INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6” x 9” black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor MI 48106-1346 USA
313/761-4700  800/521-0600
RICE UNIVERSITY

Finite–Difference Seismic Wave Modeling including Surface Topography
by
Stig Ottar Hestholm

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE Doctor of Philosophy

APPROVED, THESIS COMMITTEE:

[Signatures of committee members]

Manik Talwani, Chairman
Schlumberger Professor, Geology and Geophysics

James Wright
Associate Professor, Geology and Geophysics

William Symes
Professor, Computational and Applied Mathematics

Colin Zelt
Assistant Professor, Geology and Geophysics

Houston, Texas
April, 1999
Abstract

Finite–Difference Seismic Wave Modeling including Surface Topography

by

Stig Ottar Hestholm

I present synthetics of seismic wave propagation near free surface topography. The velocity–stress formulations of both the full elastic and viscoelastic wave equations are used, and I have derived exact boundary conditions for any arbitrary, smooth topography in terms of the particle velocities. Program codes are developed for 2 and 3 dimensions (2–D and 3–D) using finite–difference (F–D) methods for both spatial and temporal numerical discretizations. An 8th order F–D method is used inside the physical model space, and the spatial F–D order decreases gradually towards the free surface topography. The discretization of the medium equations along the side and bottom boundaries, the free surface topography boundary conditions, and the forward time stepping, are all by 2nd order F–D methods. The leap–frog technique is used for time stepping everywhere except for the memory variable equations in the viscoelastic cases, where an explicit version of the unconditionally stable Crank–Nicholson method is used.

I show synthetics applying the schemes to isotropic 2–D and 3–D media covered by topographies that are either described by analytic expressions or by real elevation data. These data are taken from an area in South–Eastern Norway that contains the
NORESS seismic receiver array. Domains up to 60×60 kilometers are used in 3-D simulations, and the applied sources are plane waves generated by a plane of Ricker type point sources. These sources represent earthquakes or teleseismic explosions. For 2-D simulations I have used both plane waves and point sources, since the larger models permissible in 2-D allow for point sources to represent earthquakes or teleseismic explosions quite well. For 2-D simulations I have also included examples using layered media with randomization by a 2-D von Kármán function with and without apparent anisotropy.

Synthetic snapshots and seismograms show Rayleigh (Rg)–waves emanating from areas of prominent topography as well as strong surface wave directivity from some topographic features. Full viscoelastic modeling with relatively low Q–values, describing near–surface sedimentary layers, exhibit intrinsic attenuation and physical dispersion of the wavefield. Results coincide with numerous observations. 3-D simulations are performed using domain decomposition parallellization implemented by Message Passing Interface (MPI).
Acknowledgments

I greatly appreciate the support of my academic advisor prof. Manik Talwani during the work of this thesis. Without his flexibility and support I would not have been able to complete the PhD at Rice. I greatly benefited from support and advises from prof. Eystein Husebye and dr. Bent Ruud (both at the Dept. of Solid Earth Physics, University of Bergen, Norway) during this thesis work. Prof. Husebye’s communicative skills and outgoing nature have been invaluable for me in my contact with several international research institutions, among them Rice University. Thanks also to my thesis and advisory committee that consisted of prof. Bill Symes, prof. Colin Zelt and prof. Jim Wright in addition to my advisor. I would like to thank prof. Dan Sorensen (Dept. of Computational and Applied Mathematics, Rice University) for his effort as a member of my advisory committee. In particular I am indebted to prof. Bill Symes because of his important role in my thesis evaluation.

I thank prof. Alan Levander (Dept. of Geology and Geophysics, Rice University) for inviting me to Rice, and I acknowledge dr. Tor Sørevik (Dept. of Informatics, University of Bergen) for support on my use of parallel machines. The Norwegian Supercomputer Committee is acknowledged for granting computer time to my PhD program; I used parallel machines located at the Dept. of Informatics, University of Bergen, over internet from Rice. I thank dr. Ove Sævareid and dr. Bjarne Herland (both at Rogaland Research, Bergen) for support on code parallellization by MPI, and dr. Peeter Akerberg (Chevron Corporation, Houston, TX), dr. Johan Robertsson (Schlumberger Cambridge Research, Cambridge, UK) and dr. Satish Pullammanappallil (William Lettis Associates, San Francisco, CA) for useful discussions. I would also like to thank dr. Aladin Kamel (Regional Information Technology & Software Engineering
Center, Heliopolis, Cairo, Egypt) for initiating my interest in seismic modeling, and I would also like to mention dr. Johnny Petersen (Rogaland Research, Bergen) as an inspirational source.

I acknowledge the Norwegian Research Council and the US–Norway Fulbright Foundation for Educational Exchange for funding my PhD-program, as well as financial support from programs led by prof. Hans Munthe-Kaas (Dept. of Informatics, University of Bergen) and dr. Tor Arne Johansen (Dept. of Solid Earth Physics, University of Bergen). Finally and importantly, I thank and appreciate Elisabeth Irgens (my daughter's mother) for taking care of my daughter Maja Hestholm Irgens in Norway during my stay at Rice, as well as my daughter's patience in staying away from me.
Contents

Abstract ii
Acknowledgments iv
Preface x

1 3-D Finite-Difference Elastic Wave Modeling including Surface Topography 1
   1.1 Summary ........................................... 1
   1.2 Introduction ..................................... 1
   1.3 Elastic Wave Modeling Formulation ............... 3
   1.4 Free Surface Boundary Conditions ................ 8
   1.5 Numerical Discretization ......................... 11
   1.6 Stability Criterion for the Surface Topography Modeling ................. 12
   1.7 P- to Rg-Scattering from Topographic Relief ............ 13
   1.8 Discussion and Conclusions ....................... 19
   1.9 Acknowledgments ................................... 20

2 3-D versus 2-D Finite-Difference Seismic Synthetics in-
   cluding Real Surface Topography 21
   2.1 Summary ........................................... 21
   2.2 Introduction ..................................... 21
   2.3 Elastic Wave Modeling Formulation ............... 26
   2.4 Numerical Discretization ......................... 29
   2.5 Numerical Simulations using Real Topography .......... 31
2.5.1 Topography Surface versus Plane Surface .................................. 32
2.5.2 3-D versus 2-D ............................................................................. 35
2.6 Discussion ......................................................................................... 46
2.7 Conclusions ..................................................................................... 48
2.8 Acknowledgments ............................................................................ 49

3 Instabilities in Applying Absorbing Boundary Conditions
to High Order Seismic Modeling Algorithms .............................. 50
3.1 Introduction ....................................................................................... 50
3.2 2-D Finite-Difference Implementation ............................................ 51
3.3 Test Cases ......................................................................................... 52
3.4 Stability of OABC ............................................................................ 53
3.5 Efficiency ......................................................................................... 55
3.6 Conclusions ..................................................................................... 61
3.7 Acknowledgments ............................................................................ 66

4 3-D Finite-Difference Viscoelastic Wave Modeling includ-
ing Surface Topography ................................................................. 67
4.1 Summary ......................................................................................... 67
4.2 Introduction ..................................................................................... 67
4.3 Viscoelastic Wave Modeling Formulation ....................................... 73
4.4 Free Surface Boundary Conditions ............................................... 79
4.5 Numerical Discretization ................................................................. 81
4.6 A geometric example ...................................................................... 84
4.7 P- to Rg-Scattering from Topographic Relief ................................. 85
4.8 Discussion ....................................................................................... 107
4.9 Conclusions .......................................................... 115
4.10 Acknowledgments .................................................. 116

5 2-D Finite-Difference Viscoelastic Wave Modeling including Surface Topography .............................................. 117
5.1 Summary ................................................................. 117
5.2 Introduction ........................................................... 118
5.3 Viscoelastic Wave Modeling Formulation ...................... 121
5.4 Free Surface Boundary Conditions .............................. 126
5.5 Numerical Discretization ............................................ 128
5.6 Scattering from a Sinusoidal Topography .................... 131
5.7 Discussion ............................................................. 151
5.8 Conclusions ........................................................... 153
5.9 Acknowledgments .................................................... 153

6 Discussion and Conclusions ......................................... 155

A Partial derivatives in 3-D medium equations ................. 168

B 3-D elastic medium equations ..................................... 170

C 3-D surface topography boundary conditions, first version ................................................................. 171

D First order system of partial differential equations from the anelastic constitutive relation ................................. 176
E 3–D viscoelastic medium equations 178
F 3–D surface topography boundary conditions 180
G Partial derivatives in 2–D medium equations 188
H 2–D viscoelastic medium equations 189
I 2–D surface topography boundary conditions 190
Preface

The solid Earth/air interface constitutes the strongest interface variation we encounter in seismic modeling. Inclusion of effects caused by the presence of surface topography will therefore be of utmost importance in seismic modeling because any rugged shapes along an interface of such strong parameter variations necessarily will have great influences on the modeled wavefield. These influences are Rg-waves and Love-waves. Rg-waves propagate along free surfaces with a sound speed slightly slower than S-waves in the medium. They occur also for plane free surfaces, but they are greatly enhanced by the presence of surface topography, the more so the stronger topography varies. This is seen in the first chapter of this thesis (3-D Finite-Difference Elastic Wave Modeling including Surface Topography). Rg-waves are also generated to a lesser extent by near-surface heterogeneities, particularly lateral medium variations. Love-waves can be generated at any medium depth, because they are waves that propagate along interfaces (they are also called head-waves). Close to a free surface there might exist interfaces of prominent parameter variations, e.g. between sedimentary layers and solid rocks, and so such waves are also often observed near free surfaces.

In addition to the fact that free surface topographies are the boundaries most affecting the wavefield in seismic modeling, they are also the ones that can be described with the smallest error margin. All other interfaces are estimated by reflection and refraction seismology (usually for wavelengths around 100 meters and above) or from ground penetrating radar data (for wavelengths less than this). What is estimated by these techniques are sound speeds and incidence angles, and so other parameters, like densities and interface locations, have to be inferred from these. The uncer-
tainty of parameters and location of other interfaces are therefore far greater than for the surface topography. Therefore this should be another reason to use it in any modeling scheme once the technology is available. Generally, its inclusion in seismic modeling (along with earth heterogeneities) causes all possible wavefield effects automatically to be present in the resulting synthetics and hence represents snapshots and seismograms much closer to reality than by excluding it.

In reflection seismology the goal is to map subsurface reflectors, i.e. interfaces. This is done on the basis of reflection coefficients between layers, which again is caused by changes of material parameters between them. In this process, sea bottom reflections and multiples in marine exploration constitute the strongest signal and has to be muted and filtered away to enhance signal–to–noise ratio from layers. On land it is the near-surface waves, mainly Rg–, and in some cases Love–waves, that constitute strong amplitude waves that are attempted filtered out and suppressed by different processing techniques. Because of this, it can be difficult to single out reflections from deeper boundaries from the complete seismogram. Generally, surface noise problems result from shallow velocity and structure irregularities. Surface related seismic noise remain a serious problem in land seismic exploration for hydrocarbons. In order to fully understand which wave portions constitute this surface noise, and thereby increase confidence in the processing performed, it is important to model these surface waves as accurately as possible. In this way an accurate representation of the surface wavefield can greatly aid in the quality of subsurface mapping. Useful techniques here would be a combination of reverse time migration and forward modeling; the latter should be used to improve the quality of starting models in the migration. Such a combination of migration and forward modeling is widely used in processing for oil recovery, and results would be improved by including the surface topography wavefield. However, surface topography modeling is generally not included in processing in the
oil industry, probably because the technology is unknown and/or no efficient way of including it in standard processing has been found. Very approximate solutions to correct for topography are used however, i.e. static corrections.

A major environmental hazard to mankind is the occurrence of large earthquakes in densely populated areas. In addition to loss of human lives in the thousands, property damages may be in the billions of dollars while a long term effect is that of retarding economical development efforts. In the past, earthquakes have caused severe damages to many cities with corresponding losses of human lives, for instance in areas adjacent to the Mediterranean Sea. Evidence of such disasters stem both from cultural (written) sources and archaeology. Quite detailed descriptions are given of past earthquake disasters 'hitting' populated areas, including use of knowledge for mitigating earthquake hazards (Mendes-Victor et al., 1975). There have also been several specific field surveys for obtaining instrumental data for site response studies; one example is described by Chaves-Garcia et al. (1996). Although the occurrence of large earthquakes cannot be prevented nor reliably predicted, seismic hazard mitigation is feasible in terms of proper emergency planning, improved building codes and other relevant measures. However, the present day earthquake hazards are more severe than ever before due to expanding population centres even in traditional earthquake prone areas.

In most earthquake disasters the damage patterns are selective in the sense that certain parts of a city may be completely destroyed while other parts suffer only very moderate damages. In the past, such effects were mostly attributed to generalized wavefield focussing/defocussing effects, which seems to be an insufficient explanation because it usually requires unrealistic velocity anomalies (Hestholm et al., 1994; Dainty, 1995). More recently, seismologists have focussed their interests on site response effects since even simple modeling schemes indicate that ground amplification
enhancements may be of the order of 2 - 10 times that of the undistorted, incoming wavefront (Geli et al., 1988; Sanchez-Sesma and Campillo, 1993; Chavies-Garcia et al., 1996). Strong Rg-waves have been observed and attributed to near-surface irregularities with particularly high propagation efficiency in low-velocity weathering layers (Levander and Hill, 1985; Ruud et al., 1993). Amplification and deamplification of propagating waves can be shown to occur at irregularities and in substantial neighborhoods around them (Sanchez-Sesma and Campillo, 1991). Alluvial filled irregularly shaped 3-D valleys with a plane free surface can also generate strong Love- and Rg-waves (Sanchez-Sesma and Luzon, 1995). Real data correspondence of simulations is particularly important in works treating earthquake hazard assessment directly (Pitarka and Irikura, 1996). Bouchon et al. (1996) employ the boundary integral method in the spatial frequency domain to model 3-D responses in elastic media of incident shear-waves polarized along the minor and major axes of a cosine-formed elliptically shaped hill. The medium is otherwise homogeneous with a plane surface. They find Rg-waves to propagate and sustain over a much longer time and distance in the direction of the long axis than in the direction of the short axis of this hill. Major conclusions from such works should be possible to infer to real life by attempting to assess relative local hazard in relevant earthquake-prone areas. However, a topography closer to actual ones should then be implemented in the modeling scheme utilized, along with plausible subsurface characteristics. This can be done using the modeling method presented in this thesis, and examples displaying scattering characteristics from 3-D real topographies are given.

Specific boundary conditions are important in seismic wave modeling mainly for modeling wavefields near a free surface on land. At other boundaries, i.e. even sharp elastic or viscoelastic medium discontinuities or, more significantly, at water/sediment interfaces, full wave equations are generally used with appropriate material parameter
values. To avoid numerical artifacts in these cases, however, it is often necessary to introduce clever ways of discretizing the medium, e.g. by using curved interfaces (Fornberg, 1988b) or rotated and/or irregular grids with interpolation between grid points (Opršal and Zahradník, 1999) with F−D methods. The medium boundary of a free surface, however, generally constitutes too strong a medium interface to achieve even stable results by merely setting the medium parameters according to their values across interfaces in the wave equations. Specific boundary conditions for free surfaces are therefore necessary. For plane surfaces, these boundary conditions are known and easily implemented with F−D methods. The rugged shape of a free surface topography, however, requires specific thought in order to be included with the F−D method because of the rectangular grid shapes employed in this method. This thesis basically addresses this issue, and I believe the presented solution to the problem is among the most competitive, if not the best, when measured according to the accuracy-to-cost ratio for complex topographies.

With finite-element (F−E) methods (Lysmer and Drake, 1972; Marfurt, 1984), triangular elements are used to align computations along interfaces, and with boundary element methods (BEM) (Mansur and Brebbia, 1982b; Mansur and Brebbia, 1982a; Hall and Robertson, 1989), line segments are used for this to calculate values at boundaries for subsequent extrapolation of calculations into the medium. An advantage of the two latter numerical discretization methods over F−D methods is that free surface topography boundary conditions can be implemented in their basic form (i.e. by the free plane surface conditions) by adapting triangles or line segments (F−E and BEM respectively) to specific topographies. However, this discretization has to be given specific thought in both methods. For the F−E method, the triangularization can be optimized when given a specific topography. However, the cost of this method is generally higher than the one used by other methods for the same accuracy. For
the BEM, discretization of each specific topography has to be considered in order to avoid instabilities. BEM has therefore so far been used only with relatively simple surface topographies, generally generated by analytic expressions (Bouchon et al., 1996; Sanchez-Sesma and Campillo, 1993). If the F–D method can be used to model arbitrary surface topography, as is shown in this work, much is gained in terms of generality and cost.

The number of methods available for modeling free surface topography was quite limited for a long time. The amount of 2–D methods, however, has become more considerable lately, as opposed to 3–D where the number of presented works are still quite low. A classic 2–D F–D work classifies various types of points along a surface topography according to each surface point’s relative positioning to others (Jih et al., 1988). A similar approach is that by Robertsson (1996), where 7 categories of surface points are classified for separate treatment in a 2–D F–D scheme. Here free surface conditions are implemented by setting the particle velocities above the surface to zero and using the 'mirror'–condition for the stress components (Levander, 1988). An interesting F–D method (Komatitsch et al., 1996) employs a complete tensorial formulation for modeling the wave equations along curved interfaces and surface topographies. It has the advantage over the method presented in this thesis of using the same amount of spatial derivatives as that of a cartesian approach (i.e. without the terms added by the chain rule in the equations for the present method) which leads to a reduction of computational cost. This advantage is more than outweighed, however, by the vast additional memory requirement for the method, which is particularly prohibitive in 3–D. An early 3–D F–D method (Frankel and Leith, 1992) achieved acceptable results by tapering the densities to zero starting at the level of the free surface, while keeping the sound speed constant. In addition to the methods by Jih et al. (1988) and Robertsson (1996), there are two other methods for modeling
real surface topography by adapting F–D rectangles to it, however these are 3–D approaches (Ohminato and Chouet, 1997; Pitarka and Irikura, 1996). The latter work simulates aftershocks of the disastrous 1995 Kobe earthquake by implementing real local topography into the approach, again using the ‘mirror’-condition on stresses (Levander, 1988). The main disadvantage of these 'staircase' approaches is their implicit presence of unwanted diffraction effects from corner points of the staggered, rectangular grids, unless an unreasonably close spatial sampling rate is used. An advantage is their unconditional stability (given fulfillment of the Courant criterion).

In Tessmer and Kosloff’s (1994) 3–D algorithm for elastic wave modeling with free surface topography, which is an extension of their 2–D approach (Tessmer et al., 1992), they transform the velocity–stress formulation of the medium equations from a curved to a rectangular grid. They use a spectral discretization horizontally and a Chebyshev discretization vertically in space. At the free topography surface, the stresses and velocities are transformed into local systems in which the vertical coordinate axis is parallel to the normal of the local surface element. The free surface conditions are then implemented by a 'characteristic' treatment of both the velocity and stress components, before they are rotated back to the original system. They show results for simple geometric configurations, but in principle any arbitrary, smooth topography can be incorporated. The method presented in this thesis is based on their method because it transforms the equations of motion from a curved to a rectangular grid. The free surface boundary conditions, however, I have developed explicitly as an exact, closed set of equations for the particle velocities.

Utilization of the F–D method is achieved in this thesis by transforming (by linear algebraic rotations) both the medium equations (which consists of the constitutive equations and the momentum conservation equations) and the free surface boundary conditions from curved grids adapted to the surface topography in a cartesian co-
ordinate system, to rectangular grids in which the F–D method can be used. Until
discretization by the F–D method is performed, both the medium equations and the
boundary conditions for the free surface topography are exact equations valid for
curved grid representation. Therefore they are independent of the F–D method or
any other numerical discretization technique. The fact that all equations are ex-
act before numerical discretization is an appealing feature in itself because it opens
possibilities of employing the derived equations with other numerical discretization
schemes besides the F–D method. Also, it minimizes approximations needed for
modeling, since the only ones used result from applying the numerical discretization
scheme. By adapting a curved grid to surface topography, I also avoid diffraction
effects typical of 'stair-case' methods that attempt to accommodate rectangular grid
cells to it. The presented method for modeling free surface topography introduces no
extra memory requirement except for the actual topography data. Furthermore, the
additional computations introduced by the method involves some extra terms in the
medium equations (by the chain rule) and a more complex set of equations to solve
for the particle velocities at the surface topography. This set of equations, however,
as discretized by second order F–Ds, involves only 3 unknowns and is solved explicitly
for each surface point, therefore there is no noticeable additional CPU–time involved
by the method. Consequently, there is no reason not to use the method in seismic
modeling as long as topography data is known.

Another advantage of the mathematical formulation of the problem involves em-
ploying the velocity–stress version (Robertsson et al., 1994; Virieux, 1986; Levander,
1988) of all equations rather than the displacement–stress version (Kelly et al., 1976;
Carcione et al., 1988a). (The velocity–stress version is arrived at by time differenti-
ating the displacement–stress version). This leads to a significant improvement in
general stability by the fact that material parameters are not differentiated spatially.
Specifically, they are not differentiated across discontinuities as opposed to what is done in the displacement-stress version. Regarding spatial discretization, using the staggered version of the F-D coefficients also leads to an order of magnitude higher accuracy than by using non-staggered coefficients with equidistant F-D methods for the same F-D operator length.

To model earthquakes and teleseismic explosions (waves propagating over distances of a couple of hundred kilometers) for realistic model sizes and frequencies in 3-D, it was necessary to use several processors in parallel for the simulations. Message Passing Interface (MPI) was used to parallelize all 3-D code versions by explicit domain decomposition. On the current SGI (Cray) Origin 2000 machine located at the Dept. of Informatics, University of Bergen, Norway, up to 24 GB run time memory distributed over 128 processors can be used, which represents excellent computer power in any research environment. Because the machine also has shared memory, several processors can be accessed either by MPI code versions or by a compiler option. The former version proved to be the faster in all cases and was therefore the one used in all 3-D simulations.

For given topographies using either real topography data or analytic expressions, results are shown for media gradually more complex, from homogeneous via layered to randomized media (in 2-D) to show respective effects on wavefield complexities. When full viscoelastic equations with relatively low $Q$-values are used in addition, yet another complexity is added to snapshots and seismograms of the wavefields. Parameters, i.e. sound speeds, densities, $Q$-values, correlation distances, RMS's, etc., are always kept within realistic bounds for different types of media in the examples shown. In 3-D I have only shown results for homogeneous media. The reason is that there are so many additional effects to show by using full 3-D wave equations with complex topography compared to 2-D media with and without topography and
3-D media with plane free surfaces, that I emphasized attempts to analyze these. Adding physical wavefield dispersion and attenuation by including full viscoelasticity in the wavefields, leads to extra wavefield effects that are yet more significant using wavefield parameters typical for exploration. The significance of such parameters on the wavefield is best displayed using 2-D simulations, because of the high demands on computational storage such examples would lead to by modeling relevant high frequencies (> 10 Hz) in 3-D.

This thesis consists of 5 papers that are either published or submitted to refereed journals. The first is '3-D Finite-Difference Elastic Wave Modeling including Surface Topography', which is a slightly modified version of the paper published in Geophysics (Hestholm and Ruud, 1998). It presents a 3-D version of the free surface topography modeling algorithm used together with curved grid versions of the velocity-stress elastic wave equations. Applications are tied to real topography from the area in South-Eastern Norway containing the 3 km aperture NORESS array of permanent seismic receiver stations, and the subsurface medium is homogenous. Plane incident waves are used to represent teleseismic explosions. The second paper employs 2-D and 3-D versions of the surface topography modeling algorithm to compare their wavefields in order to attempt to identify differences in the responses from 2-D and 3-D topography. The significance of out-of-plane scattering effects from full 3-D wavefield modeling is specifically addressed here. This paper is accepted in a proceedings volume from the 29th General Assembly of IASPEI, Thessaloniki, Greece, August 18th-28th, 1997, for the session 'Seismological Data and Practice — Beyond Year 2000' (Hestholm et al., 1999). The third paper included in the thesis is 'Instabilities in applying absorbing boundary conditions to high order seismic modeling algorithms' (Simone and Hestholm, 1998), where two methods of absorbing artificial reflections from numerical grids were analyzed and compared in 2-D (Peng
and Toksöz, 1994; Cerjan et al., 1985). The latter is the one employed in the elastic simulations in this thesis, and general features of this method are assessed. This method is an important part of the modeling algorithm I employ.

The last two papers presented as part of this thesis include two new aspects of the modeling. Firstly, an extra transformation is applied to the free surface topography boundary conditions both in 2-D and 3-D. It was used also in the second paper, and should ideally have been applied in the first one. It consists of a transformation from the curved grid in the cartesian system to a rectangular grid (by using the chain rule as for the medium equations) after the initial transformation from local, rotated coordinate systems at each surface topography point. This extra transformation leads the method to be more robust, i.e. it is stable for more steep and/or abruptly varying topographies than before. In addition, artificial grid reflections attain less relative prominence in the wavefields. Secondly, full viscoelastic equations are included in the modeling. This is pursued for the velocity–stress formulation of the full viscoelastic wave equations (Robertsson et al., 1994), based on the displacement–stress formulation (Carcione et al., 1988b; Carcione, 1993). Curved grid versions of the full viscoelastic wave equations are arrived at as for the elastic case in the first paper (Hestholm and Ruud, 1998). In computing appropriate stress and strain relaxation times based on desired $Q$–values, I use results from Blanch et al. (1995). The two last papers are entitled '3-D Finite-Difference Viscoelastic Wave Modeling including Surface Topography' and '2-D Finite-Difference Viscoelastic Wave Modeling including Surface Topography' and they are submitted to refereed journals. A general feature of both papers is the improved absorbing boundary conditions achieved by the use of a linear (or cosine) tapering of the attenuation factor $Q$ towards a low value along the grid boundaries. In the 2-D paper exploration type parameters are employed, since adequate memory is readily available in 2-D to model frequencies
greater than 10 Hz. For these frequencies it is easy to display clear differences in the modeled wavefield for elastic and viscoelastic simulations, showing obvious physical dispersion and attenuation in the latter case. In 2-D I also show effects of including layered media with and without randomization using a von Kármán realization of apparent anisotropy both for elastic and viscoelastic wavefields. In the 3-D paper focus is on displaying out-of-plane scattering resulting from using the 3-D free surface topography boundary conditions with real topography over a homogeneous medium.

Each chapter of this thesis represents a paper, and therefore inadvertently there will be a certain degree of overlap between chapters. This is particularly true for the introductory parts of each chapter, where references to many common papers occur. Since the main goal of this thesis is to analyze scattering effects from surface topography for possible application to areas as different as oil exploration and seismic hazard, the conclusionary sections from many of the chapters proclaim the same main messages as well. The same essential procedure has been used everywhere to discretize the medium equations and the boundary conditions, so the paragraphs describing this are also relatively similar. Otherwise, different governing equations are used in each chapter (i.e. combinations of 2-D/3-D, elastic/viscoelastic) and/or different aspects of scattering are emphasized.
Chapter 1

3–D Finite–Difference Elastic Wave Modeling including Surface Topography

1.1 Summary

Three-dimensional (3-D) finite–difference (F–D) modeling of seismic scattering from free surface topography has been pursued. We have developed exact 3–D free surface topography boundary conditions for the particle velocities. A velocity–stress formulation of the full elastic wave equations together with the boundary conditions has been numerically modelled by an 8th order F–D method on a staggered grid. We give a numerical stability criterion for combining the boundary conditions with curved grid wave equations, where a curved grid represents the physical medium with topography. Implementation of this stability criterion stops instabilities from arising in areas of steep and rough topographies. We have simulated scattering from teleseismic P–waves using a plane, vertically incident wavefront and real topography from a 40 \times 40 km area centered at the NORESS array of seismic receiver stations in South-Eastern Norway. Synthetic snapshots and seismograms of the wavefield show clear conversion from P– to Rg– (short period fundamental mode Rayleigh) waves in an area of rough topography approximately 10 km east of NORESS. This result is consistent with numerous observations.

1.2 Introduction

The theory presented here is a direct extension from 2–D to 3–D of the corresponding theory in Hestholm and Ruud (1994). Inclusion of topography at the free surface
of an elastic medium leads to improved modeling of near-surface scattering effects, especially those in the high frequency part of the wavefield. Relatively little has been published on the modeling of free surface topography in 2-D and even less on the modeling of 3-D surface topography. This applies both to F–D methods and any other numerical discretization method. However, a work on this problem is that of Frankel and Leith (1992) who used an F–D scheme of fourth order accuracy in space and a density taper to zero starting at the height of the free surface while keeping the medium P-velocity unaltered. They achieved stable results by multiplying the crustal density by 0.4 for the locations one grid point above the free surface, by 0.16 for the locations two grid points above the free surface, and so on. In this manner they were able to obtain reasonable modeling results for free 3-D surface topography models.

More recently, Tessmer and Kosloff expanded their procedure for elastic wave modeling with free surface topography from 2-D to 3-D (Tessmer et al., 1992; Tessmer and Kosloff, 1994). They use a spectral discretization in space, being different in the horizontal and vertical directions. The velocity-stress formulation of the elastic wave equations is transformed into a curved grid. At the free surface, the stresses and velocities are transformed into a local system in which the vertical coordinate axis is normal to the surface element. The free surface conditions are then implemented by a 'characteristic' treatment of the velocity and stress components, before rotated back to the original system. In this method, both velocities and stresses are rotated into the local system at each point of the topography surface.

In the present work explicit 3-D boundary conditions for a free surface topography are derived. The basis is the vanishing stress condition for a free surface. As in Tessmer and Kosloff (1994), a 3-D curved grid which is stretched in the vertical direction is adapted to the surface topography, i. e. the top surface of the grid co-
incides with the surface topography. A coordinate transform is then introduced for transferring the elastic, isotropic wave equations from the curved to a rectangular grid in which the numerical computations are done. At the topography surface, the velocity boundary conditions for a free surface are implemented into a local, rotated system at each point on the surface. Each of these systems has its vertical coordinate direction coinciding with the direction of the normal vector of the surface at the given point. The velocity boundary conditions are subsequently rotated back to the rectangular system. Once the boundary conditions are given in this system, the numerical discretization can be performed.

In the following paragraphs, the 3-D equivalents to the 2-D equations (Hestholm and Ruud, 1994) will be stated. In each instance, it is possible to verify coincidence between the 3-D and the 2-D case by eliminating the $\kappa$-direction (2nd coordinate direction of the computational grid) in 3-D. We can also verify the coincidence between the plane surface conditions in 3-D and the 3-D surface topography conditions applied to a plane surface. A description of the numerical discretization will be given and stability criteria for the method will be assessed. Then we present simulated scattering from teleseismic P-waves using a plane vertically incident wavefront and real topography from an area centered at the NORESS array in South-Eastern Norway. Snapshots and synthetic seismograms of the wavefield will be shown, clearly displaying Rg waves in areas of rough topography. Finally, we look at future prospects for 3-D F-D modeling, particularly in light of the recent parallelization of the seismic code.

1.3 Elastic Wave Modeling Formulation

The basic equations governing wave propagation in a continuous elastic medium are the equations of motion and the stress–strain relationship. The velocity–stress for-
mulation (Achenbach, 1975; Virieux, 1986) can be written in 3-D as

\[
\begin{align*}
\rho \frac{\partial u}{\partial t} &= \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x, \\
\rho \frac{\partial v}{\partial t} &= \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y, \\
\rho \frac{\partial w}{\partial t} &= \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z,
\end{align*}
\]

(1.1) (1.2) (1.3)

\[
\begin{align*}
\frac{\partial \sigma_{xx}}{\partial t} &= \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2\mu \frac{\partial u}{\partial x}, \\
\frac{\partial \sigma_{yy}}{\partial t} &= \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2\mu \frac{\partial v}{\partial y}, \\
\frac{\partial \sigma_{zz}}{\partial t} &= \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2\mu \frac{\partial w}{\partial z}, \\
\frac{\partial \sigma_{xy}}{\partial t} &= \mu \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \\
\frac{\partial \sigma_{xz}}{\partial t} &= \mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right), \\
\frac{\partial \sigma_{yz}}{\partial t} &= \mu \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right).
\end{align*}
\]

(1.4) (1.5) (1.6) (1.7) (1.8) (1.9)

where \(\rho\) is the density and \(\lambda\) and \(\mu\) are Lamé’s parameters. \(f_x, f_y\) and \(f_z\) are the components of the body forces, \(u, v\) and \(w\) are the particle velocity components and \(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}\) and \(\sigma_{yz}\) are the stress components.

We introduce a linear mapping from a rectangular \((\xi, \kappa, \eta)\)-system (Figure 1.1) to a curved \((x, y, z)\)-system (Figure 1.2), where both systems have positive direction upwards for the vertical coordinate. The 3-D mapping can be written as

\[
\begin{align*}
x(\xi, \kappa, \eta) &= \xi, \\
y(\xi, \kappa, \eta) &= \kappa, \\
z(\xi, \kappa, \eta) &= \frac{\eta}{\eta_{max}} z_0(\xi, \kappa),
\end{align*}
\]

(1.10) (1.11) (1.12)

where \(z_0(\xi, \kappa)\) is the topography function, and the rectangular \((\xi, \kappa, \eta)\)-system is limited by \(\xi = 0, \xi = \xi_{max}, \kappa = 0, \kappa = \kappa_{max}, \eta = 0\) and \(\eta = \eta_{max}\). For the
Figure 1.1  Rectangular system surface.
Figure 1.2  Curved system surface.
curved \((x, y, z)\)-system the degree of stretching is proportional to the distance from the bottom plane of the system \((z = 0)\). From equations (1.10)–(1.12) we get, for a differentiable function \(f(x, y, z)\),

\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x},
\]

(1.13)

\[
\frac{\partial f}{\partial y} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y},
\]

(1.14)

\[
\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial z},
\]

(1.15)

Expressions for the partial derivatives, which are needed in the medium equations, are found from equations (1.10)–(1.12) (see Appendix A),

\[
\frac{\partial \xi}{\partial x} = 1, \quad \frac{\partial \xi}{\partial y} = 0, \quad \frac{\partial \xi}{\partial z} = 0,
\]

(1.16)

\[
\frac{\partial \kappa}{\partial x} = 0, \quad \frac{\partial \kappa}{\partial y} = 1, \quad \frac{\partial \kappa}{\partial z} = 0,
\]

(1.17)

\[
A(\xi, \kappa, \eta) = \frac{\partial \eta}{\partial x} = -\frac{\eta}{z_0(\xi, \kappa)} \frac{\partial z_0(\xi, \kappa)}{\partial \xi},
\]

(1.18)

\[
B(\xi, \kappa, \eta) = \frac{\partial \eta}{\partial y} = -\frac{\eta}{z_0(\xi, \kappa)} \frac{\partial z_0(\xi, \kappa)}{\partial \kappa},
\]

(1.19)

\[
C(\xi, \kappa) = \frac{\partial \eta}{\partial z} = \frac{\eta_{\text{max}}}{z_0(\xi, \kappa)}.
\]

(1.20)

The velocity–stress formulation of the equations of motion and Hooke’s law is given in the curved \((x, y, z)\)-grid by equations (1.1)–(1.9). Expanding these by the chain rule (Appendix B) as in Hestholm and Ruud (1994), we get the medium equations in the rectangular \((\xi, \kappa, \eta)\)-system,

\[
\rho \frac{\partial u}{\partial t} = \frac{\partial \sigma_{x\xi}}{\partial \xi} + \frac{\partial \sigma_{x\eta}}{\partial \eta} + \frac{\partial \sigma_{x\kappa}}{\partial \kappa} + A(\xi, \kappa, \eta) \frac{\partial \sigma_{xy}}{\partial \eta} + B(\xi, \kappa, \eta) \frac{\partial \sigma_{xy}}{\partial \kappa} + C(\xi, \kappa) \frac{\partial \sigma_{xy}}{\partial \xi} + f_x,
\]

(1.21)

\[
\rho \frac{\partial v}{\partial t} = \frac{\partial \sigma_{y\xi}}{\partial \xi} + \frac{\partial \sigma_{y\eta}}{\partial \eta} + \frac{\partial \sigma_{y\kappa}}{\partial \kappa} + A(\xi, \kappa, \eta) \frac{\partial \sigma_{yx}}{\partial \eta} + B(\xi, \kappa, \eta) \frac{\partial \sigma_{yx}}{\partial \kappa} + C(\xi, \kappa) \frac{\partial \sigma_{yx}}{\partial \xi} + f_y,
\]

(1.22)
\[ \rho \frac{\partial w}{\partial t} = \frac{\partial \sigma_{xx}}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial \sigma_{xx}}{\partial \eta} + \frac{\partial \sigma_{yz}}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial \sigma_{yz}}{\partial \eta} \\
+ C(\xi, \kappa) \frac{\partial \sigma_{zz}}{\partial \eta} + f_z, \quad (1.23) \]

\[ \frac{\partial \sigma_{xx}}{\partial t} = (\lambda + 2\mu) \left( \frac{\partial u}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} \right) \\
+ \lambda \left( \frac{\partial v}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} + C(\xi, \kappa) \frac{\partial w}{\partial \eta} \right), \quad (1.24) \]

\[ \frac{\partial \sigma_{yy}}{\partial t} = \lambda \left( \frac{\partial u}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + C(\xi, \kappa) \frac{\partial w}{\partial \eta} \right) \\
+ (\lambda + 2\mu) \left( \frac{\partial v}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} \right), \quad (1.25) \]

\[ \frac{\partial \sigma_{zz}}{\partial t} = \lambda \left( \frac{\partial u}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} \right) \\
+ (\lambda + 2\mu) C(\xi, \kappa) \frac{\partial w}{\partial \eta}, \quad (1.26) \]

\[ \frac{\partial \sigma_{xy}}{\partial t} = \mu \left( \frac{\partial v}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} + \frac{\partial u}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} \right), \quad (1.27) \]

\[ \frac{\partial \sigma_{xz}}{\partial t} = \mu \left( \frac{\partial w}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial w}{\partial \eta} + C(\xi, \kappa) \frac{\partial u}{\partial \eta} \right), \quad (1.28) \]

\[ \frac{\partial \sigma_{yz}}{\partial t} = \mu \left( \frac{\partial w}{\partial \xi} + B(\xi, \kappa, \eta) \frac{\partial w}{\partial \eta} + C(\xi, \kappa) \frac{\partial v}{\partial \eta} \right), \quad (1.29) \]

Equations (1.21)–(1.29) are the momentum conservation equations and Hooke’s law given in the rectangular $(\xi, \kappa, \eta)$–system.

