INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor MI 48106-1346 USA
313/761-4700 800/521-0600
RICE UNIVERSITY

EXPERIMENTAL / ANALYTICAL APPROACHES TO MODELING, CALIBRATING AND OPTIMIZING SHAKING TABLE DYNAMICS FOR STRUCTURAL DYNAMIC APPLICATIONS

by

TOMASO TROMBETTI

A THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE DOCTOR OF PHILOSOPHY

APPROVED. THESIS COMMITTEE

Joel P. Conte. Director
Associate Professor of Civil Engineering

Ahmad J. Durrani. Co-Director
Professor of Civil Engineering

Pol D. Spanos
Lewis B. Ryon Professor of Mechanical and Civil Engineering

Houston, Texas
May 1998
Dedicated to Alessandra and my family, with gratitude for their support and with appreciation for patiently enduring and sharing these years of preparation with me. Their presence and encouragement made this work a reality.
ABSTRACT

Experimental / Analytical Approaches to Modeling, Calibrating and Optimizing Shaking Table Dynamics for Structural Dynamic Applications

by

Tomaso Trombetti

This thesis presents an Experimental / Analytical approach to modeling and calibrating shaking tables for structural dynamic applications. This approach was successfully applied to the shaking table recently built in the structural laboratory of the Civil Engineering Department at Rice University.

This shaking table is capable of reproducing model earthquake ground motions with a peak acceleration of 6 g's, a peak velocity of 40 inches per second, and a peak displacement of 3 inches, for a maximum payload of 1500 pounds. It has a frequency bandwidth of approximately 70 Hz and is designed to test structural specimens up to 1/5 scale. The rail/table system is mounted on a reaction mass of about 70,000 pounds consisting of three 12 ft x 12 ft x 1 ft reinforced concrete slabs, post-tensioned together and connected to the strong laboratory floor. The slip table is driven by a hydraulic actuator governed by a 407 MTS controller which employs a proportional-integral-derivative-feedforward-differential pressure algorithm to control the actuator displacement. Feedback signals are provided by two LVDT's (monitoring the slip table relative displacement and the servo valve main stage...
spool position) and by one differential pressure transducer (monitoring the actuator force).

The dynamic actuator-foundation-specimen system is modeled and analyzed by combining linear control theory and linear structural dynamics. The analytical model developed accounts for the effects of actuator oil compressibility, oil leakage in the actuator, time delay in the response of the servovalve spool to a given electrical signal, foundation flexibility, and dynamic characteristics of multi-degree-of-freedom specimens.

In order to study the actual dynamic behavior of the shaking table, the transfer function between target and actual table accelerations were identified using experimental results and spectral estimation techniques. The power spectral density of the system input and the cross power spectral density of the table input and output were estimated using the Bartlett's spectral estimation method. The experimentally-estimated table acceleration transfer functions obtained for different working conditions are correlated with their analytical counterparts. As a result of this comprehensive correlation study, a thorough understanding of the shaking table dynamics and its sensitivities to control and payload parameters is obtained. Moreover, the correlation study leads to a calibrated analytical model of the shaking table of high predictive ability. It is concluded that, in its present conditions, the Rice shaking table is able to reproduce, with a high degree of accuracy, model earthquake accelerations time histories in the frequency bandwidth from 0 to 75 Hz. Furthermore, the exhaustive analysis performed indicates that the table transfer function is not significantly affected by the presence of a large (in terms of weight) payload with a fundamental frequency up to 20 Hz. Payloads having a higher fundamental frequency do affect significantly the shaking table
performance and require a modification of the table control gain setting that can be easily obtained using the predictive analytical model of the shaking table.

The complete description of a structural dynamic experiment performed using the Rice shaking table facility is also reported herein. The object of this experimentation was twofold: (1) to verify the testing capability of the shaking table and, (2) to experimentally validate a simplified theory developed by the author, which predicts the maximum rotational response developed by seismic isolated building structures characterized by non-coincident centers of mass and rigidity, when subjected to strong earthquake ground motions.
AKNOWLEDGEMENTS

I am grateful to the many people who helped in the preparation of this thesis.

- Professor Joel P. Conte, my advisor, guided, encouraged and helped me for the successful completion this research work.
- Professor Ahmad J. Durrani, my co-advisor, supported me, helped me, and gave me maximum research freedom during both the analytical and experimental work performed at Rice University.
- Professor Pol D. Spanos, member of the thesis committee, provided insightful suggestions and encouraged me to perform the structural dynamic test described in the last part of this thesis.
- Kobie Williams provided a fundamental support to the assembly of the scaled down structural model that was tested, Professor John E. Merwin was of great help in obtaining the construction material necessary for the experimental investigation.
- Fausta Bacci helped me in the early development of the simplified method for maximum rotation estimations.

I also wish to express my deep appreciation to Professor Claudio Ceccoli for the influence he had on my academic growth. The financial assistance provided by Rice University and the Civil Engineering Department is also gratefully acknowledged.

Tomaso Trombetti
CONTENTS

PART I: ANALYTICAL DEVELOPMENTS

CHAPTER 1  INTRODUCTION ............................................. 1

1.1 BACKGROUND ................................................. 2
1.2 PROBLEM DESCRIPTION AND OBJECTIVES ............... 2
1.3 SCOPE OF WORK ............................................... 3
1.4 ORGANIZATION OF TEXT ...................................... 4

CHAPTER 2  DESCRIPTION OF THE SHAKING TABLE FACILITY.... 5

2.1 INTRODUCTION ..................................................... 6
2.2 SHAKING TABLE COMPONENTS ................................... 7
   2.2.1 Slip Table .................................................... 7
   2.2.2 Rails and Steel Base Plate ............................... 8
   2.2.3 Foundation Mass ............................................ 8
   2.2.4 Hydraulic Power Supply ................................ 8
   2.2.5 Servovalve and Actuator ................................. 9
   2.2.6 MTS 407 Controller ...................................... 10
   2.2.7 Personal Computer ....................................... 11
   2.2.8 Amplifier ..................................................... 11
2.3 ELECTRO-HYDRAULIC CONTROL SYSTEM ...................... 12
   2.3.1 Flow of the Electrical Control Signal From PC to 407 Controller 12
   2.3.2 The 407 Controller ....................................... 14
   2.3.3 The Inner Control Loop .................................. 17
   2.3.4 The Outer Control Loop .................................. 20
   2.3.5 The “Batch” Control Loop ............................... 24
2.4 DATA ACQUISITION SYSTEM ...................................... 26
   2.4.1 Accelerometers ............................................. 26
   2.4.2 Actuator LVDT .............................................. 27
   2.4.3 Dynamic Signal Acquisition Boards ................. 27
   2.4.4 Data Acquisition Software ............................. 29
2.5 CONCLUSIONS ...................................................... 30
CHAPTER 3  ANALYTICAL MODELING OF SHAKING TABLE ........ 31

3.1  INTRODUCTION

3.2  THE SERVOVALVE AND ACTUATOR MODEL
    3.2.1  Three Stage Servovalve Transfer Function
    3.2.2  “Linear” Three Stage Servovalve Transfer Function, H(t)
    3.2.3  Time Delay in Third Stage Spool Response
    3.2.4  The Oil Flow in the Actuator
    3.2.5  The Force in the Actuator
    3.2.6  Servovalve Transfer Function, S(s)
    3.2.7  Equivalent SDOF

3.3  CONTROLLER MODEL
    3.3.1  Introduction
    3.3.2  PID Gain Component, e(t)
    3.3.3  Feedforward Gain Component, x_f(t)
    3.3.4  Delta Pressure Gain Component, x_d(t)
    3.3.5  Controller Model

3.4  SYSTEM MODEL (SERVOVALVE & ACTUATOR + CONTROLLER)
    3.4.1  System Transfer Function, H(s)
    3.4.2  Results

3.5  EFFECT OF FLEXIBILITY OF FOUNDATION MASS
    3.5.1  Table Transfer Function, T(s)
    3.5.2  The Base Transfer Function B(s)
    3.5.3  The Force in the Actuator
    3.5.4  Modified System Transfer Function H'(s)
    3.5.5  Results

3.6  EFFECT OF FLEXIBLE PAYLOAD (SDOF)
    3.6.1  Introduction
    3.6.2  Payload Transfer Function, H_p(s)
    3.6.3  Payload Shear F_s(t)
    3.6.4  Modified Base Transfer Function B'(s)
    3.6.5  Force in the Actuator
    3.6.6  Modified System Transfer Function H''(s)
    3.6.7  Modified Table Transfer Function, T'(s)
    3.6.8  Difference in Table T.F. due to Damping Modeling
    3.6.9  Results

3.7  EFFECT OF FLEXIBLE PAYLOAD (MDOF)
    3.7.1  Introduction
    3.7.2  MDOF Payload Transfer Function H_p_m(s)
    3.7.3  Payload Shear
    3.7.4  Modified Base Transfer Function B''(s)
    3.7.5  Force in the Actuator
    3.7.6  Modified System Transfer Function H'''(s)
CHAPTER 4  ANALYTICAL SENSITIVITY OF TRANSFER FUNCTION 135

4.1  INTRODUCTION 136

4.2  SENSITIVITY OF TABLE TRANSFER FUNCTION TO CONTROL PARAMETERS (GAINS) 137

4.2.1  Introduction 137

4.2.2  Proportional Gain 138

4.2.3  Integral Gain 140

4.2.4  Derivative Gain (negative value of the parameters) 142

4.2.5  Derivative Gain (positive value of the parameters) 144

4.2.6  Feed Forward Gain 146

4.2.7  Delta Pressure Gain 149

4.3  SENSITIVITY OF TABLE TRANSFER FUNCTION TO PAYLOAD PARAMETERS 151

4.3.1  Sensitivity to the Weight of Rigid Payloads 152

4.3.2  Sensitivity to Payload Flexibility: Detailed Analysis 154

4.3.3  Sensitivity to Payload Flexibility: Limit Cases 165

4.3.4  Sensitivity to Payload Flexibility: General Behaviors 169

4.3.5  Sensitivity to Payload Flexibility: Transition Frequencies 172

4.3.6  Sensitivity to the Weight of Flexible Payloads 176

4.3.7  Sensitivity to MDOF Payloads 178

4.3.8  Sensitivity to MDOF Payloads: Size Effect 181

4.3.9  SDOF/MDOF Comparison 184

4.3.10  SDOF/MDOF Comparison (First Modal Frequency and Mass) 187

4.3.11  SDOF/MDOF Comparison (Higher Modal Frequencies and Masses) 193

4.3.12  Equivalence between MDOF Payload and corresponding SDOF 203

4.3.13  Sensitivity of Table Transfer Function to Payload Parameters. Conclusions 205

4.4  SENSITIVITY OF TRANSFER FUNCTION TO SERVOVALVE DELAY $\tau$ 206

4.4.1  Introduction 206

4.4.2  Small Time Delay 206

4.4.3  Medium Time Delay 209

4.4.4  Large Time Delay 211

4.5  CONCLUSIONS 213
CHAPTER 5  THEORY OF TRANSFER FUNCTION ESTIMATION .................................. 214

5.1 INTRODUCTION ........................................................................................................... 215
5.2 THE TRANSFER FUNCTION ...................................................................................... 216
  5.2.1 Definition ............................................................................................................... 216
  5.2.2 Deterministic Estimation of Transfer Function ....................................................... 219
  5.2.3 Estimation of Transfer Function by Random Excitation ......................................... 221
5.3 SPECTRAL ESTIMATION REVIEW .......................................................................... 223
  5.3.1 The Power Spectrum and the Periodogram ............................................................. 223
  5.3.2 Window Selection & Statistical Characteristics ...................................................... 231
  5.3.3 Bartlett’s Procedure .............................................................................................. 234
  5.3.4 Adopted Method of Estimation ............................................................................. 239
5.4 EXAMPLE OF SPECTRAL ESTIMATION USING ARMA MODEL .................................. 240
  5.4.1 ARMA Models ....................................................................................................... 240
  5.4.2 ARMA Model Simulation .................................................................................... 250
  5.4.3 PSD Estimation via Bartlett’s Procedure .............................................................. 251
  5.4.4 Standard Deviation of PSD Estimation ................................................................. 257
  5.4.5 Standard Deviation of Bartlett’s Estimate ............................................................ 259
  5.4.6 Effect of Different Windows Upon Bartlett’s Estimate ......................................... 264
5.5 EXAMPLE OF ERRORS IN TRANSFER FUNCTION ESTIMATION ................................. 270
  5.5.1 Component Error Estimation .............................................................................. 270
  5.5.2 Numerical Simulation ......................................................................................... 274
  5.5.3 Results .................................................................................................................. 276
5.6 CHARACTERISTICS OF ADOPTED SPECTRAL ESTIMATION METHOD ......................... 281
  5.6.1 Signals Selection ................................................................................................. 281
  5.6.2 Characteristics of Input Signals .......................................................................... 283
  5.6.3 Periodogram Estimation Procedure ................................................................... 288
5.7 CONCLUSIONS .............................................................................................................. 289

PART II: ANALYTICAL / EXPERIMENTAL CORRELATION STUDIES

CHAPTER 6  SHAKING TABLE CALIBRATION ................................................................. 290

6.1 INTRODUCTION ........................................................................................................... 291
6.2 INNER LOOP SETUP ................................................................................................. 292
  6.2.1 Introduction ......................................................................................................... 292
  6.2.2 LVDT Feedback Calibration Procedure ............................................................... 292
  6.2.3 Valve Gains Adjustment Procedure ..................................................................... 296
  6.2.4 Results ................................................................................................................. 297
6.3 OUTER LOOP SETUP, PART I: PRELIMINARY ANALYSIS VIA
  DISPLACEMENT TRANSFER FUNCTION) ........................................................................ 299
6.3.1 Introduction 299
6.3.2 Displacement T.F. Via Frequency Sweep 301
6.3.3 Displacement T. F. Via Wide-Band Deterministic Excitation 305
6.3.4 Results 308
6.3.5 Frequency Sweep Versus Wide-Band Deterministic Excitation 325
6.3.6 Conclusions 329

6.4 OUTER LOOP SETUP, PART II: DETAILED GAIN SENSITIVITY ANALYSIS VIA ACCELERATION TRANSFER FUNCTION 330
6.4.1 Introduction 330
6.4.2 Proportional Gain 333
6.4.3 Integral Gain 339
6.4.4 Derivative Gain 344
6.4.5 Feed Forward Gain 350
6.4.6 Differential Pressure Gain 356

6.5 "OPTIMAL" GAIN SETTING 362
6.6 CONCLUSIONS 370

CHAPTER 7 EFFECT OF THE PAYLOAD ON THE SHAKING TABLE CALIBRATION ........................................ 371

7.1 INTRODUCTION 372
7.2 SENSITIVITY OF TABLE TRANSFER FUNCTION TO RIGID PAYLOAD 373
7.2.1 Introduction 373
7.2.2 Sensitivity to Rigid Payload 374
7.2.3 Gain Correction: Derivative Gain 381
7.2.4 Effect of Derivative Gain & 150 lbs Rigid Payload 384
7.2.5 Effect of Derivative Gain & 450 lbs Rigid Payload 390
7.2.6 Effect of Derivative Gain & 900 lbs Rigid Payload 396
7.2.7 Gain Correction: Delta-Pressure Gain 402
7.2.8 Gain Correction: Integral Gain 406
7.2.9 Sensitivity of Table to Rigid Payload - Conclusions 412

7.3 SENSITIVITY OF TABLE TRANSFER FUNCTION TO FLEXIBLE PAYLOAD 413
7.3.1 Introduction 413
7.3.2 Comparison of the Effects of a 450 lbs Versus 900 lbs Flexible Payload 415
7.3.3 Comparison of the Effects of a 450 lbs Rigid Versus Flexible Payload 421
7.3.4 Gain Corrections for the 450 lbs Flexible Payload 427
7.3.5 Comparison of the Effects of a 900 lbs Rigid Versus Flexible Payload 433
7.3.6 Gain Corrections for the 900 lbs Flexible Payload (Gain Increase) 438
7.3.7 Gain Corrections for the 900 lbs Flexible Payload (Gain Decrease) 444
7.3.8 Sensitivity of Shaking Table Transfer Function to Flexible Payload - Conclusions 450

7.4 CONCLUSIONS 452
CHAPTER 8  CORRELATION BETWEEN ANALYTICAL AND EXPERIMENTAL SHAKING TABLE BEHAVIOR 453

8.1 INTRODUCTION 454
8.2 KNOWN, VARIABLE AND UNKNOWN PARAMETERS 455
8.3 IDENTIFICATION OF UNKNOWN SERVO-HYDRAULIC PARAMETERS 457
8.4 FOREWORD TO THE CORRELATION STUDY BETWEEN ANALYTICAL AND EXPERIMENTAL TABLE TRANSFER FUNCTIONS 460
8.5 CORRELATION BETWEEN ANALYTICAL AND EXPERIMENTAL TABLE SENSITIVITY TO PROPORTIONAL GAIN 461
  8.5.1 Proportional Gain of to 1 Volt/Volt (Fig. 8.1) 461
  8.5.2 Proportional Gain of 2 Volt/Volt 464
  8.5.3 Proportional Gain of 3 Volt/Volt (Fig. 8.4) 467
8.6 CORRELATION BETWEEN ANALYTICAL AND EXPERIMENTAL TABLE SENSITIVITY TO INTEGRAL GAIN 469
  8.6.1 Integral Gain of 20 rps (Fig. 8.5) 469
  8.6.2 Integral Gain of 40 rps (Fig. 8.6) 471
8.7 CORRELATION BETWEEN ANALYTICAL AND EXPERIMENTAL TABLE SENSITIVITY TO DERIVATIVE GAIN 473
  8.7.1 Derivative Gain of 10 milliseconds (Fig. 8.7) 473
  8.7.2 Derivative Gain of 20 milliseconds 476
8.8 CORRELATION BETWEEN ANALYTICAL AND EXPERIMENTAL TABLE SENSITIVITY TO FEED-FORWARD GAIN 479
  8.8.1 Feed Forward Gain of 20 milliseconds 479
  8.8.2 Feed Forward Gain of 30 milliseconds 483
8.9 CORRELATION BETWEEN ANALYTICAL AND EXPERIMENTAL TABLE SENSITIVITY TO DELTA-PRESSURE GAIN 486
  8.9.1 Delta Pressure Gain of 1.5 Volts/Volt (Fig. 8.14) 486
  8.9.2 Delta Pressure Gain of 3.0 Volts/Volt (Fig. 8.15) 488
8.10 CORRELATION BETWEEN ANALYTICAL AND EXPERIMENTAL TABLE BEHAVIOR UNDER THE "OPTIMAL" GAIN SETTING 490
  8.10.1 "Optimal" Gain Setting 490
8.11 CORRELATION BETWEEN ANALYTICAL AND EXPERIMENTAL TABLE SENSITIVITY TO "RIGID" PAYLOAD (RIGID MODELING) 494
8.12 CORRELATION BETWEEN ANALYTICAL AND EXPERIMENTAL TABLE SENSITIVITY TO "RIGID" PAYLOAD (PARTIALLY FLEXIBLE MODELING) 501
8.13 CORRELATION BETWEEN ANALYTICAL AND EXPERIMENTAL TABLE SENSITIVITY TO DERIVATIVE GAIN FOR "RIGID" PAYLOAD (PARTIALLY FLEXIBLE MODELING) 509
  8.13.1 150 lbs "Rigid" Payload 509
  8.13.2 450 lbs "Rigid Payload" 512
  8.13.3 900 lbs "Rigid" Payload 517
  8.13.4 Remarks 518
CHAPTER 9  ANALYTICAL SIMULATION OF
SHAKING TABLE BEHAVIOR ........................................... 553

9.1  INTRODUCTION .............................. 554
9.2  SHAKING TABLE SENSITIVITY TO CONTROL GAIN PARAMETERS ...... 555
  9.2.1  Proportional Gain .......................... 556
  9.2.2  Integral Gain .................................. 558
  9.2.3  Feed-Forward Gain .......................... 560
  9.2.4  Delta-Pressure Gain ......................... 562
9.3  SHAKING TABLE SENSITIVITY TO THE NATURAL FREQUENCY OF
AN SDOF FLEXIBLE PAYLOAD ......................... 564
  9.3.1  SDOF Payload of 150 lbs .................... 565
  9.3.2  SDOF Payload of 300 lbs .................... 568
  9.3.3  SDOF Payload of 450 lbs .................... 571
  9.3.4  SDOF Payload of 600 lbs .................... 574
  9.3.5  SDOF Payload of 900 lbs .................... 577
  9.3.6  SDOF Payload of 1500 lbs ................... 580
9.4  CONCLUSIONS ............................. 583

CHAPTER 10  SHAKING TABLE ACTUAL PERFORMANCE .............. 584

10.1  INTRODUCTION ................................ 585
10.2  SHAKING TABLE PERFORMANCE ENVELOPE ............... 586
11.3.3 Static Characteristics of the Model 675
11.3.4 Dynamic Characteristics of the Model 677
11.3.5 Comparison Between Target and Actual Model Characteristics 681
11.3.6 Fundamental Considerations for Understanding the Experimental Test Procedure and Results 682
11.3.7 Testing Procedure 683
11.4 TEST RESULTS 685
11.4.1 Actual Maximum Rotations Observed in the Dynamic Testing and Corresponding Estimations Provided by the Proposed Simplified "α" Method 685
11.4.2 Other Experimental Results 695
11.4.3 Accuracy in Earthquake Reproduction 703
11.5 CONCLUSIONS 712

CHAPTER 12 CONCLUSIONS ................................. 713

12.1 SUMMARY OF WORK 714
12.2 SUMMARY OF FINDINGS 716
12.3 FUTURE RESEARCH WORK 717

APPENDIX A ....................................................... 718

A.1 THE FLEXIBLE FOUNDATION 718
A.2 LATERAL STIFFNESS 718
  A.2.1 Determination of the Lateral Stiffness of Steel Bars Kb6 719
  A.2.2 Lateral Stiffness of the I-beam/bolts System 720
A.3 MASSES 727
A.4 MODES OF VIBRATION 728
A.5 MODAL MASSES 729
A.6 DYNAMIC PROPERTIES OF THE SHAKING TABLE FOUNDATION:
  EXPERIMENTAL RESULTS 730
  A.6.1 Modes of Vibration 730
  A.6.2 First Mode of Vibration 733
  A.6.3 Vibrations Within each Concrete Block 735
  A.6.4 Modal Damping 740

APPENDIX B ....................................................... 742

B.1 THE THREE STORY STEEL MODEL BUILT AT RICE UNIVERSITY 742
B.2 ANALYTICAL MODELING OF THE STEEL STRUCTURE AS A SHEAR BUILDING 744
B.2.1 Dynamic Characteristics of the "450 lbs Model" 745
B.2.2 Dynamic Characteristics of the "900 lbs Model" 745

B.3 Analytical Modeling Obtained Through the CAL90
    Finite Element Program 747
    B.3.1 Dynamic Characteristics of the "450 lbs Model" 749
    B.3.2 Dynamic Characteristics of the "900 lbs Model" 749

B.4 Three Dimensional Analytical Modeling 750
    B.4.1 Dynamic Characteristics of the "450 lbs" Model 752
    B.4.2 Dynamic Characteristics of the "900 lbs" Model 753
    B.4.3 Reduced System for Table Transfer Function Simulation 754
    B.4.4 Dynamic Characteristics of the Reduced "450 lbs" Model 755
    B.4.5 Dynamic Characteristics of the Reduced "900 lbs" Model 757

B.5 Experimental Dynamic Characteristics of the "450 lbs" Model 759
    B.5.1 First Lateral Mode of Vibration 762
    B.5.2 Second Lateral Mode of Vibration 765
    B.5.3 Third Lateral Mode of Vibration 768
    B.5.4 Correlation Between Experimental Results and Analytical Modeling.
        for the "450 lbs" Model 771

B.6 Experimental Dynamic Characteristics of the "900 lbs" Model 772
    B.6.1 First Mode of Vibration 775
    B.6.2 Second Mode of Vibration 778
    B.6.3 Third Mode of Vibration 782
    B.6.4 Correlation Between Experimental Results and Analytical Modeling.
        for the "900 lbs" Structure 785
    B.6.5 Coherency 786

APPENDIX C ................................................................. 788

C.1 The "Suny Buffalo" Analytical Model 788
C.2 The so-called "Reduced Suny Buffalo Model" 789

APPENDIX D ................................................................. 792

D.1 Laplace Transform 792
D.2 Differentiation 792
D.3 Integration 793
D.4 Time Delay 793
APPENDIX E ................................................................. 795
E.1 MODAL MASS ..................................................... 795
E.2 MODAL MASS OF THE FLEXIBLE TABLE FOUNDATION (BASE) 796
E.3 SHAKING TABLE TRANSFER FUNCTION SENSITIVITY TO BASE MASS 797

APPENDIX F ................................................................. 800
F.1 THE "RIGID" PAYLOAD .............................................. 800
F.2 ANALYTICAL MODELING OF THE LATERAL STIFFNESS OF THE "RIGID" PAYLOAD ..................................................... 801
F.2.1 Dynamic Characteristics of the "Rigid" Payload 802

ABREVIATIONS AND SYMBOLS ........................................... 807
A&S1 SYMBOLS VALID FOR SHAKING TABLE CHARACTERISTICS - ALL CHAPTERS 807
A&S2 SPECIAL SYMBOLS USED IN SPECTRAL ESTIMATION
  SECTIONS 5.2 AND 5.3 ................................................. 812
A&S3 SPECIAL SYMBOLS USED FOR ARMA-MODELS - SECTION 5.4 815
A&S4 SPECIAL SYMBOLS USED FOR THE SIMULATION OF THE ERRORS IN
  TABLE TRANSFER FUNCTION ESTIMATION - SECTION 5.5 816

BIBLIOGRAPHY ............................................................. 817
PART I:

ANALYTICAL DEVELOPMENTS
CHAPTER 1
INTRODUCTION
1.1 BACKGROUND

Rice University's Department of Civil Engineering recently added a state-of-the-art electro-hydraulic shaking table facility to its structural laboratory. The purpose of the shaking table is to simulate experimentally earthquake ground motions in order to study their effects on reduced scale models of buildings, towers, vibration sensitive equipments, etc. Two years of planning and development went into designing and sizing the various components of the shaking table facility. The scope of this preliminary work was to design a table able to reproduce accurately target table acceleration time histories for specific test specimen. After the assembly of the table was completed in June 1995, the process of understanding the true dynamic behavior and tuning of the shaking table started and is the main objective of this thesis.

1.2 PROBLEM DESCRIPTION AND OBJECTIVES

It is known that the reproduction through shaking table of commanded dynamic signals (e.g. target ground motions) is imperfect (Rinawi and Clough, 1991). The degree of distortion in signal reproduction depends on numerous factors such as physical system parameters (foundation flexibility, compressibility of oil column in the actuator chamber, etc.), type of electro-hydraulic control system (type of feedback signals, control gain setting, etc.), and dynamic characteristics of the test specimen. In the case of a newly assembled shaking table (as is the case at Rice University), a mere evaluation of the accuracy of the table in motion reproduction is not sufficient. In fact, it is necessary to conduct a complete study of the table sensitivity to control gains in order to determine the optimal control gain
settings which maximize the table accuracy in acceleration reproduction under different payload conditions. Furthermore, an evaluation of the table sensitivity to the payload dynamic characteristics must be performed in order to interpret properly the results obtained through shaking table tests. A complete and reliable understanding of the shaking table dynamics can be obtained only through a correlation study between analytical and experimental results.

1.3 Scope of Work

The scope of this thesis is to gain a complete understanding of the dynamic behavior of the shaking table built at Rice University and to show that it can be used effectively for structural dynamic testing.

In order to achieve this objective, the research work was subdivided into four parts. The first part involves the development of an analytical dynamic model of the shaking table. The second part is concerned with the development and numerical validation of a procedure to estimate accurately the actual transfer function of the shaking table. The third part consists of a correlation study between experimental and analytically predicted shaking table behavior and its sensitivity to both table control parameters and payload dynamic characteristics. Finally, the fourth and last part, describes and presents the results of the structural dynamics test performed dealing with lateral-torsional coupling of seismic isolated building structures.
1.4 Organization of Text

This thesis consists of ten main sections. Chapter 2 briefly describes the shaking table components and their technical characteristics. The analytical dynamic model of the shaking table dynamic is developed in Chapter 3 and accounts for proportional, integral, derivative, feedforward, differential pressure controls gains, time delay in the servovalve response, compressibility of the actuator fluid, flexibility of the foundation mass, and dynamic characteristics of the payload. In Chapter 4, the analytical model of the shaking table is used to qualitatively determine the sensitivity of the analytical table transfer function to control parameters and payload characteristics. Chapter 5 describes the method used to evaluate experimentally, via a stochastic approach, the actual table transfer function. Furthermore, this chapter presents the results of several numerical simulations used to evaluate the effectiveness of the selected method for transfer function estimation. Chapter 6 presents the experimental procedure that was followed to obtain a "near optimal" tune-up of the shaking table. Chapter 7 shows how the actual shaking table transfer function is affected by the various dynamic payload characteristics. Chapter 8 reports the results of a complete correlation study between experimental and analytical table transfer functions. Chapter 9 uses the calibrated analytical shaking table model to investigate the table transfer function sensitivity to both control gain parameters and payload characteristics. Chapter 10 compares the target shaking table performance used as design criteria to the actual one. Finally, Chapter 11 reports on the structural dynamics experiment performed using the Rice shaking table.
CHAPTER 2
DESCRIPTION OF THE SHAKEING TABLE FACILITY
2.1 INTRODUCTION

The electro-hydraulic shaking table system built at Rice University is composed of several components that are required to work smoothly and effectively together. The shaking table is unidirectional and designed to work with structural models of 1/5 scale or smaller.

In Section 2.2, the various components of the shaking table facility will be described. Section 2.3 gives a description of the functioning of the electro-hydraulic system control loops which allows a precise reproduction of the desired table motion. Section 2.4 will describe the data acquisition system used to monitor the motion of the table and to acquire significant response information about the structural model tested.

![Shaking Table Components Diagram]

Figure 2.1 Shaking Table Components
2.2 Shaking Table Components

The Rice University Shaking Table is an electro-hydraulic system, meaning that electrical signals control and regulate the hydraulic fluid at high pressure that moves the table. The major components of the shaking table facility are:

- Slip table, rails and reaction mass (Sections 2.2.1, 2.2.2, and 2.2.3);
- Hydraulic power system (Section 2.2.4);
- Servovalve and actuator (Section 2.2.5);
- Controller (Section 2.2.6);
- Personal computer and amplifier (sections 2.2.7 and 2.2.8).

2.2.1 Slip Table

The slip table is a 5 ft by 5 ft, 3 inches thick aluminum plate, with rails centered at 12 inches from the edges. The aluminum plate weighs 1,270 lbs in its final form. Aluminum was selected because of its favorable weight-to-stiffness ratio (it was preferred to magnesium, that has an even better weight-to-stiffness ratio, because of its better safety and workability. A dynamic analysis of the slip table performed with SAP90 indicated that the first mode of vibration occurs at 205.8 Hz, the second at 244.8 Hz, and the third at 291.6 Hz. Given that the frequency of operation of the shaking table is in the range between 0 and 75 Hz, the lowest natural frequency of the slip table is at about three times the highest operational frequency. This guarantees that there will be no significant interaction between the vibrations of the table and those of the structural models tested on the shaking table.
2.2.2 Rails and Steel Base Plate

The rails (Schneeberger MRB 45) are stainless steel monorails sliding on four trucks connected rigidly to the slip table. The total force necessary to move slowly the slip table - weighting 1270 lbs - has been measured to be 34 lbs which gives a coefficient of friction of 0.0283. The rails are mounted to a steel base plate to assure their precise placement. The steel base plate measures 5' by 10' is 2'' thick and weighs approximately 4000 lbs.

2.2.3 Foundation Mass

The foundation mass (also referred to as reaction mass or base) consists of three 12 ft by 12 ft by 1 ft thick reinforced concrete slabs (each one weighting around 21,600 lbs). These three slabs are post-tensioned together through 6 Dywiday "Threadbars" of 3/4 inch diameter. Each of these bars was post-tensioned to a force of 20,000 lbs. The three slabs were then connected to the strong floor of the structural laboratory at Rice University through a grid of wide flange sections (W8x35 and W8x18) welded together.

2.2.4 Hydraulic Power Supply

The pump selected is an MTS model 510.30 that is capable of providing 30 gallons per minute of hydraulic fluid flow at both low and high pressure, 150 psi (1 Mpa) and 3000 psi (21 Mpa), respectively.
2.2.5 Servovalve and Actuator

**Servovalve**
The MTS 256 servovalve is a three-stage servovalve functioning as the final element in the control loop of the servo-hydraulic system. It regulates the rate and direction of the hydraulic fluid flow to the actuator by reacting to the polarity and magnitude of the electrical input signal. The third stage of the servovalve is the main stage which comprises a large four-way spool controlling the flow to the actuator chamber. The position of the third stage spool is measured by a linear variable differential transformer (LVDT) which provides this precious information to the shaking table controller. The maximum operating pressure and standard operating pressure are both at 3,000 psi (21 Mpa). The first stage spool moves at a given frequency (called the dither frequency) to prevent the stick-slip motion of the spool. For the system acquired, the dither frequency has been set at 500 Hz. Contrary to the frequency, which is fixed, the amplitude of this anti-stick motion can be regulated through the controller. The manufacturer of the servovalve (M.T.S.) provided the following weight information for the servovalve components:

- **Servovalve (M.T.S. 256-18):** 12.0 lbs;
- **Accumulators (two M.T.S. 111.12c-03):** 48.0 lbs each.

**Actuator**
The hydraulic actuator consists of the piston rod, high-pressure fluid ports, cushions and swivel-end connections. The piston rod is made of heat treated steel alloy and is hard-chrome plated. It is equipped with a core mounted linear variable differential transducer
(LVDT) that provides information to the controller on the actuator position. The
diameter of the piston rod is 2.75 in and the effective area of the actuator chamber is
12.73 in$^2$, giving a theoretical maximum applicable force of 38.19 Kips. In reality the
actuator is guaranteed to work up to 35 kips. The maximum dynamic stroke of the
actuator is +/- 3 in, while under static conditions the maximum stroke is +/- 3.75 in. The
actuator chamber is equipped with a cushion of 0.25 in at both ends. Taking into account
the maximum dynamic stroke and the presence of the two cushions, the internal volume
of the actuator chamber can be estimated to about 101.84 in$^3$. The manufacturer of the
servovalve (M.T.S.) provided the following weight information for the actuator
components:

- Actuator (M.T.S. 244.23): 153.0 lbs;
- Pedestal Base: 27 lbs;
- Base Swivel: 66 lbs;
- Rod-end Swivel: 60 lbs.

2.2.6 MTS 407 Controller

The MTS 407 Control employs a proportional-integral-derivative-feedforward (PIDF)
displacement control algorithm. The controller is the key element of the whole system as
it is the device that provides the right voltage to the first stage of the servovalve (pilot
stage) in order to obtain a specified position of the actuator-arm/table. For more details,
the reader is referred to Section 2.3.2 devoted to the 407 controller.
2.2.7 Personal Computer

The personal computer (Gateway Pentium 2000) is the driving element of the chain as it sends to the controller, through its output D/A board, the desired displacement time history to be realized on the shaking table. The personal computer is also used to acquire data, through its input A/D boards, on the displacement and acceleration of the shaking table and/or test models for analysis of experiments.

2.2.8 Amplifier

The amplifier can be used to upscale the PC output in order to make complete use of the actuator capacity: the controller orders a full range motion of the actuator arm when it senses an input of +/- 10 Volts. The computer output boards (see Section 2.4.3) have a maximum output voltage of 2.828 Volts. In order to be able to reproduce a full range motion of the actuator-arm/table, the amplifier must multiply the computer output by a factor 4. The precision of the amplifier is +/- 10 milli-Volts.
2.3 Electro-Hydraulic Control System

The goal of the shaking table is to accurately reproduce a specified motion (acceleration, velocity and displacement) on the slip table. Since the servo-hydraulic system is a real physical system, the relationship between input (commanded motion to be reproduced) and output (reproduced motion) is non-linear. Physical phenomena, such as leakage and compressibility of the hydraulic fluid in the actuator chamber, cause these non-linearities in the servovalve and actuator response. Furthermore, the response of the model attached to the table is unpredictable and can be non-linear (in testing models in their inelastic range leading to the application of an unpredictable variable force on the actuator. As shown in Fig. 2.1, the process of reproducing a target motion on the table is a long chain of operations starting from the electrical command, $x_c$, sent by the computer to the controller and ending at the actual table motion $x_t$.

2.3.1 Flow of the Electrical Control Signal From PC to 407 Controller

The personal computer can store different displacement time histories that we may want to reproduce on the shaking table at different times. The high capacities of the adopted computer allow to transform these digital time histories into analog voltages\(^1\) with a sampling frequency of 4000 Hz or above. This allows to reproduce (theoretically) motions with frequency content of 2000 Hz or higher. This frequency limit is well above the highest significant frequency contained in earthquakes ground motions which can be estimated to be between 10 and 20 Hz. Even considering a typical 1/5 scale model.

---

\(^1\) Via a Digital to Analog Converter - DA Board.
structure tested on the shaking table, the highest significant excitation frequency lies in the range between 50 and 100 Hz scale (to satisfy the similitude requirements, the frequency scaling factor is given by the reciprocal of the geometric scaling factor). This shows that the computer characteristics are able to reproduce digitized motions that are far more accurate than the seismic engineering need. If the amplitude of the motion to be reproduced corresponds to a voltage that exceeds the output capacity of the D/A output board, the computer output (in the range +/-2.828 Volts) can be passed through the amplifier (with a magnification factor of 4) before reaching the controller, as shown in Fig. 2.2.

![Diagram](image)

**Figure 2.2**  Connections Between Personal Computer and MTS 407 Controller
2.3.2 The 407 Controller

The controller reads the voltage sent by the computer (called external input $\bar{x}_c$) in terms of displacements to be reproduced, according to the following two optional settings:

Setting 1: +/- 10 Volts of external input correspond to a desired displacement of the actuator of +/- 5 inches, with all intermediate values in the appropriate linear scale.

Setting 2: +/- 10 Volts of external input correspond to a desired displacement of the actuator of +/- 2.5 inches, with all the intermediate values in the appropriate linear scale.

As shown in Fig. 2.3, the controller works as follows:

Using the following signals as input data:

- the displacement to be reproduced, $\bar{x}_c(t)$.
- the signal $x_3(t)$ provided by the so-called “AC1 conditioner”. and
- the signal $x_{cell}(t)$ provided by the so-called “DC1 conditioner”.

the controller determines, through a series of real time computations called outer control loop, the desired valve opening signal $x_c(t)$ (i.e., voltage to send to the servovalve pilot spool).

At this point, the controller compares:

- the desired valve opening $x_c(t)$ with
- the signal $x_{3sa}(t)$ provided by the inner loop feedback conditioner
and adjusts the voltage through a series of real time computations called the *inner control loop* in order to obtain the inner-loop-corrected servovalve command $x_{ci}(t)$.

Note that:

- $x_a(t)$ is obtained through the conditioning of the signal representing the actuator arm relative displacement $x_i(t)$ and provided by the actuator mounted LVDT.

- $x_{cell}(t)$ is obtained through the conditioning of the signal representing the differential pressure $\Delta P(t)$ across the actuator piston, provided by the actuator mounted delta Pressure load cell.

- $x_{3sa}(t)$ is obtained through the conditioning of the signal representing the actual position of the third stage spool of the servovalve, provided by the servovalve mounted LVDT.
Figure 2.3 Inner and Outer Control Loops
2.3.3 The Inner Control Loop

The inner control loop consists of the series of computations that the controller performs in order to ensure that the motion of the servovalve is as commanded. The input is the voltage sent to the coil at the pilot (first-) stage of the servovalve, and the feedback is the actual measured position of the third-stage spool, the one that regulates the flow of high pressure fluid in the actuator chamber. As shown in Fig. 2.4, the inner control loop uses only Proportional and Derivative controls.

The inner loop also creates a special signal called dither signal, \( x_d(t) \) which consists of a sine wave at a frequency of approximately 500 Hz and controllable amplitude through the so-called dither motion control Gain. The dither signal \( x_d(t) \) is added to the servovalve command signal \( x_{c1}(t) \) in order to obtain the electric signal \( x_{c1d}(t) \) sent to the coil in the first-stage of the servovalve. As previously described, the dither signal is introduced in order to keep the electric spool in motion and prevent it from sticking.

The inner loop has adjustment capacities for the feedback signal in order to balance and scale it. The feedback conditioning is carried out through the following controls:

- Feedback gain to adjust the amplitude of the feedback signal in order to have the same scale as that for the input signal (±10 Volts corresponding to a full swing).

- Two levels of zero adjustment (Coarse and Fine) to adjust the zero feedback signal of the third stage spool in order to have zero feedback when the spool is at centered position.

- A phase adjustment to adjust the phase of the feedback demodulator in order to
capture the real maximum amplitude of the feedback signal.

- A feedback polarity option (normal or reversed) to adjust the polarity of the feedback. e.g. if the valve is too opened, it is necessary to send a signal with the right polarity in order to close it, and vice versa.

- A valve command polarity option (same as for the feedback polarity).
Figure 2.4 Inner Loop Diagram
2.3.4 The Outer Control Loop

The outer loop control system ensures that the motion of the actuator arm is as commanded. This control loop uses as output the voltage sent to the coil at the pilot stage of the servovalve and as feedback signal the actual measured position of the actuator arm. Using the actuator arm position as feedback signal allows to minimize the error between commanded and realized displacement time histories. Unfortunately, the acceleration being the second derivative of the displacement, a small error in displacement control can produce a large error in acceleration which is the quantity we ultimately want to have a good control on, in experimental earthquake engineering. This results in a table control loop that is not optimal for the purpose of the shaking table. A control loop that uses the acceleration of the actuator arm as feedback signal would have been more appropriate, but this type of hardware is much more expensive and typical of much larger and more sophisticated shaking table facilities. Nonetheless, the various type of control that the controller is able to perform in the outer loop are allow satisfactory control on the acceleration time history to be reproduced on the shaking table. In fact, the outer control loop uses Proportional-Integral-Derivative (PID), Feed Forward (FF) and Differential Pressure (dP) Controls as shown in Fig. 2.5.

The PID Control

PID stands for Proportional, Derivative, Integral. The PID controls use as input the difference between the reference (target) value and the output (measured) value, called the error signal $e(t)$. Proportional control can reduce error responses to external disturbances and increases the speed of response, but it allows a non-zero steady state error and it produces
a large transient overshoot. By introducing in the control algorithm a term proportional to the integral of the error, the steady state error can be eliminated, although this is realized at the expense of deterioration in the dynamic response. Addition in the control algorithm of a term proportional to the derivative of the error can damp the dynamic response. Combining these three types of control form the classical PID controller that has found wide acceptance in common control practice. Each control has an adjustable gain constant (voltage of output per voltage of input), and the PID combination is often able to provide an acceptable degree of error reduction simultaneously with acceptable stability and damping.

**FF Control**

FF control basically multiplies the derivative of the original input (commanded position of actuator arm, independently of the error value) by its gain and add the result to the conditioned input.

**dP Control**

The dP (Delta Pressure) control is much more involved. It uses as input the differential pressure existing between the two opposite faces of the actuator piston. This differential pressure is directly related to the forces applied by the actuator. This force is related both to the acceleration that the actuator is trying to impart to the table and to the dynamic response of the test structure which affects the oil pressure in the actuator chamber. This control is important to counteract forces generated by test model and to anticipate the force that will have to be generated by the actuator.
Feedback Conditioning

The feedback conditioning is carried out through the same controls as for the inner loop. namely:

- Feedback gain to adjust the amplitude of the feedback signal in order to have the same scale as that for the input signal (+/- 10 Volts corresponding to a full swing).

- Two levels of zero adjustment (Coarse and Fine) to adjust the zero feedback signal of the third stage spool in order to have zero feedback when the spool is at centered position.

- A phase adjustment to adjust the phase of the feedback demodulator in order to capture the real maximum amplitude of the feedback signal.

- A feedback polarity option (normal or reversed) to adjust the polarity of the feedback. e.g. if the valve is too opened, it is necessary to send a signal with the right polarity in order to close it, and vice versa.

- A valve command polarity option (same as above).
Figure 2.5  Outer Loop Diagram
2.3.5 The "Batch" Control Loop

If the controller settings are at the optimal values (see Section 6.5 for the so-called "optimal" value of the controller gain settings), the actual motion reproduced by the shaking table should be close to the one specified by the computer. If the level of accuracy of motion reproduction needs to be very high and the controller settings (P, I, D, FF, and dP gains) are unable to reach the desired level of accuracy, the system can be improved by modifying the input, $\vec{x}_c$, given by the computer. In fact, due to the non-linearity of this electro-hydraulic system, the motion of the reaction mass and the magnitude of the forces (that can be considerable for large models) applied by the test model to the actuator, the actual controlled motion of the table, $x_{ta}$, is expected to differ from the commanded one $\vec{x}_c$. This difference is amplified when comparing the desired table accelerations time histories, $\vec{x}_c$, with the actual table absolute acceleration time histories, $x_{ta}$, (see Fig. 2.6). In fact, a small error in displacement reproduction can be amplified into a large error in acceleration reproduction. If the tuning of the controller gains is not able to reduce this difference to an acceptable level, a solution can be provided by introducing a specific change in the original command signal sent by the computer, $\vec{x}_c$. The original signal can be conditioned in the time or frequency domain using various algorithms. In this way, a new (batch) level of control is installed. The information provided by the recorded displacement and acceleration time histories of the table are used to prepare a new, conditioned input $\vec{x}_{cc}$ to be sent to the controller. This operation can be performed iteratively until the desired degree of accuracy in table
motion is achieved. 

Figure 2.6 The "Batch" Control Loop

1. This type of "batch" control of the table has not been fully implemented and can be the subject of future research.
2.4 DATA ACQUISITION SYSTEM

External acquisition and processing of data such as relative displacement of actuator arm with respect to foundation mass, absolute acceleration of shaking table, relative displacement and acceleration response of test model are handled by a Gateway 2000 Pentium Personal Computer equipped with data acquisition and signal processing boards. This allows also for external signal generation. Acceleration response of the table and of test model are acquired via accelerometers, relative displacement of test model are acquired via linear potentiometers and table relative displacement response is acquired through the actuator mounted LVDT.

2.4.1 Accelerometers

The accelerometers used are Low Impedance Voltage Mode Accelerometers (LIVM) model 3127A made by Dytran. They contain integral impedance, converting amplifiers and have a sensitivity of 100 milli-Volts/g +/- 2% in the frequency range between 1Hz and 3000 Hz. The F.S. range is +/- 50 g. These accelerometers utilize a two plate quartz element with central pre-load, coupled to a miniature IC-Mosfet input, unity gain amplifier. The amplifier converts the high impedance open circuit voltage generated by the quartz crystals to a low impedance (< 100 Ohms) voltage signal that is coupled to the readout units (oscilloscope, spectrum analyzer and computer A/D-D/A converters) through D.C. (direct Current) blocking capacitors (10 MF usually) which separate the signal from the (approximately) +11 Volts bias existing at the central terminal of the accelerometer. These capacitors are not effective for signals at frequencies below 1 Hz.
Thus, these accelerometers do not provide a reliable response to acceleration at low frequencies, since their response is biased by a DC component that disappears slowly with time. This particular behavior leads to the need to filter out of the accelerometer response all the frequency components below 1 Hz. The accelerometers are screwed to specific mounts which are glued to the table (or test model). As stated by Dytran (manufacturer of the accelerometers), many types of adhesives can be used. We choose the Loctite 495 Superbonder in order to have a glue with a high elastic modulus, with the goal to minimize any possible action of the adhesive as an elastic medium that could false the response record.

2.4.2 Actuator LVDT

The LVDT mounted in the actuator arm provides information on the piston rod displacement. The LVDT is an electro-mechanical device that provides an output voltage which is proportional to the displacement of a movable core extension. The core extension is attached to a core mount that is secured inside the piston. The core extension is axially oriented in the LVDT coil.

2.4.3 Dynamic Signal Acquisition Boards

The Gateway 2000 P5-90 computer is equipped with two National Instrument AT-A2150 and AT-DSP2200 data acquisition plug in boards. The data acquisition characteristics of these two boards are identical. In all the experimental work described in this thesis, only the AT-A2150 boards were used to acquire data (the AT-DSP2200 board being used mainly to send the input signal to the 407 Controller). Thus, for
simplicity purposes, in the following we describe the characteristics of both data acquisition boards referring only to the AT-A2150 board. These boards digitize signals over a bandwidth from DC to 20 KHz, each one has four 16-bit analog input channels with anti-aliasing filters and 64 times oversampling delta-sigma modulating A/D converters. The analog input has software programmable AC/DC coupling and can digitize signals within a voltage range of +/- 2.828 Volts. The AT-A2150 board has both analog and real time digital filters implemented in the hardware to prevent aliasing. Input signals are first passed through analog filters to remove any signals with frequency components beyond the frequency range of the AT-A2150 board. Then, digital anti-aliasing filters automatically adjust their cutoff frequency to remove any frequency components above half the programmed sampling rate. Because of this advanced analog input design, it was not necessary to add any filters to prevent aliasing. The board has a 93 dB SNR (Signal-to-Noise-Ratio) that makes it possible to acquire wave-forms without any significant noise error. Because the board has a THD (Total Harmonic Distortion) of only -95 dB, the digitized signals are true representations of the analog signals applied to the input of the board. The AT-A2150 board has +/-0.025 dB of amplitude flatness from DC to 20 KHz. The low noise and low distortion of the AT-A2150 board are achieved by using 64 times oversampling and delta sigma modulating ADC’s. These ADC’s sample at 64 times the specified sampling rate with 1 bit resolution. A noise shaping technique then shifts quantization noise to higher frequencies. Extremely flat, linear phase low pass filters then remove this noise, divide the sample rate by 64, and increase the resolution to 16 bits. Using the delta sigma
modulating A/d converters, the AT-A2150 board is immune to the DNL (Differential Non Linearity) distortion associated with conventional data acquisition boards. The board resolution is of 16 bits corresponding to 1 in 65,536. Given a voltage range of +/- 2.828 Volt, the resolution is of 0.086303 milli-Volts.

2.4.4 Data Acquisition Software

The software used for data acquisition (and reduction) is Labview that is provided with the LabVIEW GPIB, VXI RS-232, and Data Acquisition VI Libraries that call the standard National Instrument DLL's (Dynamic Link Library) and Windows NT device drivers to acquire data from the computer boards. Labview programs are called Virtual instruments (VI's) because their appearance and operation imitate actual instruments.
2.5 CONCLUSIONS

In this chapter we described all the elements that concur to the functioning of the shaking table facility built at Rice University. All this basic information\(^1\) (dimensions, weights, functioning characteristics) will be used in the next Sections in order to develop a mathematical model of the shaking table.

---

\(^1\) For more detailed information about the shaking table components see Matt Muhlenkamp's M.S. thesis (1996).
CHAPTER 3
ANALYTICAL MODELING OF SHAKING TABLE
3.1 Introduction

In this chapter it will be developed a mathematical model of the dynamic behaviors of the shaking table (paragraphs 3.2 to 3.7). The objective of this mathematical model is to obtain a numerical simulation of the table transfer function. Once obtained an expression for the table transfer function we will perform (paragraphs 4.2 to 4.4) a complete study to understand how the table transfer function is affected by the different values of the gain parameters (that can be controlled through the table controller) and by the different type of payload.

To develop a mathematical model for the behaviors of the shaking table it will be followed the path depicted in Figure 3.1:

1) First it will be developed a mathematical model for the system composed by the three stage servovalve and the actuator, this leads to the determination of the so called “Servovalve Transfer Function $S(s)$”.

2) Then it will be developed a mathematical model for the controller that involves the determination of an expression for the so called “Force in the Actuator $F_a(s)$”. The coupling of the servovalve model $S(s)$ with the controller model give origin to the so called “Hydraulic System Transfer Function $H(s)$”;

3) Then it is taken into consideration the effect due to the flexibility of the foundation mass. Its behaviors are modeled through the so called “Base Transfer Function $B(s)$” and at this point it is defined the “Shaking Table Transfer Function $T(s)$”.

4) At last it is taken into consideration the presence on the table of a flexible payload (both a SDOF and a MDOF) which dynamic behaviors are modeled through the so called “Pay-
load Transfer Function $H_p(s)$. $H_p(s)$ is used to determine the shear force that the payload transmits to the slip table (referred as the “Base Shear $F_s(s)$”) and used to modify the expression of $S(s)$, $H(s)$, $B(s)$, and $T(s)$ in order to take into account the presence of the payload.

Figure 3.1 Analytical modeling of the Shaking Table
3.2 The Servovalve and Actuator Model

In Figure 3.3 (for more details see also Chapter 2) it is schematically represented the functioning of the Servovalve. The servovalve command $x_c$ is sent to the torque motor armature that controls the rotation of the pilot flapper. The rotation of the pilot flapper generates a differential pressure in the pilot stage ($\Delta P_p$) that controls the flow of hydraulic fluid in the second stage. The flow of hydraulic fluid in the second stage controls the position of the second stage spool that in turn controls the flow of hydraulic fluid in the third stage and the position of the third stage spool $x_{3s}$. Finally the position of the third stage spool ($x_{3s}$) controls the flow of the high pressure hydraulic fluid in the actuator load chamber $q_s$. A mathematical model for this system will be derived in the following of this paragraph.

3.2.1 Three Stage Servovalve Transfer Function

*From Servovalve Command* $x_c(t)$, *to Inner Loop Conditioned Servovalve Command* $x_{ci}(t)$

The servovalve command signal $x_c(t)$ before it is sent to the electric coil that moves the flapper (that is the first stage of the three stave servovalve), is processed by the controller through the so called "Inner Loop". The functioning of the "Inner Loop" is schematically illustrated in Figure 3.2\(^1\).

The Inner Loop control uses as input the so called "inner loop" error signal $e_i(t)$ that is obtained subtracting the magnitude of the inner loop feedback signal (conditioned feed-

---

\(^1\) In the diagram represented in Fig. 3.2 it has been neglected the signal corresponding to the dither motion. This was possible because of the high frequency at which the dither motion operates that, by definition, is out of the frequency range of work of the servovalve and actuator.
back signal, \( x_{3sa}(t) \)) from the servovalve command signal, as described by the equation:

\[
e_i(t) = x_c(t) - x_{3sa}(t)
\]  
(3.1)

or, in Laplace notation\(^1\):

\[
e_i(s) = x_c(s) - x_{3sa}(s)
\]  
(3.2)

where:

- \( e_i(t) \) is the "inner loop" error signal:
- \( x_c(t) \) is the servovalve command signal:
- \( x_{3sa}(t) \) is the conditioned feedback signal.

The inner loop feedback signal, \( x_{3sa}(t) \), is obtained from the conditioning of the electric signal representing the position of the third stage spool of the servovalve \( x_{3s}(t) \). The electric signal representing the position of the third stage spool, \( x_{3s}(t) \), is obtained from a linear variable differential transformer (LVD T) mounted on the third stage spool itself that gives a signal that is proportional to the actual position of spool. The inner loop feedback conditioner condition the feedback signal, in Laplace notation, as follows:

\[
x_{3sa}(s) = A_i(s) \cdot x_{3s}(s)
\]  
(3.3)

where:

- \( A_i(s) \) is the so called "inner loop" feedback transfer function (conditioner), and represent the effect of the inner loop conditioning upon the feedback signal representing

---

\(^1\) See Laplace notation: Appendix D.
the position of the third stage spool of the servovalve. \( A_1(s) = \frac{x_{3s}(s)}{x_{3s}(s)} \).

Substituting Eq. (3.3) in Eq. (3.2), we obtain a new expression for the inner loop error signal:

\[
e_i(s) = x_c(s) - (A_i(s) \cdot x_{3s}(s))
\]  

(3.4)

The 407 Controller in use at Rice University has been preset in order to give a conditioning equal to unity. therefore:

\[
x_{3sa}(s) = x_{3s}(s)
\]  

(3.5)

\[
A_i(s) = 1
\]  

(3.6)

As explained in Chapter 2 and shown in Figure 3.2 the inner loop processes the "inner loop" error signal, \( e_i(s) \), with a Proportional and Derivative computation. The output signal of the inner loop, \( x_{ci}(t) \), is obtained through the sum of two contributions, as expressed in the following equation:

\[
x_{ci}(t) = e_{pi}(e_i(t)) + e_{di}(e_i(t))
\]  

(3.7)

or, in Laplace notation:

\[
x_{ci}(s) = e_{pi}(e_i(s)) + e_{di}(e_i(s))
\]  

(3.8)
where:

- \( x_{cl}(t) \) is the inner loop output signal (electric current to be sent to the flapper coil);
- \( e_i(t) \) is the inner loop error signal;
- \( \varepsilon_{p_i}(e_i(t)) \) is the inner loop proportional gain electrical component;
- \( \varepsilon_{D_i}(e_i(t)) \) is the inner loop derivative gain electrical component.

The values of the electrical components of the inner loop output signal are determined as follows:

The proportional gain electrical component, \( \varepsilon_{p_i}(e_i(t)) \), is obtained by multiplying the error by the inner loop proportional constant \( K_{pro}^i \), as expressed by:

\[
\varepsilon_{p_i}(e_i(t)) = K_{pro}^i \cdot e_i(t) \tag{3.9}
\]

or, in Laplace notation:

\[
\varepsilon_{p_i}(e_i(s)) = K_{pro}^i \cdot e_i(s) \tag{3.10}
\]

The derivative gain electrical component, \( \varepsilon_{D_i}(e_i(t)) \), is obtained by multiplying the error by the inner loop derivative constant \( K_{der}^i \), as expressed by:

\[
\varepsilon_{D_i}(e_i(t)) = K_{der}^i \cdot \frac{d}{dt}e_i(t) \tag{3.11}
\]

or, in Laplace notation:

\[
\varepsilon_{D_i}(e_i(s)) = s \cdot K_{der}^i \cdot e_i(s) \tag{3.12}
\]

Substituting the expression for the proportional gain electrical component, Eq. (3.10), and
the expression for the derivative gain electrical component. Eq. (3.12). into the equation for the output signal of the inner loop. Eq. (3.8). we obtain:

\[ x_{ci}(s) = K^i_{pro} \cdot e_i(s) + s \cdot K^i_{der} \cdot e_i(s) \]  

(3.13)

where:

- \( K^i_{pro} \) is the inner loop proportional gain constant:

- \( K^i_{der} \) is the inner loop derivative gain constant.

substituting the expression for the “inner loop” error signal \( e_i(t) \) - Eq. (3.4) - we obtain:

\[ x_{ci}(s) = (K^i_{pro} + s \cdot K^i_{der}) \cdot [x_c(s) - (A_i(s) \cdot x_3(s))] \]  

(3.14)
Figure 3.2 Inner Loop diagram
From Conditioned Servo valve Command, \( x_{c1}(t) \) to the third Stage spool Position, \( x_{3s}(t) \)

The relationship between the conditioned servo valve command, \( x_{c1}(t) \) and the pilot stage differential pressure can be assumed to be linear and expressed as follows:

\[
\Delta P_p(t) = k_1 \cdot x_{c1}(t)
\]  \hspace{1cm} (3.15)

where:

* \( k_1 \) is the flapper gain;
* \( x_{c1}(t) \) is the servo valve electrical signal;
* \( \Delta P_p(t) \) is the differential pressure in the pilot spool.

Following the derivation given by Tina and Claugh, it can also be assumed a linear relationship between pressure drop induced across the pilot stage spool and the displacement of the main stage spool. This relationship can be expressed as follows:

\[
x_{3s}(t) = k_2 \cdot \Delta P_p(t)
\]  \hspace{1cm} (3.16)

where:

* \( x_{3s}(t) \) is the third (main) stage spool displacement;
* \( \Delta P_p(t) \) is the pressure drop across pilot stage spool;
* \( k_2 \) is the second stage gain factor.

Combining Eq. (3.15) and Eq. (3.16) we obtain a linear relationship between the conditioned servo valve command and the position of the third stage spool:

\[
x_{3s}(t) = k_1 \cdot k_2 \cdot x_{c1}(t)
\]  \hspace{1cm} (3.17)

Or, in Laplace notation:

\[
x_{3s}(s) = k_1 \cdot k_2 \cdot x_{c1}(s)
\]  \hspace{1cm} (3.18)
Figure 3.3  Cross Section of the Three Stage Servovalve
From the third stage spool position to the fluid flow to the actuator

Assuming a linear relationship between the third stage spool position and the fluid flow to the actuator, this relationship can be expressed as follows:

\[ q(s) = k_{xm} \cdot x_3(s) \]  

(3.19)

or, in Laplace notation:

\[ q(s) = k_{xm} \cdot x_3(s) \]

(3.20)

where:

- \( k_{xm} \) is a flow-gain coefficient;
- \( q(s) \) is the high pressure hydraulic fluid flow that ports from the servovalve to the actuator.

By substitution of Eq. (3.17) in Eq. (3.19) it can be obtained the relationship between the servovalve conditioned command and the servovalve flow to the actuator:

\[ q(s) = k_{xm} \cdot k_1 \cdot k_2 \cdot x_{ci}(t) \]

(3.21)

or, in Laplace notation:

\[ q(s) = k_{xm} \cdot k_1 \cdot k_2 \cdot x_{ci}(s) \]

(3.22)

Results

Substituting in Eq. (3.18) in the expression for the servovalve conditioned command given by Eq. (3.14), we obtain:

\[ x_{ci}(s) = (K^i_{pro} + s \cdot K^i_{der}) \cdot [x_c(s) - (A_i(s) \cdot k_1 \cdot k_2 \cdot x_{ci}(t))] \]

(3.23)

from which, developing, we obtain:

\[ x_{ci}(s) + A_i(s) \cdot k_1 \cdot k_2 \cdot x_{ci}(t) \cdot (K^i_{pro} + s \cdot K^i_{der}) = (K^i_{pro} + s \cdot K^i_{der}) \cdot x_c(s) \]

and then, by rearranging:
\[ x_{ei}(s) = \frac{(K_{pro}^i + s \cdot K_{der}^i)}{1 + A_i(s) \cdot k_1 \cdot k_2 \cdot (K_{pro}^i + s \cdot K_{der}^i)} \cdot x_e(s) \]  \hspace{1cm} (3.24)

By substituting the expression of the conditioned servo valve signal given by Eq. (3.24) into the servo valve flow equation, Eq. (3.22), we obtain:

\[ q_s(s) = k_{xm} \cdot \frac{k_1 \cdot k_2 \cdot (K_{pro}^i + s \cdot K_{der}^i)}{1 + A_i(s) \cdot k_1 \cdot k_2 \cdot (K_{pro}^i + s \cdot K_{der}^i)} \cdot x_c(s) \]  \hspace{1cm} (3.25)

For easiness of computation Eq. (3.25) can be expressed in the following notation:

\[ q_s(s) = H_t(s) \cdot x_c(s) \]  \hspace{1cm} (3.26)

where:

\[ H_t(s) = \frac{q_s(s)}{x_c(s)} = k_{xm} \cdot \frac{k_1 \cdot k_2 \cdot (K_{pro}^i + s \cdot K_{der}^i)}{1 + A_i(s) \cdot k_1 \cdot k_2 \cdot (K_{pro}^i + s \cdot K_{der}^i)} \]  \hspace{1cm} (3.27)

is the three stage servo valve Transfer Function \( H_t(s) \).
3.2.2 “Linear” Three Stage Servovalve Transfer Function, \( H_t(s) \)

“Linear” response of the three stage servovalve

The response of the servovalve is generally much more accurate than the one of the actuator mechanical system, for this reason often, in shaking table modeling, the effect of the inner loop is neglected. This simple form of three stage servovalve transfer function can be obtained from Eq. (3.27) by setting equal to zero both the feedback conditioning

\[
H_t(s) = \frac{q_s(s)}{x_c(s)} = k_x k_1 k_{xm} = k_t
\]  

(3.28)

is called the three stage servovalve Transfer Function, and

\[
k_t = k_{xm} \cdot k_1 \cdot k_2
\]

is called table gain factor.

It must be pointed out that this model of the three stage servovalve, though widely used, is fully linear and in certain cases can be not enough accurate in order to effectively represent the table functioning.

