.7%.

The state of a degenerate nonequilibrium cloud cannot generally be described by only two parameters, and uncertainty analysis is more complicated. Determination
Table B.1 Uncertainties in $N$ and $T$ contributed by from various elements. The variables are the trap oscillation frequencies $\omega$, imaging magnification $M$, imaging polarizer angle $\vartheta$, probe detuning $\Delta$, intensity $I_0$, polarization angle $\theta_p$, and propagation direction $\theta_k$. Contributions to $\sigma_T$ and $\sigma_N$ are all listed in percent, and summed in quadrature for the totals.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Uncertainty</th>
<th>$\sigma_T$</th>
<th>$\sigma_N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$</td>
<td>0.3%</td>
<td>0.6%</td>
<td>0.6%</td>
</tr>
<tr>
<td>$M$</td>
<td>1.3%</td>
<td>2.6</td>
<td>2.6</td>
</tr>
<tr>
<td>$\vartheta$</td>
<td>0.25°</td>
<td>-</td>
<td>3.3</td>
</tr>
<tr>
<td>$\Delta$</td>
<td>0.3°</td>
<td>-</td>
<td>0.8</td>
</tr>
<tr>
<td>$I_0$</td>
<td>10%</td>
<td>-</td>
<td>0.9</td>
</tr>
<tr>
<td>$\theta_p$</td>
<td>2°</td>
<td>-</td>
<td>1.1</td>
</tr>
<tr>
<td>$\theta_k$</td>
<td>2°</td>
<td>-</td>
<td>1.1</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>2.7%</td>
<td>4.7%</td>
</tr>
</tbody>
</table>

of $N$ is not so difficult, since it can still be expressed in terms of the integrated signal in the image. The arguments used in Table B.1 then apply, with only the detailed relation between $N$ and $r_G$ changing substantially. It will nonetheless remain approximately true that $N \propto r_G^2$, so $\sigma_N$ should remain at about 5%. This is consistent with the mock image analysis procedure, which correctly finds $N$ to $\sim$1%.

Systematic errors in $T$ are more difficult to evaluate. The temperature of a nonequilibrium cloud is indefinite, but a reasonable working procedure is to consider the temperature to which the cloud would thermalize if allowed to equilibrate without losses. This temperature is a function of the average energy of the trapped atoms, $\langle E \rangle$, and can therefore be determined from the fitting parameters of the image analysis. However, in the mock imaging procedure, values of $\langle E \rangle$ are not reproduced very well. Accuracy ranges from 5% to 20%, with the worst agreement just after a collapse has occurred.

Other systematic errors in the determination of $\langle E \rangle$ are smaller. The energy of the atoms is still proportional to $\langle r^2 \rangle$ by the virial theorem, and thus depends quadratically on the magnification. Although the relation between $\langle E \rangle$ and $T$ is complicated, the fact that $T$ changes more slowly with $r_G$ than does $\langle E \rangle$ means that
this effect can only reduce $\sigma_T$. The accuracy of temperature determination for an equilibrium degenerate gas is therefore 3% or better.
Appendix C

Trap Magnet Temperature Stabilization

The observed fluctuations in evaporative cooling suggest that the trap bias field is fluctuating by $\sim1$ mG. This appendix describes the efforts which have been made to stabilize this field, primarily by stabilizing the temperature of the magnets. The temperature stabilization system consists of several elements. First, the trap chamber as a whole is well-insulated, and the room temperature of the lab is stabilized using a temperature sensor located near the chamber. Second, the chamber top flange to which the trap is attached is actively stabilized. Finally, a separate control system stabilizes the trap bias magnets themselves. In addition, the microwave power used for evaporative cooling is varied during evaporation to provide a constant heat load. Effects of ambient field fluctuations are also discussed here.

A constant room temperature is maintained by an air-handler unit installed near the apparatus. This unit is digitally controlled by an Omega CN 380 thermostat, which uses a platinum RTD sensor and PID control logic. The temperature stability at the sensor is typically $\pm0.1^\circ$C. The trap chamber and ion pump are insulated by layers of fiberglass cloth and aluminum foil. At one point, temperature-stabilized water was circulated underneath the insulation, but after the heaters on the magnets were installed, this procedure was discontinued with no observed ill effect.

The trap top flange is stabilized with two Minco strip heaters, using the temperature control circuit described in [101]. The error signal on the stabilization indicates the temperature fluctuations to be a few mK.

The most important stabilization mechanism is the trap heaters. They are shown schematically in Fig. C.1. The trap is suspended from the top flange by four stainless steel tubes, which penetrate the flange and open to air. The tube wall thickness is 0.025", minimizing thermal conduction to the top flange. The tubes terminate on two
Figure C.1 Mounting scheme for trap heaters and thermistors. The trap yoke is shown suspended from the top flange by four reentrant stainless steel tubes, which provide access for the temperature control elements. The larger tubes are 3/8" OD for the heaters, while the smaller tubes are 1/4" OD for the thermistors. The tubes continue into the blocks attached to the yoke, which actually contain the elements. The heaters are Omegalux International cartridge heaters, P/N CSH-101100/120V, and the thermistors are Fenwall standard bead P/N 112-204KAJ-B01. Each element is mounted at the end of a ceramic rod for insertion into the tubes.

stainless steel blocks mounted to the trap yoke at the location of the bias magnets. One tube on each side contains a heater cartridge, and the other a thermistor sensor. In order for simple heaters to provide regulation, the trap must be kept warmer than the surrounding chamber, so the set point is \( \sim 40^\circ C \).