### 1.4 Free Surface Boundary Conditions

The 3-D free boundary conditions for the velocities at a locally horizontal surface (or in a system where the $z$–axis is parallel to the local normal vector of the surface) resulting from the vanishing stress condition can be written

\[ \frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x}, \quad (1.30) \]

\[ \frac{\partial v}{\partial z} = -\frac{\partial w}{\partial y}. \quad (1.31) \]
\[
\frac{\partial w}{\partial z} = -\frac{\lambda}{\lambda + 2\mu} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right),
\]
(1.32)

with \(x\) and \(y\) the horizontal coordinates and \(z\) the vertical coordinate. We want to apply these conditions to a topography surface. At each surface point, we introduce a local coordinate system \((x', y', z')\) in which the \(z'\)-axis coincides with the local normal vector direction of the surface. In this local system we impose the conditions (1.30)—(1.32). Once this is done, they have to be rotated back to the rectangular system \((\xi, \kappa, \eta)\) where the numerical computations can be done. This rotation is expressed by \(\vec{v} = A^{-1}\vec{v}', i.e.
\]
\[
\vec{v}' = A\vec{v},
\]
(1.33)

where \(\vec{v}\) and \(\vec{v}'\) are the particle velocity vectors in the \((\xi, \kappa, \eta)\) and \((x', y', z')\) systems respectively. \(A\) is the rotation matrix, which can be given by
\[
A = \begin{pmatrix}
\cos \theta & 0 & \sin \theta \\
-\sin \theta \sin \phi & \cos \phi & \cos \theta \sin \phi \\
-\sin \theta \cos \phi & -\sin \phi & \cos \theta \cos \phi
\end{pmatrix}.
\]
(1.34)

where \(\theta\) is the rotation angle between the \(\xi\)-axis and the local \(x'\)-axis in the \((\xi, \eta)\)-plane and \(\phi\) is the rotation angle between the \(\kappa\)-axis and the local \(y'\)-axis in the local \((y', z')\)-plane. We also have the relations \(\tan \theta = \partial z_0(\xi, \kappa)/\partial \xi\) and \(\tan \phi = \partial z_0(\xi, \kappa)/\partial \kappa \cos \theta\) (see Appendix C).

Now the calculations of the rotation from the local \((x', y', z')\)-system back to the \((\xi, \kappa, \eta)\)-system can be performed as in Appendix C. We arrive at the 3-D boundary conditions for free surface topography given in the computational \((\xi, \kappa, \eta)\)-system by
\[
\left(1 - d^2\right) \frac{\partial u}{\partial \eta} - \frac{dp}{e} \frac{\partial v}{\partial \eta} + 2d \frac{\partial w}{\partial \eta} \\
= 2d \frac{\partial u}{\partial \xi} + \frac{p}{e} \frac{\partial v}{\partial \xi} + (d^2 - 1) \frac{\partial w}{\partial \xi} + \frac{p}{e} \frac{\partial u}{\partial \kappa} + \frac{dp}{e} \frac{\partial w}{\partial \kappa},
\]
(1.35)
\[
2fp \frac{\partial u}{\partial \eta} + (p^2 - 1) \frac{\partial v}{\partial \xi} - 2ep \frac{\partial w}{\partial \eta} + 2f \frac{\partial v}{\partial \xi} + d (p^2 - 1) \frac{\partial u}{\partial \kappa} - \frac{2f \partial v}{\partial \xi} + \frac{1 - p^2}{e} \frac{\partial w}{\partial \kappa},
\]
\[
\frac{d \{1 - \zeta\}}{\partial \eta} + \frac{p}{e} \{1 - \zeta\} \frac{\partial v}{\partial \eta} - \left\{\zeta \left(\frac{d^2}{q^2} + p^2\right) + 1\right\} \frac{\partial w}{\partial \eta} + \frac{dp}{e} \{1 - \zeta\} \frac{\partial u}{\partial \kappa} + \frac{1}{e^2} \{\zeta + p^2\} \frac{\partial v}{\partial \kappa} + \frac{p}{e} \{\zeta - 1\} \frac{\partial w}{\partial \kappa},
\] (1.36)
(1.37)

with

\[
\zeta = \frac{\lambda}{\lambda + 2\mu},
\] (1.38)
\[
d = \frac{\partial z_0(\xi, \kappa)}{\partial \xi},
\] (1.39)
\[
e = \cos \left[\arctan \left(\frac{d}{\partial\xi}\right)\right],
\] (1.40)
\[
f = \sin \left[\arctan \left(\frac{d}{\partial\xi}\right)\right],
\] (1.41)
\[
p = \frac{\partial z_0(\xi, \kappa)}{\partial \kappa} e,
\] (1.42)
\[
q = \cos \left[\arctan \left(\frac{p}{\partial\kappa}\right)\right],
\] (1.43)
\[
r = \sin \left[\arctan \left(\frac{p}{\partial\kappa}\right)\right].
\] (1.44)

Equations (1.35)–(1.44) are exact 3–D boundary conditions for an arbitrary, smooth, free surface topography. They result from rotating the velocity free surface conditions from local systems at each point of the surface topography into a rectangular system (Appendix C). The vertical axis in each of the local systems is normal to the local topography. Also, the boundary conditions (1.35)–(1.44) are obviously not restricted to the F–D method or any other numerical discretization technique.
1.5 Numerical Discretization

For the numerical discretization, we refer to the corresponding paragraph in Hestholm and Ruud (1994). The same spatial and time discretization methods as in the 2-D case (Kindelan et al., 1990) are used. The schemes employ a staggered discretization stencil as in Levander (1988) of the velocity–stress formulation of the elastodynamic wave equations, which is again based on the work of Virieux (1986). An advantage of using a staggered definition of variables is that we can avoid explicit definition of the stresses at the surface topography as it suffices to define the velocities there. In order to get the velocities and stresses explicitly defined at each time step, we have to stagger the vertical velocity component $w$ one half grid length downwards. Corresponding numerical definitions of the other variables have to be implemented. Generally, $u$ is staggered one half grid length in the positive $\xi$–direction, $v$ is staggered one half grid length in the positive $\kappa$–direction, while $w$ is staggered one half grid length in the negative $\eta$–direction (downwards). The 3-D boundary conditions, equations (1.35)–(1.37) are discretized by 2nd order, staggered F–D operators (Fornberg, 1988a). Below the free surface, the central, staggered F–D method’s order is gradually increased with depth, via 4th and 6th up to 8th order, the latter is the order used inside the domain. This 8th order method is dispersion–bounded and cost–optimized (Kindelan et al., 1990). Along the artificial grid boundaries exponential damping according to Cerjan et al. (1985) is used. In a layer of 20 points along each grid boundary, the stresses and velocities are multiplied by exponentially decreasing terms towards the boundary.

To find the velocity components at the surface topography from the closed system (1.35)–(1.37), we solve it directly as a linear system with respect to the velocities at the surface as they are defined in the 2nd order vertical derivative discretizations. In this procedure, the horizontal partial derivatives are calculated one grid length or one and a half grid length below the free surface and considered known.
1.6 Stability Criterion for the Surface Topography Modeling

A numerical stability criterion for the surface topography modeling comes from the equations of motion in the medium. A necessary condition to keep a run stable is that the absolute values of the parameters $A(\xi, \kappa, \eta)$, $B(\xi, \kappa, \eta)$ and $C(\xi, \kappa)$ all be kept less than 1. This necessary condition can be written as

$$\frac{z_0(\xi, \kappa)}{\eta_{\text{max}}} > \max \left\{ \frac{1}{\| \partial z_0(\xi, \kappa) / \partial \xi \|}, \frac{1}{\| \partial z_0(\xi, \kappa) / \partial \kappa \|} \right\},$$

where $\eta_{\text{max}}$ is the total depth of the numerical $(\xi, \kappa, \eta)$–grid and $\partial z_0(\xi, \kappa) / \partial \xi$ and $\partial z_0(\xi, \kappa) / \partial \kappa$ are topography slopes. This condition must be satisfied at every point on the surface. Note that this condition can always be satisfied, if necessary, by uniformly increasing $z_0(\xi, \kappa)$ everywhere. Equivalently, $\eta_{\text{max}}$ can be decreased to satisfy the inequality.

For some combinations of sources and reliefs, this condition might not be sufficient to avoid an unstable growth right below the surface at the steepest parts of the topography. In these cases, $\eta_{\text{max}}$ should be reduced enough for stability to be maintained. By decreasing $\eta_{\text{max}}$ rather than increasing $z_0(\xi, \kappa)$ uniformly, the original physical model (represented by the curved grid) is conserved. It is a matter of experimentation, then, how much $\eta_{\text{max}}$ must be reduced in order to make wavefield simulations stable for a specific relief. The number of vertical samples must be the same in the curved and rectangular systems. This means that in order to maintain accuracy and simultaneously retain stability gained from increased curved to rectangular grid depth ratio, the vertical grid distance (vertical distance between grid points) in the numerical $(\xi, \kappa, \eta)$–system must be reduced by an appropriate factor.

For most sources/topography constellations, it turns out that $\min\{z_0(\xi, \kappa)\} \approx 3\eta_{\text{max}}$ is enough for a simulation to be stable. Therefore, this criterion might be stated
as a sufficient condition for stability in most cases. From our experiments, this order of magnitude for the ratio between the physical model and the numerical grid depth appears necessary whenever the topography data exhibits rough behaviour (large spatial second derivatives) near its steepest slopes. For rough topography without large spatial derivatives or steep topography without large second derivatives, this ratio may be smaller in order to achieve stability. In many such cases the stated necessary stability condition is also sufficient.

1.7 P– to Rg–Scattering from Topographic Relief

Since 3-D topography surfaces in general, and real reliefs in particular, are seldom used in detailed wavefield modeling, we hope that the above approach should constitute a powerful method for the F–D elastic wave modeling including surface topography. The main goal of the research performed is to improve our understanding of wave propagation and scattering phenomena for local and regional distance travel paths. An interesting case here is observationally well documented. Scattering of teleseismic P–wave energy into Rg have been extensively studied using data from the 3 km aperture NORESS array in South–Eastern Norway (Bannister et al., 1990; Gupta et al., 1993; Hedlin et al., 1991; Hedlin et al., 1994). For our 3–D F–D simulation of this phenomenon we have obtained digital elevation data for an area of 100 × 100 km centered on the NORESS array (Figure 1.3). Due to the huge computer memory requirements of 3-D F–D methods, we have so far been restricted to a model of size 40 × 40 × 35 km with 0.2 km sampling. However, this situation is now improved by parallelization of our code. In all the examples shown the incoming wave is a vertically incident plane P–wave simulating a teleseismic short period P–phase. The center frequency of the Ricker wavelet is 2.5 Hz, the P–wave velocity of the homogeneous medium is 6.0 km/s and the Poisson ratio is 0.25.
**Figure 1.3** The leftmost section shows the topography in a $100 \times 100$ km area centered at the NORESS array. The innermost $40 \times 40$ km was used in the modeling experiment. The dotted lines show the positions of the receivers in the two synthetic profiles and the circle outlines the NORESS array aperture containing sensors in a circular pattern. Labels are in kilometers and elevations are in meters above mean sea level. The black area is Lake Mjøsa (123 m above sea level).
In our present experiment the elevations were multiplied by 0.5 so that topography undulations were damped. By doing this the magnitudes of $A(\xi, \kappa, \eta)$ and $B(\xi, \kappa, \eta)$ were reduced so as to achieve stable results without the need of increasing the model depth beyond the depth of the numerical grid. Alternatively, the numerical grid depth could have been reduced or the model grid depth could have been increased by some factor. Snapshots of the wavefield (vertical component of the particle velocity) are shown in Figure 1.4. A dominant feature here is the low-frequency/long-wavelength artificial reflections from the absorbing model boundaries. Although the exponential damping technique is the most efficient absorbing boundary method we have tested, a wave incident at an angle of $90^\circ$ with the boundary is the most difficult case. Fortunately, the artificial reflections are frequency dependent, with longer wavelengths than most of the surface waves scattered by the topography, and can therefore be removed by filtering and image processing techniques. As seen from Figure 1.4 the scattered surface waves appear to radiate out from secondary point sources which coincide with areas of high topographic gradients (Figure 1.3). The dominant scattering points are along the steep valley side east of 'Bronkeberget' about 10 km east of NORESS (Bannister et al., 1990). Also in seismograms from the west–east profile P–to–Rg scattering from this area is clearly seen (Figure 1.5).

In order to quantify and locate areas of significant topographic features, we have computed different functions depending on first and second order derivatives of the topography function $z = z_0(x, y)$. First, we computed the slope, i.e., the length of the topography gradient vector as

$$|\nabla z_0(x, y)| = \left[\left(\frac{\partial z_0}{\partial x}\right)^2 + \left(\frac{\partial z_0}{\partial y}\right)^2\right]^{1/2}$$ (1.45)

This vector is useful for defining areas of strong scattering. Additionally, smooth topography is assumed in deduction of the boundary conditions. As a measure for
Figure 1.4  Snapshots of the vertical component of the particle velocity. The upper snapshots are horizontal sections at the free surface and the lower snapshots are vertical sections along a west-east profile through the center of the model. The time in the lower left corner of each horizontal section is the time from when the plane vertically incident P-wave was reflected from the surface. The straight wavefronts parallel to the model boundaries are artificial reflections from the absorbing boundaries which are seen also in the vertical sections. Note the strong scattering source located at Bronkeberget (2 km N, 10 km E) which is also highly prominent in real record analysis (Bannister et al., 1990).
Figure 1.5  Seismograms extracted at the free surface of the model. The left section shows the vertical component of the particle velocity along a 30 km long west-east profile through the center of the model. The right section shows 3-component seismograms from a receiver 3.6 km east of the center.

Topography roughness, we compute the Frobenius norm of the Hessian matrix of $z_0(x, y)$, i.e.,

\[ \| H(x, y) \|_F = \left[ \left( \frac{\partial^2 z_0}{\partial x^2} \right)^2 + \left( \frac{\partial^2 z_0}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 z_0}{\partial y \partial x} \right)^2 + \left( \frac{\partial^2 z_0}{\partial y^2} \right)^2 \right]^{1/2} \]  (1.46)

Both these functions have their largest values in the areas of strongest scattering (Figure 1.6).

Recently, we modified the code to run on parallel computers. The tool used for parallelization was MPI (Message Passing Interface), which is the first attempt of standardization to some degree of parallel programming tools. Besides speeding up computations, the parallelization also allows us to use a larger computer memory, which is essential for realistic seismic wavefield simulation. Test runs employing up to 2GBytes of memory have been performed. For a simulation of this size we used 48 hours on 48 processors on the Intel Paragon machine at the Univ. of Bergen, Norway. Increasing the number of processors to 80, which is the maximum possible number of processors on the machine's batch queue, the simulation time was reduced to a little
Figure 1.6 Length of topography gradient vector and Frobenius norm of Hessian matrix.
more than 6 hours, i.e., a speed-up factor of 8. The main reason for the significant speed-up in the latter case is that the complete 2GB of memory was distributed over real memory on the processors, so that no virtual memory had to be used, and thus paging to and from disk was avoided.

1.8 Discussion and Conclusions

Even for small scale experiments 3-D F-D synthetic wavefield analysis provides, as demonstrated here, improved insight and a better understanding of surface scattering phenomena. We find it particularly gratifying that the strong P-to-Rg scattering from the Bronkeberget hill, observed by Bannister et al. (1990) through analysis of NORESS recordings, can be realistically synthesized. Another interesting feature is that for certain spatial low-frequency variants of the local NORESS topography the Rg-wave propagation seems to change abruptly over relatively small distances. Probably, this is due to strong directionality dependence of the scattering from some topographic features. Such wavefield characteristics are sometimes observed in NORESS recordings and at the German array GERESS located in the Bavarian hills – at some sensors the Rg-phase is prominent while hardly visible at nearby sensors less than a kilometer away. In other words, also small scale crustal features may contribute to blocking effects as often observed for Rg- and Lg-phases across structural obstacles. Many of such wavefield phenomena are out of range for 2-D F-D synthetic experiments; hence our emphasis on continuous research efforts on 3-D synthetic simulation of crustal wavefield propagation.

Future directions of our research will be to increase the model size to also include the scattering areas south-east of Lake Mjøsa (Bannister et al., 1990). Furthermore, in order to compare directly with NORESS recordings it is necessary to allow for non-vertical incident waves and to use source time functions derived from teleseismic
P-beams. We also plan to implement the topography boundary conditions into 2-D and 3-D viscoelastic schemes. This will be done in order to be able to simulate a realistic attenuation of waves, in particular the strong waves generated from areas of significant surface topography. In addition, this will greatly improve the absorbing boundary conditions. Although the physical dimensions of 3-D models used in simulation will still be relatively small compared to 2-D models employed, the use of parallel computer technology opens new avenues to the study of 3-D scattering phenomena in fields like random media and corrugated interface scattering.

1.9 Acknowledgments

In developing the seismic code, the authors worked with prof. Eystein Husebye (Univ. of Bergen, dept. of Solid Earth Physics), who also helped improving the manuscript. This research was supported by the US–Norway Fulbright Foundation for Educational Exchange, The Norwegian Research Council and the Air Force Office of Scientific Research, USAF under Grant F49620-94-1-0278. S. H. would like to thank the following people for support and resources: Dr. Aladin Kamel, Dr. Patrick Gaffney and Dr. Johnny Petersen (all from the former IBM Bergen Environmental Sciences and Solutions Centre), Dr. Tor Arne Johansen (Univ. of Bergen, Dept. of Solid Earth Physics), and Dr. Hans Munthe–Kaas (Univ. of Bergen, Dept. of Informatics). S. H. also thanks M. Nafi Toksöz for the 1993 invitation to Earth Resources Laboratory, M.I.T., where the theory was developed.
Chapter 2

3–D versus 2–D Finite–Difference Seismic Synthetics including Real Surface Topography

2.1 Summary

We have pursued and compared two- and three-dimensional (3-D) finite-difference (F–D) modeling of scattering from free surface topography. A velocity–stress formulation of the full elastic wave equations are combined with exact boundary conditions for the surface topography and numerically discretized by an 8th order F–D method on a staggered grid. We have simulated scattering in 2–D and 3–D from teleseismic P–waves using a plane, vertically incident P–wave and real topography from a 60 × 60 km area including the NORESS array in South-Eastern Norway. Many field observations that are not easily explained by simpler 2–D cases are shown to better match qualitative effects from 3–D surface topography modeling. These include strong amplifications at hills, complex wave pattern caused by scattering, and directivity of scattered waves. Snapshots and seismograms show clear conversion from P– to Rg (short period fundamental mode Rayleigh) waves in an area of rough topography in the vicinity of the array site. All results are consistent with numerous observations. By parallelizing the software, possibilities have been opened for modeling with higher resolution and/or larger areas than before.

2.2 Introduction

Modeling free surface topography effects is naturally only important for seismic profiling on land, whereas in marine settings it satisfies to specify the medium parameters
without including any explicit boundary conditions. The air–solid Earth interface exhibits the strongest possible impedance contrast. For this reason alone, modeling topography along such a free surface should be important. Any irregularities along such a surface would have additional consequences for the results, the more so the stronger the gradients and/or irregularities the local topography exhibits (Hestholm and Ruud, 1998). Additionally, uncertain, or even undetermined, parameters are used when performing a seismic simulation, like seismic velocities and the associated densities. Topography data on the other hand, probably has the smallest error margins of any model parameters we use.

It is a goal for a seismic modeling experiment to achieve good match with observations. In this regard the advantage of modeling with surface topography is that implicit effects like scattering and conversion will be accounted for automatically in the wavefield synthetics. For example, plane incoming P– or S–waves cannot convert to Rg–waves when incident on a plane surface of a homogeneous medium. Also, P– to SH–phase conversion depends on surface topography. Amplification and deamplification of propagating waves can be shown to occur at irregularities and in substantial neighborhoods around them (Sanchez-Sesma and Campillo, 1991). Alluvial filled irregularly shaped valleys with a plane free surface is also seen to generate strong Love– and Rg–waves for some incident waves in 3–D (Sanchez-Sesma and Luzon, 1995). Real data correspondence of simulations has proven to be particularly important in works on earthquake hazard assessment. In urban areas housing are preferably located in hilly, 'socially prestigious' areas, often with scant attention to possible amplifications from damaging earthquakes. Recently, a study aimed at shedding light on this aspect of hazard analysis has been released (Pitarka and Irikura, 1996).

Rg–waves can mask reflections that are the basis of migration. Over the NORESS–array in South–Eastern Norway the Rg–waves seem to have amplitudes of about 10
% of those of the first teleseismic P-wave arrivals (Bannister et al., 1990; Gupta et al., 1993; Hedlin et al., 1991). By quantitatively accounting for topography effects, a dataset better suited for migration can be produced. To illustrate how this might be used in practice, consider a simulation of an earth model with known surface topography. A good correspondence with real data should be obtained to make the initial model viable. Then, using the modeling algorithm to propagate the real data backwards in time by a technique like inverse time migration, good results might be obtained (Sun and McMechan, 1992).

With the advent of digital signal recording and deployment of large aperture arrays on preCambrian bedrocks in shield areas with moderate topography, unique opportunities were given for observing wavefield complexities (Husebye and Ruud, 1989). In case of the NORSAR array (aperture ca 100 km), signal amplitudes were found to vary in the extreme by a factor of 10 across the array. In part, this can occur in a systematic manner (Haddon and Husebye, 1978). These observations were modeled in a simplistic way in terms of a deepseated (ca 100 km) structural lense producing focussing and defocussing across the array aperture. Model residues, apparently of shallow origin, were attributed to shallow crust and topography. Another puzzling feature was the strong and persistent coda levels in teleseismic recordings. The question was considered whether the coda waves were primarily due to source side scattering (S to P and Rg to P) or to the receiver side (P to S and P to Rg). This problem was solved by Bannister et al. (1990), who identified specific scattering hills in the NORESS siting area where the P to Rg conversions were most effective. In array record studies of local events P to S scattering appears to be most effective both in forward and backwards modes (Charrette, 1991; Hestholm et al., 1994; Dainty, 1995). In profiling surveys scattering effects have been recognized and modeled in 2-D in terms of near surface cavities (Imhof, 1996). Strong Rg-waves have also been
observed and attributed to near-surface irregularities with particularly high propagation efficiency in low-velocity weathering layers (Levander and Hill, 1985; Ruud et al., 1993).

Not much has been published on the modeling of free surface topography in 2-D, and even less so in 3-D. A work that considers different slope types and combinations of them explicitly in 2-D is that of Jih et al. (1988). A new 2-D approach (Robertsson, 1996) incorporates only topography portions parallel to the main axes and classifies every surface point in a way similar to Jih et al. (1988). The scheme can handle both elastic and viscoelastic schemes and implements the field variables at each of 7 categories of surface topography points separately. A 2-D method which employs a complete tensorial formulation of the wave equations for modeling curved interfaces and free surface topography is Komatitsch et al. (1996). This method has the advantage of using the same amount of spatial partial derivative calculations as for a Cartesian approach. This is 12.5% less in 2-D and 22.2% less in 3-D than the chain rule approach used in this work. However, it has the distinct disadvantage of 30% extra memory requirement in 2-D and 60% extra memory requirement in 3-D than the chain rule approach. In addition, as in the present method, most chain rule approaches use straight vertical grid lines, and therefore the memory difference will be even larger because some partial derivatives of the mapping functions vanish. Frankel and Leith’s (1992) 3-D topography modeling algorithm employs an F-D scheme of fourth order accuracy in space and uses a density taper to zero starting at the level of the free surface.

Another method for 3-D surface topography modeling of elastic media is the boundary integral method in the spatial frequency domain (Bouchon et al., 1996). The medium Green functions have to be calculated for the explicit topography surface as integrals over horizontal wavenumbers. The diffracted wavefield is the integral
over the surface topography of the Green functions times unknown source density functions. The source density functions are solved for using the conjugate gradient method. Results are shown for an incoming shear-wave polarized along the minor and major axes of a cosine-formed elliptically shaped hill. They investigate scattering effects of this hill on the wavefield propagating towards an otherwise plane surface in a homogeneous medium. Sanchez-Sesma and Campillo (1991) also use a boundary integral method to investigate topography effects. Boundary integral methods, or rather their numerical discretizations, boundary element methods, have not so far been applied to real surface topography modeling. Applications have been restricted to simpler geometrical structures. This is because too complex topography renders the method unstable.

The basis of the present method is the vanishing stress condition for a free surface. As in Tessmer and Kosloff (1994) (and earlier in Tessmer et al. (1992) in 2-D), a 3-D grid is used which is curved in the vertical direction and adapted to the surface topography, i.e. the top boundary of the grid coincides with the topography. A coordinate transform is used to transfer the elastic, isotropic wave equations from the curved to a rectangular grid in which the numerical computations are done. The velocity boundary conditions for a free surface are implemented into a local, rotated system at each point of the topography. Each of these systems has its vertical coordinate direction coinciding with the normal vector of the surface at the given point. The velocity boundary conditions are subsequently rotated back to the rectangular system. Once the boundary conditions are given in this system, the numerical discretization can be performed. New 3-D boundary conditions for the particle velocities at any arbitrary smooth topography without vertical subsections have thus been derived (Hestholm and Ruud, 1998).
In this study, we state the equations of motion and the surface topography boundary conditions for our approach. Next we give a description of the numerical discretization. Then we present simulated scattering from teleseismic P-waves using a plane vertically incident P-wave and real topography from an area that includes the NORESS array in South-Eastern Norway.

2.3 Elastic Wave Modeling Formulation

The basic equations governing wave propagation in a continuous elastic medium are the momentum conservation and the stress–strain relationship. The velocity–stress formulation (Achenbach, 1975; Virieux, 1986) can be written in 3-D as in Hestholm and Ruud (1998). We perform a linear transformation from a rectangular \((\xi, \kappa, \eta)\)-system (Figure 1.1) to a curved \((x, y, z)\)-system (Figure 1.2), where the relationship between the systems is

\[
x(\xi, \kappa, \eta) = \xi,
\]

\[
y(\xi, \kappa, \eta) = \kappa,
\]

\[
z(\xi, \kappa, \eta) = \eta \frac{z_0(\xi, \kappa)}{\eta_{\text{max}}}.
\]

The topography function \(z_0(\xi, \kappa)\) is the local height from the bottom to the surface of the curved \((x, y, z)\)-system, and the rectangular \((\xi, \kappa, \eta)\)-system is bounded by \(\xi = 0, \xi = \xi_{\text{max}}, \kappa = 0, \kappa = \kappa_{\text{max}}, \eta = 0\) and \(\eta = \eta_{\text{max}}\). By expanding the velocity–stress formulation of the elastic wave equations by the chain rule (Hestholm and Ruud, 1998), we get the equations for the wave propagation in the medium

\[
\rho \frac{\partial u}{\partial t} = \frac{\partial \sigma_{xx}}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial \sigma_{xx}}{\partial \eta} + \frac{\partial \sigma_{xy}}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial \sigma_{xy}}{\partial \eta} + C(\xi, \kappa) \frac{\partial \sigma_{xx}}{\partial \eta} + f_x,
\]

\[
\rho \frac{\partial v}{\partial t} = \frac{\partial \sigma_{xy}}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial \sigma_{xy}}{\partial \eta} + \frac{\partial \sigma_{yy}}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial \sigma_{yy}}{\partial \eta}.
\]
\[ \frac{\partial w}{\partial t} = C(\xi, \kappa) \frac{\partial \sigma_{yz}}{\partial \eta} + f_y, \quad (2.5) \]

\[ \frac{\rho}{\partial t} = \frac{\partial \sigma_{xz}}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial \sigma_{xz}}{\partial \eta} + B(\xi, \kappa, \eta) \frac{\partial \sigma_{yz}}{\partial \eta} + C(\xi, \kappa) \frac{\partial \sigma_{zz}}{\partial \eta} + f_z, \quad (2.6) \]

\[ \frac{\partial \sigma_{xx}}{\partial t} = (\lambda + 2\mu) \left( \frac{\partial u}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} \right) \]

\[ + \lambda \left( \frac{\partial v}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} + C(\xi, \kappa) \frac{\partial w}{\partial \eta} \right), \quad (2.7) \]

\[ \frac{\partial \sigma_{yy}}{\partial t} = \lambda \left( \frac{\partial u}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + C(\xi, \kappa) \frac{\partial w}{\partial \eta} \right) \]

\[ + (\lambda + 2\mu) \left( \frac{\partial v}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} \right), \quad (2.8) \]

\[ \frac{\partial \sigma_{zz}}{\partial t} = \lambda \left( \frac{\partial u}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} \right) \]

\[ + (\lambda + 2\mu) C(\xi, \kappa) \frac{\partial w}{\partial \eta}, \quad (2.9) \]

\[ \frac{\partial \sigma_{xy}}{\partial t} = \mu \left( \frac{\partial v}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} + \frac{\partial u}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} \right), \quad (2.10) \]

\[ \frac{\partial \sigma_{xz}}{\partial t} = \mu \left( \frac{\partial w}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial w}{\partial \eta} + C(\xi, \kappa) \frac{\partial u}{\partial \eta} \right), \quad (2.11) \]

\[ \frac{\partial \sigma_{yz}}{\partial t} = \mu \left( \frac{\partial w}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial w}{\partial \eta} + C(\xi, \kappa) \frac{\partial v}{\partial \eta} \right), \quad (2.12) \]

where \( A(\xi, \kappa, \eta), B(\xi, \kappa, \eta) \) and \( C(\xi, \kappa) \) are functions of the topography function \( z_0(\xi, \kappa) \), its spatial derivatives and the rectangular grid coordinates,

\[ A(\xi, \kappa, \eta) = \frac{\partial \eta}{\partial x} = -\frac{\eta}{z_0(\xi, \kappa)} \frac{\partial z_0(\xi, \kappa)}{\partial \xi}, \quad (2.13) \]

\[ B(\xi, \kappa, \eta) = \frac{\partial \eta}{\partial y} = -\frac{\eta}{z_0(\xi, \kappa)} \frac{\partial z_0(\xi, \kappa)}{\partial \kappa}, \quad (2.14) \]

\[ C(\xi, \kappa) = \frac{\partial \eta}{\partial z} = \frac{\eta_{max}}{z_0(\xi, \kappa)}. \quad (2.15) \]

The topography function \( z_0(\xi, \kappa) \) is the local height from the bottom to the surface of the curved \((x, y, z)\)-system, of which the surface is shown in Figure 1.2. \( \rho \) is the
density and $\lambda$ and $\mu$ are Lamé's parameters. $f_x$, $f_y$ and $f_z$ are the components of the body forces, and $u$, $v$ and $w$ are the particle velocity components. $\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{zz}$, $\sigma_{xy}$, $\sigma_{xz}$ and $\sigma_{yz}$ are the stress components. Equations (2.4)-(2.12) are the momentum conservation equations and Hooke's law for the curved system, now given in the rectangular $(\xi, \kappa, \eta)$-system.

The 3-D particle velocity boundary conditions for a free surface topography given in the computational $(\xi, \kappa, \eta)$-system (Hestholm, 1999) can be given by

\begin{align*}
\frac{1}{e^2} (1 + p^2) C(\xi, \kappa) \frac{\partial u}{\partial \eta} + \frac{1}{e^2} (1 + p^2) dC(\xi, \kappa) \frac{\partial w}{\partial \eta} & \\
= 2d \frac{\partial u}{\partial \xi} + \frac{p}{e} \frac{\partial v}{\partial \xi} + (d^2 - 1) \frac{\partial w}{\partial \xi} + \frac{p u}{e} \frac{\partial w}{\partial \kappa} + \frac{dp}{e} \frac{\partial w}{\partial \kappa}, & (2.16) \\
- \frac{1}{e^2} (1 + p^2) fp C(\xi, \kappa) \frac{\partial u}{\partial \eta} + \frac{1}{e^2} (1 + p^2) C(\xi, \kappa) \frac{\partial v}{\partial \eta} + \frac{1}{e} (1 + p^2) p C(\xi, \kappa) \frac{\partial w}{\partial \eta} & \\
= -2f dp \frac{\partial u}{\partial \xi} + d (1 - p^2) \frac{\partial v}{\partial \xi} + 2f p \frac{\partial w}{\partial \xi} + d (1 - p^2) \frac{\partial u}{\partial \kappa} & \\
+ 2 \frac{p}{e} \frac{\partial v}{\partial \kappa} + (p^2 - 1) \frac{\partial w}{\partial \kappa}, & (2.17)
\end{align*}

\begin{align*}
\left\{ \left( 1 + \frac{p^2}{e^2} \right) (\zeta - 1) - \zeta \left( \frac{1}{q^2} + d^2 \frac{p^2}{e^2} \right) \right\} dC(\xi, \kappa) \frac{\partial u}{\partial \eta} & \\
+ \left\{ (1 + d^2) (\zeta - 1) - \frac{1}{e^2} (\zeta + p^2) \right\} p C(\xi, \kappa) \frac{\partial v}{\partial \eta} & \\
+ \left\{ \zeta \left( \frac{d^2}{q^2} + p^2 \right) + 1 - \left( d^2 + \frac{p^2}{e^2} \right) (\zeta - 1) \right\} C(\xi, \kappa) \frac{\partial w}{\partial \eta} & \\
= - \left\{ \zeta \left( \frac{1}{q^2} + d^2 p^2 \right) + d^2 \right\} \frac{\partial u}{\partial \xi} + \frac{dp}{e} (\zeta - 1) \frac{\partial u}{\partial \kappa} + d (\zeta - 1) \frac{\partial w}{\partial \kappa} & \\
+ \frac{dp}{e} (\zeta - 1) \frac{\partial u}{\partial \kappa} - \frac{1}{e^2} (\zeta + p^2) \frac{\partial u}{\partial \kappa} - \frac{p}{e} (\zeta - 1) \frac{\partial w}{\partial \kappa}, & (2.18)
\end{align*}

using definition (2.15) and

\begin{align*}
\zeta & = \frac{\lambda}{\lambda + 2\mu}, \quad (2.19) \\
d & = \frac{\partial z_0(\xi, \kappa)}{\partial \xi} = \tan \theta, \quad (2.20) \\
e & = \cos \left[ \arctan \left( d \right) \right] = \cos \theta, \quad (2.21)
\end{align*}
\[ f = \sin \arctan (d) = \sin \theta, \quad (2.22) \]
\[ p = -\frac{\partial z_0(\xi, \kappa)}{\partial \kappa}e = \tan \phi, \quad (2.23) \]
\[ q = \cos \arctan (p) = \cos \phi, \quad (2.24) \]
\[ r = \sin \arctan (p) = \sin \phi. \quad (2.25) \]

### 2.4 Numerical Discretization

The same spatial and time discretization methods as in the 2-D case are used (Hestholm and Ruud, 1994; Kindelan et al., 1990; Sguazzero et al., 1989). The schemes employ a staggered discretization stencil of the velocity–stress formulation of the elastodynamic wave equations (Levander, 1988) based on Virieux (1986). Only the particle velocities need to be defined at the surface topography. In order to achieve explicit expressions for the particle velocities and stresses at each time step, we stagger the vertical velocity component \( w \) half a grid length downwards and the horizontal velocity components \( u \) and \( v \) half a grid length positively along the \( \xi- \) and \( \kappa- \)directions respectively. Stresses are implemented at the grid nodes or at the middle of the 'plaquettes' (the 2-D case is shown in Figure 2.1).

The 3-D boundary conditions (2.16)–(2.18) are discretized by 2nd order, staggered F–D operators (Fornberg, 1988a), from which the particle velocities at the surface are solved for from the vertical partial derivative expressions. In this procedure, the horizontal partial derivatives are calculated one grid length or one and a half grid length below the free surface and considered known from the medium equations (Figure 2.1). The resulting closed system for the surface particle velocities has a determinant which is always (for any topography slope) different from zero (Hestholm, 1999), and so the system to solve along the surface topography boundary is unconditionally stable. Moving from the free surface into the medium, the central, staggered F–D method's...
Figure 2.1  Computational staggered grid, $(\xi, \eta)$–plane.
order is gradually increased with depth, via 4th and 6th up to 8th order, which is the order used inside the domain. The 8th order method is dispersion-bound and cost-optimized (Kindelan et al., 1990). Along the artificial grid boundaries exponential damping is used (Cerjan et al., 1985). In a layer of 25 points along each grid boundary, the stresses and velocities are multiplied by exponentially decreasing terms towards the boundary. The method of Cerjan et al. (1985) was found to be among the best available for point source excitation below a free surface using high order seismic algorithms (Simone and Hestholm, 1998). Its performance was compared with the results from using the condition of Peng and Toksöz (1994), and the stability criteria of both methods were assessed and tested. Renaut and Petersen (1989) assessed and compared stability properties for a wide range of absorbing boundary conditions when used in conjunction with the seismic wave equations.

2.5 Numerical Simulations using Real Topography

For F–D modeling experiments we have used digital elevation data for an area of 60 × 60 km containing the NORESS array in South–Eastern Norway (Figure 2.2). This hilly area was chosen because of easy access to detailed topography data. Additionally, significant P to Rg scattering from specific hills is well documented from NORESS record analysis (Bannister et al., 1990). The code has been parallelized using MPI (Message Passing Interface). In this way we are able to run 3–D models with about 10^8 gridpoints on the most recent Cray Origin 2000 parallel machine. In the following examples, we display 2–D and 3–D simulations with 0.2 km grid–sampling for a 60 × 60 km topography area displayed in Figure 2.2. The most prominent topography present in the dataset is the steep Skreikampen hill immediately west of the southern part of lake Mjøsa (the long south–north oriented dark pattern in the south–west area in Figure 2.2). In all the examples shown the incoming wave is a vertically incident
plane P-wave simulating a teleseismic short period P-phase. The center frequency of the Ricker wavelet is 5 Hz, the P-wave velocity of the homogeneous medium is 6.0 km/s, the S-wave velocity is 3.46 km/s and the density is 2000 kg/m$^3$. In this section we show simulations for gradually more complex situations, with a 3-D elastic simulation using the parallel code as the final result. The outline of displaying results for the same area with gradually added complexities will illustrate the need to include these complexities in both academic applications and exploration.