3.2.3 Time Delay in Third Stage Spool Response

An improvement of the three stage servovalve model can be obtained improving the relationship between the differential pressure and the position of the third stage spool developed by Tina and Claugh Eq. (3.16):

\[
x_{3s}(t) = k_2 \cdot \Delta P_p(t)
\]

(3.16)

Introducing a time shift \( \tau \) between the moment in which a differential pressure occurs and the moment in which the third stage spool moves\(^1\) (this time shift can be physically inter-
preted as the time necessary to overcome the inertia of the spool mass). Eq. (3.16) becomes:

\[ x_{3s}(t) = k_2 \cdot \Delta P_p(t - \tau) \quad (3.29) \]

The linear relationship between the conditioned servovalve command, \( x_{ci}(t) \), and the pilot stage differential pressure \( \Delta P_p(t) \) given by Eq. (3.15) still holds:

\[ \Delta P_p(t) = k_1 \cdot x_{ci}(t) \quad (3.15) \]

And can be used to determine the value of \( \Delta P_p(t - \tau) \) as:

\[ \Delta P_p(t - \tau) = k_1 \cdot x_{ci}(t - \tau) \quad (3.30) \]

Substituting the value of \( \Delta P_p(t - \tau) \) given by Eq. (3.30) in Eq. (3.29) we obtain an expression for the relationship between the conditioned servovalve command \( x_{ci}(t) \) and the position of the third stage spool \( x_{3s}(t) \):

\[ x_{3s}(t) = k_1 \cdot k_2 \cdot x_{ci}(t - \tau) \quad (3.31) \]

That, in Laplace notation, gives:

\[ x_{3s}(s) = k_1 \cdot k_2 \cdot e^{-s\tau} \cdot x_{ci}(s) \]

By substituting the new expression for the third stage spool position given by Eq. (3.31) in the equation representing the fluid flow from the servovalve, Eq. (3.20), it can be obtained the following expression that gives the relationship between the servovalve conditioned command and the servovalve flow to the actuator:

---

1. The existence of a time delay in the servovalve response can be anticipated by dynamic considerations, and it could be observed experimentally during the table setup. See Chapter 5.
2. See Time Delay: Appendix D.
\[ q_s(s) = k_{xm} \cdot k_1 \cdot k_2 \cdot x_{ci}(s) \cdot e^{-s \cdot \tau} \]  \hspace{1cm} (3.32)

Substituting in Eq. (3.32) the expression for \( x_{ci}(s) \) given by Eq. (3.24) and following the same steps taken in paragraph 3.2.1, it is possible to obtain:

\[ q_s(s) = k_{xm} \cdot \frac{k_1 \cdot k_2 \cdot (K_{pro}^i + s \cdot K_{der}^i)}{1 + A_i(s) \cdot k_1 \cdot k_2 \cdot (K_{pro}^i + s \cdot K_{der}^i)} \cdot x_c(s) \cdot e^{-s \cdot \tau} \]  \hspace{1cm} (3.33)

Eq. (3.33) can be expressed in the usual in standard form:

\[ q_s(s) = H_i(s) \cdot x_c(s) \]  \hspace{1cm} (3.26)

where, the three stage servo valve transfer function, \( H_i(s) \), has the following new expression:

\[ H_i(s) = k_{xm} \cdot \frac{k_1 \cdot k_2 \cdot (K_{pro}^i + s \cdot K_{der}^i)}{1 + A_i(s) \cdot k_1 \cdot k_2 \cdot (K_{pro}^i + s \cdot K_{der}^i)} \cdot e^{-s \cdot \tau} \]  \hspace{1cm} (3.34)

For the simplified case in which the three stage servo valve transfer function is considered to be linear (\( A_i(s) = 0, K_{pro}^i = 1, \) and \( K_{der}^i = 0 \)), the presence of a time delay \( \tau \) in the third stage spool motion leads to the following expression of the transfer function:

\[ H_i(s) = k_i \cdot e^{-s \cdot \tau} \]  \hspace{1cm} (3.35)

The transfer function \( H_i(s) \) has been here presented in two simple forms, but for more refined analysis it can be replaced with more complicated frequency dependent expressions without affecting the validity of all the analysis that will follow.
3.2.4 The Oil Flow in the Actuator

The flow of high pressure hydraulic fluid provided by the servovalve ports into the actuator load chamber. This flow must compensate for to the increase of the chamber volume originated by the piston movement, for the flow of fluid that leaks through the actuator seals, and the amount of volume necessary to compensate for the compressibility of the oil. This leads to the following expression of the flow:

\[ q_s(t) = q_{am}(t) + q_{le}(t) + q_{com}(t) \]

or, in Laplace notation.

\[ q_s(s) = q_{am}(s) + q_{le}(s) + q_{com}(s) \]  \hspace{1cm} (3.36)

where:

- \( q_{am}(s) \) is the component of flow due to actuator motion:
- \( q_{le}(s) \) is the component of flow due to leakage:
- \( q_{com}(s) \) is the component of flow due to the fluid compression.
Component of flow due to actuator motion, \( q_{am} \):

The actuator motion requires a certain amount of fluid to flow into the actuator in order to compensate for the change in volume \( \Delta V_{am} \) shown in Figure 3.4. As the displacement of the actuator is exactly equal to the displacement of the table, \( q_{am} \) can be expressed as:

\[
q_{am}(t) = A \cdot \dot{x}_t(t)
\]

or, in Laplace notation:

\[
q_{am}(s) = s \cdot A \cdot x_t(s)
\]  

(3.37)

where:

- \( x_t(t) \) is the velocity of the table;
- \( A \) is the effective piston area.

Figure 3.4  Oil flow due to actuator motion
Component of Flow due to leakage, $q_{le}$

Assuming a linear relationship between the fluid leakage through the actuator seals and the pressure of the fluid in the actuator load chamber, the component of flow due to leakage $q_{le}(t)$ can be expressed as follows:

$$q_{le}(t) = k'_{le} \cdot \Delta P(t)$$  \hspace{1cm} (3.38)

where:

- $k'_{le}$ is the force-flow coefficient expressing the linear relationship between the pressure in the actuator and the flow of leakage fluid;
- $\Delta P(t)$ is the pressure drop across the actuator piston.

Eq. (3.38) can be expressed also in the following form:

$$q_{le}(t) = k'_{le} \cdot \frac{F_a(t)}{A}$$  \hspace{1cm} (3.39)

where:

- $F_a(t)$ is the force in the actuator;
- $A$ is the effective piston area.

Eq. (3.39) can be expressed in Laplace notation as:

$$q_{le}(s) = k_{le} \cdot F_a(s)$$  \hspace{1cm} (3.40)

where:

- $k_{le} = \frac{k'_{le}}{A}$. 
Component of flow due to compressibility of the oil $q_{com}(s)$.

When the oil in the actuator is subject to an external pressure, due to its finite bulk modulus, it undergoes a change of volume $\Delta v_{com}$. The relationship between pressure and volumetric strain for fluid is:

$$\Delta P = \beta \cdot E_v = \beta \cdot \frac{\Delta V}{V}$$

(3.41)

where:

- $\Delta P$ is the pressure change in the fluid;
- $\beta$ is the bulk modulus of the fluid;
- $E_v$ is the volumetric strain;
- $V$ is the volume of the fluid in consideration.

By rearranging Eq. (3.41), it is possible to obtain the expression for the change in volume $\Delta v_{com}$ in a fluid due to a pressure change:

$$\Delta V = \frac{\Delta P \cdot V}{\beta}$$

(3.42)

When considering the special case of the shaking table actuator (see Figure 3.5), the pressure change in the fluid $\Delta P$ and the volume of fluid compressed $V$ have the following values:

- $\Delta P$ is equal to half the force applied in the actuator divided by the total area of the actuator load cell, giving $\Delta P = \frac{F_{d}(t)}{A}$;
- the volume of fluid in consideration is equal to half of the entire volume of the actuator load chamber $V = \frac{V}{2}$. 
Substituting the expressions for $\Delta P$ and for $\psi$ in Eq. (3.42), we obtain an expression for the change of volume of the oil in the actuator $\Delta v_{\text{com}}$ due to the compression of the oil.

$$
\Delta v_{\text{com}}(t) = \frac{V}{4 \cdot \beta} \cdot \frac{F_a(t)}{A}
$$

(3.43)

where:

- $V$ is the volume of the actuator load chamber;
- $\beta$ is the bulk modulus of the oil;
- $A$ is the effective piston area;
- $F_a(t)$ is the force in the actuator.

Recalling that the fluid flow can be obtained taking the time derivative of the volume change we can write:

$$
q_{\text{com}}(t) = \frac{\partial}{\partial t} [\Delta v_{\text{com}}(t)] = \frac{\partial}{\partial t} \left[ \frac{V}{4 \cdot \beta} \cdot \frac{F_a(t)}{A} \right]
$$

where:

- $q_{\text{com}}(t)$ is the flow due to fluid compressibility.

Assuming that the volume of oil in the actuator ($V$) is constant, we obtain:

$$
q_{\text{com}}(t) = \frac{V}{4 \cdot \beta \cdot A} \cdot F_a(t)
$$

Or, in Laplace notation:

$$
q_{\text{com}}(s) = s \cdot \frac{V}{4 \cdot \beta \cdot A} \cdot F_a(s)
$$

(3.44)

where:

- $F_a(t)$ is the time derivative of the force in the actuator.
The Flow Equation

Substituting in Eq. (3.36) \( q_s(s) = q_{am}(s) + q_{le}(s) + q_{com}(s) \) - all the expressions for the fluid flow components (Equations (3.21), (3.37), (3.40), (3.44)), we can obtain the following expression:

\[
H_t(s) \cdot x_c(s) = s \cdot A \cdot x_t(s) + k_{le} \cdot F_a(s) + s \cdot \frac{V}{4 \cdot \beta \cdot A} \cdot F_a(s) \tag{3.45}
\]

Eq. (3.45) provides an expression that gives a relation between the servovalve command \( x_c \), the table displacement \( x_t \) and the force in the actuator \( F_a \).
3.2.5 The Force in the Actuator

Assuming that no friction exists between the slip table and the sliding surface, that the actuator is rigidly clamped to the ground (foundation mass perfectly rigid) and that the table is carrying no payload, the force in the actuator can be obtained as a function of the table acceleration $\ddot{x}_t$. From the free body diagram represented in Figure 3.6, it is possible to obtain the following equation that allows to express the actuator force $F_a(t)$ in terms of the table position $x_t$:

$$F_a(t) = m_t \cdot \ddot{x}_t(t)$$

Or, in Laplace notation:

$$F_a(s) = s^2 \cdot m_t \cdot x_t(s)$$

(3.46)

where:

- $\ddot{x}_t(t)$ is the second time derivative of the table displacement;
- $m_t$ is the mass of the table.

![Actuator free body diagram](image-url)
3.2.6 Servovalve Transfer Function, $S(s)$

Substituting the expression for the force in the actuator (Eq. (3.46)) into Eq. (3.45) we can finally obtain an expression that links the servovalve command to the table displacement:

$$H_t(s) \cdot x_c(s) = s \cdot x_t(s) \cdot A + s^2 \cdot k_{le} \cdot m_t \cdot x_t(s) + s^3 \cdot \frac{V}{4\beta A} \cdot m_t \cdot x_t(s)$$

Rearranging the terms, we obtain:

$$H_t(s) \cdot x_c(s) = x_t(s) \cdot \left\{ s \cdot A + s^2 \cdot k_{le} \cdot m_t + s^3 \cdot \frac{V}{4\beta A} \cdot m_t \right\}$$  \hspace{1cm} (3.47)

From which it can be derived the following expression for the table displacement:

$$x_t(s) = S(s) \cdot x_c(s)$$  \hspace{1cm} (3.48)

where $S(s)$ is the servovalve transfer function and is defined as:

$$S(s) = \frac{x_t(s)}{x_c(s)} = \frac{H_t(s)}{s^3 \cdot \left( \frac{V \cdot m_t}{4 \cdot \beta \cdot A} \right) + s^2 \cdot m_t \cdot k_{le} + s \cdot A}$$  \hspace{1cm} (3.49)

or, the expression for $H_t(s)$ (3.28) and rearranging term:

$$S(s) = \left( \frac{k_{le}}{A} \right) \cdot \frac{1}{s \cdot \left[ s^2 \cdot \left( \frac{V \cdot m_t}{4 \cdot \beta \cdot A^2} \right) + s \cdot \frac{m_t \cdot k_{le}}{A} + 1 \right]}$$  \hspace{1cm} (3.50)
3.2.7 Equivalent SDOF

The different terms of Eq. (3.50) can be interpreted in physical terms by visualizing the servovalve system as a single degree of freedom mass/spring/damper system. Consider the actuator and oil system shown in Figure 3.7. The relationship between pressure and volumetric strain is

\[ \Delta P = \beta \varepsilon_v \]  

(3.51)

where \( \Delta P \) is the fluid pressure change, \( \beta \) is the bulk modulus of the fluid, and \( \varepsilon_v \) is the volumetric strain. For a constant cross-sectional area, \( \varepsilon_v \) can be simplified to

\[ \varepsilon_v = \varepsilon_{11} \]  

(3.52)

where \( \varepsilon_{11} \) is the longitudinal strain along the cylinder axis. The force \( f_1 \) on either side of the piston required to displace the piston (originally at rest and in the center position) of one unit is:

\[ f_1 = \beta \cdot A \cdot \varepsilon_{11} = \beta A \left( \frac{1}{L/2} \right) \]  

(3.53)

The total force \( F \) that it is necessary to apply to the table in order to obtain a unit displacement is therefore equal to \( F = 2 \cdot f_1 \). Recalling the definition of lateral stiffness (force necessary to obtain a unit displacement), it is possible to define the lateral stiffness of the table as:

\[ k = 2 \cdot f_1 = \frac{4 \cdot \beta \cdot A}{L} \]  

(3.54)

Rearranging terms in Eq. (3.54), and multiplying numerator and denominator by \( A \) it can be obtained the following expression for the stiffness of the table \( k \):
\[ k = \frac{4\beta A^2}{V} \]  \hspace{1cm} (3.55)

The oil column frequency can be computed as:

\[ \omega_{oil}^2 = \frac{k}{m_t} = \frac{4\beta A^2}{Vm_t} \]  \hspace{1cm} (3.56)

It must be noticed that this expression corresponds to the term in the denominator of Eq. (3.50). This result confirms the validity of the model used to capture the behaviors of the servovalve and the actuator.

**Figure 3.7**  Equivalent Single Degree of Freedom Oscillator representing the Electro-Hydraulic Servovalve
Figure 3.8  (a) Actuator Oil Column, and (b) Equivalent Spring
3.3 Controller Model

3.3.1 Introduction

The controller provides the so called "servovalve command" $x_c(t)$. The voltage $x_c(t)$ is determined on the basis of the desired position of the table (command signal $\tilde{x}_c(t)$) and of the values of the feedback controls signals, as explained in Section 2.3.2 and shown in Figure 3.9.

In Figure 3.9 is represented how the controller determines $x_c(t)$. Following this notation the servovalve command can be written as the sum of three components:

$$x_c(t) = \varepsilon(t) + x_f(t) + x_d(t) \quad (3.57)$$

where:

- $\varepsilon(t)$ is the component due to the PID Gains;
- $x_f(t)$ is the component due to the Feed Forward Gain;
- $x_d(t)$ is the component due to the Delta Pressure Gain.

3.3.2 PID Gain Component, $\varepsilon(t)$

The Error Signal $e(t)$.

The PID gain uses as input the so called DC error signal $e(t)$ that is obtained subtracting the magnitude of the displacement feedback signal $x_d(t)$ from the magnitude of the command signal $\tilde{x}_c(t)$, as described by the equation:

$$e(t) = \tilde{x}_c(t) - x_d(t) \quad (3.58)$$
The displacement feedback signal \( x_a(t) \) is obtained from a linear variable differential transformer (LVDT) mounted on the actuator that converts the actual table displacement into a voltage. The controller multiplies the voltage of the feedback signal by a preset gain value. The preset Gain Value of the 407 Controller is equal to 1. this gives:

\[
x_a(t) = x_t(t)
\]
and therefore:

\[ e(t) = \overline{X_c} - x_t \quad (3.59) \]

The feedback transfer function \( A(s) \) has the following expression:

\[ A(s) = \frac{x_2(s)}{x_1(s)} = 1 \quad (3.60) \]

**The PID Component \( \varepsilon(t) \).**

As explained in Ch.2 and shown in Figure 3.10, the PID Gain processes the DC error \( e(t) \) with a Proportional, Derivative and Integral computation. The output signal of the PID gain - \( \varepsilon(t) \) - is obtained through the sum of three contributions, as expressed in the following equation:

\[ \varepsilon(t) = \varepsilon_p(e(t)) + \varepsilon_I(e(t)) + \varepsilon_D(e(t)) \quad (3.61) \]

where:

- \( \varepsilon(t) \) is the PID output signal;
- \( e(t) \) is the error signal;
- \( \varepsilon_p(e(t)) \) is the proportional gain component;
- \( \varepsilon_I(e(t)) \) is the integral gain component;
- \( \varepsilon_D(e(t)) \) is the derivative component.

The values of the components of the PID output signal are determined as follows:

The proportional gain component - \( \varepsilon_p(e(t)) \) - is obtained by multiplying the error by the proportional gain constant \( K_{pro} \), as expressed by:

\[ \varepsilon_p(e(t)) = K_{pro} \cdot e(t) \]

The integral gain component - \( \varepsilon_I(e(t)) \) - is obtained by multiplying the integral of the
error over one time step, or accumulated error, times the integral gain constant $K_{\text{int}}$, as expressed by:

$$\varepsilon_p(e(t)) = K_{\text{int}} \cdot \int e(t) \, dt$$

The derivative component - $\varepsilon_D(e(t))$ - is obtained by multiplying the derivative of the error, or rate of error change over one time step, by the derivative gain constant $K_{\text{der}}$, as expressed by:

$$\varepsilon_D(e(t)) = K_{\text{der}} \cdot \frac{\partial}{\partial t} e(t)$$

Substituting the values for all the components and expressing Eq. (3.61) in Laplace notation, it can be obtained the following expression for $\varepsilon(s)$:

$$\varepsilon(s) = K_{\text{pro}} \cdot e(s) + \frac{1}{s} \cdot K_{\text{int}} \cdot e(s) + s \cdot K_{\text{der}} \cdot e(s)$$  \hspace{1cm} (3.62)

For easiness of further computations it is convenient to express the PID component of the servovalve signal $\varepsilon(s)$ in the following notation:

$$\varepsilon(s) = P_{\text{PID}}(s) \cdot e(s)$$

where:

$$P_{\text{PID}}(s) = K_{\text{pro}} + \frac{1}{s} K_{\text{int}} + s K_{\text{der}}$$  \hspace{1cm} (3.63)

- $K_{\text{pro}}$ is the proportional gain:
- $K_{\text{int}}$ is the integral gain:
- $K_{\text{der}}$ is the derivative gain.
Figure 3.10 The P-I-D Gains
3.3.3 Feedforward Gain Component, $x_f(t)$

The feed-forward gain use as input the command signal $\tilde{x}_c(t)$ and multiply its derivative by the feed-forward gain constant $K_{FF}$. The feed forward component $x_f(t)$ can be obtained through following equation:

$$x_f(t) = K_{FF} \cdot \frac{\partial}{\partial t} \tilde{x}_c(t)$$  \hspace{1cm} (3.64)

where $K_{FF}$ is the feed-forward gain constant.

Expressing Eq. (3.64) in Laplace notation, we obtain:

$$x_f(s) = s \cdot K_{FF} \cdot \tilde{x}_c(s)$$  \hspace{1cm} (3.65)

For easiness of further computations it is convenient to express the Feed Forward component of the servo valve signal in the following notation:

$$x_f(s) = P_{FF}(s) \cdot \tilde{x}_c(s)$$  \hspace{1cm} (3.66)

where:

$$P_{FF}(s) = s \cdot K_{FF}$$  \hspace{1cm} (3.67)
3.3.4 Delta Pressure Gain Component, \( x_d(t) \)

The differential pressure gain use as input the differential pressure \( \Delta P(t) \) that exists across the actuator. The differential pressure \( \Delta P(t) \) is converted into the voltage \( x_{cell}(t) \) by an actuator mounted delta-P cell, as expressed in the following equation:

\[
x_{cell}(t) = k_{cell} \cdot \Delta P(t)
\]  

(3.68)

where \( k_{cell} \) is the conversion factor for pressure to voltage. This conversion factor can be adjusted through the controller. In our case the gain \( K_{cell} \) has been adjusted in order to give 0.5 millivolt of electric signal \( x_{cell}(t) \) per psi of pressure in the load cell\(^1\). This correspond to a value of \( k_{cell} \) equal to 0.0005.

The electric signal \( x_{cell}(t) \) provided by the delta pressure cell is then multiplied via the DC conditioner by the constant \( \overline{K}_{dp} \) in order to obtain the differential pressure component \( x_d(t) \) of the controller command as follows:

\[
x_d(t) = \overline{K}_{dp} \cdot x_{cell}(t)
\]

by substituting Eq. (3.68), we obtain:

\[
x_d(t) = K_{dp} \cdot \Delta P(t)
\]  

(3.69)

where \( K_{dp} = \overline{K}_{dp} \cdot k_{cell} \) is the so called Delta Pressure Gain constant.

The differential pressure \( \Delta P(t) \) is physically related to the force applied by the actuator

---

1. This calibration has been obtained by physically realizing a differential pressure of 3000 psi across the actuator, and adjusting the corresponding electric signal in order to make it equal to 1.5 Volt.
via the expression: \( \Delta P(t) = \frac{F_a(t)}{A} \) where \( A \) is the effective piston area and \( F_a(t) \) is the force in the actuator. The substitution of the expression for \( \Delta P(t) \) in Eq. (3.69) leads to the following expression:

\[
x_d(t) = \frac{K_{dp}}{A} \cdot F_a(t)
\]  

(3.70)

Substituting in Eq. (3.70) the expression of the force in the actuator (Eq. (3.46)) described in Paragraph 3.2.5 \(^1\) and passing to Laplace notation, we obtain the following expression for \( x_d(s) \):

\[
x_d(s) = \frac{s^2 \cdot K_{dp} \cdot m_i}{A} \cdot x_t(s)
\]  

(3.71)

For easiness of further computations it is convenient to express the Differential Pressure component of the servovalve signal (in Laplace notation) in the following form:

\[
x_d(s) = P_{dp}(s) \cdot x_t(s)
\]  

(3.72)

where:

\[
P_{dp}(s) = \frac{s^2 \cdot K_{dp} \cdot m_i}{A}
\]  

(3.73)

\(^1\) It is important to point out that this expression was obtained under the assumption that no friction exists between the slip table and the sliding surface, that the actuator is rigidly clamped to the ground and that the table is carrying no payload.
3.3.5 Controller Model

Finally the relationship between command signal \( x_c(t) \) and servovalve signal \( x_c(t) \) due to the Controller can be obtained by substitution of Eq. (3.63), Eq. (3.65) and Eq. (3.72) in Eq. (3.57):

\[
x_c(s) = P_{PID}(s) \cdot [x_c(s) - x_t(s)] + P_{FF}(s) \cdot \dot{x}_c(s) + P_{DP}(s) \cdot x_t(s)
\]

 restr(t) \( x_t(t) \) \( x_d(t) \)

where all the symbols have the meanings defined in the previous paragraphs.
3.4 SYSTEM MODEL (SERVOVALVE & ACTUATOR + CONTROLLER)

3.4.1 System Transfer Function, $H(s)$

The behaviors of the electro-hydraulic system that drives the shaking table can be obtained combining the model of the servovalve (described in paragraph 3.2 through Eq. (3.48)) with the model of the controller (described in paragraph 3.3 through Eq. (3.74)) and following the diagram represented in Figure 3.11.

Using Eq. (3.74) and Eq. (3.48) it is in fact possible to obtain a relationship that links the commanded input $\bar{x}_c(t)$ with the table displacement $x_t(t)$. Here it follows this derivation in Laplace notation:

Deriving $x_c(s)$ from Eq. (3.48) - $x_t(s) = S(s) \cdot x_c(s)$ - and substituting into Eq. (3.74), we obtain the following relation in terms of $\bar{x}_c(t)$ and $x_t(t)$ only:

$$x_t(s) = S(s) \cdot [\bar{x}_c(s) \cdot P_{FF}(s) + [\bar{x}_c(s) - A(s) \cdot x_t(s)] \cdot P_{PID}(s) + P_{DP}(s) \cdot x_t(s)]$$  \hspace{1cm} (3.75)

Then, rearranging the terms we obtain:

$$\{1 + S(s) \cdot [A(s) \cdot P_{PID}(s) - P_{DP}(s)]\} \cdot x_t(s) = S(s) \cdot [P_{FF}(s) + P_{PID}(s)] \cdot \bar{x}_c(s)$$  \hspace{1cm} (3.76)

From which the transfer function between external command and table displacement $H(s) = \frac{x_t(s)}{\bar{x}_c(s)}$ can be derived:

$$H(s) = \frac{x_t(s)}{\bar{x}_c(s)} = \frac{S(s)[P_{FF}(s) + P_{PID}(s)]}{1 + S(s)[A(s)P_{PID}(s) - P_{DP}(s)]}$$  \hspace{1cm} (3.77)

By substituting in Eq. (3.77) the values of the transfer functions $P$'s derived for each single component, we obtain:
\[ H(s) = \frac{x_i(s)}{x_c(s)} = \frac{S(s) \cdot \left[ s \cdot K_f + K_{pro} + \frac{1}{s} \cdot K_{int} + s \cdot K_{der} \right]}{1 + S(s) \cdot \left[ K_{pro} + \frac{1}{s} \cdot K_{int} + s \cdot K_{der} - \left( K_{dp} \cdot \frac{s^2 \cdot m_t}{A} \right) \right]} \]  

(3.78)

where all symbols have the usual meaning and \( S(s) \) is defined by Eq. (3.50).

Figure 3.11  The electro Hydraulic System
3.4.2 Results

In this chapter we show the plot of the numerical values of the transfer function of the table, as modeled in this paragraph. For a matter of simplicity it has been considered that the time delay between the servovalve command and the servovalve motion is reduced to zero. Therefore the three stage servovalve transfer function $H_i(s)$ has been assumed to be constant (equal to $k_t$), as expressed by Eq. (3.28).

To plot the transfer function the following value of the table parameters have been used:

*Values obtained from Least Square Fit:*

- Table gain factor $k_t$ equal to 330 in$^3$ / Volt sec;
- Leakage coefficient $k_{le}$ equal to $1^{-10}$ in$^3$ / psi;
- Oil bulk modulus $\beta$ equal to 98.000 psi;

*Values given:*

- Effective Actuator Area $A$ equal to 12.73 in$^2$;
- Oil column volume $V$ equal to 101.84 in$^3$;
- Table Mass $mt$ equal to 3.29 lbs sec$^2$ / in (that correspond to a table weight of 1270 lbs).

The values of $k_t$, $k_{le}$, and of $\beta$ have been selected in order to give a transfer function that is close to the actual one of the table, as it will be explained in Chapter 8.

The Gain Parameters were set as follows:

Proportional Gain $= 1$ Volt/Volt.

All other gains $= 0$.

Fig. 3.12 (a) and Fig. 3.12 (b) represent respectively the magnitude transfer function and
the phase transfer function of the table.

Fig. 3.12 (a) shows a big and sharp peak at a frequency that sits around 65-70 Hz: it corresponds to the resonance behavior of the oil column that moves the actuator. Its frequency will be mentioned hereafter as the oil column frequency, and the peak will be mentioned as the oil column peak. This peak in the magnitude transfer function is of extreme importance in the understanding of the table behaviors as divides the transfer function two different frequency areas with totally different behaviors. The oil column frequency is extremely important also to determine the behaviors of the phase transfer function. Fig. 3.12 (b) shows how the phase transfer function has an inversion of phase exactly in correspondence of the oil column frequency. Hereafter we will refer to this frequency as the inversion frequency of the phase transfer function.
Figure 3.12  The System Magnitude Transfer Function, $H(s)$:

(a) Magnitude

(b) Phase (radians)
3.5 Effect of Flexibility of Foundation Mass

3.5.1 Table Transfer Function, $T(s)$

The transfer function derived in the previous paragraph describes mathematically the relationship existing between a command signal $\bar{x}_c(t)$ and the motion of the actuator $x_t(t)$. This relationship was obtained considering the foundation mass (that provides a reaction point for the actuator) as rigidly connected to the ground. Unfortunately, the reaction mass, for its inner construction, is flexible and moves with respect to the ground. Therefore the absolute motion of the shaking table - $x_{ta}(t)$ - with respect to the ground (that is assumed to be an inertial system) will be expressed as:

$$x_{ta}(t) = x_t(t) + x_b(t) \tag{3.79}$$

where:

- $x_{ta}(t)$ is the absolute motion of the table (motion of the table with respect to the ground, that is assumed to be an inertial system);
- $x_t(t)$ is the motion of the table relative to the base;
- $x_b(t)$ is motion of the flexible base with respect to the ground.

The base flexibility affects not only the table absolute motion, but also the value of the inertia forces applied on the table mass and, consequently, the force in the actuator $F_3(t)$. The transfer function between the given input and the absolute motion of the table, for simplicity called the table transfer function $T(s)$, gets (in Laplace notation) the following new expression (called for reference $T(s)$):
$$T(s) = \frac{x_{ta}(s)}{x_c(s)} = \frac{x_b(s) + x_t(s)}{x_c(s)} = \frac{x_t(s)}{x_c(t)} \cdot \left( \frac{x_b(s)}{x_t(s)} + 1 \right) = H'(s) \cdot (B(s) + 1) \quad (3.80)$$

where:

- \( B(s) = \frac{x_b(s)}{x_t(s)} \) is the Base transfer function: its expression will be derived in the paragraph 3.5.2.
- \( H'(s) = \frac{x_t(s)}{x_c(s)} \) is the system transfer function modified in order to take into account how the motion of the base affects the value of the force in the actuator \( F_3(s) \cdot F_3(s) \) in fact enters in the computation of both the Servo valve and Controller Models. The expression of \( H'(s) \) will be derived in paragraph 3.5.4.

![Fixed Reference System (lab ground)](image)

Figure 3.13  Coordinate System: Shaking Table + Flexible Base
3.5.2 The Base Transfer Function $B(s)$

From the free body diagram shown in Figure 3.14 it can be written the following equation of motion for the flexible base:

$$m_b \cdot \ddot{x}_b(t) + c_b \cdot \dot{x}_b(t) + k_b \cdot x_b(t) = -m_t \cdot (\ddot{x}_b(t) + \dot{x}_t(t))$$  \hspace{1cm} (3.81)

$$= -F_q(t)$$

where:

- $m_b$ is the mass of the base:
- $m_t$ is the mass of the table:
- $c_b$ represent damping coefficient of the base:
- $k_b$ is the lateral stiffness of the base:
- $x_b(t)$ is the base displacement:
- $x_t(t)$ is the table displacement.

rearranging the terms in Eq. (3.81), we obtain:

$$(m_b + m_t) \cdot \ddot{x}_b(t) + c_b \cdot \dot{x}_b(t) + k_b \cdot x_b(t) = -m_t \cdot \ddot{x}_t(t)$$  \hspace{1cm} (3.82)

using Laplace notation. Eq. (3.82) becomes:

$$s^2 \cdot (m_b + m_t) \cdot x_b(s) + s \cdot c_b \cdot x_b(s) + k_b \cdot x_b(t) = -s^2 \cdot m_t \cdot x_t(s)$$

from which it can be obtained the base transfer function $B(s)$:

$$B(s) = \frac{x_b(s)}{x_t(s)} = \frac{\frac{m_t}{m_T} \cdot \frac{s^2}{s^2 + s \cdot 2 \cdot \zeta_b \cdot \omega_b + \omega_b^2}}{s^2 + s \cdot 2 \cdot \zeta_b \cdot \omega_b + \omega_b^2}$$  \hspace{1cm} (3.83)

where:
\[ m_T = m_t + m_b \] is the system total mass:

\[ \omega_b^2 = \frac{k_b}{m_T}, \text{ and } \omega_b \] is the circular natural frequency of the flexible foundation (base):

\[ 2 \cdot \omega_b \cdot \zeta_b = \frac{c_b}{m_T} \], and \( \zeta_b \) is the critical damping coefficient for the flexible foundation.

For a mathematical model of the flexible base see Chapter 7, where \( m_b, \omega_b \) and \( \zeta_b \) are computed.

Figure 3.14  Shaking Table with flexible Foundation (Base): Free Body Diagram
3.5.3 The Force in the Actuator

The motion of the foundation affects the absolute acceleration of the slip table and, consequently, also the force in the actuator $F_a(t)$ is affected. Assuming that no friction exists between the slip table and the sliding surface, and that the actuator damping is internal to the servovalve system, the force in the actuator $F_a(t)$ is equal to the mass of the table times its absolute acceleration. Following the notation of the free body diagram represented in Figure 3.15 $F_a(t)$ takes the following expression:

$$F_a(t) = m_t \cdot \ddot{x}_{ta}(t) = m_t \cdot [\ddot{x}_t(t) + \ddot{x}_b(t)] = m_t \cdot \ddot{x}_t(t) \cdot \left[ 1 + \frac{\ddot{x}_b(t)}{\ddot{x}_t(t)} \right]$$

or, in Laplace notation:

$$F_a(s) = s^2 \cdot m_t \cdot x_t(s) \cdot \left[ 1 + \frac{x_b(s)}{x_t(s)} \right] = x_t(s) \cdot s^2 \cdot m_t \cdot [1 + B(s)] \quad (3.84)$$

For simplicity of further computation it is convenient to rewrite Eq. (3.84) in the following way:

$$F_a(s) = x_t(s) \cdot s^2 \cdot m_t \cdot H_{F1}(s) \quad (3.85)$$

where $H_{F1}(s)$ is an operator defined as follows:

$$H_{F1}(s) = 1 + B(s) \quad (3.86)$$

---

1. Assumption explained in paragraph 3.2.5.
3.5.4 Modified System Transfer Function $H'(s)$

The new expression for the force in the actuator affects the system transfer function $H'(s)$ in two points:

a) The transfer function of the servo valve $S'(s)$: new expression for the flow due to leakage and for the flow due to fluid compressibility.

b) Controller Model: new expression for the Differential Pressure feedback.

a) Servovalve Model $S'(s)$.

The force in the actuator $F_3(t)$ enters in the computation of the flow due to leakage $q_{le}$ and of the flow due to compressibility of the oil $q_{com}$.

Substituting the new expression of the force in the actuator $F_3(t)$ - Eq. (3.85) - in the computation for the flow due to leakage (Eq. (3.40)), we obtain:
\[ q_{le}(s) = k_{le} \cdot F_a(s) \]  \hspace{1cm} (3.40)

we obtain:

\[ q_{le}(s) = x_t(s) \cdot s^2 \cdot m_t \cdot k_{le} \cdot H_{F1}(s) \]  \hspace{1cm} (3.87)

Substituting the new expression of the force in the actuator - Eq. (3.85) - in the computation for the flow due to the fluid compressibility (Eq. (3.44))

\[ q_{com}(s) = s \cdot \frac{V}{4 \cdot \beta \cdot A} \cdot F_a(s) \]  \hspace{1cm} (3.44)

we obtain:

\[ q_{com}(s) = x_t(s) \cdot s^3 \cdot \frac{V \cdot m_t}{4 \cdot \beta \cdot A} \cdot H_{F1}(s) \]  \hspace{1cm} (3.88)

Then, the substitution of the new expression for the flow due to leakage (Eq. (3.87)) and of the new expression for the flow due to the fluid compressibility (Eq. (3.88)) into the flow equation (Eq. (3.36))

\[ q_t(s) = q_{am}(s) + q_{le}(s) + q_{com}(s) \]  \hspace{1cm} (3.36)

leads, after some simple computations, to the following expression of the servo valve transfer function\(^1\):

\[ S'(s) = \frac{x_t(s)}{x_c(s)} = \frac{H_t(s)}{s^3 \cdot \frac{V m_t}{4 \beta A} \cdot H_{F1}(s) + s^2 m_t \cdot k_{le} \cdot H_{F1}(s) + s \cdot A} \]  \hspace{1cm} (3.89)

where \( H_t(s) \), as seen before, represents the three stage servo valve transfer function as

\(^1\) This new expression can also be obtained by direct substitution of the new expression for the force in the actuator \( F_a(t) \) - given by Eq. (3.85) - into (3.45). This procedure is faster but does not provide the physical insight given by the procedure described in this paragraph.
defined by: \( q_s(s) = H_t(s) \cdot x_c(t) \).

\textit{b) The Controller Model.}

The force in the actuator affects the signal given by the Delta Pressure feedback. Using the new expression of the force in the actuator given by Eq. (3.85) the Delta Pressure feedback signal expression (Eq. (3.70)):

\[
x_d(t) = \frac{K_{dp}}{A} \cdot F_a(t)
\]  

(3.70)

becomes:

\[
x_d(s) = \frac{K_{dp}}{A} \cdot x_t(s) \cdot s^2 \cdot m_t \cdot H_{F1}(s)
\]  

(3.90)

Or, in standard form:

\[
x_d(s) = P_{DP}(s) \cdot x_t(s) \cdot H_{F1}(s)
\]  

(3.91)

where:

\[
P_{DP}(s) = \frac{s^2 \cdot K_{dp} \cdot m_t}{A}
\]

Substituting the expression for \( x_d(s) \) - Eq. (3.91) - into the expression for the controller model (Eq. (3.57)), we obtain:

\[
x_c(t) = \epsilon(t) + x_f(t) + x_d(t)
\]  

(3.57)

performing the same computation carried on in paragraph 3.3.5. we finally obtain the
expression:

\[ x_c(s) = P_{PID}(s) \cdot [\bar{x}_c(s) - x_t(s)] + P_{FF}(s) \cdot \bar{x}_c(s) + P_{DP}(s) \cdot x_t(s) \cdot H_{F1}(s) \]  

(3.92)

**System Transfer Function (Servovalve + Controller).**

Combining the new expression of the servovalve transfer function - Eq. (3.89) - with the new expression of the controller model - Eq. (3.92) -, it can be derived the new expression of the system transfer function \( H'(s) \):

\[ H'(s) = \frac{x_t(s)}{\bar{x}_c(s)} = \frac{S'(s) \cdot [P_{FF}(S) + P_{PID}(S)]}{1 + S'(s) \cdot [A(s) \cdot P_{PID}(s) - P_{DP}(s) \cdot H_{F1}(s)]} \]  

(3.93)

Or, substituting the values of the transfer functions P's derived for each single Gain Component:

\[ H'(s) = \frac{S'(s) \cdot \left[ s \cdot K_f + K_{pro} + \frac{1}{s} \cdot K_{int} + s \cdot K_{der} \right]}{1 + S'(s) \cdot \left[ K_{pro} + \frac{1}{s} \cdot K_{int} + s \cdot K_{der} - \left( K_{dp} \cdot \frac{s^2 \cdot m_t}{A} \right) \cdot H_{F1}(s) \right]} \]  

(3.94)
3.5.5 Results

Substituting \( B(s) \) - Eq. (3.83) - and \( H'(s) \) - Eq. (3.93) - into \( T(s) = H'(s) \cdot (B(s) + 1) \), it is finally possible to obtain a table transfer function \( T(s) \) that accounts for the flexibility of the base.

Figure 3.16 (a) and Figure 3.16 (b) show respectively the magnitude and the phase of the improved table transfer function. The values selected for the dynamic parameters (\( \omega_b = 170 \text{ rad/sec} \) corresponding to approximately 27 Hz. and \( \zeta = 0.042 \)) are the ones found experimentally for the base and reported in Chapter 7. The mass of the base has been considered to be equal to 233.16 lbs sec\(^2\)/in. that correspond to an estimated base weight of 90,000 lbs\(^1\). All the other parameters have the same values used to plot the Transfer Function in paragraph 3.4.2.

Magnitude Transfer Function.

By Comparison of the transfer function of the system with and without introduction of the base flexibility - See Figure 3.16 (b) - it is evident how the base flexibility generates a small peak at a frequency close (approximately 10% lower) to the natural frequency of vibration of the flexible base.

Phase Transfer Function.

Figure 3.17 (a) shows the phase transfer function when the flexibility of the base is taken into account preparing the model for the shaking table. The flexibility of the base originates the small notch indicated by the arrow at a frequency corresponding to the natural dynamic behavior of the base. This notch indicates that the phase shift between table com-

\(^1\) for the validity of this estimation of the base mass see Appendix E.
mand and table response is increased at frequencies close to the natural one of the base.

Figure 3.17 (b) compare the phase transfer function when the flexibility of the base is taken into account and when it is neglected. It must be pointed out how the flexibility of the base not only originates the notch previously described, but also increases the value of the inversion frequency, even though of a a small amount.
Figure 3.16  Effect of the flexibility of the base upon the shaking table transfer function:
(a) Magnitude of Table Transfer Function with Flexible Base
(b) Magnitude of Table Transfer Function with and without Flexible Base
Figure 3.17  Effect of the flexibility of the base upon the shaking table transfer function:
(a) Phase of Table Transfer Function with Flexible Base
(b) Phase of Table Transfer Function with and without Flexible Base
3.6 Effect of Flexible Payload (SDOF)

3.6.1 Introduction

When a model of a considerable weight is added on the table, the response of the whole system is affected by the action of the latter. How the system is affected depends upon the weight and the rigidity of the payload. A very rigid payload behaves as if the mass of the table is increased. To account for a very rigid payload it is enough to compute the system transfer function using a value of the mass of the table determined as follows:

\[ m_t = m_{to} + m_p \]  \hspace{1cm} (3.95)

where:

- \( m_{to} \) is the mass of the bare table;
- \( m_p \) is the mass of the payload.

On the other hand, a flexible payload affects the system transfer function in a much more complicated way. In this paragraph we derive a mathematical expression for the system transfer function that takes into account the effect of a flexible payload (SDOF).

Following a formulation similar to the one used when accounting for the flexibility of the foundation - Eq. (3.80) -, the transfer function between the given input and the absolute motion of the table (see Fig. 18 (a)), when loaded with a flexible payload, can be expressed (in Laplace notation) as follows:

\[ T'(s) = \frac{x_p(s) + x_t(s)}{x_c(s)} = \frac{x_t(s)}{x_c(t)} \cdot \left( \frac{x_b(s)}{x_t(s) + 1} \right) = H'(s) \cdot (B'(s) + 1) \]  \hspace{1cm} (3.96)
where:

- \( B'(s) = \frac{x_p(s)}{x_i(s)} \) is the Base transfer function modified to take into account the effect of the flexible SDOF payload. This expression will be derived in paragraph 3.6.4.

- \( H''(s) = \frac{x_r(s)}{x_c(t)} \) is the System transfer function modified to take into account the motion of the base and the effect of the flexible SDOF payload. This expression will be derived in paragraph 3.6.6.
Figure 3.18  Shaking Table with flexible base and SDOF payload:
(a) Coordinate System
(b) Free Body Diagram
3.6.2 Payload Transfer Function, $H_p(s)$

Figure 3.18 (a) shows the coordinate system of the shaking table with a flexible SDOF payload. The absolute payload displacement can be written as:

$$x_{pa}(t) = x_b(t) + x_t(t) + x_p(t)$$  \hspace{1cm} (3.97)

where:

- $x_{pa}(t)$ is the absolute displacement of the payload;
- $x_b(t)$ is the base displacement;
- $x_t(t)$ is the table displacement with respect to the base;
- $x_p(t)$ is the relative displacement of the payload with respect to the sliding table.

To simplify this analysis, the payload is considered here as the single degree of freedom (SDOF). From the free body diagram of Figure 3.19, it can be written the differential equation of the motion of the flexible payload subject to a base acceleration equal to the absolute acceleration of the table $\ddot{x}_{ta}(t) = \ddot{x}_t(t) + \dot{x}_b(t)$:

$$m_p \cdot \ddot{x}_p(t) + c_p \cdot \dot{x}_p(t) + k_p \cdot x_p(t) = -m_p \cdot (\ddot{x}_b(t) + \dot{x}_t(t))$$  \hspace{1cm} (3.98)

where:

- $m_p$ is the mass of the payload;
- $c_p$ represent damping coefficient of the payload;
- $k_p$ is the lateral stiffness of the payload;
- $x_p$ is the payload relative displacement;
- $x_t$ is the table displacement.

Rearranging Eq. (3.98), and using the Laplace notation, we can obtain:
\[ x_p(s) = \frac{s^2 \cdot (x_b(s) + x_i(s))}{s^2 + 2 \cdot \zeta_p \cdot s \cdot \omega_p + \omega_p^2} \] (3.99)

from which, we can obtain the payload transfer function \( H_p(s) \):

\[ H_p(s) = \frac{x_p}{x_{ta}} = \frac{s^2}{s^2 + 2 \cdot \zeta_p \cdot s \cdot \omega_p + \omega_p^2} \] (3.100)

where:

- \( \zeta_p \) is the critical damping ratio: \( \frac{c_p}{m_p} = 2 \cdot \zeta_p \cdot \omega_p \).

- \( \omega_p \) is the circular natural frequency of the flexible payload: \( \frac{k_p}{m_p} = \omega_p^2 \).

Finally, the payload transfer function \( H_p(s) \) - Eq. (3.100) - can be used to express the displacement of the flexible payload as given by the following equation:

\[ x_p(s) = H_p(s) \cdot x_{ta}(s) = H_p(s) \cdot (x_i(s) + x_b(s)) \] (3.101)

![Figure 3.19 SDOF payload: Free Body Diagram](image-url)
3.6.3 Payload Shear $F_s(t)$

For a SDOF dynamic system the base shear can have different values, depending on the way damping is physically realized in the system, as it is schematically represented in Figure 3.20 (a) and Figure 3.20 (b).

*Case I*: the payload has a physically external damper.

For systems in which a physical damper is connected to the base (Figure 3.20 (a)), the base shear equals the mass of the system times its absolute acceleration:

$$F_s = -m_p \cdot \ddot{x}_p = c_p \cdot x_p + k_p \cdot x_p$$

(3.102)

by passing to Laplace notation Eq. (3.102) becomes:

$$F_s(s) = -s^2 \cdot m_p \cdot x_{pa}(s) = -s^2 \cdot m_p \cdot [x_b(s) + x_t(s) + x_p(s)]$$

substituting the expression of $x_p(s)$ given by Eq. (3.101), we obtain:

$$F_s(s) = -s^2 \cdot m_p \cdot \{x_b(s) + x_t(s) + H_p(s) \cdot [x_b(s) + x_t(s)]\}$$

Finally, rearranging the terms:

$$F_s(s) = -s^2 \cdot m_p \cdot [1 + H_p(s)] \cdot [x_b(s) + x_t(s)]$$

(3.103)

*Case II*: the payload has only internal damping.

For systems in which the damping derives from some dissipating effects that occurs inside the system itself (and no actual dampers is connected to the base, as represented in Figure 3.20 (b)) the base shear equals the SDOF displacement times the lateral stiffness of the
system, as expressed in the following equation:

$$F_s = k_p \cdot x_p$$  \hspace{1cm} (3.104)

That, by passing to Laplace notation and using Eq. (3.101), can be expressed as follows:

$$F_s(s) = k_p \cdot x_p(s) = k_p \cdot H_p(s) \cdot [x_r(s) + x_b(s)]$$  \hspace{1cm} (3.105)

Figure 3.20  Case # 1 and Case # 2. two different system of Damping:
(a)  Case # 1: Externally Damped SDOF
(b)  Case # 2: Internally Damped SDOF
3.6.4 Modified Base Transfer Function \( B'(s) \)

From the free body diagram of Figure 3.18 (b), it can be derived the differential equation of the motion of the base, when shaken by the external forces applied both by the motion of the actuator and by the shear transmitted by the flexible payload:

\[
m_b \cdot \ddot{x}_b(t) + c_b \cdot \dot{x}_b(t) + k_b \cdot x_b(t) = -m_t \cdot \ddot{x}_{ta}(t) + F_s(t)
\]

(3.106)

where \( c_b \) and \( k_b \) are the coefficients that represents the dynamic characteristics of the base.

By substituting in Eq. (3.106) one of the two expression for \( F_s(t) \) found in the previous chapter is possible to obtain a new for the base transfer function that apply to each specific case.

**Case I: the payload has a physically external damper.**

By subsisting in Eq. (3.106) the correct expression for \( F_s(t) \) - Eq. (3.103) -, the expression for \( x_{ta}(t) \) - Eq. (3.79) - and switching to Laplace notation, we obtain:

\[
s^2 \cdot m_b \cdot x_b(s) + s \cdot c_b \cdot x_b(s) + k_b \cdot x_b(s) =
\]

\[
= s^2 \cdot \{ m_t \cdot [x_b(s) + x_t(s)] + m_p \cdot [1 + H_p(s)] \cdot [x_b(s) + x_t(s)] \}
\]

(3.107)

where all the symbols have the meaning explained in the previous chapters.

Then, rearranging the terms:

\[
x_b(s) \cdot \left\{ s^2 \cdot [m_t + m_b + m_p \cdot (1 + H_p(s))] + s \cdot c_b + k_b \right\} =
\]

\[
= -s^2 \cdot x_t(s) \cdot [m_t + m_p \cdot (1 + H_p(s))]
\]

(3.108)
From which we can finally get the mathematical expression for the base transfer function $B'(s)$ that accounts for the flexibility of the payload:

$$B'(s) = \frac{x_b(s)}{x_i(s)} = \frac{-s^2 \cdot m_t \cdot \left\{ 1 + \frac{m_p}{m_t} \cdot [1 + H_p(s)] \right\}}{s^2 \cdot \left\{ m_t + m_b + m_p \cdot [1 + H_p(s)] \right\} + s \cdot c_b + k_b}$$

(3.109)

or, dividing numerator and denominator by $m_T = m_t + m_b$:

$$B'(s) = \frac{x_b(s)}{x_i(s)} = \frac{-s^2 \cdot \frac{m_t}{m_T} \cdot \left\{ 1 + \frac{m_p}{m_t} \cdot [1 + H_p(s)] \right\}}{s^2 \cdot \frac{m_t}{m_T} \cdot \left\{ 1 + \frac{m_b}{m_t} + \frac{m_p}{m_t} \cdot [1 + H_p(s)] \right\} + s \cdot 2 \cdot \zeta_b \cdot \omega_b + \omega_b^2}$$

(3.110)

where:

- $\omega_b^2 = \frac{k_b}{m_T}$, and $\omega_b$ is the circular natural frequency of the flexible foundation (base);

- $2 \cdot \omega_b \cdot \zeta_b = \frac{c_b}{m_T}$, and $\zeta_b$ is the critical damping coefficient for the flexible foundation.

*Case II: the payload has only internal damping.*

By subsisting in Eq. (3.106) the correct expression for $F_s(t)$ - Eq. (3.105) -. the expression for $x_{ta}(t)$ - Eq. (3.79) -. and switching to Laplace notation, we obtain:

$$s^2 \cdot m_b \cdot x_b(s) + s \cdot c_b \cdot x_b(s) + k_b \cdot x_b(s) =$$

$$= -s^2 \left\{ m_t \cdot [x_b(s) + x_i(s)] - \frac{k_p}{s^2} \cdot H_p(s) \cdot [x_b(s) + x_i(s)] \right\}$$

(3.111)
where all the symbols have the meaning explained in the previous chapters.

Then, rearranging the terms:

\[
x_b(s) \cdot \left\{ s^2 \cdot [m_t + m_b] + s \cdot c_b + k_b - k_p \cdot H_p(s) \right\} =
\]

\[
= -s^2 \cdot x_t(s) \cdot \left[ m_t - \frac{k_p}{s^2} \cdot H_p(s) \right]
\]

From which we can finally get the mathematical expression for the base transfer function that accounts for the flexibility of the payload \(B'(s)\):

\[
B'(s) = \frac{x_b(s)}{x_t(s)} = \frac{-s^2 \cdot m_t + k_p \cdot H_p(s)}{s^2 \cdot [m_t + m_b] + s \cdot c_b + k_b - k_p \cdot H_p(s)} \tag{3.113}
\]

or, dividing numerator and denominator by \(m_T = m_t + m_b\):

\[
B'(s) = \frac{x_b(s)}{x_t(s)} = \frac{-s^2 \cdot \frac{m_t}{m_T} + \frac{k_p}{m_T} \cdot H_p(s)}{s^2 \cdot \frac{m_t}{m_T} \cdot \left[ 1 + \frac{m_b}{m_t} \right] + s \cdot 2 \cdot \frac{c_b}{m_t} + \frac{\omega_n^2}{m_T} - \frac{k_p}{m_T} \cdot H_p(s)} \tag{3.114}
\]

where, as seen before:

- \(\omega_n^2 = \frac{k_b}{m_T}\), and \(\omega_n\) is the circular natural frequency of the flexible foundation (base);
- \(2 \cdot \omega_n \cdot \zeta_b = \frac{c_b}{m_T}\), and \(\zeta_b\) is the critical damping coefficient for the flexible foundation.
3.6.5 Force in the Actuator $F_a(t)$

As anticipated in the previous paragraphs the presence of a model on the shaking table affects the value of the force in the actuator $F_a(t)$. From the free body diagram represented in Fig. 3.21, the force in the actuator - $F_a(t)$ - can be expressed as:

$$F_a(t) = m_t \cdot \dot{x}_{ta}(t) - F_s(t)$$

As seen before, depending on the characteristics of the SDOF payload there are two different expressions for $F_s(t)$ that can be used.

**Case 1: the payload has a physically external damper.**

By passing to Laplace notation, substituting the expression for $x_{ta}(t)$ - Eq. (3.79) - and using - Eq. (3.103) - for $F_s(t)$, we obtain:

$$F_a(s) = s^2 \cdot m_t \cdot [x_t(s) + x_b(s)] + m_p \cdot [1 + H_p(s)] \cdot [x_t(s) + x_b(s)]$$

that can be developed into:

$$F_a(s) = s^2 \cdot m_t \cdot x_t(s) \cdot \left\{ 1 + \frac{x_b(s)}{x_t(s)} + \frac{m_p}{m_t} \cdot [1 + H_p(s)] \cdot \left[ 1 + \frac{x_b(s)}{x_t(s)} \right] \right\}$$

then, recalling the definition of $B'(s) = \frac{x_b(s)}{x_t(s)}$, we can finally obtain:

$$F_a(s) = s^2 \cdot m_t \cdot x_t(s) \cdot \left\{ 1 + B'(s) + \frac{m_p}{m_t} \cdot [1 + H_p(s)] \cdot [1 + B'(s)] \right\} \quad (3.115)$$

For easiness of further computation it is appropriate to write Eq. (3.115) in the following
form:

$$F_a(s) = s^2 \cdot m_t \cdot x_t(s) \cdot H_{F_2}(s) \tag{3.116}$$

where $H_{F_2}(s)$ is an operator defined as follows:

$$H_{F_2}(s) = 1 + B'(s) + \frac{m_p}{m_t} \cdot [1 + H_p(s)] \cdot [1 + B'(s)] \tag{3.117}$$

Case II: the payload has only internal damping.

By passing to Laplace notation, substituting the expression for $x_{ta}(t)$ - Eq. (3.79) - , and using - Eq. (3.105) - for $F_a(t)$, we obtain:

$$F_a(s) = s^2 \cdot m_t \cdot [x_t(s) + x_b(s)] - k_p \cdot H_p(s) \cdot [x_b(s) + x_t(s)]$$

that can be developed into:

$$F_a(s) = s^2 \cdot m_t \cdot x_t(s) \cdot \left\{ 1 + \frac{x_b(s)}{x_t(s)} - \frac{k_p}{s^2 \cdot m_t} \cdot H_p(s) \cdot \left[ 1 + \frac{x_b(s)}{x_t(s)} \right] \right\}$$

then, recalling the definition of $B'(s) = \frac{x_b(s)}{x_t(s)}$, we can finally obtain:

$$F_a(s) = s^2 \cdot m_t \cdot x_t(s) \cdot \left\{ 1 + B'(s) - \frac{k_p}{s^2 \cdot m_t} \cdot H_p(s) \cdot [1 + B'(s)] \right\} \tag{3.118}$$

For easiness of further computation it is appropriate to write Eq. (3.159) in the following form:

$$F_a(s) = s^2 \cdot m_t \cdot x_t(s) \cdot H_{F_3}(s) \tag{3.119}$$
where $H_{F3}(s)$ is an operator defined as follows:

$$H_{F3}(s) = 1 + B'(s) - \frac{k_p}{s} \cdot \frac{H_p(s)}{s} \cdot [1 + B'(s)]$$

(3.120)

Figure 3.21  Actuator Free Body Diagram
3.6.6 Modified System Transfer Function $H''(s)$

As it was the case in paragraph 3.5.4 the new expression for the force in the actuator affects the transfer function of the system in two points:

a) The transfer function of the servo valve $S(s)$: new expression for the flow due to leakage and new expression for the flow due to fluid compressibility.

b) Controller Model: new expression for the Differential Pressure feedback.

Case 1): the payload has a physically external damper.

a) Servovalve Model $S''(s)$.

Substituting the correct expression for the force in the actuator - Eq. (3.116) - in the computation for the flow due to leakage (Eq. (3.40))

$$q_{le}(s) = k_{le} \cdot F_a(s)$$

we obtain:

$$q_{le}(s) = x_t(s) \cdot s^2 \cdot m_t \cdot k_{le} \cdot H_{F2}(s)$$

Substituting the new expression for the force in the actuator - Eq. (3.116) - in the computation for the flow due to the fluid compressibility (Eq. (3.44)):

$$q_{com}(s) = s \cdot \frac{V}{4 \cdot \beta \cdot A} \cdot F_a(s)$$

we obtain:

$$q_{com}(s) = x_t(s) \cdot s^3 \cdot \frac{V \cdot m_t}{4 \cdot \beta \cdot A} \cdot H_{F2}(s)$$
Then, the substitution of the new expression for the flow due to leakage (Eq. (3.121)) and of the new expression for the flow due to the fluid compressibility (Eq. (3.122)) into the flow equation (Eq. (3.36))

\[ q_s(s) = q_{am}(s) + q_{le}(s) + q_{com}(s) \] (3.36)

leads, after some simple computations, to the following expression of the servovalve transfer function\(^1\):

\[
S''(s) = \frac{x_t(s)}{x_c(s)} = \frac{H_t(s)}{s^3 \cdot \frac{V_m}{4\beta A} \cdot H_{F2}(s) + s^2 m_t \cdot k_{le} \cdot H_{F2}(s) + s \cdot A} \tag{3.123}
\]

where \(H_t(s)\), as seen before, represents the three stage servovalve transfer function as defined by: \(q_t(s) = H_t(s) \cdot x_c(t)\).

\(b)\quad The\ Controller\ Model.\)

The new formulation for the force in the actuator changes the expression of the signal given by the Delta Pressure feedback. Substituting the new expression for the force in the actuator given by Eq. (3.116) into the Delta Pressure feedback signal expression - (Eq. (3.70)):

\[ x_d(t) = \frac{K_{dp}}{A} \cdot F_a(t) \] (3.70)

we get:

\(1.\) This new expression can also be obtained by direct substitution of the new expression for the force in the actuator \(F_a(t)\) - given by Eq. (3.116) - into (3.45). This procedure is faster but does not provide the physical insight given by the procedure described in this paragraph.
\[ x_d(t) = \frac{K_{dp}}{A} \cdot x_i(s) \cdot s^2 \cdot m_t \cdot H_{F2}(s) \]  \hspace{1cm} (3.124)

Or, in standard form:

\[ x_d(s) = P_{DP}(s) \cdot x_i(s) \cdot H_{F2}(s) \]  \hspace{1cm} (3.125)

where \( P_{DP}(s) = \frac{s^2 \cdot K_{dp} \cdot m_t}{A} \).

Substituting the expression for \( x_d(s) \) - Eq. (3.125) - into the expression for the controller model (Eq. (3.57))

\[ x_c(t) = \varepsilon(t) + x_f(t) + x_d(t) \]  \hspace{1cm} (3.57)

and performing the same computation carried on in paragraph 3.3.5, we can obtain:

\[ x_c(s) = P_{PID}(s) \cdot [\tilde{x}_c(s) - x_i(s)] + P_{FF}(s) \cdot \tilde{x}_c(s) + P_{DP}(s) \cdot x_i(s) \cdot H_{F2}(s) \]  \hspace{1cm} (3.126)

*System Transfer Function \( H''(s) \).*

Combining the new expression of the servovalve transfer function \( S''(s) \) - Eq. (3.123) - with the new expression of the controller model - Eq. (3.126) - it can be derived the new expression of the system transfer function \( H''(s) \):

\[ H''(s) = \frac{x_i(s)}{\tilde{x}_c(s)} = \frac{S''(s)[P_{FF}(S) + P_{PID}(s)]}{1 + S''(s) \cdot [A(s) \cdot P_{PID}(s) - P_{DP}(s) \cdot H_{F2}(s)]} \]  \hspace{1cm} (3.127)
Or, substituting the values of the transfer functions P's derived for each single Gain Component:

\[ H''(s) = \frac{S''(s) \cdot \left( s \cdot K_f + K_{pro} + \frac{1}{s} \cdot K_{int} + s \cdot K_{der} \right)}{1 + S''(s) \cdot \left( K_{pro} + \frac{1}{s} \cdot K_{int} + s \cdot K_{der} - \left( K_{dp} \cdot \frac{s^2 \cdot m_i}{A} \right) \cdot H_{F2}(s) \right)} \]  (3.128)
Case II: the payload has only internal damping.

a) Servovalve Model \(S''(s)\)

Substituting the correct expression for the force in the actuator - Eq. (3.119) - in the computation for the flow due to leakage (Eq. (3.40))

\[
q_{le}(s) = k_{le} \cdot F_3(s) \tag{3.40}
\]

we obtain:

\[
q_{le}(s) = x_t(s) \cdot s^2 \cdot m_t \cdot k_{le} \cdot H_F(s) \tag{3.129}
\]

Substituting the new expression for the force in the actuator - Eq. (3.119) - in the computation for the flow due to the fluid compressibility (Eq. (3.44)):

\[
q_{com}(s) = s \cdot \frac{V}{4 \cdot \beta \cdot A} \cdot F_3(s) \tag{3.44}
\]

we obtain:

\[
q_{com}(s) = x_t(s) \cdot s^3 \cdot \frac{V \cdot m_t}{4 \cdot \beta \cdot A} \cdot H_F(s) \tag{3.130}
\]

Then, the substitution of the new expression for the flow due to leakage (Eq. (3.129)) and of the new expression for the flow due to the fluid compressibility (Eq. (3.130)) into the flow equation (Eq. (3.36))

\[
q_s(s) = q_{am}(s) + q_{le}(s) + q_{com}(s) \tag{3.36}
\]
leads, after some simple computations, to the following expression of the servovalve transfer function:

\[
S''(s) = \frac{x_t(s)}{x_c(s)} = \frac{H_t(s)}{s^3 \cdot \frac{V}{4BA} \cdot H_{F3}(s) + s^2 m_t \cdot k_{te} \cdot H_{F3}(s) + s \cdot A}
\]  (3.131)

where \(H_t(s)\), as seen before, represents the three stage servovalve transfer function as defined by: \(q_s(s) = H_t(s) \cdot x_c(t)\).

\[b) \quad \text{The Controller Model.}\]

The new formulation for the force in the actuator changes the expression of the signal given by the Delta Pressure feedback. Substituting the new expression for the force in the actuator given by Eq. (3.119) into the Delta Pressure feedback signal expression - (Eq. (3.70)):

\[
x_d(t) = \frac{K_{dp}}{A} \cdot F_a(t)
\]  (3.70)

we get:

\[
x_d(t) = \frac{K_{dp}}{A} \cdot x_t(s) \cdot s^2 \cdot m_t \cdot H_{F3}(s)
\]  (3.132)

Or, in standard form:

\[
x_d(s) = P_{DP}(s) \cdot x_t(s) \cdot H_{F3}(s)
\]  (3.133)

1. This new expression can also be obtained by direct substitution of the new expression for the force in the actuator \(F_a(t)\) - given by Eq. (3.119) - into (3.45). This procedure is faster but does not provide the physical insight given by the procedure described in this paragraph.
where \( P_{DP}(s) = \frac{s^2 \cdot K_{dp} \cdot m_t}{A} \).

Substituting the expression for \( x_d(s) \) - Eq. (3.133) - into the expression for the controller model (Eq. (3.57))

\[
    \dot{x}_c(t) = \varepsilon(t) + \dot{x}_f(t) + x_d(t) \tag{3.57}
\]

and performing the same computation carried on in paragraph 3.3.5, we can obtain:

\[
x_c(s) = P_{PID}(s) \cdot (\ddot{x}_c(s) - \dot{x}_t(s)) + P_{FF}(s) \cdot \dot{x}_c(s) + P_{DP}(s) \cdot x_t(s) \cdot H_{F3}(s) \tag{3.134}
\]

**System Transfer Function.**

Combining the new expression of the servovalve transfer function \( S''(s) \) - Eq. (3.131) - with the new expression of the controller model - Eq. (3.134) -, it can be derived the new expression of the system transfer function \( H''(s) \):

\[
H''(s) = \frac{x_t(s)}{\ddot{x}_c(s)} = \frac{S''(s)[P_{FF}(s) + P_{PID}(s)]}{1 + S''(s) \cdot [A(s) \cdot P_{PID}(s) - P_{DP}(s) \cdot H_{F3}(s)]} \tag{3.135}
\]

Or, substituting the values of the transfer functions P’s derived for each single Gain Component:

\[
H''(s) = \frac{S''(s) \cdot [s \cdot K_f + K_{pro} + \frac{1}{s} \cdot K_{int} + s \cdot K_{der}]}{1 + S''(s) \cdot \left[ K_{pro} + \frac{1}{s} \cdot K_{int} + s \cdot K_{der} - \left( K_{dp} \cdot \frac{s^2 \cdot m_t}{A} \right) \cdot H_{F3}(s) \right]} \tag{3.136}
\]
3.6.7 Modified Table Transfer Function, \( T'(s) \)

As described in paragraph 3.6.1, the table Transfer Function when a flexible payload (SDOF) is applied on the table can be obtained by substituting the expressions for \( H''(s) \) and \( B'(s) \) (respectively Eq. (3.123) and Eq. (3.127) for Case # I)\(^1\) into Eq. (3.96).

\[
T'(s) = \frac{x_b(s) + x_t(s)}{x_c(s)} = \frac{x_t(s)}{x_c(t)} \left( \frac{x_b(s)}{x_t(s)} + 1 \right) = H''(s) \cdot (B'(s) + 1)
\]  

(3.96)

The results obtained are plotted in the next paragraphs.

3.6.8 Difference in Table T.F. due to different Damping Modeling

Depending on the characteristics of the dynamic SDOF applied on the table we have determined two different expression for the table transfer function, respectively referred as Case I) - when the payload has an external damper - and as Case II) - when the payload is only internally damped.

In Figure 3.22 (a) and Figure 3.22 (b) are shown respectively the magnitude and the phase transfer function for the following three situations:

- bare table;
- table loaded with a SDOF payload modeled as Case # 1;
- table loaded with a SDOF payload modeled as Case # 2;

The payload has the following characteristics:

- \( \omega_p = 62.8 \ \text{rad/sec.} \)

\(^1\) or Eq. (3.131) and Eq. (3.135) for Case # II).
• \( \zeta_p = 0.02 \):

• \( m_p = 2.356 \). that correspond to a weight of 900 lbs.

These values correspond to a SDOF with a natural frequency of 10 Hz and a mass that weights 900 lbs. These values were selected in order to have results that are comparable to the one obtained in the experimental analysis reported in Chapter 5. All the other parameters have the same values used to plot the Table Transfer Function in paragraph 3.4.2 and in paragraph 3.5.5.

**Magnitude Transfer Function (Figure 3.22 (a))**

Comparing the transfer functions for the two cases it is evident how the two cases have similar effects. The peak that rises in correspondence of the payload natural frequency is exactly the same for both cases. Only the oil column peak seems to be affected much more by *Case I* than by *Case II* that leaves the peak almost unchanged when compared to the case of the bare table.

**Phase Transfer Function (Figure 3.22 (b))**

The Transfer Function shows that no appreciable differences exist in the phase between *Case # I* and *Case # II*.