The resistances of the thermistor sensors vary by \( \sim 4\%/^\circ C \). Each thermistor makes up one element of a resistive bridge, in the circuit shown in Fig. C.2. The opposite element of the bridge is a Vishay resistor with a low temperature coefficient of 3 ppm/°C. The other two resistors are metal film resistors with tempcos of 50 ppm/°C, but are matched to within 10% so that the effective tempco of the right arm of the bridge is again a few ppm/°C. The net change in the bridge voltage is therefore roughly \( 10^4 \) times more sensitive to the thermistor temperature than to that of the remaining elements, which are mounted below the insulation on the top flange and thus stable to better than 10 mK.

The resistive bridge of C.2 is used as the detector for the temperature control
Figure C.2  Temperature sensing circuit. The resistor bridge is mounted to the top flange of the trap chamber, which is temperature stabilized. The 7812 and 7912 ICs are voltage regulators, and the LM399 is a precision voltage reference. Cascading the regulators and reference improves the stability of the bridge voltage. The set point of the temperature control circuit, $V_{\text{ref}}$, is determined by the fixed resistor values on the right arm of the bridge. Values shown are approximate.

circuit of [101]. The error signal from the circuit indicates that the thermistor temperatures fluctuate by about 10 $\mu$K peak-to-peak under ambient conditions. The sensitivity of the trap bias field to magnet temperature is $\sim$1 G/K, so the observed field fluctuations of $\sim$1 mG seem too large to be attributed to variations in the magnet temperature.

One explanation for the observed fluctuations is that the magnets are in poor thermal contact with the thermistors, and are thus insufficiently regulated. The magnet temperatures might therefore fluctuate due to radiative coupling to the chamber wall. This seems unlikely, however, because the observed field fluctuations occur on a run-to-run time scale, with a relatively low level of steady drift. Because the thermal capacitances of the trap and chamber are large and radiative coupling is weak, variations from this source would more likely exhibit the opposite form, with a large steady drift and little run-to-run fluctuation. This conclusion could be tested by installing additional thermistors on the magnets themselves. Doing so in a UHV-compatible
way is difficult, but the system could just as easily be tested under low-vacuum conditions, with the additional thermistors removed when testing is complete.

Another source of noise is the microwave radiation used to drive evaporative cooling. An estimated 50 mW of power is dissipated in the chamber, and the thermal capacitance of the trap is estimated to be \( \sim 10^3 \) J/K. A significant heating effect from the microwaves is therefore expected, and is indeed observed. To minimize this problem, the microwaves are left on at all times, with the frequency set either below or far above the trap bottom resonance when no evaporation is desired. In principle, this should provide a constant heat load. However, the antenna coupling efficiency varies considerably as a function of microwave frequency \( \Omega_T \). The measured reflection from the antenna varies between near zero to \( \sim 30\% \) during the cooling trajectory. This is observed to affect the magnet temperature, by monitoring the heater voltage required for regulation while \( \Omega_T \) is slowly varied.

To compensate for this effect, the microwave power is varied as a function of \( \Omega_T \) to provide a constant heat load. The dependence of the power on \( \Omega_T \) was determined as above, by finding the power level required to keep the stabilization heater voltages constant. This was not entirely successful, however: using the variable microwave power only reduced the variation in the heater voltage by about a factor of two, as seen in Fig. C.3. The reason for this poor performance is not clear, but may be a result of ambient temperature variation during the slow microwave scan. If this caused significant changes in the heater voltages, the determination of the required microwave power would be inaccurate. Further investigation of this problem is probably warranted.

A more accurate load stabilization technique would be to measure the transmitted power in real time, and use a feedback system to level it. The reflected power could potentially be used for this purpose, and a system based on this method was developed. Two directional couplers and PIN diodes measured the applied and reflected
Figure C.3 Effect of microwave power leveling on temperature stability. The abscissa shows the heater voltage required to maintain a constant trap temperature, as \( \Omega_T \) is slowly (~hours) scanned over the range of frequencies used in evaporative cooling. The gray curve shows the results with constant microwave power, and the black curve shows the results when the power is varied so as to provide a more constant heat load. The voltage shown is the average of that for the two heaters, and is subtracted from an arbitrary offset.

powers, and the difference was held constant. Unfortunately, this system aggravated the temperature fluctuations of the magnets, as measured by the temperature control circuit. This was possibly due to the nonlinearity of the response of the PIN diodes, which was not taken into account.

It may, however, be that the observed field fluctuations arise not from the trap but from the ambient magnetic field in the lab. This field is routinely monitored with a flux-gate magnetometer system from Applied Physics Inc. (P/N 535). No significant slowly-varying component is observed, although moving nearby metal furniture in the lab does have a significant effect. We also observe relatively large AC fields at 60 Hz and harmonics. We compensated for these fields by referencing the time of the microwave sweep to the oscillation of line voltage, but the effect of this synchronization has not been measured.

Correcting for such ambient fields is somewhat difficult. Because of the amount of
magnetic materials in and around the trap chamber, the magnetic field gradients are large, and a magnetometer reading taken outside the chamber cannot easily be related to the field at the trap. One solution is to use a set of large coils located far from the trap. Nearby magnetic materials would then presumably affect the ambient and nulling fields in the same way, and leveling the field measured at the magnetometer would level the field in nearby areas as well. Alternatively, the relation between the measured field and the field at the trap could in principal be established by applying a relatively strong uniform field to the apparatus, and comparing the response of the magnetometer to the effect on evaporative cooling. This would allow smaller and closer feedback coils to be used.