2.5.1 Topography Surface versus Plane Surface

Figure 2.3 shows snapshots for a 2-D simulation of an incident plane wave reflecting from a plane, free surface on top of a homogeneous medium of 28.8 km depth. Distances are in kilometers, so the area covers a 60 km long horizontal subsection. The left frames show the horizontal ($u$) (top) and vertical ($w$) (bottom) particle velocity component 0.12 seconds after the start of the simulation and 0.04 seconds after the reflection from the free surface of the center part of the plane wave. There is 1 second between each of the next snapshot frames. The snapshots show a clean reflection from the plane surface and artificial reflections from the top corners. All snapshots displayed in this section are scaled with respect to a maximum value determined for each simulation. This means that amplitudes can be compared directly within the same figure.

The snapshots in Figure 2.4 show the exact same times, model depth and source location/type as in Figure 2.3, but here the actual NORESS topography is added (the 60 km long west–east profile of Figure 2.2). The medium is still homogeneous, so the scattering is caused only by the topography. The strongest scattered waves are seen to radiate out from the surface at two points about 25 km and 45 km from the left edge, coinciding with areas of steep topography (see Figure 2.2). The steep hill
Figure 2.2  *Left:* Map shows the topography of the 60 × 60 km area used in the 3-D simulation. The dashed lines show the positions of the receivers in two profiles and the circle outlines the NORESS array. Labels are in kilometers and elevations are in meters above mean sea level. The black area to the south–west is Lake Mjøsa (123 m above sea level). *Right:* Topography profiles along the two lines each of 60 km length shown on the map. They are midway along the y–direction and x–direction and cover respectively the complete x– and y–dimension of the area. Horizontal axes are in kilometers and vertical axes are in meters above mean sea level.

directly west of lake Mjøsa along the profile seems to be at the edge of the present model and not contributing much to the scattering because of the absorbing boundary conditions acting there. The scattered waves are best seen on the $u$ particle velocity component because this component is not affected by the strong vertically oriented wavefront. $Rg$–waves is the dominating feature at the surface, but converted and reflected $P$– and $S$–waves are also present in the snapshots. The field propagating away from the areas of prominent topography has a relatively coherent appearance.

Figure 2.5 shows the three following snapshot times for the simulation in Figure 2.3 (plane surface). Except for the reflected surface wave seen on the $w$–component snapshots, only artificial reflections emanate from the top corners and sides. Figure 2.6 shows the same snapshots with the topography added, i.e. the simulation of Figure 2.4 for the next three times. The artificial reflections propagate further into the
Figure 2.3 Snapshots of 2-D simulation of a plane, vertically incident P-wave reflecting from a plane, free surface. The first snapshot shows the wavefield 0.04 seconds after its reflection from the surface, and the time lap between each of the next snapshots is 1 second. Distances are in kilometers, and the upper and lower series display the horizontal and vertical particle velocity components respectively.

Figure 2.4 Same as Figure 2.3, but for an irregular free surface boundary condition. The topography is that of the west–east profile of Figure 2.2.
Figure 2.5  Same as Figure 2.3 for the next three snapshots.

medium, and the scattered waves from the areas of prominent topography are seen to traverse downwards. This is most clearly seen on the $u$–component. The scattered wavefield continues to show a coherent appearance, and no wavefronts are seen after the first two clear ones propagating out slightly to the left of the middle of the model.

2.5.2  3–D versus 2–D

For the 3–D simulation we use the $60 \times 60$ km NORESS topography area (Figure 2.2, left) for a homogeneous 28.8 km deep model. Exactly the same source position and type are used as in the 2–D case, i.e. the same plane, vertically incident Gaussian.

Snapshots

The snapshots shown in Figure 2.7 are taken for the same particle velocity components ($u$ and $w$) and at exactly the same locations as the topography data for the 2–D
Figure 2.6  Same as Figure 2.4 for the next three snapshots.

Figure 2.7  Snapshots of 3-D simulation of a plane, vertically incident Ricker wavelet reflecting from an irregular free surface. The first snapshot shows the wavefield 0.04 seconds after its reflection from the surface, and the time lapse between each of the next snapshots is 1 second. Distances are in kilometers, and the upper and lower series display a horizontal (u) and vertical particle velocity component respectively. Snapshots are shown at the midpoint of the direction of the second horizontal dimension and correspond exactly to the snapshot locations of Figure 2.4 for the 2-D case.
case, i.e. an $xz$–plane midway along the $y$–direction of the $60 \times 60$ km area. The snapshots of the $w$–component from the 3–D run seems to differ slightly from the 2–D case by the stronger white band in the center of the wavefront. This must be because of an amplitude difference from geometrical spreading due to both the numerical discretization and actual physical geometrical spreading. Even if analytical geometrical spreading of a plane wave is impossible, it is important to remember that after reflection from the surface topography, the wavefront is no longer plane. As seen from Figure 2.7 the scattered surface waves appear to radiate out from secondary point sources which coincide with areas of high topographic gradients (Figure 2.2). Energy spreads out into the extra horizontal direction, therefore the amplitudes of the $w$–component, in particular, will be weaker than in 2–D. This is not apparent from the figures, since in order to visualize the scattering features, each figure is scaled according to a maximum value for that simulation. To the contrary of 2–D simulations, in 3–D the strongest amplitudes at a certain time may be from out–of–plane scattering, which will be delayed due to its longer travel path and complicate the wave pattern.

Another pronounced feature visible in the 3–D simulation is the disruption of the scattered wavefield. It appears to contain much stronger variations in its wavelengths, i.e. the spatial spectrum seems to contain more peaks than in the 2–D simulations. The total scattered wavefield has a much more complex appearance in 3–D than in 2–D. This can be seen on the wavefield pattern for both particle velocity components. 2–D simulations are expected to exhibit a greater degree of localization of the scattered wavefield than 3–D simulations (Imhof, 1996). In view of the apparent energy partitioning into frequency peaks in 3–D, this is seen to hold if physical dispersion were included (as it would be for viscoelastic wave modeling).
The increased complexity of the 3-D scattered wavefield pattern is even more apparent in the next three snapshots of the $u$ and $w$-components shown in Figure 2.8, where the scattered wavefield has propagated into most of the model. The corresponding 2-D snapshots (Figure 2.6) appear quite coherent beside the 3-D ones. Still, the clear wavefronts of the 2-D scattered field can be recognized in 3-D, but they are here more complex and interspersed by new wavefronts due to out-of-plane scattering. This phenomenon is also the reason for slightly broader wavefronts of the $u$-component and the much more complex general appearance of the 3-D scattered wavefield compared to the 2-D one.

We finally show snapshots for the remaining dimensions from the 3-D simulation. Figures 2.9 and 2.10 display snapshots of the $w$ particle velocity component for the times shown in the previous figures. The upper series in both figures show snapshots along the surface topography for the complete $60 \times 60$ km area, while the lower series is taken along the complete $60 \times 28.8$ km $yz$-plane at the midpoint along the $x$-direction.

The extra information drawn from the display of the scattered wavefield along the surface topography justifies 3-D simulations opposed to 2-D simulations. In particular, when investigating surface topography effects, the out-of-plane information is invaluable. The positions and shapes of the scattered waves can easily be traced with time in the snapshots. The strongest scattered waves can be seen to emanate from locations of strong topographic gradients. The strongest effect is seen to be caused by the steep area directly west of the southern part of lake Mjøsa (near the lower left area of the horizontal snapshots). This area causes a prominent $Rg$-wavefront that can be traced on all the snapshots. It propagates slowly towards the middle part of the topography area, where it can be seen clearly on the last snapshot. Results are consistent with the ones found by Pitarka and Irikura (1996) and Bouchon et
Figure 2.8 Same as Figure 2.7 for the next three snapshots.

Figure 2.9 The simulation displayed in Figure 2.7 for the same times, but here snapshots are shown at the surface topography and along the plane of the midpoint of the first horizontal direction (along the yz-plane). Both series display the vertical particle velocity component ($w$).
al. (1996), that elevated areas and high gradients amplify wave amplitudes and cause scattering away from these areas. The results in Figures 2.9 and 2.10 are also consistent with 3-D effects found from field experiments and from simulations (Bouchon et al., 1996), i.e. the directivity of the wavefield. The most prominent topography causes Rg-waves propagating away from it, and these waves have stronger amplitudes in certain directions. The particularly clear wavefront propagating towards the center of the area can be observed in all horizontal snapshots.

The absorbing boundary condition used in all the simulations is the exponential damping method according to Cerjan et al. (1985). Extensive tests have been done (Simone and Hestholm, 1998) to verify that this method is among the best available. Nevertheless, a plane incident wave shows the method from its most unfavorable side
in the sense that wave components in every direction hit all boundaries at all times. Hence reflections will propagate into the domain from all grid edges. This is clearly seen from Figures 2.9 and 2.10. The reflections appear as straight lines of much lower frequencies than the scattered wavefield. They might therefore be eliminated by a filtering technique, but one cannot be sure of not eliminating some physically significant wave portions at the same time.

The 2-D simulations in this section took about 5 minutes on an IBM RISC 6000 model 590 workstation. The 3-D run took 2 days and nights (when others were also using the machine) on an Intel Paragon machine using 48 processors. This is a much slower machine than the 590 workstation, but it was not possible to access enough memory on the workstation for this 3-D simulation. However, on a new Cray Origin 2000 parallel machine using a sequential version of the program on one processor and the memory of 8 processors, the time for this 3-D simulation was reduced to 4 hours when others were also using the machine. The total memory requirements of the 2-D simulation was 2.3 MB, while the size of the 3-D simulation (including all overlap domains between processors) was 763 MB.

**Seismograms**

This section will compare seismograms for corresponding 2-D and 3-D simulations. 3-D seismograms are taken from the same 3-D simulation that was used for the snapshots. Synthetics are shown along the profile shown as the dashed lines in the left part of Figure 2.2. The first recordings are taken along the west–east profile at 40 km from the south grid boundary with stations starting at 25.6 km and ending at 55 km from the west grid boundary. The second record set is along the south–north profile at 40 km from the west grid boundary with the stations distributed from 25.6 km to 55 km from the south grid boundary. There are 50 stations in each
profile and the spacing between them is 600 m. In order to compare with 2-D runs, we performed 2-D simulations using exactly the same topography data and receiver locations as along the profiles for the 3-D simulation. This means that each 2-D profile consisted of 50 stations distributed from 25.6 km to 55 km from the lower grid edge with an inter-spacing of 600 m.

Figure 2.11 compares seismograms for the vertical particle velocity component in 2-D (left part) and 3-D (right part). The receiver profile is the west–east oriented one shown in the left part of Figure 2.2. The slightly slower 3-D arrival times are due to the fact that 4 extra grid points along all edges of the computational domain are included in the model compared to the 2-D case. Inside the 3-D domain these 4 points are the overlapping layers between processors for the parallel code, but along the computational domain edges they are included in the physical model. The strong arrivals striking the rightmost receivers and spreading into the model from the right in both seismograms are the artificial reflections from the east grid boundary. The other strong arrival at all receivers after 5 seconds in the right seismogram of Figure 2.11 is the artificial reflections from the north grid boundary. The left seismogram clearly shows coherent wavetrains emanating from the areas of prominent topography. These waves can be identified as S- and Rg-waves from their velocities. Going to the 3-D case on the right part of Figure 2.11, the picture becomes much more complicated. Still the prominent topographic relief from the 2-D case can be identified as causing the largest amount of scattering, but the arrivals are more incoherent and less localized than in 2-D. The relative amplitudes of the scattered phases are greatly enhanced compared to the 2-D case. As for the snapshots, the 3-D simulation causes the wave pattern to be more irregular, which is attributed to out-of-plane topographic scattering. This is also the reason for the extra scattered arrivals in 3-D. The overall weaker amplitudes of the 3-D simulation (corresponding seismograms are scaled up
Figure 2.11  Left: Seismogram from 2-D simulation of a 60 km long topography data profile taken along the west–east oriented receiver line from the left part of Figure 2.2. The profile is 28.8 km deep and extends from 25.6 km west of the leftmost receiver to 5 km east of the rightmost receiver. The profile contains the displayed line of 50 west–east oriented receivers inter–spaced by 600 m. The vertical particle velocity component is displayed. Right: Seismogram from the 3–D simulation of the 60 × 60 × 28.8 km model displayed in the snapshots. The 50 receivers are located along the west–east oriented receiver line displayed on the left part of Figure 2.2 and inter–spaced by 600 m. Because of geometrical spreading, the amplitudes are multiplied by 3.5 to make the main arrivals the same order of magnitude as those for the 2–D case on the left. The vertical particle velocity component is displayed.
by 3.5) we attribute to geometrical spreading, which will affect all phases generated near the surface after the plane wave has been reflected. In addition, geometrical spreading will occur due to the numerical definition of the plane wave at discrete grid points. This effect will therefore weaken all 3-D arrivals compared to 2-D ones. It is also possible that more destructive interference will occur in 3-D than in 2-D because of the extra dimension of discrete point sources interacting.

In Figure 2.12 we show seismograms from 2-D (left part) and 3-D (right part) simulations for the same receivers as in Figure 2.11, but here the first horizontal particle velocity component (the one directed west–east) is displayed. The artificial boundary reflections affecting this component, i.e. the ones from the east grid boundary, are seen clearly as both P- and S-wavetrains, and after 4 seconds the artificial P-reflections from the west boundary can also be seen coming in from the left. The areas of prominent topography are seen to give rise to the strongest amplitude amplifications of scattered waves from the incident plane wave in both the 2-D and 3-D cases. Still, the appearance of the latter is different and more complicated. Additionally, extra scattered arrivals can be identified in 3-D. They are caused by out-of-plane topography. Horizontal and vertical particle velocity components have the same scaling, and so from comparing the two components in each dimension we can confirm that the scattered surface amplitudes have the same order of magnitude for all components.

In Figure 2.13 seismograms for a 2-D simulation (left) and the 3-D simulation (right) are displayed for the 50 south–north oriented receivers shown in the left part of Figure 2.2. The 60 km long 2-D profile with irregular topography is taken along this receiver line and covers the complete yz-plane of the 3-D model. The vertical particle velocity component is displayed in Figure 2.13, and 3-D results are again scaled by a constant for comparison with the 2-D results. Artificial grid reflections
Figure 2.12  Seismograms from the same 50 receivers and the same 2-D (left) and 3-D (right) simulations as shown in Figure 2.11, but here analogous horizontal particle velocity components (the west–east oriented ones) are displayed.

Figure 2.13  Left: Seismogram from 2-D simulation of a 60 km long topography data profile taken along the south–north oriented receiver line from the left part of Figure 2.2. The profile is 28.8 km deep and extends from 25.6 km south of the receiver furthest to the south to 5 km north of the receiver furthest to the north. The profile contains the displayed line of 50 south–north oriented receivers inter-spaced by 600 m. The vertical particle velocity component is displayed. Right: Seismogram from the 3-D simulation of the 60×60×28.8 km model displayed in the snapshots. The 50 receivers are located along the south–north oriented receiver line displayed on the left part of Figure 2.2 and inter-spaced by 600 m. Because of geometrical spreading, the amplitudes are multiplied by 3.5 to make the main arrivals the same order of magnitude as those for the 2-D case on the left. The vertical particle velocity component is displayed.
as seen from the east boundary for the west–east profile propagate from the north boundary, and the large arrival in 3–D after 5 seconds is an artificial reflection from the east grid boundary. In 2–D one can see that the areas of most topographic variations cause quite coherent amplitude amplifications. In 3–D the corresponding arrivals get masked from scattering from out–of–plane topography, although some of the 2–D topographic scattering pattern can be recognized on the 3–D seismogram. The 3–D topography is seen to cause scattering of much stronger relative amplitude compared to that caused by the 2–D topography.

Figure 2.14 shows seismograms for the same 2–D (left) and 3–D (right) simulations and for the same receivers as displayed in Figure 2.13, but here the south–north oriented particle velocity component is displayed. The P– and S–reflections from the north and south grid boundaries can be identified, as well as strong scattering from the areas of prominent topography. Again the 3–D case leads to a much more irregular wave pattern and extra scattered waves compared to the 2–D case, although not to the same extent as for the vertical particle velocity component.

2.6 Discussion

We have given details on our 3–D F–D scheme for computing synthetic seismograms based on the elastodynamic wave equations and produced results pertaining to the topography of the NORESS array siting area. The choice of area was motivated by easy access to topography data plus many array results exhibiting amplitude variations (Haddon and Husebye, 1978) and scattering phenomena (Bannister et al., 1990; Hedlin et al., 1994). In our synthetic experiments we focussed on two topics; (i) to demonstrate the necessity of 3–D modeling for complex topographic features and (ii) scattering features tied to prominent hills. We used both snapshots of 2–D displays and profile seismograms to illustrate the results obtained in the experiments.
Figure 2.14 Seismograms from the same 50 receivers and the same 2-D (left) and 3-D (right) simulations as shown in Figure 2.13, but here analogous horizontal particle velocity components (the south–north oriented ones) are displayed.

Firstly, the 2-D synthetics are very much simpler than the corresponding 3-D ones due to out-of-plane contributions. This is expected and easily confirmed by f-k or semblance analysis of array recordings. A few examples are given by Hestholm et al. (1994). Here loss of directionality in the Lg–coda waves is demonstrated, i.e. both forward and backward scattered wavelets arrive from any direction. Additionally, coda coherency is low or hardly significant, at levels of 0.2–0.4 units (Dainty, 1995). This is in contrast to the 2-D synthetics snapshots shown in the left parts of Figures 2.11–2.14 or from the semblance plots in Hestholm et al. (1994). Also, from Figures 2.11–2.14 we see that 3-D coda waves are characterized by larger amplitudes than what is the case for 2-D records – naturally due to out-of-plane contributions. This reflects a problem in 2-D synthetics, namely that excessive and partly unrealistic velocity perturbations (exceeding 5 %) must be introduced to match RMS coda levels in observational records (Hestholm et al., 1994; Dainty, 1995).

As observed by Bannister et al. (1990), some hills in the NORESS area literally radiate Rg–waves at regular intervals for incoming teleseismic waves of long durations. In this regard array recordings are just point observations while the synthetics in
Figures 2.4–2.10 reveal nearly symmetric Rg–radiation from some secondary sources. However, with passing time the interference of phases from a multiplicity of secondary sources becomes complex (Figure 2.9), so propagation directionality from a secondary source to a receiver will generally weaken. An observational counterpart to this result is that Rg–waves rarely propagate further than 60 km in hilly areas typical of the NORESS and GERESS (Bavaria, Germany) arrays while across the plains of northern Fennoscandia Rg–waves from explosions occasionally propagate out to 600 km.

We have quoted observations pertaining to strong wavefield amplification on top of hills in areas of rough topography. As expected, the synthetics presented here only produce moderate amplitude variations along the profiles in Figures 2.11–2.14 – less than a factor of 2. The reason is that in order to obtain stronger amplitude variations, a large slice of the wavefront has to be focussed (compressed) into a small area. This is simply not feasible with our homogeneous crustal model. The observations in Haddon and Husebye (1978) reflect wavefront 'bending' taking place more than 100 km away from the free surface sensors. This makes sense with moderate lithospheric velocity perturbations on the order of 3–5 %. Undoubtedly, some strong, localized wavefield amplifications have taken place in cities destroyed by earthquakes in view of their damage patterns.

2.7 Conclusions

It is demonstrated that 3–D F–D synthetic wavefield analysis including topography provides improved insight and a better understanding of surface scattering phenomena. Snapshots and seismograms display clear conversion to Rg–waves in areas of rough topography. Additionally, 3–D as opposed to 2–D modeling is shown to give significantly different results. The added effects from the extra complexities can contribute to explanations for deviations between numerical simulations performed on
2-D structures and 3-D scattering observed in the field. These discrepancies are connected to incoherent and unlocalized scattering, excessive amplitude amplifications and directionality dependencies. The last point can be observed in NORESS and GERESS recordings. At some sensors the Rg-phase is prominent while hardly visible at nearby sensors less than a kilometer away. Many of such wavefield phenomena are out of range for 2-D F–D synthetics. The major conclusion from the present study is that there is no substitute to 3-D synthetic wavefield modeling for obtaining an adequate comprehension of seismic wave propagation in the heterogeneous subsurface.

2.8 Acknowledgments

S. H. appreciated discussions with Dr. Satish Pullammanappallil (William Lettis Associates, San Francisco, CA). In parallelizing the code, S. H. would like to thank Bjarne Herland and Ove Sævareid (both at Rogalandsforskning, Bergen) for helpful hints. Thanks to Parallab and its leader Tor Sørevik (University of Bergen, dept. of Informatics) for use of the Intel Paragon and Cray Origin 2000 parallel machines. This research was supported by the Norwegian Research Council, the Norwegian Supercomputer Committee through a grant of computing time and the Air Force Office of Scientific Research, USAF under Grant F49620-94-1-0278.
Chapter 3

Instabilities in Applying Absorbing Boundary Conditions to High Order Seismic Modeling Algorithms

3.1 Introduction

The problem of artificial reflections from grid boundaries in the numerical discretization of elastic and acoustic wave equations has since long plagued geophysicists. Even if modern computers have made it possible to extend the the synthetics over more wavelengths, equivalent to larger propagation distances, efficient absorption methods are still needed in order to minimize interference from unwanted reflections from the numerical grid boundaries. In this study we examine applicabilities and stabilities of the Optimal Absorbing Boundary Condition (OABC) of Peng and Toksöz (1994; 1995) for 2-D and 3-D acoustic and elastic wave modeling. As a basis for comparison we use the Exponential Damping (ED) (Cerjan et al., 1985), in which velocities and stresses are multiplied by progressively decreasing terms when approaching the boundaries of the numerical grid. Peng and Toksöz (1994; 1995) emphasized the importance of stability in the choice of absorbing boundary condition. As an example, they mentioned that Emerman and Stephen (1983) demonstrated that the boundary condition of Clayton and Engquist (1977) was unstable for a wide range of elastic parameters, and further that Mahrer (1988) found the boundary condition of Reynolds (1978) to be unstable as well. Peng and Toksöz (1994) gave some examples of 3-D elastic F-D simulations where OABC was used. The employed scheme was 4th order accurate in space and 2nd order accurate in time. Elastic wave propagation in an
unbounded homogeneous medium (without a free surface), were simulated, and the results were stable. They also made a comparison between the OABC, Reynolds' (1978) and Higdon's (1990) boundary conditions and found that using the OABC led to less artificial reflections than the two other schemes. Our study may be considered a further test of absorbing boundary conditions. We therefore implement both the OABC and ED in our 2–D F–D elastic wave algorithm (Hestholm and Ruud, 1994) and check their respective performances. Our goal is to find higher order accuracy algorithms using both methods.

3.2 2–D Finite–Difference Implementation

We employ a 2–D F–D velocity–stress formulation (Hestholm and Ruud, 1994), that solves the equations governing wave propagation in an elastic isotropic medium. Following Levander (1988) and Virieux (1986) we discretize the elastodynamic equations with two staggered numerical space differentiators. Details of this numerical discretization can be found in Kindelan (1990), who developed optimal spatial finite–difference (F–D) methods based on the work of Holberg (1987). In the present work we use a method which is spatially accurate to 8th order in the interior of the computational domain. For time stepping a leap frog technique (accurate to 2nd order) is used. This allows us to achieve an upper bound of 1.5 % for the relative error of the numerical group velocity at only 3 nodes per wavelength. A schematic of our staggered grid is given in Figure 2.1.

The OABC method extrapolates values on the numerical edges of a F–D grid. We express these values as a linear combination of the wavefield at previous time steps and/or interior grids by exploiting the zeros and poles of the reflection coefficients in the complex plane. Approaching the grid edges, including the free surface, we apply successively lower order central, staggered F–D operators for discretizing the
spatial derivatives. We apply stress–free boundary conditions at the upper boundary. OABC is used at the bottom and sides of the grids. It is worthwhile to note that in the implementation of the OABC it is important to avoid using corner points of the staggered grid.

The ED method involves some restrictions on the useable part of our computational grid, since it is necessary to generate finite thickness damping strips along the grid edges. We exponentially damp velocity and stress in the strip by multiplying them by \( \exp(-\alpha \Delta)^2 \). \( \Delta \) is the number of grid points to the inner boundary of the strip, and \( \alpha \) is a constant equal to 0.015. Damping is enforced within absorption strips of 20 grid points or 7 wavelengths at 60 Hz and 1 wavelength at 10 Hz along the bottom and the sides of the grid. For both OABC and ED, the coefficients are precomputed before time extrapolation is started.

### 3.3 Test Cases

In our numerical tests the model size is 100 km by 70 km, or 40 by 30 wavelengths at the central frequency of 2.5 Hz, giving 351 grid points vertically and 501 grid points horizontally for a grid size of \( \Delta x = \Delta z = 0.2 \) km. We design two different types of models in order to compare the artificial reflections using the OABC and the ED. The first model is homogeneous with a constant P–velocity of 7.1 km/s. The second model is a multilayer realization of the crust and upper mantle consisting of several constant velocity layers (Figure 3.5). In all cases the S–wave layer velocity is related to the P–wave velocity via the Poisson ratio of \( v_P/v_S = \sqrt{3} \). Densities \( \rho \) are linear functions of the P–velocities: \( \rho = 613 + 0.328v_P \), given in kg/m\(^3\) with \( v_P \) given in m/s. The source is a Ricker wavelet (derivative of a Gaussian) in both space and time and is located 50 km from the left edge of the grid and 0.5 km below the surface.
3.4 Stability of OABC

We found that the OABC became unstable for long time simulations when using our 8th order accurate finite difference algorithm. We tried to use 8th order of accuracy in the interior of the domain combined with second order along three levels adjacent to the absorbing boundaries (thus a very sharp discontinuity in the order of accuracy). We encountered instabilities after $O(100)$ time steps, with the instability being initiated either along the absorbing boundaries or at the bottom corners of the computational grid. As already mentioned, Peng and Toksöz (1994) achieved stable results for the OABC when used with spatially 4th order accurate schemes in a 3-D homogeneous medium. Thus for a realistic test of OABC using our scheme it was natural to experiment with lowering the spatial F–D order. This led to successful runs whether using 2nd or 4th order accurate F–D operators throughout the grid; in both cases we obtained completely stable runs but with greater numerical dispersion.

We therefore modified the OABC algorithm to achieve stability in our higher order scheme, thereby obtaining higher accuracy. We solved the problem by decreasing the F–D order gradually when approaching the boundaries. We had to perform several tests in order to find the proper number of grid points near the boundaries on which to apply a certain F–D order. It seems important to use 2nd order finite differences at exactly two grid points adjacent to the boundaries. Inside this layer the number of grid points of higher order finite differences should be increased gradually with order up to where the 8th order method starts.

In our implementation we arrived at a total number of 12 grid points adjacent to the boundaries at which the F–D order were gradually lowered. We used 8th order in the interior of the grid, then 6th, 4th and 2nd order as approaching the boundaries. The following procedure was found to be optimal (in the sense that it delayed instabilities the longest) in our scheme when approaching the grid boundaries:
6 layers of 6th order F–Ds, then 4 layers of 4th order and finally 2 layers of 2nd order adjacent to the grid boundary. In this way we were able to run for at least 14000 time steps (about 60 seconds of simulated time) before a slowly growing instability started appearing along the absorbing boundaries. Simulating wavefields for such large time laps enable the most interesting wave phenomena to be adequately synthesized.

The OABC requires the zeros and poles of the reflection coefficients to be in the first quadrant of the complex plane (Peng and Toksöz, 1994; Peng and Toksöz, 1995), which leads to $k\Delta x \leq \pi/2$, where $\Delta x$ is the grid size and $k$ the wavenumber. From this it follows that $\Delta x \leq \lambda/4$, where $\lambda = 2\pi/k$ is the shortest wavelength. Therefore the OABC requires at least 4 samples per shortest wavelength to ensure stability, at least near the grid edges. In our context of a broadband Gaussian point source this cannot be ensured, since we do not have explicit control over the complete frequency range that propagates. The Ricker wavelet source is widely used in seismic simulations. However, the central frequency of the source is 2.5 Hz, which leads to a wavelength of 2.8 km for our choice of P–wave velocity in the homogeneous medium. This leads to 14.2 samples per wavelength, which is 3.5 times more than is needed for the OABC. Nevertheless, the source excites also a high-frequency part that seems to destroy stability of the OABC when used together with higher order finite differences. The high-frequency part generally needs finer sampling in order for the stability criterion to be satisfied, and instability occurs specifically in connection with higher order finite differences. Of course, an obvious solution to this problem would be to replace the Ricker wavelet with a bandlimited wavelet.

Another aspect that should be emphasized is the use of a free surface in our applications. This leads to Rg–waves propagating with much smaller speeds and wavelengths than the P– and S–waves, which consequently may violate the the stability criterion of the OABC. This problem is common with absorbing boundary conditions,
and the effect becomes even more prominent in the case of free surface topography. For reasons described, and particularly in order to be able to use realistic sources in connection with OABC, we developed the given optimal procedure of implementing OABC in our higher order F–D scheme. Following this procedure, instability is delayed until the most interesting wave phenomena have been synthesized.

3.5 Efficiency

In the first simulations the homogeneous medium model was used, using (1) the OABC (Figures 3.1 and 3.2 and (2) the ED (Figures 3.3 and 3.4. All plots were scaled according to the maximum value anywhere in the grid at each time step. If a permanent maximum value were used, reflections would be invisible. In Figure 3.1 the $L_2$-norm of the velocity vector, $\sqrt{u^2 + w^2}$, with $u$ and $w$ the horizontal and vertical particle velocities, is displayed in a snapshot 11 s after a pressure wave has been initiated. Here, the P–wave front has just reached the bottom of the model. The snapshot reveals a weak P–P reflection from the bottom, while stronger artificial P–P and P–S reflections can be seen from both sidewalls. Interestingly, the reflections from the left sidewall are stronger than the ones from the right sidewall. The reason is our staggered grid description of Figure 2.1. Since we use the same number of grid points of each field variable in each grid dimension, a different set of variables generates the left and right grid edges (Figure 2.1). Specifically, the three (of totally five) variables $\sigma_{xx}$, $\sigma_{zz}$ and $w$ are absorbed further inside the grid on the right than on the left grid edge and explains the better absorption on the right edge (Figure 3.1). Generally, in order to achieve symmetric snapshots when using OABC, symmetric grids (i.e. using the same external variables on both the left and right grid edges) would be greatly advantageous. Figure 3.1 also shows a P–SV head-wave propagating from the upper left corner only. This is because the velocity analog of the stress–free
Figure 3.1 Snapshot of the velocity vector amplitude, \( a = \sqrt{u^2 + w^2} \) (\( u \) and \( w \) the horizontal and vertical particle velocities) at \( t = 11 \) s after a pressure wave has been initiated from a Gaussian point source at depth 0.5 km below the surface. The medium is homogeneous and OABC is used.
Figure 3.2 Snapshot of the velocity vector amplitude for the same situation as in Figure 3.1 at $t = 22$ s.
min(light yellow): -0.52e-6; max(violet): 0.52e-6

\[ \Delta \sim First \ point \ of \ absorption \]

**Figure 3.3** Snapshot of the velocity vector amplitude for the same situation as in Figure 3.1, but in this case ED is used.
Figure 3.4 Snapshot of the velocity vector amplitude for the same situation as in Figure 3.3 at \( t = 22 \) s.
surface conditions (Hestholm and Ruud (1994)) is implemented in an asymmetric way according to Figure 2.1.

In Figure 3.2 the velocity vector modulus (-norm) is displayed in a snapshot 22 s after the pressure wave has been initiated. Predominantly artificial reflections can be identified here, since the dominant non-artificial wavefield has passed out of the model frame. The reflections from the S-wave are stronger than from the P-wave due to our choice of maximum P-wave absorption (Peng and Toksöz, 1994; Peng and Toksöz, 1995)). The strongest reflections are seen to come from the bottom, while the vertically oriented wave front seen near the right boundary is the P-wave reflection from the left boundary.

Figures 3.3 and 3.4 are similar to Figures 3.1 and 3.2, except that ED was applied within 20 grid points wide strips along the numerical boundaries at the bottom and sides. Clearly, the ED scheme works better than the OABC in absorbing both P-waves and S-waves. It absorbs equally well along the bottom and the sidewalls. In Figure 3.4 the non-artificial waves have passed out of the model, and only artificial grid reflections remain. Investigating amplitude values at the layer adjacent to the grid edges, these reflections were found to be weaker than the corresponding reflections with the OABC by a factor of approximately $10^{-2}$, and furthermore, the run is completely stable. Again, because of the staggering, an Rg-wave can be distinguished propagating along the surface on the right part of Figure 3.4, whereas the one on the left is too weak to be seen.

Application of lower order finite differences near boundaries introduces extra dispersion. This dispersion seems to be less well handled by the OABC than by the ED. The reason is probably that high frequencies are quickly attenuated by ED. Methods like ED (Cerjan et al., 1985) are known to absorb reflections of relatively short wavelengths better than most methods. Because of this, we see that ED absorbs both
the high-frequency part from the broadband source and the Rayleigh waves at the surface quite well. In our examples, the high-frequency part from the source seems especially problematic for the OABC to absorb. The problem may be partly due to unspecified behavior of the OABC when it is used to absorb frequency parts for which its stability criterion is not satisfied.

A homogeneous model is an oversimplification of the real Earth, and so further comparison between OABC and ED is tied to wavefield simulation using a multi-layered crust – upper mantle model shown in Figure 3.5. Velocity vector modulii (amplitudes) as in Figures 3.1–3.4 are shown in Figures 3.6 and 3.7, but now using the background model of Figure 3.5. This fact, together with the multiple additional reflections from the various layer boundaries, cause these snapshots to be less clear than those for the homogeneous medium model (Figures 3.1–3.4). Figure 3.6 shows the wavefield 22 seconds after the pressure wave was initiated, using OABC. Also here asymmetric P–P and P–S reflections from the sidewalls can be seen, together with long term effects of wave conversions and reflections from the layers. The absorption at the bottom is seen to be somewhat better than that at the sidewalls due to the staggered grid.

Figure 3.7 shows the corresponding run using ED. The absorption strips of 20 grid points' width are clearly visible along the numerical boundaries. In comparing amplitudes at locations of artificial reflections from the last two runs we found them on the average O(10) stronger from using OABC than from using ED.

3.6 Conclusions

While a naive implementation of the OABC in our 2-D F–D scheme resulted in instabilities, a simple modification of the original algorithm allowed us to delay them beyond the time of interest. We arrived at an optimal procedure for implementing
**Figure 3.5** Snapshot of the velocity vector amplitude at $t = 1.2$ s after a pressure wave has been initiated from a Gaussian point source at depth 0.5 km below the surface. The source position and the background model of the layered medium are clearly visible. The maximum P-velocity is 8.18 km/s right below the Moho. The Moho is located at 35 km depth. The closest layer above the Moho represents a P-velocity of 6.9 km/s. The next layer is the maximum velocity layer in the crust with a P-velocity of 7.1 km/s. Just below the free plane surface there is the minimum velocity layer with a P-velocity of 6.6 km/s.
Figure 3.6 Snapshot of the velocity vector amplitude at $t = 22$ s after a pressure wave has been initiated from a Gaussian point source at depth 0.5 km below the surface. The medium model is layered (Figure 3.5) and OABC is used.
min (light yellow): -0.68e-6; max (violet): 0.68e-6

\[ \Delta \sim \text{First point of absorption} \]

Figure 3.7 Snapshot of the velocity vector amplitude for the same situation as in Figure 3.6, but in this case ED is used.
Peng and Toksöz's OABC algorithm, and then compared the OABC results with those obtained by using ED.

In implementing the OABC we found the choice of spatial F–D order to be important. With broadband Gaussian point source types the stability criterion of the OABC is not fulfilled for the complete frequency range, even though stable behavior is achieved by using 2nd and 4th order spatial F–Ds throughout the grid. OABC always behaves unstably when using higher order spatial F–Ds. always behaves unstably when using higher order spatial finite differences. The order of accuracy must be decreased when approaching the numerical boundaries in order to delay the instability. In our synthetic discretization scheme an 8th order accurate spatial F–D approximation is used in the interior of the grid. In moving from the interior towards the boundaries we found the following procedure to be optimal: 6 layers of 6th order F–Ds, then 4 layers of 4th order and finally 2 layers of 2nd order adjacent to the numerical boundary. It seems clear that ED absorbs better than OABC. In a homogeneous medium, the artificial reflection amplitudes from using ED within a thin strip were smaller by a factor $10^{-2}$ compared to using OABC. A reason might be insufficient sampling of higher frequencies near boundaries using OABC, and thereby violating its stability criterion for this frequency range. Another reason is the OABC’s apparently smaller ability to handle dispersions arising from using lower order finite differences near boundaries. The mentioned drawback of additional computer storage are offset by the shorter simulation time using ED. In our 2–D applications we absorb at 13.2 % of the total number of grid points. In a 3–D application using the same grid as here with the second horizontal dimension length equal to the first, we would use 6.3 % of the grid points for absorption.
3.7 Acknowledgments

IBM Norway is acknowledged for use of its IBM RISC System/6000 550E. A. S. acknowledges support from EU's COMETT exchange program. This work was supported by the Norwegian Research Council and the Air Force Office of Scientific Research, USAF under Grant F4926-92-J-0510. For the snapshots we acknowledge the use of PGraph designed by F. Lundbo and J. Petersen. This work was part of a project with the Dept. of Solid Earth Physics, University of Bergen, represented by prof. E. Husebye and scientist dr. B. Ruud. We thank dr. C. Peng for his kind provision of the OABC source code. Thanks also to Associate Editor K. Marfurt and two anonymous reviewers for improving the manuscript.
Chapter 4

3-D Finite-Difference Viscoelastic Wave Modeling including Surface Topography

4.1 Summary

I have pursued three-dimensional (3-D) finite-difference (F-D) modeling of seismic scattering from free surface topography. Exact free surface boundary conditions for arbitrary 3-D topographies have been derived for the particle velocities. The boundary conditions are combined with a velocity-stress formulation of the full viscoelastic wave equations. A curved grid represents the physical medium and its upper boundary represents the free surface topography. The wave equations are numerically discretized by an eighth order F-D method on a staggered grid in space, and a leap-frog technique and the Crank-Nicholson method in time. I simulate scattering from teleseismic P-waves by using plane incident wavefronts and real topography from a 60 x 60 km area which includes the NORESS array of seismic receiver stations in South-Eastern Norway. Synthetic snapshots and seismograms of the wavefield show clear conversion from P- to Rg- (short period fundamental mode Rayleigh) waves in areas of rough topography, which is consistent with numerous observations. By parallelization on fast supercomputers, it is possible to model higher frequencies and/or larger areas than before.