It is to be pointed out that the differences of the two cases are small. in fact the transfer functions represented in Figure 3.22 (a) and Figure 3.22 (b) are relative to a payload with a mass that is large when compared to the one of the table (three fourths) For payload of smaller mass the differences between *Case # I* and *Case # II* are almost impossible to
detect, as shown in Figure 3.23, where it is plotted the magnitude Transfer Functions for a payload that has the same characteristics of the previous one, but that weights only 300 lbs (as opposed to 900 lbs of the previous one).

Figure 3.23  Magnitude Transfer Function for Case # I and Case # II
(payload of 300 lbs and bare table)
Figure 3.22  Table Transfer Function, Case # I. Case # II
(payload of 900 lbs and bare table)
(a) Magnitude          (b) Phase
3.6.9 Results

In this paragraph we plot the numerical values of the table transfer function for case 1: an actual damper is mounted on the payload and connected to the table base. The characteristics of the payload are the same ones seen in the previous chapter:

- $\omega_p = 62.8$ rad/sec.
- $\zeta_p = 0.02$
- $m_p = 2.356$, that correspond to a weight of 900 lbs.

These values correspond to a SDOF with a natural frequency of 10 Hz and a mass that weights 900 lbs. These values were selected in order to have results that are comparable to the one obtained in the experimental analysis reported in Chapter 5.

Magnitude Transfer Function.

Analyzing Figure 3.24 (a), where it is represented the magnitude table transfer function. it is clear how the flexible payload introduces a peak in correspondence of the natural frequency of the payload itself. In Figure 3.24 (b) are compared the transfer function of a flexible payload, the transfer function of a rigid payload of equivalent mass and the transfer function of the bare table. This figure shows clearly how the addition of a payload on the shaking table almost does not affect the frequency of the oil column resonance peak and only a small increase in its frequency and amplitude is noticeable. For more details on this behaviors see the table sensitivity analysis reported in paragraph 4.3

Phase Transfer Function.

Figure 3.25 (a) shows the phase of the transfer function when a flexible payload is
loaded on the table. Even though the mass of the payload in analysis is quite big (it is equal to three fourths of the mass of the table) the only appreciable effect on the phase is a very small notch at the payload frequency.

In Figure 3.25 (b) are compared the phase transfer functions of a flexible payload, the transfer function of a rigid payload of equivalent mass and the transfer function of the bare table. This figure shows clearly how the addition of a payload on the shaking tab basically creates only a notch in correspondence of the payload frequency. The presence of a flexible payload on the table does not affect the inversion frequency (that correspond to the oil column resonance peak) and, as it was the case for the magnitude, only a small increase in the frequency is noticeable.
Figure 3.24 Magnitude of the Table Transfer Function:

(a) Table with flexible base and flexible payload (SDOF)
(b) Table with flexible base and: bare table, flexible payload and corresponding rigid payload
Figure 3.25 Phase of the Table Transfer Function:

(a) Table with flexible base and flexible payload (SDOF)

(b) Table with flexible base and: bare table, flexible payload and corresponding rigid payload
3.7 EFFECT OF FLEXIBLE PAYLOAD (MDOF)

3.7.1 Introduction

If the payload requires to be modeled as multi degree of freedom system the Table Transfer Function derived in paragraph 3.6 for a SDOF must be modified in order to take into account the effects due to the more complicated dynamic system.

In this paragraph we derive a mathematical expression for the system transfer function that takes into account the effect of a flexible payload (SDOF).

As it was the case in paragraph 3.6 (Eq. (3.96)), the transfer function between the given input and the absolute motion of the table, when loaded with a flexible MDOF payload, can be expressed (in Laplace notation) as follows:

\[ T''(s) = \frac{x_b(s) + x_t(s)}{x_c(s)} = \frac{x_t(s)}{x_c(t)} \cdot \left( \frac{x_b(s)}{x_t(s)} + 1 \right) = H''(s) \cdot (B''(s) + 1) \]  

(3.137)

where:

- \( B''(s) = \frac{x_b(s)}{x_t(s)} \) is the Base transfer function modified to take into account the effect of the flexible MDOF payload, this expression will be derived in paragraph 3.7.4:

- \( H''(s) = \frac{x_t(s)}{x_c(t)} \) is the System transfer function modified to take into account the motion of the base and the effect of the flexible MDOF payload; this expression will be derived in paragraph 3.7.6.
3.7.2 MDOF Payload Transfer Function $H_{pm}(s)$

From the free body diagram of Figure 3.26 the differential equation of motion for the MDOF system can be written in usual matrix form as follows:

$$[m_p] \cdot \{ \ddot{x}_p \} + [c_p] \cdot \{ \dot{x}_p \} + [k_p] \cdot \{ x_p \} = -[m_p] \cdot \{ 1 \} \cdot (\dot{x}_b(t) + \ddot{x}_t(t))$$  \hspace{1cm} (3.138)

where:

- $[m_p] = \begin{bmatrix} m_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & m_N \end{bmatrix}$ is the mass matrix of the payload;

- $[c_p] = \begin{bmatrix} c_{p11} & \cdots & c_{p1N} \\ \vdots & \ddots & \vdots \\ c_{pN1} & \cdots & c_{pNN} \end{bmatrix}$ represent damping matrix of the payload;

- $[k_p] = \begin{bmatrix} k_{p11} & \cdots & k_{p1N} \\ \vdots & \ddots & \vdots \\ k_{pN1} & \cdots & k_{pNN} \end{bmatrix}$ is the lateral stiffness matrix of the payload;

- $\{ x_p(t) \} = \begin{bmatrix} x_{p1} \\ \vdots \\ x_{pN} \end{bmatrix}$ is the vector of payload relative displacements;

- $\{ 1 \} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ is a vector of ones;

- $x_b(t)$ is the base displacement;

- $x_t(t)$ is the table displacement.
Passing to the usual normal coordinates \( \{ \text{Y}(t) \} = \begin{bmatrix} \text{y}_1(t) \\ \text{y}_2(t) \\ \vdots \\ \text{y}_N(t) \end{bmatrix} \), that are related to the geometric coordinates, \( \{ x_p(t) \} \), by:

\[
\{ x_p(t) \} = [\Phi] \cdot \{ \text{Y}(t) \} = \sum_{n=1}^{N} \{ \phi_n \} \cdot \text{y}_n(t)
\]

(3.139)

where:

- \([\Phi] = \begin{bmatrix} \{ \phi_1 \} \\ \{ \phi_2 \} \\ \vdots \\ \{ \phi_n \} \end{bmatrix}\) is the matrix of mode shapes:

Substituting Eq. (3.39) in Eq. (3.138) and pre-multiplying by \([\Phi]^T\). using the orthogonality properties of \([\Phi]\) with respect to \([m_p]\) and \([k_p]\). and assuming orthonormal (classical) damping, it is possible to obtain the following matricial differential equation in which each row is de-coupled from the others:

\[
[M] \cdot \{ \ddot{\text{Y}}(t) \} + [C] \cdot \{ \dot{\text{Y}}(t) \} + [K] \cdot \{ \text{Y}(t) \} = -[\Phi]^T \cdot [m_p] \cdot \{ 1 \} \cdot (\ddot{x}_b(t) + \ddot{x}_i(t))
\]

(3.140)

or, in Laplace notation:

\[
s^2 \cdot [M] \cdot \{ \text{Y}(s) \} + s \cdot [C] \cdot \{ \text{Y}(s) \} + [K] \cdot \{ \text{Y}(s) \} = -[\Phi]^T \cdot [m_p] \cdot \{ 1 \} \cdot (\ddot{x}_b(s) + \ddot{x}_i(s))
\]

(3.141)
where:

- \([M] = [\Phi]^T \cdot [m_p] \cdot [\Phi] = \begin{bmatrix} M_1 & 0 & \ldots & 0 \\ 0 & M_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & M_N \end{bmatrix}\) is the diagonal generalized mass matrix:

- \([C] = [\Phi]^T \cdot [c_p] \cdot [\Phi] = \begin{bmatrix} C_1 & 0 & \ldots & 0 \\ 0 & C_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & C_N \end{bmatrix}\) is the diagonal generalized damping matrix:

- \([K] = [\Phi]^T \cdot [k_p] \cdot [\Phi] = \begin{bmatrix} K_1 & 0 & \ldots & 0 \\ 0 & K_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & K_N \end{bmatrix}\) is the diagonal generalized stiffness matrix:

- \(\{\phi_n\} = \begin{bmatrix} \phi_{n1} \\ \phi_{n2} \\ \vdots \\ \phi_{nN} \end{bmatrix}\) is the \(n\)-th mode shape vector.
Figure 3.26  Free Body Diagram of the MDOF
From Eq. (3.141) it can be derived the n-th modal equation of motion in Laplace notation (thanks to uncoupling):

\[
s^2 \cdot M_n \cdot y_n(s) + s \cdot C_n \cdot y_n(s) + K_n \cdot y_n(s) = -\{\phi_n\}^T \cdot [m_p] \cdot \{1\} \cdot (\dot{x}_b(s) + \dot{x}_i(s))
\]

(3.142)

from Eq. (3.142), the dynamic response of each normal coordinates can be written as:

\[
y_n(s) = \frac{\{\phi_n\}^T \cdot [m_p] \cdot \{1\}}{M_n} \cdot \frac{s^2}{s^2 + 2 \cdot \zeta_n \cdot s \cdot \omega_n + \omega_n^2} \cdot (x_b(s) + x_i(s))
\]

(3.143)

where:

- \(\omega_n^2 = \frac{K_n}{M_n}\) is the circular natural frequency of vibration of the nth mode of vibration:

- \(\zeta_n = \frac{C_n}{2 \cdot M_n \cdot \omega_n}\) is the damping ratio of the nth mode of vibration.

Using Eq. (3.139) it is possible to obtain the dynamic response in Laplace Notation of the MDOF payload in geometric coordinates as:

\[
\{x_p(s)\} = -\sum_{n=1}^{N} \{\phi_n\}^T \cdot [m_p] \cdot \{1\} \cdot \frac{s^2}{s^2 + 2 \cdot \zeta_n \cdot s \cdot \omega_n + \omega_n^2} \cdot (x_b(s) + x_i(s))
\]

(3.144)

from which, it can obtained the payload transfer function \(H_p(s)\):
\[
\{ H_{pm}(s) \} = \frac{\{ x_p(s) \}}{x_{ta}(s)} = -\sum_{n=1}^{N} \{ \phi_n \}^T \cdot \left[ \frac{\{ m_p \} \cdot \{ 1 \}}{M_n} \right] \cdot \frac{s^2}{s^2 + 2 \cdot \zeta_n \cdot s \cdot \omega_n + \omega_n^2}
\]

(3.145)

From Eq. (3.145) the displacement of the MDOF payload can be expressed as:

\[
\{ x_p(s) \} = \{ H_{pm}(s) \} \cdot x_{ta}(s)
\]

(3.146)

---

Figure 3.27  Coordinate System
3.7.3 Payload Shear $F_s(t)$

Since we are dealing with shear building models, the expression of the payload transfer function can also be used to express the shear force $F_s(t)$ that the flexible payload (through its base) transmit to the table. As it is shown in Figure 3.28, the shear force, $F_s(t)$, can be obtained as the sum of the products between the story masses $m_i$ and their respective absolute acceleration $\ddot{x}_{pa}(t)$.

$$F_s(t) = -\sum_{i=1}^{N} m_i \cdot \ddot{x}_{pa}(t) = -\{1\}^T \cdot [m_p] \cdot \{\ddot{x}_{pa}(t)\} \quad (3.147)$$

where:

- $\{\ddot{x}_{pa}(t)\} = \begin{bmatrix} \ddot{x}_{pa1}(t) \\ \vdots \\ \ddot{x}_{pan}(t) \end{bmatrix}$ is the vector of the absolute acceleration of the MDOF payload.

In Laplace notation Eq. (3.147) becomes:

$$F_s(s) = -s^2 \sum_{i=1}^{N} m_i \cdot x_{pa}(s) = -s^2 \cdot \{1\}^T \cdot [m_p] \cdot \{x_{pa}(s)\} \quad (3.148)$$

Eq. (3.79) shows that $x_{pa}(t) = x_{pl}(t) + x_{ta}(t)$, with $x_{ta}(t)$ absolute displacement of the table. Switching to vectorial notation and passing to Laplace transform, the absolute displacement of the MDOF payload can be expressed as:

$$\{x_{pa}(s)\} = \{x_{p}(s)\} + \{1\} \cdot x_{ta}(s) \quad (3.149)$$

1. For the coordinate system see Figure 3.27.
Substituting Eq. (3.149) into Eq. (3.148), we obtain:

\[ F_s(s) = -s^2 \cdot \{ I \}^T \cdot \{ m_p \} \cdot (\{ x_p(s) \} + \{ 1 \} \cdot x_{ta}(s)) \]  

(3.150)

Substituting in Eq. (3.150) the expression of \{ x_p(s) \} given by Eq. (3.146), we obtain:

\[ F_s(s) = -s^2 \cdot \{ I \}^T \cdot \{ m_p \} \cdot (\{ H_{pm}(s) \} \cdot x_{ta}(s) + \{ 1 \} \cdot x_{ta}(s)) \]

Finally, rearranging the terms, it is possible to obtain:

\[ F_s(s) = -s^2 \cdot \{ I \}^T \cdot \{ m_p \} \cdot (\{ H_{pm}(s) \} + \{ 1 \}) \cdot x_{ta}(s) \]  

(3.151)

Or, the equivalent expression:

\[ F_s(s) = -s^2 \cdot \{ I \}^T \cdot \{ m_p \} \cdot (\{ H_{pm}(s) \} + \{ 1 \}) \cdot (x_b(s) + x_t(s)) \]  

(3.152)

For simplicity of further computation it is advisable to express Eq. (3.152) in the following form:

\[ F_s(s) = -s^2 \cdot H_{pm1}(s) \cdot (x_b(s) + x_t(s)) \]  

(3.153)

where:

\[ H_{pm1}(s) = \{ I \}^T \cdot \{ m_p \} \cdot (\{ H_{pm}(s) \} + \{ 1 \}) \]
Figure 3.28  Shear Force transmitted by the MDOF payload to the shaking table
3.7.4 Modified Base Transfer Function $B''(s)$

Similarly to what has been done for the SDOF dynamic system, from the free body diagram of Figure 3.29 it can be derived the differential equation of the motion of the base:

$$m_b \cdot \ddot{x}_b(t) + c_b \cdot \dot{x}_b(t) + k_b \cdot x_b(t) = -m_t \cdot \ddot{x}_{ta}(t) + F_s(t)$$  \hspace{1cm} (3.154)

where as seen before $c_b$ and $k_b$ are the coefficients that represents the dynamic characteristics of the base. By switching to Laplace notation and substituting the expression for $x_{ta}(t)$ - Eq. (3.79) -, and the new expression for the shear $F_s(t)$ - Eq. (3.153) -, we obtain:

$$s^2 \cdot m_b \cdot x_b(s) + s \cdot c_b \cdot x_b(s) + k_b \cdot x_b(s) =$$

$$= -s^2 \cdot \left\{ m_t \cdot \left[ x_b(s) + x_t(s) \right] + H_{pm1}(s) \cdot \left[ x_b(s) + x_t(s) \right] \right\}$$

(3.155)

where all the symbols have the meaning explained in the previous chapters.

Then, rearranging the terms:

$$x_b(s) \cdot \left\{ s^2 \cdot \left[ m_t + m_b + H_{pm1}(s) \right] + s \cdot c_b + k_b \right\} =$$

$$= -s^2 \cdot x_t(s) \cdot \left[ m_t + H_{pm1}(s) \right]$$

(3.156)

From which we can finally get the mathematical expression for the base transfer function that accounts for the flexibility of the payload $B''(s)$:
\[ B''(s) = \frac{x_b(s)}{x_t(s)} = \frac{-s^2(m_t + H_{pm1}(s))}{s^2 \cdot [m_t + m_b + H_{pm1}(s)] + s \cdot c_b + k_b} \] 

or, dividing numerator and denominator by \( m_T = m_t + m_b \):

\[ B''(s) = \frac{x_b(s)}{x_t(s)} = \frac{-s^2 \left( \frac{m_t}{m_T} + \frac{H_{pm1}(s)}{m_T} \right)}{s^2 + s \cdot 2 \cdot \zeta_b \cdot \omega_b + \omega_b^2 + \frac{s^2}{m_T} \cdot H_{pm1}(s)} \]

where, as seen before:

- \( \omega_b^2 = \frac{k_b}{m_T} \), and \( \omega_b \) is the circular natural frequency of the flexible foundation (base):

- \( 2 \cdot \omega_b \cdot \zeta_b = \frac{c_b}{m_T} \), and \( \zeta_b \) is the critical damping coefficient for the flexible foundation.

![Fixed Reference System (lab ground)](image_url)

**Figure 3.29 Flexible Base Free Body Diagram**
3.7.5 Force in the Actuator $F_a(t)$

From the free body diagram represented in Figure 3.30 the force in the actuator - $F_a(t)$ - can be expressed as:

$$F_a(t) = m_t \cdot x_{ta}(t) - F_s(t)$$

As seen before, by switching to Laplace notation and substituting the expression for both $x_{ta}(t)$ - Eq. (3.79) - and $F_s(t)$ - Eq. (3.153) -, we obtain:

$$F_a(s) = s^2 \cdot \{ m_t \cdot [x_t(s) + x_h(s)] + H_{pm1}(s) \cdot [x_h(s) + x_t(s)] \}$$

that can be developed into:

$$F_a(s) = s^2 \cdot m_t \cdot x_t(s) \cdot \left\{ 1 + \frac{x_b(s)}{x_t(s)} + \frac{H_{pm1}(s)}{m_t} \cdot \left[ 1 + \frac{x_b(s)}{x_t(s)} \right] \right\}$$

then, recalling the definition of $B''(s) = \frac{x_b(s)}{x_t(s)}$, we can finally obtain:

$$F_a(s) = s^2 \cdot m_t \cdot x_t(s) \cdot \left\{ 1 + B''(s) + \frac{H_{pm1}(s)}{m_t} \cdot [1 + B''(s)] \right\} \quad (3.159)$$

For easiness of further computation it is appropriate to write Eq. (3.159) in the following form:

$$F_a(s) = s^2 \cdot m_t \cdot x_t(s) \cdot H_{F_4}(s) \quad (3.160)$$

where:

$$H_{F_4}(s) = 1 + B''(s) + \frac{H_{pm1}(s)}{m_t} \cdot [1 + B''(s)] \quad (3.161)$$
3.7.6 Modified System Transfer Function $H''''(s)$

As it was the case in paragraph 3.5.4 the new expression for the force in the actuator affects the transfer function of the system in two points:

a) The transfer function of the servovalve $S(s)$: new expression for the flow due to leakage and new expression for the flow due to fluid compressibility.

b) Controller Model: new expression for the Differential Pressure feedback.

a) Servovalve Model $S''''(s)$

Substituting the correct expression for the force in the actuator $F_a(s)$ - Eq. (3.160) - in the computation for the flow due to leakage (Eq. (3.40))

$$q_{le}(s) = k_{le} \cdot F_a(s)$$

we obtain:
\[ q_{le}(s) = x_t(s) \cdot s^2 \cdot m_t \cdot k_{le} \cdot H_{F4}(s) \] (3.162)

Substituting the new expression for the force in the actuator \( F_a(s) \) - Eq. (3.160) - in the computation for the flow due to the fluid compressibility (Eq. (3.44)):

\[ q_{com}(s) = s \cdot \frac{V}{4 \cdot \beta \cdot A} \cdot F_a(s) \] (3.44)

we obtain:

\[ q_{com}(s) = x_t(s) \cdot s^3 \cdot \frac{V \cdot m_t}{4 \cdot \beta \cdot A} \cdot H_{F4}(s) \] (3.163)

Then, the substitution of the new expression for the flow due to leakage (Eq. (3.162)) and of the new expression for the flow due to the fluid compressibility (Eq. (3.163)) into the flow equation (Eq. (3.36))

\[ q_s(s) = q_{am}(s) + q_{le}(s) + q_{com}(s) \] (3.36)

leads, after some simple computations, to the following expression of the servovalve transfer function:

\[ S''''(s) = \frac{x_t(s)}{x_c(s)} = \frac{H_t(s)}{s^3 \cdot \frac{V m_t}{4 \beta A} \cdot H_{F4}(s) + s^2 m_t \cdot k_{le} \cdot H_{F4}(s) + s \cdot A} \] (3.164)

where \( H_t(s) \), as seen before, represents the three stage servovalve transfer function as defined by: \( q_s(s) = H_t(s) \cdot x_c(t) \).

---

1. This new expression can also be obtained by direct substitution of the new expression for the force in the actuator \( F_a(t) \) - given by Eq. (3.148) - into (3.45). This procedure is faster but does not provide the physical insight given by the procedure described in this paragraph.
b) The Controller Model.

The new formulation for the force in the actuator changes the expression of the signal given by the Delta Pressure feedback. Substituting the new expression for the force in the actuator $F_3(s)$ given by Eq. (3.160) into the Delta Pressure feedback signal expression - (Eq. (3.70)):

$$x_d(t) = \frac{K_{dp}}{A} \cdot F_3(t) \quad (3.70)$$

we get:

$$x_d(t) = \frac{K_{dp}}{A} \cdot x_t(s) \cdot s^2 \cdot m_t \cdot H_{F4}(s) \quad (3.165)$$

Or, in standard form:

$$x_d(s) = P_{DP}(s) \cdot x_t(s) \cdot H_{F4}(s) \quad (3.166)$$

where $P_{DP}(s) = \frac{s^2 \cdot K_{dp} \cdot m_t}{A}$.

Substituting the expression for $x_d(s)$ - Eq. (3.166) - into the expression for the controller model (Eq. (3.57))

$$x_c(t) = \varepsilon(t) + x_t(t) + x_d(t) \quad (3.57)$$

and performing the same computation carried on in paragraph 3.3.5. we can obtain:

$$x_c(s) = P_{PID}(s) \cdot [\dot{x}_c(s) - x_t(s)] + P_{FP}(s) \cdot \ddot{x}_c(s) + P_{DP}(s) \cdot x_t(s) \cdot H_{F4}(s) \quad (3.167)$$
System Transfer Function.

Deriving a new expression for the value of \( x_c(s) \) from the new expression of the servov-valve transfer function \( S''''(s) \)\(^1\) - Eq. (3.164) - and substituting it into the new expression of the controller model - Eq. (3.167) -, it can be derived the new expression of the system transfer function \( H''''(s) \):

\[
H''''(s) = \frac{x_t(s)}{x_c(s)} = \frac{S''''(s)[P_{FF}(s) + P_{PID}(s)]}{1 + S''''(s) \cdot [A(s) \cdot P_{PID}(s) - P_{DP}(s) \cdot H_{F4}(s)]}
\]

(3.168)

Or, substituting the values of the transfer functions \( P \)'s derived for each single Gain Component:

\[
H''''(s) = \frac{S''''(s) \cdot \left[ s \cdot K_f + K_{pro} + \frac{1}{s} \cdot K_{int} + s \cdot K_{der} \right]}{1 + S''''(s) \cdot \left[ K_{pro} + \frac{1}{s} \cdot K_{int} + s \cdot K_{der} - \left( K_{dp} \cdot \frac{s^2 \cdot m_t}{A} \right) \cdot H_{F4}(s) \right]}
\]

(3.169)

where the expression for \( H_{F4}(s) \) is given by Eq. (3.161)

---

1. By definition: \( x_c(s) = \frac{x_t(s)}{S''''(s)} \).
3.7.7 Modified Table Transfer Function $T''(s)$

Following the derivation explained in paragraphs 3.5.1 and 3.6.1 the table transfer function $T''(s)$ for a MDOF can be obtained by substitution of the expressions of $H''(s)$ - Eq. (3.168) - and of $B''(s)$ - Eq. (3.158) - into Eq. (3.137):

$$T''(s) = H''(s) \cdot (B''(s) + 1)$$ (3.137)

3.7.8 Results

In this paragraph we plot the numerical values of the transfer function $T''(s)$. The modal parameters that characterize the dynamic properties of the chosen MDOF are the following:\(^{1}\):

Natural circular frequencies:

$$\omega_1 = 23.9 \, \text{radians/second}, \quad \omega_2 = 63.4 \, \text{radians/second}, \quad \omega_3 = 92.6 \, \text{radians/second}$$

Mode shapes:

$$\{\phi_1\} = \begin{bmatrix} 0.2928 \\ 0.5731 \\ 0.7654 \end{bmatrix}, \quad \{\phi_2\} = \begin{bmatrix} 0.6665 \\ 0.4576 \\ -0.5953 \end{bmatrix}, \quad \{\phi_3\} = \begin{bmatrix} 0.6914 \\ -0.6719 \\ 0.2446 \end{bmatrix}$$

Modal damping:

$$\omega_1 = 0.02, \quad \omega_2 = 0.02, \quad \omega_3 = 0.02.$$

These values are relative to the shear system depicted in Figure 3.31.

---

1. The dynamic characteristics of the MDOF payload here represented have been selected in order to match the ones of a real scale model built and tested at Rice University. For comparison with models of buildings tested in other laboratories see Appendix C.
The floor mass $m$ is equal to $m = 0.7772 \, \text{lb} \cdot \text{sec}^{-2} \text{in}^{-1}$ (corresponding to a floor weight of 300 lbs and a total model weight of 900 lbs).

And the lateral stuffiness are: $k_1 = 6097 \, \text{lb} \cdot \text{in}^{-1}, k_2 = 5226 \, \text{lb} \cdot \text{in}^{-1}, k_3 = 4355 \, \text{lb} \cdot \text{in}^{-1}$.

All the other parameters necessary to model the electro-hydraulic system and the flexible base have the values presented in paragraph 3.4.2 and in paragraph 3.5.5.

Figure 3.31 3DOF Shear Frame
Magnitude Transfer Function.

Figure 3.32 (a) shows the magnitude table transfer function when a flexible three degree of freedom payload is added on the table: the table transfer function is affected as follows:

- Three little peaks (perturbations) are originated in correspondence of each one of the payload modal frequencies.
- The oil column peak is almost not affected both in frequency and amplitude when compared to the one relative to the bare table.
- The peak in correspondence of the base natural frequency is also almost unaffected and shows only a small reduction in amplitude.

Phase Transfer Function.

Figure 3.32 (b) shows the Phase Transfer Function: three small notches appears in correspondence off the three modal frequencies of the payload. These three notches, though of small amplitude, are very clear and easier to identify than the peaks present in the magnitude of the table T.F. The inversion frequency and the other characteristics of the Phase Transfer Function are unaffected by the presence of the MDOF payload.
Figure 3.32  Table Transfer Function: bare table and table with MDOF Payload (flexible base included):

(a) Magnitude   (b) Phase
3.8 CONCLUSIONS

In this chapter it was developed a mathematical model for the table behavior. The model has the peculiar capacity to include the following real table characteristics:

- The flexibility of the reaction mass.
- The presence of a MDOF payload on the table.
- The existence of time delay $\tau$ between the servovalve command and the servovalve response.
CHAPTER 4
ANALYTICAL SENSITIVITY OF TRANSFER FUNCTION
4.1 INTRODUCTION

In this chapter we use the physics based mathematical model of the shaking table developed in chapter 3 to analyze the sensitivity of the table transfer function (magnitude and phase) to the control gain parameters and payload dynamic characteristics.

The main focus of the analysis performed in this chapter is to understand qualitatively how the table responds to different working conditions. For this reason, the sensitivity analysis was performed using control gain settings that are different from the one that will actually be used for the optimum functioning of the shaking table. In fact, the transfer function sensitivity to control gain parameters is performed by increasing, starting from zero, the value of each control gain parameter in turn. When the individual effect of one control parameter is examined, all the other control parameters (except the proportional gain) are kept equal to zero.

The transfer function sensitivity to payload characteristics is analyzed keeping all the table control gain parameters equal to zero except the proportional gain which was given a unit value. This control gain setting was selected in order to enhance the effect of the payload upon the table transfer function.

---

1. Chapter 6 analyzes quantitatively the sensitivity of the shaking table response to control gain settings and payload characteristics.

2. The proportional gain must have a value different from zero, otherwise the shaking table does not react to the input command. For this reason, when the individual effects of the other control parameters were analyzed, the proportional gain was always kept equal to one Volt/Volt.
4.2 Sensitivity of Table Transfer Function to Control Parameters (Gains)

4.2.1 Introduction

In this section we analyze the response of the table to the different gains ($K_{pro}$, $K_{int}$, $K_{der}$, $K_{ff}$ and $K_{dp}$) using the mathematical expression for the table transfer function derived in section 3.5 and given in Eq. (3.93). This expression takes into account the flexibility of the base, but does not consider the presence of any payload on the table. Therefore the results presented in this section are for the bare table condition only. The analysis is conducted referring to the various table control parameters by their names used in common practice.

For this reason,

$K_{pro}$ will be referred to as Proportional Gain (P-gain) - analyzed in section 4.2.2;

$K_{int}$ will be referred to as Integral Gain (I-gain) - analyzed in section 4.2.3;

$K_{der}$ will be referred to as Derivative Gain (D-gain) - analyzed in section 4.2.4 and 4.2.5;

$K_{ff}$ will be referred to as Feed Forward Gain (FF-gain) - analyzed in section 4.2.6;

$K_{dp}$ will be referred to as Delta Pressure Gain (dP-gain) - analyzed in section 4.2.7.

Except for the control gain parameters which have different values in each section of this analysis, all the other parameters entering the formulation of the table transfer function have constant values equal to the ones defined and used in sections 3.4.2 and 3.5.5.

---

1. More specifically: parameters expressing the characteristics of the servovalve and actuator ($k_t = 330$, $k_f = 10^{-10}$, $\beta = 98.000 \text{ lbs/in}^2$) and parameters expressing the characteristics of the flexible base ($m_b$, $\omega_b$ and $\zeta_b$).
4.2.2 Proportional Gain

Figure 4.1 (a) and Figure 4.1 (b) show the magnitude and phase, respectively, of the table transfer function for the following gain settings:

P-Gain = 1, 2, 3, 4, 5, 6:
All other control gains = 0.

Magnitude Transfer Function:

It is observed that the main effect of increasing the proportional gain is to raise the value of magnitude of the transfer function in the low and intermediate frequency range (from 0 to 60 Hz). The amplitude and frequency of the oil column peak is not affected by the increase of the proportional gain, while the peak due to the flexibility of the base increases its value with increasing values of the proportional gain.

Phase Transfer Function:

An increase in the proportional gain also improves the phase transfer function by reducing the phase shift in the low and intermediate frequency range. The inversion frequency (at the oil column frequency) is not affected at all by the proportional gain.
Figure 4.1  Effect of the Proportional Gain upon the table transfer function:
P-gain = 1, 2, 3, 4, 5, 6 V/V

(a) Magnitude  (b) Phase (radians)
4.2.3 Integral Gain

Figure 4.2(a) and in Figure 4.2(b) show the magnitude and phase, respectively, of the transfer function for the following gain settings:

I-Gain = 0, 20, 40, 60;
P-Gain = 1. All other control gains = 0.

Magnitude Transfer Function:

The main effect of increasing the integral gain is to raise the value of the transfer function in the low frequency range (from 0 to 30 Hz). Moreover, an increase in the integral gain increases slightly the value of the oil column peak, while reducing slightly its frequency.

Phase Transfer Function:

An increase in the proportional gain improves the phase response of the table in the very low frequency range, but increases the phase shift in the intermediate frequency range thus spoiling the performance of the table in this frequency range.
Figure 4.2  Effect of the Integral Gain upon the table transfer function:
I-gain = 0, 20, 40, 60 radians per second
(a) Magnitude  (b) Phase (radians)
4.2.4 Derivative Gain (Negative Value of the Parameter)

Figure 4.3 (a) and Figure 4.3 (b) show the magnitude and phase, respectively, of the table transfer function for the following gain settings:

D-Gain = 0, 10, 20, 30, 40: ¹
P-Gain = 1. All other gains = 0.

Magnitude Transfer Function:

The main effect of an increase in the derivative gain is to increase substantially the value of the oil column peak and shift in its frequency towards lower values. Moreover, an increase in the D-Gain raises and thus improves the value of the table transfer function in the intermediate frequency range (5 to 40 Hz).

Phase Transfer Function:

Analysis of Figure 4.3 (b) reveals that an increase in the derivative gain increases significantly the phase shift between table command and table response in the low and intermediate frequency range. For frequencies above the oil column frequency (high frequency range), the phase shift on the other hand is significantly reduced as the derivative gain is increased. The inversion frequency does not change with increasing derivative gain, which suggests that the shifting of the oil column frequency noticed in the magnitude transfer function in reality is a broadening of the peak towards lower frequencies.

¹ In order to obtain results which are consistent with the table measured response behavior, these values of the D-Gain had to be set negative (i.e., $K_{der} = -D$).
Figure 4.3  Effect of the Derivative Gain (Negative value of the parameter) upon the table transfer function: D-gain = 0, 10, 20, 30, 40 milliseconds

(a) Magnitude       (b) Phase (radians)
4.2.5 Derivative Gain (Positive Value of the Parameter)

Figure 4.4 (a) and Figure 4.4 (b) show the magnitude and phase, respectively, of the table transfer function for the following gain settings:

D-Gain = 0, 20, 30, 40: 1
P-Gain = 1. All other gains = 0.

*Magnitude Transfer Function:*

The main effect of an increase in the derivative gain is to increase substantially the amplitude of the oil column peak and shift its frequency towards higher values. Moreover, an increase in the D-Gain raises (thus improving slightly) the value of the transfer function in the intermediate frequency range (from 20 to 60 Hz).

*Phase Transfer Function:*

Figure 4.4 (b) indicates that an increase in the derivative gain improves dramatically the phase response of the table in the low and intermediate frequency range. This effect is particularly evident for frequencies just below the oil column frequency. For frequencies above the oil column frequency (high frequency range) the phase shift increases as the derivative gain increases. The inversion frequency does not change with increasing the derivative gain, thus suggesting that the shifting of the oil column frequency noticed in the magnitude transfer function in reality is a broadening of the peak towards higher frequencies.

---

1. In this section, the derivative gain parameter is taken as positive, which is consistent with the derived model of the shaking table system but does not reflect the actual table behavior.
Figure 4.4  Effect of the Derivative Gain (Positive value of the parameter) upon the table transfer function: D-gain = 0, 20, 30, 40 milliseconds

(a) Magnitude  (b) Phase (radians)
4.2.6 Feed Forward Gain

Figure 4.5 (a) and Figure 4.5 (b) show the magnitude and the phase, respectively, of the transfer function for the following gain settings:

F Gain = 0, 20, 30, 40;
P Gain = 1. All other gains = 0.

Magnitude Transfer Function:

The main effect that an increase in the feed forward gain has upon the magnitude of the table transfer function is an increase of the height and width of the oil column peak. Notice that the oil column frequency, on the other hand, is not affected by the feed forward gain. An increase in the feed forward gain also raised moderately the magnitude of the transfer function in the intermediate frequency range (from 20 to 60 Hz).

Phase Transfer Function:

The effect of the feed forward gain on the phase of the transfer function in the low, intermediate and high frequency ranges is opposite to the effect of the derivative gain\(^1\): an increase in the feed forward gain improves dramatically the phase performance of the table in the low and intermediate frequencies. This effect is especially evident for frequencies just below the oil column frequency for which the phase shift is almost reduced to zero. Also for frequencies above the inversion frequency, an increase in feed forward gain increases the phase shift. Notice that an increase in the feed forward gain reduces the inversion frequency. This behavior does not match the fact that the oil column frequency (determined from the magnitude of the transfer function) is unaffected by the feed forward

---

\(^1\) Obtained with negative values of the derivative gain. Positive values of the derivative gain give results that are similar to the ones obtained for the feed forward gain.
gain. The shift in the inversion frequency is anyway quite small and occurs only when the feed forward gain is introduced (20 milliseconds of feed forward gain as compared to a null feed forward gain). Any further increase in the feed forward gain does not shift the inversion frequency any more.
Figure 4.5  Effect of the Feed Forward Gain upon the table transfer function:

FF-gain = 0, 20, 30, 40 milliseconds

(a) Magnitude    (b) Phase (radians)
4.2.7 Delta Pressure Gain

Figure 4.6 (a) and Figure 4.6 (b) show the magnitude and the phase, respectively of the transfer function for the following gain settings:

dP-Gain = 0, 1, 2, 3, 4;
P-Gain = 1, all other gains = 0.

Magnitude Transfer Function:

Figure 4.6 (a) clearly indicates that increase in the delta pressure gain reduces the magnitude of the oil column peak without affecting its frequency at all. No other effect of the dP-Gain can be noticed (no effect on the low, intermediate and high frequency ranges).

Phase Transfer Function:

As for the magnitude of the transfer function, the delta pressure gain has its main effect around the oil column frequency where an increase in the delta pressure gain reduces the phase shift thus improving the table performance. The inversion frequency is not affected by the delta pressure gain at all.
Figure 4.6  Effect of the Delta Pressure Gain upon the table transfer function:
\[ dP = 0, 1, 2, 3, 4 \text{ V/V} \]
(a) Magnitude    (b) Phase (radians)
4.3 SENSITIVITY OF TABLE TRANSFER FUNCTION TO PAYLOAD PARAMETERS

In this section, we analyze the effect of rigid and flexible payloads upon the table transfer function. As flexible payloads are characterized not only by their weight but also by their natural frequencies of vibration, in order to fully understand the response of the table when loaded with a flexible payload, two different analyses are performed: a flexibility analysis (to study the effect of the payload natural frequencies for a given payload weight) and a weight analysis (to study the effect of payload weight for a given payload natural frequency).

All the parameters necessary to evaluate the table transfer functions presented in this section have the values defined and used in sections 3.4.2, 3.5.5 and 3.6.9. The gain setting (P-Gain = 1, all other control Gains = 0) has been chosen as "loose" (small value of proportional gain), undamped (D-Gain = 0) and uncorrected (I-Gain = FF-Gain = DP Gain = 0), in order to enhance the effect of the flexible payload on the table transfer function. For this reason, certain resonances and strong effects of the payload on the table transfer function are present only under the assumptions of this analysis and do not reflect the effect of the payload upon the table under usage conditions (for this analysis, see Chapter 5).
4.3.1 Sensitivity to the Weight of Rigid Payloads

Figure 4.7 (a) and Figure 4.7 (b) show the magnitude and the phase, respectively, of the table transfer function for rigid payloads with the following weights:

\[ W = 150, 300, 450, 600 \text{ and } 900 \text{ lbs.} \]

Magnitude Transfer Function:

An increase in the mass of the rigid payload shifts the frequency of the oil column peak towards lower frequencies without affecting the amplitude of the peak itself. This shift toward lower frequencies appears to increase linearly with the payload weight. A small increase in the magnitude of the table transfer function in the intermediate frequency range (20 to 60 Hz) as also observed as the payload weight is increased.

Phase Transfer Function:

The addition of heavier and heavier rigid payloads on the table affects the phase of the table transfer function by shifting the inversion frequency toward lower values. This behavior matches that of the oil column frequency, thus confirming the link between the oil column frequency (noticeable from the magnitude of the transfer function) and the inversion frequency (noticeable from the phase transfer function). It is also noticed that the notch due to the base flexibility increases slightly with increasing payload.
Figure 4.7  Sensitivity of Table Transfer Function to Rigid Payload:

\[ W = 150, 300, 450, 600, 900 \text{ lbs added} \]

(a) Magnitude  (b) Phase (Radians)
4.3.2 Sensitivity to Payload Flexibility: Detailed Analysis

When payloads of different flexibility are present on the shaking table, the table transfer function is affected in various different ways for which a simple behavior trend cannot be identified. Nonetheless, these various effects can be categorized into three groups of behavior. Each group is defined by all payloads with a natural frequency falling within a specific frequency range. We will first present the effect of payload having a low natural frequency, then the effect of a payload having a natural frequency falling in the intermediate range, and at last the effect of a payload having a high natural frequency. Payloads with low and high natural frequencies impact very differently the shaking table transfer function. Payloads with a natural frequency falling in the intermediate frequency range produce a 'transition behavior' of the table transfer function between the other two extremes. In this analysis, we consider a SDOF payload having a constant weight of 300 lbs, a damping ratio of 2% and a varying natural frequency.

a) Payloads with a low natural frequency (from 0 to 40 Hz range): Increasing Effect

Figure 4.8 (a) and Figure 4.8 (b) show the magnitude and phase, respectively, of the table transfer function for flexible payloads having the following natural frequencies:

\[ f_p = 10, 20, 30, 40 \text{ Hz}. \]

Magnitude Transfer Function:

From Fig. 4.8 (a) it is observed that the presence of a flexible payload characterized by a low natural frequency (lower than 55 Hz) affects the magnitude of the table transfer function in the following ways:

- A perturbation of the table transfer function appears at the natural frequency of the
payload. This perturbation is characterized by an increase in the value of the transfer function (peak) for frequencies slightly smaller than the payload natural frequency followed by a decrease in the transfer function (valley) at the payload natural frequency. From now on, we will refer to this perturbations as the payload peak and payload valley.

- The payload peak grows both in amplitude and frequency as the natural frequency of the payload increases. It is also noticed that as the payload natural frequency increases, the frequency distance between peak and valley increases.
- The payload valley increases in depth with increasing natural frequency of the payload. The frequency of the payload valley always corresponds to the payload natural frequency.
- The frequency of the oil column peak is slightly shifted towards higher frequencies and its amplitude is slightly increased.

In summary, we can say that for payload with a low natural frequency, an increase in the payload natural frequency enhances the peak and the valley, their frequency distance, and increases the oil column peak as well as its frequency.

*Phase Transfer Function:*

The way the phase of table transfer function is affected by a flexible payload with a natural frequency falling within the low frequency range reflects the way the magnitude of the table transfer function is affected, namely:

- A sharp perturbation (valley) appears at the natural frequency of the payload, meaning that the phase shift of the table response is increased by the presence of a flexible payload in the neighborhood of the payload natural frequency.
- The inversion frequency is slightly shifted towards higher frequencies, the frequency shift increasing with increasing payload natural frequency (as was the case for the oil column frequency in the magnitude of the transfer function).
- The valley at the natural frequency of the payload increases in depth as the payload
natural frequency approaches the oil column frequency.

It is also noticed that the peak corresponding to the base natural frequency is only marginally affected (small increase) by the presence of a flexible payload with a low natural frequency. Only the payload with a natural frequency of 30 Hz increases the peak due to base flexibility, thus suggesting the presence of a dynamic interaction between the flexible payload and the flexible base (that has a natural frequency of approximately 27-30 Hz).
Figure 4.8  Effect of different payload natural frequencies upon the table transfer function: payload having low natural frequencies - 0 to 40 Hz - (payload weight = 300 lbs)

(a) Magnitude  (b) Phase (Radians)
b) **Payloads with a medium natural frequency (from 40 to 65-70 Hz): Transition**

Figure 4.9 (a) and Figure 4.9 (b) show the magnitude and the phase, respectively, of the table transfer function for flexible payloads with the following natural frequencies:

\[ f_p = 45, 55, 65, 70 \text{ Hz}. \]

Notice that the boundaries (45 -70 Hz) of the above payload natural frequency range correspond to the oil column frequency of the table with a rigid payload of 300 lbs and the oil column frequency of the base table.

**Magnitude Transfer Function:**

From Fig. 4.9(a), it is observed that a flexible payload characterized by a natural frequency falling in the intermediate frequency range of response of the table affects the magnitude of the table transfer function in the following ways:

- The payload peak amplitude keeps increasing in magnitude for payload natural frequencies up to 60 Hz reaching very high values (2.5 to 3.0) which make it larger than the oil column peak. For higher payload natural frequencies the behavior changes and the amplitude of the payload peak starts to decrease. The rate of increase in the frequency of the payload peak decreases, thus increasing the frequency distance between the payload peak and valley.

- The depth of the payload valley increases more and more, thus dropping the magnitude of table transfer function to values close to zero for a payload natural frequency higher than 50 Hz. The bottom of the valley still corresponds to the payload natural frequency.

- The amplitude of the oil column peak keeps increasing for payload natural frequen-
cies up to 65 Hz, reaching very high values (2.5 to 3.0). For higher payload natural frequencies the behavior changes and the amplitude of the oil column peak starts to decrease. The oil column frequency is subject to a dramatic increase as the payload natural frequency grows.

In summary we can say that for a payload with a natural frequency falling in the intermediate frequency range, an increase in the payload natural frequency increases the magnitude of the payload peak and of the oil column peak, thus creating two distinct (and very high) peaks of comparable size. The frequencies of these two peaks are located below and above the the oil column frequencies of the table for a rigid payload of same weight and for a bare table. The valley at the payload natural frequency reduces the amplitude of the table transfer function to values close to zero. For payload natural frequencies higher than 65 Hz (which correspond to the oil column frequency for a rigid payload of same weight), these two peaks start to decrease.

**Phase Transfer Function:**

The phase of the transfer function is affected in the following ways by the presence of a flexible payload with a natural frequency falling in the intermediate frequency range:

- For a payload natural frequency below 60 Hz, a phase inversion appears corresponding to the payload peak, while the oil column inversion frequency is shifted towards higher values.

- For a payload natural frequency above 60 Hz, it seems that the oil column inversion frequency is shifted towards lower values, while the payload peak inversion frequency is shifted toward higher values.
This behavior suggests that for a payload in the intermediate frequency range, the following changes occur:

a) As the payload natural frequency is increased, the former payload peak becomes progressively an oil column peak at a frequency lower than the oil column frequency for a rigid payload of identical mass.

b) As the payload natural frequency is increased, the former oil column frequency becomes progressively a payload peak that now occurs after and not before the valley, thus originating a valley-peak system.
Figure 4.9  Effect of different payload natural frequencies upon the table transfer function: payload having medium natural frequencies - 40 to 70 Hz - (payload weight = 300 lbs)

(a) Magnitude  (b) Phase (Radians)
c) **Payloads with a high natural frequency (above 80 Hz): Decreasing Effect**

Fig. 4.10 (a) and Fig. 4.10 (b) show respectively the magnitude and phase, respectively, of the table transfer function for flexible payloads with the following natural frequencies:

\[ f_p = 80, 90, 100, 110 \text{ Hz}. \]

**Magnitude Transfer Function:**

From Fig. 4.10 (b), it is observed that the presence of a flexible payload with a high natural frequency (above 80 - 90 Hz, which corresponds to the natural frequency of the oil column frequency for the bare table) affects the table transfer function in the following way:

- A valley is still present due to the payload natural frequency.
- The peak associated with the valley now occurs after the valley, taking the role of what was before the oil column peak. The amplitude of this peak decreases as the payload becomes more and more rigid and for a payload natural frequency above 100 Hz it almost disappears.
- The valley is still located at the payload frequency and brings the magnitude of the table transfer function close to zero.
- The former (i.e., for low payload natural frequency) payload peak now becomes an oil column peak occurring at a frequency lower than the oil column frequency for a rigid payload of same mass. The amplitude of this peak decreases progressively as this peak approaches the oil column peak for a rigid payload.
- The former oil column peak strongly decreases in amplitude and for a payload with a natural frequency of about 100 - 110 Hz, it almost disappears. The frequency of the former oil column peak increases with the payload natural frequency and appears to be approximately 20% higher than the payload natural frequency.
**Phase Transfer Function:**

The phase of the transfer function is affected by flexible payloads with a high natural frequency (say above 100 Hz) in a very complicated way:

- The phase transfer function displays three inversion frequencies. A first inversion occurs at a frequency close to the oil column frequency. A second inversion occurs at the payload natural frequency (corresponding to the valley in the magnitude transfer function). Finally a third inversion occurs at a frequency approximately 20% higher than the payload natural frequency (corresponding to the payload peak in the magnitude transfer function).

- As the stiffness of the payload increases, the first inversion frequency approaches the inversion frequency for a rigid payload of same mass.

- The notch both in the phase and magnitude transfer function is not affected at all by the rigidity of the flexible payload.
Figure 4.10  Effect of different payload natural frequencies upon the table transfer function: payload having high natural frequencies - above 80 Hz - (Payload weight = 300 lbs)

(a) Magnitude     (b) Phase (Radians)
4.3.3 Sensitivity to Payload Flexibility: Limit Cases

*High Stiffness Payload*

When the flexible payload becomes more and more rigid, the corresponding table transfer function approaches the one for a rigid payload with the same mass. This fact is clearly shown in Fig. 4.11 (a) and Fig. 4.11 (b), where the magnitude and phase transfer functions of a flexible payload with a natural frequency of 1000 Hz are compared with the corresponding transfer functions for a rigid payload of equivalent mass (600 lbs).

*Magnitude Transfer Function:*

Fig. 4.11 (a) shows that the magnitude of the table transfer function for a very stiff payload with a natural frequency of 1000 Hz coincides with the table transfer function for a rigid payload of equivalent mass.

*Phase Transfer Function:*

Fig. 4.11 (b) shows that the phase transfer function for a very stiff payload with a natural frequency of 1000 Hz coincides with the phase transfer function for a rigid payload of same mass.
Figure 4.11  Limit of Table Transfer Function as payload becomes stiffer and stiffer
(a) Magnitude   (b) Phase (Radians)
**Low Stiffness Payload**

When the flexible payload becomes more and more flexible the corresponding table transfer function approaches the table transfer function for the bare shaking table. This fact is clearly shown in Figure 4.11 and Figure 4.11(b) where the table magnitude and phase transfer functions for a flexible payload with a natural frequency of 0.001 Hz are compared with those for the bare table.

**Magnitude Transfer Function:**

Fig. 4.12 (a) shows that the magnitude of the table transfer function for a very flexible payload with a natural frequency of 0.001 Hz coincides that for the bare table.

**Phase Transfer Function:**

Fig. 4.12 (b) shows that the phase transfer function of a very flexible payload with a natural frequency of 0.001 Hz coincides with that for the bare table.
Figure 4.12  Limit of Table Transfer Function as Payload becomes more and more flexible

(a) Magnitude    (b) Phase (Radians)
4.3.4 Sensitivity to Payload Flexibility: General Behavior

As shown in previous sections, the effect of a flexible payload upon the table transfer function depends highly on the payload natural frequency as compared to the oil column frequency for base table condition. For payload natural frequencies inferior to the ones of the transition range we have the following effects:

1) the frequency and amplitude of oil column peak increased compared to the ones for a rigid payload of the same mass;

2) a small peak appears at a frequency just below the payload natural frequency.

For payload natural frequencies higher than the ones of the transition range, we have the following effects:

1) the frequency and amplitude of the oil column peak appear to be increased and decreased, respectively, compared to those for a rigid payload of same mass;

2) a small peak appears at a frequency just above the payload natural frequency.

In both cases (payload natural frequencies lower and higher than the ones of the transition range), a notch (valley) appears centered at the payload natural frequency.

For payload natural frequencies that fall within the transition range, it is possible to identify a hybrid behavior with two separate peaks:

1) one at a frequency below the oil column frequency for a rigid payload of same mass;

2) another peak at a frequency above the oil column frequency for bare table condition.

Fig. 4.13 (a) represents a three dimensional plot of the magnitude transfer function of the shaking table loaded with a flexible payload weighting 600 lbs with varying natural frequency. This plot shows clearly the peak that arises in the transfer function at a frequency
close to the payload natural frequency (it is the series of crests that goes from the upper left corner to the lower right corner). It is also clear that when this line of peaks intersects the oil column peak zone, a strong resonance phenomenon occurs and two very high peaks are generated: one at a frequency lower than the oil column peak for very stiff payload and the other at a frequency higher than the oil column peak for very flexible payloads. It is also observed that the oil column peak occurs at two distinct frequencies for payloads having very high stiffness (very high natural frequency) and payloads having very low stiffness (very low natural frequency).

Fig. 4.13 (b) is a contour plot of Fig. 4.13 (a), with contour lines set at the following levels: 0.1, 0.2, 0.5 and 0.9. From this contour plot, we can observe the notch (valley) that is located close to the payload natural frequency. The valley can be clearly seen as running from the bottom left corner of the plot to the upper right corner. We can also see clearly that the oil column peak appears at two different frequencies for very flexible and very stiff payloads. The plot also shows clearly the transition zone in which two high oil column peaks coexist: this transition zone spreads between 50 and 80 Hz.
Figure 4.13  3D Plot: Magnitude of the Transfer Function Vs. Payload Natural Frequency

(a) 3-Dimensional Plot  (b) Contour Plot (lines at: 0.1, 0.2, 0.5, 0.9)
4.3.5 Sensitivity to Payload Flexibility: Transition Frequencies

Given the identified behavior of the table transfer function, it is of extreme importance to exactly identify the transition frequency range. This range has been shown numerically (through the analysis of different simulated table responses) to depend upon the weight of the flexible payload. As anticipated from the previous section, for a flexible payload weighting 300 lbs., the transition range spans approximately between 50 Hz, that corresponds to the oil column frequency of the table for a rigid payload of 300 lbs. and 70 Hz, that corresponds to the oil column frequency for the bare table condition. Analyses using flexible payloads of different weights indicated that the transition frequency range can always be anticipated to fall between the oil column frequency for a rigid payload of same mass and the oil column frequency for the bare table condition. Fig. 4.14, Fig. 4.15 and Fig. 4.16 show the table transfer functions (magnitude and phase) for flexible payloads of varying natural frequency, but a constant weight of 600 lbs (as opposed to a weight of 300 lbs used for the analysis presented in section 4.3.2). Analysis of these plots show clearly that in this case the transition frequency range spans between approximately 40 Hz and approximately 65 - 70 Hz, which correspond to the oil column frequency for a rigid payload weighting 600 lbs. and the oil column frequency of the bare table, respectively.
Figure 4.14 Effect of different payload natural frequencies upon the table transfer function: payload having "low" natural frequencies - 0 to 30 Hz - (weight = 600 lbs)

(a) Magnitude   (b) Phase (Radians)
Figure 4.15 Effect of different payload natural frequencies upon the table transfer function: payload having "medium" natural frequencies - 40 to 55 Hz - (weight = 600 lbs)

(a) Magnitude  (b) Phase (Radians)
Figure 4.16 Effect of different payload natural frequencies upon the table transfer fctn.: payload having "high" natural frequencies - above 70 Hz - (weight = 600 lbs)

(a) Magnitude  (b) Phase (Radians)
4.3.6 Sensitivity to the Weight of Flexible Payloads

Fig. 4.17 (a) and Fig. 4.17 (b) show the magnitude and phase, respectively of the table transfer function for flexible payloads with the following weights:

\[ W = 150, 300, 450, 600 \text{ and } 900 \text{ lbs}. \]

All the flexible payloads were considered to have the same natural frequency of 20.0 Hz and a damping ratio of 0.02 (these values were selected to obtain results that are comparable with the analysis of the model presented in Appendix B).

*Magnitude Transfer Function:*

Differently from the case of a rigid payload, an increase in the mass of a flexible payload does not simply shift the frequency of the oil column peak towards smaller frequencies. An increase in the mass of a flexible payload enhances the effects of the flexible payload itself upon the table transfer function. So as the weight of the payload increases, the payload peak in the table transfer function increases, and both the amplitude and frequency of the oil column peak increase, but by a lesser amount.

*Phase Transfer Function:*

Similarly, the presence on the table of a flexible payload of increasing weight enhances the already known effect that a flexible payload has on the phase of the table transfer function: the notch at the payload natural frequency increases in depth, and the inversion frequency increases slightly.
Figure 4.17  Sensitivity of Table Transfer Function to weight of Flexible Payload (SDOF)

(a) Magnitude  (b) Phase (Radians)
4.3.7 Sensitivity to MDOF Payloads

**MDOF: Overall Effect**

Fig. 4.18 (a) and Fig. 4.18 (b) show the magnitude and phase, respectively, of the table transfer function when the table is loaded with an MDOF payload. The key characteristics of the MDOF payload (described in more detail in section 3.7.8) are:

- weight per floor: 300 lbs (three floors, for a total weight of 900 lbs);
- modal frequencies: \( f_1 = 5.97 \) Hz, \( f_2 = 15.8 \) Hz, \( f_3 = 23.16 \) Hz.

Figures 4.18 (a) and (b) also show, for comparison purposes, the table transfer functions corresponding to the SDOF payloads of 300 lbs and natural frequency equal to the modal frequencies of the MDOF payload.

**Magnitude Transfer Function:**

Fig. 4.18 (a) shows that the presence of an MDOF payload on the shaking table originates three peaks corresponding to the three modal frequencies. Comparison with the effect of the corresponding modal SDOF payloads shows that the effect of the MDOF payload is much smaller than the effect of the SDOF payloads except for the first modal frequency. In this case, the two effects are of comparable magnitude.

**Phase Transfer Function:**

Fig. 4.18 (b) shows that the presence of an MDOF payload on the shaking table creates three notches corresponding to the three modal frequencies. Comparison with the effects of the corresponding modal SDOF payloads shows that these notches are much smaller than the ones produced by the modal SDOF payloads except for the first modal frequency. In this case, the two effects are comparable. Notice the differences between the effects of
the MDOF payload and those of the modal SDOF payloads increase for higher modes.
Figure 4.18  Sensitivity of Table Transfer Fctn. to MDOF payload: MDOF Payload and corresponding SDOF Payloads (SDOF mass = 1/3 MDOF mass = 300 lbs, SDOF natural frequencies = MDOF modal frequencies: \( f_1 = 5.97 \) Hz, \( f_2 = 15.8 \) Hz, \( f_3 = 23.16 \) Hz)

(a) Magnitude    (b) Phase (Radians)
4.3.8 Sensitivity to MDOF Payloads: Size Effect

Figures 4.19 (a) and 4.19 (b) compares the magnitude and phase of the table transfer function for the MDOF payload having the characteristics described in the previous section with those for an MDOF payload having the masses and stiffness increased by a factor 16 (referred to as "Modified" MDOF as opposed to the "Original" MDOF). The characteristics of the "Modified" MDOF payload are as follows:

- weight per floor: 4800 lbs;
- lateral stiffness: $k_1 = 97.552$ lbs/inch, $k_2 = 83.616$ lbs/inch, $k_3 = 69.680$ lbs/inch.

The modal frequencies are the same as for the "Original" MDOF Payload:

- modal frequencies: $f_1 = 5.97$ Hz, $f_2 = 15.8$ Hz, $f_3 = 23.16$ Hz.

The "Modified" MDOF payload is a very large payload and represents an extreme case: its mass is 12 times the mass of the slip table itself and 3 times the maximum table capacity (1500 lbs). This payload has been selected in order to enhance the effects of the MDOF payload which have been shown to be quite small in the previous section.

Magnitude Transfer Function (M.T.F):

Fig. 4.19 (a) shows that the "Modified" MDOF payload has a much larger effect on the table transfer function than the "Original" MDOF payload. Specifically, the difference (increase in the peaks of the M.T.F.) is largest at the first modal frequency. At the second and third modal frequencies the peaks, for the "Modified" and the "Original" MDOF payloads are closer. This indicates that an MDOF payload has its main effect upon the M.T.F. at the first modal frequency. A sharp increase in the oil column frequency is also noticed, but the amplitude of the oil column peak is unaffected.
Phase Transfer Function:

Fig. 4.19 (b) shows that the difference of the effects of the “Original” and “Modified” MDOF payloads upon the phase of the table transfer function is similar to the difference of the effect upon the magnitude of the table transfer function. Notice that the notch (effect of the MDOF payload upon the table transfer function) corresponding to the first modal frequency is larger than those corresponding to the higher modal frequencies. This difference is not as marked as for the magnitude. Notice also an increase in the inversion frequency that corresponds to the increase in the oil column frequency noticed in the magnitude.

In summary, it can be stated that the table transfer function is not only sensitive to the ratio of mass and stiffness of the MDOF payload, but also to the absolute value of the payload mass and stiffness.
Figure 4.19 Sensitivity to Size Effect of MDOF Payload: "Original" MDOF and "Modified" MDOF (stiffness and masses increased 16 times, same mode shapes and modal frequencies as the "Original")

(a) Magnitude  (b) Phase (Radians)
4.3.9 SDOF/MDOF Comparison

Figs. 4.20 (a) and 4.20 (b) show the magnitude and phase, respectively, of the table transfer function for the “Modified” MDOF\(^1\) payload and of its corresponding modal SDOF payloads. The corresponding modal SDOF payloads have been determined as follows:

An SDOF payload having the same natural frequency of the mode shape to reproduce and having a mass equal to one third of the total mass of the \(^2\) model (the model was a three-degree-of-freedom system)

Magnitude Transfer Function:

Fig. 4.20 shows clearly that the effects of the “Modified” MDOF and of the corresponding modal SDOF payloads are almost the same for the first modal frequency, confirming what had already been found for the “Original” MDOF payload. It is also noticed that the “Modified” MDOF payload originates a peak-and-valley profile that is centered exactly at the modal frequency, while the corresponding modal SDOF payloads have the valley centered at the modal frequency. It is interesting to notice that the oil column frequency for the “Modified” payload sits in the middle of the oil column frequencies for the modal SDOF payloads. The amplitude of the oil column peak is almost unaffected by the presence of the “Modified” MDOF payload which is different for the modal SDOF payloads that increase sharply the oil column peak amplitude.

Phase Transfer Function

Figure 4.20 (b) shows clearly that the effect of the “Modified” MDOF payload and of its

---

1. We chose to compare the effects of MDOF and SDOF payload for the “modified” MDOF since its effect upon the table transfer function are pronounced.
2. The masses used for the analysis performed in this section were selected empirically. In the next section (4.3.10), we will see the effect of use of modal masses.
corresponding modal SDOF payloads upon the phase of the table transfer function. are almost the same at the first modal frequency, and that the effect of the “Modified” MDOF become smaller and smaller for higher modal frequencies. This confirms what already had been noticed for the magnitude. It is also be noticed that for the modal SDOF payloads corresponding to the second and third modes, the table transfer function displays a peak before the valley. This peak is not present for the MDOF payload.
Figure 4.20  Sensitivity of Table Transfer Function to MDOF: "Modified" MDOF and corresponding SDOF: \( w = 4800 \) lbs/floor. \( f_1 = 5.97 \) Hz, \( f_2 = 15.8 \) Hz, \( f_3 = 23.16 \) Hz

(a) Magnitude  (b) Phase (Radians)
4.3.10 SDOF/MDOF Comparison (First Modal Frequency and Mass)

Figs. 4.21 and 4.22 show the comparison between the table transfer function for an MDOF payload and the table transfer function for an SDOF payload having a natural frequency equal to the first modal frequency of the MDOF payload and a mass equal to the first modal mass\(^1\) of the MDOF payload. This comparison was done in order to check if the effect of an MDOF payload upon the table transfer function can be approximated well by the effect of an SDOF payload having similar characteristics.

In Fig. 4.21 the comparison is shown for the “Original” MDOF payload described in section 3.7.8. This MDOF payload is a good approximation of typical structural models that will be tested on the shaking table\(^2\). As shown in section 4.3.8, this model is of small size and its effects upon the table transfer function are small. For this reason, in Fig. 4.22 the comparison is shown for the much larger “Modified” MDOF payload described in section 4.3.8 which even though it is not representative of typical models tested on the table, gives an enhanced picture of the way a payload affects the table transfer function.

Comparison between the “Original” MDOF and the equivalent SDOF (having first modal frequency and mass)

Magnitude of the table transfer function:

Fig. 4.21 (a) shows that the table transfer function for the “Equivalent” SDOF payload perfectly matches the table transfer function for the “Original” MDOF payload at the first

\(^1\) For a definition of modal mass see Appendix E.

\(^2\) A real model having similar characteristics has already been built and tested at Rice University.
modal frequency: the peak is perfectly matched both in frequency and amplitude. Also the
two table transfer functions match perfectly up to the second modal frequency. Obviously
the table transfer function for the “Equivalent” SDOF payload does not match exactly the
one for the “Original” MDOF at the second and third modal frequencies. It is at the oil
column peak that the two table transfer functions differ the most. The oil column peak for
the “Equivalent” SDOF payload is higher and occurs at a slightly lower frequency than the
“Original” MDOF payload.

This result is in agreement with what had been found from the approximate comparison
carried out in section 4.3.9: an SDOF payload having the same first natural frequency as
the MDOF payload provides a table transfer function that matches closely the one for the
MDOF payload for frequencies close to the first natural frequency, but does not match pre-
cisely the oil column peak.

*Phase of the table transfer function:*

Fig. 4.21 (b) confirms for the phase what has already been found for the magnitude: the
phase transfer function for an MDOF payload and the phase transfer function for the cor-
responding first modal SDOF payload match very well for frequencies up to the second
modal frequency. At the second and third modal frequencies obviously the two phase
transfer functions do not match precisely. At the oil column frequency the two phase trans-
fer functions are slightly offset.
Figure 4.21  Sensitivity of Table Transfer Function to MDOF: “Original” MDOF and its corresponding SDOF having the first modal mass: modal mass = 2.0682 lb-sec^2/in (equivalent to 798.3 lbs = 88.7% of the MDOF total mass)  (a) Magnitude  (b) Phase (Radians)
Comparison between “Modified” MDOF and the equivalent SDOF (having first modal frequency and mass)

The “Modified” MDOF has an exaggerated effect upon the table transfer function, and therefore in this analysis it is much easier than in the previous one to evaluate accuracy of the approximation of an MDOF payload by an “Equivalent” SDOF payload.

Magnitude of the table transfer function:

Fig. 4.22 (a) shows the magnitude of the two table transfer functions: for the “Modified” MDOF and “Equivalent” SDOF payloads. This analysis confirms what has been found in the case of the “Original” MDOF payload: the table transfer function for the “Modified” MDOF payload and the one for the “Equivalent” SDOF payload match perfectly at the first modal frequency. Obviously they differ at the second and third modal frequencies, but, recalling that the effect of an MDOF payload upon the table transfer function becomes smaller and smaller at high modal frequencies, it can be concluded that up to the third modal frequency, the two transfer match reasonably well.

At the oil column peak, the two transfer functions are very different: the one relative to the “Equivalent” SDOF payload is much higher (about three times) than the one for the MDOF payload and its frequency is shifted towards lower frequencies.