Both of these schemes require the sources of ambient field fluctuations to be distant, so that they can be accurately simulated by a uniform applied field. The actual gradients in the ambient field need not be small, but they must be constant in time. A pair of magnetometers could be used to determine whether or not this is the case.

An alternative to active compensation of the ambient field is the passive reduction of it. We have attempted this to some extent, by identifying and eliminating or shielding nearby magnetic sources. Further reduction could be obtained by shielding the trap chamber with \( \mu \)-metal. Because of the irregular shape of the chamber, this would be an expensive and time-consuming operation. It is, however, the standard procedure in magnetic shielding problems, and may be the most promising.
Appendix D
Frequency Stabilization Systems

The experiments described here rely on having stable and accurate laser frequencies. Two servo systems are used to provide this stability, and are detailed here.

D.1 Heat-Pipe Lock

A Coherent 699 dye laser is used to generate the Zeeman slowing, deflection, and collimation beams, and is locked to a $^7$Li heat-pipe via a Doppler-free saturated absorption measurement. The lock uses a high-frequency modulation technique, similar to the Pound-Drever scheme commonly used to lock a laser to a cavity. An illustration of the optical layout is shown in Fig. D.1.

The laser first passes through an acousto-optical modulator (AOM), which allows the laser to be offset from resonance if needed. Only the deflected beam is used by the lock. The beam is split by a glass plate into a relatively strong pump beam with a power of $\sim 5$ mW, and a weaker probe beam of $\sim 300$ $\mu$W. The beams are each $\sim 3$ mm diameter. The pump beam polarization is rotated by 90° using a half-wave plate, and then directed into the heat pipe by a polarizing beam splitter (PBS).

![Diagram of heat-pipe lock](image)

**Figure D.1** Optical layout for heat-pipe lock.
The probe beam is frequency modulated by an electro-optic modulator (EOM), operating at $\sim 10$ MHz. Details of the EOM are given below. With 1-2 W of radio-frequency drive, it splits the laser power approximately evenly between the carrier and the two first-order sidebands. Ideally, it introduces no intensity modulation, but the effects of its nonideality are discussed below.

The probe beam is then directed into the heat pipe, propagating counter to the pump beam. The vapor pressure of $^7$Li in the heat pipe is adjusted using the temperature, and is typically set to give $\sim 50\%$ absorption on resonance. When the laser frequency is at the zero-velocity resonance, the transmission of the probe will be enhanced due to saturation by the pump. Because the linewidth of the transition is comparable to the probe modulation frequency, when the laser is slightly off resonance, the red and blue sidebands are attenuated by different amounts. This introduces intensity modulation in the probe beam at the 10 MHz sideband frequency.

The intensity modulation is coherent with that of the EOM drive, and the relative phase of the two varies from $-180^\circ$ to $+180^\circ$ as the laser passes through resonance. A dc error signal can therefore be obtained by measuring the intensity modulation with a photodiode and mixing it with the EOM drive, as shown in Fig. D.2. The photodiode is from Thorlabs, P/N DET-2SI, and is shielded with aluminum foil to minimize noise from electrical pickup. It is amplified by 60 db using a Miteq AU-4A-0110 signal amplifier. The mixer is from Minicircuits, P/N ZAD-3H. Its IF output is filtered with a homemade low-pass circuit.

The error signals obtained when the laser frequency is swept across resonance are shown in Fig. D.3. In Fig. D.3(a), an 8.5 GHz sweep was used, and the Doppler-broadened absorption profile can be seen, along with the saturated absorption peaks at resonance. The central narrow feature is the crossover resonance. Figure D.3(b) shows a 500 MHz scan across the $F = 2$ transition, to which the laser is locked. A feedback signal is generated from the error signal, using the circuit shown in Fig. D.4.
**Figure D.2** RF components for heat-pipe lock. The directional coupler splits off a -10 db fraction of the power to drive the mixer. In the final low-pass filter, inductance values are in $\mu$H and capacitances in nF.

**Figure D.3** Saturated absorption signals from heat pipe lock. With the laser unlocked, it was scanned across the $^7$Li resonance, and the error signal recorded. In (a), the laser was scanned 8.5 GHz, while in (b) it was scanned 500 MHz across the $F = 2$ transition, which is the peak on the left in (a).
Figure D.4 Feedback circuit for heat-pipe lock. Unlabeled resistors are 15 kΩ, unlabeled capacitors are 100 pF, and operational amplifiers are LF411. All capacitors shown are polystyrene. In addition, a 1 μF tantalum capacitor to ground is used on each op-amp power supply pin. Note that the lock output and the integrator zero are controlled by the same switch.
In practice, the best stability is observed with no proportional gain.

The 10 MHz EOM uses a LiNbO$_3$ crystal as the capacitor in an LC oscillator circuit. The inductor consists of 12 loops of wire wrapped on a 2.8 cm-diameter Teflon spool, with a spacing of 0.7 mm/turn. The crystal is 5 cm long, 4 mm wide, and 2 mm high, with the electric field applied across the short dimension and the laser propagating through the long dimension. Its dc capacitance is $\sim$40 pF. A 20 pF mica capacitor is placed in parallel with the crystal, to attempt to improve the quality factor of the oscillator. Similar performance was observed with and without the additional capacitor.