4.2 Introduction

Modeling free surface topography effects is mostly important for seismic profiling on land, whereas in marine settings it usually satisfies to specify medium parameters
without including explicit boundary conditions. However, topography can be impo-
tant for marine seisms as rough water surfaces can produce scattering that reduces
the signal-to-noise ratio. It is, however, the air-solid Earth interface that exhibits
the strongest possible impedance contrasts. Modeling topography along such a free
surface is important for these reasons alone. Any irregularities along such a surface
would have consequences for the results, the more so the stronger the gradients and/or
irregularities the local topography exhibits (Hestholm and Ruud, 1998). Additionally,
uncertain, or even undetermined parameters are used when performing a seismic sim-
ulation, such as seismic velocities and densities. Topographic data probably has the
smallest error margins of any model parameters we use on the other hand. Also,
whenever topographic data is known, it might as well be used in seismic modeling.
Performing wave modeling with topography as in the method described in this paper
involves a few extra terms in the medium equations along with more complicated, but
still explicit, boundary conditions for the particle velocities. Except for the topogra-
phy data itself, no extra memory is required. In addition, the difference in simulation
cost is negligible.

The advantage of modeling with surface topography is that implicit effects like
scattering and conversion will be accounted for automatically in the wavefield synthet-
ics. For example, plane P- or S-waves incident on a plane surface of a homogeneous
medium cannot convert to Rg-waves. Also, P- to SH-phase conversion depends
on surface topography. Amplification and deamplification of propagating waves can
be shown to occur at irregularities and in substantial neighborhoods around them
(Sanchez-Sesma and Campillo, 1991). Alluvial filled irregularly shaped valleys with a
plane free surface can generate strong Love- and Rg-waves for some incident waves in
3-D (Sanchez-Sesma and Luzon, 1995). Correspondence with real data is particularly
important in earthquake hazard assessment (Pitarka and Irikura, 1996).
Rg-waves can mask reflections that are the basis of migration. Over the NORESS-array in South-Eastern Norway the Rg-waves have amplitudes of about 10% of those of the first teleseismic P-wave arrivals, see Bannister, Husebye and Ruud (1990), Gupta, Lynnes and Wagner (1993), Hedlin, Minster and Orcutt (1991). By quantitatively accounting for these topography effects, a dataset better suited for migration can be produced. To illustrate how this might be used in practice, consider a simulation of an initial earth model with known surface topography. A good correspondence with real data should be obtained to make the initial model viable. Then, using the topography modeling algorithm to propagate the real data backwards in time by a technique like inverse time migration, good results might be obtained (Sun and McMechan, 1992).

The literature on free surface topography modeling is relatively limited in 2-D, and even more so in 3-D. A work considering different slope types and combinations of them explicitly in 2-D is that of Jih, McLaughlin and Der (1988). Another 2-D approach (Robertsson, 1996) classifies every surface point in a way similar to Jih et al. (1988). The scheme can handle both elastic and viscoelastic schemes and implements the field variables at each of 7 categories of surface topography points separately. Then the free surface conditions are accounted for by setting all velocities above the surface to zero and using a 'mirror-condition' on the normal stress components (Levander, 1988). Another work that models surface topography by F-D in a 'stair-case' manner, but using 3-D, is that by Ohminato and Chouet (1997). An interesting 2-D method considers a complete tensorial formulation of the wave equations for modeling curved interfaces and free surface topography, see Komatitsch, Coutel and Mora (1996). This method gives very accurate results and has the advantage of using the same amount of spatial partial derivative calculations as for a cartesian approach. This is 12.5% less in 2-D and 22.2% less in 3-D than the chain
rule approach used in this work. However, it has the distinct disadvantage of 30% extra memory requirement in 2-D and 60% extra memory requirement in 3-D than the general chain rule approach. Additionally, the memory difference will often be even larger because some partial derivatives vanish in the mapping functions due to straight vertical grid lines in most chain rule approaches, as is also the case in the present work. Another procedure is due to Frankel and Leith (1992), who use a 3-D topography modeling algorithm employing a density taper to zero starting at the level of the free surface while keeping the P-velocity unaltered.

The boundary integral method, or rather its numerical discretization, the boundary element method, have been used so far only to model relatively simple geometrical features. The reason is that the discretization applied to each structure must be given specific thought to avoid numerical instabilities. A space-frequency domain representation, see Bouchon, Schultz and Toksöz (1996), calculates the Green functions in the medium for the explicit topography surface under consideration as integrals over horizontal wavenumbers. The diffracted wavefield is the integral over the surface topography of the Green functions times unknown source density functions. These functions are solved for by the conjugate gradient method. Results are shown for an incident shear-wave polarized along the minor and major axes of a cosine-formed elliptically shaped hill, and they investigate scattering effects of this hill on the wavefield propagating towards an otherwise plane surface in a homogeneous medium. Sanchez-Sesma and Campillo (1991) also use a boundary integral method to investigate topography effects.

Tessmer and Kosloff (1994) extended their 2-D procedure, see Tessmer, Kosloff and Behle (1992), for elastic wave modeling with free surface topography to 3-D, transforming the velocity-stress formulation of the medium equations from a curved to a rectangular grid. They used a spectral discretization horizontally and a Chebyshev
one vertically in space. At the free topography surface, the stresses and velocities are transformed into local systems in which the vertical coordinate axis is parallel to the normal of the local surface element. The free surface conditions are then implemented by a 'characteristic' treatment of both the velocity and stress components, before they are rotated back to the original system. They show results for simple geometrical configurations, but in principle any arbitrary, smooth topography can be incorporated. The present method is based on their method because it transforms the equations of motion from a curved to a rectangular grid. The free surface boundary conditions, however, are developed in the present work explicitly as an exact, closed set of equations for the 3-D particle velocities.

Day and Minster (1984) were the first to replace the convolutional stress–strain relation for a linear viscoelastic solid (Christensen, 1982) with a set of first order partial differential equations through their Padé approximant method. For constant $Q$ over a predetermined frequency range their approximation yields excellent results. Carcione, Kosloff and Kosloff (1988b), however, were the first to represent the convolutional form of the constitutive equations by a first order system exactly. It was already done for the viscoacoustic case in Carcione, Kosloff and Kosloff (1988a) and further developed in Carcione (1993) for 2-D and 3-D displacement–stress formulations of the viscoelastic wave equations. A pseudospectral method was used here for spatial discretization along with a Chebyshev method in time, see Tal-Ezer, Carcione and Kosloff (1990), to minimize numerical dispersion. Robertsson, Blanch and Symes (1994) extended the method of Carcione et al. (1988b) by exactly representing the convolutional stress–strain relation in the 2-D and 3-D viscoelastic wave equations by a system of first order partial differential equations for the particle velocities and stresses. They used spatial fourth and temporal second order F–D discretizations. Employing these formulations, Blanch, Robertsson and Symes (1995), proposed cost-
efficient procedures for modeling constant $Q$ over a predefined frequency range. The immediate advantage of using velocity–stress formulations is that we do not differentiate material parameters across discontinuities (Virieux, 1986). Hence, when incorporating our topography boundary conditions in viscoelastic schemes, it is the velocity–stress formulation of Robertsson et al. (1994) and the $Q$–modeling procedure of Blanch et al. (1995) which I use as starting points.

As in Tessmer and Kosloff (1994) and Hestholm and Ruud (1998), a 3–D grid which is curved in the vertical direction is adapted to the surface topography, i.e. the top surface of the grid coincides with the surface topography. This method was originally proposed to adapt grids to interior interfaces using pseudospectral spatial derivatives (Fornberg, 1988b). I transform the velocity–stress formulation of the viscoelastic, isotropic wave equations (Robertsson et al., 1994) from the curved to a rectangular grid in which the numerical computations are done. At the topography surface, the velocity boundary conditions for a free surface are implemented into a local, rotated system at each point of the surface. Each of these systems has its vertical coordinate direction coinciding with the direction of the normal vector of the surface at the given point. The velocity boundary conditions are then rotated back to the cartesian system before finally they are transformed into curved grid equations in a rectangular grid. Once the boundary conditions are given in this grid, the numerical discretization can be performed.

In the following paragraphs, I give the connection between the convolutional stress–strain relation for a linear viscoelastic solid and its first-order partial differential equation representation (Carcione et al., 1988b) of velocities and stresses (Robertsson et al., 1994). The viscoelastic equivalents to the velocity–stress formulation of the elastic curved grid wave equations (Hestholm and Ruud, 1998; Tessmer and Kosloff, 1994) will be stated. I show the derivation of free surface topography
boundary conditions as an exact closed system for the particle velocities and give a description of its numerical solution. Next I show a simulation comparison with a geometric example for viscoelastic and elastic cases. Then I present simulated scattering results from teleseismic P-waves using a plane wavefront incident on real topography containing the NORESS array in South-Eastern Norway. Snapshots and synthetic seismograms of both elastic and viscoelastic wavefields clearly display Rg-waves in areas of rough topography.

4.3 Viscoelastic Wave Modeling Formulation

The basic hypothesis of viscoelastic theory is that the current value of the stress tensor depends upon the history of the strain tensor. This hypothesis can be described as

\[ \sigma_{ij} = G_{ijkl} \dot{\varepsilon}_{kl} = \dot{G}_{ijkl} \varepsilon_{kl}, \]

(Christensen, 1982) for a linear isotropic material. \( \dot{\varepsilon}_{ij}(t) \) denotes time convolution, and it transforms each strain history, \( \dot{\varepsilon}_{ij}(t) \), into the current stress value, \( \sigma_{ij}(t) \). The dot denotes time differentiation and \( G \) is a fourth order tensor of time called the relaxation function. For a homogeneous material \( G \) collapses into two independent functions. This is an assumption which is often used also for inhomogeneous materials. Each of these functions is often assumed to have the form of a standard linear solid

\[ G(t) = M_R \left( 1 - \sum_{\ell=1}^{L} \left( 1 - \frac{\tau_{\varepsilon \ell}}{\tau_{\sigma \ell}} \right) e^{-t/\tau_{\varepsilon \ell}} \right) \theta(t) \]

(Carcione et al., 1988b; Blanch et al., 1995). \( M_R \) is the relaxation modulus of the medium and \( \theta(t) \) is the Heaviside function. The relaxation function \( G(t) \) is equivalent to \( L \) standard linear solids connected in parallel. Each standard linear solid describes a dashpot and a spring in series in parallel with a spring. \( \tau_{\varepsilon \ell} \) and \( \tau_{\sigma \ell} \) are the stress and strain relaxation times of the \( \ell \)th mechanism.
Following Robertsson et al. (1994), the velocity–stress formulation of the viscoelastic wave equations can be derived from the constitutive relation (4.1) (Appendix D).

In 3-D with one relaxation mechanism (one standard linear solid) this can be written

\[
\frac{\partial u}{\partial t} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + f_x, \quad (4.3)
\]

\[
\frac{\partial v}{\partial t} = \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + f_y, \quad (4.4)
\]

\[
\frac{\partial w}{\partial t} = \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + f_z, \quad (4.5)
\]

\[
\frac{\partial \sigma_{xx}}{\partial t} = \pi \frac{\tau_e^P}{\tau_\sigma} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - 2\mu \frac{\tau_e^S}{\tau_\sigma} \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + r_{xx}, \quad (4.6)
\]

\[
\frac{\partial \sigma_{yy}}{\partial t} = \pi \frac{\tau_e^P}{\tau_\sigma} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - 2\mu \frac{\tau_e^S}{\tau_\sigma} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) + r_{yy}, \quad (4.7)
\]

\[
\frac{\partial \sigma_{zz}}{\partial t} = \pi \frac{\tau_e^P}{\tau_\sigma} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - 2\mu \frac{\tau_e^S}{\tau_\sigma} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + r_{zz}, \quad (4.8)
\]

\[
\frac{\partial \sigma_{xy}}{\partial t} = \mu \frac{\tau_e^S}{\tau_\sigma} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + r_{xy}, \quad (4.9)
\]

\[
\frac{\partial \sigma_{xz}}{\partial t} = \mu \frac{\tau_e^S}{\tau_\sigma} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) + r_{xz}, \quad (4.10)
\]

\[
\frac{\partial \sigma_{yz}}{\partial t} = \mu \frac{\tau_e^S}{\tau_\sigma} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) + r_{yz}, \quad (4.11)
\]

\[
\frac{\partial r_{xx}}{\partial t} = -\frac{1}{\tau_\sigma} \left\{ r_{xx} + \pi \left( \frac{\tau_e^P}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - 2\mu \left( \frac{\tau_e^S}{\tau_\sigma} - 1 \right) \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right\}, \quad (4.12)
\]

\[
\frac{\partial r_{yy}}{\partial t} = -\frac{1}{\tau_\sigma} \left\{ r_{yy} + \pi \left( \frac{\tau_e^P}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - 2\mu \left( \frac{\tau_e^S}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) \right\}, \quad (4.13)
\]

\[
\frac{\partial r_{zz}}{\partial t} = -\frac{1}{\tau_\sigma} \left\{ r_{zz} + \pi \left( \frac{\tau_e^P}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - 2\mu \left( \frac{\tau_e^S}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right\}, \quad (4.14)
\]

\[
\frac{\partial r_{xy}}{\partial t} = -\frac{1}{\tau_\sigma} \left\{ r_{xy} + \mu \left( \frac{\tau_e^S}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right\}, \quad (4.15)
\]
\[
\frac{\partial \tau_{xx}}{\partial t} = -\frac{1}{\tau_\sigma} \left\{ \tau_{xx} + \mu \left( \frac{\tau^S_\varepsilon}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \right\}, \tag{4.16}
\]
\[
\frac{\partial \tau_{yz}}{\partial t} = -\frac{1}{\tau_\sigma} \left\{ \tau_{yz} + \mu \left( \frac{\tau^S_\varepsilon}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial y} \right) \right\}, \tag{4.17}
\]
where \( \rho \) is the density, \( \pi \) is the relaxation modulus for P-waves, \( \pi = \lambda + 2\mu \) (\( \lambda \) and \( \mu \) the Lamé parameters), and \( \mu \) is the relaxation modulus for S-waves as in the elastic case. \( \tau^P_\varepsilon \) and \( \tau^S_\varepsilon \) are the strain relaxation times for P- and S-waves respectively, and \( \tau_\sigma \) is the stress relaxation time. The same \( \tau_\sigma \) can be used both for P- and S-waves (Blanch et al., 1995). \( f_x, f_y \) and \( f_z \) are the components of the body forces, \( u, v \) and \( w \) are the particle velocity components and \( \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz} \) and \( \sigma_{yz} \) are the stress components. \( \tau_{xx}, \tau_{yy}, \tau_{zz}, \tau_{xy}, \tau_{xz} \) and \( \tau_{yz} \) are the components of the memory variables. These are the equations governing wave propagation in a linear isotropic viscoelastic medium, and they are the equations of motion, the stress–strain relation and the memory variable equations.

I introduce a linear mapping from a rectangular \((\xi, \kappa, \eta)\)-system (Figure 4.1) to a curved grid in the \((x, y, z)\)-system (Figure 4.2), where both systems have positive direction upwards for the vertical coordinate. The 3-D mapping can be written as
\[
x(\xi, \kappa, \eta) = \xi, \tag{4.18}
\]
\[
y(\xi, \kappa, \eta) = \kappa, \tag{4.19}
\]
\[
z(\xi, \kappa, \eta) = \frac{\eta}{\eta_{\text{max}}} z_0(\xi, \kappa), \tag{4.20}
\]
where \( z_0(\xi, \kappa) \) is the topography function, and the rectangular \((\xi, \kappa, \eta)\)-system is bounded by \( \xi = 0, \xi = \xi_{\text{max}}, \kappa = 0, \kappa = \kappa_{\text{max}}, \eta = 0 \) and \( \eta = \eta_{\text{max}} \). For the curved grid in the \((x, y, z)\)-system the extent of stretching is proportional to the distance from the bottom plane of the grid \((z = 0)\). From equations (4.18)–(4.20) we get, for a differentiable function \( f(x, y, z) \),
\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x}, \tag{4.21}
\]
Figure 4.1  Rectangular grid surface.

Figure 4.2  Curved grid surface.
\[
\begin{align*}
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial \kappa} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y}, \\
\frac{\partial f}{\partial z} &= \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial z},
\end{align*}
\] (4.22)

Expressions for the partial derivatives, which are needed in the medium equations, are found from equations (4.18)–(4.20) (see Appendix A),

\[
\begin{align*}
\frac{\partial \xi}{\partial x} &= 1, & \frac{\partial \xi}{\partial y} &= 0, & \frac{\partial \xi}{\partial z} &= 0, \\
\frac{\partial \kappa}{\partial x} &= 0, & \frac{\partial \kappa}{\partial y} &= 1, & \frac{\partial \kappa}{\partial z} &= 0,
\end{align*}
\] (4.24)

\[
\begin{align*}
A(\xi, \kappa, \eta) &= \frac{\partial \eta}{\partial x} = -\frac{\eta}{z_0(\xi, \kappa)} \frac{\partial z_0(\xi, \kappa)}{\partial \xi}, \\
B(\xi, \kappa, \eta) &= \frac{\partial \eta}{\partial y} = -\frac{\eta}{z_0(\xi, \kappa)} \frac{\partial z_0(\xi, \kappa)}{\partial \kappa}, \\
C(\xi, \kappa) &= \frac{\partial \eta}{\partial z} = \frac{\eta_{max}}{z_0(\xi, \kappa)}.
\end{align*}
\] (4.26)

The velocity–stress formulation of the equations of motion, Hooke's law and the memory variable equations is given in the curved grid in the \((x, y, z)\)–system by equations (4.3)–(4.17). Expanding these by the chain rule (Appendix E) as for the elastic cases in Hestholm and Ruud (1998; 1994) and substituting for \(\partial \eta/\partial x\), \(\partial \eta/\partial y\) and \(\partial \eta/\partial z\), I get the medium equations with one relaxation mechanism in the rectangular \((\xi, \kappa, \eta)\)–system,

\[
\rho \frac{\partial u}{\partial t} = \frac{\partial \sigma_{xx}}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial \sigma_{xx}}{\partial \eta} + \frac{\partial \sigma_{xy}}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial \sigma_{xy}}{\partial \eta}
\]
\[
+ C(\xi, \kappa) \frac{\partial \sigma_{zz}}{\partial \eta} + f_x, \tag{4.29}
\]

\[
\rho \frac{\partial v}{\partial t} = \frac{\partial \sigma_{xy}}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial \sigma_{xy}}{\partial \eta} + \frac{\partial \sigma_{yy}}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial \sigma_{yy}}{\partial \eta}
\]
\[
+ C(\xi, \kappa) \frac{\partial \sigma_{yz}}{\partial \eta} + f_y, \tag{4.30}
\]

\[
\rho \frac{\partial w}{\partial t} = \frac{\partial \sigma_{zz}}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial \sigma_{zz}}{\partial \eta} + \frac{\partial \sigma_{yz}}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial \sigma_{yz}}{\partial \eta}
\]
\[
+ C(\xi, \kappa) \frac{\partial \sigma_{zz}}{\partial \eta} + f_z. \tag{4.31}
\]
\[ \frac{\partial \sigma_{xx}}{\partial t} = \frac{\tau_{e}^{S}}{\tau_{e}} \left( \frac{\partial u}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} + C(\xi, \kappa) \frac{\partial w}{\partial \eta} \right) \\
- 2\mu \left( \frac{\tau_{e}^{S}}{\tau_{e}} - 1 \right) \left( \frac{\partial u}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + C(\xi, \kappa) \frac{\partial w}{\partial \eta} \right) + r_{xx}, \tag{4.32} \]

\[ \frac{\partial \sigma_{yy}}{\partial t} = \frac{\tau_{e}^{S}}{\tau_{e}} \left( \frac{\partial u}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} + C(\xi, \kappa) \frac{\partial w}{\partial \eta} \right) \\
- 2\mu \left( \frac{\tau_{e}^{S}}{\tau_{e}} - 1 \right) \left( \frac{\partial u}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + C(\xi, \kappa) \frac{\partial w}{\partial \eta} \right) + r_{yy}, \tag{4.33} \]

\[ \frac{\partial \sigma_{zz}}{\partial t} = \frac{\tau_{e}^{S}}{\tau_{e}} \left( \frac{\partial u}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} + C(\xi, \kappa) \frac{\partial w}{\partial \eta} \right) \\
- 2\mu \left( \frac{\tau_{e}^{S}}{\tau_{e}} - 1 \right) \left( \frac{\partial u}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + C(\xi, \kappa) \frac{\partial w}{\partial \eta} \right) + r_{zz}, \tag{4.34} \]

\[ \frac{\partial \sigma_{xy}}{\partial t} = \frac{\tau_{e}^{S}}{\tau_{e}} \left( \frac{\partial u}{\partial \xi} + B(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \kappa} + A(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} \right) + r_{xy}, \tag{4.35} \]

\[ \frac{\partial \sigma_{xz}}{\partial t} = \mu \left( \frac{\tau_{e}^{S}}{\tau_{e}} - 1 \right) \left( \frac{\partial u}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + A(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} \right) + r_{xz}, \tag{4.36} \]

\[ \frac{\partial \sigma_{yz}}{\partial t} = \mu \left( \frac{\tau_{e}^{S}}{\tau_{e}} - 1 \right) \left( \frac{\partial u}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + A(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} \right) + r_{yz}, \tag{4.37} \]

\[ \frac{\partial \tau_{xx}}{\partial t} = -\frac{1}{\tau_{e}} \left\{ \tau_{xx} + \pi \left( \frac{\tau_{e}^{S}}{\tau_{e}} - 1 \right) \left[ \frac{\partial u}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \kappa} \right. \right. \\
\left. \left. + B(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} + C(\xi, \kappa, \eta) \frac{\partial w}{\partial \eta} \right] \right\}, \tag{4.38} \]

\[ \frac{\partial \tau_{yy}}{\partial t} = -\frac{1}{\tau_{e}} \left\{ r_{yy} + \pi \left( \frac{\tau_{e}^{S}}{\tau_{e}} - 1 \right) \left[ \frac{\partial u}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \kappa} \right. \right. \\
\left. \left. + B(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} + C(\xi, \kappa, \eta) \frac{\partial w}{\partial \eta} \right] \right\}, \tag{4.39} \]

\[ \frac{\partial \tau_{zz}}{\partial t} = -\frac{1}{\tau_{e}} \left\{ r_{zz} + \pi \left( \frac{\tau_{e}^{S}}{\tau_{e}} - 1 \right) \left[ \frac{\partial u}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \kappa} \right. \right. \\
\left. \left. + B(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} + C(\xi, \kappa, \eta) \frac{\partial w}{\partial \eta} \right] \right\}. \]
\[-2\mu \left( \frac{\tau_s}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} \right) \right], \quad (4.40)
\]
\[
\frac{\partial r_{xy}}{\partial t} = -\frac{1}{\tau_\sigma} \left( r_{xy} + \mu \left( \frac{\tau_s}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} \right) \right), \quad (4.41)
\]
\[
\frac{\partial r_{xz}}{\partial t} = -\frac{1}{\tau_\sigma} \left( r_{xz} + \mu \left( \frac{\tau_s}{\tau_\sigma} - 1 \right) \left( C(\xi, \kappa, \eta) \frac{\partial u}{\partial \eta} + \frac{\partial w}{\partial \xi} + A(\xi, \kappa, \eta) \frac{\partial w}{\partial \eta} \right) \right), \quad (4.42)
\]
\[
\frac{\partial r_{yz}}{\partial t} = -\frac{1}{\tau_\sigma} \left( r_{yz} + \mu \left( \frac{\tau_s}{\tau_\sigma} - 1 \right) \left( C(\xi, \kappa, \eta) \frac{\partial v}{\partial \eta} + \frac{\partial w}{\partial \kappa} + B(\xi, \kappa, \eta) \frac{\partial w}{\partial \eta} \right) \right). \quad (4.43)
\]

Equations (4.29)–(4.43) are the momentum conservation equations, Hooke's law and the memory variable equations given in the rectangular \((\xi, \kappa, \eta)\)-grid.

### 4.4 Free Surface Boundary Conditions

The 3-D free boundary conditions for the particle velocities at a locally horizontal surface (or in a system where the \(z\)-axis is parallel to the local normal vector of the surface) resulting from the vanishing stress condition can be written

\[
\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad (4.44)
\]
\[
\frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}, \quad (4.45)
\]
\[
\frac{\partial w}{\partial z} = -\frac{\lambda}{\lambda + 2\mu} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right), \quad (4.46)
\]

with \(x\) and \(y\) the horizontal coordinates and \(z\) the vertical coordinate. We want to apply these conditions to a topography surface. At each surface point, I introduce a local coordinate system \((x', y', z')\) in which the \(z'\)-axis coincides with the local normal vector direction of the surface. In this local system I impose the conditions (4.44)–(4.46). Once this is done, they have to be rotated back to the cartesian \((x, y, z)\)-system. This rotation is expressed by \(\tilde{\vartheta} = A^{-1}\tilde{\vartheta}'\), i.e.

\[
\tilde{\vartheta}' = A\tilde{\vartheta}, \quad (4.47)
\]
where \( \mathbf{v} \) and \( \mathbf{v'} \) are the particle velocity vectors in the \((x, y, z)\)- and the \((x', y', z')\)-systems respectively. \( \mathbf{A} \) is the rotation matrix, which can be given by

\[
\mathbf{A} = \begin{pmatrix}
\cos \theta & 0 & \sin \theta \\
-\sin \theta \sin \phi & \cos \phi & \cos \theta \sin \phi \\
-\sin \theta \cos \phi & -\sin \phi & \cos \theta \cos \phi
\end{pmatrix},
\]

(4.48)

where \( \theta \) is the rotation angle between the \(x\)-axis and the local \(x'\)-axis in the \((x, z)\)-plane and \( \phi \) is the rotation angle between the \(y\)-axis and the local \(y'\)-axis in the local \((y', z')\)-plane. We also have the relations \( \tan \theta = \partial z_0(\xi, \kappa)/\partial \xi \) and \( \tan \phi = \partial z_0(\xi, \kappa)/\partial \kappa \cos \theta \), now expressed in the computational \((\xi, \kappa, \eta)\)-grid (see Appendix F).

The calculations of the rotation from the local \((x', y', z')\)-system back to the cartesian \((x, y, z)\)-system can be performed as in Appendix F. Once this is done, a transformation from the cartesian \((x, y, z)\)-system to the computational \((\xi, \kappa, \eta)\)-grid should be performed (Appendix F) to express the boundary conditions as curved grid equations in the rectangular, computational \((\xi, \kappa, \eta)\)-grid. This is achieved by using the chain rule in the same way as was done for the medium equations. I arrive at the 3-D boundary conditions for free surface topography given in the computational \((\xi, \kappa, \eta)\)-grid by

\[
\frac{1}{e^2} \left( 1 + p^2 \right) C(\xi, \kappa) \frac{\partial u}{\partial \eta} + \frac{1}{e^2} \left( 1 + p^2 \right) dC(\xi, \kappa) \frac{\partial w}{\partial \eta} = 2d \frac{\partial u}{\partial \xi} + \frac{p \partial v}{e \partial \kappa} + \left( d^2 - 1 \right) \frac{\partial w}{\partial \xi} + \frac{p \partial u}{e \partial \kappa} + \frac{dp \partial w}{e \partial \kappa},
\]

(4.49)

\[
-\frac{1}{e^2} \left( 1 + p^2 \right) f p C(\xi, \kappa) \frac{\partial u}{\partial \eta} + \frac{1}{e^2} \left( 1 + p^2 \right) C(\xi, \kappa) \frac{\partial v}{\partial \eta} + \frac{1}{e} \left( 1 + p^2 \right) p C(\xi, \kappa) \frac{\partial w}{\partial \eta} = -2f dp \frac{\partial u}{\partial \xi} + d \left( 1 - p^2 \right) \frac{\partial v}{\partial \xi} + 2fp \frac{\partial w}{\partial \xi} + d \left( 1 - p^2 \right) \frac{\partial u}{\partial \kappa}
\]

\[
+ \frac{2p \partial v}{e \partial \kappa} + \frac{p \partial v}{e \partial \kappa} + \left( p^2 - 1 \right) \frac{\partial w}{\partial \kappa},
\]

(4.50)

\[
-\frac{d}{e^2} \left( 1 + p^2 \right) C(\xi, \kappa) \frac{\partial u}{\partial \eta} - \frac{p}{e^3} \left( 1 + p^2 \right) C(\xi, \kappa) \frac{\partial v}{\partial \eta} + \frac{1}{e^2} \left( 1 + p^2 \right) C(\xi, \kappa) \frac{\partial w}{\partial \eta} = - \left\{ \zeta \left( 1 + \frac{p^2}{e^2} \right) + d^2 \right\} \frac{\partial u}{\partial \xi} + \frac{dp}{e} \left( \zeta - 1 \right) \frac{\partial v}{\partial \xi} - d \left( \zeta - 1 \right) \frac{\partial w}{\partial \xi}
\]
\[ + \frac{dp}{e} (\zeta - 1) \frac{\partial u}{\partial \kappa} - \frac{1}{e^2} (\zeta + p^2) \frac{\partial v}{\partial \kappa} - \frac{p}{e} (\zeta - 1) \frac{\partial w}{\partial \kappa}, \]

using definition (4.28) and

\[ \zeta = \frac{\lambda}{\lambda + 2\mu}, \]  
\[ d = \frac{\partial z_0(\xi, \kappa)}{\partial \xi} = \tan \theta, \]  
\[ e = \cos [\arctan (d)] = \cos \theta, \]  
\[ f = \sin [\arctan (d)] = \sin \theta, \]  
\[ p = \frac{\partial z_0(\xi, \kappa)}{\partial \kappa} e = \tan \phi. \]

Equations (4.49)–(4.56) are exact 3–D boundary conditions for an arbitrary, smooth, free surface topography. They result from rotating the velocity free surface conditions from local systems at each point of the surface topography into a cartesian system before transformation into curved grid equations in the computational grid (Appendix F). The boundary conditions (4.49)–(4.56) are obviously not restricted to the F–D method or any other numerical discretization technique.

### 4.5 Numerical Discretization

To discretize the viscoelastic wave equations (4.29)–(4.43), high–order, cost–optimized F–D operators were used, see Kindelan, Kamel and Sguazzero (1990). Their method of optimal F–D coefficients from minimization of the total simulation cost under the constraint of a predefined maximum numerical dispersion is based on the work by Holberg (1987). The schemes employ a staggered discretization stencil of the velocity–stress formulation of the viscoelastic wave equations as was employed for the elastodynamic wave equations in Levander (1988) and Virieux (1986). An advantage of using a staggered definition of variables is that we can avoid explicit definition of stresses at the surface topography as it suffices to define the velocities there. In
order to get the velocities and stresses explicitly defined at each time step, I stagger the vertical velocity component $w$ one half grid length downwards. Generally, $u$ is staggered one half grid length in the positive $\xi$–direction, $v$ is staggered one half grid length in the positive $\kappa$–direction, while $w$ is staggered one half grid length in the negative $\eta$–direction (downwards). The stresses and the memory variables are defined either at the grid points ($\sigma_{xx}$, $\sigma_{yy}$, $\sigma_{zz}$, $r_{xx}$, $r_{yy}$ and $r_{zz}$), or at the midpoint of the grid rectangles ($\sigma_{xy}$, $\sigma_{xz}$, $\sigma_{yz}$, $r_{xy}$, $r_{xz}$ and $r_{yz}$), i.e. one half grid length positively in each of the directions of their indices. The 3–D boundary conditions, equations (4.49)–(4.51) are discretized by second order, staggered F–D operators. Below the free surface, the central, staggered F–D method’s order (Fornberg, 1988a) is gradually increased with depth, via fourth and sixth up to eighth order; the latter is the method used inside the domain and it is dispersion–bounded and cost–optimized (Kindelan et al., 1990).

To extrapolate velocities and stresses in time, the (second order) leap–frog technique is used. The equations for the memory variables $r_{ij}$ might become stiff for $\tau_\sigma$ small compared to the time step $\Delta t$, therefore the Crank–Nicholson stiff solver was used to propagate the memory variables in time (Robertsson et al., 1994). Since the memory variable equations are first order ordinary differential equations, the usually implicit Crank–Nicholson scheme becomes explicit, and only marginally more expensive than conventional explicit schemes. The Crank–Nicholson scheme is unconditionally stable.

Blanch et al. (1995) describe a method for approximating a constant $Q$ over a predetermined frequency interval for an arbitrary number of $L$ standard linear solids. However, both in the present work and in Robertsson et al. (1994), only one standard linear solid (relaxation mechanism) is used, and then a simpler method can be employed. Looking at the curves for $Q$ versus frequency for one relaxation mechanism (Blanch et al., 1995), we see that these curves are symmetric about a minimum
Q. The method I use is to set the desired Q equal to this minimum value for the central frequency of the source, and calculate \( \tau_\sigma \), \( \tau_\epsilon^P \) and \( \tau_\epsilon^S \) accordingly; possibly with a different \( Q \) for P– and S–waves. This procedure was used in Robertsson et al. (1994) and ensures a symmetric behaviour of the attenuation and dispersion about the source’s central frequency. The expressions for the relaxation times resulting from this principle are

\[
\tau_\sigma = \frac{1}{\omega} \left( \sqrt{1 + \frac{1}{Q_P^2}} - \frac{1}{Q_P} \right), \tag{4.57}
\]

\[
\tau_\epsilon^P = \frac{1}{\omega^2 \tau_\sigma}, \tag{4.58}
\]

\[
\tau_\epsilon^S = \frac{1 + \omega \tau_\epsilon Q_S}{\omega Q_S - \omega^2 \tau_\sigma}. \tag{4.59}
\]

with \( \omega = 2\pi f \) where \( f \) is the central frequency, and \( Q_P \) and \( Q_S \) are \( Q \)-values for the P– and S–waves respectively.

For the viscoelastic modeling examples I found the following procedure to work best to absorb waves along the grid boundaries: First I calculated \( \tau_\sigma \), \( \tau_\epsilon^P \) and \( \tau_\epsilon^S \) from the above formulas with \( Q_P = 2.5 \) and \( Q_S = 2.0 \). Then I explicitly set \( \tau_\sigma \) to a very high value, i.e. \( \tau_\sigma \sim 100 \); thereby attaining a much higher order of magnitude of the stress relaxation time compared to the strain relaxation times. For the elastic examples shown I use exponential damping according to Cerjan et al. (1985). The stresses and velocities are multiplied by exponentially decreasing terms towards the artificial boundaries. The thickness of the absorbing layers was 30 grid points both for the viscoelastic and the elastic examples with real topography, which corresponds to 2.5 times the dominant wavelength of the source. In the viscoelastic case, 20 of these grid points were used for a linear tapering of the relaxation times towards their values along the boundaries.

To find the particle velocity components at the surface topography from the closed system (4.49)–(4.51), I solve it directly and simultaneously as a linear system with
respect to the velocities at the surface as they are defined in the second order vertical derivative discretizations. In this procedure, the horizontal partial derivatives are calculated one grid length and one and a half grid length below the free surface and considered known from the medium equations. The resulting linear system of equations for the unknown particle velocities at the free surface has determinant

\[ D = \frac{1}{e^6} (1 + p^2)^4 (1 + d^2) C^3(\xi, \kappa), \quad (4.60) \]

using definitions (4.28) and (4.52)–(4.56). This determinant is always positive and its minimum value is \( C^3(\xi, \kappa) \), which occurs for a plane surface. Therefore this numerical solution of boundary conditions (4.49)–(4.51) is unconditionally stable.

4.6 A geometric example

Figures 4.3–4.5 show snapshots of the initial shot point of a Ricker P–source with a central frequency of 5 Hz in the focus of a 3–D parabola (Figure 4.3) and reflected waves from its surface (t=1.4 seconds) for an elastic (Figure 4.4) and a corresponding viscoelastic case (Figure 4.5). The snapshots are taken along the vertical diagonal (from small \( \xi \) and \( \kappa \) to large \( \xi \) and \( \kappa \)) and the vertical velocity component \( w \) is displayed. According to the analytic solution the reflected wave in the elastic case should be perfectly plane and in the viscoelastic case it should have a plane appearance. This is seen to be the case in Figures 4.4 and 4.5. The P– and S–wave velocities of the homogeneous medium is 5.0 and 2.89 km/s respectively and the density is 2000 kg/m³. The viscoelastic \( Q \)–value is 20 both for P– and S–waves and the rectangular grid in each case has size 127 X 127 X 127 with a uniform grid distance of 0.1 km. The maximum medium height (at the top of the parabola) is 12.6 km and the focal point is 3 km below it (the focal point defines the curvature of the parabola). The central frequency of 5 Hz is high enough in these cases for a visible difference between
the elastic and viscoelastic cases to appear. The intrinsic attenuation causes the viscoelastic reflection to have a smoother appearance and the physical dispersion leads to a broader wavefront. This dispersion combined with the high medium velocity leads the viscoelastic wavefront to propagate noticeably further than the corresponding elastic wavefront in the same time. The snapshots of Figures 4.4 and 4.5 are scaled individually according to their maximum value to enhance the features of each case.

4.7 P- to Rg–Scattering from Topographic Relief

For the present 3-D F–D simulations I have used digital elevation data from an area of 60 × 60 km containing the 3 km aperture NORESS array in South–Eastern Norway (Figure 4.6). This hilly area was chosen because of easy access to detailed topographic data. Additionally, significant P to Rg scattering from specific hills is well documented from NORESS record analysis (Bannister et al., 1990; Gupta et al., 1993; Hedlin et al., 1991). More recent analysis is found in Hedlin, Minster
Figure 4.4  Reflected wave from the parabolic free surface \((t = 1.4 \text{ s})\) in the elastic case. Diagonal vertical snapshot.