Phase of the table transfer function:

Fig. 4.22 (b) shows the phase of the two table transfer functions: “Modified” MDOF payload and “Equivalent” SDOF payload. Its analysis confirms what was found for the magnitude of the table transfer function: good match between the two table transfer functions for frequencies up to the second modal frequency, and significant differences at the inversion
frequency (oil column frequency). The inversion frequency is shifted toward lower frequencies.
Figure 4.22  Sensitivity of Table Transfer Function to MDOF: "Modified" MDOF and its corresponding SDOF having the first modal mass: modal mass = 33.09 lb-sec^2/in (equivalent to 12,773 lbs = 88.7% of the MDOF total mass)  
(a) Magnitude  
(b) Phase (Radians)
4.3.11 SDOF/MDOF Comparison (Higher Modal Frequencies and Masses)

In the previous section (4.3.10), we compared the table transfer function of an MDOF payload with the one of an SDOF payload having a natural frequency equal to the first modal frequency and a mass equal to the first modal mass. As the transfer function of a shaking table supporting an MDOF payload is mainly influenced by its first modal characteristics, it was found that the table transfer function for an MDOF payload can be approximated by the one relative to the equivalent SDOF payload, with the exception of the oil column peak region that basically remains unchanged with respect to the one for the bare table condition.

In this section, we compare the effect upon the table transfer function of an MDOF payload with the effect of an equivalent SDOF payload having dynamic characteristics similar to the higher modes of vibration of the MDOF payload.

*Comparison between the "Original" MDOF and the corresponding SDOF (having second modal frequency and mass)*

*Magnitude of the table transfer function:*

Fig. 4.23 (a) shows the perfect match at the second modal frequency between the magnitude of the table transfer function for MDOF payload and the magnitude of the table transfer function for the corresponding SDOF payload. Obviously, the two transfer functions differ at the first and third modal frequencies. The oil column peak for the SDOF payload is slightly higher and appears at a lower frequency than the one for the MDOF payload (as was the case for the SDOF payload having natural frequency and mass to match the first
mode of vibration of the MDOF payload).

*Phase of the table transfer function:*

As is the case for the magnitude, Fig. 4.23 (b) shows that the phase of the table transfer function for the MDOF payload and the phase of the table transfer function for the corresponding SDOF payload match perfectly at the second modal frequency. Obviously, at the first and third modal frequencies there is no match between the two table transfer functions. The inversion frequency for the SDOF payload occurs at a frequency slightly lower than the one for the MDOF payload.
Figure 4.23  Sensitivity of Table Transfer Function to MDOF: “Original” MDOF table transfer function and table transfer function of its corresponding SDOF having the second modal frequency and mass: modal mass = 0.2124 lb·sec^2/in (equivalent to 82.0 lbs = 9.11% of the MDOF total mass)  (a) Magnitude  (b) Phase (Radians)
Comparison between the "Original" MDOF and the corresponding SDOF (having third modal frequency and mass)

For the SDOF payload having a natural frequency and mass equal to the third modal frequency of the MDOF payload we have obtained results similar to the ones obtained for the SDOF payload equivalent to the second mode of vibration of the MDOF payload. This confirms the validity of the observations made in the preceding section.

Magnitude of the table transfer function:

Fig. 4.24 (a) shows the perfect match at the third modal frequency between the magnitude of the table transfer function for MDOF payload and the magnitude of the table transfer function for the corresponding SDOF payload. Obviously, the two transfer functions differ at the first and second modal frequencies. The oil column peak for the SDOF payload is slightly higher and appears at a lower frequency than the one for the MDOF payload table transfer function (as was the case for the SDOF payload having natural frequency and mass to match the first mode of vibration of the MDOF payload).

Phase of the table transfer function:

As is the case for the magnitude. Fig. 4.24 (b) shows that the phase of the table transfer function for the MDOF payload and the phase of the table transfer function of the corresponding SDOF payload match perfectly at the third modal frequency. Obviously, at the first and second modal frequencies there is no match between the phase of the two table transfer functions. The inversion frequency for the SDOF payload occurs at a frequency slightly lower than the one for the MDOF payload.
Figure 4.24 Sensitivity of Table Transfer Function to MDOF: "Original" MDOF table transfer function and table transfer function of its corresponding SDOF having the third modal frequency and mass: modal mass = 0.0516 lb-sec$^2$/in (equivalent to 19.7 lbs = 2.18% of the MDOF total mass)

(a) Magnitude  (b) Phase (Radians)
Comparison between the “Modified” MDOF and the corresponding SDOF (having second modal frequency and mass)

In this section, we perform the same comparison as the one presented in the preceding two sections, but using as a reference the “Modified” MDOF payload which, having a very large mass affects considerably the table transfer function.

Magnitude of the table transfer function:

Fig. 4.25 (a) shows the good match at the second modal frequency between the magnitude of the table transfer function for the “Modified” MDOF payload and the magnitude of the table transfer function for the SDOF payload whose characteristics correspond to the second vibration mode of the MDOF payload. In the case of the “Modified” MDOF payload, the matching between the MDOF and the corresponding SDOF payloads is not as good as it was for the “Original” MDOF payload: at the second modal frequency the table transfer function for the SDOF payload is much larger than the transfer function for MDOF payload. As was the case for the “Original” MDOF payload the two transfer functions differ at the first and third modal frequencies. In this case, the oil column peak for the SDOF payload is much higher (~20%) and occurs at a lower frequency (~10%) than the one for the MDOF payload.

Phase of the table transfer function:

Differently from the magnitude, Fig. 4.25 (b) shows that the phase of the table transfer function for the MDOF payload and the phase of the table transfer function for the corresponding SDOF payload match perfectly at the second modal frequency. Obviously, at the
other modes of vibration (first and third) there is no match between the phase of the two table transfer functions. As was the case for the magnitude, the inversion frequency of the table transfer function for the corresponding SDOF payload occurs at a lower frequency (~10%) than for the MDOF payload.
Figure 4.25 Sensitivity of Table Transfer Function to MDOF: “Modified” MDOF table transfer function and table transfer function of its corresponding SDOF having the second modal frequency and mass: modal mass = 3.3992 lb-sec^2/in (equivalent to 1312 lbs = 9.11% of the MDOF total mass)  
(a) Magnitude  (b) Phase (Radians)
Comparison between the "Modified" MDOF and the corresponding SDOF (having third modal frequency and mass).

Magnitude of the table transfer function:

Different from the case of the SDOF payload corresponding to the second mode of the "Modified" MDOF payload, and the case of the SDOF payload corresponding to the third mode of the "Original "MDOF (in both cases the table transfer functions of the MDOF and of the SDOF payloads had a good match at the specific modal frequency), the magnitude of the table transfer functions for the SDOF payload corresponding to the third mode of the "Modified" MDOF payload is not very accurate at the third modal frequency. As Fig. 4.26 (a) shows, the magnitude of the table transfer function for the SDOF payload is much larger (~50%) than the one for the MDOF payload. As expected, the two transfer functions differ significantly at the first and second modal frequencies. The oil column peak for the SDOF payload is higher (~15%) and occurs at a lower frequency (~10%) than for the MDOF payload.

Phase of the table transfer function:

As is the case for the magnitude, the phase of the table transfer function for the SDOF payload corresponding to the third modal frequency is larger (~30%) than the one for the MDOF payload. As expected, at the other modes of vibration (first and second) there is no match between the phase of the two table transfer functions. As was the case for the magnitude, the inversion frequency of the table transfer function for the SDOF payload occurs at a lower frequency (~15%) than for the MDOF payload.
Figure 4.26  Sensitivity of Table Transfer Function to MDOF: "Modified" MDOF

table transfer function and table transfer function of its corresponding
SDOF having the third modal frequency and mass: modal mass = 0.815
lb-sec^2/in (equivalent to 312 lbs = 2.18% of the MDOF total mass)
(a) Magnitude  (b) Phase (Radians)
4.3.12 Equivalence between MDOF Payload and corresponding SDOF

In summary, it can be stated that the presence of an MDOF payload on the shaking table affects the table transfer function in a way that is similar to the superposition of the effects of three separate SDOF payloads: each one having a natural frequency and a mass corresponding to the modal ones. The "importance" of the effect of the different SDOF payloads upon the transfer function of the bare table is not the same. The effect of the SDOF payload corresponding to the first mode of vibration can be added entirely to the transfer function for the bare table. The effects of the SDOF payloads corresponding to the higher modes of vibration must be decreased as the modes get higher before adding them to the transfer function for the bare table. The decrease factor is larger than the modal participation factors. Regarding the effect of the MDOF payload upon the oil column peak, our numerical simulation indicates that both its amplitude and frequency are not significantly affected by the presence of an MDOF payload.

Fig. 4.27 shows the table transfer function for the "Modified" MDOF payload and the table transfer function for three SDOF payloads corresponding to its three modes of vibration.
Figure 4.27  Sensitivity of Table Transfer Function to MDOF: "Modified" MDOF table transfer function and table transfer function of its corresponding SDOF having the third modal frequency and mass: modal mass = 0.815 lb·sec²/in (equivalent to 312 lbs = 2.18% of the MDOF total mass)

(a) Magnitude  (b) Phase (Radians)
4.3.13 Sensitivity of Table Transfer Function to Payload Parameters, Conclusions

For the practical use of the shaking table, it is of extreme importance to understand the effect of an MDOF payload upon the table transfer function. From the analyses conducted in this chapter it can be concluded that the main effect of an MDOF payload can be approximated without significant error with the effect of an SDOF payload having a mass corresponding to the first modal mass of the MDOF payload and a natural frequency equal to the first modal frequency. The analysis of the influence of an SDOF payload has shown that the effect of a flexible payload upon the table transfer function is very limited for:

1) payloads having a small mass (less than 300 lbs) regardless of their natural frequency;

2) payload having a large mass (more than 300 lbs) but a natural frequency below 40 - 50 Hz.

This leads to the conclusion that the presence on the shaking table of payloads for which the table has been designed (weight up to 1500 lbs and first natural frequency up to 10-20 Hz) should not affect greatly the performance of the table itself. Moreover, the whole analysis of the influence of payload characteristics upon the table transfer function was performed using a very “weak” set of table control gain parameters. This “weak” set of control gain parameters was chosen in order to enhance the effect of the payload upon the shaking table transfer function for better understanding. Therefore, the influence of a payload upon the table transfer function will be significantly smaller when a “stronger” set of control gain parameters is used.
4.4 Sensitivity of Transfer Function to Servovalve Delay $\tau$

4.4.1 Introduction

In this section, we analyze the influence of the servovalve time delay $\tau$ that can be introduced in the analytical model of the table transfer function. This parameter has been introduced in order to take into account the time delay between the instant at which the electrical signal carrying the servovalve command $X_c$ reaches the servovalve coil that moves the first stage servovalve pool and the instant at which the third stage spool moves according to the commanded displacement. This time shift has been experimentally detected by monitoring the opening of the valve vs. the valve command. The fact that this time delay physically exist has suggested the need to include it in the transfer function model, as explained in section 3.2.3. The value of the time delay reported here is always expressed in milliseconds.

4.4.2 Small Time Delay

In this section, we show the effect of a small time delay upon the table transfer function. Small time delays of the order of 1 to 3 milliseconds have been found to be the one actually existing in the Rice shaking table.

Magnitude Transfer Function:

Fig. 4.28 (a) shows that an increase of the time delay $\tau$ from 0 to 3 milliseconds increases the amplitude (up to $3.5 \div 4$ from 1) of the oil column peak and shifts its frequency toward lower values.
Phase Transfer Function:

Fig. 4.28 (b) shows that an increase of the time delay $\tau$ increases the phase shifts in the low-intermediate frequency range (below the inversion frequency), while it reduces the phase shifts in the high frequency range (above the inversion frequency). The increase in the phase shift seems to be linear with $\tau$. Although the oil column frequency is shifted toward lower values, the analysis of the phase transfer function shows that the inversion frequency is not affected by an increase in the time delay.
Figure 4.28  Sensitivity of Table Transfer Function to Time delay $\tau$ in the servovalve response. Small delay: $\tau = 0, 1, 2, 3$ milliseconds.

(a) Magnitude of Table Transfer Fctn. $T(s)$

(b) Phase of Table Transfer Fctn. $T(s)$ - (Radians)
4.4.3 Medium Time Delay

**Magnitude Transfer Function:**

Fig. 4.29 (a) shows that the increase of the time delay $\tau$ from 0 to 10 milliseconds increases the amplitude (up to $3.4 \div 5$ from 1) of the oil column peak. The frequency of the oil column peak does not follow a clear behavior: for time delay up to 5 milliseconds it is decreased, while for larger values of the time delay it is increased.

**Phase Transfer Function:**

Fig. 4.29 (b) shows that an increase in the time delay $\tau$ results in larger phase shifts in the low-intermediate frequency range (below the inversion frequency). This increase in the phase shift is so strong that an inversion occurs also before the inversion frequency. For frequencies above the inversion frequency, the phase shift is reduced by an increase of the time delay $\tau$. 
Figure 4.29  Sensitivity of Table Transfer Function to Time delay $\tau$ in the servovalve response. Medium delay: $\tau = 0, 5, 10$ milliseconds
(a) Magnitude of Table Transfer Fctn. $T(s)$
(b) Phase of Table Transfer Fctn. $T(s)$ - (Radians)
4.4.4 Large Time Delay

Magnitude Transfer Function:

Fig. 4.30 (a) shows that an increase of the time delay $\tau$ from 0 to 40 milliseconds increases the amplitude of the oil column peak. Since these amplitudes of the oil column peak are smaller than the ones found for a time delay of 5 to 10 milliseconds, it can be concluded that the increase in the oil column peak amplitude is not monotonic with $\tau$, but that has a "bouncing behavior" (increasing and decreasing). The oil column frequency does not follow a clear trend: for a time delay up to 30 milliseconds, it decreases, but for larger time delays it increases. It is also observed that large values of time delay increase the table transfer function at low frequencies as was found for the integral control gain effect. These peaks can reach pretty high values as is the case for a time delay of 40 milliseconds for which this low frequency peak reaches a value of approximately 2.5.

Phase Transfer Function:

Fig. 4.29 (b) shows that large values of the time delay $\tau$ have an effect similar to the one observed for medium values of the time delay. An increase of the time delay $\tau$ increases the phase shifts in the low-intermediate frequency range (below the inversion frequency). This increase in the phase shift is so strong that an inversion occurs also before the inversion frequency.
Figure 4.30  Sensitivity of Table Transfer Function to Time delay $\tau$ in the servovalve response. Large delay: $\tau = 0, 20, 30, 40$ milliseconds

(a) Magnitude of Table Transfer Fctn. $T(s)$

(b) Phase of Table Transfer Fctn. $T(s)$ - (Radians)
4.5 CONCLUSIONS

In this chapter, we exercise the physics based mathematical model of the shaking table system developed in chapter 3 to fully understand the response of the table at the different values of the control gains which can be set through the servovalve controller. This study proved to be extremely important in order to understand how to determine the different control gain settings. A good setting of the table gains is fundamental to obtain a transfer function that is as close as possible to one over the widest possible frequency range.

The effect of SDOF and MDOF payloads on the shaking table performance has also been studied. It was found that, for regular payloads (weight and frequencies within the range for which the table has been designed), the performance of the table, although influenced by the presence of the payload, is not spoiled and the table should guarantee an acceptable degree of accuracy in reproducing commanded displacement time histories.
CHAPTER 5
THEORY OF TRANSFER FUNCTION ESTIMATION
5.1 INTRODUCTION

To study the actual performances of the shaking table it has been decided to experimentally determine the table transfer functions. Transfer functions give an indication on the capacity of the system to reproduce the commanded signal over a given frequency bandwidth (in our case spanning from 0 to 1000 Hz). Moreover, the transfer functions (or, more specifically, their magnitude and phase components) have the additional advantages of being simple to estimate from real data and of being the quantities that have been used to define the mathematical model of the table. As will be shown in Chapters 5 and 6 the results obtained experimentally proved the validity of the method chosen. This chapter will focus on the method adopted to estimate the table transfer function. Moreover, a series of numerical simulations will be performed in order to determine the accuracy of the estimated table transfer functions thus determined.
5.2 THE TRANSFER FUNCTION

5.2.1 Definition

General input-output relationships in the frequency domain can be obtained for linear time-invariant systems\(^1\) through the so called Transfer Function. Even though the electro-hydraulic shaking table built at Rice University is highly non linear for its hydraulic nature, a linear approximation proved to be quite effective in capture its behavior.

Figure 5.1 shows a linear time invariant system. The input excitation time history can be defined as \(f(t)\), while the output response time history is \(x(t)\). Let's suppose to apply to a linear system a deterministic sinusoidal excitation \(f(t)\) of frequency \(\omega\) having the expression:

\[
f(t) = F_0 \cdot e^{i \omega t}
\]

where:

- \(F_0\) is the complex input vector (characteristics in terms of amplitude and phase of the sinusoidal input);
- \(i\) is the imaginary unit;
- \(\omega\) is the circular natural frequency of the excitation.

Then, the response of the system \(x(t)\) is also sinusoidal and has the same frequency of the

1. **Definition of a linear system:**

Suppose that \(V2(t)\) is the response of a system to a stimulus \(V1(t)\), and that \(W2(t)\) is the response to \(W1(t)\). Then the system is said to be linear if the response to \(V1(t) + W1(t)\) is \(V2(t) + W2(t)\), irrespective of the choice of \(V1(t)\) and \(W1(t)\). Time invariance means that the response to \(V1(t-T)\) is \(V2(t-T)\) for all \(T\) and \(V1(t)\).
excitation, as expressed by:

\[ x(t) = X_0 \cdot e^{i \omega t} = H(\omega) \cdot F_0 \cdot e^{i \omega t} \]  

(5.2)

where:

- \( X_0 \) is the complex output vector (characteristics in terms of amplitude and phase of the sinusoidal output):

- \( H(\omega) \) is the so called System Transfer Function.

\[ X_0 = H(\omega) \cdot F_0 \]  

(5.3)

\( H(\omega) \) is a complex frequency response function that alters both magnitude and phase of the output vector \( X_0 \) relative to the input vector \( F_0 \) and depends upon the circular frequency \( \omega \).

From deterministic frequency domain analysis it is known how all periodic input (also other than sinusoidal) can be decomposed in continue or discrete frequency components.

For linear systems Eq. (5.3) retains its validity for each and every frequency and can be rewritten\(^1\) in the following form:

\[ x(\omega) = H(\omega) \cdot F(\omega) \]  

(5.4)

In an equivalent way, also Eq. (5.4) can be rewritten as:

\[ H(\omega) = \frac{X(\omega)}{F(\omega)} \]  

(5.5)

where:

- \( X(\omega) \) is the frequency spectra of the output \( x(t) \);

\(^1\) For continuous frequency components.
- \( F(\omega) \) is the frequency spectra of the input \( f(t) \).

Eq. (5.5) shows that the units of \( H(\omega) \) are those of output \( x(t) \) divided by those of the input \( f(t) \). In the experimental analysis of the shaking table both inputs and output have been calibrated in order to have the same units so the Table Transfer Function is a pure number.

![Block diagram of a general linear dynamic system](image-url)

Figure 5.1  Block diagram of a general linear dynamic system
5.2.2 Deterministic Estimation of Transfer Function

Eq. (5.5) suggests that experimental values of the Transfer Function can be obtained by measuring for each input frequency the amplitude of both input and output and their relative phase shift: this approach is commonly referred in experimental dynamics as “Frequency Sweep”. On the other hand Eq. (5.4) and Eq. (5.5) suggest that the functional form of $H(\omega)$ can be obtained by measuring both input and output frequency spectra. These measurement can be done on either a discrete or a continuous frequency basis as long as magnitude and phase information are preserved. This other technique is commonly referred as “Frequency Domain approach”.

**Frequency Sweep**

Using the frequency sweep approach is very simple: it is sufficient to send to the system a sinusoidal input of a given amplitude and measure the amplitude and phase of the response. This procedure must be repeated for each frequency that need to be determined. This approach has proved experimentally to lead to transfer functions that have a smooth shape and are unaffected by spectral estimation errors. On the other hand it has been noticed that this method can lead to different results depending on the level of excitation (this fact is due to the non-linearities in the actual system). Another shortcoming of this method lies its experimental effort: in order to have a good frequency resolution in the Transfer Function it is necessary to experimentally excite the dynamic system for a very high number of different frequencies. Furthermore, depending on the physical characteristics of the system, it has been found experimentally that
certain frequencies of inputs are impossible give to be given to the system as inputs (the
dynamic behaviors of the system make it impossible to drive the system at frequencies
close but not equal to its natural frequencies of vibration).

Due to these shortcomings the frequency sweep method has been used only to obtain
some preliminary Table Transfer Function having a low frequency resolution. The
transfer function determined in this way (see Ch. 5) proved to be effective in giving an
overall idea of the table performances but lacked frequency resolution and precision.

*Frequency Domain Approach (Fourier or Wide-Band-Deterministic-Excitation based
  technique)*

In order to use the Frequency Domain Approach it is necessary to measure the both the
system input and output and to compute their respective frequency spectra. This compu-
tation may require a certain computation effort (if performed digitally) but provide the
knowledge of the transfer function over the whole frequency range with a single test.

There is no need to repeat the measurement for each frequency as it was the case of the
frequency sweep: this represent the main advantage of this method over the frequency
sweep. Another positive feature of this method is that the problem of frequency input
reproduction is totally eliminated. On the other hand, this approach has proved experiemen-
tally to lead to transfer functions that have a very rough shape originated by spectral esti-
mation errors.
5.2.3 Estimation of Transfer Function by Random Excitation

In the random excitation approach, a stochastic excitation (instead of a determined input) is sent to the system in order to estimate its transfer function.

When any of the elements of the linear system represented in Figure 5.1 is random, then, also the system output \( x(t) \) is random. In almost all problems encountered by structural dynamicists only random excitation or of major concern, while the others parameters of the system and the initial conditions can be reasonably assumed to be deterministic. Limiting our attentions to such problems, we can replace the deterministic forcing input \( f(t) \) with a random process \( F(t) \). As a consequence of this assumption, also the system output \( X(t) \) is a random process. Restricting the analysis to random processes inputs and outputs that are at least weakly stationary, the system response is analogous to the steady response in the deterministic vibration theory. Under all these assumptions the probabilistic theory of structural dynamics gives the following expression for the square of the magnitude of the transfer function:

\[
|H(\omega)|^2 = \frac{\Phi_{XX}(\omega)}{\Phi_{FF}(\omega)}
\]  

(5.6)

where:

- \( \Phi_{XX}(\omega) \) is the power spectral density of the output \( X(t) \);
- \( \Phi_{FF}(\omega) \) is the power spectral density of the input \( F(t) \);
- \( |H(\omega)|^2 \) is the square of the magnitude of the system transfer function, sometimes

1. Strictly speaking this is meaningful only if the derivative of \( X(t) \) exists in one of the following definitions: convergence with probability one, convergence improbability, convergence in distribution, convergence in mean square. See Y.K. Lin, Probabilistic Theory of Structural Dynamics.
referred as transmittancy function or system function.

Eq. (5.6) can provide informations on the magnitude of the transfer function but not on its phase, the probabilistic theory of structural dynamics provides us also with another expression for the system transfer function (input-output relationship):

$$H(\omega) = \frac{\Phi_{yX}(\omega)}{\Phi_{y\gamma}(\omega)}$$  \hspace{1cm} (5.7)

where $\Phi_{yX}(\omega)$ is the Cross spectral density between the input and output.

Eq. (5.7) provides an expression to evaluate the system transfer function when the input use to excite the system is a stochastic process. In the actual testing of the shaking table a white noise has been used as input. Moreover, the power spectral density of the input and cross power spectral density between input and output are not known and can be estimated following the procedures described in paragraph 5.3.3.
5.3 Spectral Estimation Review

In paragraph 5.2.3 it has been explained how in order to determine the table transfer function via a stochastic approach it is necessary to know the power spectral density of the system input and cross spectral density between system input and output. As the inputs and outputs are real signal (single time-limited realizations of a stochastic process) only an estimation (called periodogram) of their power spectra can be obtained from their recordings. In this paragraph it is presented a brief review of spectral estimation theory and a description of the Bartlett's procedure adopted to estimate the power spectra necessary to obtain the table transfer functions.

5.3.1 The Power Spectrum and the Periodogram

A key characteristic to evaluate the validity of the estimation of a power spectral density (or Power Spectrum) of a real signal is the spectral resolution. The resolution of any spectral estimate is the degree to it can express spectral detail. Resolution generally depends on two factors: the variance of the estimate and the degree of spectral smoothing implicit in the spectral estimation algorithm

Let \( r(t) \) denote a zero mean, stationary stochastic sequence or discrete time series (as it is our system input\(^1\) and system output).

The covariance function \( K_r(m) \) of the process \( r(t) \) is defined as:

---

\(^1\) The system input is a zero mean white noise. All the theory presented in this paragraph is valid for zero mean stationary stochastic processes. The theory presented in paragraph 5.2.3 to determine the table transfer function was developed for stochastic processes that are at least weakly stationary (including therefore zero mean stationary processes). For this reason the stochastic definition of table transfer function given in paragraph 5.2.3 still retains its validity.
\[ K_r(m) = E[r(l) \cdot r(l + m)] \quad (5.8) \]

in which \( E[\ldots] \) denotes the expectation or ensemble-average operator and \( l \) is a dummy time index:

\[ x(l) = x(t = l \cdot \Delta t) \]

where \( \Delta t \) is the sampling time interval.

The power spectrum of the process \( r(t) \) is defined as \( \Phi_{rr}(\omega) \).

The power spectrum and the covariance function of the process are related to each other through the following relationships:

\[ \Phi_{rr}(\omega) = \sum_{m = -\infty}^{\infty} K_r(m) \cdot e^{-i \cdot \omega \cdot m} \quad (5.9) \]

\[ K_r(m) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} \Phi_{rr}(\omega) \cdot e^{i \cdot \omega \cdot m} \, d\omega \quad (5.10) \]

In practical analysis the signal \( r(t) \) is observed over a finite interval of time only. Let "L" be the total duration (total number of samples) of the observation time and "l" the relative time index of observation, the time interval is then \([0, L-1]\). The covariance function \( K_r(m) \) can be estimated by approximating the averaging implicit in its definition\(^1\) and it is given by the following expression:

\[ \hat{K}_r(m) = \frac{1}{L} \cdot \sum_{l = 0}^{L - |m| - 1} r(l) \cdot r(l + m) \quad (5.11) \]

---

1. See Eq. (5.8).
where "m" is the time lag and has the following limitations: $0 \leq |m| \leq L - 1$.

The upper limits of the summation represented by Eq. (5.11) arises because of the data's finite extent, and can be explained as follows: the signal $r(l + m)$ exists only for values of the index $l$ ranging from $-m$ to $L - m - 1$.

The limits on the lag $m$ arises because beyond the $L-1$ lag value, this simple estimate provides no value.

The expected value of the estimate expressed by Eq. (5.11), can be shown to be:

$$E[\hat{K}_r(m)] = \left(1 - \frac{|m|}{L}\right) \cdot K_r(m) \quad (5.12)$$

From Eq. (5.12), it is clear how the estimate of $\hat{K}_r(m)$ is biased, but asymptotically ($\lim_{L \to \infty}$) unbiased. More important, bias differ for each lag, the smaller lags having much less bias than the larger ones. This effect is traced to the number of signal values that contribute to the estimate for a given lag: for example a given lag $m$ we have $L-m$ terms. Usually the average is computed by summing a few terms, then dividing by the number of terms in the sum. The estimate of the covariance function given by Eq. (5.12) normalizes the sum by $L$, no matter how many terms contribute to the sum and this choice leads to the bias.

One way to obtain an estimation of the power spectrum (often referred as the Periodogram\(^1\)) is to compute the Fourier Transform of the covariance function's estimate.

\(^1\) Periodogram - is an estimate of the Power Spectral Density (PSD).
$$\hat{\Phi}_{rr}(\omega) = \sum_{m = -(L-1)}^{L-1} \hat{K}_r(m) \cdot e^{-i \omega \cdot m}$$

(5.13)

This spectral estimate can be related directly to the observations by substituting the expression for the covariance estimate (Eq. (5.11)), thus obtaining the following expression:

$$\hat{\Phi}_{rr}(\omega) = \frac{1}{L} \cdot \sum_{m = -(L-1)}^{L-1} \sum_{l = 0}^{L-|m|-1} r(l) \cdot r(l + m) \cdot e^{-i \omega \cdot m}$$

(5.14)

Where the summation limits results from the finite extent of the observations.

We can express the actual data available by replacing the signal \(r(l)\) by its windowed signal, as expressed by:

$$r(l) \rightarrow w_L^r(l) \cdot r(l)$$

(5.15)

where \(w_L^r(l)\) is the rectangular or "boxcar" window of duration \(L\) thus defined:

$$w_L^r(l) = \begin{cases} 1 & 0 \leq l \leq L - 1 \\ 0 & \text{otherwise} \end{cases}$$

Substituting Eq. (5.15) in Eq. (5.14) the limits on the summation now become infinite and it can be obtained the following expression for the estimate of the power spectra:

$$\hat{\Phi}_{rr}(\omega) = \frac{1}{L} \cdot \sum_{m = -\infty}^{\infty} \sum_{l = -\infty}^{\infty} w_L^r(l) \cdot w_L^r(l + m) \cdot r(l) \cdot r(l + m) \cdot e^{-i \omega \cdot m}$$

(5.16)

A simple manipulation of the summation yields to the following insightful re-expression:
\[ \Phi_{rr}(\omega) = \frac{1}{L} \cdot \sum_{l=-\infty}^{\infty} \left[ w^R_L(l) \cdot r(l) \cdot e^{i \omega l} \right] \sum_{m=-\infty}^{\infty} w^R_L(l+m) \cdot r(l+m) \cdot e^{-i \omega (l+m)} \]  

(5.17)

Due to the stationarity of \( r(l) \), the value of square brackets in Eq. (5.17) does not depend on \( l \), and is the Fourier Transform of the windowed signal \( R_L(\omega) \) as expressed by the following expression:

\[ R_L(\omega) = \sum_{\xi=-\infty}^{\infty} w^R_L(\xi) \cdot r(\xi) \cdot e^{-i \omega \xi} \]  

(5.18)

By substituting Eq. (5.18) in Eq. (5.17) and bringing \( R_L(\omega) \) (a constant) outside the first summation, we obtain:

\[ \Phi_{rr}(\omega) = \frac{1}{L} \cdot R_L(\omega) \cdot \sum_{l=-\infty}^{\infty} w^R_L(l) \cdot r(l) \cdot e^{i \omega l} \]  

(5.19)

Comparing the summation in Eq. (5.19) with the expression of \( R_L(\omega) \) given by Eq. (5.18), it can be seen how the summation gives the complex conjugate of \( R_L(\omega) \):

\[ \hat{R}_L(\omega) = \sum_{\xi=-\infty}^{\infty} w^R_L(\xi) \cdot r(\xi) \cdot e^{i \omega \xi} \]  

(5.20)

Substituting Eq. (5.20) in Eq. (5.19) it results that the power spectral estimate\(^1\) of the process \( r(l) \) is the product between the windowed signal's Fourier Transform and its conjugate, as expressed by:

---

\(^1\) Or Periodogram.
\[
\Phi_{rr}(\omega) = \frac{1}{L} \cdot |R_L(\omega)|^2
\]  
(5.21)

The expression given by (5.21), called the periodogram, is biased since it is the Fourier Transform of a biased estimate of the covariance function.

The expected value of the periodogram (estimation of the power spectral density) equals the Fourier Transform of the triangularly windowed covariance function and can be derived as follows

\[
E[\Phi_{rr}(\omega)]
\]  
(5.22)

Substituting in Eq. (5.22) the expression of \( \Phi_{rr}(\omega) \) given by Eq. (5.13), we obtain:

\[
E[\Phi_{rr}(\omega)] = \sum_{m = -(L-1)}^{L-1} E[\hat{K}_r(m)] \cdot e^{-i \cdot \omega \cdot m}
\]  
(5.23)

Then, substituting in Eq. (5.23) the expression of \( E[\hat{K}_r(m)] \) given by Eq. (5.12), we obtain:

\[
E[\Phi_{rr}(\omega)] = \sum_{m = -(L-1)}^{L-1} \left( 1 - \frac{|m|}{L} \right) \cdot K_r(m) \cdot e^{-i \cdot \omega \cdot m}
\]  
(5.24)

or,

\[
E[\Phi_{rr}(\omega)] = \sum_{m = -(L-1)}^{L-1} \tilde{w}_{2L-1}^T(m) \cdot K_r(m) \cdot e^{-i \cdot \omega \cdot m}
\]  
(5.25)

where:

- \( \tilde{w}_{2L-1}(m) = 1 - \frac{|m|}{L} \)  
(5.26)

with \( m = -L, \ldots, L \) is a triangular window referred in literature as a triangular Bartlett
Window and is the window function \( w_L^q(1) \) in lag domain\(^1\).

The lag domain window expressed by (5.26) has the specific triangular shape and has its effect as a consequence of the rectangular window \( w_L^q(1) \) applied to the data.

By substituting in (5.25) the expression of \( K_r(m) \) given by Eq. (5.10):

\[
K_r(m) = \frac{1}{2 \pi} \int_{-\pi}^{\pi} \Phi_{rr}(\omega) \cdot e^{i \omega m} d\omega
\]  

(5.10)

we obtain:

\[
E[\hat{\Phi}_{rr}(\omega)] = \frac{1}{2 \pi} \int_{-\pi}^{\pi} \Phi_{rr}(\alpha) \cdot \sum_{m = -(L-1)}^{L-1} \tilde{w}_{2L-1}^T(m) \cdot e^{-i(\omega - \alpha)m} d\alpha
\]  

(5.27)

Recalling the definition of the window function \( \tilde{w}_{2L-1}^T(m) \), the limits in the summation of Eq. (5.27) can be replaced by \(-\infty\) and \(+\infty\), thus leading to the following expression:

\[
E[\hat{\Phi}_{rr}(\omega)] = \frac{1}{2 \pi} \int_{-\pi}^{\pi} \Phi_{rr}(\alpha) \cdot \sum_{m = -\infty}^{\infty} \tilde{w}_{2L-1}^T(m) \cdot e^{-i(\omega - \alpha)m} d\alpha
\]  

(5.28)

Where it can be recognized that the summation leads to the Fourier Transform of the window function in the lag domain \( \tilde{w}_{2L-1}^T(m) \), as expressed by the following equation:

\[
\tilde{W}_{2L-1}^T(\omega - \alpha) = \sum_{m = -\infty}^{\infty} \tilde{w}_{2L-1}^T(m) \cdot e^{-i(\omega - \alpha)m}
\]  

(5.29)

Substituting Eq. (5.29) in Eq. (5.28), it is possible see how the expected value of the

---

\(^1\) Given a certain function \( f(t) \), its corresponding function in the lag domain \( f(m) \) is the self-convolution of the function \( f(t) \).
Periodogram is the convolution of the actual power density spectrum with the Bartlett window's Fourier Transform.

\[ E[\Phi_{rr}(\omega)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{rr}(\alpha) \cdot \tilde{W}^T_{2L-1}(\omega - \alpha) d\alpha \]  

(5.30)

or:

\[ E[\Phi_{rr}(\omega)] = \Phi_{rr}(\omega) \otimes \tilde{W}^T_{2L-1}(\omega) \]

Where:

- \( \otimes \) represent convolution:

- \( \tilde{W}^T_{2L-1}(\omega) \) is the Fourier Transform of the Bartlett window and is known as the Fejer kernel:

\[ \tilde{W}^T_{2L-1}(\omega) = \frac{1}{L} \cdot \left( \frac{\sin(\omega \cdot L)}{\sin(\omega)} \right)^2 = (\text{sinc}(\omega))^2. \]
5.3.2 Window Selection & Statistical Characteristics

These results derived for the rectangular window can be easily generalized to other window shapes: applying a window function $w_L(l)$ to the data (time domain) is equivalent to applying the corresponding window function in lag domain $\tilde{w}_{2L-1}(m)$ to the covariance function, where the lag domain$^1$ window equals the autocorrelation function of the data window function. The relationships between the data window $w_L(l)$ and the periodogram expected value $E[\hat{\Phi}_{rr}(\omega)]$ can be summarized as follows:

$$\tilde{w}_{2L-1}(m) = \frac{1}{L} \sum_{l = -\infty}^{\infty} w_L(l) \cdot w_L(l + m)$$  (5.31)

Eq. (5.31) represents the convolution of two general windows.

$$\tilde{W}_{2L-1}(\omega) = \sum_{m = -\infty}^{\infty} \tilde{w}_{2L-1}(m) \cdot e^{-i\omega m}$$  (5.32)

Eq. (5.32) represents the Fourier Transform of the convolution between two windows.

$$E[\hat{\Phi}_{rr}(\omega)] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_{rr}(\alpha) \cdot \tilde{W}_{2L-1}(\omega - \alpha) d\alpha$$  (5.33)

where:

- $w_L(l)$ is the window function in the time domain (window shape);
- $\tilde{w}_{2L-1}(m)$ is the window function in the lag domain;
- $\tilde{W}_{2L-1}(\omega)$ is the Fourier Transform of the window function in the lag domain.$^2$

---

1. Sometimes referred also as covariance-domain.
2. Given a certain function f(t), its corresponding function in the lag domain f(m) is the self-convolution of the function f(t).
From the formulation given by Eq. (5.33) it is clear that the actual power spectrum \( \Phi_{re}(\omega) \) is convoluted with the so called kernel \( \tilde{\mathcal{W}}_{2L-1}(\omega) \). The resulting Periodogram is therefore a smoothed estimate of the actual power spectrum. As a consequence, only those aspects of the spectrum which vary over a range of frequencies wider than the width of the kernel will be noticeable\(^1\). This width can be defined much as a main-lobe as the distance between the smallest zeros surrounding the origin. For the Fejer kernel this width is \( 4\pi L \). The Periodogram's bias has a very complicated structure, being neither additive nor multiplicative. The relation of the periodogram's expected value to the convolution of the actual spectrum with the magnitude squared of the window's Fourier Transform (Eq. (5.33)) expresses the smoothing characteristics of periodogram-based spectral estimates. Using a language gleaned from conventional beamforming, the Fejer kernel has a very narrow mainlobe and large ripples (sidelobes) elsewhere. It is possible to use window shapes other than the rectangular to improve these smoothing characteristics. Anyway all windows produces sidelobes of varying structure that affects the spectral leakage: influence on the estimate at a particular frequency by remote portions of the spectrum. Among other types of windows, it is worth to mention the Hanning window and the 3-Terms Blackmann Harris that have been used to find the best estimation procedure to be adopted for the determination of the shaking table performances.

Despite the expected value of the Periodogram is a reasonable approximation of the

---

1. As we will see later the aspect of frequency resolution is not the main issue for the estimation of table behavior as the power spectral density of the underlying processes are smooth.
signal's power spectrum and despite the fact that the estimate of the covariance function is asymptotically unbiased and consistent, the periodogram unhappily does not converge to the power spectrum: in fact the variance of the power spectrum does not tend to zero as the number of observations increases. For example, the periodogram asymptotic variance, for Gaussian signals, tends at any frequency toward the square of the power spectrum density.

\[ \lim_{L \to \infty} \text{Var}[\hat{\Phi}_r(\omega)] \approx \Phi_r^2(\omega) \]  

(5.34)

Because the asymptotic variance is not zero, the periodogram is not a consistent estimate of the power spectrum. In estimation theory, the periodogram is perhaps the most famous example of an inconsistent, yet asymptotically unbiased estimator. The reason for which the periodogram does not converge lies in the expression of the power spectrum as an integral transformation of the estimated covariance function. The variance of the covariance estimate is given approximately by \(^2\) (Jenkins and Watts, 1968):

\[ \text{Var}[\hat{\Phi}_r(m)] = \frac{1}{L} \cdot \sum_{k=-L-1}^{L-1-m} \left( 1 - \frac{m+|k|}{L} \right) \cdot [K_r^2(k) + K_r(k + m) \cdot K_r(k - m)] \]  

(5.35)

From Eq. (5.35) it is possible to find that the covariance function's estimate $\hat{\Phi}_r(m)$ at

---

1. Consistent: (estimation theory) \( \lim_{N \to \infty} \text{Var}[\hat{\Phi}_r(\omega)] = 0 \).

2. The reason for the approximation is the need for the fourth moment of the stochastic sequence \( r(t) \). This computation is dependent on the signal's joint amplitude distribution. This approximation, derived under a Gaussian assumption, also applies to other distributions.
larger lags has a larger variance than at smaller lags. These large terms are added in the Fourier sum and their total variability is never compensated by the normalization factor. This flaw in the periodogram is a key that can allow to find alternate spectral estimate that do converge.

### 5.3.3 Bartlett’s Procedure

To obtain a consistent spectral estimate from the empirical covariance function, either higher lags must not be used in computations, or more data should contribute toward their average. With the latter idea in mind, it is possible to segment the total data of \( L \) observations into \( K \) series of equal duration \( M \), as expressed by:

\[
L = K \cdot M
\]  
\[(5.36)\]

As developed by Bartlett (1948) a periodogram estimate can be obtained by averaging the spectral estimation computed from each segment, as expressed by:

\[
\Phi_{rr}^B(\omega, k) = \frac{1}{K} \cdot \sum_{k = 0}^{K - 1} \Phi_{rr}^{(k)}(\omega)
\]  
\[(5.37)\]

where:

\[
\Phi_{rr}^{(k)}(\omega) = \frac{1}{M} \cdot |R_{MM}^{(k)}(\omega)|^2
\]  
\[(5.38)\]

\[
R_{MM}^{(k)}(\omega) = \sum_{l = 0}^{M - 1} w_L(1) \cdot r(k \cdot M + l) \cdot e^{-i \omega l}
\]  
\[(5.39)\]

It is important to point out how Eq. (5.38) and Eq. (5.39) are the equivalent expression of Eq. (5.21) and Eq. (5.18), but that accounts for the segmentation of the observation.

In this case the Periodogram are computed from each segment: conceptually, any
spectral estimate could be averaged in this way. Assuming that the segments are mutually independent, the variance of the Bartlett estimate equals the sum of the individual variances divided by $K$, as expressed by:

$$\text{Var} \left[ \hat{\Phi}_r^B(\omega, k) \right] \approx \frac{1}{K} \cdot \Phi_{rr}(\omega) \quad (5.40)$$

From Eq. (5.40) it can be obtained the following expression for the standard deviation of the power spectrum estimate:

$$\sigma_{\Phi_{rr}}(\omega) \approx \frac{1}{\sqrt{K}} \cdot \Phi_{rr}(\omega) \quad (5.41)$$

Thus, as $L \to \infty$, maintaining a fixed duration $M$ of segments, implies that $K \to \infty$, so that Bartlett spectral estimate is consistent.

When windows other than the rectangular one are used, data aligned with the ends of our somewhat arbitrarily selected segments will not contribute to the spectral estimate. Segments overlapping can be used in order to obtain more segments from a given set of data. This increased number of does not mean that the variance of the estimate is reduced proportionally. The variance of the spectra resulting from Bartlett’s procedure is inversely proportional to the number of statistically independent segments.

It is interesting to point out that, while the variance is proportional to $1/K$, the width\(^1\) of the smoothing kernel is proportional to $1/M$ (where $M$ is the duration of the segment and $K$ is the number of segments). Figure 5.2 and shows how the width of kernels decreases as $M$ increases (longer and longer sampling segments).

---

1. The ideal kernel is a Dirac delta function: zero width $\to$ no leakage $\to$ no smoothing $\to$ no bias.
The product of the variance for the width of smoothing kernel is thus proportional to $1/L$ which is inversely proportional to the amount of data available.

$$(\text{variance}) \cdot (\text{width of smoothing kernel}) \approx \frac{1}{L} \quad (5.42)$$

Eq. (5.42) express the fundamental trade-off between variance and leakage of Bartlett (classical) spectral estimation:

"The product of the degree of smoothing of the spectral estimate and of the estimate variance is a constant" (Don Johnson. 1991).

This means that in statistical terms the Bartlett's estimate is consistent, but it is not asymptotically unbiased because of spectral smoothing.
Figure 5.2  
Width of smoothing kernel for different duration of the sampling segment. 
Sampling frequency = 400 Hz (window #1: Number of points = $2^8$ (256),
total time duration = 0.64 sec, $\Delta f = 1.5625$ Hz; window #2: Number of
points = $2^{10}$ (1024), total time duration = 2.56 sec. $\Delta f = 0.3906$ Hz)
Figure 5.3  Width of "sinc" function for different duration of the sampling segment. sampling frequency = 10 Hz (window # 1: Number of points = $2^8$ (256), total time duration = 25.6 sec. $\Delta f = 0.0391$ Hz. window # 2: Number of points = $2^{10}$ (1024), total time duration = 102.4 sec. $\Delta f = 0.0098$ Hz)
5.3.4 Adopted Method of Estimation

This review explained how from the experimental data recorded from the table behaviors it is only possible to obtain a Periodogram that is clearly biased compared to the power spectrum. In order to reduce the bias it has been decided to use the Bartlett’s estimation procedure with non overlapping segments.

The actual method adopted uses statistically independent realizations of the same random process instead of segments of the same realization. Statistically there is no difference between the use of non overlapping windows of one realization or the use of many statistical independent realizations. The use of a number of statistically independent realizations was selected because it gives some computational advantages: the vectors containing recorded data can have a smaller size and therefore are easier to be handled. Moreover, in actual experimentations the recorded signal are generally stored in a buffer location that has a limited capacity. The use of many records having the maximum buffer capacity allowed us to both have a high sampling frequency (to avoid aliasing problems) and to have a long segment duration (to improve the frequency resolution). These advantages proved to be of extreme importance for easiness of actual experimentations and the use of many realizations was preferred. For this reason in the next paragraphs when referring to the Bartlett’s estimate we will mention to the number of averages (or number of s.i. realizations) instead of number of averaging windows.

As a final remark it is to be noticed that the power spectra that we want to estimate (table input and table output) are supposed to be smooth curves, and therefore our estimation is not heavily affected by the smoothing effect of the window kernel.
5.4 Example of Spectral Estimation using ARMA Model

An ARMA(2,1) model was used to simulate different realizations of a random process of a known underlying process. The interest of the arma model rely in the fact that they are capable to reproduce single realizations of a known random process. This allowed us to understand relationship existing between single realizations and the underlying random process. This simulation has been used to evaluate the effect of the Bartlett estimation procedure of a random process (averaging of different realizations), and the effect of the use of different windows.

In this paragraph first we briefly describe an ARMA Model, then we use it to evaluate the effect of the Bartlett's estimation procedure upon the periodogram and its variance, finally we use it to evaluate the best smoothing window to adopt for the experimental work.

5.4.1 ARMA Models

Stationary Case

The general time-invariant ARMA model of order (p,q), abbreviated ARMA (p,q), is represented by the following stochastic linear equation:

\[ a_k - \Phi_1 \cdot a_{k-1} - \ldots - \Phi_p \cdot a_{k-p} = e_k - \theta_1 \cdot e_{k-1} - \ldots - \theta_q \cdot e_{k-q} \quad (5.43) \]

where:

- \( a_k = a(k \cdot \Delta t) \), with \( k = 0, 1, 2, \ldots \) represents the discrete realization of the random process;

- \( \Delta t \) is the sampling time interval;
• \( \{ e_k \} \) is a zero-mean discrete Gaussian white-noise of variance \( \sigma_e^2 \):

• \( \Phi_i \), with \( i = 1, \ldots, p \) are autoregressive coefficients:

• \( \theta_i \), with \( i = 1, \ldots, q \) are moving average coefficients.

By using Eq. (5.43) recursively, the ARMA(p,q) model can be expressed as a pure MA model of infinite order or ARMA(0,\( \infty \)) if the stability conditions are satisfied. The latter ensure that the influence of the driving noise \( \{ e_k \} \) in the remote past on the present value of the process \( \{ a_k \} \) becomes vanishingly small. The stability conditions are expressed in terms of the auto-regressive parameters only. Similarly, by recursive use of Eq. (5.43), the ARMA(p,q) model can be transformed into a pure AR model of infinite order or ARMA(0,\( \infty \)) provided that the invertibility conditions are satisfied. The latter warrant that the influence of the past values of the process \( \{ a_k \} \) on the present one becomes smaller as we go further in the past. The inverse situation is physically meaningless and the invertibility conditions are usually imposed when modeling physical systems. The invertibility conditions are expressed in terms of the moving average parameters only. If the stability conditions are satisfied, the output process \( \{ a_k \} \) has a finite variance and is a discrete stationary Gaussian process completely described by its mean function \( (\mu_a = 0) \) and autocovariance or autocorrelation function in the time domain or its spectrum in the frequency domain (Box, G.E.P and Jenkins, G.M., 1981). The definitions of the time and frequency domain second-order statistical properties are given next:
• Autocovariance function:

\[ \gamma_n = E[(a_k - \mu_a) \cdot (a_{k+n} - \mu_a)] \]

• Autocorrelation function:

\[ R_n = E[a_k \cdot a_{k+n}] \]

• Variance:

\[ \text{Var}[a_k] = \sigma_a^2 = \gamma_0 \]

• Autocorrelation coefficient function:

\[ \rho_n = \frac{E[(a_k - \mu_a) \cdot (a_{k+n} - \mu_a)]}{\sigma_a^2} = \frac{\gamma_n}{\gamma_0} \]

• One-sided spectrum:

\[
p(f) = 2 \cdot \sigma_a^2 \cdot \frac{1 - \sum_{j=1}^{q} \theta_j \cdot e^{-i \cdot 2 \pi \cdot j \cdot f \cdot \Delta t}}{1 - \sum_{j=1}^{p} \Phi_j \cdot e^{-i \cdot 2 \pi \cdot j \cdot f \cdot \Delta t}} \cdot \Delta t \quad (5.44)
\]

In the above, \( E[...] \) as usual represent the expectation operator and \( i = \sqrt{-1} \). If \( q - p < 0 \), the autocorrelation operator \( \rho_n \) consists of a mixture of damped exponential and/or damped sine waves (Box, G.E.P and Jenkins, G.M., 1981). The spectrum defined in Eq. (5.44) is the discrete counterpart of the power spectral density (PSD) function encountered in continuous-time random vibration theory. The mean square (or variance since \( \mu_a = 0 \)) of the ARMA process \( \{a_k\} \) can be obtained from the one-sided spectrum:

\[
\text{Var}[a_k] = \sigma_a^2 = \int_{0}^{\infty} p(f) df \quad (5.45)
\]
As indicated by Eq. (5.45), the spectrum $p(f)$ has a physical interpretation as the distribution of the average energy (or mean square) of the process over the continuous range of frequency $[0, f_{Nyq}]$, where $f_{Nyq} = \frac{1}{2 \cdot \Delta t}$ stands for the Nyquist frequency or highest frequency which can be observed in the discrete time series $\{a_k\}$.

**Physical interpretation of the ARMA model**

Under certain conditions, the ARMA processes can be interpreted as continuous-time response processes of dynamic systems uniformly sampled at time interval $\Delta t$. For example, it can be shown that the response covariance function of a white-noise excited $n$-degree-of-freedom (n-dof) continuous-time linear dynamic system discretely coincides with the covariance function of an ARMA(2n, 2n-1) discrete process. The parameters of the ARMA(2n, 2n-1) model depend on the dynamic characteristics of the covariance-equivalent n-dofs system, (i.e. natural periods and modal damping ratios) and the properties (temporal and spatial) of the white-noise excitation. These properties represents an important advantage of using discrete ARMA models instead of continuous-time models for earthquake ground motion modeling, since discrete models are easier to identify from real digital seismic data and can still be interpreted in terms of continuous-time dynamic models.
**ARMA(2,1) model**

The ARMA(2,1) model is defined by the following second order difference equation

$$a_k - \Phi_1 \cdot a_{k-1} - \Phi_2 \cdot a_{k-2} = e_k - \theta_1 \cdot e_{k-1}$$  \hspace{1cm} (5.46)

and is completely characterized by the following four parameters:

- $\Phi_1$, $\Phi_2$, $\theta_1$ and $\sigma_e^2$

The stability for the autoregressive parameters $\Phi_1$ and $\Phi_2$ corresponds to the triangular region shown in Figure 5.4 and the invertibility condition is satisfied if $|\theta_1| < 1$.

The underlying physical system corresponding to the ARMA (2,1) model is the linear viscosity damped SDOF system represented in Figure 5.5 where $m = \text{mass of the SDOF}$
oscillator, \( K = \) linear spring stiffness, \( C = \) dashpot damping coefficient, \( X(t) = \) input displacement applied separately to the spring and the dashpot in proportions \( C_s \) and \( C_d \), respectively. \( Z(t) = \) absolute displacement of the SDOF system (measured with respect to a fixed inertial coordinate frame).

The equation of motion of the SDOF system can be written as:

\[
-C \cdot [\ddot{Z}(t) - C_d \cdot \dot{X}(t)] - K \cdot [Z(t) - C_s \cdot X(t)] = m \cdot \ddot{Z}(t) \tag{5.47}
\]

By defining:

\[
\omega_g^2 = \frac{K}{m} \quad \text{and;}
\]

\[
2 \cdot \zeta_g \cdot \omega_g = \frac{C}{m},
\]

the equation of motion can be rewritten in the standard form:

\[
\ddot{Z}(t) + 2 \cdot \zeta_g \cdot \omega_g \cdot \dot{Z}(t) + \omega_g^2 \cdot Z(t) = C_s \cdot \omega_g^2 \cdot X(t) + 2 \cdot C_d \cdot \zeta_g \cdot \omega_g \cdot \dot{X}(t) \tag{5.48}
\]

It can be shown that if the input acceleration process \( \ddot{X}(t) \) is a continuous white-noise of constant power spectral density \( \phi_0 \), then the continuous response process \( \ddot{Z}(t) = \ddot{Z}(t) \) is covariance equivalent (discretely coincident) with the discrete ARMA (2,1) process.

For the continuous, as well as the discrete case, the autocovariance and autocorrelation functions (ACF) coincide since both input and output have a zero mean. The equivalence between the parameters of the discrete model \((\Phi_1, \Phi_2, \theta_1, \sigma_e^2)\) and the parameters of the continuous model \((\omega_g, \zeta_g, \frac{C_s}{C_d}, \phi_0)\) is obtained by comparing and equating, at all discrete
times \( t_n = n \cdot \Delta t \), with \( n = 1, 2, \ldots \) the autocorrelation functions of the discrete and continuous models.

In the case of the discrete ARMA (2,1) model, the autocovariance and autocorrelation functions ACF can take four different forms depending on which subregion of the stability region the autoregressive parameters belong to. These four subregions are referred to as Zones I, Zone II, Zone III, Zone IV, and are shown in Figure 5.4.

For the continuous model, both underdamped \((\zeta_g < 1)\) and overdamped \((\zeta_g > 1)\) cases have been considered. Coincidence of the continuous and discrete ACFs at all discrete times \((t_n = n \cdot \Delta t, (n = 1, 2, \ldots))\) can exist only in Zones I and Zone II. The terms \((-1)^n\) and \((-1)^{n+1}\) prevent that discrete coincidence in Zones III and IV. It is also to be noticed that Zones I and Zone II correspond to the underdamped and overdamped case, respectively. The one-to-one mapping between the continuous and discrete parameters is summarized in Tables 7.1 and 7.2, where are reported the formulas for the conversion relative to Zone I (that correspond to the underdamped case). The results for Zones II, III, and IV can be found in (Conte et al. 1992). This mapping depends on \(\Delta t\), the sampling time interval.

In the underdamped case, uniqueness of the discrete representation in warranted provided that \(\Delta t\) is sufficiently small, so that the damped natural frequency
\[
(\omega_d = \omega_g \cdot \sqrt{1 - \zeta_g^2})
\]
is smaller than the highest frequency \(f_{Nyq} = \frac{1}{2 \cdot \Delta t}\), known as the Nyquist frequency.
As indicated in Tables 7.1 and 7.2, the relationship between the variance of the input
discrete white-noise and the power spectral density $\phi_0$ of the continuous input white-
noise is obtained by equating the variances of the discrete and continuous output
processes.

Figure 5.5  Underlying physical system
Table 7.1 Continuous to Discrete parameter conversion for Zone I
(Underdamped Case: $\zeta_g < 1$

<table>
<thead>
<tr>
<th>Expression</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi_1$</td>
<td>$2 \cdot e^{-\zeta_g \cdot \omega_g \cdot \Delta t} \cdot \cos(\omega_g \cdot \sqrt{1 - \zeta_g^2} \cdot \Delta t)$</td>
</tr>
<tr>
<td>$\Phi_2$</td>
<td>$-e^{-\zeta_g \cdot \omega_g \cdot \Delta t}$</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>solution of: $\theta_1^2 + \frac{2 \cdot \rho_1 \cdot \Phi_1 - \Phi_1^2 + \Phi_2^2 - 1}{\Phi_1 - \rho_1 \cdot (1 - \Phi_2)} \cdot \theta_1 + 1 = 0$ with $</td>
</tr>
<tr>
<td>$\rho_1$</td>
<td>$\rho(\Delta t) = \frac{1}{2} \cdot \frac{(C_s/C_d)^2 - 4 \cdot \zeta_g^2}{(C_s/C_d)^2 + 4 \cdot \zeta_g^2 \cdot \sqrt{1 - \zeta_g^2}} \cdot \frac{\zeta_g}{\sqrt{-(\Phi_1^2 + 4 \cdot \Phi_2) + \frac{1}{2} \cdot \Phi_1}}$</td>
</tr>
<tr>
<td>$0 &lt; \omega_g \cdot \sqrt{1 - \zeta_g^2} \leq \frac{\pi}{\Delta t}$</td>
<td>Nyquist Frequency</td>
</tr>
<tr>
<td>$(1 - \Phi_2) \cdot (1 + \theta_1^2) - 2 \cdot \Phi_1 \cdot \theta_1$</td>
<td>$\sigma_e^2 = \frac{\pi \cdot \phi_0 \cdot \omega_g}{2} \cdot \frac{\omega_g}{\zeta_g} \cdot (C_s^2 + 4 \cdot C_d^2 \cdot \zeta_g^2)$</td>
</tr>
<tr>
<td>$(1 + \Phi_2) \cdot [(1 - \Phi_2)^2 - \Phi_1^2]$</td>
<td></td>
</tr>
</tbody>
</table>
Table 7.2 Discrete to Continuous parameter conversion for Zone I
(Underdamped Case: $\zeta_g < 1$

<table>
<thead>
<tr>
<th>Equation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_g = \frac{1}{2 \cdot \Delta t} \cdot \sqrt{[\ln(-\Phi_2)]^2 + 4 \cdot \lambda_d^2}$</td>
<td></td>
</tr>
<tr>
<td>$\zeta_g = \frac{-\ln(-\Phi_2)}{\sqrt{[\ln(-\Phi_2)]^2 + 4 \cdot \lambda_d^2}}$</td>
<td></td>
</tr>
<tr>
<td>$(\frac{C_s}{C_d})^2 = 4 \cdot \zeta_g \cdot \frac{\zeta_g + \sqrt{1 - \zeta_g^2} \cdot \tan(\mu_d)}{\zeta_g - \sqrt{1 - \zeta_g^2} \cdot \tan(\mu_d)}$</td>
<td></td>
</tr>
<tr>
<td>$\lambda_d = \arccos\left(\frac{\Phi_1}{2 \cdot \sqrt{-\Phi_2}}\right)$, with $0 \leq \lambda_d \leq \pi$</td>
<td></td>
</tr>
<tr>
<td>$\mu_d = \arctan\left(\frac{2 \cdot \rho_1 - \Phi_1}{\sqrt{-(\Phi_1^2 + 4 \cdot \Phi_2)}}\right)$, with $-\frac{\pi}{2} \leq \mu_d \leq \frac{\pi}{2}$</td>
<td></td>
</tr>
<tr>
<td>$\rho_1 = \frac{\Phi_1 \cdot (1 + \theta_1^2 - \theta_1 \cdot \Phi_1) - \theta_1 \cdot (1 - \Phi_1^2)}{(1 - \Phi_2) \cdot (1 + \theta_1^2 - \theta_1 \cdot \Phi_1) - \theta_1 \cdot \Phi_1 \cdot (1 + \Phi_2)}$</td>
<td></td>
</tr>
</tbody>
</table>
5.4.2 ARMA Model Simulation

The ARMA Model described in the previous paragraphs was used to simulate different realizations of a Random Process. The advantage of using an ARMA Model rely in the fact that it can easily provide single realizations of the stochastic process as well as the PSD (Power Spectral Density) of the Underlying Process. This simulation was used to obtain different realization of a Random Process in order to check the effectiveness of the Bartlett's Procedure for spectral estimate and in order to evaluate the best smoothing window to be adopted.

In order to perform a simulation that can provide insightful information about the shaking table transfer function it has been decided to select a random process which underlying process has a power spectra that is similar (in the magnitude outline only) to the transfer function of the shaking table itself. To obtain such a result, the characteristics for the underlying physical system have been chose to be the following:

- Natural frequency of the system equal to 80 Hz. this frequency was chosen because the estimated oil column frequency of the actuator is between 80 - 90 Hz: $\omega_g = 160 \cdot \pi$;
- Damping Ratio of 50%. this value has been estimated to e in this range due to the high damping characteristics of the hydraulic system: $\zeta_g = 0.5$;
- $\frac{C_S}{C_d} = 1$;
- $C_S = 0.01$, this value was selected in order to have the output of the ARMA
Model that spans the range $-2 \leq \text{output} \leq 2$ in order to match the $-2 \leq \ldots \leq 2$ range of the table acceleration:

- $\Delta t = 0.00025$ seconds, in order to match the actual sampling frequency $f_s$ of 4000 Hz used in the experiments.

Given these values of the continuous model, the corresponding parameters for the Arma Model are were found to be:

- $\Phi_1 = 1.8671$;
- $\Phi_2 = -0.8819$;
- $\Theta_1 = 0.8817$;
- $\sigma_c^2 = 0.033949$.

N.B. These values obviously satisfy the relation $\Phi_1^2 + 4 \cdot \Phi_2 < 0$.

Furthermore, in order to have this simulation as close as possible to the actual experimentation, it has been selected to use a window frame having the same characteristics of the one used for in practical study:

- window length $= 2^{14}$ points ($N = 16384$);
- sampling frequency $f_s = 4000$ Hz.

These values gives a frequency resolution of the power spectrum $\Delta f = 0.244140625$ Hz.$^1$

\[ 1. \quad \Delta f = \frac{1}{N \cdot \Delta t} \]
5.4.3 PSD Estimation via Bartlett’s Procedure

In this paragraph we use the ARMA model in order to compare the Bartlett estimated PSD (Periodogram) with the actual PSD Value. The Bartlett’s procedure suggests to compute the PSD of a Random Process through the use of a moving window. As explained in paragraph 5.3.4 for our experimentation instead to use a moving window across a single Random Process Realization, it was selected to use a number of Statistically Independent (S.I) Realizations of the same Random Process. For a matter of consistency with the method adopted in the actual experimentation also in the ARMA Model simulation it was selected to use a number of Statistically Independent (S.I) Realizations of the same Random Process.

From the analysis of Figure 5.6, Figure 5.7, and Figure 5.8 it is clear how increasing the number of averages results in a better estimation of the PSD.

Figure 5.6 represents the PSD estimation obtained from one record: it is clear how the Periodogram is very scattered around the actual value of the exact P.S.D. of the Underlying Process: it is almost impossible to identify the actual value of the P.S.D starting from its estimation.

In Figure 5.7 is plotted the underlying power spectrum of the ARMA Model\(^1\) and the results of some Bartlett’s estimation obtained using different number of averaging windows (averages):

- estimation using 2 windows:

- estimation using 132 windows:

---

1. N.B. the ARMA Model has been selected in order to simulate the actual values found for the table transfer function.
- estimation using 256 windows.

The analysis of this figure shows how increasing the number of averaging dramatically improves the estimation of the power spectrum. Nonetheless it is also clear how when the number of averages becomes higher than a certain value the improvement on the estimation becomes smaller and smaller. In fact, passing from 132 averages to 256 averages does not bring a great improvement on the estimation despite the fact that the number of averages is more than doubled.

In Figure 5.8 are represented the estimations of the PSD that can be obtained from 256 averages and the use of various smoothing windows (boxcar, Hanning, and Blackmann Harris). All the estimations are very close to the exact value of the PSD of the Underlying Process and none of them can be identified as giving better performances than the other. This result proves the validity of the Bartlett's procedure for spectral estimation. Nonetheless the 256 averages is a very high number and it is impossible to use as many records for actual experimentation.
Figure 5.6  Exact PSD and inconsistent estimation of the PSD obtained from the Fourier analysis of one realization
Figure 5.7 Influence of the number of averaging windows (2, 132, 256) on spectral estimation based on Bartlett's method
Figure 5.8  Power spectrum estimation using 256 averaging windows
5.4.4 Standard Deviation of PSD Estimation

The theory of spectral estimation predict that the Standard Deviation of an estimated Periodogram tends to be equal to the PSD itself as the duration of the recorded signal tend to infinity (Eq. (5.34)).

$$\lim_{L \to \infty} \text{Var}[\hat{\Phi}_{rr}(\omega)] \sim \Phi_{rr}^2(\omega)$$  \hspace{1cm} (5.34)

From Eq. (5.34) it can be obtained:

$$\lim_{L \to \infty} \sigma_{\Phi_{rr}(\omega)} \sim \Phi_{rr}(\omega)$$  \hspace{1cm} (5.49)

An estimation of the real (inaccessible) value of $\sigma_{\Phi_{rr}(\omega)}$, referred as $\hat{\sigma}_{\Phi_{rr}(\omega)}$, can be obtained from the analysis of different P.S.D. estimations $\hat{\Phi}_{rr}(\omega)$ (periodograms).

To verify Eq. (5.49) in Figure 5.9 are plotted the following functions:

- the exact PSD of the Underlying Process $\Phi_{rr}(\omega)$;
- the estimation of the Standard Deviation $\hat{\sigma}_{\Phi_{rr}(\omega)}$ of the periodogram (obtained from 256 periodograms).

From this figure it is clear how the two plots are almost coincident, this proving the validity of our model.
Figure 5.9  Standard deviation of the periodogram (estimated from 256 averaging windows)
5.4.5 Standard Deviation of Bartlett's Estimate

The Bartlett's procedure, on the other hand, states that the Standard Deviation of the PSD estimated via Bartlett's procedure decreases with the square root of the number of averages used to compute the estimate as expressed by (Eq. (5.41)):

\[ \sigma_{\Phi_r^B}(\omega) \propto \frac{1}{\sqrt{K}} \cdot \Phi_{rr}(\omega) \]  \hspace{1cm} (5.41)

where \( K \) is the number of statistically independent windows used in the averaging.