The quality of the oscillator is $\omega/\Delta\omega \approx 20$, but the LC resonance is complicated by the presence of piezoelectric resonances in the crystal. These cause the circuit response to vary erratically as the drive frequency is tuned across the LC resonance, with a sensitivity of $\sim$100 kHz. The magnitude of the variations was observed to depend on the oscillator used to drive the circuit, which is surprising since the same amplifier, Minicircuits P/N ZHL-3A, was used in all cases. It may be that the piezoelectric resonances respond sensitively to harmonics of the drive frequency, thus demanding a spectrally pure drive. A Hewlett-Packard 8601A signal generator is currently being used, and gives good results. Similar performance was observed with a Stanford Reseach Systems DS345 function generator, but two cheaper oscillators from BK Precision Instruments were virtually unusable.

The dispersion curves such as in Fig. D.3(b) are generally offset from zero. This is caused by a variety of sources, including the Doppler-broadened profile on which the signal sits, electrical pickup on the photodiode from the LC oscillator, and intensity modulation of the probe beam by the EOM, and spectral impurity in the drive. The effect of the Doppler profile can be seen in Fig. D.3, where the $F = 2$ saturated absorption curve is lowered and the $F = 1$ curve is raised. Evidence for the effect of spectral impurity is the fact that the offsets depended on the source used, as above. To
counteract these effects, a variable offset voltage is included in Fig. D.4. In principle, the noise sources or the offset voltage itself might drift, but the atom-trapping signal in the experiment indicates that the drift is less than the sensitivity of $\sim 1$ MHz. As measured by the error signal, the short term stability of the laser lock is $\sim 500$ kHz rms.

D.2 Heterodyne Lock

The trapping and probing beams are generated by a Spectra-Physics 370D laser, and are detuned by $\sim 1.4$ GHz from the $F = 2$ resonance. In order to accurately reference this laser to the atomic transition, a heterodyne measurement is made between it and the Coherent 699 laser which is locked to the heat pipe. However, the large detuning produces too fast a beat note to measure easily, so the 699 beam is first modulated by a high-frequency traveling wave EOM. By driving the EOM at $\sim 1.35$ GHz, a 50 MHz beat note is obtained, which can be readily detected and counted. The optical layout of the system is shown in Fig. D.5.

The EOM is driven by a Hewlett-Packard 8690A sweep oscillator, with an 8693B plug-in. This signal is amplified to $\sim 2$ W by a Trontech P1022-40 amplifier. The EOM design itself is discussed elsewhere [102].

The heterodyne signal is detected with a Thor Labs DET2-SI photodiode, amplified with a Minicircuits ZFL-2000 amplifier, and counted using a Hewlett-Packard

![Figure D.5 Heterodyne lock schematic.](image-url)
5328A counter. The counter communicates over an IEEE bus with a personal computer, which numerically generates a feedback signal and applies it to the laser with a digital-to-analog output. The computer implements a integral-control feedback algorithm.

The program providing this service is named "lock", and is located in the directory /lab/source/beam/hellock. It is a DOS program, compiled using Quick C. It is run by simply entering lock at the command line. The program then prompts for the heterodyne frequency to be maintained, and a "slope" parameter which sets the integration time constant. Typical values are 50 MHz and $2 \times 10^{-8}$, respectively.

The implementation of the lock program is straightforward, save for one element. Because the counter uses an out-of-date version of the IEEE protocol, modern drivers cannot communicate with it in the standard way. In the Keithley-Metrabyte IEEE library, the function Send1 must be used, with the syntax

\[
\text{Send1}(0,"<\text{Command String}>","/n'!END);\]

where \text{<Command String>} is the command sequence to be sent to the device.
Appendix E
Programs

The results discussed in this thesis depend heavily on several fairly complex computer programs. The actual source code listings are given elsewhere, but this appendix describes how the programs can be modified, compiled, and run.

All of the programs are written in C for Unix, specifically the IBM AIX operating system with the XLC compiler. No special efforts were made to ensure their portability to other operating systems or compilers. The programs are all located in subdirectories of the /lab/source/beam directory, and are generally organized with source code in a source directory and results in a data directory. The polr program is an exception, since the data files it produces are stored with the image data, in the \lab\data hierarchy. Note that a brief description of all the programs described here can be obtained by running the executable on the command line with no arguments. Also, many README.txt files contain detailed information about the contents of the directory in which they reside.

The programs are compiled using the Unix “make” facility. The source code is spread among several .c files, which are separately compiled and linked. The make program handles this process, and keeps track of which files are current and which should be recompiled. Compilation is accomplished by running

```
make -f makefile,
```

where the makefile is an instruction set located with the source code. It can be identified because its name begins with “make”, for example, makeopt.

E.1 Optimization of Evaporative Cooling

The program is run using the syntax

```
opt N_{start} N_{stop} T_{start}(\mu K) \ n_{grid} \ fileroot \ [-b \ G_{1}^{-1}(s)] \ [-t \ 2G_{2} \ (cm^{2}/s)],
```

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where $N_{\text{start}}$ is the initial number of atoms in the trap, $N_{\text{stop}}$ is the number of atoms at which the simulation halts, $T_{\text{start}}$ is the initial temperature, $n_{\text{grid}}$ is the number of energy grid points to use, and "fileroot" is an identifier used in the data files created by the run. Typically, $n_{\text{grid}} = 200$, and "fileroot" has the form "optXX" with sequentially increasing number XX. The options listed in brackets "[...]") can be omitted, in which case default values are used. The default value of the density-independent loss rate $G_1$ is $1.7 \times 10^{-3}$ s$^{-1}$, and of the dipolar decay rate constant $G_2$ is $1 \times 10^{-14}$ cm$^3$/s. Note that the program takes the loss coefficient $2G_2$ rather than the rate constant itself.