Figure 4.5  Reflected wave from the parabolic free surface \((t = 1.4 \text{ s})\) in the viscoelastic case. Diagonal vertical snapshot.
and Orcutt (1994). The most prominent topography present in the dataset is the steep Skreikampen hill immediately west of the southern part of lake Mjøsa (the long south–north oriented blue pattern in the south–west area of Figure 4.6). In the first examples the source wave is a vertically incident plane P-wave simulating a teleseismic short period P-phase. The center frequency of the plane Ricker wavelet (point Ricker sources implemented along a plane) is 2.5 Hz, the P- and S-wave velocities of the homogeneous medium is 6.0 km/s and 3.46 km/s respectively and the density is 2000 kg/m$^3$. I set $Q = 250$ for P-waves and $Q = 200$ for S-waves. I use a uniform grid sampling of 0.2 km and a total grid size of $300 \times 300 \times 100$. The viscoelastic simulation corresponds to a total computer memory of 1.4 GB using domain decomposition by Message Passing Interface (MPI) and took about 1 hour on 18 processors using the SGI (Cray) Origin 2000 parallel machine located at the Dept. of Informatics, University of Bergen, Norway. The corresponding elastic simulation used a memory of slightly more than 1 GB and took fifteen minutes less on the same number of processors. Using domain decomposition parallelization via MPI on this machine enables me to run 3-D models with about $10^8$ grid points, as opposed to about $10^7$ on earlier machines. The sizes of the present simulations are far below these maximum sizes.

The plane teleseismic P-wave is vertically incident on the topography and released at its level. The first snapshot time of Figure 4.7 is $t=0.5$ seconds and there is 1 second between each of the snapshots displaying the vertical velocity component $w$ at the surface topography (upper series) and along the west–east topography profile of Figure 4.6 (lower series). Figure 4.8 are the same snapshots for the corresponding elastic case. An absorption thickness of 30 grid points is used along the grid boundaries in both the viscoelastic and elastic cases, and exponential damping according to Cerjan et al. (1985) is used with an absorption constant of 0.05 in the elastic case. As
Figure 4.6  Left: Map shows the topography of the $60 \times 60$ km area used in the 3-D simulations. The dashed lines show the positions of the receivers in two profiles and the circle outlines the NORESS array. Labels are in kilometers. The blue area to the south-west is Lake Mjøsa (123 m above sea level). Right: Topography profiles along two lines each of 60 km length shown on the map. They are midway along the $y$-direction and $x$-direction and cover respectively the complete $x$- and $y$-dimension of the area. Horizontal axes are in kilometers and vertical axes are in meters above mean sea level.
Figure 4.7 Snapshots of the vertical velocity component at various times after a plane wave is released at the surface topography of Figure 4.6. The upper series of the viscoelastic simulation is taken at the surface topography and the lower series is taken along the west–east topography profile of Figure 4.6.
expected, this case leads to artificial boundary reflections of stronger amplitudes than the viscoelastic case. In addition, the strongest grid reflections in the viscoelastic case propagate into the domain at a later stage than they do in the elastic case. Although the exponential damping technique is the most efficient absorbing boundary method I have tested (Simone and Hestholm, 1998), a wave incident at an angle of 90° with the boundary is the most difficult case. It is apparent that absorption layers of low $Q$ along the grid boundaries in a viscoelastic scheme performs superior to exponential damping in such a thin layer. Nevertheless, both cases generate clear grid boundary reflections. The reason is that the present situation of a plane wave reaching out to all grid edges is a worst case scenario for grid reflection since wave components in every direction reach all grid edges along the surface immediately. Even so I prefer to include these examples both to show each wave modeling scheme's ability to absorb every incident wave component and to illustrate the behaviour of the topography boundary conditions at every location of the real topography as soon as possible. Figures 4.7 and 4.8 show a circular wave pattern of Rg–waves emanating from the area of prominent topography at 30 km from the south and 40 km from the west grid edges. The scattered surface waves appear to radiate out from secondary point sources which coincide with areas of high topographic gradients. This scattering originates from the steep valley side east of Bronkeberget about 10 km east of NORESS (Bannister et al., 1990). However, the surface wave with the highest amplitude in these examples is a slowly moving Rg–wave propagating from the south–west corner of the area into the domain. The steepest topography of the model can be found in this area close to Lake Mjøsa (the Skreikampen hill). P–to–Rg scattering in this area was observed by Bannister et al. (1990), and can now be realistically synthesized.

Figures 4.9 and 4.10 show respectively the next three snapshots of the viscoelastic simulation of Figure 4.7 and the elastic simulation of Figure 4.8. The viscoelastic grid
Figure 4.8  Snapshots of the vertical velocity component at various times after a plane wave is released at the surface topography of Figure 4.6. The first series of the elastic simulation is taken at the surface topography and the second series is taken along the west–east topography profile of Figure 4.6.
reflections from the thin absorbing strips are clearly seen as straight lines, although the elastic grid reflections are more dominant and propagate faster into the domain. Features of scattering can be seen in both cases although the viscoelastic case gives more information due to slower grid reflections. Most clearly we see the strong Rg-wave which continues to propagate away from the area of prominent topography in the south–west corner towards the center of the domain. We also see the circular wave pattern propagating further out from the area 30 km from the south and 40 km from the west grid edges.

Using the simulation time of 5.5 seconds there is no noticeable difference between the viscoelastic and elastic cases for the present parameters. The $Q$-values (250 for P- and 200 for S-waves), are low combined with the high (6 km/s) P-velocity. The applied center frequency of 2.5 Hz then leads to such long wavelengths that there are no visible attenuation and dispersion of the wavefield for the times and scales shown. Using $A(x) = A_0 \exp\{-\pi f x/(cQ)\}$ (Aki and Richards, 1980), where $f$ is the central frequency, $c$ is the wave speed, $A(x)$ is the wave amplitude at travel distance $x$ and $A_0$ is the wave amplitude at travel distance 0, leads to attenuations of Rg-waves of 10 % after 2.75 seconds and 20 % after 5.5 seconds of simulation time. This attenuation is unnoticeable in Figures 4.7 and 4.9. In addition, from the dispersion curves in Robertsson et al. (1994), the present parameters should lead to an approximate dispersion of 2 %, which after 5.5 seconds of simulation time of Rg-waves corresponds to a maximum dispersion distance of 0.37 km. It is clear that these quantities are unnoticeable at the present scales and parameters, and so the scattering will look the same for the elastic and the viscoelastic cases.

Therefore clearly, the predominant argument for viscoelastic rather than elastic modeling for teleseismic distances and earthquake simulations is the improved absorbing boundary conditions along the grid edges. Figures 4.11 and 4.12 show color
Figure 4.9  The next three snapshots following the ones in Figure 4.7, i.e. the vertical velocity component at the surface topography (top) and along the west–east topography profile of Figure 4.6 (bottom).
Figure 4.10 The next three snapshots following the ones in Figure 4.8, i.e. the vertical velocity component at the surface topography (top) and along the west–east topography profile of Figure 4.6 (bottom).
snapshots of the viscoelastic and elastic cases respectively, corresponding to the vertical snapshots (lower series) of Figures 4.7 and 4.9 (viscoelastic case) and Figures 4.8 and 4.10 (elastic case). As before, each vertical snapshot is taken along the west–east topography profile (along the $xz$–plane) of Figure 4.6, but now also the horizontal particle velocity component $u$ is displayed in the right series of Figures 4.11 and 4.12. We see as before much more prominent grid reflections in the elastic case (Figure 4.12). Only six colors are used in these figures, therefore the small numerical dispersion in front of the wavefront is magnified unrealistically, but other aspects are shown more clearly in these snapshots. The same number of colors are used in the corresponding snapshots of the vertical particle velocity component $w$ along the surface topography in Figures 4.13 and 4.14, corresponding respectively to the viscoelastic case of Figures 4.7 and 4.9 (upper series) and the elastic case of Figures 4.8 and 4.10 (upper series). The color displays show some of the $R_g$–wave scattering clearer than before.

I also show some seismograms from the simulations taken along the receiver profiles of Figure 4.6. Figure 4.15 is the second horizontal particle velocity component $v$ displayed along the west–east oriented receiver profile of Figure 4.6 from the viscoelastic simulation. The seismogram shows scattering from the topography, but after 3 seconds of simulation time boundary reflections from the north grid boundary tend to dominate.

Figures 4.16 and 4.17 are the vertical particle velocity component $w$ along the south–north oriented receiver profile of Figure 4.6 of the viscoelastic and elastic simulations respectively. Here the earlier and stronger boundary reflections in the elastic case is obvious, and we see clear irregular and very localized scattering from the surface topography. In the corresponding seismogram in the viscoelastic case for the first horizontal particle velocity component $u$ (Figure 4.18), the scattering exhibits a
Figure 4.11 Snapshots of the vertical velocity component $w$ (left series) and the first horizontal velocity component $u$ (right series) at various times after a plane wave is released at the surface topography of Figure 4.6. The snapshots of the viscoelastic simulation is taken along the west–east topography profile of Figure 4.6.
Figure 4.12  Snapshots of the vertical velocity component $w$ (left series) and the first horizontal velocity component $u$ (right series) at various times after a plane wave is released at the surface topography of Figure 4.6. The snapshots of the elastic simulation is taken along the west-east topography profile of Figure 4.6.
Figure 4.13 Snapshots of the vertical velocity component \( u \) at various times after a plane wave is released at the surface topography of Figure 4.6. The snapshots of the viscoelastic simulation is taken along the surface topography.
Figure 4.14 Snapshots of the vertical velocity component $w$ at various times after a plane wave is released at the surface topography of Figure 4.6. The snapshots of the elastic simulation is taken along the surface topography.
Figure 4.15  Seismogram of the second horizontal particle velocity component $v$ for the viscoelastic simulation of Figures 4.7 and 4.9 along the west–east oriented receiver profile of Figure 4.6.
broader pattern and a stronger coherency affecting many more receivers at a time. This is expected since Rg-waves travel predominantly horizontally along the surface. However, grid reflections from the east boundary dominate after 3 seconds of simulation time. Since the medium is homogeneous, the scattering we see is only due to the 3-D topography. Previous work (Hestholm et al., 1999) has shown that a great amount of scattering is due to out-of-plane effects from 3-D topography as 2-D synthetics leads to much simpler seismograms.

In the final example I show a viscelastic simulation using an incident dipping plane wave towards the same surface topography area using the same grid parameters and dimensions and the same homogeneous medium as before. The example is included to represent an incoming wave from an earthquake or a teleseismic explosion in a more realistic way than in the last example. The first three snapshots along the surface and along the vertical \(xz\)-plane of the \(w\) particle velocity component are displayed in Figure 4.19 and the three next ones in Figure 4.20. The plane wave has the same small negative dip in both the \(x\)- and \(y\)-directions. This case leads to less grid boundary reflections than in the case of the vertically incident plane wave. We see that the same areas as before, i.e. the one 30 km from the south and 40 km from the west grid edges as well as the one near the south-west corner, are giving rise to the most prominent topography scattering. In this case, however, the scattering from the first of these areas has not reached as far as it did for the horizontal plane wave at the same times because in this case the plane wave was not initiated at the surface. On the other hand, the dipped plane wave gives rise to scattering of stronger amplitudes than for the vertically incident (horizontal) plane wave. The snapshots are scaled relative to each other, and the surface topography scattering of Figures 4.19 and 4.20 appears clearer than before. In the last snapshots (at 5.5 seconds) the bottom grid boundary reflection reaches the surface.
Figure 4.16  Seismogram of the vertical particle velocity component $w$ for the viscoelastic simulation of Figures 4.7 and 4.9 along the south–north oriented receiver profile of Figure 4.6.
Figure 4.17  Seismogram of the vertical particle velocity component $w$ for the elastic simulation of Figures 4.8 and 4.10 along the south–north oriented receiver profile of Figure 4.6.
Figure 4.18  Seismogram of the first horizontal particle velocity component \( u \) for the viscoelastic simulation of Figures 4.7 and 4.9 along the south–north oriented receiver profile of Figure 4.6.
Figure 4.19  Snapshots of the vertical velocity component at various times after a dipping plane wave is released near the surface topography of Figure 4.6. The upper series of the viscoelastic simulation is taken at the surface topography and the lower series is taken along the west-east topography profile of Figure 4.6.
Figure 4.20  The next three snapshots following the ones in Figure 4.19, i.e. the vertical velocity component at the surface topography (top) and along the west–east topography profile of Figure 4.6 (bottom).
Figures 4.21 and 4.22 are the corresponding six-color snapshots to the simulation of Figures 4.19 and 4.20. Figure 4.21 shows the $xz$-snapshots of the velocity component $u$ in addition in the right series. In some ways Figures 4.21 and 4.22 give a clearer picture of the Rg scattering and the propagation of the plane wave reflection than Figures 4.19 and 4.20. They are scaled relative to the previous color snapshots of Figures 4.11–4.14 and exhibit stronger scattering amplitudes.

Figures 4.23–4.25 are seismograms from the last simulation of the three particle velocity components $u$, $v$ and $w$ along the south–north oriented receiver profile of Figure 4.6. Apart from the plane wave reflection these seismograms exhibit predominantly surface topography scattering. The horizontal velocity component scattering is more coherent than the $w$ scattering, although $u$ more so than $v$.

4.8 Discussion

In most landseismic profiling there is one or more low-velocity layers near the surface. 2-D investigations with the aim of quantifying the scattering from near-surface irregularities were performed (Hill and Levander, 1984; Levander and Hill, 1985). Irregularities were introduced laterally along the bottom boundary of a surface low-velocity layer. Comparing results of a plane, vertically incident wave upon such a layer and a layer with a plane lower boundary, energy of large amplitude motion is confirmed to get trapped near the free (plane) surface in the case of lateral corrugation of the lower boundary. The roughness induces a strong resonant coupling of incident P-waves to Rg modes. Irregularities along the lower boundary are shown to cause large amount of energy trapped into Rg modes near the free surface. This energy will stay strong and be the dominant feature of the wavefield after the main pulse has radiated out of the model. Similar results were obtained from a shallowly buried thin layer with random velocity perturbations. When such strong energy concentra-
Figure 4.21 Snapshots of the vertical velocity component \( w \) (left series) and the first horizontal velocity component \( u \) (right series) at various times after a dipping plane wave is released near the surface topography of Figure 4.6. The snapshots of the viscoelastic simulation is taken along the west–east topography profile of Figure 4.6.
Figure 4.22 Snapshots of the vertical velocity component $w$ at various times after a dipping plane wave is released near the surface topography of Figure 4.6. The snapshots of the viscoelastic simulation is taken along the surface topography.
Figure 4.23  Seismogram of the first horizontal particle velocity component $u$ for the viscoelastic simulation of Figures 4.19 and 4.20 along the south–north oriented receiver profile of Figure 4.6.
Figure 4.24  Seismogram of the second horizontal particle velocity component $v$ for the viscoelastic simulation of Figures 4.19 and 4.20 along the south–north oriented receiver profile of Figure 4.6.
Figure 4.25  Seismogram of the vertical particle velocity component $w$ for the viscoelastic simulation of Figures 4.19 and 4.20 along the south–north oriented receiver profile of Figure 4.6.
tion along the free surface can be caused by irregularities along the bottom boundary of a low-velocity layer along the surface, one would believe that irregularities along the much stronger discontinuities of a free surface would cause even stronger energy trapping and conversions to Rg modes in the presence of such a low-velocity layer. These effects are not seen in the case of a plane surface and a homogeneous layer with smooth boundaries. In this case, energy escapes from the near-surface layer and is radiated back into the half-space after reverberation within the layer (Levander and Hill, 1985). In the present work I show effects of surface topographies without inclusion of near-surface low-velocity layers in order to attempt to isolate effects of free surface topography. 2-D F-D elastic simulations with topography, random media perturbations of the crust and upper mantle and low-velocity layers near the free surface have nevertheless been performed using the present method (Hestholm and Ruud, 1994; Hestholm et al., 1994). Analysis of results is found in Ruud, Husebye and Hestholm (1993). At any rate, even in a homogeneous medium scattering from surface topography is quite effective, as is shown in this work.

Discussion of possible topography scattering in a dataset assembled by Conoco in West Texas has been pursued (Imhof, 1996). Although Imhof claims (Chapter 6) that for records shot on top of the mesas, topography can be discarded as a major scattering mechanism because of the very smooth topography there, he mentions that the record with the largest amount of 'noise' is the one shot across the roughest part of the mesa. To do F-D simulations with topography he uses the method of Jih et al. (1988) resulting in relatively simple snapshots, although modeling only of incident Rg-waves along the surface was performed. He attributes most of the scattering to near-surface cavities, but if simulations were to be done across the complete area of field data, including the debris-filled valleys, topography effects surely will have great significance on the results.
Effects of topography are important for earthquake hazard assessment, and crustal studies have been pursued with this goal using central frequencies of 1–10 Hz. The topography causes the amplitudes of seismic waves from earthquakes to vary significantly locally. The spectral content of the waves will accordingly be affected significantly from the topography (Pitarka and Irikura, 1996). An F–D algorithm for 'stair–case' modeling of 3–D surface topography on top of an elastic medium is given in this work and used to model incident waves on simple topography structures and at the KOB–JMA site near Kobe, Japan. Low–frequency (1–3 Hz) amplification is attributed to local topography effects. High–frequency deamplification results from deconstructive interference of scattered waves. Results agree well with ground motion recordings of aftershocks at the site. As already mentioned, Bouchon et al. (1996) use the boundary element method in the wavenumber domain, and investigate the effect of a hill on the ground motion from an earthquake. The hill is cosine–shaped with an elliptical base of ratio 2 to 1. Amplitude amplifications are consistently found to occur at and near the top of the hill over a broad range of frequencies. The results show that amplifications are higher and stay higher over a much broader frequency band for incident shear waves polarized along the minor axis of the ellipse as opposed to incident shear waves polarized along the longitudinal axis. A strong directivity of scattered wavefield away from the topographic feature is also confirmed, exhibiting strong waves propagating away along the minor axis of the ellipse while almost no scattering occurs along the longitudinal direction of it. This work is particularly interesting in that it assesses scattering effects (including frequency and directionality dependencies) from a simple topographic structure for incident S–waves with different polarities. Theoretical and numerical results consistently predict amplification at topography ridge crests, but nevertheless almost always systematically underestimate actual amplifications observed in the field. Among the explanations for this is
the 2-D nature of the topographic geometries assumed in calculations. Seismic responses from hills exhibit 3-D behaviour and there is a current lack of 3-D theoretical investigations.

Bannister et al. (1990) observe that some hills in the NORESS area literally radiate Rg-waves at regular intervals for incident teleseismic waves of long durations. Array recordings are just point observations in this regard while the synthetic snapshots among Figures 4.7-4.25 reveal nearly symmetric Rg-radiation from such secondary sources. However, this radiation from a multiplicity of secondary sources becomes complex with time, so propagation directionality from a secondary source to a receiver can be very local and will also generally weaken with distance. Observationally this is confirmed by the fact that Rg-waves rarely propagate further than 60 km in hilly areas typical of the NORESS and GERESS (Bavaria, Germany) arrays while across the plains of northern Fennoscandia Rg-waves from explosions occasionally propagate out to 600 km.

4.9 Conclusions

Exact boundary conditions for free surface topography combined with the viscoelastic wave equations in the velocity-stress formulation give realistic scattering and complex wave pattern due to out-of-plane effects from 3-D topography. I demonstrate clear P-to-Rg scattering from prominent topography and I find it gratifying that the strong P-to-Rg scattering from the Skreikampen and Bronkeberget hills, observed by Bannister et al. (1990) through analysis of NORESS recordings, can be realistically synthesized. Another interesting phenomenon is that abrupt changes of Rg-wavefield occur over relatively small distances. This is due to strong directivity of scattering from some topographic features as well as destructive interference from a multiplicity of secondary sources. Such wavefield characteristics are sometimes ob-
served in recordings – at some sensors the Rg–phase is prominent while hardly visible at nearby sensors less than a kilometer away. Even in simulations of earthquakes and teleseismic explosions, where parameters close to elastic cases are used, it is advantageous to use the viscoelastic code with a very high $Q$ because of its superior absorbing boundary conditions along the grid edges. At longer time laps and/or lower $Q$–values it also gives realistic intrinsic attenuation and physical dispersion of all waves.

4.10 Acknowledgments

I greatly appreciated discussions with dr. Bent Ruud and prof. Eystein Husebye (both at the University of Bergen, Norway, Dept. of Solid Earth Physics). I also had useful discussions with dr. Johan Robertsson (Schlumberger Cambridge Research, Cambridge, England) and dr. Peeter Akerberg (Rice University, Houston, TX, USA, Dept. of Geology and Geophysics). Thanks to dr. Johnny Petersen (Rogalandsforskning, Bergen) for the idea of the analytic example and the IBM RS 6000 pixel graphics displaying it. I acknowledge Bjarne Herland and dr. Ove Sævareid (both at Rogalandsforskning, Bergen) for support on MPI and parallellization of the code. Thanks to dr. Tor Sørevik (Parallab, University of Bergen, Dept. of Informatics) for his support on the Cray Origin 2000 parallel machine. This research was supported by the Norwegian Research Council and by the Norwegian Supercomputer Committee through a grant of computing time. I would like to thank dr. Aladin Kamel (Regional Information Technology & Software Engineering Center, Heliopolis, Cairo, Egypt) for igniting my interest in numerical seismic modeling and prof. Manik Talwani (Rice University, Dept. of Geology and Geophysics) for general support.
Chapter 5

2-D Finite-Difference Viscoelastic Wave Modeling including Surface Topography

5.1 Summary

We have pursued two-dimensional (2-D) finite-difference (F-D) modeling of seismic scattering from free surface topography. Exact free surface boundary conditions for the particle velocities have been derived for arbitrary 2-D topographies. The boundary conditions are combined with a velocity-stress formulation of the full viscoelastic wave equations. A curved grid represents the physical medium and its upper boundary represents the free surface topography. The wave equations are numerically discretized by an 8th order F-D method on a staggered grid in space, and a leap-frog technique and the Crank-Nicholson method in time.

To demonstrate capabilities of the surface topography modeling technique, we simulate incident point sources with a sinusoidal topography in seismic media of increasing complexities. We present results using parameters typical of exploration surveys with topography and heterogeneous media. Topography on homogeneous media is shown to generate significant scattering. We show additional effects of layering in the medium, with and without randomization using a von Kármán realization of apparent anisotropy. Synthetic snapshots and seismograms indicate that prominent surface topography can cause back-scattering, wave conversions and complex wave-pattern which are usually discussed in terms of inter-crust heterogeneities.
5.2 Introduction

Because the air–solid Earth interface exhibit the strongest possible impedance contrasts, modeling topography along such a free surface should be important. Any irregularities along such a surface would have additional consequences for the results (Hestholm and Ruud, 1998). Additionally, topography data probably has the smallest error margins of any modeling parameters we use. We might as well use topography data in seismic modeling whenever it is known because wave modeling with topography as in the method described in this paper involves just a few extra terms in the medium equations, along with more complicated but explicit boundary conditions for the particle velocities. Except for the topography data itself, no extra memory is required, and the difference in simulation cost is negligible. The immediate advantage of modeling with surface topography is that implicit effects like scattering and conversion will be accounted for automatically in the wavefield synthetics. For example, plane P– or S–waves incident on a plane surface of a homogeneous medium cannot convert to Rg–waves, whereas P– to SH–phase conversion depends on surface topography. Amplification and deamplification of propagating waves can be shown to occur at irregularities and in substantial neighborhoods around them (Sanchez-Sesma and Campillo, 1991). Correspondence with real data is particularly important in earthquake hazard assessment (Pitarka and Irikura, 1996).

A few works are available for the modeling of free surface topography. A work using F–D modeling and considering different slope types and combinations of them explicitly is that of Jih et al. (1988). Another 2–D approach (Robertsson, 1996) incorporates only topography portions parallel to the main axes and classifies every surface point in a way similar to Jih et al. (1988). A 2–D method that considers a complete tensorial formulation of the wave equations for modeling curved interfaces and free surface topography is that of Komatitsch et al. (1996). This method
gives very accurate results but has the distinct disadvantage of 30% extra memory requirement in 2-D and 60% extra memory requirement in 3-D than the general chain rule approach used in this work. The boundary integral method, or rather its numerical discretization, the boundary element method, have been used so far only to model relatively simple geometrical features. This is to avoid numerical instabilities. A space–frequency domain representation is due to Bouchon et al. (1996), and Sanchez–Sesma and Campillo (1991) also use a boundary integral method to investigate topography effects.

Tessmer and Kosloff (1994) extended their 2-D procedure (Tessmer et al., 1992) for elastic wave modeling with free surface topography to 3-D, transforming the velocity–stress formulation of the medium equations from a curved to a rectangular grid. They used a spectral discretization horizontally and a Chebyshev one vertically in space. At the free topography surface, the stresses and velocities are transformed into local systems in which the vertical coordinate axis is parallel to the normal of the local surface element. The free surface conditions are then implemented by a 'characteristic' treatment of both the velocity and stress components, before they are rotated back to the original system. They show results for simple geometrical configurations, but in principle any arbitrary, smooth topography can be incorporated. The present method is based on their method because it transforms the equations of motion from a curved to a rectangular grid. The free surface boundary conditions, however, are developed here explicitly as an exact, closed set of equations for the 2-D particle velocities. 3-D equivalent formulations to the present work is given in Hestholm (1999).

Day and Minster (1984) replaced the convolutional constitutional relation for a linear viscoelastic solid (Christensen, 1982) with a set of first order partial differential equations through their Padè approximant method. For constant $Q$ over a predetermined frequency range their approximation yields excellent results. Carcione et
al. (1988b), however, were first to represent the convolutional form of the constitutive equations by a first order system exactly. It was done initially for the viscoacoustic case in Carcione et al. (1988a) and further developed in Carcione (1993) for 2-D and 3-D displacement-stress formulations of the viscoelastic wave equations. A pseudospectral method was used here for spatial discretization along with a Chebyshev method in time (Tal-Ezer et al., 1990) to minimize numerical dispersion. Robertsson et al. (1994) extended the method of Carcione et al. (1988b) by exactly representing the convolutional stress-strain relation in the 2-D and 3-D viscoelastic wave equations by a system of first order partial differential equations for the particle velocities and stresses. They used spatial fourth and temporal second order F-D discretizations. Employing these formulations, Blanch et al. (1995), proposed cost-efficient procedures for modeling constant $Q$ over a predefined frequency range. The immediate advantage of using velocity-stress formulations is that we do not differentiate material parameters across discontinuities (Virieux, 1986). Hence we use the velocity-stress formulation of Robertsson et al. (1994) and the Q-modeling procedure of Blanch et al. (1995) as a starting point when incorporating our topography boundary conditions in viscoelastic schemes.

As in Tessmer et al. (1992) and Hestholm and Ruud (1994), a 2-D grid which is curved in the vertical direction is adapted to the surface topography, i.e. the top surface of the grid coincides with the surface topography. This method was originally proposed to adapt grids to interior interfaces using pseudospectral spatial derivatives (Fornberg, 1988b). We transform the velocity-stress formulation of the viscoelastic, isotropic wave equations (Robertsson et al., 1994) from the curved to a rectangular grid in which the numerical computations are done. At the topography surface, the velocity boundary conditions for a free surface are implemented into a local, rotated system at each point of the surface. Each of these systems has its vertical coordinate
direction coinciding with the direction of the normal vector of the surface at the given point. The velocity boundary conditions are then rotated back to the cartesian system before finally they are transformed into curved grid equations in a rectangular grid. Once the boundary conditions are given in this grid, the numerical discretization can be performed.

In the following paragraphs, we give the connection between the convolutional stress–strain relation for a linear viscoelastic solid and its first–order partial differential equation representation (Carcione et al., 1988b) of velocities and stresses (Robertsson et al., 1994). The viscoelastic equivalents to the curved grid elastic wave equations (Hestholm and Ruud, 1994; Tessmer et al., 1992) will be stated. We show the derivation of free surface topography boundary conditions as an exact closed system for the particle velocities and give a description of our numerical solution. Next we show simulation comparisons between viscoelastic and elastic cases using a Ricker point source in media of increasing complexities. Simulations using a sinusoidal topography are shown with a homogeneous medium and a layered medium. We also show results with added complexities from a von Kármán realization of apparent anisotropy, using different parameters in each layer. Synthetics show generation of Rg–waves from peaks and troughs as well as incoherent wave pattern for complex media.

5.3 Viscoelastic Wave Modeling Formulation

The basic hypothesis of viscoelastic theory is that the current value of the stress tensor depends upon the history of the strain tensor. This hypothesis can be described as

$$\sigma_{ij} = G_{ijkl} * \dot{\varepsilon}_{kl} = \dot{G}_{ijkl} * \varepsilon_{kl},$$  \hspace{1cm} (5.1)
(Christensen, 1982) for a linear isotropic material. * denotes time convolution, and it transforms each strain history, $\dot{e}_{ij}(t)$, into the current stress value, $\sigma_{ij}(t)$. The dot denotes time differentiation and $G$ is a fourth order tensor of time called the relaxation function. For a homogeneous material $G$ collapses into two independent functions. This is an assumption which is often used also for inhomogeneous materials. Each of these functions is often assumed to have the form of a standard linear solid

$$G(t) = M_R \left(1 - \sum_{\ell=1}^{L} \left(1 - \frac{\tau_{\sigma\ell}}{\tau_{\sigma\ell}} \right) e^{-t/\tau_{\sigma\ell}} \right) \theta(t)$$  \hspace{1cm} (5.2)

(Carcione et al., 1988b; Blanch et al., 1995). $M_R$ is the relaxation modulus of the medium and $\theta(t)$ is the Heaviside function. The relaxation function $G(t)$ is equivalent to $L$ standard linear solids connected in parallel. Each standard linear solid describes a dashpot and a spring in series in parallel with a spring. $\tau_{\sigma\ell}$ and $\tau_{\epsilon\ell}$ are the stress and strain relaxation times of the $\ell$th mechanism.

Following Robertsson et al. (1994), the velocity–stress formulation of the viscoelastic wave equations can be derived from the constitutive relation (5.1) (Appendix D). In 2-D with one relaxation mechanism, i.e., one standard linear solid (SLS) this can be written as

$$\rho \frac{\partial u}{\partial t} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} + f_x,$$  \hspace{1cm} (5.3)

$$\rho \frac{\partial w}{\partial t} = \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} + f_z,$$  \hspace{1cm} (5.4)

$$\frac{\partial \sigma_{xx}}{\partial t} = \pi \frac{\tau_e}{\tau_\sigma} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) - 2\mu \frac{\tau_e}{\tau_\sigma} \frac{\partial w}{\partial z} + r_{xx},$$  \hspace{1cm} (5.5)

$$\frac{\partial \sigma_{xz}}{\partial t} = \pi \frac{\tau_e}{\tau_\sigma} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) - 2\mu \frac{\tau_e}{\tau_\sigma} \frac{\partial u}{\partial x} + r_{xz},$$  \hspace{1cm} (5.6)

$$\frac{\partial \sigma_{zz}}{\partial t} = \mu \frac{\tau_e}{\tau_\sigma} \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) + r_{zz},$$  \hspace{1cm} (5.7)

$$\frac{\partial r_{xx}}{\partial t} = - \frac{1}{\tau_\sigma} \left[ r_{xx} + \pi \left( \frac{\tau_e}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) - 2\mu \left( \frac{\tau_e}{\tau_\sigma} - 1 \right) \frac{\partial w}{\partial z} \right],$$  \hspace{1cm} (5.8)

$$\frac{\partial r_{zz}}{\partial t} = - \frac{1}{\tau_\sigma} \left[ r_{zz} + \pi \left( \frac{\tau_e}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) - 2\mu \left( \frac{\tau_e}{\tau_\sigma} - 1 \right) \frac{\partial u}{\partial x} \right],$$  \hspace{1cm} (5.9)
\[
\frac{\partial r_{zz}}{\partial t} = -\frac{1}{\tau_\sigma} \left[ r_{zz} + \mu \left( \frac{\tau_\sigma^{S}}{\tau_\sigma} - 1 \right) \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) \right].
\] (5.10)

where \( \rho \) is the density, \( \pi \) is the relaxation modulus for P-waves corresponding to \( \lambda + 2\mu \) (\( \lambda \) and \( \mu \) the Lamé parameters) in the elastic case and \( \mu \) is the relaxation modulus for S-waves corresponding to \( \mu \) in the elastic case. \( \tau_\sigma^{P} \) and \( \tau_\sigma^{S} \) are the strain relaxation times for P- and S-waves respectively, and \( \tau_\sigma \) is the stress relaxation time. The same \( \tau_\sigma \) can be used both for P- and S-waves (Blanch et al., 1995). \( f_x \) and \( f_z \) are the components of the body forces, \( u \) and \( w \) are the particle velocity components and \( \sigma_{xx}, \sigma_{zz} \) and \( \sigma_{xz} \) are the stress components. \( r_{xx}, r_{zz} \) and \( r_{xz} \) are the components of the memory variables. These are the equations governing wave propagation in a linear isotropic viscoelastic medium, and they are the equations of motion, the stress-strain relation and the memory variable equations.

I introduce a linear mapping from a rectangular \((\xi, \eta)\)-system (Figure 5.1) to a curved grid in the \((x, z)\)-system (Figure 5.2), where both grids have positive direction upwards for the vertical coordinate. The 2-D mapping can be written as

\[
x(\xi, \eta) = \xi,
\]
\[
z(\xi, \eta) = \frac{\eta}{\eta_{max}} z_0(\xi),
\]

where \( z_0(\xi) \) is the topography function, and the rectangular \((\xi, \eta)\)-system is bounded by \( \xi = 0, \xi = \xi_{max}, \eta = 0 \) and \( \eta = \eta_{max} \). For the curved grid in the \((x, z)\)-system the extent of stretching is proportional to the distance from the bottom plane of the grid \( (z = 0) \). From equations (5.11)–(5.12) we get, for a differentiable function \( f(x, z) \),

\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial x},
\]
\[
\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \eta} \frac{\partial \xi}{\partial z}.
\]

(5.13) (5.14)
Figure 5.1 Rectangular \((\xi, \eta)\)-system.

Figure 5.2 Curved grid in the \((x, z)\)-system.
Expressions for the partial derivatives, which are needed in the medium equations, are found from equations (5.11)–(5.12) (see Appendix G),

\[
\frac{\partial \xi}{\partial x} = 1, \quad \frac{\partial \xi}{\partial z} = 0, \quad (5.15)
\]

\[
A(\xi, \eta) = \frac{\partial \eta}{\partial x} = -\frac{\eta}{\xi_0(\xi)} \frac{\partial \xi_0(\xi)}{\partial \xi}, \quad (5.16)
\]

\[
B(\xi) = \frac{\partial \eta}{\partial z} = \frac{\eta_{\text{max}}}{\xi_0(\xi)} \quad (5.17)
\]

The velocity–stress formulation of the equations of motion, Hooke’s law and the memory variable equations is given in the curved grid in the \((x, z)\)-system by equations (5.3)–(5.10). Expanding these by the chain rule (Appendix H) as for the elastic cases in Hestholm and Ruud (1994; 1998) and substituting for \(\partial \eta/\partial x\) and \(\partial \eta/\partial z\), we get the medium equations with one relaxation mechanism in the rectangular \((\xi, \eta)\)-system,

\[
\rho \frac{\partial u}{\partial t} = \frac{\partial \sigma_{xx}}{\partial \xi} + A(\xi, \eta) \frac{\partial \sigma_{xx}}{\partial \eta} + B(\xi) \frac{\partial \sigma_{xx}}{\partial \eta} + f_x, \quad (5.18)
\]

\[
\rho \frac{\partial w}{\partial t} = \frac{\partial \sigma_{zz}}{\partial \xi} + A(\xi, \eta) \frac{\partial \sigma_{zz}}{\partial \eta} + B(\xi) \frac{\partial \sigma_{zz}}{\partial \eta} + f_z, \quad (5.19)
\]

\[
\frac{\partial \sigma_{xx}}{\partial t} = \pi \frac{\tau^P}{\tau_\sigma} \left( \frac{\partial u}{\partial \xi} + A(\xi, \eta) \frac{\partial u}{\partial \eta} + B(\xi) \frac{\partial w}{\partial \eta} \right) - 2\mu \frac{\tau^S}{\tau_\sigma} B(\xi) \frac{\partial w}{\partial \eta} + r_{xx}, \quad (5.20)
\]

\[
\frac{\partial \sigma_{zz}}{\partial t} = \pi \frac{\tau^P}{\tau_\sigma} \left( \frac{\partial u}{\partial \xi} + A(\xi, \eta) \frac{\partial u}{\partial \eta} + B(\xi) \frac{\partial w}{\partial \eta} \right) - 2\mu \frac{\tau^S}{\tau_\sigma} \left( \frac{\partial u}{\partial \xi} + A(\xi, \eta) \frac{\partial u}{\partial \eta} + B(\xi) \frac{\partial w}{\partial \eta} \right) + r_{zz}, \quad (5.21)
\]

\[
\frac{\partial \sigma_{xz}}{\partial t} = \mu \frac{\tau^S}{\tau_\sigma} \left( \frac{\partial w}{\partial \xi} + B(\xi) \frac{\partial u}{\partial \eta} + A(\xi, \eta) \frac{\partial w}{\partial \eta} \right) + r_{xz}, \quad (5.22)
\]

\[
\frac{\partial r_{xx}}{\partial t} = -\frac{1}{\tau_\sigma} \left[ r_{xx} + \pi \left( \frac{\tau^P}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial \xi} + A(\xi, \eta) \frac{\partial u}{\partial \eta} + B(\xi) \frac{\partial w}{\partial \eta} \right) \right. \left. - 2\mu \left( \frac{\tau^S}{\tau_\sigma} - 1 \right) B(\xi) \frac{\partial w}{\partial \eta} \right], \quad (5.23)
\]

\[
\frac{\partial r_{zz}}{\partial t} = -\frac{1}{\tau_\sigma} \left[ r_{zz} + \pi \left( \frac{\tau^P}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial \xi} + A(\xi, \eta) \frac{\partial u}{\partial \eta} + B(\xi) \frac{\partial w}{\partial \eta} \right) \right. \left. - 2\mu \left( \frac{\tau^S}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial \xi} + A(\xi, \eta) \frac{\partial u}{\partial \eta} + B(\xi) \frac{\partial w}{\partial \eta} \right) \right], \quad (5.24)
\]
\begin{align}
-2\mu \left( \frac{r_z}{r_\sigma} - 1 \right) \left( \frac{\partial u}{\partial \xi} + A(\xi, \eta) \frac{\partial u}{\partial \eta} \right) \right],
\end{align}
(5.24)
\begin{align}
\frac{\partial r_{xz}}{\partial t} = -\frac{1}{r_\sigma} \left[ r_{xz} + \mu \left( \frac{r_z}{r_\sigma} - 1 \right) \left( \frac{\partial w}{\partial \xi} + B(\xi) \frac{\partial u}{\partial \eta} + A(\xi, \eta) \frac{\partial w}{\partial \eta} \right) \right].
\end{align}
(5.25)

Equations (5.18)-(5.25) are the momentum conservation equations, Hooke's law and the memory variable equations given in the rectangular \((\xi, \eta)\)-system.