ARMA models are very useful to simulate (in a relatively simple and fast way) a large number of Random Process realizations. This simulation allowed us to easily construct a large statistical population made of many Bartlett's estimations of PSD: each Bartlett's estimation was referred in paragraph 5.3.3 as \( \Phi_{rr}^B(\omega, K) \), where \( K \) is the number of statistically independent windows used in the averaging.

From the population of \( \Phi_{rr}^B(\omega, k) \) it is possible to estimate the Standard Deviation of the Bartlett's estimates \( \hat{\sigma}_{\Phi_r^B}(\omega) \).

In Figure 5.10 are plotted the exact values of the standard deviation of the Bartlett's estimation of the PSD (\( \sigma_{\Phi_r^B}(\omega) \)) obtained for \( K \) equal to 8, 16, 32 and 64 averages.

In Figure 5.11 are plotted the estimations of the standard deviations of the Bartlett's estimation of the PSD (\( \hat{\sigma}_{\Phi_r^B}(\omega) \)) obtained for \( K \) equal to 8, 16, 32 and 64 averages.

Comparing Figure 5.10 and Figure 5.11, it is clear how the experimental estimation of the standard deviations follows what predicted by the theory (Eq. (5.41)).
Figure 5.10 and Figure 5.11 show also how by increasing the number of averages the standard deviation of the estimate decreases with the square root of the number of averages, as expressed by the following equation:

\[
\frac{\sigma_{\Phi_\text{fr}(\omega, K_1)}}{\sigma_{\Phi_\text{fr}(\omega, K_2)}} = \sqrt{\frac{K_2}{K_1}}
\] (5.50)

where \(k_1\) and \(k_2\) are the number of statistically independent windows (averaging windows) used to compute the Bartlett’s estimation of the P.S.D.

In Figure 5.12 are compared the Standard Deviation of the periodogram PSD estimate \(\hat{\sigma}_{\Phi_\text{fr}(\omega)}\) with the standard deviation of the Bartlett’s estimation of PSD computed with 11 averages \(\hat{\sigma}_{\Phi_\text{fr}(\omega, 11)}(\omega)\). \(K = 11\) has been selected as 11 is the number of S.I. Realizations that are used in the actual shaking table testing. The number of averaging windows had to be limited in order to maintain a good frequency resolution without having to store large amount of data and \(k = 11\) proved to be a fair compromise (a high sampling frequency was necessary to limit aliasing and to have a PSD that cover a reasonable Frequency Range).
Figure 5.10 Standard deviation of the estimation done with different number of averaging windows
Figure 5.11  Standard Deviation of the estimations obtained with 8, 16, 32, 64 averaging windows
Figure 5.12  Standard deviation of the estimation of the PSD done without averaging and using 11 averaging windows
5.4.6 Effect of Different Windows Upon Bartlett’s Estimate

We have seen in the previous paragraphs, how averaging over an increasing number of windows allows to obtain better PSD Estimates. We know that to prevent aliasing and alleviate the error implicit in discrete time analysis it is convenient to use special windows. After investigating several Windows types available in literature, we focused our attention to the usual Boxcar, the Hanning Window and the 3-Terms Blackmann Harris. The ARMA Model simulation here performed showed how the use of different windows does not affect the spectral estimate in a dramatic way.

In Figure 5.8 are plotted the estimation obtained with 256 iterations and the use of these three different windows. The results are very similar and it is impossible to affirm that one window allows a better estimate than another. Nonetheless, by comparing how the spectral estimate reach a value close to the actual one by increasing the number of average used, it was possible to find some interesting behaviors.

Figure 5.13 shows how the estimation of the value of the PSD (and its standard deviation) at a frequency of 10 Hz improves as more and more windows are used in the Bartlett’s estimation.

In Figure 5.14, Figure 5.15 and Figure 5.16 are compared the ways in which spectral estimates made with different windows get close to the actual PSD, as the number of averaging windows increases. It is evident how, independently of the type of smoothing window used, it is necessary to have a high number of averaging windows in order to reach a good PSD approximation. However, it can be easily noticed that, when only a small number of windows is used, the Blackmann Harris Windows performs better than
the other ones. This characteristic of the blackman Harris window has been noticed for many frequencies analyzed and it is particularly evident for the plot reported in that refers to a frequency of Hz. This behavior is at the base of the choice to use a 3-Terms Blackmann Harris Window throughout all of our experimentations.
Figure 5.13  Convergence of estimation and its standard deviation to the right value with increasing number of averaging windows
Figure 5.14  Effect of different windows in spectral estimation: convergence to the 10 Hz PSD value
Figure 5.15  Effect of different windows in spectral estimation: convergence to the 50 Hz PSD value
Figure 5.16 Effect of different windows in spectral estimation: convergence to the 100 Hz PSD value
5.5 Example of Errors in Transfer Function Estimation

To compute an Acceleration Transfer Function using the devices available in our Shaking Table Facility, errors occur due to the unavoidable physical characteristics of the system. The sources of possible errors were identified in:

- Sensitivity Limits of the Accelerometers;
- Electric noise (line noise) in the cables (for both input and output records);
- Precision Limits (digitization error) of the input/output Boards.

To study the effect of these possible sources of errors, first an analysis of each error source is carried out in order to have a quantitative estimation. Then a computer simulation is conducted in order to evaluate how these errors affect the final evaluation of the Transfer Function.

5.5.1 Component Error Estimation

Measurement Noise (Accelerometers Error)

The Low Impedance Voltage Mode Accelerometers used in the shaking table are calibrated by the manufacturer. They have a sensitivity of 100 mV/g +/- 2%.

We used this given value to estimate the error that can be possibly originated by these instruments. The perturbed output of the accelerometer (called $\ddot{X}_a(t)$) has been simulated by adding to the actual acceleration a uniformly distributed white noise of amplitude equal to 2% of the acceleration itself; this addition was done according to the following formula:

$$\ddot{X}_a(t) = \ddot{X}_i(t) \cdot (1 + 0.02 \cdot W(t)) \tag{5.51}$$
where:

- $\ddot{X}_a(t)$ is the output of the accelerometer (acceleration) affected by the accelerometer error;

- $\ddot{X}_l(t)$ is the actual acceleration;

- $W(t)$ is a white noise uniformly distributed between -1 and +1.

Following Eq. (5.51) each acceleration value recorded by the accelerometer is affected by a random error of amplitude 2% of the value itself.

*Electrical noise induced in the cables: Line Noise*

This error is connected with electric currents that are running through the cables used to connect accelerometers and acquisition boards. These currents are present in the cable, probably due to induced currents and can be noticed when all the equipments are switched on but no signal is sent from the accelerometers.

The amplitude of these currents was estimated, by a simple oscilloscope analysis, to be in the range of 4 - 5 milliVolts, with peaks not exceeding the 10 millivolts.

A frequency analysis of these currents showed a rather uniform spectrum with a relatively small peak at 60 Hz.

For a matter of simplicity these currents were simulated as a uniform white noise, therefore neglecting the 60 Hz. frequency component. The maximum amplitude of the white noise (called $A_i$ and $A_o$ for the input or output lines respectively) was varied in the analysis between 0 and 10 milliVolts. This white noise error of variable amplitude
was simply added to each line in analysis in accordance with the following equation:

$$L_a(t) = L(t) + A_{i10} \cdot W(t)$$  \hspace{1cm} (5.52)

where:

- $L_a(t)$ is the line signal affected by the line noise;
- $L(t)$ is the noiseless line signal (unaffected by line noise);
- $A_{i10}$ is the maximum amplitude of the line noise;
- $W(t)$ is a white noise uniformly distributed between -1 and 1.

**Digitization Error (Input/Output Boards)**

The output board (Digital to Analog, D/A) convert the digital data stored in the computer (displacement to be reproduced, already converted in terms of equivalent voltage) into a physical voltage to be sent to the controller. The error that can be originated from this digital to analog conversion does not disturb the computation of the transfer function as we decided to compute it as the relationship existing between physical electric input $\tilde{x}_c$ sent to the controller (recorded via an acquisition board) and the output given by the accelerometer fixed on the table (see Figure 5.17). For this reason the output conversion error is totally neglected in this simulation.

The A/D boards (or A/D converters) convert physical electric signal into digital data to be processed by the computer. This error is guaranteed by the manufacturer of the boards to be less than 0.0863 milliVolts (16 bits resolution over a range of +/- 2.828 Volts). This resolution is much higher than the error induced by the line noise (estimated to be
approximately +/- 5 milliVolts) and by the precision of the accelerometers (+/- 2 milliVolt). The difference between these errors is more than one order of magnitude and this allowed us to neglect also this possible source of error.

Figure 5.17 Diagram of the input and output acquisition for experimental determination of the table transfer function
5.5.2 Numerical Simulation

To perform the simulation that is suitable to evaluate the effect of these equipment errors upon the table transfer function we performed a numerical dynamic analysis of a Single-Degree-of-Freedom linear system with natural frequency and damping characteristics representative of the seismic table system. The transfer function between the acceleration applied at the base of the SDOF and its total acceleration is:

$$|H(\omega)| = \frac{\sqrt{1 + (2 \cdot \zeta \cdot \beta)^2}}{\sqrt{(1 - \beta^2)^2 + (2 \cdot \zeta \cdot \beta)^2}}$$

where:

- \( \beta = \frac{\omega}{\omega_n} \);
- \( \zeta \) is the coefficient of critical damping, the value used for this simulation is 50%:
- \( \omega_n \) is the natural circular frequency of the dynamic system. The value of \( \omega_n \) used in the numerical simulation is equal to 251.32 rpm, that correspond to a natural frequency of 80 Hz (this value was selected because on the base of the estimated oil column frequency).

The simulated transfer function was found by solving the dynamic system for the acceleration (through a frequency domain analysis), and by adding the errors as illustrated in Figure 5.18: line noises are added both to the input and to the output line. an accelerometer error is added to the actual table acceleration in order to simulate the lack of precision of the accelerometer.

In order to make the simulation as close as possible to the actual working conditions of
the table, the discrete time interval used in this numerical simulation was assumed to be the same of the sampling frequency used in real records (4 KHz). Furthermore, the acceleration time histories used to excite the numerical dynamic system were the same as those applied later to the actual shaking table.

![Diagram of Sources of Error](image)

Figure 5.18  Sources of error
5.5.3 Results

The results of this analysis showed that the transfer function is affected mainly by the line noise.

In Figure 5.19 and Figure 5.20 are plotted the T.F when the following noises are accounted for:

- Accelerometers measurement error of 2% (+/- 2 mV/100mV (g));
- line noise of 10 millivolts (Figure 5.19), line noise = 0 (Figure 5.20);
- digitization error = 0.

Figure 5.20 shows that when the line noise is equal to 0, the estimated T. F. almost coincide with the theoretical one, this allows to affirm that the error in the transfer function due to the accelerometer is small. Figure 5.19 shows a very scattered transfer function. The computed T.F. found experimentally in the research are very similar to the one represented in Figure 5.19 as it can be seen in Figure 5.21 where it is represented an experimental transfer function obtained without any averaging.

This result confirm the importance of the line noise and confirming the validity of the assumption (line noise amplitude) made in the simulation. Figure 5.22 shows the value of different transfer functions obtained for various values of line noise.
Figure 5.19  10 millivolts line noise
Figure 5.20  0 millivolts line noise
Figure 5.21  Actual table Transfer function obtained without any averaging: the scattering may be originated by line noise
Figure 5.22  Simulated transfer function obtained for different values of line noise
5.6 Characteristics of Adopted Spectral Estimation Method

In this paragraph we briefly give all the necessary informations about the procedures that were actually adopted to experimentally compute the transfer function of the shaking table.

5.6.1 Signals Selection

*Inputs and outputs*

In paragraph 5.2.1, we defined the Transfer Function as the ratio between the system output, and the system input. Given the physical characteristics of our shaking table the available inputs and outputs for practical purposes are the following:

*Inputs:*

- Table displacement input: record of the displacement command signal, analog electrical signal $\tilde{x}_c(t)$ \(^1\), sent to the table.

- Table acceleration input $\ddot{x}_c(t)$: double time derivative (obtained numerically in the frequency domain) of the recorded displacement command signal. $\tilde{x}_c(t)$. sent to the table.

*Outputs:*

- table displacement time history $x_i(t)$: obtained from LVDT attached to the

---

1. For better understanding see also Ch. 3 and Ch. 2. This notation is consistent with the one used in Ch.3.
actuator;

- table acceleration time history $\ddot{x}_t(t)$ obtained by double time numerical derivation of $x_t(t)$;

- table acceleration time history $\ddot{x}_a(t)$ obtained from the accelerometers installed on the table.

**Transfer Functions**

Using the different inputs and outputs available it is possible to compute the three different estimations of the table transfer function here described below and referred to as $H_1(\omega)$, $H_2(\omega)$, $H_3(\omega)$:

$$H_1(\omega): \text{Input} = \ddot{x}_c(t) \quad \text{Output} = x_t(t) \quad (5.53)$$

$$H_2(\omega): \text{Input} = \dddot{x}_c(t) \quad \text{Output} = \ddot{x}_t(t) \quad (5.54)$$

$$H_3(\omega): \text{Input} = \ddot{x}_c(t) \quad \text{Output} = \dddot{x}_a(t) \quad (5.55)$$

All these three transfer functions in an ideal case should coincide. However, experimentally $H_1(\omega)$, $H_2(\omega)$ and $H_3(\omega)$ may differ.

$H_1(\omega)$ and $H_2(\omega)$ were found to be practically identical, as expected from theory.

On the other hand, $H_3(\omega)$ proved to be quite different from $H_1(\omega)$ and $H_2(\omega)$.

This result was expected, as the actuator-mounted LVDT senses only the relative displacement between the actuator and the reaction mass. Any motion of the reaction mass (and its related acceleration) is therefore neglected in transfer functions $H_1(\omega)$ and
Given the fact that the purpose of a shaking table is the reproduction of a specific absolute acceleration signal, we focused our attention on the best estimation of \( H_3(\omega) \) (referred as acceleration transfer function). Transfer functions like \( H_1(\omega) \) and \( H_2(\omega) \) (referred as displacement transfer function) were used only to perform preliminary analysis of the table behavior thanks to their smooth behavior.

### 5.6.2 Characteristics of Input Signals

A key point in computing a transfer function is the selection of an appropriate input both for acceleration or displacement. As the goal of transfer function computation is to obtain the response of the structure at various frequencies, the basic idea that is commonly followed is to provide an input that has power spectrum as flat as possible: same amount of energy for all frequencies. For this reason it was decided to use as inputs a uniform white noises.

*Displacement Transfer Function Input*

To compute the displacement transfer function we used as displacement inputs a series of uniform white noises with uniform distributions which maximum amplitude had to be limited in order to be physically reproducible on the table: high amplitude white noises are difficult to be reproduced accurately since they are very jagged\(^1\). The amplitude of

\(^1\) Moreover, being the acceleration the second derivative of the displacement a white noise displacement function gives a corresponding acceleration that has a power spectrum that increases with the square of the frequency. In order to have acceleration that do not exceed the table limits, the amplitude of the white noise displacement has to be limited. As shown in Chapter 5 even considering a small value for the maximum displacement amplitude of the white noise, the table is not able to reproduce the motion for high frequency (above 50 Hz).
these motions were limited to +/- 0.1% of the maximum shaking table capacity, that is +/- 0.005 inches (+/- 1.25 mm). due to the characteristics of the table controller (it accepts as input an electric signal representing the displacement to be reproduced), the displacement white noises can be sent directly to the controller.

*Acceleration Transfer Function Input*

Uniform white noises were selected as accelerations to be reproduced (in order to have a uniform distribution of the energy over frequency). In order to obtain the displacement that correspond to the selected acceleration (as explained before the controller accepts as inputs only displacements to be reproduced), a double integration had to be performed. The integration was carried on in the frequency domain. The displacement thus obtained, were then corrected in order to have zero initial, zero final displacement and to prevent discontinuity due to initial velocity different from zero.

The correction was carried as follows:

For a given acceleration time history \( \ddot{x}_1(t) \), whose corresponding Fourier Transform \( \ddot{X}_1(\omega) \) has the desired flatness characteristics (uniform white noise), it can determined by double time integration a time history \( x_1(t) \), whose second derivative is equal to \( \ddot{x}_1(t) \). Any other time history \( x'_1(t) \) having the following expression:

\[
x'_1(t) = x_1(t) + a + b \cdot t
\]

(5.56)

where "a" and "b" are arbitrary constants.

will still have the same second derivative of \( x_1(t) \), with all its properties.
\[ \ddot{x}'_i(t) = \ddot{x}_i(t) \]  

(5.57)

Therefore the fourier transform \( \hat{\ddot{x}}_i(\omega) \) of the second derivative of any function \( x'_i(t) \) will be equal to \( \hat{\ddot{x}}_i(\omega) \):

\[ \hat{\ddot{x}}_i(\omega) = \hat{\ddot{x}}_i(\omega) \]  

(5.58)

Each white noise was corrected using specific values of the parameters "a's" and "b's". These parameters were determined by imposing the initial and final table displacements to be equal to zero:

\[
\begin{align*}
\frac{x'_i(0)}{} &= 0 \\
\frac{x'_i(t_f)}{} &= 0
\end{align*}
\]  

(5.59)

where \( t_f \) is the final time of the white noise.

Reproducing time histories determined in this way can generate some problems due to the non zero value of the initial velocity. A solution to this problem was found in the following correction.

The accelerometers used to acquire the table accelerations are not sensitive to any acceleration at frequency below 1 Hz and for this reason we are forced to neglect the transfer function for quasi-static acceleration. Therefore the addition of a static component to the original white noise acceleration time history will not affect the peculiar frequency characteristics of the input. For this reason any function having the following form:

\[ \ddot{x}'_i(t) = \ddot{x}_i(t) + c \]  

(5.60)
for small values of \( c \) has spectral characteristics that for practical purposes are as good as the ones of \( \ddot{x}_i(t) \).

By double integration of \( \dddot{x}_i(t) \), we obtain the following displacement input:

\[
x'_i(t) = x_i(t) + a + b \cdot t + c \cdot t^2 \tag{5.61}
\]

where "a", "b" and "c" are arbitrary constants.

The displacement input \( x'_i(t) \) thus obtained has the same salient spectral characteristics of any \( x_i(t) \) obtained by double integration of \( \ddot{x}_i(t) \).

The parameters "a", "b" and "c" present in (5.61), can be determined by imposing zero initial displacement, zero initial velocity and zero final displacement:

\[
\begin{align*}
x'_i(0) &= 0 \\
x'_i(t_f) &= 0 \\
x'_i(0) &= 0
\end{align*}
\]

or, alternatively zero initial displacement, zero final displacement and the initial and final value to be equal:

\[
\begin{align*}
x'_i(0) &= 0 \\
x'_i(t_f) &= 0 \\
x'_i(0) &= x'_i(t_f)
\end{align*}
\]

Anyway the first type of correction was never used to obtain displacement inputs for experimentation because the correction thus obtained gave displacement inputs that often were out of the maximum available displacement range.

It was then decided to reproduce the same white noise three times in a row and to record
only the acceleration obtained in the second reproduction. In this way, instead of having two conditions to be respected (initial and final velocity equal to zero), we had only one (initial velocity to be equal to final velocity).

Furthermore experimental analysis showed that the error connected with the initial velocity different from zero is very small that can be neglected.

Using the integration and correction procedure described above, we prepared 100 different displacement time histories, each one corresponding to a different uniform white noise acceleration time history uniformly distributed between +2 g's and -2 g's. These white noises were used to estimate the transfer function based on random excitation.
5.6.3 Periodogram Estimation Procedure

In order to obtain a good estimation of both the input and the output power spectra, the analysis and the simulation described in paragraphs 5.3, 5.4, and 5.5. suggested the use of the Bartlett's procedure with the following characteristics:

- the PSD estimates were obtained based on averaging over statistically independent time records;
- eleven (11) s.i. records were used for averaging;
- a 3-Terms Blackman Harris Window was applied to window each record;
- each record had a duration of \(4.0906\) seconds \(\left(2^{14}/4000\right)\) and was sampled at 4000 Hz.

Each segment was recorded independently from the other following this procedure: the input relative to the segment was repeated three times and only the second repetition was recorded. This in order to:

- have the response of the table in stationary conditions;
- avoid discontinuity for nonzero initial and final velocity.
5.7 CONCLUSIONS

Due to the its capacity to provide an overall picture of the system behavior with only one experiment, it was decided to use the estimation of the table transfer function based on random excitation, as opposed to frequency sweep. This approach has its shortcomings. In effect measuring the behaviors at all frequencies through the response to a single input give the following problems. It happens the frequency content of the input for a specific frequency can be close to zero, while the response detected at the same frequency can have a finite value (noise in the line, frequency leakage etc.) so that the ratio of the two blows up to high values. However this effect is averaged out when multiple windows are used. The non parametric spectral estimation method results in transfer function that have an irregular shape and sometime are not consistent (randomness of line noise and of frequency estimation error - leakage). To overcome this indeterminacy it was decided to perform a series of transfer function estimations and then to compute the mean of the results thus obtained as theorized by Bartlett. To reduce the spectral leakage (bias and smoothing) we adopted the Blackman Harris 3-Terms windows.
VOLUME II

EXPERIMENTAL / ANALYTICAL APPROACHES TO MODELING.
CALIBRATING AND OPTIMIZING SHAKING TABLE DYNAMICS
FOR STRUCTURAL DYNAMIC APPLICATIONS

by

TOMASO TROMBETTI

Houston, Texas
May 1998
PART II:

ANALYTICAL / EXPERIMENTAL CORRELATION STUDIES
CHAPTER 6
SHAKING TABLE CALIBRATION
6.1 INTRODUCTION

In this Chapter we describe the steps taken in order to optimize the response of the shaking table (shaking table tune-up). In order to make the bare shaking table reach its optimal performances (magnitude of the table transfer function as close as possible to unity and phase of the table transfer function as close as possible to zero) a calibration of the gain values for the inner loop and outer loop had to be performed.

Section 6.2 describes the procedure followed in order to calibrate the inner loop.

Section 6.3 describes the preliminary analysis conducted via the displacement transfer function in order to find the values of the gain parameters of the outer loop (called control parameters) that give an acceptable\(^1\) response of the table.

Section 6.4 describes the experimental sensitivity of the table acceleration transfer function upon the outer loop control parameter and quantifies their effect upon the table acceleration transfer function.

Section 6.5 describes the combinations of control gains (called “optimal” gain settings) that have proven experimentally to give the best table performances.

---

1. Acceptable = stable response of the table, with magnitude of the transfer function not too different from unity.
6.2 INNER LOOP SETUP

6.2.1 Introduction

The inner loop setup is the procedure that allows the control user to find the optimal inner loop gains to obtain the best results in term of servovalve response to a given command.

To achieve a good response of the servovalve in a wide range of frequency (0 to 200 Hz) and flow (0 to 30 gallons per minute) the following two steps were taken: first an LVDT feedback calibration had to be performed\(^1\), then a fine adjustment for the Control Gain had to be found.

6.2.2 LVDT Feedback Calibration Procedure

To perform the calibration of the LVDT feedback signal, the following step were taken:

1) De-Activation of the outer loop.

To be sure to send the desired command straight to the servovalve, without any intervention of the outer loop we had to:

- set the AC conditioner Gain of the outer loop (amplification. \( A(s) \). of the voltage feedback given by the Actuator position LVDT) to zero;
- set all the gains of the outer loop, except the proportional gain, to zero.

The system now is in the situation displayed in Figure 6.1.

2) Send the high pressure pilot fluid to the three stage servovalve, but do not send any fluid to the actuator.

\(^1\) This is the inner loop LVDT, this LVDT provides the controller with informations on the position of the third stage spool of the servovalve.
Figure 6.1  De-activation of the Outer Loop
This realizes a situation for which high pressure fluid is sent to the first-and-second-stage of the servovalve, but no fluid flows out of the third stage into the actuator chamber. To realize this odd situation it is necessary to have the system in full operation with zero pressure in the main hose (that brings the high pressure oil to the third stage of the servovalve). This is possible only by physically disconnecting (unscrewing) the electrical cables that opens the solenoid that let the main stage fluid flow from the manifold to the servovalve. The hydraulic system is now in the situation described by Figure 6.2.

3) Move the servovalve spool to its absolute maximum displacement.

To realize this situation it is necessary to reverse the valve polarity and to set all the parameters of the servovalve inner loop (Drive) to zero except the proportional gain (i.e. $K_{der}^1 = 0$). This setup results in a servovalve command that pushes the spool position further and further away from the null (center) position until the spool reaches one end position and stops because it is physically impossible to move any further.

4) Adjust the phase of the feedback demodulator (inner loop conditioner) until we have a maximum value of the LVDT feedback.

5) Adjust the LVDT feedback gain (inner loop conditioner) until the magnitude of the inner loop feedback signal is about +/- 10 Volts.

6) Make the third (main) stage spool move between its two extreme positions. This is possible by sending a square wave.

7) Adjust the “zero” of the feedback signal by tuning the “Zero” parameter (inner loop conditioner) so that the spool feedback is symmetric about zero (0) Volt.

---

1. The motion of the spool to one end is due to the reversal of polarity.
Figure 6.2  Main pressure solenoid disconnected, oil pressure only in the first two stage of the servovalve
6.2.3 Valve Gains Adjustment Procedure

To find the gain setting of the inner loop parameters ($K_{pro}^i$ and $K_{der}^i$) that produce a good response of the servovalve and in order to protect at the same time the LVDT feedback calibration, it was necessary to take the following steps:

- keep the gain setting of the inner loop conditioner ("Phase", "Feedback gain", "Zero"...) found for the LVDT feedback calibration unchanged;
- switch the Valve polarity back to its normal value, so that the spool once again follows the inner loop command.

At this point it was possible to send all types of command to the valve and monitor its response through the LVDT feedback. To ensure a correct and stable response of the valve for a broad range of frequencies, an oscilloscope was used to display the response versus the command (sine wave).

To perform the inner loop gain adjustment, a sinusoidal input was sent to the valve and, while monitoring its response through the oscilloscope, the following steps were taken in sequence:

- the proportional gain, $K_{pro}^i$, and the derivative gain, $K_{der}^i$, were set equal to zero (0).
- the proportional gain, $K_{pro}^i$, was increased in order to:
  have the response Vs. command signal describe a line on the oscilloscope (as opposed to an ellipse)\(^1\), for input frequencies up to 20 - 30 Hz;
  have a stable elongated ellipse for input frequencies up to 150 - 200 Hz.

---

1. i.e. the response is without delay and preserves the amplitude when compared to the command.
the derivative gain (often referred as "damping") was increased in order to reduce the amplitude of the response until the oscilloscope displays a horizontal ellipse with a vertical-to-horizontal axis ratio equal to 2/3, for input frequencies up to 150 - 200 Hz.;

the inner loop gain setup (proportional and derivative gains: $K_{pro}^i$ and $K_{der}^i$) found in this way are checked for stability as the input frequency is increased up to 300 - 400 Hz.

6.2.4 Results

This procedure allowed us to "see" the actual response of the valve at different frequencies and to verify the degree of accuracy of the valve response. To have a complete "picture" of the valve response we repeated the procedure described in paragraph 6.2.3 for different values of the valve opening (using different amplitude of the commanded sine wave). The results were very satisfactory. it was possible to find a gain setting that allows the servovalve to perform linearly in the frequency range from 0 to 30 Hz. and to perform at least in a totally stable manner for frequencies well above the ones that this shaking table was designed to reproduce.

The servovalve and actuator manufacturer suggests also the following alternative procedure for inner loop gains calibration:

send a square wave. and adjust the proportional gain $K_{pro}^i$ (keeping the derivative

---

1. i.e., without time delay and reserving the command amplitude.
2. The oil column resonance limits the performances of the shaking table at a maximum frequency of approximately 70 Hz.
gain equal to zero) until the monitored valve response shows a small overshoot (say 30% to 40%);

- for a fixed proportional gain, increase the derivative gain in order to eliminate the overshoot.

This second procedure is much less refined than the first one (adopted here), as it is possible to find many sets of the gain parameters which satisfy the two conditions. For the sake of security we decided to perform this alternative calibration as a check for the values determined according to the first procedure. The inner loop control gains finally adopted showed a good response of the servo valve with little or null overshoot.
6.3 Outer Loop Setup, Part I: Preliminary Analysis via Displacement Transfer Function

6.3.1 Introduction

The importance of the outer loop setup.

The correct tuning of the outer loop setup is the most important step in a shaking table preparation. In fact, during this procedure we assure that the shaking table is able to reproduce the commanded time histories with the degree of precision desired. The first problem encountered was to find a suitable way to determine the degree of accuracy of the table in reproducing the desired motion. We decided to follow the usual procedure that uses the transfer function as the parameter for table performance evaluation. It is important to point out that the search for the optimal gains setup (to get a transfer function as close as possible to unity\(^1\)) led to the complete knowledge of the table behavior. In fact, during these steps we had the chance not only to determine the response of the table to different outer loop gain settings, but also to identify the intrinsic table dynamics that is not possible to control through gain adjustments. These preliminary experiments led to a preliminary system identification of the seismic table.

Displacement and Acceleration Transfer Function

This paragraph is the first of two that deals with the study of the Transfer Function of Shaking Table. In this chapter we use displacement transfer function. As opposed to the acceleration transfer function used in the next chapter, to perform a preliminary analysis of

\(^1\) See paragraph 5.2.1
the table transfer function (T.F.). The displacement transfer function is not the perfect tool to identify how a shaking table is able to reproduce given time histories (the acceleration T.F. is much more appropriate\(^1\)). Nonetheless, given the preliminary characteristics of this analysis, we decided to use the displacement transfer function for its repeatability features that allowed us to perform a deterministic estimation of the transfer function as opposed to the stochastic estimation necessary for acceleration transfer function. In fact, the advantage of the displacement T.F over the Acceleration T.F. lies in the consistency of its results that are unaffected by the random error of accelerometers and experimentally show a very high repeatability. On the other hand, the displacement transfer function (being based on the relative displacement between the actuator and the reaction mass) does not take into account any possible motion of the reaction mass on which the slip table sits\(^2\). Furthermore a small error in displacement reproduction can grow into a large error in acceleration reproduction. Nonetheless the results obtained from the displacement transfer function used in this preliminary analysis proved to be quite good and (as we will show in more details later) allowed us to identify a range of values for the optimal setting for the proportional and derivative gain.

---

1. A large proportion of the structural specimen which will be tested on the table will fall in the acceleration-sensitive range of the spectra.
2. Following the notation used in the mathematical modeling of the shaking table behaviors explained in Chapter 3, the displacement transfer function correspond to the so called system transfer function $H(s)$. 
6.3.2 Displacement T.F. Via Frequency Sweep

*Characteristics of input and output*

In order to determine the table transfer function via Frequency sweep the following procedure has been used.

A sinusoidal wave was sent to the table using the function generation capability of the MTS 407 controller. The amplitude of the sine wave was limited to 1% (corresponding to 0.05") of the maximum displacement range of the actuator, and kept constant for all the frequencies analyzed. The amplitude of motion were limited to small values in order not to exceed the table performance limits (see Chapter 7) for the whole frequency range considered. The frequencies analyzed span the range from 1 Hz to around 110 Hz, with a frequency interval of approximately 1 Hz. Frequencies were swept at an interval of approximately 1 Hz, as smaller frequency intervals would have made the procedure too time consuming. During the actual experimentation, the frequency interval of one (1) Hz could not be maintained constant throughout the whole frequency range swept. When the system was excited at frequencies close to its resonant ones\(^1\) it was extremely hard to maintain the table in stable conditions as the amplitude of the table response showed a tendency to increase with time. For this reason, at these frequencies it was not possible to identify a reliable and stable amplitude of the table response, and therefore the table transfer function could not be determined. The magnitude of the table displacement transfer function obtained with the frequency sweep procedure is shown in Figure 6.3 (a).

---

1. Later identified at around 27 Hz and around 70 Hz, for the reaction mass flexible system and for the oil column system respectively.
Data acquisition and processing

In order to estimate the transfer function at a given frequency, the amplitude of the input (displacement to be reproduced on the table) and the amplitude of the output (relative motion of the actuator) were measured from the voltage information available from the MTS 407 controller: electric signal of the external command \( \tilde{x}_c \) and of the actuator mounted LVDT. The controller, in fact, is able to provide electric signals representing both the input (electrical signal proportional to the actuator motion to be reproduced) and the output (LVDT feedback signal relative to the actual position of the actuator with respect to the base). These voltages (input and output) were converted to digital data (simultaneously and at a sampling frequency of 4000 Hz) through the A/D converter of the AT/A2150 board installed in the computer. Figure 6.3 (b) compasses the commanded and the actual table displacement.

Using the recorded data, the amplitude of both input and output sinusoidal wave were estimated as follows:

\[
\text{Amplitude} = \frac{D_{\text{max}} - D_{\text{min}}}{2}
\]  

(6.1)

where:

\[
D_{\text{max}} = \frac{1}{N} \cdot \sum_{i=1}^{N} \text{LocalMaxima}_i(\text{Signal})
\]  

(6.2)

---

1. These electric signals are available from the controller and are commonly referred as "monitor" signals.
\begin{equation}
D_{\text{min}} = \frac{1}{N} \sum_{i=1}^{N} \text{LocalMinima}_i(\text{Signal})
\end{equation}

in which \(N\) denotes the number of local max and min over which the average is based. usually \(N = 10\):

LocalMaxima and LocalMinima (series of subsequent local maxima and minima) are built in functions of the of the data acquisition software (Labview).

An attempt to estimate the phase of the transfer function through the angle of the ellipse that is originated in the plot of the table response Vs. the table input, was also carried out.

This ellipse is showed in Figure 6.3 (c). The results of this type of estimation were considered unsatisfactory and therefore no phase transfer functions were computed at this stage of the research.
Figure 6.3  Table Displacement Transfer Function via Frequency Sweep:
(a) Magnitude of Displacement Transfer Function; (b) Time histories of Input, $\tilde{x}_c$ and Output, $x_t$; (c) Output Vs. Input
6.3.3 Displacement T. F. Via Wide-Band Deterministic Excitation

*Characteristics of input and output*

A uniformly distributed white noise is sent from the computer to the controller as a displacement time history to be reproduced. The maximum amplitude of the uniform white noise had to be limited to 0.01 Volts (corresponding to a maximum actuator extension of 0.005") due to the physical difficulties in realizing a highly jagged displacement time history. Although a continuous-time white-noise is physically unrealizable due to its infinite variance. A discrete-time white-noise provides a good approximation to a continuous-time white-noise and, having finite variance, is therefore physically realizable.

*Data acquisition and processing*

The displacement time history actually realized on the shaking table is monitored through the feedback signal provided by the actuator mounted LVDT (it measures the relative displacement between the actuator position and the reaction mass). As usual, both signal (input and output) were digitized and recorded simultaneously by the computer acquisition board at the sampling frequency of 4KHz. These two signal are used to computed the T.F. following the cross spectrum procedure described in Chapter 4.

Both the commanded and realized table displacement time histories are shown in Figure 6.4 (a): it is clear how it is impossible for the table to follow rigorously the input, nonetheless the estimated T.F. derived from these data, and showed in Figure 6.4 (b), is quite good. Considering the very small displacement reproduced on the table, the accuracy of the estimation of the table transfer function is very good and confirms the validity of the table transfer function evaluation by wide band deterministic excitation\(^1\). This result gives a
clear indication of how accurate system identification results can be obtained based on small accelerations and displacements. The wide-band-excitation technique gave good results also in the estimation of the phase of the table transfer function, as it is shown in Figure 6.4 (c).

The smoother results obtained through the frequency sweep procedure would suggest to use this procedure for all types of transfer function estimation. The higher speed and the easiness of realization of the FFT Based technique led us to prefer the latter one for the more extensive analysis carried on in paragraph 5.2.3.

1. The complete definition of this type of estimation is: Fourier-Transform-based wide band deterministic estimation technique.
Figure 6.4 Table Displacement Transfer Function determined via White Noise Excitation: (a) time histories of Input, $\tilde{x}_c$, and Output, $x_i$; (b) Magnitude of the table Transfer Function $H(s)$; (c) Phase of the table Transfer Function $H(S)$
6.3.4 Results

In this paragraph we show the results of an investigation performed through the use of the displacement transfer function in order to get some knowledge about the response of the table to different gain settings. This investigation allowed us to identify the range of values of the various gain that give acceptable table performance\(^1\). Furthermore this investigation gave some interesting insight about the sensitivity of the table transfer function to the control gain parameters. Next, the effects on the table performances of the proportional gain (P-gain in common practice, and \(K_{pro}\) in the mathematical model of the table explained in Chapter 3), derivative gain (D-gain in common practice, and \(K_{dp}\) in the mathematical model of the table), feed-forward gain (FF-gain in common practice, and \(K_{ff}\) in the mathematical model of the table) and delta-pressure gain (dP-gain in common practice, and \(K_{dp}\) in the mathematical model of the table).

*Proportional Gain*

Fig. 6.5. Fig. 6.6. Fig. 6.7 show the table displacement transfer functions determined for the following gain parameters:

\[
\begin{align*}
\text{P-gain} &= 2.5 \quad \text{Vs.} \quad 4.5 \text{ Volt/Volt;} \\
\text{I-gain} &= 1.5 \text{ radians per second (rps);} \\
\text{D-gain} &= 10 \text{ millisecond;} \\
\text{FF-gain} &= 0 \text{ millisecond;} \\
\text{dP-gain} &= 1 \text{ Volt/Volt.}
\end{align*}
\]

---

\(^1\) Acceptable in terms of having a table transfer function that has magnitude not too different from the optimal value of unity and a response that is stable.
Results obtained by Frequency Sweep method

The results presented in Fig. 6.5 show how this set of gain parameters produces an acceptable table transfer function only up to 20 - 30 Hz. From these two graphs it is evident that an increase in the proportional gain is able to improve moderately the response of the table in the medium frequency range (between 30 and 50 Hz approximately) by raising the value of the transfer function from values that are around 0.2 to values that are around 0.45.\(^1\) On the other hand the increase in the proportional gain heightens the resonant peak located at 20 - 22 Hz, rising the value of the transfer function well above unity (up to 2.00 - 2.50). This result indicates clearly that the solution to be found in order to obtain a table transfer function that is as close as possible to the ideal one (constant value of one throughout the whole frequency range) will involve the correct selection of the setting of the other parameters (I-gain, D-gain, FF-gain and dP-gain), and not the adjustment of the proportional gain only.

Results obtained by wide-band-deterministic-excitation method

Fig. 6.6 and Fig. 6.7 display respectively the magnitude and the phase of the table transfer function determined for the identical gain condition presented in Fig. 6.5 (P gain = 2.5 and 4.5 V/V). These two graphs confirm the trend identified using the frequency sweep: an increase in proportional gain improve the response at intermediate frequencies but amplifies the resonant peak around 20 Hz.

Note that the wide-band-deterministic-excitation method is not able to capture the spectral

---

1. The optimal value of the table transfer function is equal to one (in order to have a perfect reproduction on the table of the amplitude of the given signal).
peak corresponding to the oil column frequency.¹

From this preliminary analysis it was possible to learn that the best gain setting, in order to have a transfer function as close as possible to one, has the proportional gain values in the range that spans between 2.00 Volt/Volt to 4.50 Volt/Volt.

¹ The explanation for this shortcoming of the WBDEM is the following: The commanded discrete white noise displacement contains significant energy at oil column frequency. However, the corresponding level of acceleration at the oil column frequency exceed the force capacity of the actuator. Therefore, the reproduced table displacement is low pass filtered by the shaking table.
Figure 6.5 Proportional Gain Effect upon the magnitude of the displacement transfer function (estimation obtained via frequency sweep method):

(a) Proportional gain = 2.5 Volt/Volt
(b) Proportional gain = 4.5 Volt/Volt
Figure 6.6 Proportional Gain Effect upon the magnitude of the displacement transfer function (estimation obtained via wide-band-deterministic-excitation method):

(a) Proportional gain = 2.5 Volt/Volt
(b) Proportional gain = 4.5 Volt/Volt
Figure 6.7  Proportional Gain Effect upon the phase of the displacement transfer function (estimation obtained via wide-band-deterministic-excitation method):

(a)  Proportional gain = 2.5 Volt/Volt
(b)  Proportional gain = 4.5 Volt/Volt
**Derivative Gain**

Fig. 6.8 and Fig. 6.9 show the table displacement transfer functions determined for the following gain parameters:

- \( P\)-gain = 4.5 Volt/Volt:

- \( I\)-gain = 2.2 radians per second (rps):

- \( D\)-gain = 10 millisecond, 15 millisecond, 20 millisecond, 25 millisecond:

- \( FF\)-gain = 0 millisecond:

- \( dP\)-gain = 1 Volt/Volt.

**Results obtained from the Frequency Sweep Method**

Fig. 6.8 shows the different values of the magnitude of the table transfer function for values of \( D\)-gain increasing from 10 milliseconds to 15, 20, and 25 milliseconds. It is that an increase in the derivative gain raises the values of the table transfer function for frequencies ranging between 30 and 60 Hz, without increasing in an appreciable way the peaks of the transfer function. This characteristic of the derivative gain is of extreme importance as it gives a possibility (at least for the displacement) to correct (raise) the table transfer function in the middle frequency range without spoiling (amplifying) it at the resonance peaks (as it is the case with the proportional gain). This results suggests that the derivative gain plays a key role in the tuning of the shaking table, even more so than the role played by the proportional gain. Therefore, in the more accurate gain sensitivity analysis performed in the Section 6.4 particular attention will be given to the derivative gain. In practical table setup the \( D\)-gain showed to be the most important of all gain factors.
This analysis indicated that the optimal value of the derivative gain has to be searched in the range of values between 15 and 20 milliseconds. It is important to point out that this observed response of the table to the derivative gain, as well as the suggested range of optimal setting for the D-gain parameter, will be confirmed by the detailed analysis performed on the acceleration transfer function in Section 6.4.4. This shows that the shaking table behavior determined by the displacement transfer function, even though approximated, can be extremely valid.

*Results obtained by Wide-Band Deterministic Excitation Method*

This analysis confirms the results obtained with the frequency sweep method: an increase in the derivative gain parameter increases the table transfer function in its middle frequency range. Fig. 6.9 shows the magnitude of the displacement transfer function determined by discrete white noise excitation, for the same gain settings used in the frequency sweep method. These transfer functions display the characteristic jaggedness typical of FFT based experimental transfer functions.
Figure 6.8  Effect of derivative gain upon the magnitude of the displacement transfer function determined via frequency sweep
Figure 6.9  Effect of derivative gain upon the magnitude of the displacement transfer function determined via wide-band deterministic excitation
Feed Forward Gain

Fig. 6.10 shows the magnitude of the displacement transfer function determined for the following gain parameter values:

\begin{align*}
P\text{-gain} &= 2.2 \text{ Volt/Volt;} \\
I\text{-gain} &= 2.2 \text{ radians per second (rps);} \\
D\text{-gain} &= 15 \text{ millisec;} \\
\text{FF-gain} &= 2 \text{ millisec., 4 millisec., 6 millisec., 10 millisec.;} \\
dP\text{-gain} &= 1 \text{ Volt/Volt.}
\end{align*}

This gain setting has been selected on the basis of the results obtained for the derivative gain analysis presented in the previous section. In this analysis, as it was the case for the P-gain and D-gain in the previous sections, we try to identify a range of values for the feed-forward gain in order to improve the table transfer function and make it closer to the optimal value of unity.

Results obtained by Wide-Band Deterministic Excitation Method

Fig. 6.10 shows how increasing the feed-forward gain from 2 milliseconds to 10 milliseconds improves the displacement transfer function in the medium range. The plots in Fig. 6.10 show how the magnitude of the transfer function increases in the frequency range between 30 and 50 Hz, passing from approximately 0.75 to approximately 1. Unfortunately, this increase in the magnitude of the transfer function in the medium frequency range is accompanied by an amplification of the two spectral peaks located at approximately 27 and 50 Hz. From this analysis, it can be concluded that the optimal value of the feed forward gain has to be searched in the range of values between 6 and 10 milliseconds.
Figure 6.10 Effect of feed-forward gain upon the magnitude of the displacement transfer function determined via wide-band deterministic excitation
**Delta Pressure Gain**

Fig. 6.11 and Fig. 6.12 show the magnitude of the displacement transfer function determined for the following gain parameters values:

P-gain = 5.0 Volts/Volt;
I-gain = 2.2 radians per second (rps);
D-gain = 25 millisecond;
FF-gain = 0 millisecond;

\( dP\text{-gain} = 0, 2, 4 \text{ Volt/Volt}. \)

The role of the \( dP\)-gain feedback is to reduce the resonant peak at the oil column frequency, and for this reason it is referred also as “electronic” damping. As already noticed earlier, it appears that the wide-band deterministic excitation method is not able to capture the resonant peak at the oil column frequency when a discrete white noise is used as displacement time history to be reproduced\(^1\).

Even with the limitation explained earlier, in this analysis we tried to identify the range of values and the polarity\(^2\) of the delta pressure gain, by monitoring the peak at approximately 50 Hz\(^3\).

**Results obtained by Wide-Band Deterministic Excitation Method**

Fig. 6.11 shows the effect of an increase of the delta pressure feedback on the magnitude of the table transfer function, when the polarity of this control is set to its regular value

---

1. On the other hand, the WBDEM proved to be able to capture the oil column frequency when, as done in Sections 6.4, 7.2 and 7.3, a discrete white noise is used as the acceleration time history to be reproduced.
2. The delta pressure gain can be controlled in its polarity in order to have an action of the controller that damps out the actuator resonance instead of enhancing it.
3. As explained earlier this peak might not be exactly the oil column peak.
(normal). It is noticed that an increase in the value of the delta pressure gain does not decrease peak at 50 Hz. On the contrary, an increase in the dP gain from a value of zero (0) to a value of two (2) Volt/Volt slightly increases the value of this spectral peak from a value of approximately 3.3 to a value of approximately 3.5. Increasing the delta-P gain to a value of four (4) Volt/Volt gives even worse results: in this case the value of the spectral peak reaches a value close to eleven (11). This result suggested that the polarity of the dP gain should be inverted.

The inversion of the dP-gain polarity gave surprising results as well. Fig. 6.12 show the plots of the magnitude of the displacement transfer function for the same gain parameters values as those in Fig. 6.11, but for an inverted polarity of the dP gain. This figure shows that an increase in the dP-gain from a value of zero (0) to a value of two (2) Volt/Volt slightly decreases the height of the 50 Hz spectral peak from a value of 3.3 to a value of approximately 3.0. This reduction, however, is very limited, especially if compared with the reduction in the oil column peak predicted by the numerical simulation performed in chapter 3. This indicates that the 50 Hz peak is unlikely to be the oil column peak. An increase of the value of the dP-gain to a value of 4. brings the height of the 50 Hz peak to a value of approximately 11, as was the case with the normal polarity.

These contradictory results were interpreted as follows:

- The dP gain parameter has almost no influence on the displacement transfer function for the frequency range from 0 to 40 Hz.
- Due to actuator saturation it is not possible to capture the oil column peak and therefore observe its dependence on the dP gain parameter.
• The small reduction in the height of the 50 Hz spectral peak obtained with the inverted polarity suggested that the inverted polarity is able to mitigate the oil column peak.

• The fact that for values of the delta pressure gain equal to 4, for both valve polarity settings (regular or inverted), the height of the 50 Hz spectral peak increases to large values (much higher than the one obtained when the delta pressure gain = 0), has been interpreted as an index that the system is become unstable.

This investigation suggested that the optimal value of the dP-gain that has to be searched in the range between 0 and 2 - 3 Volt/Volt and that the correct polarity for the dP gain is the inverted one.
Normal Polarity

Magnitude of Table Transfer Function $H(s)$

(a) $dP$-gain = 0 Volt/Volt

Magnitude of Table Transfer Function $H(s)$

(b) $dP$-gain = 2 Volt/Volt

Magnitude of Table Transfer Function $H(s)$

(c) $dP$-gain = 4 Volt/Volt

Figure 6.11 Effect of the Delta Pressure Gain upon the magnitude of the displacement transfer function (normal polarity)
Inverted Polarity

Magnitude of Table Transfer Function $H(s)$

(a) $dP$-gain $= 0$ Volt/Volt

Hz

Magnitude of Table Transfer Function $H(s)$

(b) $dP$-gain $= 2$ Volt/Volt

Hz

Magnitude of Table Transfer Function $H(s)$

(c) $dP$-gain $= 4$ Volt/Volt

Hz

Figure 6.12 Effect of the Delta Pressure Gain upon the magnitude of the displacement transfer function (inverted polarity)
6.3.5 Frequency Sweep Versus Wide-Band Deterministic Excitation

The frequency sweep is the approach that seems to catch better the actual transfer function without being significantly affected by any type of error. The wide-band excitation approach leads to results that in substance are comparable but that are spoiled by the high sensitivity of this method to the errors induced by the line currents. For this reason the transfer functions obtained by the wide-band-excitation method are not as "clear" (clean smooth lines) as the ones obtained by frequency sweep. This difference is clear by comparing the displacement transfer functions represented in Fig. 6.13 and Fig. 6.14. Both figures represent plots of the table transfer function obtained for the same values of the table gain settings. The transfer functions showed in Fig. 6.13 were determined by frequency sweep: notice how the two plots are very smooth and almost identical. This is a clear indication of:

- the accuracy of the transfer function determined by frequency sweep;
- the high repeatability of the frequency sweep due to the absence of random errors in its determination.

The transfer functions showed in Fig. 6.14 were determined by wide-band excitation: notice how the two plots have a very jagged outline with multiple peaks (spikes) occurring at different frequencies for different experimentations (plot). This is a clear indication of:

- the smaller accuracy of wide-band-excitation estimated transfer functions, when compared with those obtained by frequency sweep;
- the lack of a strong repeatability of the transfer function estimation obtained by

---

1. More precisely the gain setting used is the following: P-gain = 3.5 Volt/Volt, I-gain = 2.2 radians per second, D-gain = 1.5 millisecond, FF-gain = 0, dP-gain = 1.0 Volt/Volt.
wide-band excitation.

The bad repeatability characteristics of the displacement transfer function obtained through wide-band excitation are probably due to the so called "line noise". The line noise has been defined in the error simulation conducted in Chapter 4.

The wide-band excitation based transfer functions have a rough outline that in a certain way hides the true characteristics of the system. These characteristics, on the other hand, are clearly evident in the transfer functions determined through frequency sweep. Despite this shortcoming, the transfer functions determined through wide-band excitation have a certain advantage with respect to those determined through frequency sweep. In fact, one estimation of the transfer function over the entire frequency range obtained by wide-band excitation takes the same experimental time of the evaluation of the transfer function at a single frequency when using the frequency sweep. As an example, the experimental effort required to produce the results showed in Fig. 6.13 is about one hundred (100) times larger than the one required to produce the equivalent results represented in Fig. 6.14. This remarkable difference has motivated the decision to use wide-band excitation method instead of the frequency sweep for the more detailed analysis of the table behavior carried out in the next Section.

In order to improve the smoothness of the table transfer functions determined through wide band excitation, it was decided to use the Bartlett's procedure for spectral estimation\(^1\) with an average over eleven (11) windows. This particular number of windows was selected as the best compromise between a high number of windows needed for a good

---

1. See Chapter 4.
spectral estimation, and the small number of averages necessary to maintain the experimental time within acceptable limits.

**Repeatability**

Magnitude of Table Transfer Function $H(s)$

![Graph](image)

Hz

Magnitude of Table Transfer Function $H(s)$

![Graph](image)

Hz

Figure 6.13 Frequency Sweep: two different estimations of the displacement transfer function obtained for the following gain setting:

$P$-gain = 3.5 V/V, $I$-gain = 2.2 rps, $D$-gain = 1.5 msec., $FF$-gain = 0 msec., $dP$-gain = 1.0 V/V.
Repeatability

Magnitude of Table Transfer Function $H(s)$

Hz

Magnitude of Table Transfer Function $H(s)$

Hz

Figure 6.14  Wide-band-deterministic-excitation: two different estimations of the displacement transfer function obtained for the following gain setting:

$P$-gain = 3.5 $V/V$, $I$-gain = 2.2 rps, $D$-gain = 1.5 msec., $FF$-gain = 0 msec., $dP$-gain = 1.0 $V/V$ (same setting as the one used in the frequency sweep determination of Fig. 6.13)
6.3.6 Conclusions

From this preliminary analysis conducted via displacement transfer function, it was possible to obtain the following indications on the value of the controller gains:

- the optimal value of the proportional gain has to be searched in the range between 2 and 4.5 Volt/Volt;
- the optimal value of the derivative gain has to be searched in the range between 10 and 25 millisecond;
- the optimal value of the feed forward gain has to be searched in the range between 5 and 15 millisecond;
- the optimal value of the dP gain has to be searched in the range between 0 and 2 Volt/Volt, and the polarity must be “inverted”.

Furthermore, this analysis showed the capacity of the transfer functions, determined with wide-band excitation to capture satisfactorily the dynamic table behavior, although with some limitation in terms of accuracy.
6.4 OUTER LOOP SETUP, PART II: DETAILED GAIN SENSITIVITY ANALYSIS VIA ACCELERATION TRANSFER FUNCTION

6.4.1 Introduction

In this Section we will use the more accurate acceleration transfer function to analyze how the table respond to the different gain settings (Sections 6.4.2 - 6.4.6). The knowledge of how the table responds to different gains is then used in order to determine the gain setting that gives the optimal accuracy in the table response (Section 6.5). The table acceleration transfer functions $T(s)^1$ will be estimated using the Bartlett’s estimation procedure described in Chapter 4. The maximum acceleration to be reproduced on the table for this estimation is set equal to $2g$'s. This value proved experimentally to be the best one in order to identify the table behavior within its performance limits and accounting for the similitude requirements.

As already mentioned, the table transfer function will be studied trough its two components: the Magnitude of the T.F. (that graphically represent how an acceleration amplitude is changed by the system) and the Phase T.F. (that graphically represent how an acceleration phase is changed by the system). The “reference” or “target” value for the Magnitude T.F. is unity: exact reproduction of the acceleration amplitude for all frequencies. The “reference” or “target” value for the Phase T.F. is zero: no phase shift in the reproduction of the acceleration for all frequencies.

---

1. Due to the characteristics of acceleration transfer function, all the transfer functions experimentally determined in this paragraph correspond to the so called “table transfer function $T(s)$”, as defined in Chapter 3.
Magnitude T.F.

In the analysis of the magnitude of the table acceleration transfer function (for simplicity referred as magnitude of T.F., or M.T.F) it is possible to identify three different behaviors in three different frequency ranges: the low frequency range (0 to 25 - 30 Hz), the middle frequency range (30 to 60 - 70 Hz) and the high frequency range (above 70 Hz). The delimitation of these three frequency ranges is based on the following peaks:

- peak corresponding to the natural frequency of the reaction mass system:
- peak corresponding to the oil column frequency.

In general for small values of the gains, the M.T.F. has values below unity except for these peaks. An increase in the gains tends to rise the M.T.F. to values closer to unity but on the other side it tends also to heightens the magnitude of these peaks.

In order to have a better evaluation of the M.T.F. two parameters were defined and analyzed in addition to its graphic representation:

- the so-called Root Mean Square of the Error (R.M.S.) and
- the Maximum absolute value of the Error (Max).

The error was defined as the difference between the experimental M.T.F. and the target unity T.F. The error was computed over the frequency range spanning between 0.5 Hz. and 120 Hz.

Phase T.F.

In general the P.T.F. shows a shift that varies linearly with increasing frequencies. The different gain settings affect the rate at which the phase shift between commanded and
actual table acceleration increases with the increasing frequencies of the spectrum (slope of the phase T.F. plot). An indication of the value of this rate can be easily obtained from the frequency at which a shift of $-\pi$ occurs.
6.4.2 Proportional Gain

The shaking table performances were studied for the following values of the control gains:

\[
P\text{-gain} = \text{variable from 1 to 6 V/V: 1, 2, 3, 4, 5, 6 Volt/Volt};
\]

I-gain = 0 rps;

D-gain = 0 millisecond;

FF-gain = 0 millisecond;

dP-gain = 0 Volt/Volt.

Magnitude of the table transfer function (M.T.F.)

Fig. 6.15 shows the M.T.F. for increasing values of the proportional gain from 1 to 6 Volt/Volt. The main effect that an increase in the Proportional Gain has upon the table behavior was found experimentally to be a raise in the value of the Magnitude of the Transfer Function. The increase of the M.T.F. is particularly pronounced at the location of the two resonant peaks of the system.

Fig. 6.16 and Fig. 6.17, show that when increasing the proportional gain from a value of 1 Volt/olt to a value of 2 - 3 V/V, both the R.M.S and the Max value of the Error decrease. At the contrary, when the value of the proportional gain exceed 2 - 3 Volt/Volt, the R.M.S and the Max value of the Error start to increase.

This fact can be simply explained as follows: the initial increase of the proportional gain raises the M.T.F. in intermediate frequency range to values close to unity. thus reducing both R.M.S and Max value of the error. However, a further increase in the Proportional
gain literally blows up the table response at the resonant frequencies of the system. This increase in the amplitude of the resonant peaks contribute more to the error measures than the error reduction contributed by the raise of the M.T.F. in the intermediate frequency range. These results indicate that the optimal setting for the P gain has to be searched between the values of 1.5 and 2.5 Volt/Volt.

*Phase of the table transfer function (P.T.F.)*

An increase in the proportional gain (P-gain) has the effect to increase the value of frequency at which a - $\pi$ phase shift occurs. Fig. 6.18 clearly show how this frequency increases from a value around 15 Hz to values around 40 Hz, as the Proportional gain increases from 1 Volt/Volt to 6 Volt/Volt.
Figure 6.15  Effect of the proportional gain upon the Magnitude of the table acceleration transfer function
Figure 6.16  Effect of the proportional gain upon the table transfer function: Root Mean Square of the error of the magnitude of the table transfer function
Figure 6.17  Effect of the proportional gain upon the Maximum value of the error of the magnitude of the table acceleration transfer function
Figure 6.18  Effect of the proportional gain upon the Phase of the table acceleration transfer function
6.4.3 Integral Gain

The shaking table performances were studied for the following values of the control gains:

\[ \begin{align*}
\text{P-gain} & = 1 \text{ Volt/Volt;} \\
\text{I-gain} & = \text{variable from 0 to 40 rps (radians per second): 0, 20, 30, 40 rps;} \\
\text{D-gain} & = 0 \text{ millisecond;} \\
\text{FF-gain} & = 0 \text{ millisecond;} \\
\text{dP-gain} & = 0 \text{ Volt/Volt.}
\end{align*} \]

*Magnitude of the table transfer function (M.T.F.)*

Fig. 6.19 shows the M.T.F. for increasing values of the integral gain from 0 to 40 rps. The experimental analysis of the table response showed how increasing in the integral gain raises the Magnitude of Table Transfer Function mainly in the very low frequency range: notice how the increase of the M.T.F. in the low frequency range can be extremely high and can originate a new peak in the table transfer function. It is very interesting the fact that an increase in this gain does not raise the M.T.F in correspondence of the resonant peaks as it is a common with the other gains analyzed. The analysis of the R.M.S. and the Max. value of the Error (Fig. 6.20 and Fig. 6.21) indicates that the optimal value for the I-gain is to be searched in the interval between 20 and 40 rps.

*Phase of the table transfer function (P.T.F.)*

An increase in the I-gain has no significant effect upon the Phase T.F. as indicated by Fig. 6.22.
Figure 6.19 Effect of the integral gain upon the magnitude of the table acceleration transfer function
Figure 6.20  Effect of the integral gain upon the Root Mean Square of the error of Magnitude of the table acceleration transfer function
Figure 6.21  Effect of the integral gain upon the Maximum value of the error of the magnitude of the table acceleration transfer function
Figure 6.22 Effect of the integral gain upon the phase of the table acceleration transfer function
6.4.4 Derivative Gain
The shaking table performances were studied for the following values of the control gains:

\[
P\text{-gain} = 1 \text{ Volt/Volt;}
\]

\[
I\text{-gain} = 0 \text{ rps;}
\]

\[
D\text{-gain} = \text{variable from 0 to 40 milliseconds: 0, 10, 20, 30, 40 milliseconds;}
\]

\[
FF\text{-gain} = 0 \text{ millisecond;}
\]

\[
dP\text{-gain} = 0 \text{ Volt/Volt.}
\]

Magnitude of the table transfer function (M.T.F.)
Fig. 6.23 shows the M.T.F. for increasing values of the derivative gain from 0 to 40 milliseconds. The experimental analysis of the table response showed that the main effect of an increment in the derivative gain is to raise the M.T.F. in the intermediate frequency range. Furthermore, it is very interesting to observe that an increase in the derivative gain amplifies the oil column peak and shifts its frequency towards lower frequencies. The so-called “reaction mass peak” is unaffected by the derivative gain. The overall effect of an increase in the derivative gain is the creation of a very wide oil column peak that spans frequencies between approximately 40 and 70 Hz. The Frequency corresponding to the apex of the oil column peak is lowered from 67 Hz to 53 Hz approximately, as the derivative gain increases from 0 to 40 milliseconds. The analysis of the R.M.S. and the Max. value of the Error (Fig. 6.24 and Fig. 6.25) indicates that the optimal value for the derivative gain is to be searched in the values located around 20 milliseconds.
Phase of the table transfer function (P.T.F.)

As shown in Fig. 6.26, an increase in the derivative gain has a dramatic effect upon the phase of the T.F. In fact, passing from a derivative gain equal to zero to a derivative gain equal to 10 milliseconds (the smallest non zero value considered in this analysis) shifts the frequency at which a - π phase shift occurs from approximately 20 Hz to approximately 60 Hz. This is the biggest improvement in the phase of the T.F. given by a single gain. Further increases in the derivative gain have practically no effect upon the Phase T.F.
Figure 6.23  Effect of the derivative gain upon the Magnitude of the acceleration table transfer function
Figure 6.24  Effect of the derivative gain upon the Root Mean Square of the error of the magnitude of the table acceleration transfer function
Figure 6.25  Effect of the derivative gain upon the Maximum value of the error of the magnitude of the table acceleration transfer function
Figure 6.26  Effect of the derivative gain upon the Phase table acceleration transfer function
6.4.5 Feed Forward Gain

The shaking table performances were studied for the following values of the control gains:

- **P-gain** $= 1$ Volt/Volt;
- **I-gain** $= 0$ rps;
- **D-gain** $= 0$ millisecond;
- **FF-gain** $= \text{variable from 0 to 40 milliseconds: 0, 20, 30, 40 milliseconds} ;$
- **dP-gain** $= 0$ Volt/Volt.

**Magnitude of the table transfer function (M.T.F.)**

Fig. 6.27 shows the M.T.F. for increasing values of the feed-forward gain from 0 to 40 milliseconds. The experimental analysis of the table response showed how an increment in the feed-forward gain raises the magnitude of the table transfer function uniformly over the whole frequency range. Notice how increasing the feed-forward gain neither blows up the resonant peaks (as in the case of the proportional gain and derivative gain) nor create new peaks (as in the case of the integral gain). The overall effect of the Feed Forward Gain increase is an uplift of the M.T.F. that repeats its pattern almost unaltered. It is observed that an increase in the feed-forward gain improves the performances of the M.T.F. in the low and intermediate frequency range (where the M.T.F. is less than unity), but spoils the performances of the M.T.F. in correspondence of the oil column frequency (where the M.T.F. is already greater than unity). The analysis of the R.M.S. and the Max. value of the error (Fig. 6.28 and Fig. 6.29) indicates that the optimal value
for the feed-forward gain has to be searched in the range of values around 20 milliseconds, which is similar to the case of the derivative gain.

_Phase of the table transfer function (P.T.F.)_

Fig. 6.30 shows how the P.T.F. varies with increasing values of the feed-forward gain from 0 to 40 milliseconds. The feed forward gain has a dramatic effect upon the phase of the T.F. This effect is very similar to the one observed in the case of the derivative gain. Increasing the value of the feed-forward gain from 0 to 10 milliseconds (the smallest non zero value considered in this analysis) shifts the frequency at which a - π phase shift occurs from approximately 18 Hz to approximately 60 Hz. As for the derivative gain, further increases of the feed-forward gain have basically no effect upon the phase of the T.F.
Figure 6.27 Effect of the feed forward gain upon the Magnitude of the table acceleration transfer function
Figure 6.28  Effect of the feed forward gain upon the Root Mean Square of the error of the magnitude of the table acceleration transfer function
Figure 6.29  Effect of the feed forward gain upon the Maximum value of the error of the magnitude of the table acceleration transfer function
Figure 6.30  Effect of the feed forward gain upon the Phase of the table acceleration transfer function
6.4.6 Differential Pressure Gain

The shaking table performances were studied for the following values of the control gains:

P-gain = 2 Volt/Volt\(^1\);

I-gain = 0 rps;

D-gain = 0 millisecond;

FF-gain = 0 millisecond;

dP-gain = variable from 0 to 3 V/V: 0, 1, 2, 4 Volt/Volt.

Magnitude of the table transfer function (M.T.F.)

Fig. 6.31 shows the M.T.F. for increasing values of the differential-pressure gain from 0 to 4 Volt/Volt. It is observed experimentally that increasing the differential-pressure gain (dP-gain), decreases the magnitude of the transfer function at the oil column frequency. Because of this property the dP-gain is sometimes referred to as “electronic” damping. This feature is particularly welcomed in order to decrease the high peaks that are generated by large values of the proportional and derivative gain. The dP-gain has basically no other effect upon the magnitude transfer function at other frequencies. The analysis of the R.M.S. and of the Max. value of the error (Fig. 6.32 and Fig. 6.33) indicates that the magnitude of the T.F. improves more and more as the value of the dP-

---

1. In this analysis, a value of the proportional gain equal to two (2) Volt/Volt was used instead of the usual value of one (1) Volt/Volt used for the other gains analyses. This choice was made in order to show more clearly the mitigating effect of the dP-gain on the magnitude of the T.F. around the oil column frequency. A value of the proportional gain equal to one V/V would not have generated a resonant peak high enough to appreciate its reduction.
gain increases. Nonetheless the optimum value of the dP-gain has is around the value of three Volts/Volts.