When opt is run, it produces four files, named <fileroot>.dat, <fileroot>.log, <fileroot>.dist, and <fileroot>.eta. The .dat file is the most important. It reports, as a function of time, the number of atoms remaining, their average energy and peak density, and the optimized value of the evaporation cutoff, $E_T$. This file is used to generate the evaporation trajectory for the experiment, as described below. The .log file records the time of the run and values for the various parameters used, and notes the times when the energy grid is rescaled. The .dist file contains the complete energy distribution of the cloud at the end of the run, so that the simulation can be continued later if desired. Finally, the .eta file lists the value of $\eta_T = E_T/\langle E \rangle$ as a function of time, which is related to the efficiency with which evaporation is occurring.

Two steps are involved in creating the data file needed for the experiment. First, the values of $E_T$ in the .dat file are converted to frequency units using the program evaptraj. In addition to the conversion, the format is changed to that used in the experiment, in which each evaporation frequency is listed together with the time, in ms, that the frequency should be applied. The program and its source are located in the opt/data directory. Its syntax is

```
evaptraj infile outfile $T_{\text{start}}(\mu\text{K})$,
```
where "infile" is the full name of a .dat file, "outfile" is the name of the file to be produced, and $T_{\text{start}}$ is the temperature at which the cooling trajectory is to begin. Typically $T_{\text{start}} = 250 \, \mu K$, but the simulation is started with $T = 500 \, \mu K$ so that if hotter clouds are ever obtained in the experiment, new trajectories can be created simply by re-running evaptraj.

After running evaptraj, it is necessary to add the microwave amplitude information used to help stabilize the trap temperature. This is accomplished using the program evapamp1, located in /lab/source/beam/rfstab. Its syntax is

\[ \text{evapamp1 infile P_{num}(dbm)}, \]

where "infile" is the file previously produced by evaptraj and $P_{\text{nom}}$ is the nominal value of the microwave power applied to the antenna during cooling. Typically, $P_{\text{nom}} = 15 \, \text{dbm}$ is used. The output of evapamp1 is a file named evapdat.in. This should be renamed using the form <fileroot>_pXXX.in, where "fileroot" is the original file name used by opt and "XXX" denotes $P_{\text{nom}}$ in tenths of a dbm. So typically, "XXX" is "150". The actual file to be used by the ACK experiment-control software must be named evapdat.in, and placed in the appropriate directory.

Many optimized trajectories have been obtained under a variety of conditions, and are all indexed in the file opt/data/INDEX.txt. It is very useful to keep this file up to date as further runs are made.

**E.2 Quantum Boltzmann Equation**

The full QBE simulation program is called qboltz, with an executable abbreviated to qb. It is run in much the same way as opt, as

\[ \text{qb N_{start} N_{stop} T_{start}(\mu K) n_{grid} fileroot}, \]

but has a larger set of options:
-b $G_1^{-1}$ (s)
-t 2$G_2$ (cm$^3$/s)
-s Random number seed
-z Time step (s)
-i Start time.

The loss rate options are identical to those of opt, including the extra factor of 2 in -t. The random number seed is an integer defining the sequence used to check when collapses occur, and to determine the number of atoms remaining after a collapse when this number is not fixed. It defaults to one. The time step is the interval at which the program puts out data, and defaults to 0.2 s.

The -i option is used to continue the simulation from a previous run. To do so, the .dist file (q.v.) from the earlier run must be present in the current directory, with the name infile.dist. If the -i flag is present, the program searches infile.dist for the entry closest to the specified start time. This distribution is then used as the initial condition for the current run. The -i option has several uses. It can be used to continue a run to longer times, go over a portion of a run again with finer step size, or to repeat a series of collapses with a different random number sequence.

In addition to the above run-time options, there are several aspects of the program that must be modified before compiling. Chief of these is the evaporation trajectory and microwave sweep, which are determined in the source file cutoff.c. The function get_cutoff sets $E_T$ as a function of time. In the same file, the function perturb implements the probing perturbation, if desired. Comments in these functions indicate how they work. In addition, the file qb.c contains the constant declarations GAUSS_COLL, N_COLL and W_COLL. If GAUSS_COLL is defined, the collapse sets $N_0$ to a random number, with a Gaussian distribution of mean N_COLL and rms width W_COLL. If GAUSS_COLL is not defined, exactly N_COLL atoms are left in the condensate after each collapse.
The convention for the fileroot parameter has been to use the form qbXXX, with XXX a unique sequential identifier for the output files. As with opt, qb produces several files, and the central one is <fileroot>.dat which contains \( N, \langle E \rangle \) in \( \mu \text{K} \), and \( N_0 \) as functions of time in seconds. The logfile <fileroot>.log records parameters for the run, times at which the energy grid is rescaled, and times of collapses.

The .dist file is somewhat different. First, it is produced as screen output, which should be redirected to a file if desired, or /dev/null if not of interest. The files produced are quite large, as they contain the full set of values \( f(E) \) at each time step. This is required for later continuation from an arbitrary point, and for analysis of the distributions with the mockimg program.