\section{5.4 Free Surface Boundary Conditions}

The 2-D free boundary conditions for the velocities at a locally horizontal surface (or in a system where the \(z\)-axis is parallel to the local normal vector of the surface) resulting from the vanishing stress condition can be written
\begin{align}
\frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x},
\end{align}
(5.26)
\begin{align}
\frac{\partial w}{\partial z} = -\frac{\lambda}{\lambda + 2\mu} \left( \frac{\partial u}{\partial x} \right),
\end{align}
(5.27)
with \(x\) the horizontal coordinate and \(z\) the vertical coordinate. We want to apply these conditions to a topography surface. At each surface point, we introduce a local coordinate system \((x', z')\) in which the \(z'\)-axis coincides with the local normal vector direction of the surface. In this local system we impose the conditions (5.26)-(5.27). Once this is done, they have to be rotated back to the cartesian \((x, z)\)-system. This rotation is expressed by \(\bar{\bar{v}} = A^{-1} \bar{v}'\), i.e.
\begin{align}
\bar{v}' = A \bar{v},
\end{align}
(5.28)
where \(\bar{v}\) and \(\bar{v}'\) are the particle velocity vectors in the \((x, z)\)- and the \((x', z')\)-systems respectively. \(A\) is the rotation matrix, which can be given by
\begin{align}
A = \begin{pmatrix}
\cos \phi & \sin \phi \\
-\sin \phi & \cos \phi
\end{pmatrix},
\end{align}
(5.29)
where $\phi$ is the rotation angle between the $x$–axis and the local $x'$–axis in the $(x, z)$–plane. We also have the relation $\tan \phi = \partial z_0(\xi)/\partial \xi$, now expressed in the computational $(\xi, \eta)$–grid.

The calculations of the rotation from the local $(x', z')$–system back to the cartesian $(x, z)$–system can be performed as in Appendix I. Once this is done, a transformation from the cartesian $(x, z)$–system to the computational $(\xi, \eta)$–grid should be performed (Appendix I) to express the boundary conditions for a curved grid in the rectangular, computational $(\xi, \eta)$–grid. This is achieved by using the chain rule in the same way as was done for the medium equations. We arrive at the 2–D boundary conditions for free surface topography given in the computational $(\xi, \eta)$–grid by

$$
\left(1 + \left[\frac{\partial z_0(\xi)}{\partial \xi}\right]^2\right)B(\xi) \frac{\partial u}{\partial \eta} + \frac{\partial z_0(\xi)}{\partial \xi} \left(1 + \left[\frac{\partial z_0(\xi)}{\partial \xi}\right]^2\right)B(\xi) \frac{\partial w}{\partial \eta}
$$

$$
= \frac{2}{\lambda + 2\mu} \frac{\partial z_0(\xi)}{\partial \xi} \left(1 + \left[\frac{\partial z_0(\xi)}{\partial \xi}\right]^2\right)B(\xi) \frac{\partial u}{\partial \eta} + \left(1 - \frac{\lambda}{\lambda + 2\mu}\right) \frac{\partial w}{\partial \xi},
$$

(5.30)

$$
\frac{\partial z_0(\xi)}{\partial \xi} \left(1 + \left[\frac{\partial z_0(\xi)}{\partial \xi}\right]^2\right)B(\xi) \frac{\partial u}{\partial \eta} + \left(1 + \left[\frac{\partial z_0(\xi)}{\partial \xi}\right]^2\right)B(\xi) \frac{\partial w}{\partial \eta}
$$

(5.31)

using definition (5.17).

Equations (5.30)–(5.31) are exact 2–D boundary conditions for an arbitrary, smooth, free surface topography. They result from rotating the velocity free surface conditions from local systems at each point of the surface topography into a cartesian system before transformation into curved grid equations expressed in the computational grid (Appendix I). We note that the conditions coincide with the plane surface conditions (5.26)–(5.27) in the case of a plane surface ($z_0(\xi) = \text{constant}$). The boundary conditions (5.30)–(5.31) are obviously not restricted to the F–D method or any other numerical discretization technique.
5.5 Numerical Discretization

To discretize the viscoelastic wave equations (5.18)–(5.25), high-order, cost-optimized F–D operators (Kindelan et al., 1990) were used. Their method of optimal F–D coefficients from minimization of the total simulation cost under the constraint of a predefined maximum numerical dispersion is based on the work by Holberg (1987). The schemes employ a staggered discretization stencil of the velocity–stress formulation of the viscoelastic wave equations as was employed for the elastodynamic wave equations in Levander (1988) and Virieux (1986). An advantage of using a staggered definition of variables is that we can avoid explicit definition of stresses at the surface topography as it suffices to define the velocities there. In order to express velocities and stresses explicitly at each time step, we generally stagger \( u \) one half grid length in the positive \( \xi \)-direction while \( w \) is staggered one half grid length in the negative \( \eta \)-direction (downwards). The stresses and the memory variables are defined either at the grid points \( (\sigma_{xx}, \sigma_{zz}, r_{xx}, \text{and} r_{zz}) \), or at the midpoint of the "plaquettes" \( (\sigma_{xz} \text{ and } r_{xz}) \), i.e. one half grid length positively in each of the directions of their indices.

The 2-D boundary conditions, equations (5.30)–(5.31) are discretized by second order, staggered F–D operators. Below the free surface, the central, staggered F–D method’s order (Fornberg, 1988a) is gradually increased with depth, via fourth and sixth up to eighth order; the latter is the method used inside the domain and it is dispersion-bounded and cost-optimized (Kindelan et al., 1990).

To extrapolate velocities and stresses in time, the (second order) leap-frog technique is used. The equations for the memory variables \( r_{ij} \) might become stiff for \( \tau_\sigma \) small compared to the time step \( \Delta t \), therefore the Crank–Nicholson stiff solver was used to propagate the memory variables in time (Robertsson et al., 1994). Since the memory variable equations are first order ordinary differential equations, the usually implicit Crank–Nicholson scheme becomes explicit, and only marginally more
expensive than conventional explicit schemes. The Crank–Nicholson scheme is unconditionally stable in time.

Blanch et al. (1995) describe a method for approximating a constant $Q$ over a predetermined frequency interval for an arbitrary number of $L$ standard linear solids. However, both in the present work and in Robertsson et al. (1994), only one standard linear solid (relaxation mechanism) is used, and then a simpler method can be employed. Looking at the curves for $Q$ versus frequency for one relaxation mechanism (Blanch et al., 1995), we see that these curves are symmetric about a minimum $Q$. The method used here is to set the desired $Q$ equal to this minimum value for the central frequency of the source, and calculate $\tau_\sigma$, $\tau_\varepsilon^P$ and $\tau_\varepsilon^S$ accordingly; possibly with a different $Q$ for P– and S–waves. This procedure was also used in Robertsson et al. (1994) and ensures a symmetric behavior of the attenuation and dispersion about the source’s central frequency. The expressions for the relaxation times resulting from this principle are

\[
\begin{align*}
\tau_\sigma &= \frac{1}{\omega} \left( \sqrt{1 + \frac{1}{Q_P^2}} - \frac{1}{Q_P} \right), \\
\tau_\varepsilon^P &= \frac{1}{\omega^2 \tau_\sigma}, \\
\tau_\varepsilon^S &= \frac{1 + \omega \tau_\sigma Q_S}{\omega Q_S - \omega^2 \tau_\sigma}.
\end{align*}
\]  

(5.32)  
(5.33)  
(5.34)  

with $\omega = 2\pi f$ where $f$ is the central frequency, and $Q_P$ and $Q_S$ are $Q$–values for the P– and S–waves respectively.

For the viscoelastic modeling examples we found the following procedure to work best to absorb waves along the grid boundaries: First we calculated $\tau_\sigma$, $\tau_\varepsilon^P$ and $\tau_\varepsilon^S$ from the above formulas with $Q_P = 2.5$ and $Q_S = 2.0$. Then we explicitly set $\tau_\sigma$ to a very high value, i.e. $\tau_\sigma \sim 100$; thereby attaining a much higher order of magnitude of the stress relaxation time compared to the strain relaxation times. For the elastic examples shown we use exponential damping according to Cerjan et al. (1985). The
stresses and velocities are multiplied by exponentially decreasing terms towards the artificial boundaries. The thickness of the absorbing layers was 100 grid points both for the viscoelastic and the elastic examples, which corresponds to 2.5 times the dominant wavelength of the source. In the viscoelastic case, we found a cosine-taper of the relaxation times towards their values along the boundaries to work best.

To find the particle velocity components at the surface topography from the closed system (5.30)–(5.31), we discretize it by second order F–Ds from which we solve for the particle velocities at the surface. We solve for the upper values in the terms of the vertical derivatives, while their lower values as well as the horizontal derivatives are considered known from calculations in the previous time step and are calculated either from the medium equations or the boundary conditions. This procedure leads to an explicit system of equations for the particle velocities at the free surface which has determinant

\[
D = \left[ \left( 1 + \tan^2 \phi \right)^2 + \tan^2 \phi \left( 1 + \tan^2 \phi \right)^2 \right] B^2(\xi) > 0 \forall \phi \quad (5.35)
\]

where \( \phi \) is the topography slope angle with the horizontal and definition (5.17) is used.

We see that the minimum value of \( D \) is \( B^2(\xi) \) and occurs for a plane surface. Therefore the system we solve for the numerical solution of boundary conditions (5.30)–(5.31) is unconditionally stable. Hence the method should be a robust and accurate technique for F–D wave modeling with topography in elastic and viscoelastic media.

Some aspects that are important in the implementation of the boundary conditions (5.30)–(5.31), are that they should be staggered consistently with the staggering of the wave equations inside the medium, i.e. the implementation of each spatial partial derivative should be staggered either in the positive or negative direction according to their staggering in the wave equations. We also found it advantageous to implement the conditions (5.30)–(5.31) first (or last) in the time loop of the codes, thereby
ensuring the use of variables from the same time step. The condition

\[ \text{mean} \{ z_0(\xi), \xi \in [\xi_{\text{min}}, \xi_{\text{max}}] \} \approx z_{\text{max}}, \quad z_{\text{max}} = (nz - 1)dz \]  

(5.36)

should be satisfied, where \( nz \) is the number of grid points vertically and \( dz \) is the vertical grid distance. It means that the average curved grid depth below the real topography should approximate the rectangular, computational grid depth. This is a requirement one would expect both from an accuracy and stability viewpoint. Finally, for numerical stability, it turns out that the total model depth should be large enough that the topographic undulations do not exceed a certain percentage of its size. This percentage, however, is dependent on the topography used.

### 5.6 Scattering from a Sinusoidal Topography

In the present examples we have used a sinusoidal topography with amplitude of 92 m from top to bottom and period of 500 m. It corresponds to a maximum slope of 30° along its relief. A Ricker point source with central frequency of 25 Hz is located 2000 m below the free topography surface in the middle of the domain and excites both a P- and an S-wave. The domain has width of 5000 m and the average depth is also 5000 m. The uniform grid distance of the rectangular numerical grid is 5 m and so it is discretized by \( 1000 \times 1000 \) grid points. The snapshots of Figures 5.3–5.5 display this situation for a homogeneous medium of P-velocity equal to 5000 m/s, S-velocity equal to 2887 m/s and density equal to 2000 kg/m³. Figure 5.3 shows the sinusoidal topography over the homogeneous medium, the shot location and the P- and S-waves propagating out from it at 0.01 and 0.28 seconds after excitation. In this figure as well as all subsequent ones, the vertical particle velocity component \( w \) is displayed to the left and the horizontal particle velocity component \( u \) is displayed to the right. Figures 5.4 and 5.5 then show the corresponding viscoelastic and elastic
cases respectively, after 0.54 and 0.8 seconds. The viscoelastic case has a uniform $Q$ of 250 both for $P$- and $S$-waves, and uses a cosinus–taper towards $Q$-values of 2.5 for $P$-waves and 2.0 for $S$-waves at the grid boundaries in strips of 100 grid points. The elastic case uses strips of the same thickness to multiply all field variables by exponentially decreasing terms toward the boundaries (Cerjan et al., 1985) using an absorption constant of 0.06. We found that by using this method in strips of 50 grid points also in the viscoelastic cases in addition to decreasing $Q$-values toward grid boundaries, we achieved greatly improved results. The simulations were done on an IBM RISC 6000/590 workstation and took approximately 3.2 and 2.5 hours using 81 MB and 57 MB for the viscoelastic and elastic simulations respectively.

We achieved smaller reflection amplitudes from the grid boundaries by the combination procedure used in the viscoelastic case (Figure 5.4) than in the elastic case (Figure 5.5) as is seen from the grid reflections from the incident $P$-wavefront in each case. $P$ to $P$ and $P$ to $S$ wavetrains reflected from the sinusoidal surface are clearly seen in each simulation at $t=0.54$ seconds. These wavetrains can then be followed in each case at $t=0.8$ seconds, with, as expected, the $u$-component containing the most prominent $P$ to $P$ and the $u$-component the most prominent $P$ to $S$ reflections. Additionally, we see scattered reflections from the incident $S$-waves from the sinusoidal topography. Both for the incident $P$- and $S$-waves, some of the resulting waves are seen to propagate along the surface and hence are Rayleigh– (Rg) waves. Apart from slightly clearer appearances of reflected waves in the viscoelastic case, we see very little difference between the snapshots in each case. This is because of individual scaling of snapshots of each simulation, which is used to visualize wavefronts using the present color table. However, amplitudes of the two simulations can be directly compared in the seismograms of Figures 5.6 (viscoelastic simulation) and 5.7 (elastic simulation) because they are scaled according to the same maximum amplitude.
Figure 5.3 Snapshots of the vertical (left series) and horizontal (right series) particle velocity component at $t=0.01$ and $t=0.28$ seconds after a Ricker point source with central frequency of 25 Hz is released 2000 m below a sinusoidal surface topography with maximum dip of 30° and period of 500 m. The domain size is 5000 m x 5000 m and the medium is homogeneous.
Figure 5.4 Snapshots for a viscoelastic simulation of the situation in Figure 5.3 at $t=0.54$ and 0.8 seconds. Homogeneous medium of $v_p = 5000$ m/s and $v_s = 2887$ m/s and $Q = 250$ for both P- and S-waves.
Figure 5.5 Snapshots for the corresponding elastic simulation of Figure 5.4.
For the present exploration type of parameters we can see how much amplitudes are damped by a uniform $Q$–factor of 250. Using $A(x) = A_0 \exp\left\{ -\pi f x / (cQ) \right\}$ (Aki and Richards, 1980), where $f$ is the central frequency, $c$ is the wave speed, $A(x)$ is the wave amplitude at travel distance $x$ and $A_0$ is the wave amplitude at travel distance 0, leads to an attenuation of 75 % of the first direct P–wave arrival at the surface through this layered medium. In both simulations it is interesting to notice that it is the peaks rather than the troughs of the sinusoidal topography that generate most Rg–waves. We see them as the slowest moving wavetrains propagating left and right from each peak. The fastest moving wavetrains are P–wave reflections generated at each trough. These are the wavetrains that are seen clearest in Figure 5.6, where the Rg–wavetrains are dispersed strongly and gradually more so with time. In the $w$–component seismograms (not shown), the Rg–wavetrains from each peak are more prominent and have stronger amplitudes than the P–wave reflections along the surface.

Figures 5.8 and 5.9 are snapshots for the same topography, Ricker source, shot location and times as Figures 5.4 and 5.5 for the viscoelastic and elastic cases respectively, but in these cases there is an added near–surface low-velocity layer of P–velocity 1500 m/s and S–velocity 866 m/s. This model is shown in Figure 5.10. In the viscoelastic case this low–velocity layer has a low $Q$–value of 20 in order to display how sediments will affect the wave pattern. Comparing with the simulations of Figures 5.4 and 5.5, there is a considerable delay of the P–wavefront on its way to the surface after having passed through the sinusoidal interface between the fast and the slow domain. This is seen in Figures 5.8 and 5.9 at $t=0.54$ seconds. The resulting P to S transmission is naturally seen predominantly in the $u$–component (right snapshot series). At $t=0.8$ seconds there are three scattered wavetrains behind the circular S–wavefront from the source. The first of these wavetrains are the P to P
Figure 5.6  Seismogram of the horizontal particle velocity component \( u \) for the viscoelastic simulation of Figure 5.4.
Figure 5.7 Seismogram of the horizontal particle velocity component $u$ for the elastic simulation of Figure 5.5.
Figure 5.8  Snapshots for a viscoelastic simulation of the situation in Figure 5.3 at $t=0.54$ and 0.8 seconds. The layered medium used is shown in Figure 5.10 with a low $Q$-value of 20 in the near-surface low-velocity layer.
Figure 5.9  Snapshots for an elastic simulation of the situation in Figure 5.3 at t=0.54 and 0.8 seconds. The layered medium used is shown in Figure 5.10.
Figure 5.10  Layered medium model used for the simulations in Figures 5.8 and 5.9. The lower medium (dark blue) has P- and S-velocities of 5000 and 2887 m/s respectively, with an added near-surface low-velocity layer (light blue) of P- and S-velocities of 1500 and 866 m/s. The density is 2000 kg/m$^3$ uniformly. On top there is vacuum (green) over a sinusoidal surface topography of maximum dip of 30$^\circ$ and period 500 m. In the viscoelastic simulations, the $Q$-values are 250 and 20 respectively in the lower (dark blue) and upper (light blue) layer both for P- and S-waves.
reflected waves from the sinusoidal interface. The second wavetrain is a combination of P to S and S to P reflections from this interface. Since P to S reflections are more efficient, this wavetrain is clearest seen on the u-component. The third wavetrain are S to S reflections from the interface and therefore again seen most clearly on the u-component.

Inside the low-velocity layer, the wavetrain that has propagated furthest is seen on the w-component and can be identified as the reflected P-waves from the surface topography. Transmitted P to S and S to P waves from the lower to the upper layer have barely reached the surface topography at t=0.8 seconds. In any case, the low-velocity layer works to trap transmitted and reflected waves near the surface, although the waves are expected to be attenuated fast in the viscoelastic simulation of a near-surface sedimentary layer in Figure 5.8. In these snapshots there is no difference in the appearance of the viscoelastic and elastic wavetrains, except possibly for a smoother shape of the scattered waves in the viscoelastic case. At t=0.8 seconds both cases as well as both particle velocity components show Rg-waves propagating away from the middle towards the edges of the model along the surface topography. The exponential damping technique used in the elastic simulation is seen to work very well, but when it is used in combination with our procedure for viscoelastic damping, the results are even better, as they should be. This is seen from Figure 5.4 where the artificial reflections from the incident P-wave on the grid boundaries are hardly visible. P-wave reflections and waves incident on grid boundaries by a small angle are the most difficult for viscoelastic damping to absorb. Even so, the cosinus-taper of decreasing Q-values combined with exponential damping over 50 grid points is seen to work well.

Seismograms for these simulations are shown in Figures 5.11 and 5.12 for the viscoelastic and elastic cases respectively. The slow Rg-waves seen to originate now
Figure 5.11 Seismogram of the horizontal particle velocity component $u$ for the viscoelastic simulation of Figure 5.8.
Figure 5.12  Seismogram of the horizontal particle velocity component $u$ for the elastic simulation of Figure 5.9.
predominantly from troughs rather than peaks in the elastic case (Figure 5.12) are seen to be masked almost completely by dispersed wave arrivals in the viscoelastic case (Figure 5.11). This parameter set of typical exploration type (higher frequencies and closer sampling along with short wavelengths) make for maximal difference between elastic and viscoelastic simulations, as viscoelastic damping and dispersion increase with the number of wavelengths traversed. Even if the first arrivals in each case can be compared and identified, they are more pronounced in the viscoelastic case, and it seems that the complete seismogram is slightly shifted towards smaller arrival times for the low $Q$-value that we use near the surface. This is combined with a clear attenuation of first arrival amplitudes as can be seen because the seismograms are scaled with respect to the same maximum amplitude.

Figures 5.13–5.16 are snapshots and seismograms corresponding to Figures 5.8–5.12 using the same layered model of Figure 5.10 and the same $Q$–values in the viscoelastic case, but with added randomization using a von Kármán realization of apparent anisotropy. The parameters used are von Kármán function orders of 0 and 0.5 respectively for the lower and upper layer (an order of 0 represents a self similar medium and an order of 0.5 a medium with an exponential correlation function (Frankel and Clayton, 1986)), a uniform correlation distance of 12.5 m and a uniform RMS of 4 % for the velocities in each layer. Additionally, we have used an apparent velocity anisotropy of 4:1 in the horizontal versus the vertical direction in the lower layer only.

Comparing the snapshots of Figures 5.13 and 5.14 with the ones in Figures 5.8 and 5.9 we see remainders of wave portions at every location that wavefronts have propagated across. The uniform correlation distance for the velocity perturbations is 2.5 times the sampling distance and the dominant wavelength is 4.8 times that correlation distance, hence the resulting small features in the wavefield. One can argue
Figure 5.13 Snapshots for a viscoelastic simulation of the situation in Figure 5.3 at $t=0.54$ and 0.8 seconds. The layered medium used is shown in Figure 5.10 with a low $Q$-value of 20 in the near-surface low-velocity layer.

Both layers are additionally randomized by a von Kármán realization of apparent anisotropy, using a von Kármán order of 0 and 0.5 in the lower and upper layer respectively, a uniform correlation distance of 12.5 m, an RMS of 4% for the velocities in both layers and an apparent velocity anisotropy of 4:1 in the horizontal versus the vertical direction in the lower layer only.
Figure 5.14  Snapshots for an elastic simulation of the situation in Figure 5.3 at $t=0.54$ and $0.8$ seconds. The layered medium used is shown in Figure 5.10. Both layers are additionally randomized by a von Kármán realization of apparent anisotropy, using a von Kármán order of 0 and 0.5 in the lower and upper layer respectively, a uniform correlation distance of 12.5 m, an RMS of 4 % for the velocities in both layers and an apparent velocity anisotropy of 4:1 in the horizontal versus the vertical direction in the lower layer only.
Figure 5.15  Seismogram of the horizontal particle velocity component $u$ for the viscoelastic simulation of Figure 5.13.
**Figure 5.16** Seismogram of the horizontal particle velocity component $u$ for the elastic simulation of Figure 5.14.
that this sampling rate (2.5) of the medium velocity perturbations is too sparse to sufficiently sample this type of medium, but this is a general problem in wave modeling of random medium realizations. The wavefronts of the random medium are slightly more irregular and dominated by heterogeneous small scale features, in particular the waves scattered off from the sinusoidal near-surface interface. However, it is in the seismograms that the most significant differences occur by including random variations in the layered medium. This is clearly seen by comparing Figures 5.16 and 5.12. The first arrivals are very similar, but the subsequent P-waves are masked and some of them are very hard to identify in the random medium seismogram because of the wavetrain from the small features. Then going to the corresponding viscoelastic cases, we see that the strongly dispersed, but regular arrivals in the layered medium (Figure 5.11) are disrupted by smaller wavelengths scattered from random heterogeneities (Figure 5.15). The first arrivals however, look quite similar, although slightly more incoherent in amplitudes and phases in the random medium case.

Then comparing the elastic and viscoelastic random medium seismograms (Figure 5.16 with Figure 5.15) we observe the generally weaker amplitudes of the viscoelastic case, which is in agreement with previous elastic/viscoelastic simulations. Because the elastic and viscoelastic layered cases (Figures 5.12 and 5.11) show fundamentally different features (the viscoelastic case of Figure 5.11 shows regularly dispersed arrivals that continue long after the first arrivals), it is expected and confirmed from these seismograms that the character of the wavefields are very different. Both seismograms show highly incoherent and irregular waves, but the elastic case contains seemingly more distinct arrivals, especially in the first time following the first arrivals. The layered viscoelastic seismogram (Figure 5.11) shows a highly dispersed wavefield where arrivals seem to continue over a long time. An added random realization would therefore lead to a seismogram with similar features over time containing a more ir-
regular wave pattern (Figure 5.15). The simpler elastic layered case (Figure 5.12) dominated by fewer arrivals and stronger amplitudes leads to a complicated seismogram in the random case (Figure 5.16) where more distinct arrivals can be identified when compared with the viscoelastic case.

\section*{5.7 Discussion}

There is usually one or more low-velocity layers near the surface in landseismic profiling. 2-D investigations with the aim of quantifying the scattering from near-surface irregularities were performed (Hill and Levander, 1984; Levander and Hill, 1985). Irregularities were introduced laterally along the bottom boundary of a surface low-velocity layer. Comparing results of a plane, vertically incident wave upon such a layer and a layer with a plane lower boundary, energy of large amplitude motion is confirmed to get trapped near the free (plane) surface in the case of lateral corrugation of the lower boundary. Irregularities along the lower boundary are shown to cause large amount of energy trapped into Rg modes near the free surface. Similar results were obtained from a shallowly buried thin layer with random velocity perturbations. When such strong energy concentration along the free surface can be caused by irregularities along the bottom boundary of a low-velocity layer along the surface, one would believe that irregularities along the much stronger discontinuities of a free surface would cause even stronger conversions to Rg modes in the presence of such a low-velocity layer. 2-D F-D elastic simulations with topography, random media perturbations of the crust and upper mantle and low-velocity layers near the free surface have been performed (Hestholm and Ruud, 1994; Hestholm et al., 1994; Ruud et al., 1993).

In Hestholm (1999), investigations have been carried out using the present method in 3-D to model teleseismic waves with real surface topography located in the vicin-
ity of the NORESS array in South–Eastern Norway. Scattering effects are modeled here that have not been synthesized before, both because of the lack of an accurate method and because of the capacity of machines previously used. Using the low frequencies and long wavelengths associated with parameters typical for earthquake simulations, there is very small or no visible physical attenuation or dispersion in the viscoelastic modeling. Therefore the predominant advantage of using this code is the vastly improved absorbing boundary conditions even for elastic simulations using a very high $Q$–value. Bannister et al. (1990) observe that some hills in the NORESS area literally radiate Rg–waves at regular intervals for incident teleseismic waves of long durations. Array recordings are just point observations in this regard while the synthetic snapshots in Hestholm and Ruud (1998), for example, reveal nearly symmetric Rg–radiation from such secondary sources. However, this radiation from a multiplicity of secondary sources becomes complex with time, so propagation directionality from a secondary source to a receiver can be very local and will generally weaken with distance. This is confirmed by the fact that Rg–waves rarely propagate further than 60 km in hilly areas typical of the NORESS and GERESS (Bavaria, Germany) arrays while across the plains of northern Fennoscandia Rg–waves from explosions occasionally propagate out to 600 km. By the synthetic examples in the present work we have used the 2–D version of the algorithm to show simulations for parameters typical of exploration surveys. For these parameters the significance of included physical attenuation and dispersion in viscoelastic modeling is noticeable and considerable because of the associated high frequencies and short wavelengths compared to earthquake modeling. In both types of simulations though, the viscoelastic code gives clearly less artificial reflections from grid boundaries.
5.8 Conclusions

Exact boundary conditions for free surface topography have been derived and combined with the velocity–stress formulation of the viscoelastic wave equations in a curved grid. This is seen to give realistic scattering from a sinusoidal topography that is used with a homogeneous medium, with a layered medium and with the layered medium randomized by a von Kármán realization of apparent velocity anisotropy. The results are shown in snapshots and seismograms using a Ricker point source located 2 km below the free surface. The exploration type of parameters that are used in the simulations, i.e. a central frequency of 25 Hz and a grid distance of 5 m, are confirmed to give rise to significant attenuation of wave amplitudes and dispersion of waveforms both when used in conjunction with a $Q$–value of 250 and a $Q$–value of 20 for both P– and S–waves. This is expected and confirmed by comparisons with corresponding elastic simulations. In this regard it is crucial to include viscoelasticity in modeling of exploration surveys, as opposed to modeling of earthquakes and teleseismic explosions, where the associated long wavelengths and low frequencies lead to seismograms that are virtually indistinguishable from the corresponding elastic ones. However, also in these types of simulations, where parameters close to elastic cases are used, it is advantageous to use the viscoelastic code with a very high $Q$ because of its superior absorbing boundary conditions along the grid edges. At longer time laps and/or lower $Q$–values it also gives valuable extra information compared with elastic modeling.

5.9 Acknowledgments

We greatly appreciated discussions with prof. Eystein Husebye (University of Bergen, Norway, Dept. of Solid Earth Physics), dr. Johan Robertsson (Schlumberger Cambridge
Research, Cambridge, England) and dr. Peeter Akerberg (Rice University, Houston, TX, USA, Dept. of Geology and Geophysics). This research was supported by the Norwegian Research Council and by the Norwegian Supercomputer Committee through a grant of computing time. We would like to thank prof. Manik Talwani (Rice University, Dept. of Geology and Geophysics) for his support.
Chapter 6

Discussion and Conclusions

The main goal of this thesis is to improve our understanding of wave propagation, signal amplification and scattering phenomena for local and regional distance travel paths. As a means for this, I derived exact boundary conditions for arbitrary free surface topography in 2-D and 3-D elastic and viscoelastic media. The boundary conditions are expressed as a closed system for the particle velocities and hence are commensurate with velocity–stress versions of the full wave equations. A curved grid represents the physical medium and its upper boundary represents the free surface topography. The wave equations are transformed to curved grid equations given in a rectangular grid and are numerically discretized by an 8th order F–D method on a staggered grid in space. A leap–frog technique and the Crank–Nicholson method are used in time. The latter is an unconditionally stable method which is used in an explicit version in the viscoelastic case to time–propagate the equations for the memory variables, which may occasionally become stiff. The spatial F–D order inside the medium is gradually decreased towards the free surface. I use second order F–Ds to discretize the boundary conditions. These are solved for the upper values in the terms of the vertical derivatives, while their lower values as well as the horizontal derivatives are considered known from calculations in the previous time step. This procedure leads to an explicit, unconditionally stable system of equations for the particle velocities at the free surface. Hence the method is a robust and accurate technique for F–D wave modeling with surface topography in seismic media.

The viscoelastic codes exhibit improved absorbing boundaries along the numerical grid edges compared to the elastic codes, as well as visible attenuation and physical
dispersion in the wavefield inherent in viscoelastic modeling. Even in simulations of
earthquakes and teleseismic explosions, where parameters close to elastic cases are
used, it is advantageous to use the viscoelastic code with a very high $Q$ because of its
superior absorbing boundary conditions along the grid edges. I simulate plane waves
and Ricker point sources in 2–D and plane waves in 3–D to represent earthquakes and
teleseismic explosions incident on real topography. Additionally, a 'geometric' 3–D
equation in the form of a parabolic topography is presented as well as a sinusoidal
topography in 2–D to simulate exploration examples (frequencies $> 10$ Hz). Here
I show additional effects of layers in the medium, with and without randomization
using von Kármán realizations of apparent anisotropy. Real topography on homoge-
neous media is shown to generate significant scattering, particularly in 3–D, where
all out–of–plane effects are included. The real topography used is a $60 \times 60$ km
area which includes the $3$ km aperture NORESS array of seismic receiver stations in
South-Eastern Norway. Synthetic snapshots and seismograms indicate that promi-
nent surface topography can cause back–scattering, mode conversions and complex
wave pattern, which are usually discussed in terms of inter–crust heterogeneities.

Significant P–to–Rg scattering from specific hills is well documented from NORESS
record analysis (Bannister et al., 1990; Gupta et al., 1993; Hedlin et al., 1994). Synthetics show scattered surface waves that appear to radiate out from secondary
point sources which coincide with areas of high topographic gradients. The clearest
scattering of this type originates from the steep valley side east of Bronkeberget about
$10$ km east of NORESS (Bannister et al., 1990). However, the surface wave with the
highest amplitude in the examples presented is a slowly moving Rg–wave propagating
from the south–west corner of the area into the domain. The steepest topography
of the model can be found in this area close to Lake Mjøsa (the Skreikampen hill).
P–to–Rg scattering in this area was observed by Bannister et al. (1990), and can now
be realistically synthesized. Comparing with the same simulation using a vertically incident plane P-wave, the case of a dipping incident plane wave gives rise to scattering of stronger amplitudes, showing the significance of source angles of incidence for earthquake wave amplitude amplification. Bannister et al. (1990) observe that some hills in the NORESS area literally radiate Rg-waves at regular intervals for incident teleseismic waves of long durations. Array recordings are just point observations in this regard while presented synthetic snapshots reveal nearly symmetric Rg-radiation from such secondary sources. However, this radiation from a multiplicity of secondary sources becomes complex with time, so propagation directionality from a secondary source to a receiver can be very local and will also generally weaken with distance. Observationally this is confirmed by the fact that Rg-waves rarely propagate further than 60 km in hilly areas typical of the NORESS and GERESS (Bavaria, Germany) arrays while across the plains of northern Fennoscandia Rg-waves from explosions occasionally propagate out to 600 km.

It is clear from the above that the exact boundary conditions for free surface topography combined with full seismic wave equations in the velocity-stress formulation give realistic scattering and complex wave pattern due to out-of-plane effects from 3-D topography. I demonstrate clear P-to-Rg scattering from prominent topography and I find it gratifying that the strong scattering of this type from the Skreikampen and Bronkeberget hills can be realistically synthesized. Another interesting phenomenon is that abrupt changes of the Rg-wavefield occur over relatively small distances. The most prominent example of this is shown by the Rg-wavefront emanating from the south-west corner of the 60 × 60 km NORESS-area (the Skreikampen hill) into the domain. The phenomenon is due to strong directivity of scattering from some topographic features as well as destructive interference from a multiplicity of secondary sources. Such wavefield characteristics are sometimes observed in record-
ings – at some sensors the Rg–phase is prominent while hardly visible at nearby sensors less than a kilometer away. In other words, small scale crustal features may also contribute to blocking effects as often observed for Rg– and Lg–phases. This is also compatible with damage pattern and macroseismic observations. Many of such wavefield phenomena are out of range for 2–D F–D synthetic experiments; hence my emphasis on research efforts on 3–D synthetic simulation of crustal wavefield propagation.

I feel confident that the basic tool presented in this thesis can handle 3–D site response simulations for realistic models typical of populated, earthquake–prone areas as well as simulating teleseismic explosions. In this regard it may be appropriate to quote a recent site response study by Pitarka and Irikura (1996), namely (p. 989), 'the three–dimensional numerical methods can predict the effects of any irregular structure perfectly. However, such methods are very expensive and not feasible unless a supercomputer or parallel processor is available.' I have modified the code to run on parallel computers, using MPI (Message Passing Interface) for parallelization. MPI is the first attempt of standardization to some degree of parallel programming tools. Besides speeding up computations, the parallelization also allows me to use a larger total computer memory, which is essential for realistic site response simulation. Using the SGI (Cray) Origin 2000 parallel machine located at the University of Bergen, Dept. of Informatics, it is also possible to access all available memory from one processor, i.e. the sequential program version can be used to access all processors for parallel runs. Hence I was able to make an explicit comparison between the performances of a tool–based, explicitly parallelized code and a compiler parallelized code, of which I always found the former to be the faster. It was therefore the version used in all 3–D simulations in this work. Using domain decomposition parallelization via MPI on this machine enables me to run 3–D models with about $10^8$ grid points,
as opposed to about $10^7$ on earlier machines. The sizes of the presented simulations, however, are far below these maximum sizes, but still much larger than would have been possible to simulate using any workstation.
References


— 1982b, Numerical implementation of the boundary element method for two-dimensional transient scalar wave propagation problems; in Applied Mathematical Modelling, 6, 299–306.