*Phase of the table transfer function (P.T.F.)*

As in the case of the integral gain, an increase in the dP-gain has no substantial effect upon the phase of the T.F., as shown in Fig. 6.34 where phase of the T.F. obtained for the different values of the dP-gain are overlaps.

---

1. The limitation of the optimal value of the dP-gain to values not in excess of 4 Volt/Volt, derives from the stability limits found while conducting the preliminary table study via the displacement transfer function (Section 5.2). The frequency sweep analysis showed how values of the dP-gain equal or greater than 4 render the response of the table unstable. This instability could not be caught using wide-band acceleration input.
Figure 6.31  Effect of the delta pressure gain upon the Magnitude of the table acceleration transfer function (M.T.F.)
Figure 6.32 Effect of the delta pressure gain upon the Root Mean Square of the error of the M.T.F.
Figure 6.33  Effect of the delta pressure gain upon the Maximum value of the error of the M.T.F.
Figure 6.34: Effect of the delta pressure gain upon the Phase of the table acceleration transfer function (P.T.F.)
6.5 "Optimal" Gain Setting

Based upon the results obtained with the sensitivity analysis of the table transfer function upon the different control gains illustrated in the previous sections, a search for an "optimal" gain setting was performed.

The objective of this search is to find a set of control gain parameters that minimizes both the R.M.S. and the Maximum value of the error of the magnitude of the table acceleration transfer function as compared to unity. The search was conducted by manually adjusting the gains to different values and then estimating the T.F. with all its characteristics. After a long trial and error process of fine tuning all the parameters, four different gain settings emerged as giving the best table performances.

Gain Setting # 1:

P-Gain = 1.81 V/V;

I-Gain = 40 rps;

D-Gain = 18 milliseconds;

FF-Gain = 18 milliseconds;

dP-Gain = 3.0 V/V.
**Gain Setting # 2:**

P-Gain   =  1.81  V/V;
I-Gain   =   40  rps;
D-Gain   =  20  milliseconds;
FF-Gain  =  16  milliseconds;
dP-Gain  =  3.0  V/V.

**Gain Setting # 3:**

P-Gain   =  1.81  V/V;
I-Gain   =   18  rps;
D-Gain   =  12  milliseconds;
FF-Gain  =  18  milliseconds;
dP-Gain  =  2.6  V/V.

**Gain Setting # 4:**

P-Gain   =  1.81  V/V;
I-Gain   =  20  rps;
D-Gain   =  12  milliseconds;
FF-Gain  =  14  milliseconds;
dP-Gain  =  2.0  V/V.
Given the importance of these "optimum" gain settings, the table transfer function has been evaluated by averaging the input power-spectra and input-output cross-power spectra over 20 statistically independent time windows (instead of the usual 11), each one sampled at the usual 4KHz sampling frequency and consisting of $2^{14}$ points, thus giving a frequency resolution in the periodogram of 0.244 Hz.

**Magnitude of the table transfer function (M.T.F.)**

Fig. 6.35 shows the Magnitude T.F. for all the four different gain settings: **Gain Setting # 1, Gain Setting # 2, Gain Setting # 3, Gain Setting # 4**.

*Setting # 1* proves to be the best in the intermediate and high frequency ranges as this gain setting is able to maintain the magnitude of the table T.F. around the value of one almost continuously until a frequency around 70 Hz. The other gain settings, even if they perform better than *Setting # 1* in the intermediate frequency range (30 to 60 Hz) show a sharp drop (well below unity) in the magnitude of the table transfer function for frequencies larger than approximately 60 Hz. *Setting # 4* gives the best table response only up to a frequency of 35 Hz.

The analysis of the value of the R.M.S. of the error for the different gain settings (Fig. 6.36) confirms the results obtained visually and according to which *Setting # 1* gives the best overall table performances.

The analysis of the Maximum value of the error (Fig. 6.37) indicates that *Setting # 4* gives the smallest maximum error, thus confirming that *Setting # 1* and *Setting # 4* are the best ones.
Phase of the table transfer function (P.T.F.)

Fig. 6.38 indicates that, in terms of phase transfer function, Setting # 1 and Setting # 2 are better than Setting # 3 and Setting # 4. Indeed, the first two cases shows a - π shift at frequencies around 60 Hz, while the last two cases show the same shift at frequencies around 50 Hz.

Conclusions

From this analysis it can be concluded that the best overall table performance is obtained with gain Setting # 1. Setting # 4 has a better performance for frequencies up to 40 Hz and has a significantly smaller (than Setting # 1) resonant peak in correspondence of the natural frequency of the reaction mass.
Figure 6.35  Optimal Gain Setting: Magnitude of the table transfer function for different gain settings ("optimal gain" Setting # 1, # 2, # 3, and # 4)
Figure 6.36: Optimal Gain Setting: Root Mean Square of the error of the magnitude of the table transfer function for different values gain settings ("optimal gain" settings #1, #2, #3, and #4)
Figure 6.37  Optimal Gain Setting: Maximum value of the error of the magnitude of the table transfer function for different gain settings ("optimal gain" Settings # 1, # 2, # 3, and # 4)
Figure 6.38  Optimal Gain Setting: Phase of the table transfer function for different gain settings ("optimal gain" Settings # 1, # 2, # 3, and # 4)
6.6 CONCLUSIONS

In this Chapter we described the steps taken in order to obtain a perfect calibration of the response of the shaking table. Section 6.5 describes the values of the control parameters experimentally identified as the ones that give the best table performances.

The experimental sensitivity of the shaking table to control parameters proved to match the results obtained in the mathematical qualitative sensitivity simulation performed in Section 4.
CHAPTER 7
EFFECT OF THE PAYLOAD ON
THE SHAKING TABLE CALIBRATION
7.1 INTRODUCTION

In this Chapter we describe the effect of the different payload characteristics upon the shaking table transfer function.

Sections 7.2 and 7.3 describe respectively the table sensitivity to very rigid and to flexible payload having different weights.

Given the fact that in both cases the presence of a payload affects the table response, in both sections it is carried out an attempt to find new sets of gain parameters that give a table transfer function with a magnitude as close as possible to unity ("optimal" table response).
7.2 SENSITIVITY OF TABLE TRANSFER FUNCTION TO RIGID PAYLOAD

7.2.1 Introduction

This paragraph investigates the effect of rigid payloads on the shaking table behavior\textsuperscript{1}.

The objectives of this Section are:

- to investigate whether and how the table transfer function is affected by the addition of large rigid payloads;
- to find how the optimal setting of the gain parameters has to be changed in order to keep the transfer function at its best performances.

The results obtained showed that a rigid payload produces an increase in the magnitude of the T.F. in the intermediate frequency range. These experimental results seemed to indicate that the actual influence of rigid payloads is larger than the one predicted by the mathematical model of the shaking table developed in Chapter 3.

An experimental investigation showed that, in order to keep the table transfer function in its best condition, it is necessary to modify the value of the derivative gain mainly.

It is worth noting that this sensitivity analysis considered the addition to the table of very large weights (up to 900 lbs), that are close to the maximum payload selected in the design of the shaking table (1500 lbs). It is expected that the majority of future experimentations performed on the table will involve models of smaller weights.

\textsuperscript{1} The effect of flexible payload upon the table transfer function are studied in Section 7.3.
7.2.2 Sensitivity to Rigid Payload

Using the gain setting identified as the “optimal” one (Set #1 described in Section 6.5), the transfer function of the shaking table was estimated for the following conditions:

- Bare table;
- 150 lbs rigid payload rigidly bolted (in a symmetric position) on the shaking table;
- 300 lbs rigid payload rigidly bolted (in a symmetric position) on the shaking table;
- 450 lbs rigid payload rigidly bolted (in a symmetric position) on the shaking table;
- 600 lbs rigid payload rigidly bolted (in a symmetric position) on the shaking table;
- 900 lbs rigid payload rigidly bolted (in a symmetric position) on the shaking table.

Magnitude of the table transfer function (M.T.F.)

Fig. 7.1 shows the magnitude of the transfer function of the table loaded with the following rigid payloads: 0 lb, 150 lbs, 300 lbs, 450 lbs, 600 lbs, 900 lbs. This figure indicates that as rigid payloads of increasing weight are added on the table, the magnitude of the T.F. raises more and more in the intermediate frequency range (producing a wide peak) and, as expected analytically, the oil column peak is shifted toward smaller and smaller frequencies. The largest increase of the M.T.F. in the intermediate frequency range occurs with the first addition of a 150 lbs rigid payload (the value of the M.T.F. increases from approximately 1.4 to 3.5). The addition of heavier payloads (300 lbs, 450 lbs, up to 600 lbs) does not increase the amplitude of newly formed peak proportionally: the maximum height of the peak grows approximately from 3.5 to 4.5. Only when a 900 lbs rigid payload is added on the table,
the height of the newly originated peak shows a considerable increase from approximately 4.5 to 6.8.

A close look at the magnitude of the T.F. indicates clearly that this new peak in the intermediate frequency range does not correspond to a shifted oil column peak. The latter is still identifiable at frequency of 55 Hz. The decrease in the natural frequency of the oil column peak from approximately 68 to 55 Hz is in good agreement with the results predicted by the mathematical model of the table developed in Chapter 3. According to this model, the oil column frequency should decrease from approximately 70 to 55 Hz as the weight of the rigid payload increases from 0 to 900 lbs\textsuperscript{1}.

Fig. 7.2 and Fig. 7.3 show that the R.M.S. and the Maximum value of the error of the M.T.F. as a function of the weight of the rigid payload. These figures display a bi-linear behavior with large slopes and small and large weights and a reduced slope for intermediate weights. This bi-linear trend reflects the prior observation that the largest change in the M.T.F. occur as the weight of the rigid payload increases from 0 to 150 lbs and from 600 to 900 lbs.

It is noteworthy that the sensitivity of the M.T.F. to the addition of rigid payloads is qualitatively similar to the sensitivity of the M.T.F. to increasing values of the derivative gain.

\begin{table}
\begin{tabular}{|c|c|c|}
\hline
Weight of rigid payload & Experimental nat. freq. & Theoretical nat. freq. \\
\hline
0 lbs & 68 Hz & 70 Hz \\
150 lbs & 62 Hz & 65 Hz \\
300 lbs & 60 Hz & 62 Hz \\
450 lbs & 59 Hz & 60 Hz \\
600 lbs & 58 Hz & 58 Hz \\
900 lbs & 56 Hz & 55 Hz \\
\hline
\end{tabular}
\end{table}
Phase of the table transfer function (P.T.F.)

Figure 7.4 shows the phase of the table transfer function as rigid payloads of increasing weights are added to the table: 0 lbs, 150 lbs, 300 lbs, 450 lbs, 600 lbs, 900 lbs.

From these plots, it is observed that the phase transfer function deteriorates progressively as heavier and heavier payloads are added onto the table. For the bare table, the frequency corresponding to a -π phase shift occurs at approximately 55 Hz. For the table loaded with a rigid payload of 150 lbs, the frequency corresponding to a -π phase shift occurs at approximately 42 Hz (showing a decrease of approximately 10 to 15 Hz with respect to the bare table case). The addition of heavier rigid payloads (300 lbs, 450 lbs and 600 lbs) do not produce a large decrease in the -π phase shift frequency, that decreases almost linearly from approximately 42 to 40 Hz. The addition of a 900 lbs rigid payload produces a larger relative decrease in the frequency at which -π shift occurs bringing it down at approximately 35 Hz.
Figure 7.1  Effect of rigid payload of increasing weights upon the Magnitude of the table transfer function
Figure 7.2  Effect of rigid payload of increasing weights upon Root Mean Square of the error of the magnitude of the table transfer function
Figure 7.3  Effect of rigid payload of increasing weights upon Maximum value of the error of the magnitude of the table transfer function
Figure 7.4  Effect of rigid payload of increasing weights on the Phase of the table transfer function
7.2.3 Gain Correction: Derivative Gain

In the previous Section, we showed how the addition of a rigid payload on the shaking table changes dramatically its transfer function. As a rigid payload of increasing weight is attached rigidly to the table, the oil column frequency decreases (as predicted analytically) and a broad peak rises in the intermediate frequency range. In order to bring back the magnitude of the transfer function to a value close to unity (optimal value) it was decided to investigate which corrections to the control gain settings have to be applied.

As the effect of the increasing of the weight of the rigid payload on the M.T.F. is similar to the effect of an increase in the derivative gain\(^1\), it was decided to investigate whether decreasing the derivative gain could lower the M.T.F. of the loaded table back to its optimal value. The results of this investigation (described in Sections 7.2.4 to 7.2.6) show that changes in the derivative gain are able to reduce the M.T.F. in the intermediate frequency range. In order to have a complete understanding of the effect of the derivative gain upon the M.T.F. of the loaded table, it was decided to find a new "optimal" value of the derivative gain for rigid payloads weighting 150 - 300 - 450 - 600 - and 900 lbs. The objective of this investigation is to experimentally determine a chart indicating the optimal value of the derivative gain as a function of the weight of the rigid payload.

The final results indicate that in order to optimize the table performance, the derivative gain has to be reduced\(^2\) according to the smooth curve represented in Fig. 7.5. Notice

\(^1\) Both the mathematical and experimental analysis of the table sensitivity to the gain parameters showed how an increase in D-gain and FF-gain raises the value of the M.T.F. in the intermediate frequency range.

\(^2\) With respect to the "optimal" setting found for the bare table.
how even relatively “light” rigid payloads (weighting only 150 lbs) require a drastic reduction of the derivative gain (from approximately 18 to 10 milliseconds). The reduction in the value of the derivative gain is not linear with respect to the weight of the rigid payload. As heavier and heavier rigid payloads are attached to the table, the decrease in the value of the derivative gain becomes smaller and smaller. The curve representing the optimal value of the derivative gain levels off at about 5 milliseconds for rigid payload weighting more than 450 lbs.

In the next three sections (7.2.4 to 7.2.6), we will report the analyses conducted in order to find the “optimal” value of the derivative gain when the shaking table is loaded with rigid payload weighting 150 lbs, 450 lbs and 900 lbs.

---

1. The chart represented in Fig. 7.5 shows the optimal value of the D-gain found for rigid payloads weighting 150 lbs, 300 lbs, 450 lbs, 600 lbs and 900 lbs. The next three sections report the complete analysis performed for rigid payloads weighting 150 lbs, 450 lbs, and 900 lbs only; this was done for the sake of conciseness.
Figure 7.5  "Optimal" value of the derivative gain as a function of the weight of the rigid payload
7.2.4 Effect of Derivative Gain & 150 lbs Rigid Payload

Magnitude of the table transfer function (M.T.F.)

The results of this analysis indicate that a reduction of the value of the derivative gain from 18 to 10 milliseconds is able to reduce the M.T.F. in the intermediate frequency range. Figure 7.6 shows how this decrease in the derivative gain produces a final shape of the M.T.F. that is not very different from the one obtained with the optimal gain setting and the bare table. It is observed that the shift of the oil column frequency toward smaller frequencies (from approximately 68 Hz for the bare table to 62 Hz for the loaded table) results in a loss of the frequency bandwidth\(^1\). No derivative gain correction is able to compensate for this loss.

The R.M.S. and Max value of the error of the M.T.F. plotted in Figs. 7.7 and 7.8 show that this decrease in derivative gain is able to reduce both error norms. Nonetheless, this reduction in the derivative gain is not able to reduce the R.M.S. and Max value of the error to the level of the error norms obtained for the bare table with "optimal" gain setting.

Phase of the table transfer function (P.T.F.)

Contrary to the results obtained for the magnitude of the T.F., a reduction of the derivative gain does not compensate for the error introduced in the phase of the T.F. by the addition of a rigid payload. Fig. 7.9 shows that a 150 lbs rigid payload reduces the frequency at which the phase shift of - \( \pi \) occurs from about 55 Hz to 44 Hz. No change

\(^1\) Frequency Bandwidth = frequency beyond which the M.T.F. drops dramatically below 0.7 (-3dB).
in the derivative gain can significantly improve the P.T.F. the reduction of the derivative gain from 18 millisecond to 10 millisecond is able to increase this frequency only to approximately 44 - 46 Hz.
Figure 7.6  Effect of the derivative gain upon the **Magnitude** of the transfer function of the table loaded with a 150 lbs rigid payload
Figure 7.7  Effect of the derivative gain upon the **Root Mean Square of the error** of the transfer function of the table loaded with a 150 lbs rigid payload
Figure 7.8  Effect of the derivative gain upon the Maximum value of the error of the transfer function of the table loaded with a 150 lbs rigid payload
Figure 7.9: Effect of the derivative gain upon the Phase of the transfer function of the table loaded with a 150 lbs rigid payload
7.2.5 Effect of Derivative Gain & 450 lbs Rigid Payload

Magnitude of the table transfer function (M.T.F.)

The effect of the reduction of the derivative gain upon the M.T.F. of the shaking table loaded with a 450 lbs rigid payload, confirms the results obtained for the 150 lbs rigid payload. A reduction of the derivative gain is able to reduce the amplitude of the M.T.F. in the intermediate frequency range, thus improving the performance of the shaking table. Fig. 7.10 shows that decreasing the derivative gain from 18 to 5 and 6 milliseconds, it is possible to obtain an M.T.F. that is not too different from the one obtained for the bare table and the "optimal" gain setting. As in the case of the 150 lbs rigid payload, no correction in the derivative gain is able to compensate for the reduction of the bandwidth of the shaking table due to the lowering of the oil column frequency. Unlike the 150 lbs rigid payload, the shift of the oil column frequency is also accompanied by an amplitude reduction of the oil column peak from about 1.5 to 1.0.

Fig. 7.11 shows how the R.M.S. of the error of the M.T.F. decreases with decreasing value of the derivative gain. A D-gain of 5 milliseconds reduces the R.M.S of the error to approximately 0.44, a value much smaller than 0.92 obtained with a D-gain of 18 milliseconds, but still much larger than the value of 0.2 obtained with the bare table under the "optimal" gain setting.

The plot of the Maximum value of the error for different values of the Derivative Gain represented in Fig. 7.12, shows that a derivative gain of 6 milliseconds gives the best table performances. In this case, the Max value of the error is about 1.2, which is very close to the value of 1.1 found for the bare table with the "optimal" setting.
Phase of the table transfer function (P.T.F.)

The effect of a decrease in the derivative gain upon the phase of the T.F. of the table loaded with a 450 lbs rigid payload is similar to the results obtained for the 150 lbs rigid payload. From Fig. 7.13, it is observed that a reduction of the derivative gain does not improve the performance of the phase of the T.F. The addition upon the shaking table of a 450 lbs rigid payload shifts the frequency at which a phase shift of $-\pi$ radians occurs from about 55 Hz to 42 - 43 Hz. No decrease in the derivative gain is able to shift this frequency back to 55 Hz.
Figure 7.10  Effect of the derivative gain upon the Magnitude of the transfer function of the table loaded with a 450 lbs rigid payload
Figure 7.11  Effect of the derivative gain upon the Root Mean Square of the error of the transfer function of the table loaded with a 450 lbs rigid payload
Figure 7.12  Effect of the derivative gain upon the Maximum value of the error of the transfer function of the table loaded with a 450 lbs rigid payload
Figure 7.13  Effect of the derivative gain upon the Phase of the transfer function of the table loaded with a 450 lbs rigid payload
7.2.6 Effect of Derivative Gain & 900 lbs Rigid Payload

Magnitude of the table transfer function (M.T.F.)

The results obtained experimentally for the shaking table loaded with a rigid payload weighting 900 lbs are similar to those obtained for rigid payloads weighting 150 lbs and 450 lbs. A reduction in the derivative gain enables to improve the M.T.F. in the intermediate frequency range. Fig. 7.14 shows the effect of decreasing the derivative gain upon the M.T.F. In order to restore the M.T.F. close to the one corresponding to the bare table with optimal gain setting, it was necessary to set the D-gain equal to 5 milliseconds. Notice that this value of the derivative gain is exactly the same as that for a 450 lbs rigid payload. This indicates that the correction in the derivative gain is not monotonically decreasing with the weight of the rigid payload.

As was the case for lighter rigid payloads, the frequency bandwidth of the loaded shaking table is reduced as compared to the bare table condition with "optimal" gains. Furthermore, as previously noticed for the 450 lbs rigid payload, the amplitude of the oil column peak is also significantly reduced by the addition of the rigid weight. No correction of the derivative gain is able to compensate for either the frequency shift and amplitude reduction of the oil column peaks of the table T.F.

Figs. 7.15 and 7.16 display the plot of the R.M.S. and Maximum value of the error for different values of the derivative gain. These plots confirm the results obtained visually in the analysis of the M.T.F. (Fig. 7.14), according to which the optimum value of the D-gain lies between 5 and 6 milliseconds. Again, as for lighter rigid payloads, the reduction in the derivative gain improves both R.M.S. and Max. values of the error, but is not able
to lower them at the levels found for the bare table with the optimal gain setting.

*Phase of the table transfer function (P.T.F.)*

The analysis performed with a 900 lbs rigid payload added on the table confirmed the results found with the other smaller rigid payloads. A reduction of the derivative gain does not affect the phase of the table T.F. Fig. 7.17 shows clearly how the P.T.F. is not affected by decreasing the value of the derivative gain. The frequency at which a phase shift of $-\pi$ radians occurs is at around 40 Hz for all D-gain values.
Figure 7.14  Effect of the derivative gain upon the **Magnitude** of the transfer function of the table loaded with a 900 lbs rigid payload
Figure 7.15  Effect of the derivative gain upon the Root Mean Square of the error of the transfer function of the table loaded with a 900 lbs rigid payload
Figure 7.16  Effect of the derivative gain upon the Maximum value of the error of the transfer function of the table loaded with a 900 lbs rigid payload.
Figure 7.17  Effect of the derivative gain upon the Phase of the transfer function of the table loaded with a 900 lbs rigid payload
7.2.7 Gain Correction: Delta-Pressure Gain

In the preceding chapter it has been shown that the derivative gain plays a key role in the correction of the magnitude of the table T.F. when a given rigid payload is clamped to the table. From the mathematical modeling of the shaking table it is known that other gains such as the delta-pressure gain and integral gain are able to modify the amplitude of the table T.F. The effect (raising or lowering the M.T.F in specific frequency regions) of the delta-pressure gain was experimentally tested to check if it can be used to compensate the changes in the table M.T.F. caused by the presence of a rigid payload upon the table. The results of this experimental test are reported in this chapter and show that, once the derivative gain is modified as explained in Section 7.2.3, the delta-pressure gain is able to provide only a marginal improvement on the table M.T.F. This experimental test was conducted with the table loaded with a rigid payload weighting 900 lbs.

Figs. 7.18 and 7.19 show the effect of the dP-gain upon the table acceleration transfer function. The gain setting is the "optimal" one for bare table condition, with the derivative gain corrected as follows in order to account for the 900 lbs rigid payload (see Section 7.2.3):

\[
\begin{align*}
\text{P-gain} & = 1.81 \text{ Volt/Volt;} \\
\text{I-gain} & = 40 \text{ rps;} \\
\text{D-gain} & = 5 \text{ milliseconds;} \\
\text{FF-gain} & = 18 \text{ milliseconds;} \\
\text{dP-gain} & = 3 \text{ Volt/Volt, 2 Volt/Volt, 1 Volt/Volt.}
\end{align*}
\]
Magnitude of the table transfer function (M.T.F.)

The value of the delta-pressure gain was reduced as an attempt to improve the M.T.F. in the high frequency range (55 to 90 Hz), in which the response of the loaded table proved to be well below the optimal M.T.F. value of one. From the known damping effects of the DP-gain upon the oil column peak it was deduced that a reduction in the value of this gain could potentially help to increase the table M.T.F. in the neighborhood of the oil column frequency. Fig. 7.18 shows that a decrease in the delta-pressure gain has practically no effect upon the M.T.F around the oil column frequency. Only a slight decrease of the M.T.F. around 20 Hz is noticeable. Therefore, the value of dP = 1 Volt/Volt is favored for the loaded table condition.

Phase of the table transfer function (P.T.F.)

Fig. 7.19 shows that a change in the value of the dP-gain does not affect the phase of the table transfer function.
Figure 7.18 Effect of the differential-pressure gain upon the Magnitude of the transfer function of the table loaded with a 900 lbs rigid payload.
Figure 7.19  Effect of the differential-pressure gain upon the Phase of the transfer function of the table loaded with a 900 lbs rigid payload.
7.2.8 Gain Correction: Integral Gain

This section deals with the results of the experimental tests conducted in order to investigate the capability of the integral gain in improving the table T.F. It was found that the integral gain is able to improve marginally the M.T.F. in the low frequency range. These experimental tests were conducted with the table loaded with a rigid payload weighting 900 lbs.

Figs. 7.20 to 7.23 show the effect of the integral gain upon the table acceleration transfer function. The gain setting is the "optimal" one for bare table condition, with the derivative and delta-pressure gains corrected for the 900 lbs rigid payload as explained in sections 7.2.3 and 7.2.7 respectively:

\[\begin{align*}
\text{P-gain} & = 1.81 \text{ Volt/Volt;} \\
\text{I-gain} & = 40 \text{ rps, 20 rps, 10 rps, 0 rps;} \\
\text{D-gain} & = 5 \text{ milliseconds;} \\
\text{FF-gain} & = 18 \text{ milliseconds;} \\
\text{dP-gain} & = 1 \text{ Volt/Volt.}
\end{align*}\]

**Magnitude of the table transfer function (M.T.F.)**

Fig. 7.20 shows that a reduction in the value of the integral gain is able to reduce the M.T.F. in the low frequency range (0 to 15 Hz) from about 1.45 to 1.15.

The plots of the R.M.S. and Maximum value of the M.T.F. error given in Figs. 7.21 and 7.22, indicate that a reduction in the I-gain has a beneficial effect upon the table M.T.F.
The R.M.S. drops from 0.48 to 0.44 as the I-gain is decreased from 40 to 0 rps. The
Maximum value of the error drops from 1.32 to 0.9 as the I-gain is decreased from 40 to
20 rps.
It is noticed, however, that these improvements on the table M.T.F. remain marginal
when compared with those obtained by changing the derivative gain.

*Phase of the table transfer function (P.T.F.)*

Fig. 7.23 shows that the integral gain has no effect on the phase of the table T.F.
Figure 7.20: Effect of the integral gain upon the Magnitude of the transfer function of the table loaded with a 900 lbs rigid payload.
Figure 7.21  Effect of the integral gain upon Root Mean Square of the error of the transfer function of the table loaded with a 900 lbs rigid payload
Figure 7.22: Effect of the integral gain upon the **Maximum value of the error** of the transfer function of the table loaded with a 900 lbs rigid payload.
Figure 7.23: Effect of the integral gain upon the **Phase** of the transfer function of the table loaded with a 900 lbs rigid payload
7.2.9 Sensitivity of Table to Rigid Payload - Conclusions

From the experimental study conducted to investigate the effects of the derivative, delta-pressure and integral gains in mitigating the degrading of the table M.T.F. caused by the addition of a rigid payload, it is concluded that only the derivative gain is effective. Starting from the optimal gain setting for the bare table condition, the derivative gain should be adjusted according to the charts in Fig. 7.5.

Concerning the table P.T.F., it is concluded that the addition of a rigid payload degrades the phase performance of the table (the frequency at which the $-\pi$ phase shift occurs is generally lowered from 60 to 40 Hz). The experimental results indicated that the control gains are unable to compensate for this degradation.
7.3 Sensitivity of Table Transfer Function to Flexible Payload

7.3.1 Introduction

In this paragraph we analyze the effect of the addition on the shaking table of two different flexible payloads (one weighting approximately 450 lbs and the other 900 lbs). The flexible payload (described in detail in Appendix B) is a three story steel moment resisting frame 26' deep, 54' wide, and 7 ft. 6' tall. This model steel frame, which weighs approximately 70 lbs. was loaded as follows:

- for the so-called “450 lbs flexible payload”:

  three concrete blocks, of approximately 150 lbs each were fixed “rigidly” to the three floors (one block per floor). This resulted in a total test payload of approximately 500 lbs.

- for the so-called “900 lbs flexible payload”:

  six concrete blocks of approximately 150 lbs each were “rigidly” fixed to the three floors of the frame model (two blocks per floor). This resulted in a total test payload of approximately 970 lbs.

Given the fact that the model remains the same for both cases, the natural frequencies of vibration of the two models are different\(^1\): \(f_1 = 6.74\) Hz, \(f_2 = 15.91\) Hz, \(f_3 = 19.22\) Hz for the 450 lbs flexible payload; 900 lbs flex. \(f_1 = 5.15\) Hz, \(f_2 = 11.80\) Hz, \(f_3 = 14.14\) Hz for the

\(^1\) These theoretical values are derived in Appendix B, where they are compared with the experimental results.
900 lbs flexible payload.

In the first part of this section the effects of these two flexible test payloads on the table transfer function and its deviation from the ideal unit transfer function are analyzed. The second part will deal with the gain corrections that have to be applied to the “optimal” gain setting found for bare table condition in order to reset the table transfer function as close to unity as possible.
7.3.2 Comparison of the Effects of a 450 lbs Versus 900 lbs Flexible Payload

The analysis of the effects of the 450 lbs and 900 lbs frame models upon the table transfer function reported in this chapter has been conducted using the optimal gain setting found for bare table condition.

Magnitude of the table transfer function (M.T.F.)

Fig. 7.24 shows that the 450 lbs and the 900 lbs frame models affect the table M.T.F. in a similar way:

- both models produce a spike in the M.T.F. corresponding to their first modal frequency\(^1\). the theoretical prediction of which are 5.15 Hz and 6.74 Hz for the 900 and 450 lbs models respectively (See Appendix B):

- both structural models increase the amplitude of the M.T.F. (as compared to the M.T.F. of the bare table) in the neighborhood of the oil column frequency (the oil column peak increases approximately from 1.5 to 3.0 for the 450 lbs model and to 4.0 for the 900 lbs model) without affecting the oil column frequency\(^1\):

- the presence of both structural models produce a decrease well below unity (approximately 0.5) of the M.T.F. in the intermediate frequency range. This drop in the M.T.F. will be referred later as valley.

Fig. 7.25 shows that the presence of a flexible payload increases the R.M.S. of the error of the M.T.F. from a value of 0.20 for the bare table to a value of approximately 0.45 for

---

\(^1\) This result agrees well with the theoretical predictions presented in Chapter 8 (see Figs. 8.35 and 8.37)
the 450 lbs model and approximately 0.60 for the 900 lbs model.

Fig. 7.26 indicates that the presence of a flexible payload increases the Max value of the error of the M.T.F. from a value of approximately 1.2 for the bare table to a value of approximately 2.0 for the 450 lbs model, and approximately 3.1 for the 900 lbs payload.

The results described above show that the "optimal" gain setting found for the bare table condition does not provide a satisfactory table performance in the presence of a test structure with a total mass representing an important fraction of the maximum design payload. Therefore a search for a new "optimal" gain setting had to be carried out for each flexible model.

*Phase of the table transfer function (P.T.F.)*

Fig. 7.27 shows that the presence of a flexible payload does not change significantly the P.T.F. Notice that, as predicted by the mathematical model\(^1\), both flexible payloads produce a notch in the P.T.F. at the fundamental frequency of the test structure. The notch caused by the 900 lbs model is deeper than that due to the 450 lbs model. The presence of these two flexible payloads also produce a noticeable increase in the phase shift in the frequency range between 10 and 30 Hz.

In summary, the overall behavior of the P.T.F. is much less affected by the presence of flexible payloads than the M.T.F. therefore, the gain correction should focus on improving the shape of the M.T.F.

---

1. See Chapter 8.
Figure 7.24  Effect of 450 lbs and 900 lbs flexible payloads upon the Magnitude of the table transfer function
Figure 7.25  Effect of 450 lbs and 900 lbs flexible payload upon the Root Mean Square of the error of the magnitude of the table transfer function
Figure 7.26  Effect of 450 lbs and 900 lbs flexible payload upon the Max. value of the error of the magnitude of the table transfer function
Figure 7.27  Effect of 450 lbs and 900 lbs flexible payload upon the Phase of the table transfer function
7.3.3 Comparison of the Effects of a 450 lbs Rigid Versus Flexible Payload

The goal of this experiment is to compare the effects of a 450 lbs flexible payload with those of a 450 lbs rigid payload on the table transfer function when the "optimal" gains obtained for the bare table condition are used.

Magnitude of the table transfer function (M.T.F.)

Fig. 7.28 shows that the addition on the table of a 450 lbs flexible payload produces a valley in the M.T.F. located in the same intermediate frequency range where the rigid payload produces a wide peak (see Sections 7.2.2 and 7.2.4). Notice that the valley created by the flexible payload is less pronounced than the peak due to the rigid payload. This fact is confirmed quantitatively by the plots of the R.M.S. and Max. value of the M.T.F. error shown in Figs. 7.29 and 7.30, respectively. The value of the R.M.S. error increases from approximately 0.20 for the bare table, to 0.44 for the flexible payload and to 0.92 for the rigid payload. The Max. value of the error increases from 1.1 for the bare table to 1.9 for the flexible payload and to 3.9 for the rigid payload. These results confirm quantitatively that the table M.T.F. is much less affected by a flexible payload than by a rigid one.

Phase of the table transfer function (P.T.F.)

Fig. 7.31 displays the phase of the T.F. for the bare table, the table with a 450 lbs rigid payload and the table with a 450 lbs flexible payload. It is observed that up to 30 Hz, the rigid payload does not affect much the P.T.F. of the bare table, while the flexible payload
does, as already noticed before. For frequencies above 30 Hz, the opposite effect is found, namely that the flexible payload does not affect much the P.T.F. of the table, while the rigid payload lowers the so-called inversion frequency (frequency at which a $\pi$ phase shift is reached) from about 5 to 40 Hz.
Figure 7.28  Effect of the 450 lbs rigid and flexible payload (MDOF) upon the Magnitude of the table transfer function
Figure 7.29  Effect of the 450 lbs rigid and flexible payload (MDOF) upon the Root Mean Square of the error of M.T.F.
Figure 7.30 Effect of a 450 lbs rigid and flexible payload (MDOF) upon the Maximum value of the error of the M.T.F.
Figure 7.31  Effect of a 450 lbs rigid and flexible payload (MDOF) upon the Phase of the table transfer function
7.3.4 Gain Corrections for the 450 lbs Flexible Payload

As explained in the previous Section, the addition of a flexible payload to the shaking table produces a wide valley in the table M.T.F. in the intermediate frequency range. In order to increase the value of the M.T.F. in this frequency range to bring it back to values closer to unity, the effectiveness of the Derivative and Feed Forward gains has been investigated\(^1\).

Starting from the "optimal" gain setting found for the bare table condition (P-gain = 1.81 Volt/Volt, I-gain = 40 milliseconds, dP-gain = 3 Volt/Volt), the following values of the D-gain and FF-gain were examined:

\[
\text{FF-gain} = \text{D-gain} = 18 \text{ milliseconds (optimal gain setting for bare table condition)}:
\]

\[
\text{FF-gain} = \text{D-gain} = 21 \text{ milliseconds}:
\]

\[
\text{FF-gain} = \text{D-gain} = 24 \text{ milliseconds}:
\]

\[
\text{FF-gain} = \text{D-gain} = 34 \text{ milliseconds}.
\]

Magnitude of the table transfer function (M.T.F.)

Fig. 7.32 shows that an increase in the derivative and feed-forward gains is able to compensate almost completely for the decrease in the table M.T.F. in the intermediate frequency range due to the flexible payload. However, Fig. 7.32 also shows that an increase in the derivative and feed forward gains considerably amplifies the oil column

---

1. Both the mathematical and experimental analyses of the table sensitivity to the gain parameters indicated that an increase in D-gain and FF-gain raises the value of the M.T.F. in the intermediate frequency range.
peak and lowers its frequency (as predicted by the mathematical model of the shaking table presented in Chapters 3 and 4).

Figure 7.33 indicates that the RMS error of the M.T.F. increases with increasing values of the derivative and feed forward gains. The RMS error increases from about 0.44 to about 1.18 as the FF-gain and the D-gain vary from 18 to 34 milliseconds. This result indicates that the amplification of the oil column peak caused by increasing values of the derivative and feed-forward gains produces an increase of the RMS error of the table M.T.F. that is larger than the reduction of the RMS error due to the beneficial effects of increasing values of the D-gain and FF-gain in the intermediate frequency region. This result is confirmed in Fig. 7.34 which shows that the Max error of the table M.T.F. increases from about 1.9 to about 6.2 when the FF and D-gain vary from 18 to 34 milliseconds.

*Phase of the table transfer function (P.T.F.)*

From Figure 7.35, it is observed that the Derivative and Feed-Forward gains does not have only a minor effect on the phase transfer function.
Figure 7.32  Effect of different gain settings upon the Magnitude of the transfer function of the table loaded with a 450 lbs flexible payload (modal natural frequencies: $f_1 = 8.4$ Hz, $f_2 = 22.4$, $f_3 = 32.7$)
Figure 7.33  Effect of different FF- and D-gain values upon Root Mean Square of the error of the M.T.F. of the table loaded with a 450 lbs flexible payload.
Figure 7.34 Effect of different FF- and D-gain values upon Maximum value of the error of the M.T.F. of the table loaded with a 450 lbs flexible payload
Figure 7.35  Effect of different FF- and D-gain values upon the Phase of the transfer function of the table loaded with a 450 lbs flexible payload (modal natural frequencies: $f_1 = 8.4$ Hz, $f_2 = 22.4$, $f_3 = 32.7$ Hz)
7.3.5 Comparison of the Effects of a 900 lbs Rigid Versus Flexible Payload

The objective of this experiment is to compare the effects of a 900 lbs flexible payload with those of a 900 lbs rigid payload on the table transfer function when the "optimal" gains obtained for the bare table condition are used. The results obtained here for the 900 lbs flexible versus rigid payload are very similar to those obtained in the previous section comparing the effects on the table performance of a 450 lbs flexible versus rigid payload.

Magnitude of the table transfer function (M.T.F.)

Fig. 7.36 shows that a 900 lbs flexible payload reduces the magnitude of the table T.F. in the intermediate frequency range as in the case of the 450 lbs model. Also, as for the 450 lbs payload, this reduction in the M.T.F. caused by the flexible payload is much smaller in size than the amplification of the M.T.F. in the intermediate frequency range due to a rigid payload of same total weight.

Figs. 7.37 and 7.38 show the RMS and Max values of the M.T.F. error respectively. Notice that a flexible payload affects the table M.T.F much less than a rigid payload of the same weight.

Phase of the table transfer function (P.T.F.)

Fig. 7.39 also indicates that the phase of the table transfer function is much less affected by a flexible payload than a rigid payload of the same weight.
Figure 7.36  Effect of a 900 lbs rigid versus flexible payload (MDOF) upon the Magnitude of the table transfer function
Figure 7.37  Effect of a 900 lbs rigid versus flexible payload (MDOF) upon the Root Mean Square of the error of the table M.T.F.
Figure 7.38  Effect of a 900 lbs rigid versus flexible payload (MDOF) upon the Maximum error of the table M.T.F.
Figure 7.39  Effect of a 900 lbs rigid versus flexible payload (MDOF) upon the Phase of the table transfer function
7.3.6 Gain Corrections for the 900 lbs Flexible Payload (Gain Increase)

In order to improve the magnitude of the table transfer function (M.T.F.), more specifically to raise the M.T.F. in the intermediate frequency range, we investigated experimentally the effectiveness of increasing the derivative (D-) and feed-forward (FF-) gains\(^1\).

Starting from the "optimal" gain setting found for the bare table condition (P-gain = 1.81 Volt/Volt, I-gain = 40 milliseconds, dP-gain = 3 Volt/Volt), the following D-gain and FF-gain values were tested:

\[
\begin{align*}
D\text{-gain} & = FF\text{-gain} = 18 \text{ millisec. (optimal gain setting for bare table condition)} \\
D\text{-gain} & = FF\text{-gain} = 21 \text{ millisec.} \\
D\text{-gain} & = FF\text{-gain} = 24 \text{ millisec.} \\
D\text{-gain} & = FF\text{-gain} = 34 \text{ millisec.}
\end{align*}
\]

**Magnitude of the table transfer function (M.T.F.)**

Fig. 7.40 shows that, as in the case of the 450 lbs flexible payload, a simultaneous increase in the derivative and feed-forward gains is able to raise the magnitude of the table T.F. in the intermediate frequency range up to a value close to unity. But, unfortunately, this improvement of the M.T.F. in the intermediate frequency range is accompanied by a significant increase in the amplitude of the oil column peak. Any

---
\(^1\) Both the mathematical and experimental analyses of the table sensitivity to the gain parameters indicated that an increase in D-gain and FF-gain raises the value of the M.T.F. in the intermediate frequency range.
attempt to reduce the amplitude of this peak using the delta-pressure gain proved to be ineffective.

Figs. 7.41 and 7.42 show that both the RMS error and Maximum error of the M.T.F. increase as the derivative and feed-forward gains increase from 18 to 34 milliseconds. Thus, the increase in the M.T.F. error due to the amplification of the oil column peak dominates the reduction in the M.T.F. error obtained in the intermediate frequency range. Therefore, this type of gain correction (increase of the derivative and feed forward gains) proves to be effective only if the type of motion to be reproduced on the shaking table has a frequency content below approximately 40 Hz and if the response of the test specimen to base input motions above 40 Hz does not compromise the validity of the experiment.

Phase of the table transfer function (P.T.F.)

Fig. 7.43 indicates that the phase T.F. is not affected significantly by an increase of the derivative and of the feed-forward gains, except for a small improvement in the frequency range between 5 and 25 Hz. It is noticed that the notch in correspondence to the payload first natural frequency (at approximately 5 Hz) remains unaffected by any change in the derivative and feed-forward gains.
Figure 7.40  Effect of D- and FF- gains increase upon the Magnitude of the T.F. of the table loaded with a 900 lbs flexible payload
Figure 7.41  Effect of D- and FF- gains increase upon the Root Mean Square error of the M.T.F. of the table loaded with a 900 lbs flexible payload.
Figure 7.42  Effect of D- and FF-gains increase upon the Maximum error of the M.T.F. of the table loaded with a 900 lbs flexible payload
Figure 7.43  Effect of D- and FF-gains increase upon the Phase of the T.F. of the table loaded with a 900 lbs flexible payload
7.3.7 Gain Corrections for the 900 lbs Flexible Payload (Gain Decrease)

The previous section showed that an increase in the FF-gain and D-gain as effective in improving the shaking table performance only up to about 40 Hz. This section investigates the effect upon the table T.F. of a simultaneous decrease of these same gains.

Starting from the “optimal” gain setting found for bare table condition (P-gain = 1.81 Volt/Volt, I-gain = 40 milliseconds, dP-gain = 3 Volt/Volt), the D-gain and FF-gain were modified as follows:

\[
\begin{align*}
\text{D-gain} &= \text{FF-gain} = 18 \text{ millisec. (optimal gain setting for bare table condition)} \\
\text{D-gain} &= \text{FF-gain} = 16 \text{ millisec.} \\
\text{D-gain} &= \text{FF-gain} = 5 \text{ millisec.}
\end{align*}
\]

Magnitude of the table transfer function (M.T.F.)

Fig. 7.44 shows that a small decrease in the Derivative and Feed-Forward gains (from 18 to 16 milliseconds) is able to raise the M.T.F. in the intermediate frequency range without producing the undesired amplification of the oil column peak (as was the case when increasing the same gains), but actually reducing its amplitude. Further reduction in the derivative gain proved to be ineffective. Fig. 7.44 shows that for a value of the D-gain and FF-gain equal to 5 milliseconds (which corresponds to the corrected value of the D-gain obtained for the 900 lbs rigid payload, see Section 7.2.6 and Fig. 7.5) the M.T.F. in the intermediate frequency range is much lower than for D-gain and FF-gain value of 18 milliseconds (starting value). Figs. 7.45 and 7.46 show that a small decrease
in the derivative and feed-forward gains is able to reduce both the R.M.S and Max error of the M.T.F.

Larger reduction in the derivative and feed-forward gains is able to reduce only the Max error of the M.T.F., while the R.M.S. error of the M.T.F. increases.

*Phase of the table transfer function (P.T.F.)*

Figure 7.47 shows how a small decrease from 18 to 16 milliseconds in the derivative and feed-forward gains does not affect the phase performance of the shaking table. Larger decreases in the derivative and feed-forward gains (reduction to 5 milliseconds), on the other hand, increase the phase shift in the frequency range between 5 and 25 Hz.
Figure 7.44  Effect of D- and FF- gains decrease upon the Magnitude of the T.F. of the table loaded with a 900 lbs flexible payload
Figure 7.45  Effect of D- and FF- gains decrease upon the: **Root Mean Square error**

of the M.T.F. of the table loaded with a 900 lbs flexible payload.
Figure 7.46  Effect of D- and FF- gains decrease upon the Maximum error of the M.T.F. of the table loaded with a 900 lbs flexible payload
Figure 7.47  Effect of D- and FF- gains decrease upon the Phase of the T.F. of the table loaded with a 900 lbs flexible payload
7.3.8 Sensitivity of Shaking Table Transfer Function to Flexible Payload - Conclusions

The addition of a relatively heavy (compared to the design maximum payload) flexible payload on the shaking table tuned according to the optimal gain setting found for bare table condition, has a smaller effect upon the magnitude and phase transfer functions than the one produced by a rigid payload of equivalent total weight. The overall effect can be summarized as follows:

Magnitude Transfer Function

- reduction, in the neighborhood of 50% of the M.T.F. in the intermediate frequency range (see Fig. 7.24):

- a very narrow peak at the location of the first natural frequency of the test specimen;

- amplification of the oil column peak with the total weight of the test specimen.

Phase Transfer Function

- appearance of a narrow notch at the location of the first natural frequency of the test specimen;

- moderate deterioration (~ 30%) of the table performance in the frequency range from 5 to 30 Hz.
Gain Corrections

Unlike the case of a rigid payload, for a flexible payload it was not possible to identify a clear trend of gain corrections as a function of the total weight of the test specimen. This indicates that the specific gain correction to be applied depends not only on the specimen weight, but also on its dynamic characteristics. In general, improvements of table performance in the intermediate frequency and in the neighborhood of the oil column frequency could not be obtained simultaneously (trade-off).
7.4 CONCLUSIONS

In general, it can be stated that as payload of increasing mass are tested on the shaking table, a correction in the Derivative and/or Feed-Forward "optimal" gain values has to be implemented in order to improve the table performance. For rigid payloads, this correction can be obtained from the chart provided in Fig. 7.5. For flexible payloads, it was not possible to identify a precise pattern of gain correction as simply a function of the specimen weight. The "optimum" gain correction depends specifically on the dynamics characteristics of the test specimen at hand.
CHAPTER 8
CORRELATION BETWEEN ANALYTICAL AND EXPERIMENTAL
SHAKING TABLE BEHAVIOR
8.1 INTRODUCTION

This chapter presents we perform a correlation study between the table transfer functions obtained experimentally and their analytical counterparts obtained using the analytical model developed in Chapter 3.

In the first part of the chapter (Sections 8.2 through 8.4), we describe the results of a least square fitting used to determine the values of the servo-hydraulic parameters which were not known physically a priori.

The second part of the chapter (Sections 8.5 through 8.9) describes the ability of the analytical model capture the sensitivity of the shaking table transfer function to the various control gains (proportional, integral, derivative, feed-forward, delta-pressure gains).

Finally, the last part of the chapter (Sections 8.10 through 8.16) describes the ability of the analytical model to capture the sensitivity of the shaking table transfer function to the various payload characteristics (rigid payloads, flexible payloads, SDOF, MDOF, etc.).
8.2 Known, Variable and Unknown Parameters

The analytical model of the table transfer function contains several parameters that can be divided into three groups.

(1) **Known System parameters:** Effective Area of the piston (A), volume of the actuator chamber (V), mass of the slip table including actuator arm and swivel head (M).

(2) **Variable control parameters:** Proportional gain (P), integral gain (I), derivative gain (D), feed-forward gain (FF) and delta-pressure gain (d-P)\(^1\).

(3) **Unknown servo-hydraulic parameters:** Table response coefficient (K_t), leakage coefficient (K_l), effective oil bulk modulus (β) and servovalve time delay (τ).

The **known system parameters** relate to geometrical and physical properties of the system which are precisely known a priori.

The **variable control parameters** can be set precisely by the operator of the shaking table through the controller. The performances behavior of the shaking table depends on the choice of the control parameters.

The **unknown parameters** characterizing the servo-hydraulic system relate to physical

---

\(^1\) The symbols P, I, D, FF and d-P, used here to refer to the proportional, integral, derivative, feed-forward and delta pressure gains, are the one commonly used in control theory. However, in the analytical model developed in Chapter 3 these gains are referred to as \(K_{pro}\), \(K_{int}\), \(K_{der}\), \(K_{ff}\) and \(K_{dp}\) respectively.
response characteristics of the system that are physically not known a priori. The leakage coefficient, table response coefficient (see Section 3.2.2) and servovalve time delay depend on the internal characteristics of the servovalve and actuator. The oil bulk modulus can be estimated to be around 100,000 psi., however, due to temperature changes and finite stiffness of the walls of the actuator chamber, it is very difficult to make an a priori accurate estimation of the effective oil bulk modulus.

A fourth group of parameters could be defined which represent the proportionality constants between the control gain parameters in the analytical model and their counterparts in the actual controller. For example, in the analytical model the delta pressure gain, \( K_{dp} \), relates a physical differential pressure quantity to a correction in displacement (units of \( K_{dp} \) are in/ksi). whereas in the actual controller, the delta pressure gain, \( \bar{K}_{dp} \), converts an electrical signal representing the differential pressure across the actuator piston into another electrical signal representing the displacement correction to be applied to the actuator arm (units of \( \bar{K}_{dp} \) are volt/volt). The proportionality constant between \( K_{dp} \) and \( \bar{K}_{dp} \) was determined through least square fit between analytical and experimental results for the transfer function and found to be \( \frac{\bar{K}_{dp}}{K_{dp}} = 45.0 \). Similarly for the feed forward, \( K_{ff} \), and derivative, \( K_{der} \), gains it was found that \( \frac{\bar{K}_{ff}}{K_{ff}} = 1.85 \) and \( \frac{\bar{K}_{der}}{K_{der}} = 1.90 \). For the proportional, \( K_{pro} \), and integral, \( K_{int} \), gains, it was found that \( \bar{K}_{pro} = K_{pro} \) and \( \bar{K}_{int} = K_{int} \).
8.3 Identification of Unknown Servo-Hydraulic Parameters

In order to perform a sensitivity analysis of the table behavior with respect to the control parameters, it is necessary to have a reasonable estimate of the unknown servo-hydraulic parameters, namely $K_t$, $K_i$, $\beta$, $\tau$. For this purpose, we performed a least square fit between the experimental table transfer function obtained from various dynamic tests of the table and the analytical table transfer function. The least square fit was carried out by minimizing the euclidean ($L_2$) norm of the difference between the magnitude of the experimental and analytical transfer functions over the frequency range between 0 and 120 Hz. Due to the extreme jaggedness of the estimated phase transfer function, no attempt of fitting the experimental and analytical phase transfer functions were made. The approach described above leads to acceptable results thanks to the Bode's Gain Phase Relationship (Bode, H. W., 1965) which states that for any stable minimum-phase system, the phase of $G(j\omega)$ is uniquely related to the magnitude of $G(j\omega)$.

The least square optimization results obtained indicated that it is impossible to find a unique set of optimum servo-hydraulic parameters ($K_t$, $K_i$, $\beta$ and $\tau$) that applies to all table working conditions. This is probably due to the non-linearities of the servo-hydraulic system. Thus, it was necessary to several different sets of optimum servo-hydraulic parameters corresponding to different testing conditions. It was found that the

---

1. The minimization was carried out using the "fmins" Matlab function. This function uses the Nelder-Mead simplex search. At each step of the search a new point in or near the current simplex is generated. The function value at the new point is compared with the function's value at the vertices of the simplex and, usually, one of the vertices is replaced by the new point, giving a new simplex. This step is repeated until the diameter of the simplex is less than the specified tolerance. The tolerance used in this analysis is 0.01. For more details, see (Nelder and Mead 1987, Dennis and Woods 1987).
servo-hydraulic parameters are sensitive mainly to the value of the derivative gain and to the payload characteristics. The following sets of optimum servo-hydraulic parameters were identified as sufficient to model, within engineering accuracy, the shaking table behavior over a wide range of working conditions.

### Table 8.1

<table>
<thead>
<tr>
<th>D-gain = 0 millisecond</th>
<th>D-gain = 10 millisecond</th>
<th>D-gain = 18 millisecond</th>
<th>D-gain = 20 millisecond</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>K_t</strong> = 340 in^2/sec</td>
<td><strong>K_t</strong> = 322 in^2/sec</td>
<td><strong>K_t</strong> = 322 in^2/sec</td>
<td><strong>K_t</strong> = 312 in^2/sec</td>
</tr>
<tr>
<td><strong>β</strong> = 100.000 psi</td>
<td><strong>β</strong> = 85.000 psi</td>
<td><strong>β</strong> = 73.000 psi</td>
<td><strong>β</strong> = 70.000 psi</td>
</tr>
<tr>
<td><strong>K_l</strong> = 0.0013 in^1/sec</td>
<td><strong>K_l</strong> = 0.0013 in^1/sec</td>
<td><strong>K_l</strong> = 0.0013 in^1/sec</td>
<td><strong>K_l</strong> = 0.0013 in^1/sec</td>
</tr>
<tr>
<td><strong>τ</strong> = 12.0 millisecond</td>
<td><strong>τ</strong> = 13.0 millisecond</td>
<td><strong>τ</strong> = 13.5 millisecond</td>
<td><strong>τ</strong> = 14.0 millisecond</td>
</tr>
</tbody>
</table>

| name of parameter set: | d0 | name of parameter set: | d10 | name of parameter set: | d18 | name of parameter set: | d20 |

### Table 8.2

Shaking Table with payloads varying between 500 and 100 lbs (D-gain = 18 millisecond)

<table>
<thead>
<tr>
<th>“Flexible” Payloads (fund. freq. &lt; 50 Hz)</th>
<th>“Semi-Rigid” Payloads (50 Hz &lt; fund. freq. &lt; 50 Hz)</th>
<th>“Rigid” Payloads (fund. freq. &gt; 100 Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>K_t</strong> = 340 in^2/sec</td>
<td><strong>K_t</strong> = 387 in^2/sec</td>
<td><strong>K_t</strong> = 370 in^2/sec</td>
</tr>
<tr>
<td><strong>β</strong> = 100.000 psi</td>
<td><strong>β</strong> = 74.000 psi</td>
<td><strong>β</strong> = 76.000 psi</td>
</tr>
<tr>
<td><strong>K_l</strong> = 0.0013 in^1/sec</td>
<td><strong>K_l</strong> = 0.0019 in^1/sec</td>
<td><strong>K_l</strong> = 0.0013 in^1/sec</td>
</tr>
<tr>
<td><strong>τ</strong> = 12.0 millisecond</td>
<td><strong>τ</strong> = 10.3 millisecond</td>
<td><strong>τ</strong> = 10.42 millisecond</td>
</tr>
</tbody>
</table>

| name of parameter set: | pf | name of parameter set: | pf | name of parameter set: | pr |

| name of parameter set: | pf | name of parameter set: | pf | name of parameter set: | pr |
Table 8.1 shows the dependence of the optimum values of the servo-hydraulic parameters upon the value of the derivative gain, D-gain. It is possible to identify a trend in the variation of these parameters with the D-gain. An increase in the D-gain from 0 to 20 milliseconds decreases the value of both the table response coefficient $K_t$ (from 340 to 312: 8% reduction) and the effective oil bulk modulus $\beta$ (from 100,000 to 70,000 psi: 30% reduction). An increase in the D-gain from 0 to 20 milliseconds increases the value of the servovalve delay $\tau$ from 12 to 14 milliseconds (16.5% increase). The variations of the servo-hydraulic parameters appear to be close to linear and suggest the existence of a possible law that links these parameter values to the D-gain. The leakage coefficient $K_l$ appears to be independent of the D-gain.

Table 8.2 shows the dependence of the optimum values of the servo-hydraulic parameters upon the "rigidity" of the table payload as measured by its fundamental natural frequency. It is not possible to identify a clear trend in the variation of these parameters for increasing rigidity of the payload. An increase in the payload rigidity first increases and then decreases the value of both the table response coefficient $K_t$ (340-387-370) and the leakage coefficient $K_l$ (0.0013-0.0019-0.0013). On the other hand, the effective oil bulk modulus $\beta$ and the servovalve time delay $\tau$ first decrease and then increase in value for increasing stiffness of the table payload. These "hill" and "valley" trends of parameter variations seem to indicate that these parameters are affected by the dynamic interaction between test specimen and oil column frequency.
8.4 **Foreword to the Correlation Study Between Analytical and Experimental Table Transfer Functions**

All the figures presented in the following sections contain a plot of both the analytical table transfer function for the servovalve time delay $\tau$ and that which does not account for $\tau$. It must be pointed out that the analytical table transfer function that does not account for the servovalve time delay $\tau$ uses the same set of servo-hydraulic parameters (except, obviously, for the value of $\tau$) obtained by least square fitting the model which accounts for time delay $\tau$. For this reason, it cannot be guaranteed that the servo-hydraulic parameters used in the analytical model without time delay $\tau$ are the ones that least square fit the analytical transfer function without time delay to the experimental one. Nevertheless in our thorough investigation of the table transfer function sensitivity to the servo-hydraulic parameters, it was found that the analytical model without time delay is not able to fit the experimental results as well as the analytical model which accounts for the servovalve time delay.
8.5 Correlation Between Analytical and Experimental Table Sensitivity to Proportional Gain

In this section, we show the correlation obtained between the experimental\(^1\) and the analytical table transfer functions for bare table conditions under different values of the proportional gain. The set of servo-hydraulic parameters used in the analytical model is the one referred to as "d0" in Table 8.1. This set was used for all values of the proportional gain, and the high degree of correlation obtained confirms the assumed independence of the servo-hydraulic parameters with respect to the proportional gain.

8.5.1 Proportional Gain of to 1 Volt/Volt (Fig. 8.1)

Magnitude of the table transfer function (M.T.F.)

The analytical table transfer function obtained with a nonzero time delay (solid line) is able to match very well the experimental transfer function. In the low frequency range (between 0 and 27 Hz), the analytical model with time delay gives much better results than the one without time delay.

Phase of the table transfer function (P.T.F.)

As is the case for the magnitude transfer function, the analytical model with time delay matches much better the experimental phase transfer function. This improved correlation confirms the validity of introducing the servo-valve time delay as an important model parameter. Notice that although the least square fit of the analytical to the experimental

\(^1\) The experimental table transfer functions have been obtained following the estimation procedure described in Chapter 5.
transfer function was performed based on only the magnitude transfer function. There is satisfactory agreement between the analytical (with time delay) and experimental phase transfer function, especially in the low frequency range. It is also observed that both the analytical and experimental table transfer functions exhibit a downward curvature between 0 and 20 Hz.
Figure 8.1  Table Transfer Function for **P-gain = 1 Volt/Volt**, all other gains = 0:
(a) Magnitude transfer function (b) Phase transfer function
(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
8.5.2 Proportional Gain of 2 Volt/Volt

Magnitude of the table transfer function (M.T.F.) (Fig. 8.2)

In this case also the analytical model obtained with non-zero servovalve time delay shows a high predictive capability. It is very interesting to notice how this analytical model is able to capture the “hump” originated by the proportional gain between 7 and 17 Hz. This “hump” is clearly not present for the P-gain = 1 volt/volt as shown in Fig. 8.1 (a). Moreover, this “hump”, obtained experimentally, cannot be physically related neither to the oil column peak nor to the natural frequency of vibration of the flexible foundation. This analytical model suggests that this “hump” is due to some interaction between the servovalve time delay and the dynamic characteristics of the shaking table system. As shown in Fig. 8.2 (b), the analytical model without time delay gives totally unreliable results for this value of the proportional gain.

Phase of the table transfer function (P.T.F.) (Fig. 8.3)

Fig. 8.3 shows that the analytical and experimental phase transfer function are also in good agreement. Notice that the analytical model with non-zero time delay is also able to capture correctly the change in curvature of the experimental phase transfer function occurring between 0 and 20 Hz.
Figure 8.2  Magnitude of Table Transfer Function for P-gain = 2 Volts/Volt, all other gains = 0 (solid line = analytical T.F. with non-zero time delay τ, dotted line = analytical T.F. with zero time delay τ, dashed line = experimental T.F.)
Figure 8.3  Phase of Table Transfer Function for $P\text{-gain} = 2 \text{ Volts/Volt}$, all other gains = 0 (solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. without time delay $\tau$, dashed line = experimental T.F.)
8.5.3 Proportional Gain of 3 Volt/Volt (Fig. 8.4)

*Magnitude of the table transfer function (M.T.F.*)

In this case also, the analytical model with a non-zero servovalve time delay $\tau$ shows good agreement with the experimental results. All three table transfer function peaks are well captured: the first peak at 18 Hz (due to servovalve time delay), the second peak at 26 Hz (due to the foundation flexibility) and the third peak at 70 Hz (oil column frequency). The amplitude of this last peak is overestimated. However, this analytical model proves far superior to the one with zero time delay which is not able to capture the first two peaks at all.

*Phase of the table transfer function (P.T.F.)*

The analytical model with non-zero servovalve time delay also shows excellent agreement with the experimental results.
Figure 8.4 Table Transfer Function for **P-gain = 3 Volts/Volt**, all other gains = 0:
(a) Magnitude transfer function (b) Phase transfer function
(solid line = analytical T.F. with non-zero time delay \( \tau \), dotted line = analytical T.F. with zero time delay \( \tau \), dashed line = experimental T.F.)
8.6 Correlation Between Analytical and Experimental Table Sensitivity to Integral Gain

In this section, we show the correlation obtained between the experimental and the analytical table transfer functions for bare table conditions and for different values of the integral gain. The set of servo-hydraulic parameters used in the analytical model is the one referred to as “d0” in Table 8.1. This set was used for all values of the integral gain, and the high degree of correlation obtained confirms the assumed independence of the servo-hydraulic parameters with respect to the integral gain.

8.6.1 Integral Gain of 20 rps (Fig. 8.5)

Magnitude of the table transfer function (M.T.F.)

As was the case for the proportional gain, the analytical model with non-zero servo-valve time delay \( \tau \) is in very good agreement with the experimental results. The analytical model is able to capture very well both the peak created by the integral gain at approximately 10 Hz and the oil column peak. The analytical model with a zero servo-valve time delay \( \tau \) underestimates the peak at 10 Hz and overestimates the oil column peak.

Phase of the table transfer function (P.T.F.)

The analytical model with non-zero servo-valve time delay \( \tau \) agrees much better with the experimental results than the one with zero time delay. Even though the agreement between analytical and experimental results can be considered as good, it is inferior to that obtained for the different values of the proportional gain.
Figure 8.5  Table Transfer Function for **I-gain = 20 rps** (P-gain = 1 Volt/Volt, all other gains = 0):

(a) Magnitude transfer function  (b) Phase transfer function
(solid line = analytical T.F. with non-zero time delay \( \tau \), dotted line = analytical T.F. with zero time delay \( \tau \), dashed line = experimental T.F.)
8.6.2 Integral Gain of 40 rps (Fig. 8.6)

Magnitude of the table transfer function (M.T.F.)

For an integral gain of 40 rps, we also obtain a very good agreement between analytical and experimental results. The analytical model with non-zero servovalve time delay $\tau$ perfectly captures very well all the peaks of the experimental transfer function. As was the case for I-gain = 20 rps, the analytical model with zero time delay underestimates the peak at 10 Hz and overestimate the oil column peak.

Phase of the table transfer function (P.T.F.)

The analytical model with non-zero servovalve time delay $\tau$ predicts very well the experimental results. Notice how well it captures the change in curvature between 10 and 20 Hz. The analytical model with zero servovalve time delay has a much lower predictive capability.
Figure 8.6  Table Transfer Function for **I-gain = 40 rps** (P-gain = 1 Volt/Volt, all other gains = 0):

(a) Magnitude transfer function  
(b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
8.7 CORRELATION BETWEEN ANALYTICAL AND EXPERIMENTAL TABLE SENSITIVITY TO DERIVATIVE GAIN

In this section, we show the correlation obtained between the experimental and the analytical table transfer functions for bare table condition and for different values of the derivative gain. The set of servo-hydraulic parameters used in the analytical model has to be adapted to the value of the derivative gain as shown in Table 8.1. The dependence of the servo-hydraulic parameters on the derivative gain was obtained by least square fitting the analytical table transfer function to the experimental one (see Section 8.3).

8.7.1 Derivative Gain of 10 milliseconds (Fig. 8.7)

The set of servo-hydraulic parameters referred to as “d10” in Table 8.1 was found to maximize the correlation between the analytical and experimental table transfer function for a derivative gain of 10 milliseconds. This optimum correlation is presented below.

Magnitude of the table transfer function (M.T.F.)

The analytical model with non-zero servovalve time delay $\tau$ given in Table 8.1 is in very good agreement with the experimental results as shown in Fig. 8.7 (a). On the other hand, the analytical model with a zero servovalve time delay $\tau$ does not capture satisfactorily the experimental behavior of the shaking table. It underestimates the magnitude of the table transfer function in the frequency range between 0 and 40 Hz and overestimates the oil column peak.
Phase of the table transfer function (P.T.F.)

In this case, the analytical model with non-zero servovalve time delay τ does not give as good results as for the P- and I- gains. It can be seen that the experimental P.T.F. lays between the ones predicted by the two analytical models (with zero and non-zero value of the servo-valve time delay τ).
Table Transfer Function for D-gain = 10 milliseconds (P-gain = 1 Volt/Volt, all other gains = 0):

(a) Magnitude transfer function (b) Phase transfer function
(solid line = analytical T.F. with non-zero time delay τ, dotted line = analytical T.F. with zero time delay τ, dashed line = experimental T.F.)
8.7.2 Derivative Gain of 20 milliseconds

The results presented below correspond to the set of servo-hydraulic parameters referred to as "d20" in Table 8.1 for a derivative gain equal to 20 milliseconds.