### E.3 Simulated Image Analysis

The program mockimg is used to generate false CCD images based on the results of the QBE model. These images can then be analyzed with the polzr analysis program, and the results compared to the known input. The syntax is

\[
\text{mock } \theta(\text{deg}) \Delta(\text{MHz}) \ I_0/I_{\text{sat}} \text{ size fileroot.}
\]

The parameters \( \theta, \Delta, \) and \( I_0 \) describe, respectively, the imaging polarizer angle, the probe detuning, and the probe intensity. The images produced will be "size" pixel across. Because a fast-Fourier transform technique is used, the image size must be a power of two.

The program uses two input files. The first, in<fileroot>.dat contains the population values \( f(E) \), while the second, lens.dat contains the lens aberration parameters to be used for convolving the image. Two image files are produced, amock<root>.pmi and nmock<root>.pmi, which simulate the signal and background images required for analysis. These files can be directly analyzed with polzr. The input file containing \( f(E) \) can be generated from the .dist files of qb using the program makein. Its syntax is
makein in.dist time(s),

where "in.dist" should be the name of the source .dist file, and "time" is the time value for the distribution to be extracted.

E.4 Image Analysis

The polzs program provides a general environment for analysis of polarization phase-contrast images. It is started by entering polzs on the command line by itself, which generates an explanation of the program and listing of subcommands, followed by a prompt. Entering a period "." at the prompt exits the program, and a question mark "?" redisplays the help information. The input redirect symbol "<" followed by a file name reads command input from that file.

Parameters are set by entering an equals sign "="; followed by a one-letter code and the parameter value or values. The current list of parameters and their codes is included in the help information. When polzs is started, it searches for a file defaults.plz; if it exists, default parameter values are read from it using the same syntax with which they are entered from the prompt.

Any other input string is interpreted as the file root of an input file to be analyzed. Three images are required: an image of the cloud, a background image of the probe, and a blank reference image which gives the pixel values generated by the camera in the absence of signal.

Most of the parameters are self-explanatory, but one deserves further comment. The fitting mode of the program is set by entering =z value, with the help information listing the possible mode values. Several choices are available: No fit at all can be performed, in which case radial profiles of the data will still be generated and output. A nondegenerate cloud can be fit to a Gaussian function. An equilibrium fit can be performed, in which the parameter A is set to 1, with either a Gaussian blur or the full coherent convolution to represent the effects of imaging resolution. A similar
pair of choices is available for a nonequilibrium fit, in which \( A \) is variable. Finally, a nonequilibrium fit with a Gaussian blur can be performed which also calculates the uncertainty in \( N_0 \). The values obtained are appended to an output file \texttt{uncert.dat}. Currently, this option is not available with the coherent convolution, since the calculation would be very slow and the appropriate Gaussian blur yields identical results. Nonetheless, incorporation of this option would probably be worthwhile.

The results of the fit are printed to the screen, and also to a log file \texttt{polzr.log}. Commands setting parameter values are also echoed to the log file, so the entire session can be reconstructed from the log. The radial profiles of the data and fit functions are output to a separate file \texttt{angle<fileroot>.dat}.

Only one compile-time option need generally be considered, and that is whether the condensate size should be allowed to vary as a function of \( N_0 \) to reflect the compression due to mean-field interactions. It is available only for the coherent-convolution fits, and is set in the source file \texttt{chisqrft.c} by defining the macro \texttt{VAR_CSIZE}.

In the \texttt{polzr} directory, a subdirectory \texttt{accessories} contains programs for fitting the imaging aberration coefficients to images of an optical fiber or bare condensate. The fits are generated by the programs \texttt{fitfiber} and \texttt{fitbare}, respectively. Their use is straightforward, and explained by the syntax information available on the command line.
Appendix F
Data files

The tables herein list the data and simulation files used throughout the thesis. Unless noted otherwise, data files are referenced to /lab/data/1998beam and simulation files to /sackett/proj/evap/qboltz/data.

Notes:

(1) Only images with $5 \times 10^3 < N < 3.5 \times 10^4$ are used.
(2) 7/17 analysis.
(3) 9/23 analysis.
(4) Fit to nonequilibrium distribution.
(5) Fit to equilibrium distribution.
(6) Same data used for Fig. 4.17, with fit to equilibrium distribution.
(7) Average of files 675, 727, 743, 777, 796, 810, 827, 831, 835 and 859.
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Table F.1  Data files
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Table F.2 QBE simulation files
Appendix G
List of Symbols

In the following, $X$ stands for various other symbols, and never occurs itself in the text. Consistency with conventional notation sometimes requires symbols to have multiple meanings. The intended use should be clear from context. Greek letters are listed after latin letters.