Sguazzero, P., Kindelan, M., and Kamel, A., 1989, Dispersion–bounded numerical integration of the elastodynamic equations: ICOSAHOM Conference, Como, Italy,


Appendix A

Partial derivatives in 3–D medium equations

For the medium equations we need \( \partial \eta / \partial x \), \( \partial \eta / \partial y \) and \( \partial \eta / \partial z \). They are found from

\[
\begin{align*}
\frac{\partial x}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial x}{\partial \kappa} \frac{\partial \kappa}{\partial x} + \frac{\partial x}{\partial \eta} \frac{\partial \eta}{\partial x} &= 1, \\
\frac{\partial y}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial y}{\partial \kappa} \frac{\partial \kappa}{\partial y} + \frac{\partial y}{\partial \eta} \frac{\partial \eta}{\partial y} &= 0, \\
\frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial z}{\partial \kappa} \frac{\partial \kappa}{\partial z} + \frac{\partial z}{\partial \eta} \frac{\partial \eta}{\partial z} &= 0,
\end{align*}
\]

(A.1)

(A.2)

(A.3)

\[
\begin{align*}
\frac{\partial x}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial x}{\partial \kappa} \frac{\partial \kappa}{\partial y} + \frac{\partial x}{\partial \eta} \frac{\partial \eta}{\partial y} &= 0, \\
\frac{\partial y}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial y}{\partial \kappa} \frac{\partial \kappa}{\partial y} + \frac{\partial y}{\partial \eta} \frac{\partial \eta}{\partial y} &= 1, \\
\frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \kappa} \frac{\partial \kappa}{\partial y} + \frac{\partial z}{\partial \eta} \frac{\partial \eta}{\partial y} &= 0,
\end{align*}
\]

(A.4)

(A.5)

(A.6)

\[
\begin{align*}
\frac{\partial x}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial x}{\partial \kappa} \frac{\partial \kappa}{\partial z} + \frac{\partial x}{\partial \eta} \frac{\partial \eta}{\partial z} &= 0, \\
\frac{\partial y}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial y}{\partial \kappa} \frac{\partial \kappa}{\partial z} + \frac{\partial y}{\partial \eta} \frac{\partial \eta}{\partial z} &= 0, \\
\frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial z}{\partial \kappa} \frac{\partial \kappa}{\partial z} + \frac{\partial z}{\partial \eta} \frac{\partial \eta}{\partial z} &= 1.
\end{align*}
\]

(A.7)

(A.8)

(A.9)

This leads to

\[
\begin{align*}
\frac{\partial \eta}{\partial x} &= \left( \frac{\partial y}{\partial \xi} \frac{\partial z}{\partial \kappa} - \frac{\partial z}{\partial \xi} \frac{\partial y}{\partial \kappa} \right) / J, \\
\frac{\partial \eta}{\partial y} &= \left( \frac{\partial z}{\partial \xi} \frac{\partial x}{\partial \kappa} - \frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \kappa} \right) / J, \\
\frac{\partial \eta}{\partial z} &= \left( \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \kappa} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \kappa} \right) / J,
\end{align*}
\]

(A.10)

(A.11)

(A.12)
where
\[
J = \frac{\partial x}{\partial \xi} \left( \frac{\partial y \partial z}{\partial \kappa \partial \eta} - \frac{\partial z \partial y}{\partial \kappa \partial \eta} \right) - \frac{\partial x}{\partial \kappa} \left( \frac{\partial y \partial z}{\partial \xi \partial \eta} - \frac{\partial y \partial z}{\partial \eta \partial \xi} \right) + \frac{\partial x}{\partial \eta} \left( \frac{\partial y \partial z}{\partial \xi \partial \kappa} - \frac{\partial z \partial y}{\partial \xi \partial \kappa} \right). \tag{A.13}
\]

With our choice of mapping functions, equations (1.10)-(1.12), (equations (4.18)-(4.20)) we get
\[
\begin{align*}
\frac{\partial x}{\partial \xi} &= 1, & \frac{\partial x}{\partial \kappa} &= 0, & \frac{\partial x}{\partial \eta} &= 0, \\
\frac{\partial y}{\partial \xi} &= 0, & \frac{\partial y}{\partial \kappa} &= 1, & \frac{\partial y}{\partial \eta} &= 0,
\end{align*} \tag{A.14}
\]
\[
\frac{\partial z}{\partial \xi} = \frac{\eta}{\eta_{\text{max}}} \frac{\partial z_0(\xi, \kappa)}{\partial \xi}, \quad \frac{\partial z}{\partial \kappa} = \frac{\eta}{\eta_{\text{max}}} \frac{\partial z_0(\xi, \kappa)}{\partial \kappa}, \quad \frac{\partial z}{\partial \eta} = \frac{z_0(\xi, \kappa)}{\eta_{\text{max}}} \tag{A.16}
\]
and
\[
J = \frac{\partial z}{\partial \eta} = \frac{z_0(\xi, \kappa)}{\eta_{\text{max}}}. \tag{A.17}
\]

From this we get the expressions (1.16)-(1.20) (alternatively (4.24)-(4.28)).
Appendix B

3-D elastic medium equations

Applying the chain rule to equations (1.1)–(1.9) and using the properties of equations (1.13)–(1.15) leads to

\[
\begin{align*}
\frac{\partial u}{\partial t} & = \frac{\partial \sigma_{xx}}{\partial \xi} + \frac{\partial \sigma_{xx}}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial \sigma_{xy}}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial \sigma_{zz}}{\partial \eta} \frac{\partial \eta}{\partial z} + f_z, \\
\frac{\partial v}{\partial t} & = \frac{\partial \sigma_{xy}}{\partial \xi} + \frac{\partial \sigma_{xy}}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial \sigma_{yy}}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial \sigma_{yz}}{\partial \eta} \frac{\partial \eta}{\partial z} + f_y, \\
\frac{\partial w}{\partial t} & = \frac{\partial \sigma_{xz}}{\partial \xi} + \frac{\partial \sigma_{xz}}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial \sigma_{yz}}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial \sigma_{zz}}{\partial \eta} \frac{\partial \eta}{\partial z} + f_z, \\
\frac{\partial \sigma_{xx}}{\partial t} & = \lambda \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) + 2\mu \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} \right), \\
\frac{\partial \sigma_{xy}}{\partial t} & = \lambda \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) + 2\mu \left( \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} \right), \\
\frac{\partial \sigma_{zz}}{\partial t} & = \lambda \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) + 2\mu \left( \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right), \\
\frac{\partial \sigma_{xy}}{\partial t} & = \mu \left( \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} \right), \\
\frac{\partial \sigma_{xz}}{\partial t} & = \mu \left( \frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} \right), \\
\frac{\partial \sigma_{yz}}{\partial t} & = \mu \left( \frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} \right).
\end{align*}
\]

Substituting for \(\partial \eta/\partial x\), \(\partial \eta/\partial y\) and \(\partial \eta/\partial z\) from equations (1.16)–(1.20), we get the medium equations (1.21)–(1.29).
Appendix C

3–D surface topography boundary conditions, first version

Assume a velocity vector \( \vec{v} \) with components \( u, v \) and \( w \) is given in the \( (\xi, \kappa, \eta) \)–coordinate system with basis vectors \( \vec{i}, \vec{j} \) and \( \vec{k} \). This system is then rotated through angles \( (\theta, \phi) \) into a new \( (x', y', z') \)–coordinate system with basis vectors \( \vec{i}', \vec{j}' \) and \( \vec{k}' \). \( \theta \) is the rotation angle between the \( \xi \)–axis and the \( x' \)–axis in the \( (\xi, \eta) \)–plane. \( \phi \) is the rotation angle between the \( \kappa \)–axis and the \( y' \)–axis in the \( (y', z') \)–plane. In this new system the vector \( \vec{v} \) is denoted by \( \vec{v}' \) with components \( u', v' \) and \( w' \). Then we have the relationships

\[
\begin{pmatrix}
\vec{i}' \\
\vec{j}' \\
\vec{k}'
\end{pmatrix} = \mathbf{A} \begin{pmatrix}
\vec{i} \\
\vec{j} \\
\vec{k}
\end{pmatrix},
\]

where \( \mathbf{A} \) is the rotation matrix, given by equation (1.34). Correspondingly,

\[
\begin{pmatrix}
\vec{i} \\
\vec{j} \\
\vec{k}
\end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix}
\vec{i}' \\
\vec{j}' \\
\vec{k}'
\end{pmatrix} = \mathbf{A}^T \begin{pmatrix}
\vec{i}' \\
\vec{j}' \\
\vec{k}'
\end{pmatrix},
\]

where \( \mathbf{A}^{-1} \) and \( \mathbf{A}^T \) are equal and the inverse and transposed of \( \mathbf{A} \) respectively.

Using \( |\vec{n}| = \sqrt{\left( \frac{\partial z_0}{\partial \xi} \right)^2 + \left( \frac{\partial z_0}{\partial \kappa} \right)^2 + 1} \), a unit normal vector to a surface topography element can be written as

\[
\vec{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{1}{|\vec{n}|} \left( -\frac{\partial z_0(\xi, \kappa)}{\partial \xi}, -\frac{\partial z_0(\xi, \kappa)}{\partial \kappa}, 1 \right)^T
\]

\[
= (-\cos \phi \sin \theta, -\sin \phi, \cos \phi \cos \theta)^T
\]

with our choice of rotation angles. From this we get

\[
\frac{\tan \phi}{\cos \theta} = \frac{\partial z_0(\xi, \kappa)}{\partial \kappa}, \text{ i.e. } \tan \phi = \frac{\partial z_0(\xi, \kappa)}{\partial \kappa} \cos \theta,
\]

\( \text{(C.4)} \)
\[
\cos \phi = \cos \left[ \arctan \left( \frac{\partial z_0(\xi, \kappa)}{\partial \kappa} \cos \theta \right) \right]
\]
(C.5)

and
\[
\sin \phi = \sin \left[ \arctan \left( \frac{\partial z_0(\xi, \kappa)}{\partial \kappa} \cos \theta \right) \right].
\]
(C.6)

The coordinate transformation for \( \vec{v} \) is given by \( \vec{v} = A^{-1} \vec{v}' \), or \( \vec{v}' = A \vec{v} \). Componentwise this is
\[
u' = (\cos \theta)u + (\sin \theta)w,
\]
(C.7)
\[
u' = -(\sin \theta \sin \phi)u + (\cos \theta)u + (\cos \theta \sin \phi)w,
\]
(C.8)
\[
w' = -(\sin \theta \cos \phi)u - (\sin \phi)u + (\cos \theta \cos \phi)w.
\]
(C.9)

Applying the chain rule to a differentiable function \( f \), we then get
\[
\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial x'} + \frac{\partial f}{\partial \kappa} \frac{\partial \kappa}{\partial x'} = \frac{\partial f}{\partial \xi} \cos \theta + \frac{\partial f}{\partial \eta} \sin \theta,
\]
(C.10)
\[
\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial y'} + \frac{\partial f}{\partial \kappa} \frac{\partial \kappa}{\partial y'} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial y'}
\]
\[
= \frac{\partial f}{\partial \xi} (-\sin \theta \sin \phi) + \frac{\partial f}{\partial \kappa} \cos \phi + \frac{\partial f}{\partial \eta} \cos \theta \sin \phi,
\]
(C.11)
\[
\frac{\partial f}{\partial z'} = \frac{\partial f}{\partial \xi} \frac{\partial \xi}{\partial z'} + \frac{\partial f}{\partial \kappa} \frac{\partial \kappa}{\partial z'} + \frac{\partial f}{\partial \eta} \frac{\partial \eta}{\partial z'}
\]
\[
= \frac{\partial f}{\partial \xi} (-\sin \theta \cos \phi) + \frac{\partial f}{\partial \kappa} (-\sin \phi) + \frac{\partial f}{\partial \eta} \cos \theta \cos \phi.
\]
(C.12)

The last equalities are obtained from equation (C.2). The free boundary conditions (1.30)–(1.32) for the velocities have to be enforced in the local \((x', y', z')\)-system, where the \(z'\)-axis is normal to the surface at the local point, i.e.
\[
\frac{\partial u'}{\partial z'} = -\frac{\partial w'}{\partial x'}, \quad \frac{\partial w'}{\partial z'} = -\frac{\partial \nu'}{\partial y'}, \quad \frac{\partial w'}{\partial z'} = -\frac{\lambda}{\lambda + 2\mu} \left( \frac{\partial u'}{\partial x'} + \frac{\partial \nu'}{\partial y'} \right),
\]
(C.13–15)
If the chain rule is applied as above, we get

\[
\frac{\partial u'}{\partial \xi}(-\sin \theta \cos \phi) + \frac{\partial u'}{\partial \kappa}(-\sin \phi) + \frac{\partial u'}{\partial \eta} \cos \theta \cos \phi
\]

\[
= -\frac{\partial w'}{\partial \xi} \cos \theta - \frac{\partial w'}{\partial \eta} \sin \theta,
\]

(C.16)

\[
\frac{\partial v'}{\partial \xi}(-\sin \theta \cos \phi) + \frac{\partial v'}{\partial \kappa}(-\sin \phi) + \frac{\partial v'}{\partial \eta} \cos \theta \cos \phi
\]

\[
= -\frac{\partial w'}{\partial \xi}(-\sin \theta \sin \phi) - \frac{\partial w'}{\partial \kappa} \cos \phi - \frac{\partial w'}{\partial \eta} \cos \theta \sin \phi,
\]

(C.17)

\[
\frac{\partial w'}{\partial \xi}(-\sin \theta \cos \phi) + \frac{\partial w'}{\partial \kappa}(-\sin \phi) + \frac{\partial w'}{\partial \eta} \cos \theta \cos \phi
\]

\[
= -\frac{\lambda}{\lambda + 2\mu} \left( \frac{\partial w'}{\partial \xi} \cos \theta + \frac{\partial w'}{\partial \eta} \sin \theta + \frac{\partial v'}{\partial \xi}(-\sin \theta \sin \phi)
\]

\[
+ \frac{\partial v'}{\partial \kappa} \cos \phi + \frac{\partial v'}{\partial \eta} \cos \theta \sin \phi \right).
\]

(C.18)

Now we apply the above expressions for \( u', v' \) and \( w' \) that were obtained from the rotation. This leads to

\[
\begin{bmatrix}
\cos \theta \frac{\partial u}{\partial \xi} + \sin \theta \frac{\partial w}{\partial \xi} \\
\cos \theta \frac{\partial u}{\partial \kappa} + \sin \theta \frac{\partial w}{\partial \kappa} \\
\cos \theta \frac{\partial u}{\partial \eta} + \sin \theta \frac{\partial w}{\partial \eta}
\end{bmatrix} (-\sin \theta \cos \phi)
\]

\[
+ \begin{bmatrix}
\sin \theta \cos \phi \frac{\partial u}{\partial \xi} + \sin \phi \frac{\partial v}{\partial \xi} - \cos \theta \cos \phi \frac{\partial w}{\partial \xi} \\
\sin \theta \cos \phi \frac{\partial u}{\partial \kappa} + \sin \phi \frac{\partial v}{\partial \kappa} - \cos \theta \cos \phi \frac{\partial w}{\partial \kappa} \\
\sin \theta \cos \phi \frac{\partial u}{\partial \eta} + \sin \phi \frac{\partial v}{\partial \eta} - \cos \theta \cos \phi \frac{\partial w}{\partial \eta}
\end{bmatrix} \cos \theta
\]

\[
+ \begin{bmatrix}
-\sin \theta \sin \phi \frac{\partial u}{\partial \xi} + \cos \phi \frac{\partial v}{\partial \xi} + \cos \theta \sin \phi \frac{\partial w}{\partial \xi} \\
-\sin \theta \sin \phi \frac{\partial u}{\partial \kappa} + \cos \phi \frac{\partial v}{\partial \kappa} + \cos \theta \sin \phi \frac{\partial w}{\partial \kappa} \\
-\sin \theta \sin \phi \frac{\partial u}{\partial \eta} + \cos \phi \frac{\partial v}{\partial \eta} + \cos \theta \sin \phi \frac{\partial w}{\partial \eta}
\end{bmatrix} (-\sin \theta \cos \phi)
\]

\[
+ \begin{bmatrix}
-\sin \theta \sin \phi \frac{\partial u}{\partial \xi} + \cos \phi \frac{\partial v}{\partial \xi} + \cos \theta \sin \phi \frac{\partial w}{\partial \xi} \\
-\sin \theta \sin \phi \frac{\partial u}{\partial \kappa} + \cos \phi \frac{\partial v}{\partial \kappa} + \cos \theta \sin \phi \frac{\partial w}{\partial \kappa} \\
-\sin \theta \sin \phi \frac{\partial u}{\partial \eta} + \cos \phi \frac{\partial v}{\partial \eta} + \cos \theta \sin \phi \frac{\partial w}{\partial \eta}
\end{bmatrix} (-\sin \phi)
\]

\[
+ \begin{bmatrix}
-\sin \theta \sin \phi \frac{\partial u}{\partial \xi} + \cos \phi \frac{\partial v}{\partial \xi} + \cos \theta \sin \phi \frac{\partial w}{\partial \xi} \\
-\sin \theta \sin \phi \frac{\partial u}{\partial \kappa} + \cos \phi \frac{\partial v}{\partial \kappa} + \cos \theta \sin \phi \frac{\partial w}{\partial \kappa} \\
-\sin \theta \sin \phi \frac{\partial u}{\partial \eta} + \cos \phi \frac{\partial v}{\partial \eta} + \cos \theta \sin \phi \frac{\partial w}{\partial \eta}
\end{bmatrix} \cos \theta \cos \phi
\]

\[
= \begin{bmatrix}
\sin \theta \cos \phi \frac{\partial u}{\partial \xi} + \sin \phi \frac{\partial v}{\partial \xi} - \cos \theta \cos \phi \frac{\partial w}{\partial \xi} \\
\sin \theta \cos \phi \frac{\partial u}{\partial \kappa} + \sin \phi \frac{\partial v}{\partial \kappa} - \cos \theta \cos \phi \frac{\partial w}{\partial \kappa}
\end{bmatrix} (-\sin \theta \cos \phi)
\]

(C.19)
\begin{align*}
&+ \left[ \sin \theta \cos \phi \frac{\partial u}{\partial \kappa} + \sin \phi \frac{\partial v}{\partial \kappa} - \cos \theta \cos \phi \frac{\partial w}{\partial \kappa} \right] \cos \phi \\
&+ \left[ \sin \theta \cos \phi \frac{\partial u}{\partial \eta} + \sin \phi \frac{\partial v}{\partial \eta} - \cos \theta \cos \phi \frac{\partial w}{\partial \eta} \right] \cos \theta \sin \phi, \quad (C.20) \\
&+ \left[ -\sin \theta \cos \phi \frac{\partial u}{\partial \xi} - \sin \phi \frac{\partial v}{\partial \xi} + \cos \theta \cos \phi \frac{\partial w}{\partial \xi} \right] (-\sin \theta \cos \phi) \\
&+ \left[ -\sin \theta \cos \phi \frac{\partial u}{\partial \kappa} - \sin \phi \frac{\partial v}{\partial \kappa} + \cos \theta \cos \phi \frac{\partial w}{\partial \kappa} \right] (-\sin \phi) \\
&+ \left[ -\sin \theta \cos \phi \frac{\partial u}{\partial \eta} - \sin \phi \frac{\partial v}{\partial \eta} + \cos \theta \cos \phi \frac{\partial w}{\partial \eta} \right] \cos \theta \cos \phi \\
&= -\frac{\lambda}{\lambda + 2\mu} \left\{ \left[ \cos \phi \frac{\partial u}{\partial \xi} + \sin \theta \frac{\partial w}{\partial \xi} \right] \cos \theta \\
&+ \left[ \cos \theta \frac{\partial u}{\partial \eta} + \sin \theta \frac{\partial w}{\partial \eta} \right] \sin \theta \\
&+ \left[ -\sin \theta \cos \phi \frac{\partial u}{\partial \xi} + \cos \phi \frac{\partial v}{\partial \xi} + \cos \theta \sin \phi \frac{\partial w}{\partial \xi} \right] (-\sin \theta \cos \phi) \\
&+ \left[ -\sin \theta \cos \phi \frac{\partial u}{\partial \kappa} + \cos \phi \frac{\partial v}{\partial \kappa} + \cos \theta \sin \phi \frac{\partial w}{\partial \kappa} \right] \cos \phi \\
&+ \left[ -\sin \theta \cos \phi \frac{\partial u}{\partial \eta} + \cos \phi \frac{\partial v}{\partial \eta} + \cos \theta \sin \phi \frac{\partial w}{\partial \eta} \right] \cos \theta \sin \phi \right\}. \quad (C.21)
\end{align*}

We divide the first of these equations by \( \cos^2 \theta \cos \phi \), the second by \( \cos \theta \cos^2 \phi \) and the third by \( \cos^2 \theta \cos^2 \phi \). These divisions assume \( \theta \neq \pi/2 \) and \( \phi \neq \pi/2 \). This means that the topography cannot have vertical sections along the planes of rotation, i.e. the topography function must be single-valued. This is a reasonable assumption given that the topography function \( z_0(\xi, \kappa) \) is assumed to be smooth. After restructuring, the three equations become

\begin{align*}
&\left(1 - \tan^2 \theta\right) \frac{\partial u}{\partial \eta} - \tan \theta \frac{\partial v}{\partial \eta} + 2 \tan \theta \frac{\partial w}{\partial \eta} \\
&= 2 \tan \theta \frac{\partial u}{\partial \xi} + \frac{\tan \phi \frac{\partial v}{\partial \xi} + (\tan^2 \theta - 1) \frac{\partial w}{\partial \xi} + \frac{\tan \phi \frac{\partial u}{\partial \kappa} + \tan \theta \tan \phi \frac{\partial w}{\partial \kappa}}{\cos \theta} + \frac{\tan \theta \tan \phi \frac{\partial w}{\partial \kappa}}{\cos \theta}, \quad (C.22) \\
&-2 \sin \theta \tan \phi \frac{\partial u}{\partial \eta} + (1 - \tan^2 \phi) \frac{\partial v}{\partial \eta} + 2 \cos \theta \tan \phi \frac{\partial w}{\partial \eta} \\
&= -2 \sin \theta \tan \theta \tan \phi \frac{\partial u}{\partial \xi} + \tan \theta \left(1 - \tan^2 \phi\right) \frac{\partial v}{\partial \xi} + 2 \sin \theta \tan \phi \frac{\partial w}{\partial \xi}
\end{align*}
\[ + \tan \theta \left(1 - \tan^2 \phi\right) \frac{\partial u}{\partial \kappa} + 2 \frac{\tan \phi}{\cos \theta} \frac{\partial v}{\partial \kappa} + \left(\tan^2 \phi - 1\right) \frac{\partial w}{\partial \kappa}, \quad (C.23) \]

\[ \tan \theta \left\{ \frac{\lambda}{\lambda + 2\mu} \left(\frac{1}{\cos^2 \phi} - \tan^2 \phi\right) - 1 \right\} \frac{\partial u}{\partial \eta} + \frac{\tan \phi}{\cos \theta} \left\{ \frac{\lambda}{\lambda + 2\mu} - 1 \right\} \frac{\partial v}{\partial \eta} \]

\[ + \left\{ \frac{\lambda}{\lambda + 2\mu} \left(\frac{\tan^2 \theta}{\cos^2 \phi} + \tan^2 \phi\right) + 1 \right\} \frac{\partial w}{\partial \eta} \]

\[ = \left\{ \frac{\lambda}{\lambda + 2\mu} \left(\frac{1}{\cos^2 \phi} + \tan^2 \theta \tan^2 \phi\right) - \tan^2 \theta \right\} \frac{\partial u}{\partial \xi} + \frac{\tan \theta}{\cos \theta} \tan \phi \left\{ \frac{\lambda}{\lambda + 2\mu} - 1 \right\} \frac{\partial v}{\partial \xi} \]

\[ + \tan \theta \left\{ \frac{\lambda}{\lambda + 2\mu} \left(\tan^2 \phi - \frac{1}{\cos^2 \phi}\right) + 1 \right\} \frac{\partial w}{\partial \xi} + \frac{\tan \theta}{\cos \theta} \tan \phi \left\{ \frac{\lambda}{\lambda + 2\mu} - 1 \right\} \frac{\partial u}{\partial \kappa} \]

\[ - \frac{\lambda}{\cos^2 \theta} \left\{ \frac{\lambda}{\lambda + 2\mu} + \tan^2 \phi \right\} \frac{\partial v}{\partial \kappa} + \frac{\tan \phi}{\cos \theta} \left\{ 1 - \frac{\lambda}{\lambda + 2\mu} \right\} \frac{\partial w}{\partial \kappa}. \quad (C.24) \]

Using the relations \( \tan \theta = \partial z_0(\xi, \kappa)/\partial \xi \) and \( \tan \phi = \partial z_0(\xi, \kappa)/\partial \kappa \cos \theta \) together with the definitions (1.38)--(1.44) leads to the closed set of velocity boundary conditions at a free surface topography (1.35)--(1.37).
Appendix D

First order system of partial differential equations from the anelastic constitutive relation

The constitutive relation for a linear viscoelastic isotropic homogeneous medium is (Christensen, 1982)

$$\sigma_{ij} = \dot{\Lambda} * \delta_{ij} \varepsilon_{kk} + 2\dot{M} * \varepsilon_{ij}, \quad (D.1)$$

where Einstein's summation convention is used. $\Lambda$ and $2M$ are the two independent functions resulting from the fourth-order tensor $G_{ijkl}$ in equation (4.1) for a homogeneous medium. We define

$$\Pi = \Lambda + 2M \quad (D.2)$$

and use the standard linear solid model for $\Pi$ and $M$, i.e.

$$\Pi = \pi \left(1 - \sum_{\ell=1}^{L} \left(1 - \frac{\tau_{\ell\ell}^{P}}{\tau_{\sigma\ell}}\right) e^{-t/\tau_{\sigma\ell}}\right) \theta(t), \quad (D.3)$$

$$M = \mu \left(1 - \sum_{\ell=1}^{L} \left(1 - \frac{\tau_{\ell\ell}^{S}}{\tau_{\sigma\ell}}\right) e^{-t/\tau_{\sigma\ell}}\right) \theta(t), \quad (D.4)$$

where $\pi = \lambda + 2\mu$ and $\lambda$ and $\mu$ are the elastic Lamé parameters. $\Pi$ and $M$ depend respectively on the strain relaxation time for P-waves, $\tau_{\ell\ell}^{P}$, and the strain relaxation time for S-waves, $\tau_{\ell\ell}^{S}$. This allows for independent definition of $Q$ for P- and S-waves. Using the time derivative of the definition of strain,

$$\dot{\varepsilon}_{ij} = \frac{1}{2} (\partial_{i}v_{j} + \partial_{j}v_{i}), \quad (D.5)$$

we get from the constitutive relation,

$$\ddot{\sigma}_{ii} = (\dddot{\Pi} - 2\ddot{M}) * \partial_{k}v_{k} + 2\dot{M} * \partial_{i}v_{i}, \quad (D.6)$$

$$\ddot{\sigma}_{ij} = \dot{M} * (\partial_{i}v_{j} + \partial_{j}v_{i}), \quad i \neq j. \quad (D.7)$$
Now we insert the standard linear solid expressions for $\Pi$ and $M$ and perform the time differentiation for each of their factors. Then we obtain

\begin{align}
\dot{\sigma}_{ii} &= \left\{ \pi \left[ 1 - \sum_{\ell=1}^{L} \left( 1 - \frac{\tau_{el}^P}{\tau_{sol}} \right) \right] - 2\mu \left[ 1 - \sum_{\ell=1}^{L} \left( 1 - \frac{\tau_{el}^S}{\tau_{sol}} \right) \right] \right\} \partial_k v_k \\
&\quad + 2\mu \left\{ 1 - \sum_{\ell=1}^{L} \left( 1 - \frac{\tau_{el}^S}{\tau_{sol}} \right) \right\} \partial_i v_i + \sum_{\ell=1}^{L} r_{i\ell i}, \tag{D.8}
\end{align}

\begin{align}
\dot{\sigma}_{ij} &= \mu \left\{ 1 - \sum_{\ell=1}^{L} \left( 1 - \frac{\tau_{el}^S}{\tau_{sol}} \right) \right\} \left( \partial_i v_j + \partial_j v_i \right) + \sum_{\ell=1}^{L} r_{ij \ell}, \quad i \neq j, \tag{D.9}
\end{align}

with

\begin{align}
r_{i\ell i} &= \left\{ \pi \frac{1}{\tau_{sol}} \left( 1 - \frac{\tau_{el}^P}{\tau_{sol}} \right) - 2\mu \frac{1}{\tau_{sol}} \left( 1 - \frac{\tau_{el}^S}{\tau_{sol}} \right) \right\} e^{-t/\tau_{sol}} \theta(t) * \partial_k v_k \\
&\quad + 2\mu \frac{1}{\tau_{sol}} \left( 1 - \frac{\tau_{el}^S}{\tau_{sol}} \right) e^{-t/\tau_{sol}} \theta(t) * \partial_i v_i, \quad 1 \leq \ell \leq L, \tag{D.10}
\end{align}

\begin{align}
r_{ij \ell} &= \mu \frac{1}{\tau_{sol}} \left( 1 - \frac{\tau_{el}^S}{\tau_{sol}} \right) e^{-t/\tau_{sol}} \theta(t) * \left( \partial_i v_j + \partial_j v_i \right), \quad i \neq j, \quad 1 \leq \ell \leq L, \tag{D.11}
\end{align}

being the memory variables. We obtain the first order system for them by evaluating the time derivatives the same way as was done in the previous step,

\begin{align}
\dot{r}_{i\ell i} &= -\frac{1}{\tau_{sol}} \left\{ r_{i\ell i} + \left( \pi \left( \frac{\tau_{el}^P}{\tau_{sol}} - 1 \right) - 2\mu \left( \frac{\tau_{el}^S}{\tau_{sol}} - 1 \right) \right) \partial_k v_k + 2\mu \left( \frac{\tau_{el}^S}{\tau_{sol}} - 1 \right) \partial_i v_i \right\}, \\
&\quad 1 \leq \ell \leq L, \tag{D.12}
\end{align}

\begin{align}
\dot{r}_{ij \ell} &= -\frac{1}{\tau_{sol}} \left\{ r_{ij \ell} + \mu \left( \frac{\tau_{el}^S}{\tau_{sol}} - 1 \right) \left( \partial_i v_j + \partial_j v_i \right) \right\}, \quad i \neq j, \quad 1 \leq \ell \leq L. \tag{D.13}
\end{align}

The momentum conservation equation completes the viscoelastic wave equations,

\begin{align}
\rho \ddot{u}_i = \partial_j \sigma_{ij} + f_i, \tag{D.14}
\end{align}

$f_i$ being the volume force and $\rho$ the density. The 3–D case for $L = 1$ (one standard linear solid) yields equations (4.3)–(4.17). The 2–D case for $L = 1$ yields equations (5.3)–(5.10).
Appendix E

3–D viscoelastic medium equations

Applying the chain rule to equations (4.3)–(4.17) and using the properties of equations (4.21)–(4.23) leads to

\[
\frac{\partial u}{\partial t} = \rho \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial \sigma_{xx}}{\partial \eta} \frac{\partial u}{\partial x} + \frac{\partial \sigma_{xy}}{\partial \eta} \frac{\partial u}{\partial y} + \frac{\partial \sigma_{xz}}{\partial \eta} \frac{\partial u}{\partial z} + f_x, \quad (E.1)
\]

\[
\frac{\partial v}{\partial t} = \rho \frac{\partial^2 v}{\partial \xi^2} + \frac{\partial \sigma_{xy}}{\partial \eta} \frac{\partial v}{\partial x} + \frac{\partial \sigma_{yy}}{\partial \eta} \frac{\partial v}{\partial y} + \frac{\partial \sigma_{yz}}{\partial \eta} \frac{\partial v}{\partial z} + f_y, \quad (E.2)
\]

\[
\frac{\partial w}{\partial t} = \rho \frac{\partial^2 w}{\partial \xi^2} + \frac{\partial \sigma_{xz}}{\partial \eta} \frac{\partial w}{\partial x} + \frac{\partial \sigma_{yz}}{\partial \eta} \frac{\partial w}{\partial y} + \frac{\partial \sigma_{zz}}{\partial \eta} \frac{\partial w}{\partial z} + f_z, \quad (E.3)
\]

\[
\frac{\partial \sigma_{xx}}{\partial t} = \pi \frac{\tau_e P}{\tau_g} \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right)
- 2 \mu \frac{\tau_s P}{\tau_g} \left( \frac{\partial u}{\partial \kappa} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) + r_{xx}, \quad (E.4)
\]

\[
\frac{\partial \sigma_{yy}}{\partial t} = \pi \frac{\tau_e P}{\tau_g} \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right)
- 2 \mu \frac{\tau_s P}{\tau_g} \left( \frac{\partial u}{\partial \kappa} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) + r_{yy}, \quad (E.5)
\]

\[
\frac{\partial \sigma_{zz}}{\partial t} = \pi \frac{\tau_e P}{\tau_g} \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right)
- 2 \mu \frac{\tau_s P}{\tau_g} \left( \frac{\partial u}{\partial \kappa} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) + r_{zz}, \quad (E.6)
\]

\[
\frac{\partial \sigma_{xy}}{\partial t} = \mu \frac{\tau_e P}{\tau_g} \left( \frac{\partial u}{\partial \kappa} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \kappa} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial z} \right) + r_{xy}, \quad (E.7)
\]

\[
\frac{\partial \sigma_{xz}}{\partial t} = \mu \frac{\tau_e P}{\tau_g} \left( \frac{\partial u}{\partial \kappa} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \kappa} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) + r_{xz}, \quad (E.8)
\]

\[
\frac{\partial \sigma_{yz}}{\partial t} = \mu \frac{\tau_e P}{\tau_g} \left( \frac{\partial v}{\partial \kappa} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \kappa} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) + r_{yz}, \quad (E.9)
\]

\[
\frac{\partial r_{xx}}{\partial t} = -\frac{1}{\tau_g} \left( r_{xx} + \tau_e \left( \frac{\tau_e P}{\tau_g} - 1 \right) \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) \right)
\]
\[
\frac{\partial r_{yy}}{\partial t} = -\frac{1}{\tau_\sigma} \left\{ r_{yy} + \pi \left( \frac{r_{\pi}}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) \right\},
\]

\[
\frac{\partial r_{zz}}{\partial t} = -\frac{1}{\tau_\sigma} \left\{ r_{zz} + \pi \left( \frac{r_{\pi}}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) \right\},
\]

\[
\frac{\partial r_{xy}}{\partial t} = -\frac{1}{\tau_\sigma} \left\{ r_{xy} + \mu \left( \frac{r_{\mu}}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) \right\},
\]

\[
\frac{\partial r_{zx}}{\partial t} = -\frac{1}{\tau_\sigma} \left\{ r_{zx} + \mu \left( \frac{r_{\mu}}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) \right\},
\]

\[
\frac{\partial r_{yz}}{\partial t} = -\frac{1}{\tau_\sigma} \left\{ r_{yz} + \mu \left( \frac{r_{\mu}}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta} \frac{\partial \eta}{\partial y} + \frac{\partial w}{\partial \xi} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) \right\},
\]

Substituting for \( \partial \eta/\partial x \), \( \partial \eta/\partial y \) and \( \partial \eta/\partial z \) from equations (4.24)-(4.28), we get the medium equations (4.29)-(4.43).
Appendix F

3-D surface topography boundary conditions

Assume a velocity vector \( \vec{v} \) with components \( u, v \) and \( w \) is given in a cartesian \((x, y, z)\)-coordinate system with basis vectors \( \vec{i}, \vec{j} \) and \( \vec{k} \). This system is then rotated through angles \((\theta, \phi)\) into a new \((x', y', z')\)-coordinate system with basis vectors \( \vec{i}', \vec{j}' \) and \( \vec{k}' \). \( \theta \) is the rotation angle between the \(x\)-axis and the \(x'\)-axis in the \((x, z)\)-plane. \( \phi \) is the rotation angle between the \(y\)-axis and the \(y'\)-axis in the \((y', z')\)-plane. In this new system the vector \( \vec{v} \) is denoted by \( \vec{v}' \) with components \( u', v' \) and \( w' \). Then we have the relationships

\[
\begin{pmatrix}
\vec{i}' \\
\vec{j}' \\
\vec{k}'
\end{pmatrix} = A
\begin{pmatrix}
\vec{i} \\
\vec{j} \\
\vec{k}
\end{pmatrix},
\]

where \( A \) is the rotation matrix, given by equation (4.48). Correspondingly,

\[
\begin{pmatrix}
\vec{i} \\
\vec{j} \\
\vec{k}
\end{pmatrix} = A^{-1}
\begin{pmatrix}
\vec{i}' \\
\vec{j}' \\
\vec{k}'
\end{pmatrix} = A^{T}
\begin{pmatrix}
\vec{j}' \\
\vec{k}'
\end{pmatrix},
\]

where \( A^{-1} \) and \( A^{T} \) are equal (since \( A \) is orthogonal) and the inverse and transposed of \( A \) respectively.