**Magnitude of the table transfer function (M.T.F.) (Fig. 8.8 (a))**

For this value of the derivative gain, the analytical model with non-zero servovalve time delay τ is also in very good agreement with the experimental results as shown in Fig. 8.8 (a). On the other hand, the analytical model with a zero servovalve time delay τ does not capture satisfactorily the experimental results, namely it underestimates the magnitude of the table transfer function in the frequency range between 0 and 50 Hz and overestimates the oil column peak.

**Phase of the table transfer function (P.T.F.) (Fig. 8.8(b))**

A for the derivative gain of 10 milliseconds, the analytical model with non-zero servovalve time delay τ does not give as good results as for the P- and I- gains. It can be seen that the experimental P.T.F. lays between the ones predicted by the two analytical models (with zero and non-zero servovalve time delay τ).
Figure 8.8  Magnitude of Table Transfer Function for $D$-gain = 20 milliseconds ($P$-gain = 1 Volt/Volt, all other gains = 0)

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.9  Phase of Table Transfer Function for **D-gain = 20 milliseconds** (P-gain = 1 Volt/Volt, all other gains = 0)

(solid line = analytical T.F. with non-zero time delay \( \tau \), dotted line = analytical T.F. with zero time delay \( \tau \), dashed line = experimental T.F.)
8.8 Correlation Between Analytical and Experimental Table Sensitivity to Feed-Forward Gain

In this section, we show the correlation obtained between the experimental and the analytical table transfer function for bare table condition and for different values of the feed-forward gain. The set of servo-hydraulic parameters used in the analytical model is the one referred to as "d0" in Table 8.1. This set was used for all values of the feed-forward gain, and the good correlation obtained between experimental and analytical results confirms the assumed independence of the servo-hydraulic parameters with respect to the feed-forward gain.

8.8.1 Feed Forward Gain of 20 milliseconds

Magnitude of the table transfer function (M.T.F.) (Fig. 8.10)

The analytical model with non-zero servovalve time delay $\tau$ is in satisfactory agreement with the experimental results as shown in Fig. 8.10. The agreement between the experimental and analytical M.T.F.'s, although satisfactory, is not as close as for the other control parameters.

As was the case for the P-, I-, and D- control gains, the analytical model with zero servovalve time delay $\tau$ does not capture the experimental behavior of the shaking table as well as well as the model accounting for the servovalve time delay $\tau$. It underestimates the magnitude of the table transfer function in the frequency range between 0 and 40 Hz and overestimates the oil column peak.
Phase of the table transfer function (P.T.F.) (Fig. 8.11)

As was the case with the derivative gain, the experimental P.T.F. lays between the ones predicted by the two analytical models (with zero and non-zero servo valve time delay τ).
Figure 8.10 Magnitude of Table Transfer Function for **FF-gain = 20 milliseconds** (P-gain = 1 Volt/Volt, all other gains = 0)

(solid line = analytical T.F. with non-zero time delay τ, dotted line = analytical T.F. with zero time delay τ, dashed line = experimental T.F.)
Figure 8.11  Phase of Table Transfer Function for FF-gain = 20 milliseconds (P-gain = 1 Volt/Volt. all other gains = 0)
(solid line = analytical T.F. with non-zero time delay τ. dotted line = analytical T.F. with zero time delay τ, dashed line = experimental T.F.)
8.8.2 Feed Forward Gain of 30 milliseconds

*Magnitude of the table transfer function (M.T.F.) (Fig. 8.12)*

As in the case of the feed-forward gain equal to 20 milliseconds, the analytical model with the no-zero servovalve time delay \( \tau \) is in good agreement with the experimental results. The agreement between the experimental and analytical M.T.F.'s, although satisfactory, is not as close as for the other control parameters.

As was the case for the P-, I- and D- control gains, the analytical model with zero servovalve time delay \( \tau \) does not capture the experimental behavior of the shaking table as well as the model accounting for the servovalve time delay \( \tau \). It underestimates the magnitude of the table transfer function in the frequency range between 0 and 40 Hz and overestimates the oil column peak.

*Phase of the table transfer function (P.T.F.) (Fig. 8.13)*

As was the case for the derivative gain, the experimental P.T.F. lays between the ones predicted by the two analytical models (with zero and non-zero servovalve time delay \( \tau \)).
Figure 8.12  Magnitude of Table Transfer Function for FF-gain = 30 milliseconds (P-gain = 1 Volt/Volt. all other gains = 0)
(solid line = analytical T.F. with non-zero time delay \(\tau\). dotted line = analytical T.F. with zero time delay \(\tau\), dashed line = experimental T.F.)
Figure 8.13  Phase of Table Transfer Function for **FF-gain = 30 milliseconds** (P-gain = 1 Volt/Volt, all other gains = 0)

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
8.9 Correlation Between Analytical and Experimental Table Sensitivity to Delta-Pressure Gain

In this section we show the correlation obtained between the experimental and the analytical table transfer functions for bare table condition and for different values of the delta-pressure gain. The set of servo-hydraulic parameters used in the analytical model is the one referred to as "d0" in Table 8.1. This set was used for all values of the delta-pressure gain, and the high degree of correlation obtained confirms the assumed independence of the servo-hydraulic parameters with respect to the delta-pressure gain.

8.9.1 Delta Pressure Gain of 1.5 Volts/Volt (Fig. 8.14)

Magnitude of the Table Transfer Function (M.T.F.)

The analytical model with non-zero servovalve time delay \( \tau \) is in excellent agreement with the experimental results as shown in Fig. 8.14. As was the case for the derivative and feed forward gains, the analytical model with zero servovalve time delay \( \tau \) underestimates the magnitude of the table transfer function in the frequency range between 0 and 30 Hz and overestimates the oil column peak.

Phase of the Table Transfer Function (P.T.F.)

The analytical model with non-zero servovalve time delay \( \tau \) is in better agreement with the experimental results than the analytical model with zero servovalve time delay \( \tau \).
Figure 8.14 Table Transfer Function for **dP-gain = 1.5 Volts/Volt** (P-gain = 1 Volt/Volt, all other gains = 0):

(a) Magnitude transfer function (b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay \( \tau \), dotted line = analytical T.F. with zero time delay \( \tau \), dashed line = experimental T.F.)
8.9.2 Delta Pressure Gain of 3.0 Volts/Volt (Fig. 8.15)

Magnitude of the table transfer function (M.T.F.)

The analytical model with non-zero servovalve time delay $\tau$ is in good agreement with the experimental results as shown in Fig. 8.15 (a). The analytical model with zero servovalve time delay $\tau$ underestimates the magnitude of the table transfer function in the frequency range between 0 and 30 Hz but captures well the experimental the oil column peak.

Phase of the table transfer function (P.T.F.)

The experimental P.T.F. lays between the ones predicted by the two analytical models (with zero and non-zero servovalve time delay $\tau$).
Figure 8.15 Table Transfer Function for **dP-gain = 3.0 Volts/Volt** (P-gain = 1 Volt/Volt, all other gains = 0):

(a) Magnitude transfer function (b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
8.10 **Correlation Between Analytical and Experimental Table Behavior Under the "Optimal" Gain Setting**

In this section, we show the correlation obtained between the experimental and the analytical table transfer functions under the optimal gain setting found for bare table condition. The set of servo-hydraulic parameters used in the analytical model is the one referred to as "d18" in Table 8.1, since it corresponds to the value of the derivative gain (18 milliseconds) of the optimal gain setting.

### 8.10.1 "Optimal" Gain Setting

The 'optimal' gain setting for the bare table condition has been found experimentally to be (see Section 6.5):

- **P-Gain** = 1.81 Volts/Volt
- **I-Gain** = 40 rps
- **D-Gain** = 18 milliseconds
- **FF-Gain** = 18 milliseconds
- **dP-Gain** = -3 Volts/Volts

*Magnitude of the table transfer function (M.T.F.) (Fig. 8.16)*

The analytical model with non-zero servovalve time delay $\tau$ matches very well the experimental results as shown in Fig. 8.16. It is noticed that the analytical model is able to capture all the peaks present in the experimental M.T.F. There is a perfect matching of the first peak at about 5 Hz. The peak-valley sequence due to the foundation flexibility at about 27 Hz is also extremely well captured by the analytical model. The analytical model with zero servovalve time delay $\tau$ does not capture the experimental behavior as well as the
model with non-zero servovalve time delay. As shown in Fig. 8.16 (b), it largely underestimates the M.T.F in the 0 to 50 Hz frequency range and largely overestimates the amplitude of the oil column peak.

*Phase of the table transfer function (P.T.F.) (Fig. 8.17)*

Here, the analytical model with non-zero servovalve time delay $\tau$ does not give a better prediction of the experimental results than the analytical model with zero servovalve time delay. The experimental P.T.F. lays between the two analytical predictions (with zero and non-zero servovalve time delay $\tau$) which are further apart than in the cases analyzed in previous sections.
Figure 8.16  Magnitude of Table Transfer Function for the "Optimal" gain setting
(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.17  Phase of Table Transfer Function for the "Optimal" gain setting
(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
8.11 Correlation Between Analytical and Experimental Table Sensitivity to “Rigid” Payload (Rigid Modeling)

In this section we show the correlation obtained between the experimental and the analytical table transfer functions for the shaking table loaded with a “rigid” payload consisting of a series of concrete blocks of approximately 150 lbs each and clamped to the slip table through 7/16” bolts. These payloads were included in the analytical model as totally rigid masses added to the mass of the slip table.

The control parameters used to determine the present correlation correspond to the optimal gain setting for bare table condition, see Section 6.5. The set of servo-hydraulic parameters used in the analytical model is the one referred to as “pr” in Table 8.2. This set was selected because it provided the least square fit between analytical and experimental transfer functions of the table loaded with very rigid (stiff) payload, see Section 8.3.

Figs 8.18 to 8.22 show the comparison between the experimental and analytical table transfer functions for the shaking table loaded with “rigid” payloads of 150, 300, 450, 600 and 900 lbs, respectively.

Magnitude of the table transfer function (M.T.F.)

The analytical model with a non-zero servovalve time delay \( \tau \) is in very good agreement with the experimental results for all “rigid” payloads considered. Notice that the analytical model captures very well the first three peaks of the M.T.F. (6-7 Hz, 27 Hz and oil column frequency). Furthermore the analytical model is able to capture very well the apparent

---

1. See Appendix F.
change in the oil column frequency which varies from about 42 Hz for 150 lbs of "rigid" payload.

The analytical model with zero servo valve time delay $\tau$ does not capture the actual shaking table behavior as well as well as the model accounting for the servo valve time delay $\tau$. In general, it underestimates the magnitude of the table transfer function in the frequency range between 0 and 50 Hz and overestimates the amplitude of the oil column peak (except for the cases of 600 and 900 lbs “rigid” payload). Furthermore the analytical model with zero servo valve time delay overestimates the oil column frequency.

*Phase of the table transfer function (P.T.F.)*

In general, the analytical model with non-zero servo valve time delay $\tau$ does not give a better prediction of the experimental results than the analytical model with zero servo valve time delay. In fact, the experimental P.T.F. lays between the ones predicted by the two analytical models (with zero and non-zero servo valve time delay $\tau$). However, it is noticed that the analytical model with non-zero time delay does capture better both the inversion frequency and the upwards-downwards curvature changes of the P.T.F. in the frequency range between 0 and 35 Hz, especially in Fig. 8.22.
Figure 8.18  Table Transfer Function. **Rigid Payload of 150 lbs:**

(a) Magnitude transfer function (b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.19  Table Transfer Function. **Rigid Payload of 300 lbs:**

(a) Magnitude transfer function (b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay τ, dotted line = analytical T.F. with zero time delay τ, dashed line = experimental T.F.)
Figure 8.20  Table Transfer Function. **Rigid Payload of 450 lbs:**

(a) Magnitude transfer function (b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.21  Table Transfer Function. **Rigid Payload of 600 lbs:**
(a) Magnitude transfer function  (b) Phase transfer function
(solid line = analytical T.F. with non-zero time delay τ, dotted line = analytical T.F. with zero time delay τ, dashed line = experimental T.F.)
Figure 8.22  Table Transfer Function. **Rigid Payload of 900 lbs:**

(a) Magnitude transfer function (b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay \( \tau \), dotted line = analytical T.F. with zero time delay \( \tau \), dashed line = experimental T.F.)
8.12 Correlation Between Analytical and Experimental Table Sensitivity to "Rigid" Payload (Partially Flexible Modeling)

As in section 8.11, the present section deals with the correlation between the experimental and the analytical table transfer functions for the shaking table loaded with a "rigid" payload\(^1\). Here we use an improved analytical model of the shaking table transfer function that takes into account the fact that the clamping of the concrete blocks to the table is not perfectly rigid. In this analytical model (referred to as partially flexible modeling), the concrete blocks are modeled as a combination of a SDOF system (having a natural frequency of about 55 Hz) and a rigid weight perfectly clamped to the slip table. Appendix F gives the rationale and the details of this modeling.

The control parameters used in this present correlation study correspond to the optimal gain setting found for bare table condition, see Section 6.5. The set of servo-hydraulic parameters used in the analytical model is the one referred to as "pfr" in Table 8.2. This set was selected because it provided the least square fit between analytical and experimental table transfer functions of the table loaded with a very rigid (stiff) payload, see Section 8.3.

Figs. 8.23 to 8.27 show the comparison between the experimental and the improved analytical table transfer functions for "rigid" payloads of 150, 300, 450, 600, and 900 lbs. respectively.

---

1. To load the table with a very stiff payloads, different concrete blocks (of about 150 lbs each) were fixed to the table through 7/16" bolts.
Magnitude of the table transfer function (M.T.F.)

The analytical model with a non-zero servovalve time delay \( \tau \) gives improves the matching between experimental and analytical results for all "rigid" payload weights. Notice that, as was already the case with the totally rigid modeling reported in Section 8.11, this improved analytical model captures very well the first three peaks of the M.T.F. (1 at 6-7 Hz, 27 Hz and oil column frequency) and the change in the oil column frequency from about 45 Hz for a 150 lbs "rigid" payload to about 36 Hz for a 900 lbs "rigid" payload. A unique characteristic of this improved analytical model is that it is able to capture very well the two peaks into which the oil column peak splits. see CHAPTER 4. The experimental M.T.F. indicates that, due to the presence of a "rigid" payload, the oil column peak is split into two peaks. One, of larger amplitude and lower frequency occurs at approximately from 35 to 45 Hz, depending on the weight, the other, of relatively smaller amplitude, occurs at a frequency 15 to 20 Hz higher. The improved matching between analytical and experimental M.T.F. suggests that the "rigid" payload actually behaves as a flexible SDOF system.

The analytical model with zero servovalve time delay \( \tau \) does not capture the experimental behavior as well as the model accounting for the servovalve time delay \( \tau \). In general, it underestimates the magnitude of the table transfer function in the frequency range from 0 to 50 Hz and it does capture the splitting in two of the oil column peak but with the wrong amplitudes and frequencies.
Phase of the table transfer function (P.T.F.)

In general, the analytical model with non-zero servovalve time delay τ does not give a better approximation of the experimental results than the analytical model with τ = 0 even though it captures very well the frequency at which a - π phase shifts occurs (this result is particularly good for the 900 lbs payload). The experimental P.T.F., in fact, lays between the ones predicted by the two analytical models (with zero and non-zero servovalve time delay τ). It is to be noticed, however, that the analytical model with non-zero τ does capture the inversion in the P.T.F. curvature that occurs at about 30 - 40 Hz. This good matching is particularly evident in the case of the 900 lbs payload.
Figure 8.23  Table Transfer Function. **Rigid Payload of 150 lbs** (partially flexible analytical modeling):

(a) Magnitude transfer function  
(b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.24  Table Transfer Function. **Rigid Payload of 300 lbs** (partially flexible analytical modeling):

(a) Magnitude transfer function  (b) Phase transfer function
(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.25  Table Transfer Function, **Rigid Payload of 450 lbs** (partially flexible analytical modeling):

(a) Magnitude transfer function (b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.26 Table Transfer Function, **Rigid Payload of 600 lbs** (partially flexible analytical modeling):

(a) Magnitude transfer function  
(b) Phase transfer function  
(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.27  Table Transfer Function. **Rigid Payload of 900 lbs** (partially flexible analytical modeling):

(a) Magnitude transfer function (b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
8.13 Correlation Between Analytical and Experimental Table Sensitivity to Derivative Gain for "Rigid" Payload (Partially Flexible Modeling)

In Sections 7.2.4, 7.2.5, and 7.2.6, we analyzed the effect of a reduction in the derivative gain upon the experimental T.F. of the shaking table loaded with a "rigid" payload of 150, 450 and 900 lbs, respectively. This Section shows how the correlation obtained between the experimental and the analytical table transfer function of the shaking table under these conditions. The analytical model used is the improved one (corresponding to a partially flexible modeling of the "rigid" payload) described in Section 8.12 and in Appendix F.

The table control parameters used in this experiment are the "optimal" ones for bare table conditions, except for the derivative gain that is progressively reduced. The set of servo-hydraulic parameters used in the analytical model is the one referred to as "pfr" in Table 8.2. This set was selected because it provide the best least square fit to the actual behavior of the table loaded with a semi rigid payload.

8.13.1 150 lbs "Rigid" Payload

Fig. 8.28 compares the experimental and analytical table transfer function for the shaking table loaded with a "rigid" payloads of 150 lbs and the derivative gain reduced to 10 milliseconds (from an initial value of 18 milliseconds). The experimental and analytical table transfer functions for a derivative gain of 18 milliseconds were shown in Fig. 8.23.

Magnitude of the table transfer function (M.T.F.)

The analytical model with non-zero servovalve time delay $\tau$ agrees well with the experi-
mental transfer function. It is observed that the analytical model captures very well the peaks of the M.T.F. at 16Hz (due to integral gain), 27 Hz (due to base flexibility) and the two peaks resulting from the splitting of the oil column peak due to partial flexibility of "rigid" payload. Clearly, the analytical model with zero servo-valve time delay does not provide a good correlation with the experimental results, although it is able to capture the basics characteristics. It underestimates the magnitude of the table transfer function in the frequency range between 0 and 50 Hz and overestimates the amplitude of the second split oil column peak.

*Phase of the table transfer function (P.T.F.)*

As for the M.T.F., the analytical model with non-zero servo-valve time delay $\tau$ agrees better with the experimental results than the analytical model with zero time delay. In the frequency range between 0 and 40 Hz, the experimental P.T.F. lays between the two analytical models, while above 40 Hz, the analytical model accounting for servo-valve time delays gives better results.
Figure 8.28 Table Transfer Function. **Rigid Payload of 150 lbs, Derivative Gain Reduced to 10 milliseconds** (partially flexible analytical modeling):

(a) Magnitude transfer function  (b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
8.13.2 450 lbs “Rigid Payload”

Figs. 8.29, 8.30, and 8.31 compare the experimental and analytical table transfer functions of the shaking table loaded with a “rigid” payloads of 450 lbs and the derivative gain reduced to 10, 6, and 5 milliseconds (from an initial value of 18 milliseconds). The experimental and analytical table transfer functions for a derivative gain of 18 milliseconds were shown in Fig. 8.25.

Magnitude of the table transfer function (M.T.F.)

For all three values of the derivative gain the analytical model with non-zero servovalve time delay $\tau$ is in good agreement with the experimental transfer function. In this case also, the analytical model captures well the various peaks of the experimental table transfer function. It is noticed, however that the agreement between analytical and experimental results in the low frequency range (0 to 40 Hz) is not as good as in the higher frequency range (> 40 Hz) and as in the case of the 150 lbs “rigid” payload. Also, it appears that the correlation worsens progressively as the derivative gain is reduced.

It is also clear that the analytical model with zero servovalve time delay $\tau$ is not able to capture the experimental behavior as well as well as the one accounting for the servovalve time delay $\tau$. Again, it underestimates the magnitude of the table transfer function in the frequency range between 0 and 50 Hz and overestimates the amplitude of the second split oil column peak.
Phase of the table transfer function (P.T.F.)

As for the M.T.F., the analytical model with non-zero servovalve time delay $\tau$ agrees better with the experimental results than the analytical model with zero time delay. In the frequency range between 0 and 40 Hz, the experimental P.T.F. lays between the two analytical models, while above 40 Hz, the analytical model accounting for servo-valve time delays gives better results.
Figure 8.29  Table Transfer Function. **Rigid Payload of 450 lbs, Derivative Gain Reduced to 10 milliseconds** (partially flexible analytical modeling):

(a) Magnitude transfer function (b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.30  Table Transfer Function. **Rigid Payload of 450 lbs, Derivative Gain Reduced to 6 milliseconds** (partially flexible analytical modeling):

(a) Magnitude transfer function (b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.31 Table Transfer Function. **Rigid Payload of 450 lbs, Derivative Gain Reduced to 5 milliseconds** (partially flexible analytical modeling):

(a) Magnitude transfer function (b) Phase transfer function
(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
8.13.3 900 lbs “Rigid” Payload

Figs. 8.32, 8.33, and 8.34 compare the experimental and analytical table transfer functions of the shaking table loaded with a “rigid” payload of 900 lbs and the derivative gain reduced to 10, 6, and 5 milliseconds (from an initial value of 18 milliseconds). The experimental and analytical table transfer functions for a derivative gain of 18 milliseconds were shown in Fig. 8.27.

Magnitude of the table transfer function (M.T.F.)

For all three values of the derivative gain, the analytical model with non-zero servovalve time delay $\tau$ agrees reasonably well with the experimental results. The analytical model captures well the location of the peaks of the M.T.F., but it underestimates their amplitude in the frequency range between 20 and 50 Hz and overestimates the amplitude of the highest frequency peak. Notice that the quality of the match between analytical and experimental results is inferior to that observed for the lighter payloads of 150 and 450 lbs. It is worth mentioning that the same set of servo-hydraulic parameters have been used for the various “rigid” payloads and derivative gain values.

In this case also, the analytical model with zero servovalve time delay $\tau$ is inferior to that accounting for the servovalve time delay $\tau$. Its underestimation of the table M.T.F. between 0 and 40 Hz and overestimation of it above 50 Hz are much more pronounced than in the case of the analytical model accounting for the servovalve time delay.
Phase of the table transfer function (P.T.F.)

Overall, the analytical model accounting for servo-valve time delay is in better agreement with the experimental results than the analytical model with zero servo-valve time delay. In the frequency range from 0 to 30 Hz, the experimental P.T.F. lies between the two analytical P.T.F.'s, but for higher frequencies, the analytical model accounting for servo-valve time delay is more accurate. It captures well the peak at about 58 Hz which is due to the partial flexibility of the "rigid" payload. This agreement confirms the validity of the modeling assumption for the "rigid" payload.

8.13.4 Remarks

It is known, from the analysis presented in Sections 7.2.4 to 7.2.6, that the table M.T.F. is very sensitive to the derivative gain. Furthermore, it was also found in Section 8.3 that the servo-hydraulic parameters (obtained through least square fitting) depend on the derivative gain. The fact that the analytical model accounting for servo-valve time delay gives a good agreement with experimental results throughout Section 8.13 although a constant set of servo-hydraulic parameters was used confirms further the validity of the analytical model developed for the shaking table.
Figure 8.32 Table Transfer Function. **Rigid Payload of 900 lbs, Derivative Gain Reduced to 10 milliseconds** (partially flexible analytical modeling):

(a) Magnitude transfer function  
(b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.33  Table Transfer Function. **Rigid Payload of 900 lbs, Derivative Gain Reduced to 6 milliseconds** (partially flexible analytical modeling):

(a) Magnitude transfer function (b) Phase transfer function
(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.34  Table Transfer Function. **Rigid Payload of 900 lbs, Derivative Gain Reduced to 5 milliseconds** (partially flexible analytical modeling):

(a) Magnitude transfer function  
(b) Phase transfer function  
(solid line = analytical T.F. with non-zero time delay \( \tau \), dotted line = analytical T.F. with zero time delay \( \tau \), dashed line = experimental T.F.)
8.14 **Effects of Flexible Payload (3-DOF): Correlation Between Analytical and Experimental Results**

This Section shows the correlation obtained between the analytical and the experimental table transfer functions for the shaking table loaded with the "flexible" payload described in Appendix B. This flexible payload consisted of a reduced scale 3-story single bay steel moment resisting frame\(^1\). Its dynamic characteristics (mass matrix, mode shapes and modal frequencies) to be included in the analytical model of the shaking table with an MDOF payload were obtained using the CAL90 computer program and are described in Section B.3 of Appendix B.

The table control parameters used in this experiment are the "optimal" ones obtained for bare table condition. The set of servo-hydraulic parameters used in the analytical model is the one referred to as "pf" in Table 8.2. This set was selected because it least-square fitted the experimental M.T.F. of the table loaded with a relatively flexible payload (natural frequencies below 50 Hz). In Section 8.3, it was found that the servo-hydraulic parameters used in the model of the table depend on the characteristics of the payload, especially its fundamental frequency and its total mass. In this analysis, however, it was decided to use the same set of servo-hydraulic parameters, referred to as "pf", for both payloads (450 and 900 lbs). Therefore, the correlation obtained between analytical and experimental table transfer functions for the 450 and 900 lbs flexible payloads is not as good as if payload mass dependent servo-valve parameters were used.

---

1. See Section 3.7 "Effect of a MDOF Upon the Shaking Table Transfer Function".
8.14.1 Flexible MDOF Payload of 900 lbs

Figs. 8.35 and 8.36 show the comparison between the experimental and analytical table transfer functions for the shaking table loaded with a flexible payload of 900 lbs.

Magnitude of the table transfer function (M.T.F.)

The analytical model with non-zero servovalve time delay $\tau$ agrees very well with the experimental results. Notice that the analytical model captures very well the three peaks in the M.T.F. which correspond to the modal frequencies of the payload, namely about 5.16 and 24 Hz. The peak in due to the flexibility of the foundation and the oil column peak are also well captured with the analytical model with some underestimation of the amplitude of the oil column peak.

It is observed that the analytical model with zero servovalve time delay $\tau$ does not capture the experimental behavior as well as the one accounting for the servovalve time delay $\tau$. It underestimate the magnitude of the table transfer function in the frequency range between 0 and 50 Hz and overestimates the amplitude of the oil column peak.

Phase of the table transfer function (P.T.F.)

Overall, the analytical model with non-zero servovalve time delay $\tau$ gives a better approximation of the experimental results. It is noticed that the analytical model with non-zero time delay does capture very well the troughs in the P.T.F. due to the payload modal frequencies.
Figure 8.35  Magnitude of Table Transfer Function. **Flexible Payload (3 DOF) of 900 lbs.** "Optimal" gain setting

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.36  Phase of Table Transfer Function. **Flexible Payload (3 DOF) of 900 lbs.**

"Optimal" gain setting

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
8.14.2 Flexible MDOF Payload of 450 lbs

Figs. 8.37 and 8.38 show the comparison between the experimental and analytical table transfer functions for the shaking table loaded with a flexible payload of 450 lbs.

Magnitude of the table transfer function (M.T.F.)

The analytical model with non-zero servovalve time delay $\tau$ is in very good agreement with the experimental results, although not quite as good as for the 900 lbs flexible payload in the frequency range between 30 and 50 Hz. Notice the small shift between the analytically predicted and experimentally observed first peak of the M.T.F. due to the fundamental frequency of the flexible payload. It is also observed that the analytical model accounting for the servo-valve time delay captures very well the amplitude of the oil column peak.

The analytical model with zero servovalve time delay $\tau$ does not capture the experimental behavior as well as the one accounting for the servovalve time delay $\tau$. It underestimate the magnitude of the table transfer function in the frequency range between 0 and 50 Hz and widely overestimates the amplitude of the oil column peak.

Phase of the table transfer function (P.T.F.)

The correlation between the analytical and experimental P.T.F. is similar to that observed in the case of 900 lbs flexible payload.
Figure 8.37  Magnitude of Table Transfer Function. **Flexible (3 DOF) Payload of 450 lbs.** "Optimal" gain setting
(solid line = analytical T.F. with non-zero time delay τ, dotted line = analytical T.F. with zero time delay τ, dashed line = experimental T.F.)
Figure 8.38  Phase of Table Transfer Function. **Flexible (3 DOF) Payload of 450 lbs.**

"Optimal" gain setting

(solid line = analytical T.F. with non-zero time delay $\tau$. dotted line = analytical T.F. with zero time delay $\tau$. dashed line = experimental T.F.)
8.14.3 Remarks

Figs. 8.35 though 8.38 show a peak-trough sequence at a frequency of about 55 Hz, indicating another source of flexibility. This flexibility could be due to the imperfect clamping of the concrete blocks on the model frame (see Section 8.12 and Appendix F) or to the unmodeled flexibility of the beams supporting the concrete blocks.
8.15 **Correlation Between Analytical and Experimental Table Sensitivity to Derivative and Feed-Forward Gain for Table Loaded With 450 lbs Flexible Payload (3-DOF)**

In section 7.3.4, we analyzed the effect of a simultaneous increase of the derivative and feed-forward gains upon the experimental transfer function of the shaking table loaded with a flexible payload of 450 lbs. This Section examines the correlation obtained between the experimental and the analytical table transfer functions under the same conditions. The analytical model of the flexible payload is the CAL90 Model described in Section B.3 of Appendix B.

The table control parameters used in this experiment are the "optimal" ones obtained for bare table conditions except for the derivative and feed-forward gains that are varied.

The set of servo-hydraulic parameters used in the analytical model is the one referred to as "pf" in Table 8.2. This set was selected because it was least square fitting the experimental behavior of the table loaded with a flexible payload.
8.15.1 Derivative and Feed-Forward Gains Increased to 21, 24, 34 Milliseconds

Magnitude of the table transfer function (M.T.F.)

Figs. 8.39, 8.41, and 8.43 compare the experimental and analytical M.T.F.s of the shaking table loaded with a flexible payload of 450 lbs and with both the derivative and feed-forward gains increased to 21, 24 and 34 milliseconds (from an initial value of 18 milliseconds).

For all three values of the derivative and feed-forward gains, the analytical model with non-zero servovalve time delay \( \tau \) is in good agreement with the experimental results although it does not capture precisely the position along the frequency axis and the magnitude of the peaks due to the modal frequencies of the flexible payload. The other peaks of the M.T.F. (at 5 Hz due to integral gain, at 27 Hz due to foundation flexibility and oil column peak at about 67 Hz) are however well captured.

The analytical model with zero servovalve time delay \( \tau \) does not capture the experimental behavior as well as the one accounting for the servovalve time delay \( \tau \). As usual, it under-estimates the magnitude of the table transfer function in the frequency range between 0 and 40 Hz and overestimates the amplitude of the M.T.F. in the neighborhood of the oil column peak.

Phase of the table transfer function (P.T.F.)

Figs. 8.40, 8.42, and 8.44 compare the experimental and analytical P.T.F.s of the shaking table loaded with a flexible payload of 450 lbs and with both the derivative and feed-forward gains increased to 21, 24 and 34 milliseconds (from an initial value of 18 millisecond-
onds). In the frequency range between 0 and 20 hz the experimental P.T.F. lies between the analytical models with and without servo-valve time delay. for higher frequencies, the analytical model accounting for servo-valve time delay seems to approximate better the experimental results, although not very well.
Figure 8.39  Magnitude of Table Transfer Function, **Flexible Payload of 450 lbs, Derivative Gain Increased to 21 milliseconds**

(solid line = analytical T.F. with non-zero time delay \( \tau \), dotted line = analytical T.F. with zero time delay \( \tau \), dashed line = experimental T.F.)
Figure 8.40  Phase of Table Transfer Function. **Flexible Payload of 450 lbs, Derivative Gain Increased to 21 milliseconds**

(solid line = analytical T.F. with non-zero time delay τ, dotted line = analytical T.F. with zero time delay τ, dashed line = experimental T.F.)
Figure 8.41 Magnitude of Table Transfer Function. **Flexible Payload of 450 lbs, Derivative Gain Increased to 24 milliseconds**

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.42  Phase of Table Transfer Function. **Flexible Payload of 450 lbs. Derivative Gain Increased to 24 milliseconds**

(solid line = analytical T.F. with non-zero time delay \(\tau\), dotted line = analytical T.F. with zero time delay \(\tau\), dashed line = experimental T.F.)
Figure 8.43 Magnitude of Table Transfer Function. **Flexible Payload of 450 lbs, Derivative Gain Increased to 34 milliseconds**

(solid line = analytical T.F. with non-zero time delay \( \tau \), dotted line = analytical T.F. with zero time delay \( \tau \), dashed line = experimental T.F.)
Figure 8.44  Phase of Table Transfer Function. **Flexible Payload of 450 lbs, Derivative Gain Increased to 34 milliseconds**

(solid line = analytical T.F. with non-zero time delay \( \tau \). dotted line = analytical T.F. with zero time delay \( \tau \). dashed line = experimental T.F.)
8.16 **Correlation Between Analytical and Experimental Table Sensitivity to Derivative and Feed-Forward Gains for Table Loaded With 900 Lbs Flexible Payload (3-DOF)**

In sections 7.3.6. and 7.3.7. we analyzed the effects of reducing and increasing simultaneously the derivative and feed-forward gains upon the experimental transfer function of the shaking table loaded with a flexible payload of 900 lbs. This Section examines the correlation obtained between the experimental and the analytical table transfer functions under the same conditions. The analytical model of the flexible payload is the CAL90 Model described in Section B.3 of Appendix B.

The table control parameters used in this experiment are the “optimal” ones obtained for bare table condition, except for the derivative and feed-forward gains which are varied. The set of servo-hydraulic parameters used in the analytical model is the one referred to as “pf” in Table 8.2. This set was selected because it was least-square fitting the experimental behavior of the table loaded with a flexible payload.
8.16.1 Effect of Reducing Both the Derivative and Feed-Forward Gains

Magnitude of the table transfer function (M.T.F.)

Figs. 8.45 and 8.47 compare the experimental and analytical M.T.F.s of the shaking table loaded with a flexible payloads of 900 lbs and with both the derivative and feed-forward gains reduced simultaneously to 16. and 5 milliseconds (from an initial value of 18 milliseconds). For both values of the derivative and feed-forward gains, the analytical model with non-zero servovalve time delay $\tau$ agrees well with the experimental results. This analytical model captures well both the overall outline of the M.T.F. and the peaks due to the modal frequencies of the flexible payload, especially for the gains set to 16 milliseconds. The analytical model with zero servovalve time delay $\tau$ does not capture the experimental behavior as well as the one accounting for the servovalve time delay $\tau$. As usual, it underestimates the magnitude of the table transfer function in the frequency range between 0 and 50 Hz and overestimates the amplitude of the M.T.F. in the neighborhood of the oil column frequency.

Phase of the table transfer function (P.T.F.)

Figs. 8.46 and 8.47 compare the experimental and analytical P.T.F.s of the shaking table loaded with a flexible payload of 900 lbs and with both the derivative and feed-forward gains decreased simultaneously to 16 and 5 milliseconds (from an initial value of 18 milliseconds). Overall, the analytical model with non-zero servovalve time delay $\tau$ gives a better approximation, although not of the same level of approximation found for the M.T.F., of the experimental results than the analytical model with zero servo-valve time delay.
Figure 8.45  Magnitude of Table Transfer Function. **Flexible Payload of 900 lbs, Derivative Gain Reduced to 16 milliseconds**

(solid line = analytical T.F. with non-zero time delay τ, dotted line = analytical T.F. with zero time delay τ, dashed line = experimental T.F.)
Figure 8.46  Phase of Table Transfer Function. **Flexible Payload of 900 lbs, Derivative Gain Reduced to 16 milliseconds**

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.47 Table Transfer Function. **Flexible Payload of 900 lbs, Derivative Gain Reduced to 5 milliseconds**

(a) Magnitude transfer function (b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
8.16.2 Effect of Increasing Both the Derivative and Feed-Forward Gain

Magnitude of the table transfer function (M.T.F.)

Figs. 8.48, 8.49 and 8.51 compare the experimental and analytical M.T.Fs of the shaking table loaded with a flexible payload of 900 lbs and with both the derivative and feed-forward gains increased simultaneously to 21, 24 and 34 milliseconds (from an initial value of 18 milliseconds). For these values of the derivative and feed-forward gains, the analytical model with non-zero servovalve time delay $\tau$ agrees very well with the experimental results. Notice the splitting of the experimental oil column peak in two peaks, the first one of which is not captured by the analytical model. As already mentioned (see Section 8.14.3), this splitting could be due to the imperfect clamping of the concrete blocks or to the unmodeled flexibility of the beams supporting the blocks. It is observed that the peaks due to the modal frequencies of the flexible payload are captured very accurately, especially with the gains set to 34 milliseconds. As usual the analytical model with zero servovalve time delay $\tau$ does not capture the experimental behavior as well as the one accounting for the servovalve time delay $\tau$. It underestimates the magnitude of the table transfer function in the frequency range between 0 and 50 Hz and overestimates the amplitude of the M.T.F. in the neighborhood of the oil column frequency.

Phase of the table transfer function (P.T.F.)

Figs. 8.48, 8.50 and 8.52 compare the experimental and analytical P.T.Fs of the shaking table loaded with a flexible payload of 900 lbs and with both the derivative and feed-forward gains increased to 21, 24 and 34 milliseconds (from an initial value of 18 millisec-
onds). Once again, overall the analytical model with non-zero servovalve time delay $\tau$ gives a better approximation, although not very good, of the experimental results than the analytical model with zero servo-valve time delay. It is noticed that the analytical model accounting for the servovalve time delay $\tau$ captures very well the trough of the P.T.F. due to the first modal frequency of the MDOF payload.
Figure 8.48  Table Transfer Function. **Flexible Payload of 900 lbs, Derivative Gain Increased to 21 milliseconds**

(a) Magnitude transfer function  (b) Phase transfer function

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.49 Magnitude of Table Transfer Function. **Flexible Payload of 900 lbs, Derivative Gain Increased to 24 milliseconds**

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.50  Phase of Table Transfer Function. **Flexible Payload of 900 lbs, Derivative Gain Increased to 24 milliseconds**

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.51 Magnitude of Table Transfer Function. **Flexible Payload of 900 lbs, Derivative Gain Increased to 34 milliseconds**

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
Figure 8.52 Phase of Table Transfer Function. **Flexible Payload of 900 lbs. Derivative Gain Increased to 34 milliseconds**

(solid line = analytical T.F. with non-zero time delay $\tau$, dotted line = analytical T.F. with zero time delay $\tau$, dashed line = experimental T.F.)
8.16.3 Remarks

It is known, from the analysis presented in Sections 7.3.4, 7.3.6 and 7.3.7, that a change in the derivative gain modify significantly the shape of the M.T.F. The fact that, for a constant set of servo-hydraulic parameters, the analytical model accounting for the servovalve time delay $\tau$ is able to capture the change in shape of the experimental M.T.F. confirms the validity of the analytical shaking table model developed here.
8.17 CONCLUSIONS

The correlation results presented in this chapter indicates that the analytical shaking table model developed in Chapter 3 is able to capture very well the actual table behavior under a wide range of working conditions. Sections 8.5 through 8.9 showed how the analytical model accounting for the servovalve time delay $\tau$ is able to capture the sensitivity of the table behavior to the various control gains both qualitatively and quantitatively.

Sections 8.11, 8.12 and 8.14 showed that the analytical shaking table model is able to capture the actual effects of rigid and flexible payloads upon the shaking table transfer function.

Sections 8.13, 8.15 and 8.16 showed that the shaking table analytical model is able to capture the sensitivity to control gain values of the shaking table loaded with rigid and flexible payloads.
CHAPTER 9
ANALYTICAL SIMULATION OF SHAKING TABLE BEHAVIOR
9.1 Introduction

In this chapter we use the analytical model of the shaking table to simulate the sensitivity of its transfer function to control gain parameters and to payload characteristics under realistic working conditions. This chapter differs from the similar sensitivity analysis carried out in Chapter 4 for the following reasons: here the sensitivity analysis is performed based on a carefully calibrated analytical model of the shaking table, while in Chapter 4, the sensitivity analysis was essentially qualitative since both the table control gains and servo-hydraulic parameters were not set to their "optimal" and least square fitted values, respectively.

Section 9.2 presents the results on the sensitivity of the table transfer function to the table control gains. While Section 9.3 reports on the sensitivity of the table transfer function to the total mass and to the dynamic characteristics of the payload.
9.2 SHAKING TABLE SENSITIVITY TO CONTROL GAIN PARAMETERS

This Section investigates the effects upon the calibrated analytical table transfer function of the different control gain parameters. These effects are presented through three dimensional plots (plotting the table transfer function versus the value of a specific control gain parameter).

In order to obtain the table sensitivity to the control gain parameters for realistic working conditions, the gain setting around which the sensitivity analysis is performed corresponds to the "optimal" one for bare table condition determined in Section 6.5. One control gain parameter at a time is perturbed in order to evaluate its effect on the shaking table transfer function. The perturbation of the control gain parameter covers a range centered on the "optimal" value.

The servo-hydraulic parameters used here are the one that were found to least square fit the actual table transfer function under "optimal" control gains for bare table condition. This set of servo-hydraulic parameters is the set referred to as d18 in Table 8.1.

The sensitivity of the shaking table transfer function to control gain parameters has been carried out for the proportional, integral, feed forward and delta pressure gains. This sensitivity analysis could not be performed for the derivative gain since, as shown in Chapter 8, the derivative gain affects significantly the values of the servo-hydraulic parameters which best fit the actual table transfer function and which cannot be found analytically.
9.2.1 Proportional Gain

Fig. 9.1 shows the effects upon the shaking table transfer function of a variation in the value of the proportional gain around its "optimal" setting (P = 1.81 Volts/Volt) when all the other gains are at their "optimal" value. It is observed that for a range of values of the P-gain between 1.2 and 2.5 Volts/Volt, the magnitude of the table transfer function is close to unity in the frequency range from 0 to approximately 65 Hz.

Values of P-gain smaller than 1.2 Volts/Volt produce a high and narrow peak at very low frequencies (between 0 and 7 Hz) and to a deep valley between 10 and 20 Hz.

Values of the P-gain higher than 2.5 Volts/Volt give rise to a very high and wide peak at approximately 16 Hz and to a significant increase of the amplitude of the oil column peak (at approximately 65 Hz).
Figure 9.1  Analytical Table Transfer Function with “optimal” Gain Setting and Least Square Fitted Servo-Hydraulic parameters: Effect of Increasing the Proportional Gain
9.2.2 Integral Gain

Fig. 9.2 shows the effects upon the shaking table transfer function of a variation in the value of the integral gain around its "optimal" setting (I-gain = 40 rps), when all the other gains are set at their "optimal" values. The 3-D plots indicates that an increase in the integral gain give rise to a narrow peak at around 4 to 5 Hz. The height of this peaks increases with the value of the I-gain. Notice that no other appreciable effects on the shaking table transfer function can be detected.

The amplitude of the low frequency peak remains acceptable limits for values of the I-gain up to approximately 60 rps. This gives a range of suitable values for the I-gain between 0 and 60 rps.
Figure 9.2  Analytical Table Transfer Function with "optimal" Gain Setting and Least Square Fitted Servo-Hydraulic parameters: **Effect of Increasing the Integral Gain**
9.2.3 Feed-Forward Gain

Fig. 9.3 shows the effects upon the shaking table transfer function of a variation in the value of the feed-forward gain around its "optimal" setting (FF-gain = 18 milli-seconds), when all the other gains are set at their "optimal" value.

The three dimensional plots indicate that an increase in the feed-forward gain produces a quasi linear increase in the value of the table transfer function in the intermediate frequency range between 20 and 70 Hz.

It is noticed that both the amplitude of the oil column peak and the peak due to the foundation flexibility increase simultaneously and quasi linearly for an increasing value of the feed-forward gain. Given the above sensitivity property of the table transfer function, an acceptable range for the feed-forward gain is [0 - 40] milli-seconds.
Figure 9.3  Analytical Table Transfer Function with "optimal" Gain Setting and Least Square Fitted Servo-Hydraulic parameters: Effect of Increasing the Feed-Forward Gain
9.2.4 Delta-Pressure Gain

Fig. 9.4 shows the effects upon the shaking table transfer function of a variation in the value of the delta-pressure gain around its "optimal" setting (dP-gain = -3 Volts/Volt), when all the other gains are set at their "optimal" value.

The three-dimensional plots of the table transfer function show that an increase (in absolute value, since the delta pressure gain used here has an inverted polarity) in the value of the delta-pressure gain has the only effect of reducing the amplitude of the oil column peak without affecting any other part of the shaking table transfer function. This is a clear indication that the delta-pressure gain can be used to control and reduce the negative resonance effect associated with the oil column peak.

The plots suggest that a suitable range for the delta-pressure gain is between -2 and -4 Volts/Volt.
Figure 9.4  Analytical Table Transfer Function with "optimal" Gain Setting and Least Square Fitted Servo-Hydraulic parameters: Effect of Increasing the Delta-Pressure Gain
9.3 **SHAKING TABLE SENSITIVITY TO THE NATURAL FREQUENCY OF AN SDOF FLEXIBLE PAYLOAD**

This Section looks at the effects upon the table transfer function of the natural frequency of a flexible SDOF payload. These effects are presented in the form of three-dimensional and contour plots of the table transfer function versus the natural frequency of the payload.

In simulating the table sensitivity to the payload natural frequency, the control gain setting was set to the "optimal" setting for bare table condition (see Section 8.10). The servo-hydraulic parameters used in this simulation are the ones referred to as "d18" in Table 8.1.

The sensitivity of the shaking table transfer function to payload natural frequency was obtained for a payload weight of 150, 300, 450, 600, 900 and 1500 lbs.
9.3.1 SDOF Payload of 150 lbs

The effects upon the shaking table transfer function of an 150 lbs flexible SDOF payload with varying natural frequency are shown in Figs. 9.5 and 9.6. As anticipated in the general analysis performed in Chapter 4, a deep valley occurs near the payload natural frequency.

SDOF payloads having a natural frequency up to approximately 25 Hz do not affect significantly the table transfer function except for the aforementioned valley at the location of the payload natural frequency. Notice that payloads having a natural frequency between 40 and 80 Hz give rise to a very high and narrow peak above the payload natural frequency (see Section 4.3). Payloads having their natural frequency above approximately 80 Hz do not introduce any significant valley and peak in the shaking table transfer function. The frequency bandwidth of the table for very rigid payloads (natural frequency of 120 Hz and above) is 75 Hz, down from a value of 80 Hz for the bare table condition.

The peak-valley sequence due to the foundation flexibility is not affected by the presence of a flexible payload.
Figure 9.5  Analytical Simulation of the Table Transfer Function with "optimal" Gain Setting for bare table condition and corresponding values of the Servo-Hydraulic Parameters: Effects of an 150 lbs SDOF Payload of Varying natural Frequency.
Figure 9.6  Contour Plot of the Analytical Simulation of the Table Transfer Function with “optimal” Gain Setting for bare table condition and corresponding values of the Servo-Hydraulic Parameters: Effects of an 150 lbs SDOF Payload of Varying natural Frequency.
9.3.2 SDOF Payload of 300 lbs

The effects upon the shaking table transfer function of a 300 lbs flexible SDOF payload with varying natural frequency are shown in Figs. 9.7 and 9.8. As for the 150 lbs case, a deep valley occurs near the payload natural frequency.

As was the case for the 150 lbs payload, SDOF payloads having a natural frequency up to approximately 25 Hz do not affect significantly the table transfer function. Payloads having their natural frequency between 40 and 80 Hz produce a very high and narrow peak above the payload natural frequency. In this case the high peak region spans between 70 and 100 Hz, as opposed to between 70 and 90 Hz for the 150 lbs payload. Payloads having their natural frequency above approximately 90 Hz do not introduce any significant valley and peak in the shaking table transfer function. The frequency bandwidth of the shaking table for very rigid payload (natural frequency of 120 Hz and above) of 67 Hz, down from a value of 80 Hz for the bare table condition.

The peak-valley sequence due to the foundation flexibility is not affected by the presence of the payload, except when the payload natural frequency is in the proximity of the natural frequency of the flexible foundation.
Figure 9.7 Analytical Simulation of the Table Transfer Function with “optimal” Gain Setting for bare table condition and corresponding values of the Servo-Hydraulic Parameters: Effects of a 300 lbs SDOF Payload of Varying natural Frequency.
Figure 9.8  Contour Plot of the Analytical Simulation of the Table Transfer Function with “optimal” Gain Setting for bare table condition and corresponding values of the Servo-Hydraulic Parameters: Effects of an 150 lbs SDOF Payload of Varying natural Frequency.
9.3.3 SDOF Payload of 450 lbs

The effects upon the shaking table transfer function of a 450 lbs flexible SDOF payload with varying natural frequency are shown in Figs. 9.9 and 9.10. As for the previous cases, a deep valley occurs near the payload natural frequency.

Payloads having their natural frequency up to 20 Hz do not affect significantly the shaking table transfer function. This frequency range is down from 25 Hz for the lighter payloads previously examined, indicating that heavier payloads are more prone to spoiling the table transfer function. Payloads having their natural frequency between 40 and 80 Hz give rise to a very high and narrow peak above the payload natural frequency. The frequency range in which this high peak occurs is between 70 and 105 Hz, wider than for lighter payloads.

Payload having their natural frequency above approximately 90 Hz do not introduce any significant valley-peak sequence in the shaking table transfer function.

The frequency bandwidth of the shaking table for very rigid payload (natural frequency of 120 Hz and above) is 60 Hz, down from a value of 80 Hz for the bare table condition.

The peak-valley sequence due to the foundation flexibility is not affected by the presence of the payload, except when the payload natural frequency is in the proximity of the natural frequency of the flexible foundation.
Figure 9.9  Analytical Simulation of the Table Transfer Function with "optimal" Gain Setting for bare table condition and corresponding values of the Servo-Hydraulic Parameters: Effects of a 450 lbs SDOF Payload of Varying natural Frequency.
Figure 9.10  Contour Plot of the Analytical Simulation of the Table Transfer Function with "optimal" Gain Setting for bare table condition and corresponding values of the Servo-Hydraulic Parameters: Effects of a 450 lbs SDOF Payload of Varying natural Frequency.
9.3.4 SDOF Payload of 600 lbs

The effects upon the shaking table transfer function of a 600 lbs flexible SDOF payload with varying natural frequency are shown in Figs. 9.11 and 9.12. As for the previous cases, a deep valley occurs near the payload natural frequency. Payloads having their natural frequency up to 20 Hz do not affect significantly the shaking table transfer function as was the case for the 450 lbs payload. Payloads having their natural frequency between 35 and 85 Hz give rise to a very high and narrow peak above the payload natural frequency. This natural frequency range is broadened with respect to the one observed for lighter payloads. The frequency range in which this high peak occurs is between 70 and 105 Hz, which is similar to the case of 450 lbs payload. Payloads having their natural frequency above approximately 90 Hz do not introduce any significant peak-valley sequence in the shaking table transfer function. The frequency bandwidth of the shaking table for very rigid payload (natural frequency of 120 Hz and above) of 58 Hz, down from a value of 80 Hz for the bare table condition.

The peak-valley sequence due to the foundation flexibility is amplified by the presence of the payload having a natural frequency above 20 Hz. Furthermore this peak-valley sequence is amplified when the payload natural frequency is in the neighborhood of the foundation natural frequency.
Figure 9.11  Analytical Simulation of the Table Transfer Function with “optimal” Gain Setting for bare table condition and corresponding values of the Servo-Hydraulic Parameters: Effects of a 600 lbs SDOF Payload of Varying natural Frequency.
Figure 9.12 Contour Plot of the Analytical Simulation of the Table Transfer Function with “optimal” Gain Setting for bare table condition and corresponding values of the Servo-Hydraulic Parameters: Effects of a 600 lbs SDOF Payload of Varying natural Frequency.
9.3.5 SDOF Payload of 900 lbs

The effects upon the shaking table transfer function of a 900 lbs flexible SDOF payload with varying natural frequency are shown in Figs. 9.13 and 9.14. As for the previous cases, a deep valley occurs near the payload natural frequency.

Payloads with their natural frequency up to 18 Hz do not affect significantly the shaking table transfer function. Payloads having their natural frequency between 20 and 85 Hz produce a narrow peak above the payload natural frequency. This natural frequency range is widened with respect to the one observed for lighter payloads. The frequency range in which this high peak occurs is between 70 and 105 Hz, which is similar to the case of 450 and 600 lbs payloads. Payloads having their natural frequency above approximately 90 Hz do not introduce any significant peak-valley sequence in the shaking table transfer function. The frequency bandwidth of the shaking table for very rigid payload (natural frequency of 120 Hz and above) of 52 Hz, down from a value of 90 Hz for the bare table condition.

The peak-valley sequence due to the foundation flexibility is amplified by the presence of the payload having a natural frequency above 20 Hz. Furthermore this peak-valley sequence is amplified when the payload natural frequency is in the neighborhood of the foundation natural frequency.
Figure 9.13  Analytical Simulation of the Table Transfer Function with “optimal” Gain Setting for bare table condition and corresponding values of the Servo-Hydraulic Parameters: Effects of a 900 lbs SDOF Payload of Varying natural Frequency.
Figure 9.14  Contour Plot of the Analytical Simulation of the Table Transfer Function with "optimal" Gain Setting for bare table condition and corresponding values of the Servo-Hydraulic Parameters: Effects of a 900 lbs SDOF Payload of Varying natural Frequency.
9.3.6 SDOF Payload of 1500 lbs

The effects upon the shaking table transfer function of a 1500 lbs flexible SDOF payload with varying natural frequency are shown in Figs. 9.15 and 9.16. As for the previous cases, a deep valley occurs near the payload natural frequency.

Payloads with their natural frequency up to 10 Hz do not affect significantly the shaking table transfer function. Payloads having natural frequencies between 20 and 80 Hz produce a very high narrow peak above the payload natural frequency. The frequency range in which this high peak occurs is between 70 and 105 Hz, similar to the cases of 450, 600, and 900 lbs payload. Payloads having their natural frequency above approximately 90 Hz do not introduce any significant peak-valley sequence in the shaking table transfer function. The frequency bandwidth of the shaking table for very rigid payload (natural frequency of 120 Hz and above) is 48 Hz, down from a value of 90 Hz for the bare table condition.

The peak-valley sequence due to the foundation flexibility is amplified by the presence of the payload having a natural frequency above 20 Hz. This peak reaches a value of approximately 2.2 (up from a value of approximately 1.9 for the 900 lbs SDOF flexible payload, 1.7 for the 600 lbs SDOF flexible payload and approximately 1.4 for the 450, 300 and 150 lbs SDOF flexible payloads). Furthermore this peak-valley sequence is even more amplified when the payload natural frequency is in the neighborhood of the foundation natural frequency of 27 Hz.
Figure 9.15  Analytical Simulation of the Table Transfer Function with “optimal” Gain Setting for bare table condition and corresponding values of the Servo-Hydraulic Parameters: Effects of a 1500 lbs SDOF Payload of Varying natural Frequency.
Figure 9.16  Contour Plot of the Analytical Simulation of the Table Transfer Function with "optimal" Gain Setting for bare table condition and corresponding values of the Servo-Hydraulic Parameters: Effects of a 1500 lbs SDOF Payload of Varying natural Frequency.
9.4 CONCLUSIONS

In this chapter, we presented the analytical simulation of the sensitivity of the table transfer function to control gain parameters and to payload characteristics.

This analysis was performed in order to simulate as accurately as possible the actual behavior of the shaking table under realistic working conditions. The results obtained show that, under normal operating conditions, the magnitude of the shaking table transfer function remains quite close to unity over the frequency range between 0 and 70 Hz.

The 3d and contour plots of the magnitude of the shaking table transfer function presented in this chapter are a valuable tool to predict the sensitivity of the actual shaking table to control and payload parameters.
10.1 **INTRODUCTION**

This Chapter investigates the actual performance of the shaking table in terms of performance envelope and oil column frequency. The experimental results for these quantities are compared with the corresponding theoretical predictions. The good agreement between the actual performance and the target performance envelope used for design indicates that the as built Rice shaking table is able to perform up to the design specifications.
10.2 SHAKING TABLE PERFORMANCE ENVELOPE

Fig. 10.1 compares the tri-partite graph of the actual table performance envelope with that of the theoretical performance envelope obtained from the maximum actuator span, the maximum oil flow from the pump, and the maximum force that the actuator can apply to the slip table\(^1\). The graph indicates that the actual and theoretical performance envelope of the table are in good agreement. The small difference found between the actual and theoretical performance envelope are well within the precision limits of the instruments and method used to evaluate the table performance envelope. It is pointed out that no data were collected regarding the response of the table at frequencies below 1 Hz since it was not possible to monitor the slip table acceleration due to the sensitivity limits of the accelerometers\(^2\). Furthermore, it is noticed that, in accordance with the theoretical model, the table limits up to approximately 50 Hz are controlled by the flow capacity of the pump and above this frequency by the maximum actuator force applicable. To measure the table performance limits at a given frequency, a series of sinusoidal motions of increasing amplitude were imposed upon the table. During the application of these harmonic motions, we monitored the value of the oil pressure in the manifold (to check for limits in the oil flow from the pump) and the amplitude of the actual motion (to check for an actuator force limit). For sinusoidal motions up to 50 Hz, the slip table was able to reproduce entirely the imposed amplitude of motion, and the only limitation was due by a pressure drop in the manifold. For harmonic motions above 50 Hz the table found its limitation due to its

---

2. The accelerometers do not provide reliable monitoring of accelerations at frequencies below 1 Hz.
inability to reproduce the imposed amplitude of motion: an increase in amplitude of the commanded signal did not produce an increase in amplitude of the realized motion.

The maximum table span, maximum table velocity, and maximum table acceleration versus frequency are shown in Figs. 10.2, 10.3, and 10.4, respectively.
Figure 10.1 Tri-Partite Graph of Shaking Table Performance

(solid line = analytical performance envelope

dashed line = experimental performance envelope)
Figure 10.2  Maximum Table Span Versus Frequency

(solid line = experimental performance envelope
dashed line = analytical performance envelope)
Figure 10.3 Maximum Table Velocity Versus Frequency
(solid line = experimental performance envelope
dashed line = analytical performance envelope)
Figure 10.4  Maximum Table Acceleration Versus Frequency
(solid line = experimental performance envelope
dashed line = analytical performance envelope)
10.3 Experimental Determination of the Oil Column Frequency

To determine experimentally the oil column frequency of the shaking table we performed the following:

- an accelerometer was rigidly attached to the slip table;
- the slip table was hit with a heavy hammer;
- the power spectrum of the recorded acceleration time histories was computed.

These three steps were repeated for different operating conditions:

(1) with high pressure oil filling the actuator, and control loop activated.
(2) with high pressure oil filling the actuator and control loop cisactivated.
(3) with low pressure in the actuator and control loop disactivated.
(4) with no pressure in the actuator and control loop disactivated.

These experiments were conducted in order to check any influence of the oil pressure or of the electrical control loops upon the oil column frequency. The test with no pressure in the actuator was performed in order to better identify other sources of free vibration of the shaking table which are not related to the oil column resonance phenomenon.

The oil column circular frequency can be estimated theoretically using Eq. (3.56)

\[ \omega_{oil}^2 = \frac{k}{m_t} = \frac{4\beta A^3}{V m_t} \quad (3.56) \]

which gives the following oil column frequency:

\[ f_{oil} = \frac{1}{2\pi \sqrt{m_t}} \sqrt{\frac{k}{m_t}} = \frac{A}{\pi \sqrt{\beta V m_t}} \quad (10.1) \]

where:
• A is the effective cross sectional area of the actuator (\(= 12.73 \text{ in}^2\));

• \(\beta\) is the oil bulk modulus (\(= 100,000 \text{ psi}\));

• \(V\) is the volume of oil in the actuator (\(= 101.84 \text{ in}^3\));

• \(m_t\) is the total mass of the slip table (including the piston rod and the end swivel)

\[
m_t = \frac{1260 + 60}{386.4} = \frac{1320}{386.4} = 3.416 \text{ lbs-sec}^2/\text{in}
\]

Substituting the above parameter values in Eq. (10.1) one finds the following prediction for the oil column frequency:

\[
f_{oil} = \frac{12.73}{\pi} \cdot \sqrt{\frac{100,000}{101.84(3.416)}} = 68.7 \text{ Hz}
\]
10.3.1 Oil at High Pressure in the Actuator

Fig. 10.5 shows the power spectrum of the free vibration of the slip table hit by a heavy hammer as high pressure oil was filling the actuator and the control loop was activated in order to maintain the table at its zero position\(^1\). The plot indicates that the estimated oil column frequency is at approximately 65 Hz. This value is only slightly smaller (4\%) than the one predicted using Eq. (3.56). This close agreement confirms that the shaking table behaves as predicted during the design phase.

Fig. 10.6 shows the power spectrum of the free vibration response of the shaking table in the same conditions as in Fig. 10.5, except that in this case the control loop is deactivated. Inspection of Fig. 10.6 gives an estimated oil column frequency of 64 Hz. The difference between this estimation and the one obtained with the control loop activated is within the resolution limits of the estimation method and therefore it can be concluded that the control loop has no influence on the value of the oil column frequency.

Fig. 10.7 shows the power spectrum of the free vibration response of the shaking table in the same conditions as in Fig. 10.6, except that in this case only low pressure oil fills the actuator. The oil column frequency can be estimated at 67 Hz. This value is very close to the one obtained with high pressure oil filling the actuator and almost coincides with the value predicted by Eq. (3.56).

\(1\). The control gain setting used during this experiment is the optimal gain setting for bare table condition (see Section 6.5).
Figure 10.5  Power Spectrum of the free acceleration response of the slip table hit by a heavy hammer. As oil at high pressure fills the actuator and the electro-hydraulic control loop is activated.
Figure 10.6  Power Spectrum of the free acceleration response of the slip table hit by a heavy hammer. as oil at high pressure fills the actuator and the electro-hydraulic control loop is disactivated
Figure 10.7  Power Spectrum of the free acceleration response of the slip table hit by a heavy hammer. As oil at low pressure fills the actuator and the electro-hydraulic control loop is activated.
10.3.2 No Oil Pressure the Actuator

Figs. 10.8 and 10.9 show the power spectra of the slip table free vibrations when the actuator is not filled with pressurized oil. This experiment was performed in order to identify other sources of free vibrations that can occur in the slip table and that are not related to the oil column resonance phenomenon.

Fig. 10.8 shows the results obtained when the slip table was hit with the hammer. A sharp spectral peak is observed at about 27 Hz, which corresponds to the natural frequency of vibration of the flexible base\(^1\). Notice that the previous sharp resolution of the oil column frequency is lost and replaced by a wide peak centered at around 60 Hz.

Fig. 10.9 shows the power spectrum of the free vibration response of the slip table after the top concrete slab of the foundation mass was hit with the hammer. The resonant peak due to the foundation flexibility appears clearly at 27 Hz. We can also observe a very narrow peak at 60 Hz (probably due to line noise) and a broader peak centered at around 100 Hz whose origin has not been identified.

\(^1\) See Appendix A.
Figure 10.8  Power Spectrum of the free acceleration response of the slip table hit by a heavy hammer, as no oil flows to the actuator and the electro-hydraulic control loop is activated.
Figure 10.9  Power Spectrum of the free acceleration response of the slip table as the top concrete slab of the foundation mass is hit by a heavy hammer, and as no oil flows to the actuator and the electro-hydraulic control loop is activated.
10.4 CONCLUSIONS

The actual table performance presented in this Chapter confirms that the shaking table is performing according to its design criteria and specifications. Thus, this dynamic testing facility can be used reliably for the type of experiments it was designed for.
VOLUME III

EXPERIMENTAL / ANALYTICAL APPROACHES TO MODELING, CALIBRATING AND OPTIMIZING SHAKING TABLE DYNAMICS FOR STRUCTURAL DYNAMIC APPLICATIONS

by

TOMASO TROMBETTI

Houston, Texas
May 1998
PART III:

SHAKING TABLE TEST
CHAPTER 11
DYNAMIC TESTING OF LATERAL-TORSIONAL COUPLING IN SEISMIC ISOLATED STRUCTURES
11.1 INTRODUCTION

This Chapter presents the structural dynamic experiments that were performed on the Rice Shaking Table. The objective of these experiments was twofold: a) to validate the testing capabilities of the new shaking table itself; and b) to provide experimental support for a simplified theory which predicts the maximum rotation developed in seismic isolated buildings characterized by non-coincident centers of mass and rigidity, when subject to strong earthquake ground motions.

This chapter is divided in three parts. The simplified theory for predicting the torsional effects in eccentric seismic isolated buildings is described in the first part. The characteristics of the scaled building model and the testing procedures are described in the second part. Finally, in the third part, the actual experimental results are presented and compared with their theoretical counterparts.
11.2 ALATERAL-TORSIONAL COUPLING IN SEISMIC ISOLATED STRUCTURES

11.2.1 Introduction

The ratio between the maximum rotation and the maximum longitudinal displacement developed by an eccentric system (i.e., system with non-coincident centers of mass and rigidity) in free vibration is identified as a key factor controlling the dynamic response of such a system under forced excitation.

The importance of this factor in understanding the dynamic response behavior of eccentric systems is two-fold:

a) it can be used to assess for a given dynamic system its tendency to develop rotational deformations under external dynamic excitations (e.g., earthquakes, earth tremors, winds, machine vibrations, ...);

b) it can be used to compute an estimate of the maximum rotation developed by a given eccentric system under a specific excitation.

The simplified procedure to predict the maximum rotation is presented and its results are compared with those obtained from a full numerical simulation of the dynamic response.

11.2.2 Background

The dynamic behavior of eccentric structures has been the object of extensive past research work investigating the problem both in linear and non-linear domains. These studies pointed to the existence of a number of still unresolved problems hampering the full comprehension of torsional phenomena in structures. In some cases, even for fairly regular structures, it is difficult to define a reliable value for the eccentricity of the center
of mass with respect to the center of rigidity (Goel and Chopra, 1990).

The research performed up to date can be subdivided into three categories:

(1) Investigation of the elastic response of three-dimensional laterally-torsionally coupled systems (Boroschek and Mahin, 1991; Chandler and Hutchinson, 1987; Lu and Hall, 1992; Constantinou et al., 1991 and 1993; Papageorgiou and Lin, 1989). This group of studies focuses on the fundamentals of the dynamic behavior of eccentric systems and on the identification of the key parameters governing the dynamic response of such systems.