\begin{itemize}
\item $'$ Denotes a modified value
\item $\cdot$ Denotes a unit vector or an operator
\item $\nabla$ Gradient operator
\item $A$ Amplitude or prefactor of an expression
\item $\mathcal{A}$ Nonequilibrium distribution parameter
\item $A_P$ Area of pixel
\item $A_v$ Photoassociation overlap parameter
\item $a$ Scattering length
\item $B$ $|B|$
\item $\mathbf{B}$ Magnetic field
\item $B_X$ Expansion coefficient for $B$
\item $C$ Normalization constant
\item $C_X$ Expansion coefficient for $U$ or $W$
\item $D$ Diameter of magnetic drive coil
\item $D_v$ Côté's value for overlap integral
\item $d\Omega$ Differential element of solid angle
\item $dl$ Integration element along line parallel to probe propagation
\item $E$ Energy
\item $\mathbf{E}$ Electric field
\item $(E)$ Average energy of atoms in trap
\item $E_0$ Zero-point energy
\item $E_T$ Trap depth set by evaporative cooling
\item $E_c$ Energy cutoff imposed by microwave sweep
\item $E_m$ Energy parameter used in optical density expression
\item $E_v$ Binding energy for molecular level $v$
\item $F$ Atom flux in a beam
\item $F$ Hyperfine quantum number
\item $\mathcal{F}$ State population re-expressed, $f(E)/(1 + f(E))$
\item $f$ State population
\item $f$ Scattering amplitude
\item $f_n$ Values of state population on a grid
\item $G$ Green's function for atom scattering or imaging ($\equiv$ point-spread function)
\item $G$ Loss rate from trap
\item $G$ Sum of geometric series
\item $G_X$ Loss rate constant
\item $g$ Density of states
\item $H$ Hamiltonian
\item $h$ Definite integral used in QBE
\item $\hbar$ Planck's constant divided by $2\pi$
\item $I$ Laser intensity
\end{itemize}
\( I \) Elastic collision integral
\( \mathcal{I} \) Identity operator
\( I_P \) Photoassociation laser power
\( I_s \) Signal intensity
\( I_{\text{sat}} \) Saturation intensity
\( i \) Dummy discrete variable (i.e., in sums)
\( j \) Grid spacing \( \delta E / h\omega \) used in solving QBE
\( j \) Dummy discrete variable (i.e., in sums)
\( j \) Probability current
\( k \) Wave vector of atoms or light
\( k_B \) Boltzmann's constant
\( L \) Angular momentum
\( L \) Distance from trap to magnetic drive coil
\( l \) Angular momentum quantum number
\( \ell \) Size of condensate
\( \ell_0 \) Condensate size for ideal gas in harmonic trap, \( = (\hbar / m\omega)^{1/2} \)
\( M \) Magnification of imaging system
\( m \) Atomic mass
\( m \) Azimuthal quantum number
\( m_\text{r} \) Reduced mass
\( m \) Magnetic moment of an atom
\( m^* \) Effective mass of condensate for variational calculation
\( m_F \) Magnetic quantum number for total spin
\( m_I \) Magnetic quantum number for nuclear spin
\( m_I \) Magnetic quantum number for electron angular momentum
\( N \) Number of objects, usually atoms
\( N_0 \) Condensate occupation number
\( N_0' \) Condensate number measured in previous experimental run
\( N_R \) Mean number of atoms remaining after collapse
\( N_c \) Critical number for BEC, \( = 1.202(k_B T / h\omega)^3 \)
\( N_f \) Final value of \( N \)
\( N_i \) Initial value of \( N \)
\( N_m \) Stability limit for condensate with \( a < 0 \)
\( n \) Atomic density
\( \bar{n} \) Mean atomic density
\( n_s \) Number of radial bins generated from image
\( n_s \) Number of photons scattered by an atom during probe pulse
\( P \) Probability of observing an given condensate number
\( P_H \) Probability of a set of identical Bose coins to all be "H"
\( P_R \) Probability for a number of atoms to remain after collapse
\( P_S \) Probability for condensate to fill without collapsing
\( p \) \( |p| \)
\( p \) Momentum
\( p_H \) Probability of a single Bose coin being "H"
\( Q \) Function used to find \( N_m \) in asymmetric trap
\( q \ell/\ell_0 \)  
Distance between two colliding atoms

\( R_0 \)  
Condensate filling rate

\( R_A \)  
Atomic light-scattering rate

\( R_C \)  
Elastic collision rate

\( R_H \)  
Heating rate

\( R_P \)  
Photoassociation rate

\( R_U \)  
Range of \( U \)

\( R_s \)  
Light-scattering rate

\( R_{\text{loss}} \)  
Total loss rate from photoassociation beam

\( r \)  
Radial distance or separation, \(|r|\)

\( r \)  
Position

\( r_p \)  
Atom-pair distance satisfying 
\( R_U \ll r_p \ll \Lambda \)

\( r_t \)  
Classical turning point of bound state

\( S \)  
Reduced image signal

\( S/N \)  
Signal-to-noise ratio

\( s \)  
Signal measured on a pixel

\( s \)  
Scaled position \((\omega_x z, \omega_y y, \omega_z z)\)

\( T \)  
Temperature

\( T^{2B} \)  
Zero-energy limit of T-matrix, 
\[ = 4\pi h^2 a/m \]

\( T_c \)  
Critical temperature for BEC

\( t \)  
Time

\( t_0 \)  
Constant describing evaporation trajectory

\( t_p \)  
Transmission of phase plate for phase-contrast imaging

\( U \)  
Molecular potential

\( \bar{U} \)  
Pseudo-potential

\( u_1 \)  
Alternate fit parameter for \( N(t) \)

\( u_2 \)  
Alternate fit parameter for \( N(t) \)

\( u_l \)  
Radial wavefunction

\( V \)  
Trap potential

\( V^* \)  
Anti-trap potential

\( v \)  
Velocity

\( v \)  
Vibrational quantum number of a molecular state

\( v_B \)  
Parameter describing tunneling barrier of condensate

\( v_R \)  
Recoil velocity from spontaneous emission

\( W \)  
Phase error of imaging system

\( w \)  
Variation in number of atoms remaining after collapse

\( X_1 \)  
Initial value of \( X(t) \)