Using the computational grid coordinates \((\xi, \kappa, \eta)\) instead of the cartesian \((x, y, z)\)-system and \( |\vec{n}| = \sqrt{\left(\frac{\partial z_0}{\partial \xi}\right)^2 + \left(\frac{\partial z_0}{\partial \kappa}\right)^2 + 1} \), a unit normal vector to a surface topography element can be written as

\[
\vec{\bar{n}} = \frac{\vec{n}}{|\vec{n}|} = \frac{1}{|\vec{n}|}
\begin{pmatrix}
-\frac{\partial z_0(\xi, \kappa)}{\partial \xi}, -\frac{\partial z_0(\xi, \kappa)}{\partial \kappa}, 1
\end{pmatrix}^{T}
\]

\[
= (-\cos \phi \sin \theta, -\sin \phi, \cos \phi \cos \theta)^{T}
\]

\[(F.3)\]
with our choice of rotation angles. From this we get
\[
\frac{\tan \phi}{\cos \theta} = \frac{\partial z_0(\xi, \kappa)}{\partial \kappa}, \quad \text{i.e.} \quad \tan \phi = \frac{\partial z_0(\xi, \kappa)}{\partial \kappa} \cos \theta, \quad \text{(F.4)}
\]
\[
\cos \phi = \cos \left[ \arctan \left( \frac{\partial z_0(\xi, \kappa)}{\partial \kappa} \cos \theta \right) \right], \quad \text{(F.5)}
\]
\[
\text{and} \quad \sin \phi = \sin \left[ \arctan \left( \frac{\partial z_0(\xi, \kappa)}{\partial \kappa} \cos \theta \right) \right]. \quad \text{(F.6)}
\]

The coordinate transformation for \( \vec{v} \) is given by \( \vec{v} = A^{-1} \vec{v}' \), or \( \vec{v}' = A \vec{v} \). Componentwise this is
\[
u' = (\cos \theta) u + (\sin \theta) w, \quad \text{(F.7)}
\]
\[
v' = -(\sin \theta \sin \phi) u + (\cos \phi) v + (\cos \theta \sin \phi) w, \quad \text{(F.8)}
\]
\[
w' = -(\sin \theta \cos \phi) u - (\sin \phi) v + (\cos \theta \cos \phi) w. \quad \text{(F.9)}
\]

Applying the chain rule to a differentiable function \( f \), we get
\[
\frac{\partial f}{\partial x'} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial x'} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x'} = \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial z} \sin \theta, \quad \text{(F.10)}
\]
\[
\frac{\partial f}{\partial y'} = \frac{\partial f}{\partial x} \frac{\partial y}{\partial y'} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial y'} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial y'} + \frac{\partial f}{\partial z'} \frac{\partial z}{\partial y'} \]
\[
= \frac{\partial f}{\partial x} (-\sin \theta \sin \phi) + \frac{\partial f}{\partial y} \cos \phi + \frac{\partial f}{\partial z} \cos \theta \sin \phi, \quad \text{(F.11)}
\]
\[
\frac{\partial f}{\partial z'} = \frac{\partial f}{\partial x} \frac{\partial z}{\partial z'} + \frac{\partial f}{\partial y} \frac{\partial z}{\partial z'} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial z'} \]
\[
= \frac{\partial f}{\partial x} (-\sin \theta \cos \phi) + \frac{\partial f}{\partial y} (-\sin \phi) + \frac{\partial f}{\partial z} \cos \theta \cos \phi. \quad \text{(F.12)}
\]

The last equalities are obtained from equation (F.2). The free boundary conditions (4.44)-(4.46) for the velocities have to be enforced in the local \((x', y', z')\)-system, where the \(z'\)-axis is normal to the surface at the local point, i.e.
\[
\frac{\partial u'}{\partial z'} = -\frac{\partial u'}{\partial x'}, \quad \text{(F.13)}
\]
\[
\frac{\partial v'}{\partial z'} = -\frac{\partial v'}{\partial y'}, \quad \text{(F.14)}
\]
\[
\frac{\partial w'}{\partial z'} = -\frac{\lambda}{\lambda + 2\mu} \left( \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} \right). \quad \text{(F.15)}
\]
If the chain rule is applied as above, we get

\[
\frac{\partial u'}{\partial x}(-\sin \theta \cos \phi) + \frac{\partial u'}{\partial y}(-\sin \phi) + \frac{\partial u'}{\partial z} \cos \theta \cos \phi \\
= -\frac{\partial w'}{\partial x} \cos \theta - \frac{\partial w'}{\partial z} \sin \theta, \quad (F.16)
\]

\[
\frac{\partial v'}{\partial x}(-\sin \theta \cos \phi) + \frac{\partial v'}{\partial y}(-\sin \phi) + \frac{\partial v'}{\partial z} \cos \theta \cos \phi \\
= -\frac{\partial w'}{\partial x}(-\sin \theta \sin \phi) - \frac{\partial w'}{\partial y} \cos \phi - \frac{\partial w'}{\partial z} \cos \theta \sin \phi, \quad (F.17)
\]

\[
\frac{\partial w'}{\partial x}(-\sin \theta \cos \phi) + \frac{\partial w'}{\partial y}(-\sin \phi) + \frac{\partial w'}{\partial z} \cos \theta \cos \phi \\
= -\frac{\lambda}{\lambda + 2\mu} \left( \frac{\partial u'}{\partial x} \cos \theta + \frac{\partial u'}{\partial x} \sin \theta + \frac{\partial v'}{\partial y}(-\sin \theta \sin \phi) \\
+ \frac{\partial v'}{\partial y} \cos \phi + \frac{\partial v'}{\partial z} \cos \theta \sin \phi \right). \quad (F.18)
\]

Now we apply the above expressions for \( u', v' \) and \( w' \) that were obtained from the rotation. This leads to

\[
\begin{bmatrix}
\cos \theta \frac{\partial u}{\partial x} + \sin \theta \frac{\partial w}{\partial x}
\end{bmatrix}(-\sin \theta \cos \phi) \\
+ \begin{bmatrix}
\cos \theta \frac{\partial u}{\partial y} + \sin \theta \frac{\partial w}{\partial y}
\end{bmatrix}(-\sin \phi) \\
+ \begin{bmatrix}
\cos \theta \frac{\partial u}{\partial z} + \sin \theta \frac{\partial w}{\partial z}
\end{bmatrix} \cos \theta \cos \phi
\]

\[
= \begin{bmatrix}
\sin \theta \cos \phi \frac{\partial u}{\partial x} + \sin \phi \frac{\partial u}{\partial x} - \cos \theta \cos \phi \frac{\partial w}{\partial x}
\end{bmatrix} \cos \theta \\
+ \begin{bmatrix}
\sin \theta \cos \phi \frac{\partial u}{\partial z} + \sin \phi \frac{\partial v}{\partial z} - \cos \theta \cos \phi \frac{\partial w}{\partial z}
\end{bmatrix} \sin \theta, \quad (F.19)
\]

\[
\begin{bmatrix}
-\sin \theta \sin \phi \frac{\partial u}{\partial x} + \cos \phi \frac{\partial v}{\partial x} + \cos \theta \sin \phi \frac{\partial w}{\partial x}
\end{bmatrix}(-\sin \theta \cos \phi) \\
+ \begin{bmatrix}
-\sin \theta \sin \phi \frac{\partial u}{\partial y} + \cos \phi \frac{\partial v}{\partial y} + \cos \theta \sin \phi \frac{\partial w}{\partial y}
\end{bmatrix}(-\sin \phi) \\
+ \begin{bmatrix}
-\sin \theta \sin \phi \frac{\partial u}{\partial z} + \cos \phi \frac{\partial v}{\partial z} + \cos \theta \sin \phi \frac{\partial w}{\partial z}
\end{bmatrix} \cos \theta \cos \phi \\
= \begin{bmatrix}
\sin \theta \cos \phi \frac{\partial u}{\partial x} + \sin \phi \frac{\partial v}{\partial x} - \cos \theta \cos \phi \frac{\partial w}{\partial x}
\end{bmatrix}(-\sin \theta \sin \phi)
\]
\[ + \left[ \sin \theta \cos \phi \frac{\partial u}{\partial y} + \sin \phi \frac{\partial v}{\partial y} - \cos \theta \cos \phi \frac{\partial w}{\partial y} \right] \cos \phi \]
\[ + \left[ \sin \theta \cos \phi \frac{\partial u}{\partial z} + \sin \phi \frac{\partial v}{\partial z} - \cos \theta \cos \phi \frac{\partial w}{\partial z} \right] \cos \theta \sin \phi, \quad (F.20) \]
\[ - \sin \theta \cos \phi \frac{\partial u}{\partial x} - \sin \phi \frac{\partial v}{\partial x} + \cos \theta \cos \phi \frac{\partial w}{\partial x} \right] (- \sin \theta \cos \phi) \]
\[ + \left[ - \sin \theta \cos \phi \frac{\partial u}{\partial y} - \sin \phi \frac{\partial v}{\partial y} + \cos \theta \cos \phi \frac{\partial w}{\partial y} \right] (- \sin \phi) \]
\[ + \left[ - \sin \theta \cos \phi \frac{\partial u}{\partial z} - \sin \phi \frac{\partial v}{\partial z} + \cos \theta \cos \phi \frac{\partial w}{\partial z} \right] \cos \theta \sin \phi \]
\[ = - \frac{\lambda}{\lambda + 2\mu} \left\{ \cos \theta \frac{\partial u}{\partial x} + \sin \theta \frac{\partial w}{\partial x} \right\} \cos \theta \]
\[ + \left[ \cos \theta \frac{\partial u}{\partial z} + \sin \theta \frac{\partial w}{\partial z} \right] \sin \theta \]
\[ + \left[ - \sin \theta \sin \phi \frac{\partial u}{\partial x} + \cos \phi \frac{\partial v}{\partial x} + \cos \theta \sin \phi \frac{\partial w}{\partial x} \right] (- \sin \theta \sin \phi) \]
\[ + \left[ - \sin \theta \sin \phi \frac{\partial u}{\partial y} + \cos \phi \frac{\partial v}{\partial y} + \cos \theta \sin \phi \frac{\partial w}{\partial y} \right] \cos \phi \]
\[ + \left[ - \sin \theta \sin \phi \frac{\partial u}{\partial z} + \cos \phi \frac{\partial v}{\partial z} + \cos \theta \sin \phi \frac{\partial w}{\partial z} \right] \cos \theta \sin \phi \right\}. \quad (F.21) \]

We divide the first of these equations by \( \cos^2 \theta \cos \phi \), the second by \( \cos \theta \cos^2 \phi \) and the third by \( \cos^2 \theta \cos^2 \phi \). These divisions assume \( \theta \neq \pm \pi/2 \) and \( \phi \neq \pm \pi/2 \). This means that the topography cannot have vertical sections along the planes of rotation, i.e. the topography function must be single-valued. This is a reasonable assumption given that the topography function \( z_0(\xi, \kappa) \) is assumed to be smooth. After restructuring, the three equations become
\[ \left( 1 - \tan^2 \theta \right) \frac{\partial u}{\partial z} - \frac{\tan \theta}{\cos \theta} \tan \phi \frac{\partial v}{\partial z} + 2 \tan \theta \frac{\partial w}{\partial z} \]
\[ = 2 \tan \theta \frac{\partial u}{\partial x} + \frac{\tan \phi}{\cos \theta} \frac{\partial v}{\partial x} + \left( \tan^2 \theta - 1 \right) \frac{\partial w}{\partial x} + \frac{\tan \phi}{\cos \theta} \frac{\partial u}{\partial y} + \frac{\tan \theta}{\cos \theta} \frac{\partial w}{\partial y}, \quad (F.22) \]
\[ -2 \sin \theta \tan \phi \frac{\partial u}{\partial z} + \left( 1 - \tan^2 \phi \right) \frac{\partial v}{\partial z} + 2 \cos \theta \tan \phi \frac{\partial w}{\partial z} \]
\[ = -2 \sin \theta \tan \theta \tan \phi \frac{\partial u}{\partial x} + \tan \theta \left( 1 - \tan^2 \phi \right) \frac{\partial v}{\partial x} + 2 \sin \theta \tan \phi \frac{\partial w}{\partial x} \]
\[ + \tan \theta \left( 1 - \tan^2 \phi \right) \frac{\partial u}{\partial y} + 2 \frac{\tan \phi}{\cos \theta} \frac{\partial v}{\partial y} + \left( \tan^2 \phi - 1 \right) \frac{\partial w}{\partial y}, \] (F.23)

\[ \tan \theta \left\{ \frac{\lambda}{\lambda + 2\mu} \left( \frac{1}{\cos^2 \phi} - \tan^2 \phi \right) - 1 \right\} \frac{\partial u}{\partial z} + \frac{\tan \phi}{\cos \theta} \left\{ \frac{\lambda}{\lambda + 2\mu} - 1 \right\} \frac{\partial v}{\partial z} \\
+ \left\{ \frac{\lambda}{\lambda + 2\mu} \left( \tan^2 \phi \frac{\cos^2 \phi}{\cos^2 \phi} + \tan^2 \phi \right) + 1 \right\} \frac{\partial w}{\partial z} \] 

\[ = \left\{ \frac{\lambda}{\lambda + 2\mu} \left( \frac{1}{\cos^2 \phi} + \tan^2 \phi \tan^2 \phi \right) - \tan^2 \phi \right\} \frac{\partial u}{\partial x} + \frac{\tan \theta}{\cos \theta} \tan \phi \left\{ \frac{\lambda}{\lambda + 2\mu} - 1 \right\} \frac{\partial v}{\partial x} \\
+ \tan \theta \left\{ \frac{\lambda}{\lambda + 2\mu} \left( \tan^2 \phi - \frac{1}{\cos^2 \phi} \right) + 1 \right\} \frac{\partial w}{\partial x} + \frac{\tan \theta}{\cos \theta} \tan \phi \left\{ \frac{\lambda}{\lambda + 2\mu} - 1 \right\} \frac{\partial u}{\partial y} \\
- \frac{1}{\cos^2 \theta} \left\{ \frac{\lambda}{\lambda + 2\mu} + \tan^2 \phi \right\} \frac{\partial v}{\partial y} + \frac{\tan \phi}{\cos \theta} \left\{ 1 - \frac{\lambda}{\lambda + 2\mu} \right\} \frac{\partial w}{\partial y}. \] (F.24)

The equations are now given in the cartesian \((x, y, z)\)-system. However, as for the medium equations, we require that the cartesian equations in \(x, y\) and \(z\) be valid inside the curved grid whose surface coincides with the surface topography in the cartesian system. Then we must find the appearance of these equations in the rectangular \((\xi, \kappa, \eta)\)-grid where the numerical computations are performed. This results in curved grid equations given in the \((\xi, \kappa, \eta)\)-grid. We must therefore apply the chain rule to the boundary conditions in the same way as was done for the medium equations.

At the free surface \(\eta = \eta_{\text{max}}\), and with \(\partial z_0(\xi, \kappa)/\partial \xi = \tan \theta\) and \(\partial z_0(\xi, \kappa)/\partial \kappa = \tan \phi/\cos \theta\) (now using the computational \((\xi, \kappa, \eta)\)-grid), equations (4.26) and (4.27) become

\[ A(\xi, \kappa, \eta) = \frac{\partial \eta}{\partial \xi} = -\frac{\eta_{\text{max}}}{z_0(\xi, \kappa)} \tan \theta = -C(\xi, \kappa) \tan \theta, \] (F.25)

\[ B(\xi, \kappa, \eta) = \frac{\partial \eta}{\partial \eta} = -\frac{\eta_{\text{max}}}{z_0(\xi, \kappa)} \tan \phi = -C(\xi, \kappa) \tan \phi. \] (F.26)

The equations (4.21)-(4.23) for a differentiable function \(f(x, y, z)\) at the surface then become

\[ \frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} - C(\xi, \kappa) \tan \theta \frac{\partial f}{\partial \eta}, \] (F.27)
\[
\begin{align*}
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial \kappa} - C(\xi, \kappa) \frac{\tan \phi}{\cos \theta} \frac{\partial f}{\partial \eta}, \\
\frac{\partial f}{\partial z} &= C(\xi, \kappa) \frac{\partial f}{\partial \eta}.
\end{align*}
\]

(F.28)  
(F.29)

When this is used in the boundary conditions, together with \(\zeta = \lambda/(\lambda + 2\mu)\), we obtain

\[
\begin{align*}
(1 - \tan^2 \theta) C(\xi, \kappa) \frac{\partial u}{\partial \eta} - \frac{\tan \theta}{\cos \theta} \tan \phi C(\xi, \kappa) \frac{\partial v}{\partial \eta} + 2 \tan \theta C(\xi, \kappa) \frac{\partial w}{\partial \eta} \\
= 2 \tan \theta \left( \frac{\partial u}{\partial \xi} - C(\xi, \kappa) \tan \theta \frac{\partial u}{\partial \eta} \right) + \frac{\tan \phi}{\cos \theta} \left( \frac{\partial v}{\partial \xi} - C(\xi, \kappa) \tan \theta \frac{\partial v}{\partial \eta} \right) \\
+ (\tan^2 \theta - 1) \left( \frac{\partial w}{\partial \xi} - C(\xi, \kappa) \tan \theta \frac{\partial w}{\partial \eta} \right) + \frac{\tan \phi}{\cos \theta} \left( \frac{\partial u}{\partial \kappa} - C(\xi, \kappa) \tan \phi \frac{\partial u}{\partial \eta} \right) \\
+ \frac{\tan \theta}{\cos \theta} \tan \phi \left( \frac{\partial w}{\partial \kappa} - C(\xi, \kappa) \tan \phi \frac{\partial w}{\partial \eta} \right),
\end{align*}
\]

(F.30)

\[
\begin{align*}
-2 \sin \theta \tan \phi C(\xi, \kappa) \frac{\partial u}{\partial \eta} + (1 - \tan^2 \phi) C(\xi, \kappa) \frac{\partial v}{\partial \eta} + 2 \cos \theta \tan \phi C(\xi, \kappa) \frac{\partial w}{\partial \eta} \\
= -2 \sin \theta \tan \theta \tan \phi \left( \frac{\partial u}{\partial \xi} - C(\xi, \kappa) \tan \theta \frac{\partial u}{\partial \eta} \right) \\
+ \tan \phi \left( 1 - \tan^2 \phi \right) \left( \frac{\partial v}{\partial \xi} - C(\xi, \kappa) \tan \theta \frac{\partial v}{\partial \eta} \right) \\
+ 2 \sin \theta \tan \phi \left( \frac{\partial w}{\partial \xi} - C(\xi, \kappa) \tan \theta \frac{\partial w}{\partial \eta} \right) \\
+ \tan \phi \left( 1 - \tan^2 \phi \right) \left( \frac{\partial u}{\partial \kappa} - C(\xi, \kappa) \tan \phi \frac{\partial u}{\partial \eta} \right) + 2 \tan \phi \left( \frac{\partial v}{\partial \kappa} - C(\xi, \kappa) \tan \phi \frac{\partial v}{\partial \eta} \right) \\
+ (\tan^2 \phi - 1) \left( \frac{\partial w}{\partial \kappa} - C(\xi, \kappa) \tan \phi \frac{\partial w}{\partial \eta} \right),
\end{align*}
\]

(F.31)

\[
\tan \theta \{ \zeta - 1 \} C(\xi, \kappa) \frac{\partial u}{\partial \eta} + \frac{\tan \phi}{\cos \theta} \{ \zeta - 1 \} C(\xi, \kappa) \frac{\partial v}{\partial \eta} \\
+ \left\{ \zeta \left( \frac{\tan^2 \theta}{\cos^2 \phi} + \tan^2 \phi \right) + 1 \right\} C(\xi, \kappa) \frac{\partial w}{\partial \eta} \\
= \left\{ - \zeta \left( \frac{1}{\cos^2 \phi} + \tan^2 \theta \tan^2 \phi \right) - \tan^2 \theta \right\} \left( \frac{\partial u}{\partial \xi} - C(\xi, \kappa) \tan \theta \frac{\partial u}{\partial \eta} \right) \\
+ \frac{\tan \theta}{\cos \theta} \tan \phi \{ \zeta - 1 \} \left( \frac{\partial v}{\partial \xi} - C(\xi, \kappa) \tan \theta \frac{\partial v}{\partial \eta} \right) \\
+ \frac{\tan \theta}{\cos \theta} \tan \phi \{ \zeta - 1 \} \left( \frac{\partial w}{\partial \xi} - C(\xi, \kappa) \tan \theta \frac{\partial w}{\partial \eta} \right).
\]
\[- \tan \theta \{ \zeta - 1 \} \left( \frac{\partial w}{\partial \xi} - C(\xi, \kappa) \tan \theta \frac{\partial w}{\partial \eta} \right) \]
\[+ \frac{\tan \theta}{\cos \theta} \tan \phi \{ \zeta - 1 \} \left( \frac{\partial u}{\partial \kappa} - C(\xi, \kappa) \tan \phi \frac{\partial u}{\cos \theta \partial \eta} \right) \]
\[- \frac{1}{\cos^2 \theta} \left\{ \zeta + \tan^2 \phi \right\} \left( \frac{\partial v}{\partial \kappa} - C(\xi, \kappa) \tan \phi \frac{\partial v}{\cos \theta \partial \eta} \right) \]
\[- \frac{\tan \phi}{\cos \theta} \{ \zeta - 1 \} \left( \frac{\partial w}{\partial \kappa} - C(\xi, \kappa) \tan \phi \frac{\partial w}{\cos \theta \partial \eta} \right) \]

\[(F.32)\]

Rearranging terms and using the trigonometric simplifications

\[2 \cos \theta + 2 \sin \theta \tan \theta + \frac{1}{\cos \theta} (\tan^2 \phi - 1) = \frac{1}{\cos \theta} (1 + \tan^2 \phi) , \quad (F.33)\]
\[-2 - 2 \tan^2 \theta + \frac{1}{\cos^2 \theta} (1 - \tan^2 \phi) = -\frac{1}{\cos^2 \theta} (1 + \tan^2 \phi) , \quad (F.34)\]
\[1 + \tan^2 \theta + \frac{\tan^2 \phi}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} (1 + \tan^2 \phi) , \quad (F.35)\]
\[(1 + \tan^2 \theta) (1 - \tan^2 \phi) + 2 \frac{\tan^2 \phi}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} (1 + \tan^2 \phi) . \quad (F.36)\]

lead to the equations

\[\frac{1}{\cos^2 \theta} (1 + \tan^2 \phi) C(\xi, \kappa) \frac{\partial u}{\partial \eta} + \frac{1}{\cos^2 \theta} (1 + \tan^2 \phi) \tan \theta C(\xi, \kappa) \frac{\partial w}{\partial \eta} \]
\[= 2 \tan \theta \frac{\partial u}{\partial \xi} + \frac{\tan \phi \partial v}{\cos \theta \partial \xi} + (\tan^2 \theta - 1) \frac{\partial w}{\partial \xi} + \frac{\tan \phi \partial u}{\cos \theta \partial \kappa} + \tan \theta \tan \phi \frac{\partial w}{\partial \kappa} , \quad (F.37)\]
\[-\frac{1}{\cos^2 \theta} (1 + \tan^2 \phi) \sin \theta \tan \phi C(\xi, \kappa) \frac{\partial u}{\partial \eta} + \frac{1}{\cos^2 \theta} (1 + \tan^2 \phi) C(\xi, \kappa) \frac{\partial v}{\partial \eta} \]
\[+ \frac{1}{\cos \theta} (1 + \tan^2 \phi) \tan \phi C(\xi, \kappa) \frac{\partial w}{\partial \eta} \]
\[= -2 \sin \theta \tan \theta \tan \phi \frac{\partial u}{\partial \xi} + \tan \theta (1 - \tan^2 \phi) \frac{\partial v}{\partial \xi} + 2 \sin \theta \tan \phi \frac{\partial w}{\partial \xi} \]
\[+ \tan \theta (1 - \tan^2 \phi) \frac{\partial u}{\partial \kappa} + 2 \frac{\tan \phi \partial v}{\cos \theta \partial \kappa} + (\tan^2 \phi - 1) \frac{\partial w}{\partial \kappa} , \quad (F.38)\]
\[\left\{ \left( \frac{1}{\cos^2 \theta} + \tan^2 \theta \tan^2 \phi \right) \right\} \tan \theta C(\xi, \kappa) \frac{\partial u}{\partial \eta} \]
\[+ \left\{ \left( \frac{1}{\cos^2 \theta} + \tan^2 \phi \right) \right\} \frac{\tan \phi \partial v}{\cos \theta C(\xi, \kappa) \frac{\partial v}{\partial \eta}} \]
\[+ \left\{ \frac{\tan^2 \theta}{\cos^2 \theta} + \tan^2 \phi \right\} + 1 - \left( \tan^2 \theta + \frac{\tan^2 \phi}{\cos^2 \theta} \right) (\zeta - 1) \right\} \right\}
\[C(\xi, \kappa) \frac{\partial w}{\partial \eta} \]
\[
\begin{align*}
&= \left\{ -\zeta \left( \frac{1}{\cos^2 \phi} + \tan^2 \theta \tan^2 \phi \right) - \tan^2 \theta \right\} \frac{\partial u}{\partial \xi} + \frac{\tan \theta}{\cos \theta} \tan \phi (\zeta - 1) \frac{\partial v}{\partial \xi} \\
&\quad - \tan \theta (\zeta - 1) \frac{\partial w}{\partial \xi} + \frac{\tan \theta}{\cos \theta} \tan \phi (\zeta - 1) \frac{\partial u}{\partial \kappa} - \frac{1}{\cos^2 \theta} (\zeta + \tan^2 \phi) \frac{\partial v}{\partial \kappa} \\
&\quad - \frac{\tan \phi}{\cos \theta} (\zeta - 1) \frac{\partial w}{\partial \kappa}.
\end{align*}
\]

(F.39)

Finally, I simplify expressions by using the trigonometric relations

\[
1 + \frac{\tan^2 \phi}{\cos^2 \theta} = \frac{1}{\cos^2 \phi} + \tan^2 \theta \tan^2 \phi,
\]

(F.40)

\[
\frac{\tan^2 \theta}{\cos^2 \phi} + \tan^2 \phi = \frac{\tan^2 \phi}{\cos^2 \theta} + \tan^2 \theta,
\]

(F.41)

use the relations \(\tan \theta = \partial z_0(\xi, \kappa)/\partial \xi\), \(\tan \phi = \partial z_0(\xi, \kappa)/\partial \kappa \cos \theta\) and the definitions (4.52)–(4.56). This leads to the closed set of boundary conditions for the particle velocities at a free surface topography (4.49)–(4.51).
Appendix G

Partial derivatives in 2-D medium equations

For the medium equations we need $\partial \eta / \partial z$ and $\partial \eta / \partial z$. They are found from

$$\frac{\partial x}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial x}{\partial \eta} \frac{\partial \eta}{\partial x} = 1, \quad \frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial z}{\partial \eta} \frac{\partial \eta}{\partial x} = 0, \quad (G.1)$$

$$\frac{\partial x}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial x}{\partial \eta} \frac{\partial \eta}{\partial z} = 0, \quad \frac{\partial z}{\partial \xi} \frac{\partial \xi}{\partial z} + \frac{\partial z}{\partial \eta} \frac{\partial \eta}{\partial z} = 1. \quad (G.2)$$

This leads to

$$\frac{\partial \eta}{\partial x} = -\frac{\partial z}{\partial \xi} / J, \quad \frac{\partial \eta}{\partial z} = \frac{\partial x}{\partial \xi} / J, \quad (G.3)$$

where

$$J = \frac{\partial x}{\partial \xi} \frac{\partial z}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial z}{\partial \xi}. \quad (G.4)$$

With our choice of mapping functions, equations (5.11)–(5.12), we get

$$\frac{\partial x}{\partial \xi} = 1, \quad \frac{\partial x}{\partial \eta} = 0, \quad (G.5)$$

$$\frac{\partial z}{\partial \xi} = \frac{\eta}{\eta_{max}} \frac{\partial z_0(\xi)}{\partial \xi}, \quad \frac{\partial z}{\partial \eta} = \frac{z_0(\xi)}{\eta_{max}} \quad (G.6)$$

and

$$J = \frac{\partial z}{\partial \eta} = \frac{z_0(\xi)}{\eta_{max}}. \quad (G.7)$$

From this we get the expressions (5.15)–(5.17).
Appendix H

2–D viscoelastic medium equations

Applying the chain rule to equations (5.3)–(5.10) and using the properties of equations (5.13)–(5.14) leads to

\[
\rho \frac{\partial u}{\partial t} = \frac{\partial \sigma_{xx}}{\partial \xi} + \frac{\partial \sigma_{xx}}{\partial \eta} + \frac{\partial \sigma_{xx}}{\partial z} + f_x, \quad (H.1)
\]

\[
\rho \frac{\partial w}{\partial t} = \frac{\partial \sigma_{xx}}{\partial \xi} + \frac{\partial \sigma_{xx}}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial \sigma_{xx}}{\partial \eta} \frac{\partial \eta}{\partial z} + f_z, \quad (H.2)
\]

\[
\frac{\partial \sigma_{xx}}{\partial t} = \pi \frac{\tau_\sigma}{\tau_\sigma} \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) - 2\mu \frac{\tau_\sigma}{\tau_\sigma} \frac{\partial w}{\partial \eta} + r_{xx}, \quad (H.3)
\]

\[
\frac{\partial \sigma_{zz}}{\partial t} = \pi \frac{\tau_\sigma}{\tau_\sigma} \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) - 2\mu \frac{\tau_\sigma}{\tau_\sigma} \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \right) + r_{zz}, \quad (H.4)
\]

\[
\frac{\partial \sigma_{xz}}{\partial t} = \mu \frac{\tau_\sigma}{\tau_\sigma} \left( \frac{\partial w}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \eta} \right) + r_{xz}, \quad (H.5)
\]

\[
\frac{\partial r_{xx}}{\partial t} = -\frac{1}{\tau_\sigma} \left[ r_{xx} + \pi \left( \frac{\tau_\sigma}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial w}{\partial \eta} \frac{\partial \eta}{\partial z} \right) - 2\mu \left( \frac{\tau_\sigma}{\tau_\sigma} - 1 \right) \frac{\partial w}{\partial \eta} \right], \quad (H.6)
\]

\[
\frac{\partial r_{zz}}{\partial t} = -\frac{1}{\tau_\sigma} \left[ r_{zz} + \pi \left( \frac{\tau_\sigma}{\tau_\sigma} - 1 \right) \left( \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \eta} \right) - 2\mu \left( \frac{\tau_\sigma}{\tau_\sigma} - 1 \right) \frac{\partial w}{\partial \eta} \right], \quad (H.7)
\]

\[
\frac{\partial r_{xz}}{\partial t} = -\frac{1}{\tau_\sigma} \left[ r_{xz} + \mu \left( \frac{\tau_\sigma}{\tau_\sigma} - 1 \right) \left( \frac{\partial w}{\partial \xi} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial z} + \frac{\partial w}{\partial \eta} \right) \right]. \quad (H.8)
\]

Substituting for \( \partial \eta/\partial x \) and \( \partial \eta/\partial z \) from equations (5.15)–(5.17), we get the medium equations (5.18)–(5.25).
Appendix I

2–D surface topography boundary conditions

Assume a velocity vector \( \vec{\nu} \) with components \( u \) and \( w \) is given in a cartesian \((x, z)\)–coordinate system with basis vectors \( \vec{i} \) and \( \vec{j} \). This system is then rotated an angle \( \phi \) into a new \((x', z')\)–coordinate system with basis vectors \( \vec{i}' \) and \( \vec{j}' \). \( \phi \) is the rotation angle between the \( x \)–axis and the \( x' \)–axis in the \((x, z)\)–plane. In this new system the vector \( \vec{\nu} \) is denoted by \( \vec{\nu}' \) with components \( u' \) and \( w' \). Then we have the relationships

\[
\begin{pmatrix}
\vec{i}' \\
\vec{j}'
\end{pmatrix} = \mathbf{A}
\begin{pmatrix}
\vec{i} \\
\vec{j}
\end{pmatrix},
\]

(I.1)

where \( \mathbf{A} \) is the rotation matrix, given by equation (5.29). Correspondingly,

\[
\begin{pmatrix}
\vec{i} \\
\vec{j}
\end{pmatrix} = \mathbf{A}^{-1}
\begin{pmatrix}
\vec{i}' \\
\vec{j}'
\end{pmatrix} = \mathbf{A}^T
\begin{pmatrix}
\vec{\nu}' \\
\vec{j}'
\end{pmatrix},
\]

(I.2)

where \( \mathbf{A}^{-1} \) and \( \mathbf{A}^T \) are equal (since \( \mathbf{A} \) is orthogonal) and the inverse and transposed of \( \mathbf{A} \) respectively.

The coordinate transformation for \( \vec{\nu} \) is given by \( \vec{\nu} = \mathbf{A}^{-1} \vec{\nu}' \), or \( \vec{\nu}' = \mathbf{A} \vec{\nu} \). Componentwise this is

\[
\begin{align*}
  u' &= (\cos \phi)u + (\sin \phi)w, \\
  w' &= -(\sin \phi)u + (\cos \phi)w.
\end{align*}
\]

(I.3)

Applying the chain rule to a differentiable function \( f \), we get

\[
\begin{align*}
  \frac{\partial f}{\partial x'} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial x'} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial x'} = \frac{\partial f}{\partial x} \cos \phi + \frac{\partial f}{\partial z} \sin \phi, \\
  \frac{\partial f}{\partial z'} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial z'} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial z'} \\
  &= \frac{\partial f}{\partial x} (-\sin \phi) + \frac{\partial f}{\partial z} \cos \phi.
\end{align*}
\]

(I.5)
The last equalities are obtained from equation (I.2). The free boundary conditions (5.26)–(5.27) for the velocities have to be enforced in the local \( (x', z') \)-system, where the \( z' \)-axis is normal to the surface at the local point, i.e.

\[
\frac{\partial u'}{\partial z'} = \frac{\partial w'}{\partial x'}, \\
\frac{\partial w'}{\partial z'} = \frac{\lambda}{\lambda + 2\mu} \frac{\partial u'}{\partial x'}. \tag{I.7}
\]

If the chain rule is applied as above, we get

\[
\frac{\partial u'}{\partial x}(-\sin \phi) + \frac{\partial u'}{\partial z} \cos \phi = -\frac{\partial w'}{\partial x} \cos \phi - \frac{\partial w'}{\partial z} \sin \phi, \tag{I.9}
\]

\[
\frac{\partial w'}{\partial x}(-\sin \phi) + \frac{\partial w'}{\partial z} \cos \phi = -\frac{\lambda}{\lambda + 2\mu} \left( \frac{\partial u'}{\partial x} \cos \phi + \frac{\partial u'}{\partial z} \sin \phi \right). \tag{I.10}
\]

Now we apply the above expressions for \( u' \) and \( w' \) that were obtained from the rotation. This leads to

\[
-\sin \phi \cos \phi \frac{\partial u}{\partial x} - \sin^2 \phi \frac{\partial w}{\partial x} + \cos^2 \phi \frac{\partial u}{\partial z} + \sin \phi \cos \phi \frac{\partial w}{\partial z} = \sin \phi \cos \phi \frac{\partial u}{\partial x} - \cos^2 \phi \frac{\partial w}{\partial x} + \sin^2 \phi \frac{\partial u}{\partial z} - \sin \phi \cos \phi \frac{\partial w}{\partial z}, \tag{I.11}
\]

\[
\sin^2 \phi \frac{\partial u}{\partial x} - \sin \phi \cos \phi \frac{\partial w}{\partial x} - \sin \phi \cos \phi \frac{\partial u}{\partial z} + \cos^2 \phi \frac{\partial w}{\partial z} = -\frac{\lambda}{\lambda + 2\mu} \left\{ \cos^2 \phi \frac{\partial u}{\partial x} + \sin \phi \cos \phi \frac{\partial w}{\partial x} + \sin \phi \cos \phi \frac{\partial u}{\partial z} + \sin^2 \phi \frac{\partial w}{\partial z} \right\}. \tag{I.12}
\]

We divide both equations by \( \cos^2 \phi \). This division requires the assumption \( \phi \neq \pm \pi/2 \).

This means that the topography cannot have vertical sections along the plane of rotation, i.e. the topography function \( z_0(\xi) \) must be single–valued. Given the condition of smoothness for \( z_0(\xi) \), this is a reasonable assumption. After restructuring, the equations become

\[
(1 - \tan^2 \phi) \frac{\partial u}{\partial z} + 2 \tan \phi \frac{\partial w}{\partial z} = 2 \tan \phi \frac{\partial u}{\partial x} + \left( \tan^2 \phi - 1 \right) \frac{\partial w}{\partial x}, \tag{I.13}
\]

\[
\tan \phi \left( \frac{\lambda}{\lambda + 2\mu} - 1 \right) \frac{\partial u}{\partial z} + \left( \frac{\lambda}{\lambda + 2\mu} \tan^2 \phi + 1 \right) \frac{\partial w}{\partial z} = -\left( \frac{\lambda}{\lambda + 2\mu} + \tan^2 \phi \right) \frac{\partial u}{\partial x} + \tan \phi \left( 1 - \frac{\lambda}{\lambda + 2\mu} \right) \frac{\partial w}{\partial x}. \tag{I.14}
\]
The equations are now given in the cartesian \((x, z)\)-system. However, as for the medium equations, we require that the cartesian equations in \(x\) and \(z\) be valid inside the curved grid whose surface coincides with the surface topography in the cartesian system. Then we must find the appearance of these equations in the rectangular \((\xi, \eta)\)-grid where the numerical computations are performed. This results in curved grid equations given in the \((\xi, \eta)\)-grid. We must therefore apply the chain rule to the boundary conditions in the same way as was done for the medium equations.

At the free surface \(\eta = \eta_{\text{max}}\), and with \(\partial z_0(\xi)/\partial \xi = \tan \phi\) (now using the computational \((\xi, \eta)\)-grid), equation (5.16) becomes

\[
A(\xi, \eta) = \frac{\partial \eta}{\partial x} = -\frac{\eta_{\text{max}}}{z_0(\xi)} \tan \phi = -B(\xi) \tan \phi. \tag{I.15}
\]

The equations (5.13)-(5.14) for a differentiable function \(f(x, z)\) at the surface then become

\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} - B(\xi) \tan \phi \frac{\partial f}{\partial \eta}, \tag{I.16}
\]

\[
\frac{\partial f}{\partial z} = B(\xi) \frac{\partial f}{\partial \eta}. \tag{I.17}
\]

When this is used in the boundary conditions, we obtain

\[
(1 - \tan^2 \phi) B(\xi) \frac{\partial u}{\partial \eta} + 2 \tan \phi B(\xi) \frac{\partial w}{\partial \eta}
\]

\[
= 2 \tan \phi \left( \frac{\partial u}{\partial \xi} - B(\xi) \tan \phi \frac{\partial u}{\partial \eta} \right) + \left( \tan^2 \phi - 1 \right) \left( \frac{\partial w}{\partial \xi} - B(\xi) \tan \phi \frac{\partial w}{\partial \eta} \right) \tag{I.18}
\]

\[
\tan \phi \left( \frac{\lambda}{\lambda + 2\mu} - 1 \right) B(\xi) \frac{\partial u}{\partial \eta} + \left( \frac{\lambda}{\lambda + 2\mu} \tan^2 \phi + 1 \right) B(\xi) \frac{\partial w}{\partial \eta}
\]

\[
= - \left( \frac{\lambda}{\lambda + 2\mu} + \tan^2 \phi \right) \left( \frac{\partial u}{\partial \xi} - B(\xi) \tan \phi \frac{\partial u}{\partial \eta} \right)
\]

\[
+ \tan \phi \left( 1 - \frac{\lambda}{\lambda + 2\mu} \right) \left( \frac{\partial w}{\partial \xi} - B(\xi) \tan \phi \frac{\partial w}{\partial \eta} \right). \tag{I.19}
\]

Rearranging terms and simplifying coefficients lead to

\[
(1 + \tan^2 \phi) B(\xi) \frac{\partial u}{\partial \eta} + \tan \phi \left( 1 + \tan^2 \phi \right) B(\xi) \frac{\partial w}{\partial \eta}
\]
\[
= 2 \tan \phi \frac{\partial u}{\partial \xi} + (\tan^2 \phi - 1) \frac{\partial w}{\partial \xi}, \quad (I.20)
\]

\[
= -\frac{\lambda}{\lambda + 2\mu} \tan^2 \phi \frac{\partial u}{\partial \xi} + \tan \phi \left(1 - \frac{\lambda}{\lambda + 2\mu}\right) \frac{\partial w}{\partial \xi}. \quad (I.21)
\]

Using the equality \( \tan \phi = \partial z_0(\xi)/\partial \xi \), leads to the closed set of boundary conditions for the particle velocities at a free surface topography (5.30)-(5.31).