\( x \)  
Cartesian position coordinate

\( x \)  
Dummy continuous variable (i.e., in integrals)

\( x' \)  
Coordinate in image plane

\( x_c \)  
Coordinate of center of image

\( Y_{lm} \)  
Spherical harmonics

\( y \)  
Cartesian position coordinate

\( y' \)  
Coordinate in image plane

\( y_c \)  
Coordinate of center of image

\( Z \)  
Fugacity

\( z \)  
Axial position coordinate; direction of bias field in trap

\( z \)  
Dummy continuous variable (i.e., in integrals)

\( z' \)  
Coordinate normal to image plane

\( \alpha \)  
Resonant optical density

\( \beta \)  
\( 1/k_B T \)

\( \beta \)  
Dimensionless interaction strength parameter

\( \tilde{\beta} \)  
Photoassociation overlap parameter
\( \beta_s \) Light shift parameter
\( \chi^2 \) Goodness-of-fit parameter
\( \chi_p^2 \) Reduced goodness-of-fit parameter, 
\[ = \frac{\chi^2}{\nu} \]
\( \Delta \) Detuning of laser from atomic resonance
\( \Delta \varepsilon \) Molecular level spacing
\( \Delta \rho \) Detuning of photoassociation laser from atomic resonance
\( \Delta X \) Change or spread in quantity \( X \)
\( \delta X \) Variation, spread, or step in quantity \( X \)
\( \delta_s \) Light shift of photoassociation resonance
\( \delta(\cdot) \) Dirac delta-function
\( \epsilon \) Trap asymmetry \( \omega_p/\omega_z \)
\( \hat{\epsilon} \) Polarization vector of laser beam
\( \epsilon_0 \) Asymmetry of condensate, 
\[ = (\omega_p/\omega_z)^{1/2} \]
\( \varepsilon \) Efficiency of evaporative cooling
\( \varepsilon_q \) Quantum efficiency of CCD camera
\( \eta \) Figure-of-merit for photoassociation experiment
\( \eta_c \) \( E_c/k_B T \) for microwave sweep
\( \eta_T \) \( E_T/\langle E \rangle \) for evaporative cooling
\( \Gamma \) Natural linewidth of atomic transition
\( \Gamma_0 \) Decay rate of condensate due to tunneling
\( \Gamma_C \) Total decay rate of condensate
\( \Gamma_T \) Decay rate of condensate due to thermal excitation
\( \gamma_X \) Exponent in fit for \( X(t) \)
\( \gamma_p \) Logarithmic derivative of wave function at \( r_p \)
\( \gamma_{ij} \) Coupling between states \( i \) and \( j \)
\( \kappa \) Fit parameter for equilibration experiment
\( \kappa_N \) Alternate fit parameter for \( N(t) \)
\( \kappa_X \) Fitting parameter for \( X(t) \)
\( \Lambda \) deBroglie wavelength
\( \lambda \) Wavelength of light
\( \mu \) Chemical potential
\( \mu_B \) Bohr magneton
\( \nu \) Number of degrees of freedom in fit
\( \Omega \) Rabi frequency
\( \Omega_F \) Rabi frequency in notation of Fedichev et al., \( = \Omega/2^{1/2} \)
\( \Omega_T \) Microwave frequency for evaporative cooling
\( \omega \) Trap oscillation frequency, in angular units; specifically \( (\omega_p/\omega_z)^{1/3} \)
\( \omega_B \) Frequency of breathing mode of condensate
\( \omega_M \) Trap modulation frequency
\( \omega_X \) Trap oscillation frequency along coordinate \( X \)
\( \Phi \) Many-body wave function
\( \phi \) Phase shift of probe laser induced by atoms
\( \phi_0 \) Wave function during collision, unperturbed by photoassociation
\( \varphi \) Azimuthal angle in spherical coordinates
\( \pi^{-} \) Denotes transition with no change in angular momentum
\( \Psi \) Time-dependent wave function
\( \psi \) Wave function
\( \psi_o \) Excited state molecular wave function
\( \rho \) Radial coordinate in cylindrical symmetry
\( \varrho \) Ratio of \( N \) to \( N_c \)
\( \Sigma \) Lens aperture
\( \sigma \) Elastic scattering cross section
\( \sigma^+ \) Denotes transition in which angular momentum increases by \( \hbar \)
\( \sigma_L \) Light-scattering cross section
\( \sigma_X \) Uncertainty in quantity \( X \)
\( \tau \) Free evolution time of atom cloud
\( \tau_p \) Duration of probe pulse
\( \Theta \) Overlap integral for photoassociation
\( \theta \) Polar angle in spherical coordinates
\( \psi \) Polarizer angle for phase contrast imaging
\( \Upsilon \) Portion of \( \mathcal{I} \) integral involving distribution function \( f \)
\( \Xi \) Imaging perturbation coefficient
\( \xi \) Polarization overlap for photoassociation transition
\( \zeta \) Asymmetry of imaged cloud,
\[ = 3/(1 + 2\varepsilon) \]
\( \zeta_0 \) Asymmetry of imaged condensate,
\[ = 3/(1 + 2\varepsilon_0) \]
\( \zeta_v \) Detuning of photoassociation laser from molecular resonance
References


55. E. Aboundanzieri and P. Nordlander. Rice University, unpublished.


102. R. G. Hulet. Rice University, unpublished.