(2) Investigation of the inelastic dynamic response of three-dimensional eccentric structural systems (De Stefano, Faella and Ramasco, 1993; Kelly, 1990 and 1993; Constantinou et al., 1993; Goel and Chopra, 1990; Tso and Ying, 1990). This body of research focuses mainly on the importance of the localization of the instantaneous center of rigidity and its dependence on the deformation state of the inelastic eccentric system.

(3) Analysis and evaluation of code provisions for torsional effects. This group of studies mainly focused on the determination of simplified methods of torsional analysis (Lu and Hall, 1992; Chandler and Hutchinson, 1987; Goel and Chopra, 1993).

For seismic isolated structures, the analysis of their torsional behavior can be fairly simplified. In fact:

- The most common seismic isolators are cylindrical elements whose lateral rigidity is generally well known and independent of the direction of deformation.

- It has been shown (Kelly, 1993) that the dynamic behavior of seismic isolators can
be fairly well captured through a simplified linear analysis (in spite of their specific non-linear characteristics).

- Under seismic excitation, the deformations of a seismic isolated structure are localized mainly in the seismic isolators and are only marginally influenced by the interaction of the superstructure (Nagarajaiah, Reinhorn and Constantinou, 1993; Kelly, 1993).

For all of the above reasons, the analysis of seismic isolated structures can be reduced to that of a one story three-dimensional linear structural system. Given that the maximum lateral deformation is the fundamental design parameter in various design codes, it is essential to develop a simplified method for the determination of the maximum incremental lateral deformation due to torsional effects.

A recent paper (Nagarajaiah and Constantinou, 1993) confirmed the small influence of the superstructure upon the maximum deformation of base isolators and showed the effects upon the dynamic response of base isolated systems of a few parameters characterizing the dynamic behavior of eccentric structures. These results, though, are presented for a specific structural system and are not developed into a general purpose theory. The simplified approach proposed here is applicable to a wide range of seismic isolated systems and enables to understand qualitatively and estimate quantitatively the lateral-torsional response of specific systems. The results of the present work, based upon the study of the free vibration response of eccentric systems, lead to a compact algebraic formulation which allows to quantify the influence of various key structural parameters upon the maximum rotation and translation response parameters of the base isolated structure.
11.2.3 The Eccentric Dynamic System and its Modes of Vibration

A three-dimensional one story linear dynamic system characterized by non-coincident centers of mass and stiffness is considered. The three degrees of freedom used in the formulation of the equation of motion are as shown Fig. 11.1. Notice that these degree of freedom are those of the center of mass.

![Diagram of the coordinate system](image)

Figure 11.1 The coordinate system

Under the following hypotheses.

- the lateral stiffness of each base isolator does not depend on the direction of deformation:

- the rotation $u_\theta$ developed under seismic excitation are limited to values such that

$$ u_\theta \approx \sin(u_\theta) \approx \tan(u_\theta) $$

the displacements of the dynamic system under consideration are governed by the following set of simultaneous differential equations (Bacci, Ceccoli and Trombetti, 1994):
where:

\[ u_x(t), u_y(t), u_\theta(t) \] = displacements along the x- and y- directions and rotation along the z-axis, respectively:

\[ m \] = total mass of the super-structure (i.e., total mass resting over the base isolators):

\[ I_p \] = polar mass moment of inertia of the superstructure:

\[ \rho = \frac{I_p}{m} \] = radius of gyration of the superstructure:

\[ [C] \] = damping matrix:

\[ \omega_L = \sqrt{\frac{k}{m}} \] = uncoupled lateral (longitudinal or transversal) natural circular frequency of vibration:\n
\[ \omega_\theta = \sqrt{\frac{k_{\theta\theta}}{I_p}} \] = uncoupled rotational natural circular frequency:

\[ \gamma = \frac{\omega_\theta}{\omega_L} = \sqrt{\frac{k_{\theta\theta}}{\rho^2 \cdot k}} \] = ratio of the rotational to the longitudinal uncoupled natural

---

1. \( \omega_L \) is the longitudinal natural circular frequency of a structure similar to the one considered, but that has coincident centers of mass and rigidity.
circular frequencies$^1$:

\[ k_{\theta \theta} = \text{uncoupled rotational stiffness of the base isolation system:} \]

\[ k = \text{uncoupled lateral stiffness of the base isolation system:} \]

\[ D_e = \rho \sqrt{12} = \text{"equivalent diagonal" of the system}^2: \]

\[ E_x \cdot E_y = \text{eccentricity of the center of stiffness with respect to the center of mass in the x- and y- directions, respectively:} \]

\[ e_x = \frac{E_x}{D_e}, \quad e_y = \frac{E_y}{D_e} = \text{relative eccentricities in the x- and y- directions, respectively:} \]

\[ p_x(t), \quad p_y(t), \quad p_\theta(t) = \text{respectively: external forces applied along the "x", "y" directions and external couples.} \]

Given the linear nature of the problem, it is possible, for undamped or classically damped systems, to uncouple the equations of motion (11.1) through modal decomposition. The solution of the eigenvalue problem (i.e., undamped free vibrations) gives the following results for the natural circular frequencies$^3$:

\[ \Omega_1 = \left( \frac{\omega_1}{\omega_L} \right)^2 = \frac{1}{2} \cdot \left\{ 1 + \gamma^2 - \sqrt{\left( \gamma^2 - 1 \right)^2 + 48 \cdot e^2} \right\} \quad (11.2) \]

---

1. This ratio, generally larger than one, tends to one as the number of seismic isolators increases.

2. The "equivalent diagonal" is a useful standard measure for the planar dimensions of the system. It is equal to $\sqrt{12}$ times the radius of inertia of the system. This length, for continuous systems of rectangular or square shape coincides with the actual diagonal length of the system.

3. The undamped natural circular frequencies are normalized as defined in Eq. (11.2) a, b. and c. for convenience purposes in the analysis of eccentric dynamic systems that follows.
\[ \Omega_2 = \left( \frac{\omega_2}{\omega_L} \right)^2 = 1 \quad \text{(11.2) b} \]

\[ \Omega_3 = \left( \frac{\omega_3}{\omega_L} \right)^2 = \frac{1}{2} \left\{ 1 + \gamma^2 + \sqrt{(\gamma^2 - 1)^2 + 48 \cdot e^2} \right\} \quad \text{(11.2) c} \]

where

\[ e^2 = e_x^2 + e_y^2. \]

For small values of \( \gamma \) (\( \gamma^2 = 1 \)) which is the case for most seismic isolated structures and small \( e \), Eqs. (11.2) a, b and c can be approximated using \( \sqrt{a^2 + b} \equiv a + \frac{b}{2 \cdot a} \), into the following expressions:

\[ \Omega_1 \equiv 1 - \frac{12 \cdot e^2}{\gamma^2 - 1} \quad \text{(11.3) a} \]

\[ \Omega_2 = 1 \quad \text{(11.3) b} \]

\[ \Omega_3 \equiv \gamma^2 + \frac{12 \cdot e^2}{\gamma^2 - 1} \quad \text{(11.3) c} \]

Inspection of Eqs. (11.3) a, b, d. and numerical evaluation of Eqs. (11.2) a, b, c for the most common values of the \( \gamma \) and \( e \) parameters (namely, \( 1.1 \leq \gamma \leq 1.4 \) and \( 0.02 \leq e \leq 0.22 \)) show that \( \Omega_1 \) is generally close to one, while \( \Omega_3 \) can be quite higher. These considerations are clearly illustrated in Fig. 11.2, where the values of \( \Omega_1 \), \( \Omega_2 \) and \( \Omega_3 \) are plotted versus parameters \( e \) and \( \gamma \). Thus, it appears that the first and second natural modes of vibration of the system have closely spaced natural circular frequencies. It is
important to point out that $\omega_2$ corresponds to the uncoupled longitudinal natural circular frequency $\omega_L$, and therefore we have that $\omega_1 \equiv \omega_2 = \omega_L$. As a result, it is anticipated that the longitudinal natural circular frequency of vibration of the uncoupled system is an important parameter characterizing the eccentric system.

![Diagram](image)

Figure 11.2 Values of $\Omega_1$, $\Omega_2$, $\Omega_3$ versus "e" and "y"

The solution of the eigenvalue undamped free vibration problem also gives the following mode shapes:

$$\{\Phi_1\} = \begin{bmatrix} e_y \sqrt{12} \\ l - \Omega_1 \\ e_x \sqrt{12} \\ \Omega_1 - l \\ 1 \end{bmatrix}$$

(11.4) a
\[\{\Phi_2\} = \begin{bmatrix} e_x \\ e_y \\ 1 \\ 0 \end{bmatrix}\]  \hspace{1cm} (11.4) b

\[\{\Phi_3\} = \begin{bmatrix} e_y \cdot \sqrt{\frac{1}{12}} \\ \frac{1}{1 - \Omega_3} \\ e_x \cdot \sqrt{\frac{1}{12}} \\ \frac{\Omega_3 - 1}{1 - \Omega_3} \\ 1 \end{bmatrix}\]  \hspace{1cm} (11.4) c

Inspection of the analytical expressions for the three mode shapes in Eqs. (11.4) a. b. c show that in the first and third modes, the translations are coupled with the rotations, while the second mode shape is purely translational in the direction defined by the centers of mass and stiffness.
11.2.4 Free Vibrations of the Undamped System

The free vibration response of undamped eccentric system (from a given initial deformation along the y-direction) is given by the following expressions:

\[ u_y = a \cdot \frac{e_x^2}{e^2} \left\{ \frac{(1 - \Omega_3)}{(\Omega_1 - \Omega_3)} \cdot \cos(\omega_1 t) + \frac{e_y^2}{e_x^2} \cdot \cos(\omega_2 t) + \frac{\Omega_1 - 1}{(\Omega_1 - \Omega_3)} \cdot \cos(\omega_3 t) \right\} \] (11.5) a

\[ u_x = a \cdot \frac{e_x}{e^2} \cdot \frac{e_y}{e^2} \left\{ \frac{\Omega_3 - 1}{(\Omega_1 - \Omega_3)} \cdot \cos(\omega_1 t) + (1 - \Omega_1) \cdot \cos(\omega_2 t) + \frac{\Omega_1 - 1}{(\Omega_1 - \Omega_3)} \cdot \cos(\omega_3 t) \right\} \] (11.5) b

\[ u_\theta = \frac{a}{\sqrt{12} \cdot \rho} \cdot \frac{e_x}{e^2} \cdot \frac{\Omega_3 - 1}{(\Omega_3 - \Omega_1)} \cdot \left\{ \cos(\omega_1 t) - \cos(\omega_3 t) \right\} \] (11.5) c

Where “a” denotes the initial imposed translation in the y-direction.

Given this initial deformation state, the x-axis will be referred to as the transversal direction, while the y-axis will be referred to as the longitudinal direction.

Above it was shown that for the most common values of the “e” and “γ” parameters, the normalized quantity Ω_1 is close to unity. Therefore, the following approximation can be made:

\[ \frac{\Omega_3 - 1}{\Omega_3 - \Omega_1} \approx 1 \] (11.6)

Substitution of the above approximation in Eqs. (11.5) a, b, c, leads to the following expressions for the three components of the free vibration response:

\[ u_y = a \cdot \frac{e_x^2}{e^2} \left\{ \cos(\omega_1 t) + \frac{e_y^2}{e_x^2} \cdot \cos(\omega_2 t) + A_3 \cdot \cos(\omega_3 t) \right\} \] (11.7) a

\[ u_x = a \cdot \frac{e_x}{e^2} \cdot \frac{e_y}{e^2} \cdot \left\{ \cos(\omega_1 t) - \cos(\omega_3 t) \right\} \] (11.7) b
\[ u_\theta = a \cdot \frac{1}{\sqrt{12} \cdot \rho} \cdot \frac{e^2}{e^2} \cdot (\Omega_1 - 1) \cdot \{ \cos(\omega_1 t) - \cos(\omega_3 t) \} \quad (11.7) \ c \]

where

\[ A_3 = \frac{(\Omega_1 - 1)}{(\Omega_1 - \Omega_3)} \]

is a quantity that, for most common values of the "e" and "\gamma" parameters, is smaller than one as shown in Fig. 11.3.

Inspection of the expressions in Eqs. (11.7) a, b, c leads to the following observations:

- The longitudinal displacement, transversal displacement and rotation are linearly proportional to the initial deformation a:

- In order to have a non-zero displacement response of the center of mass in the transversal direction (perpendicular to the direction of initial deformation), the structure must have eccentricities in both the x- and y-directions. A zero eccentricity in any of the x- and y-directions will give a zero transversal displacement.

Figure 11.3  Values of \( A_3 \) versus the "e" and "\gamma" parameters
through the term 
\[ \mathbf{e}_x \cdot \mathbf{e}_y \] in Eq. (11.5) c:

- For a given value of transversal eccentricity \( e_x \), the system develops the largest rotation when the longitudinal eccentricity is null (\( e_y = 0 \)). This can be seen by looking at the factor
\[ \frac{e_x}{e_x^2 + e_y^2} \]
which controls the amplitude of the torsional response \( u_\theta(t) \):

- The longitudinal displacement, transversal displacement and rotation responses consist of the sum of trigonometric functions of various amplitudes and circular frequencies. The sum of two trigonometric functions of similar amplitudes and different frequencies produce a harmonic with modulated amplitude known as the beating phenomenon as described in Section 11.2.17.

### 11.2.5 Amplitude Modulation in the Free Vibration Response of the Undamped Systems - Beating Phenomenon

Analysis of Eq. (11.7) a, b, and c in the light of the observations reported in section 11.2.17, leads to the following observations about the rotational, transversal and longitudinal components of the free vibration of the system.

**Rotation Response** \( u_\theta(t) \)

The algebraic expressions for the rotation \( u_\theta(t) \) in Eqs. (11.5) c and (11.7) c are given as the sum of two harmonics of equal amplitude, which results in an harmonic (fast mode) whose amplitude (envelope) is modulated by a full sine wave (slow mode). The fast mode
has the circular frequency $\omega_{h\theta} = \frac{\omega_1 + \omega_3}{2}$, while the slow mode has the circular frequency $\omega_{m\theta} = \frac{\omega_1 - \omega_3}{2}$. These two circular frequencies give the following periods of vibration: $T_{h\theta} = \frac{2 \cdot \pi}{\omega_{h\theta}}$ and $T_{m\theta} = \frac{2 \cdot \pi}{\omega_{m\theta}}$.

**Transversal Displacement Response $u_x(t)$**

The algebraic expression for the transversal displacement response, $u_x(t)$, is given as the sum of three harmonics. Two of them (of circular frequencies equal to $\omega_1$ and $\omega_2$) have a unit amplitude, while the third one (of circular frequency equal to $\omega_3$) has an amplitude equal to $A_3$. As shown previously, in general $A_3$ assumes values well below unity. Neglecting the effects of this third harmonic (as its contribution to the total transversal motion can be seen as a small perturbation), the transversal displacement $u_x(t)$ is given by the sum of two harmonics of equal amplitude. Again, this results in a harmonic (fast mode) with its amplitude modulated as a full sine wave (slow mode). In this case, the circular frequency of the fast mode is equal to $\omega_h = \frac{\omega_1 + \omega_2}{2}$, while that of the slow mode is $\omega_m = \frac{\omega_1 - \omega_2}{2}$. The corresponding periods of vibration for the fast and slow modes are:

$T_h = \frac{2 \cdot \pi}{\omega_h}$ and $T_m = \frac{2 \cdot \pi}{\omega_m}$, respectively. The example time history of the transversal displacement response $u_x(t)$ shown in Fig. 11.4 clearly illustrates the amplitude modula-
tion (beating phenomenon) and the natural periods $T_h$ and $T_m$.

**Longitudinal Displacement Response $u_y(t)$**

The algebraic expression for the longitudinal displacement response $u_y(t)$ is very similar to that for the transversal displacement response $u_x(t)$. The only difference resides in the fact that the amplitude of the second harmonic (of circular frequency $\omega_2$) is in this case equal to $\left(\frac{e_y}{e_x}\right)^2$ instead of unity. Thus, only in the case of equal longitudinal and transversal eccentricities ($E_x = E_y$ or $e_x = e_y$), the first two harmonics have equal amplitude and their sum results in a fast mode whose envelope (slow mode) is modulated as a full sine wave, as was the case for the rotation and transversal displacement responses. When the

![Graph](image)

**Figure 11.4 Transversal displacements $u_x$**

(Undamped free vibrations of a structure having $\gamma = 1.2$, $D_e = 28m$.

$e_x = e_y = 0.1$)
longitudinal eccentricity $e_y$ is smaller than the transversal eccentricity $e_x$. Or, in the limit case when $e_y = 0$, the longitudinal displacement response is composed of a fast mode whose envelope is only partially modulated (for more details, see Section 11.2.17). In other words, in this case the slow mode is not a full sine wave. However, it is still a periodic function, but it does not have nodes of zero amplitude. Fig. 11.5 shows the longitudinal displacement time history of an eccentric system having $e_y = e_x$ and for which the envelope is fully modulated.

$$T_m$$

$${ }_{1}$$

$${ }_{0.5}$$

$${ }_{-0.5}$$

$${ }_{-1}$$

$${ }_{T_h}$$

Figure 11.5 Longitudinal displacement response $u_y(t)$

(Undamped free vibration of a structure having $\gamma = 1.2, D_e = 28m, e_x = e_y = 0.1$)

*Considerations on the values of the natural periods of the fast modes*

The fast mode of the rotation response of eccentric systems in undamped free vibration has a circular frequency referred to as $\omega_{h0}$. The fast modes of the longitudinal and trans-
versal displacement responses have a common circular frequency referred to as $\omega_h$.

Recalling the definition of $\omega_h$ and the observations on the values of $\omega_1$, $\omega_2$ and $\omega_3$ made in Section 11.2.3, it follows that $\omega_h = \omega_L$ and that $\omega_{h \theta}$ is always larger than $\omega_h$. $\omega_{h \theta} > \omega_h$. Observation of the results plotted in Fig. 11.2 indicates that in the majority of cases $\omega_h < \omega_{h \theta} < 1.5 \omega_h$. Consequently, the fast modes of the longitudinal, transversal, and rotational responses have circular frequencies not too far from $\omega_L$.

Considerations on the values of the natural periods of the slow modes

In the case of equal longitudinal and transversal eccentricities ($e_y = e_x$), the circular frequencies of the displacement and rotation slow modes ($\omega_m$ and $\omega_{m \theta}$) have the following values: $\omega_m = \frac{\omega_1 - \omega_2}{2}$ and $\omega_{m \theta} = \frac{\omega_1 - \omega_3}{2}$, respectively.

It has already been shown that $\omega_1 = \omega_2 = \omega_L$, while $\omega_3$ assumes a relatively larger value. Therefore the longitudinal and transversal displacement responses have a slow mode circular frequency which is much smaller than that of the corresponding fast mode ($\omega_m \ll \omega_h$). This explains why, in the majority of cases, the dynamic longitudinal and transversal responses go through many fast mode oscillations to complete one modulation cycle of their envelope. This fact appears clearly in Figs. 11.4 and 11.5 where approximately nineteen transversal and longitudinal oscillations are necessary to develop a complete envelope modulation.
On the other hand, the rotation response has a slow mode circular frequency much larger than that of the longitudinal and transversal displacement responses. i.e., \( \omega_{m\theta} \gg \omega_m \). As already observed, the circular frequencies of the rotation, longitudinal and transversal displacement responses are not far apart (\( \omega_h < \omega_{h\theta} < 1.5 \omega_h \)). This explains why, in the majority of cases, it takes only a few cycles of the fast mode of the rotation response to go through a complete modulation cycle of its envelope. Thus, the modulation of the envelope (slow mode) is much faster for the rotational response than for the longitudinal and transversal displacement responses. This fact will play a key role in understanding the dynamic behavior of eccentric systems.

11.2.6 Longitudinal and Transversal Displacements in Free Vibration of Undamped Eccentric Systems

Fig. 11.6 shows the time evolution of the longitudinal displacement \( u_y(t) \) versus the transversal displacement \( u_x(t) \) for an undamped eccentric system in free vibration. These plots represent the exact solutions given in Eqs. (11.5) a and b. It ise observed that initially the system oscillates mainly in the direction of the initial displacement (i.e., longitudinal), and then the system starts to follow an ellipsoidal path that, with increasing number of cycles, elongates more and more in the transversal direction while shrinking more and more in the longitudinal direction. At some point of the response process, the longitudinal and transversal axes of the ellipsoidal path are equal and the structure center of mass follows a path which is almost circular. This evolution of the response continues until the system oscillates almost entirely along the transversal direction. At this point the oscillation of the sys-
term will evolve back towards the longitudinal direction. Given the undamped nature of the
dynamic system assumed here, this process will repeat itself indefinitely. This description
of the longitudinal and transversal displacements explains how the amplitude modulation
(referred to as slow mode or beating phenomenon) observed in the longitudinal and trans-
versal displacement responses corresponds to the cyclic change in direction of the free
oscillations (from longitudinal to transversal and vice versa). Fig. 11.6 shows how the
change in direction of the free oscillations require as many oscillation cycles as already
noted in Section 11.2.5 when analyzing the circular frequencies of the longitudinal and
transversal slow modes.
Figure 11.6  Time evolution of $u_y$ versus $u_x$

(Undamped free vibration of a structure having $\gamma = 1.7$. $D_c = 28 m$.

$e_x = e_y = 0.05$)
11.2.7 Rotation in the Free Vibration of Undamped Eccentric Systems

*Time evolution of rotation response*

Figs. 11.7a and b. show the time evolution of the rotation $u_\theta(t)$ versus the longitudinal displacement $u_y(t)$ of an undamped eccentric system in free vibration. These results were obtained from the exact solutions given in Eqs. (11.5) a and c. A close inspection of these

![Diagram of rotation evolution](image)

Figure 11.7 Time evolution of $u_\theta$ versus $u_y$

(Undamped free vibrations of a structure having $\gamma = 1.7$, $D_e = 28m$.

$e_x = 0.05$, $e_y = 0$)
Figure 11.8  Time evolution of $u_\theta$ versus $u_y$

(Undamped free vibration of a structure having $\gamma = 1.2 \cdot D_c = 28 \text{m}$.

$e_\chi = 0.05, e_\psi = 0$)

plots reveals that for every longitudinal cycle of vibration, the rotation response of the eccentric dynamic system reaches a maximum value $u_{\theta, \text{max}}$ which occurs almost simultaneously with the maximum longitudinal displacement response $u_{y, \text{max}}$. Comparing Fig. 11.7, which corresponds to an eccentric system with $\gamma = 1.7$, with Fig. 11.8, corresponding to an eccentric system with $\gamma = 1.2$, it can observed that eccentric structures charac-
terized by large values of $\gamma$ are faster, in terms of number of longitudinal rotations, in developing cycles close to the maximum one, $u_{\theta, \text{max}}$, than structures with lower values of $\gamma$.

*Observations on the values of the period ratio* $\frac{T_{m\theta}}{T_h}$

Fig. 11.9 shows the ratio between the natural period of the slow mode (or envelope modulation) of the rotation, $T_{m\theta}$, and the natural period of the fast mode of the longitudinal vibration, $T_h$.

Recalling that:

- $\omega_h \equiv \omega_L$, and therefore $T_h \equiv T_L$;
- $\frac{T_{m\theta}}{T_L} = \frac{\omega_L}{\omega_{m\theta}}$;
- $\left(\frac{\omega_{m\theta}}{\omega_L}\right)^2 = \Omega_1 + \Omega_3 - 2 \cdot \sqrt{\Omega_1 \cdot \Omega_3}$;

the value of the period ratio $\frac{T_{m\theta}}{T_h}$ can be approximated as follows:

$$\frac{T_{m\theta}}{T_h} \approx \frac{T_{m\theta}}{T_L} = \frac{\omega_L}{\omega_{m\theta}} = (\Omega_1 + \Omega_3 - 2 \cdot \sqrt{\Omega_1 \cdot \Omega_3})^{-\frac{1}{2}}$$ (11.8)

Eqs. (11.2) a, b and c provide a closed-form (exact) solution for $\Omega_1$, $\Omega_2$, and $\Omega_3$ and therefore appropriate closed-form solution for $\frac{T_{m\theta}}{T_L}$. 
Fig. 11.9 plots the exact closed form solution of $\frac{T_{m\theta}}{T_L}$ and indicates that, for the most common values of the "e" and "γ" parameters, the natural period of the slow mode of the rotation response, $T_{m\theta}$, is close to that of the fast mode of the longitudinal response, $T_L$. Only for a restricted range of values of the "e" and "γ" parameters, namely for small values of "e" and "γ" $T_{m\theta}$ reaches values that are approximately four times $T_L$. It is therefore clear that, for the majority of real eccentric structures, a full modulation of the amplitude of the rotation response is developed at every cycle of longitudinal oscillatory response. Only structures characterized by small values of "e" and "γ" require approximately four cycles of longitudinal vibration in order to develop a full modulation on the amplitude of the rotation response. Nonetheless, it is recalled that the slow mode of a beating phenomenon reaches its maximum (envelope maximum) twice during one period of

![Graph](image_url)

Figure 11.9 $\frac{T_{m\theta}}{T_L}$ versus the parameters "e" and "γ"
the slow mode. As a result, structures characterized by small "e" and "γ" parameters
develop also rotations close to the maximum one $u_{0, \text{max}}$ at almost every longitudinal
cycle. This result gives analytical support to the observations made in Section 11.2.7 based
on numerical simulation results.

In summary, the free vibration rotation response of undamped eccentric systems is charac-
terized by a fast evolution of its slow mode. This implies that a rotation close to the maxi-
mum one, $u_{0, \text{max}}$, is developed for almost every cycle of longitudinal response and almost
simultaneously with the maximum longitudinal response. On the contrary, longitudinal
and transversal displacement responses display a slower evolution of their slow mode and
their relative maxima occur at very different times. Thus, no useful behavior pattern
between longitudinal and transversal vibration responses exists.

11.2.8 Free Vibration Rotation Response of Damped Eccentric Systems

For damped eccentric systems, the maximum displacement and rotation responses cannot
be expressed in simple closed-form anymore. For this reason the investigation on the lon-
gitudinal, transversal and rotational components of the response of damped eccentric sys-
tems was carried out first through numerical simulation and then through shaking table
experimentation. Both numerical and experimental results obtained indicate that damped
eccentric structures follow behavioral patterns similar to those found for undamped eccen-
tric structures. Namely, a rotation close to the maximum one is reached when the longitu-
dinal response of the system is at one of its maxima. In the numerical simulation, damping
effects were modeled using Rayleigh's damping matrices, and caused the rotation
response to decrease in time. The reduction in the maximum rotation response (as compared to the undamped case) developed by damped eccentric systems in free vibration is larger in structures having small values of the parameter "$\gamma$". This is due to the fact that for structures characterized by small values of "$\gamma$", it takes a relatively larger number of longitudinal oscillations to develop the maximum rotation $u_{\theta,\text{max}}$. This is clearly illustrated in Fig. 11.10, where damped and undamped free vibrations of structures characterized by $\gamma = 1.2$ and $\gamma = 1.7$, respectively, are compared. From this figure, it is also noticed that the response behavior of eccentric damped systems exhibit the same trends identified for undamped systems. The reduction in the amplitude of the maximum rotation developed due to damping is estimated to be in the range from 20 to 40%.
Figure 11.10 Time evolution of rotation $u_{\theta}$ versus $u_{\gamma}$: free vibrations of an undamped (dashed line) and 5% damped (solid line) structures

($D_e = 28\text{m}$, $e_x = 0.05$, $e_y = 0$)
11.2.9 Ratio Between Maximum Rotation and Maximum Longitudinal Displacement in Free Vibration of Damped and Undamped Eccentric Systems

In the previous sections, it was found that almost once per longitudinal oscillation, the system develops simultaneously a rotation and longitudinal response close to their respective maxima. This behavior suggested that the maximum rotation and longitudinal displacement responses might be strongly correlated representing a basic characteristic of the dynamic behavior of eccentric structures. It was thus decided to investigate further the value of the ratio between the maximum rotation and maximum longitudinal displacement developed by eccentric structures in free vibration from a given initial deformation. From here on, this ratio is expressed as

$$\left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right)_{fv} \quad (11.9)$$

where:

$$\theta_{\text{max}} = \text{maximum rotation developed by the structure (previously indicated as } u_{\theta_{\text{max}}})$$

$$Y_{\text{max}} = \text{maximum longitudinal displacement (previously indicated as } u_{Y_{\text{max}}}).$$

For undamped eccentric structures, the ratio $$\left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right)_{fv}$$ can be expressed in closed-form using Eqs. (11.5) a, and c. For the special case of null longitudinal (y-direction) eccentricity of particular interest since it is the condition for which the system develops the maximum rotation, see Section 11.2.4. the ratio $$\left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right)_{fv}$$ can be expressed as:

$$\left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right)_{fv} = \left| \frac{u_{\theta_{\text{max}}}}{a} \right| = \frac{2}{\rho \cdot \sqrt[3]{12}} \cdot e \cdot (\Omega_1 - 1) \cdot \frac{\Omega_3 - 1}{\Omega_3 - \Omega_1} = \frac{4 \cdot e \cdot \sqrt{3}}{\rho \cdot \sqrt{(\gamma^2 - 1)^2 + 48 \cdot e^2}} \quad (11.10)$$
where \( a \) = the initial longitudinal deformation, and \( e \) = the relative transversal eccentricity

\[
e = \frac{E_x}{D_c}.
\]

From the combined analysis of Eq. (11.10) and its plot shown in Fig. 11.11, it can be observed that for the same initial longitudinal deformation, eccentric systems can develop very different values of maximum rotation, depending on the system characteristics. Eq. (11.10), indicates that the maximum rotation response is inversely proportional to the radius of gyration, \( \rho \), of the structure. Furthermore, Fig. 11.11 suggests that the maximum rotation developed is larger for systems having values of \( \gamma \) close to unity (i.e., systems with similar uncoupled lateral and torsional periods of vibration). Fig. 11.11 suggests also

Figure 11.11 \( \left| \frac{u_{\theta_{\max}}}{a} \right| \) for undamped eccentric structures with \( \rho = 1 \) m \((e_y = 0)\)
that the ratio \( \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}}_{\text{fv}} \right) \) smoothly increases with increasing value of the relative transversal eccentricity "e". Overall, it is observed that in the system parameter space there is a smooth transition from systems developing a small maximum rotation response to those developing a large rotation response for a given initial deformation.

A numerical simulation was carried out for damped eccentric structures. The results obtained for the special case \( e_y = 0 \) are presented in Fig. 11.12 and show that the depen-

Figure 11.12 \( \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}}_{\text{fv}} \right) \) for 5% damped eccentric structures with various values of 

\[ D_e \, (e_x = 0) \]
dence of \( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \) upon structural parameters is similar to that found for undamped structures. As expected, the value of the maximum rotation for a given initial displacement diminishes with increasing values of damping. Fig. 11.12 shows the results obtained for structures with 5% damping.

### 11.2.10 Ratio Between Maximum Rotation and Maximum Longitudinal Displacement in Forced Vibration of Damped and Undamped Eccentric Systems

The study of the free vibrations of eccentric systems has shown that the ratio between maximum rotation and maximum longitudinal displacement response is an important parameter characterizing the dynamic behavior of such systems. These interesting results suggested to investigate the values taken by this response ratio under forced vibration. Thus a comprehensive numerical investigation was performed in order to analyze the value taken by the ratio \( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \) in eccentric systems subjected to various external dynamic excitations. This numerical investigation has been carried out not only for a variety of external inputs, but also for a wide range of structural parameters. In fact, this investigation considered structures in the following parameter range:

- \( 0.02 \cdot D_e \leq e \leq 0.22 \cdot D_e \) (with \( e = e_x \cdot e_y = 0 \)):
- \( 1,1 \leq \gamma \leq 2 \):
- \( 28 \text{m} \leq D_e \leq 70 \text{m} \):
- damping ratio comprised between 5 and 15% of the critical value.

Section 11.2.12 will explain why other parameters were not taken into consideration as
they do not influence the response ratio under consideration.

\[
\frac{\theta_{\text{max}}}{Y_{\text{max}}} \quad \text{Rad/m}
\]

\[
\begin{align*}
(1) & \quad \text{Parkfield} \\
(2) & \quad \text{El Centro} \\
(3) & \quad \text{Friuli Breginj} \\
(4) & \quad \text{Montenegro Bar} \\
(5) & \quad \text{Montenegro Petrovac} \\
(6) & \quad \text{Mexico City (\(\gamma = 1.2\), \(D_e = 28\)m)}
\end{align*}
\]

Figure 11.13: \(\frac{\theta_{\text{max}}}{Y_{\text{max}}}\) versus “\(e\)” for structures subjected to the following earthquakes:

Values of \(\frac{\theta_{\text{max}}}{Y_{\text{max}}}\) under seismic excitation

Numerical simulation of the dynamic response of eccentric structures subjected to the following earthquake ground motions were performed:

Parkfield 1966, El Centro 1940, Montenegro Petrovac 1979, Montenegro Bar 1979, Friuli Breginj 1976 and Mexico City 1985. The results have shown that, with the exception of
Friuli Breginj, the value of the response ratio \( \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right) \) obtained for earthquake excitation always remains close to the value obtained for free vibration condition:

\[
\left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right)_{\text{eqke - exc}} \equiv \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right)_{fv} \tag{11.11}
\]

Some of the numerical results obtained are represented graphically in Figs. 11.13 through 11.15 for a damping ratio of 5, 10, and 15%, respectively. Notice that, in Figs. 11.13 through 11.15 the average value of the ratio \( \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right) \) over the selected set of earthquakes referred to as \( \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right)_{\text{average}} \) is close to the value \( \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right)_{fv} \) obtained for free vibration conditions.

*Values of \( \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right) \) under harmonic excitations*

In the previous section, it has been shown that, under seismic excitation, the value of the response ratio \( \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right) \) varies little from the corresponding values under free vibration. Such variation obviously depends upon the specific characteristics of the external forcing functions (e.g., frequency content). In order to better understand the dependence of \( \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right) \) on the frequency content of the dynamic excitation, the effects of simple excitations made up of two harmonics has been investigated. A structure characterized by a null longitudinal eccentricity \( (e_y = 0) \) and by a value of \( \gamma = 1.2 \) (most common value) was
Figure 11.14 $\frac{\theta_{\text{max}}}{Y_{\text{max}}}$ versus "e" for structures subjected to the following earthquakes:

1. Parkfield. 2. El Centro. 3. Friuli Breginj. 5. Montenegro Petrovac. 6. Mexico City ($\gamma = 1.2$, $D_c = 28$ m)

excited with a driving force $F(t)$ composed of a first harmonic with a natural circular frequency $\omega_L$ (= uncoupled lateral natural frequency) and amplitude $A_L$, and a second harmonic with natural circular frequency $\omega_r = \gamma \cdot \omega_L$ (= uncoupled torsional frequency) and amplitude $A_\theta$. The first harmonic was introduced in order to excite the longitudinal displacement response of the system, while the second harmonic was chosen to excite the torsional response of the system. Thus, the external driving force has the following form:

$$F(t) = A_L \cdot \cos(\omega_L \cdot t) + A_\theta \cdot \cos(\omega_r \cdot t) \quad (11.12)$$
Figure 11.15 \( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \) versus "e" for structures subjected to the following earthquakes:

(1) Parkfield, (2) El Centro, (3) Friuli Breginj, (4) Montenegro Bar, (5) Montenegro Petrovac, (6) Mexico City. (\( \gamma = 1.2, \ D_e = 28 \text{m} \))

or, further

\[
F(t) = A_L \cdot \cos(\omega_L \cdot t) + A_\theta \cdot \cos(\gamma \cdot \omega_L \cdot t)
\]

(11.12) b

Through appropriate selections of the values of the appropriate coefficients \( A_L \) and \( A_\theta \), the dependence of the response ratio \( \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right) \) upon the frequency content of the input was investigated numerically.

Fig. 11.16 shows the results obtained for \( \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right) \) as a function of the eccentricity "e". The solid lines represent the results obtained for \( A_L = 1 \) and \( A_\theta = 1, 2, 4, 10 \), while the dashed lines represent the results obtained for \( A_\theta = 1 \) and \( A_L = 2, 4, 10 \). The dashed-
Figure 11.16 \( \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right) \) versus "e" for structures subjected to driving force given by

Eq. (11.12) a \( (\gamma = 1.2, D_e = 28 \text{m}, \xi = 0.05) \)

point line gives \( \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right)_{fv} \) obtained for free vibration. Notice that the results corresponding to a driving force characterized by a value of \( \frac{A_{\theta}}{A_L} = 1 \) almost coincide with the plot of

\( \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right)_{fv} \). On the other hand, as predicted, for a driving force characterized by \( \frac{A_{\theta}}{A_L} > 1 \)

\( \left( \frac{A_{\theta}}{A_L} < 1 \right) \) the response ratio is larger (smaller) than for the free vibration condition.
11.2.11 Alpha-Method for Evaluation of Maximum Rotational Response of Eccentric Systems

The results presented in the previous sections strongly indicate that the response ratio 
\[
\frac{\theta_{\text{max}}}{Y_{\text{max}}} \quad \text{in free vibration is a stable response parameter mainly system dependent and almost excitation independent. For this reason, this ratio will be referred to from here on as the torsional factor "\(\alpha\)".}
\]

\[
\alpha = \left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right)_{fv}
\]

(11.13)

Structures characterized by large values of \(\alpha\) do have a predisposition for developing large rotations. On the other hand structures characterized by small values of \(\alpha\) are less prone to rotate when dynamically excited.

The relatively limited difference between \(\alpha\) and the value of the ratio \(\left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right)\) obtained under different types of excitations suggests the use of \(\alpha\) in estimating the maximum rotation response that a given system can develop under forced vibration (i.e., \(\left( \frac{\theta_{\text{max}}}{Y_{\text{max}}} \right) \equiv \alpha\)). Thus, the maximum rotation response can be estimated using the following formula:

\[
\theta_{\text{max}} = \alpha \cdot Y_{\text{max}}
\]

(11.14)

However, the determination of \(Y_{\text{max}}\) involves a three-dimensional dynamic analysis of the structure under investigation, thus preventing any computational advantages offered by Eq. (11.14). Nonetheless, it is possible to introduce a further simplification supported by
other researchers (Nagarajaia, Reinhorn and Constantinou, 1993). Based on the fact that the maximum longitudinal displacements response, $Y_{\text{max}}$, of an eccentric structure does not differ significantly from the maximum longitudinal displacements $Y_{\text{max-ne}}$ observed in a structure having equivalent dynamic characteristics, but no eccentricity. As an illustration, some numerical results are plotted in Fig. 11.17, which shows that for increasing values of eccentricity “e”, the maximum longitudinal displacement response does not undergo appreciable changes. From the above observation, the following approximation can be made:

$$Y_{\text{max}} \equiv Y_{\text{max-ne}}$$  \hspace{1cm} (11.15)

Figure 11.17 Maximum Longitudinal Displacements response, $Y_{\text{max}}$, versus eccentricity “e” for structures subjected to the following earthquake ground motions:

1. Parkfield, 2. El Centro, 3. Montenegro Petrovac, 4. Montenegro Bar, 5. Friuli Breginj ($\gamma = 1.225$, $D_e = 28\text{m}$, $\zeta = 0.05$)
then, the maximum rotation response of an eccentric dynamic system can be estimated using the following expression:

\[ \theta_{\text{max}} = \alpha \cdot Y_{\text{max-ne}} \quad (11.16) \]

Equation (11.16) thus becomes a powerful tool for simplified analysis of the torsional response of eccentric structures. The maximum displacement of the equivalent non-eccentric structure, \( Y_{\text{max-ne}} \), can be easily obtained as the maximum deformation of an SDOF oscillator of natural period \( T_L = \frac{2 \cdot \pi}{\omega_L} \) and mass equal to the total mass of the structure.

This SDOF will be referred to as "the equivalent oscillator". The validity of the assumption expressed by Eq. (11.15) for maximum rotations estimation has been validated through an exhaustive numerical simulation. Some of these numerical results are given in Fig. 11.18. These plots indicates the absolute and relative accuracies of the estimations of the maximum rotation response based on Eqs. (11.14) and (11.16). It is found that the two approximations are of comparable accuracy.
Figure 11.18 Maximum rotation response obtained through time history dynamical analysis of three-dimensional systems (solid lines) compared to those evaluated using Eq. (11.14) (dotted-dashed lines) and Eq. (11.16) (dashed lines)

\(\gamma = 1.225, D_e = 28\,\text{m},\) damping ratio \(\zeta = 0.05\)
11.2.12 Values of the Alpha Torsional Factor

As already mentioned, the values of the "α" torsional factor for damped eccentric structures were obtained through numerical simulations. These simulations were performed for systems having dynamic characteristics encompassing those of the majority of seismic isolated structures.

*Identification of the dynamic parameters that affect the value of "α"*

The damped¹ free vibration of an eccentric system is governed by following set of simultaneous differential equations:

\[
\begin{bmatrix}
\ddot{u}_x \\
\ddot{u}_y \\
\rho \cdot \ddot{u}_\theta
\end{bmatrix} + a_0 \cdot \begin{bmatrix}
\dot{u}_x \\
\dot{u}_y \\
\rho \cdot \dot{u}_\theta
\end{bmatrix} + a_1 \cdot \omega_L^2 \cdot \begin{bmatrix}
1 & 0 & -e_y \cdot \sqrt{12} \\
0 & 1 & e_x \cdot \sqrt{12} \\
-e_y \cdot \sqrt{12} & e_x \cdot \sqrt{12} & \gamma^2
\end{bmatrix} \cdot \begin{bmatrix}
\dot{u}_x \\
\dot{u}_y \\
\rho \cdot \dot{u}_\theta
\end{bmatrix} \\
+ \omega_L^2 \cdot \begin{bmatrix}
1 & 0 & -e_y \cdot \sqrt{12} \\
0 & 1 & e_x \cdot \sqrt{12} \\
-e_y \cdot \sqrt{12} & e_x \cdot \sqrt{12} & \gamma^2
\end{bmatrix} \cdot \begin{bmatrix}
u_x \\
u_y \\
\rho \cdot u_\theta
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \\
0 \end{bmatrix}
\]

(11.17)

with the following initial conditions (unit initial displacement along the "y" axis):

\[
\begin{bmatrix}
u_x \\
u_y \\
\rho \cdot u_\theta|_{t=0}
\end{bmatrix} = \begin{bmatrix} 0 \\
1 \\
0 \end{bmatrix}
\]

(11.18)

and where all symbol have the usual meaning and \(a_0\) and \(a_1\) are the two Rayleigh coeffi-

---

¹ Rayleigh's damping is assumed.
cients which depend on the value of the assumed modal damping ratio $\zeta$. Inspection of Eq. (11.17) shows that displacement and rotation responses depend upon the values of the following six system parameters: $\rho$, $\gamma$, $e_x$, $e_y$, $\omega_L$, $\zeta$.

From Eq. (11.17) it can be shown (Bacci, Ceccoli, Trombetti, 1994) that the value of the torsional factor "$\alpha$" is independent on the uncoupled longitudinal circular frequency $\omega_L$.

Since a) numerical investigations have shown that increasing values of the damping ratio $\zeta$ produce only a relatively small decrease in the values of "$\alpha$", and b) for conservative reasons many design codes require the assumption of minimum values of damping, only a minimal damping ratio $\zeta = 5\%$ has been considered. Finally, only the presence of transversal eccentricity ($e_y = 0$, $e_x = e$) was considered, as this condition represents the one for which structures develop the largest maximum rotation response. These three considerations have allowed to neglect the parameters $e_y$, $\omega_L$, $\zeta$ and study only the dependence of the torsional factor "$\alpha$" upon parameters $\rho$, $\gamma$ and $e$, namely

$$\alpha = \alpha(\rho, \gamma, e) \quad (11.19)$$

The numerical values of the torsional factor $\alpha$

Numerical investigation of the values of "$\alpha$" has been performed for the following ranges of values of $\rho$, $\gamma$ and $e$:

- $0.02 \cdot D_c \leq e \leq 0.22 \cdot D_c$:
- $1 \leq \gamma \leq 2$:
- $28 \text{m} \leq D_e \leq 70 \text{m}$.

This numerical investigation produced a number of graphs and tabulations of the torsional factor "α", some of which are presented in section 11.2.18. These values can be directly used in Eq. (11.16) for predicting the maximum rotation response.

To simplify the use of Eq. (11.16), a compact algebraic formulation capable of returning the value of "α" in terms of the parameters "ρ", "γ" and "e", has been sought. This search for a compact algebraic formulation has been successful, mainly due to the smooth dependence of "α" upon "ρ", "γ" and "e". By least square fitting the values of "α" obtained through time history dynamic analysis, the following algebraic expression was obtained\(^1\):

$$\alpha = f(D_e) \cdot \left\{ -0.058 \cdot e + 0.8 \cdot \frac{e^2}{\gamma^2} - 2.43 \cdot \left( \frac{e}{\gamma^2} \right)^2 \right\} \text{ [deg/m]} \quad (11.20) a$$

where

$$f(D_e) = \frac{2830}{D_e} - 1 \quad (11.20) b$$

$$e = \frac{E_s}{D_e}$$

Substituting $D_e$ in meters in Eq. (11.20) b, Eq. (11.20) a then returns the values of "α" in degrees per meter of longitudinal displacement.

The algebraic expression of the torsional factor "α" has been further specialized for structures that are characterized by a regular (following a grid) distribution of the seismic isolato-

\(^1\) This expression is valid for $0.02 \leq e \leq 0.22$, $1 \leq \gamma \leq 2$ and $28 \text{m} \leq D_e \leq 70 \text{m}$ only.
tors. a common situation in seismic isolated structures. For this type of structures, it was found, through least square fitting, that the parameter \( \gamma \) can be estimated fairly well from the formula:

\[
\gamma^2 = 1 + \frac{13}{D_e}
\]  

(11.21)

Based on the above approximation, a reduced-space least square fitting led to the following expression for the torsional factor "\( \alpha \)"

\[
\alpha = f(D_e) \cdot \{ e - 2.45 \cdot e^2 \} + 0.3 \text{ [deg/m]}
\]  

(11.22) a

where

\[
f(D_e) = \frac{1320}{D_e}
\]  

(11.22) b

Using \( D_e \) expressed in meters, Eq. (11.22) a gives "\( \alpha \)" in degree per meter of longitudinal deformation.

In Fig. 11.19 the values of "\( \alpha \)" obtained through Eq. (11.20) a are compared with the tabulated values obtained through numerical simulation.

---

1. This formulation is valid for \( 0.02 \leq c \leq 0.22 \), \( 1 \leq \gamma \leq 2 \) and \( 28 \text{m} \leq D_e \leq 70 \text{m} \) only.
Figure 11.19 Values of the torsional factor "\( \alpha \)" given by Eq. (11.20) a, compared with the values obtained through numerical simulation
11.2.13 Examples of Rotation Estimation Using the Alpha-method

Eq. (11.16) allows to estimate the maximum rotation response developed by in an eccentric structure from the knowledge of:

- the torsional factor \( \alpha \) for the structure under consideration;
- the maximum deformation of the "equivalent SDOF" defined by the total mass of the system and the uncoupled lateral natural circular frequency \( \omega_L \).

The value of the torsional factor \( \alpha \) can be obtained either from the table given in Section 11.2.17 or using one of the compact algebraic expressions presented in the previous section. The computation of the maximum deformation of the "equivalent SDOF oscillator" is fairly simple and requires, from a computational viewpoint, much less effort than a complete three-dimensional time history response analysis of the eccentric system. The maximum deformation of the equivalent SDOF can also be obtained directly from a displacement response spectrum (Chopra, 1995). Therefore the method proposed here clearly reduces the computational effort required to predict the maximum lateral and torsional response of an eccentric structure.

To verify the validity of the proposed method and its degree of accuracy, estimations obtained using Eq. (11.16) with the parameter "\( \alpha \)" provided by Eq. 11.20 were compared with those obtained through a complete three-dimensional time history dynamic analysis.

The earthquake ground motions considered in this study are: Parkfield 1966, El Centro 1940, Montenegro Petrovac 1979, Montenegro Bar 1979, Friuli Breginj 1976 and Mexico City 1985. The dynamic systems considered in this analysis were characterized by a wide
variety of eccentricities \((0 \leq e \leq 0.22)\), damping ratio \((0.05 \leq \xi \leq 0.15)\). \(\gamma = \frac{\omega_g}{\omega_L}\) \((1 \leq \gamma \leq 2)\), and "equivalent diagonal" \((20 \text{m} \leq D_e \leq 70 \text{m})\). These ranges of parameter values cover the majority of cases of structures equipped with seismic base isolation.

The results obtained show that in each case, the estimation of the maximum rotation response based on the "\(\alpha\) method" (1) is sufficiently close to that obtained via a complete three-dimensional time history dynamic analysis and (2) is comparable in terms of accuracy with the estimation obtained via the SRSS modal combination rule.

Figs. 11.20 and 11.21 compare the estimation of the maximum rotation response (in degrees) obtained using the "\(\alpha\)" method with the results obtained via a complete three-dimensional time history dynamic analysis. The degree of accuracy (considering the approximations introduced) is acceptable and the errors (in most cases conservative) are limited to 10 - 15\% except for Mexico City at large values of eccentricity.

Figs. 11.22 to 11.29 compare the maximum rotation response estimations provided by the "\(\alpha\)" method with those obtained using the SRSS method of modal combination. Notice that for large values of \(\gamma\), the "\(\alpha\)" method seems to be consistently more accurate than the SRSS modal combination method.
Figure 11.20 Maximum rotation response obtained through numerical simulation (solid lines) and through the “α” method (dashed lines) for the following earthquakes inputs: (1) Parkfield, (2) El Centro, (3) Montenegro Petrovac, (4) Montenegro Bar, (5) Friuli Breginj, (6) Mexico City ($\gamma = 1.1$, $D_e = 70\text{m}, \xi = 0.15$)
Figure 11.21 Maximum rotation response obtained through numerical simulation (solid lines) and through the “α” method (dashed lines) for the following earthquakes: (1) Parkfield, (2) El Centro, (3) Montenegro Petrovac, (4) Montenegro Bar, (5) Friuli Breginj, (6) Mexico City ($\gamma = 1.1, D_c = 28\text{m}, \xi = 0.15$)
Figure 11.22 Maximum rotation response for Parkfield earthquake excitation: three-dimensional dynamic analysis = solid line, "α" method = dashed line, SRSS modal superposition method = dotted-dashed line (γ = 1.1, D_e = 28 m)
Figure 11.23 Maximum rotation response found for *El Centro* earthquake excitation: three dimensional dynamic analysis = solid line, "α" method = dashed line, SRSS superposition = dotted-dashed line (γ = 1.1, $D_e = 28$ m)
Figure 11.24 Maximum rotation response for Montenegro Petrovac earthquake excitation: three-dimensional dynamic analysis = solid line, "α" method = dashed line, SRSS-superposition method = dotted-dashed line ($\gamma = 1.1$, $D_e = 28$ m)
Figure 11.25 Maximum rotation response for *Mexico City* earthquake excitation:
three-dimensional dynamic analysis = solid line. "α" method = dashed line, SRSS modal superposition method = dotted-dashed line
($\gamma = 1.1, D_e = 28\, \text{m}$)
Figure 11.26 Maximum rotation for *Parkfield* earthquake excitation: three-dimensional dynamic analysis = solid line. "α" method = dashed line. SRSS model superposition method = dotted-dashed line ($\gamma = 2.0$, $D_e = 28\,m$)
Figure 11.27 Maximum rotation response for *El Centro* earthquake excitation:
three-dimensional dynamic analysis = solid line, "a" method = dashed line, SRSS modal superposition method = dotted-dashed line
($\gamma = 2.0$, $D_e = 28 m$)
Figure 11.28 Maximum rotation response for Montenegro Petrovac earthquake excitation: three-dimensional dynamic analysis = solid line, “α” method = dashed line, SRSS modal superposition method = dotted-dashed line (γ = 2.0, D_e = 28 m)
Figure 11.29 Maximum rotation response for Mexico City earthquake excitation:
three-dimensional dynamic analysis = solid line, "α" method = dashed line, SRSS modal superposition method = dotted-dashed line 
(\(\gamma = 2.0, D_e = 28\text{m}\))
11.2.14 Conclusions

The proposed method allows a simplified estimation of the maximum rotation response developed by eccentric base isolated structures when subjected to earthquake ground motions. The maximum rotations are estimated (through Eq. (11.16)) starting from the knowledge of the maximum longitudinal displacement of an equivalent SDOF oscillator having the same dynamic characteristics as the seismic isolated structure but with no eccentricity and from the torsional factor “α”. The advantage of this formulation is that the maximum value of the rotation response developed by an eccentric seismic isolated structure can be predicted with reasonable accuracy without performing a full three-dimensional time history dynamic analysis of the system. The determination of the maximum deformation of the equivalent SDOF oscillator is simple and much faster (from a computational viewpoint) than a complete three-dimensional dynamic response analysis. This maximum deformation can also be obtained directly from a displacement response spectrum. The values of the torsional factor “α” can be obtained from Eqs. (11.20) a and b. or Eqs. (11.22) a and b. once the characteristics of the eccentric structure under consideration are known. The analysis performed up to date have confirmed the accuracy and robustness of the proposed simplified approach.

Furthermore, the proposed method is capable to provide a key to understanding the torsional response behavior of eccentric dynamic systems and to estimating their predisposition in developing large torsional response. In fact, it is clear that an eccentric system, with given uncoupled lateral dynamic characteristics, will develop larger or smaller torsional response depending on the value of its “α” torsional factor (larger values of “α” imply
larger possible rotation and vice versa). Through a careful tuning of the parameters upon which the torsional factor "\( \alpha \)" depends, it is possible to control the torsional response behavior of eccentric structures. From the results obtained, it can be concluded that the torsional factor "\( \alpha \)":

- decreases for increasing values of the longitudinal eccentricity (in the direction of the base excitation);
- increases for increasing values of the transversal eccentricity;
- decreases for increasing values of the radius of gyration \( \rho \) of the structure;
- increases as the "\( \gamma \)" parameter (which corresponds to the ratio of the uncoupled longitudinal and rotational natural period of vibration) tends to unity;
- decreases slightly for increasing values of damping\(^1\).

---

\(^1\) It must be recalled that the maximum longitudinal displacement, on the other hand, is greatly affected by the value of damping.
11.2.15 Appendix I: Dynamic Analysis of Eccentric Systems

The dynamic system under investigation is a three-dimensional one story linear system characterized by non-coincident center of the mass and center of rigidity. The dynamic response of this system is governed in a coordinate system with origin at the center of mass, by the following set of simultaneous differential equations:

\[
\begin{bmatrix}
  m & 0 & 0 \\
  0 & m & 0 \\
  0 & 0 & (m \cdot \rho^2)
\end{bmatrix}
\begin{bmatrix}
  \ddot{u}_x(t) \\
  \ddot{u}_y(t) \\
  \ddot{u}_\theta(t)
\end{bmatrix}
+ [C]
\begin{bmatrix}
  \ddot{u}_x(t) \\
  \ddot{u}_y(t) \\
  \ddot{u}_\theta(t)
\end{bmatrix}
= \begin{bmatrix}
  K_x & 0 & (-K_x \cdot E_y) \\
  0 & 1 & (K_y \cdot E_y) \\
  (-K_x \cdot E_y) & (K_y \cdot E_y) & K_{\theta \theta}
\end{bmatrix}
\begin{bmatrix}
  u_x(t) \\
  u_y(t) \\
  u_\theta(t)
\end{bmatrix}
= \begin{bmatrix}
  p_x(t) \\
  p_y(t) \\
  p_\theta(t)
\end{bmatrix}
\tag{11.23}
\]

where

\[ u_x(t), u_y(t), u_\theta(t) \quad = \quad \text{displacements along the "x" and "y" directions and rotations, respectively:} \]

\[ m \quad = \quad \text{total mass of the isolated structure:} \]

\[ \rho \quad = \quad \text{radius of gyration of the isolated structure:} \]

\[ [C] \quad = \quad \text{damping matrix:} \]

\[ k_{xi} \quad = \quad \text{lateral stiffness along the "x" direction of the i-th isolator:} \]

\[ k_{yi} \quad = \quad \text{lateral stiffness along the "y" direction of the i-th isolator:} \]

\[ x_i \quad = \quad \text{distance from the center of mass in the x-direction of the i-th isolator:} \]
\( y_i \) = distance from the center of mass in the "y" direction of the i-th isolator;

\[
K_x = \sum_{i=1}^{N} k_{xi}
\] = total translational stiffness along the "x" direction:

\[
K_y = \sum_{i=1}^{N} k_{yi}
\] = total translational stiffness along the "y" direction:

\[
K_{\theta\theta} = \sum_{i=1}^{N} (k_{xi} \cdot y_i^2 + k_{yi} \cdot x_i^2)
\] = torsional stiffness of the system with respect to the center of the mass:

\[
E_x = \frac{\sum_{i=1}^{N} k_{yi} \cdot x_i}{\sum_{i=1}^{N} k_{yi}}
\] = eccentricity along the "x" direction of the center of rigidity with respect to the center of mass:

\[
E_y = \frac{\sum_{i=1}^{N} k_{xi} \cdot y_i}{\sum_{i=1}^{N} k_{xi}}
\] = eccentricity along the "y" direction of the center of rigidity with respect to the center of the masses:

\( p_x(t), p_y(t), p_{\theta}(t) \) = external forces applied along the "x" and "y" directions, and external moment.
11.2.16 Appendix II: Modes of Vibration of the Undamped Eccentric System

The undamped free vibration of an eccentric system are governed by the following set of simultaneous differential equations:

\[
\begin{bmatrix}
\ddot{u}_x \\
\ddot{u}_y \\
\rho \cdot \ddot{u}_\theta
\end{bmatrix}
+ \omega_L^2 \cdot
\begin{bmatrix}
1 & 0 & (-e_y \cdot \sqrt{12}) \\
0 & 1 & (e_x \cdot \sqrt{12}) \\
(-e_y \cdot \sqrt{12}) & (e_x \cdot \sqrt{12}) & \gamma^2
\end{bmatrix}
\begin{bmatrix}
u_x(t) \\
v_y(t) \\
\rho \cdot u_\theta(t)
\end{bmatrix}
= \begin{bmatrix}0 \\ 0 \\ 0 \end{bmatrix}
\]

(11.24)

\[
\begin{bmatrix}
u_x(0) \\
\nu_y(0) \\
\rho \cdot u_\theta(0)
\end{bmatrix}
= \begin{bmatrix}0 \\ a \\ 0 \end{bmatrix}
\]

where the various symbols have the usual meaning and “a” represents the initial deformation along the y-direction.

This set of differential equations can be solved using the classical mode superposition method. Expressing the dynamic response of the system in modal coordinates, \( Y_i(t), i=1 \).

2. 3. leads to the following uncoupled modal equations:
\[
\begin{bmatrix}
\ddot{Y}_1 \\
\ddot{Y}_2 \\
\ddot{Y}_3
\end{bmatrix} + \omega_L^2 \cdot \begin{bmatrix}
\Omega_1 & 0 & 0 \\
0 & \Omega_2 & 0 \\
0 & 0 & \Omega_3
\end{bmatrix} \cdot \begin{bmatrix}
Y_1 \\
Y_2 \\
Y_3
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
Y_1 \\
Y_2 \\
Y_3
\end{bmatrix}_{t=0} = a \cdot 
\begin{bmatrix}
\frac{e_x \cdot (\Omega_1 - 1) \cdot (\Omega_3 - 1)}{\sqrt{12} \cdot e^2 \cdot (\Omega_3 - \Omega_1)} \\
\frac{e_y^3}{e^2} \\
\frac{e_x \cdot (\Omega_1 - 1) \cdot (\Omega_3 - 1)}{\sqrt{12} \cdot e^2 \cdot (\Omega_3 - \Omega_1)}
\end{bmatrix}
\]

where

\[
\begin{bmatrix}
u_x(t) \\
u_y(t) \\
\rho \cdot u_\theta(t)
\end{bmatrix} = \begin{bmatrix}\Phi_1 \{ \Phi_2 \} \{ \Phi_3 \} \end{bmatrix} \begin{bmatrix}Y_1(t) \\
Y_2(t) \\
Y_3(t)
\end{bmatrix}
\]

- \{ \Phi_1 \}, \{ \Phi_2 \}, \{ \Phi_3 \} are the eigenvectors or undamped modes of vibration defined in Section 11.2.3.

Notice that the second mode of vibration is excited only in the presence of a non-zero eccentricity "e_y" in the y-direction (longitudinal direction). On the other hand, the first and third modes of vibration are excited only if the system has a non-zero eccentricity "e_x" in the x-direction (transversal direction).
11.2.17 Appendix III: Harmonics with Amplitude Modulation (Beating Phenomenon)

It is known from trigonometry that the sum of two harmonics of equal amplitude "A" and different circular frequencies (say, $\omega_1$, and $\omega_2$) results in a periodic function of maximum amplitude "2A" characterized by a fast mode of circular frequency $\omega_h = \frac{\omega_1 + \omega_2}{2}$ and a slow mode of circular frequency $\omega_m = \frac{\omega_1 - \omega_2}{2}$ which controls the envelope of the modulated function, as expressed by:

$$A \cdot \cos(\omega_1 \cdot t) + A \cdot \cos(\omega_2 \cdot t) = 2 \cdot A \cdot \cos\left(\frac{\omega_1 - \omega_2}{2} \cdot t\right) \cdot \cos\left(\frac{\omega_1 + \omega_2}{2} \cdot t\right)$$ (11.26)

When the two harmonics have the same amplitude, the envelope follows a lobe-shaped pattern with nodes of zero amplitude, as shown in Fig. 11.30 on the other hand, when the two harmonics have different amplitudes, the slow mode modulates the envelope without nodes of zero amplitude as shown in Fig. 11.31.
Figure 11.30 Beating Phenomenon as obtained by summation of two harmonics of equal amplitude

Figure 11.31 Beating Phenomenon as obtained by summation of two harmonics of different amplitudes
11.2.18 Appendix IV: Tabulated Values of the Torsional Factor “Alpha”

This section reports in tabulated form the values of the torsional factor “\( \alpha \)” obtained through the free vibrations analysis of eccentric structures assuming Rayleigh damping with 5% of damping ratio for the first and second mode. These \( \alpha \) values, obtained through numerical simulations, are expressed in \textit{radians per meter}.

<table>
<thead>
<tr>
<th>( \epsilon = \epsilon_i )</th>
<th>( \gamma = 0.8 )</th>
<th>( \gamma = 1.0 )</th>
<th>( \gamma = 1.2 )</th>
<th>( \gamma = 1.4 )</th>
<th>( \gamma = 1.6 )</th>
<th>( \gamma = 1.8 )</th>
<th>( \gamma = 2.0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>0.0142279</td>
<td>0.0166905</td>
<td>0.0106124</td>
<td>0.0065593</td>
<td>0.0043449</td>
<td>0.0032952</td>
<td>0.0024975</td>
</tr>
<tr>
<td>4%</td>
<td>0.0262006</td>
<td>0.0288786</td>
<td>0.0201052</td>
<td>0.0128518</td>
<td>0.0086484</td>
<td>0.0065646</td>
<td>0.0049782</td>
</tr>
<tr>
<td>6%</td>
<td>0.0353746</td>
<td>0.0371671</td>
<td>0.0274427</td>
<td>0.0186132</td>
<td>0.0128684</td>
<td>0.0097822</td>
<td>0.0074253</td>
</tr>
<tr>
<td>8%</td>
<td>0.0399696</td>
<td>0.0425582</td>
<td>0.0342405</td>
<td>0.0235856</td>
<td>0.0169619</td>
<td>0.0129215</td>
<td>0.0098217</td>
</tr>
<tr>
<td>10%</td>
<td>0.0462542</td>
<td>0.0468813</td>
<td>0.0392316</td>
<td>0.0275316</td>
<td>0.0208845</td>
<td>0.0159552</td>
<td>0.0121500</td>
</tr>
<tr>
<td>12%</td>
<td>0.0506821</td>
<td>0.0475730</td>
<td>0.0419122</td>
<td>0.0317859</td>
<td>0.0245926</td>
<td>0.0188551</td>
<td>0.0143922</td>
</tr>
<tr>
<td>14%</td>
<td>0.0519828</td>
<td>0.0521316</td>
<td>0.0448365</td>
<td>0.0359977</td>
<td>0.0280459</td>
<td>0.0215918</td>
<td>0.0165299</td>
</tr>
<tr>
<td>16%</td>
<td>0.0491443</td>
<td>0.0548721</td>
<td>0.0486837</td>
<td>0.0396728</td>
<td>0.0311856</td>
<td>0.0241344</td>
<td>0.0185439</td>
</tr>
<tr>
<td>18%</td>
<td>0.0469980</td>
<td>0.0553321</td>
<td>0.0513595</td>
<td>0.0420706</td>
<td>0.0339551</td>
<td>0.0264502</td>
<td>0.0204141</td>
</tr>
<tr>
<td>20%</td>
<td>0.0530173</td>
<td>0.0529649</td>
<td>0.0526178</td>
<td>0.0449897</td>
<td>0.0362931</td>
<td>0.0285044</td>
<td>0.0221197</td>
</tr>
<tr>
<td>22%</td>
<td>0.0593714</td>
<td>0.0470683</td>
<td>0.0521717</td>
<td>0.0464204</td>
<td>0.0381321</td>
<td>0.0302597</td>
<td>0.0236384</td>
</tr>
</tbody>
</table>
Table 11.2 Radius of Gyration $\rho = 15 \cdot \sqrt{2}$

<table>
<thead>
<tr>
<th>$c = c_x$</th>
<th>$\gamma = 0.8$</th>
<th>$\gamma = 1.0$</th>
<th>$\gamma = 1.2$</th>
<th>$\gamma = 1.4$</th>
<th>$\gamma = 1.6$</th>
<th>$\gamma = 1.8$</th>
<th>$\gamma = 2.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>0.0094853</td>
<td>0.0111270</td>
<td>0.0070749</td>
<td>0.0043778</td>
<td>0.0028996</td>
<td>0.0021968</td>
<td>0.0016649</td>
</tr>
<tr>
<td>4%</td>
<td>0.0174671</td>
<td>0.0192524</td>
<td>0.0134035</td>
<td>0.0085678</td>
<td>0.0057656</td>
<td>0.0043764</td>
<td>0.0033188</td>
</tr>
<tr>
<td>6%</td>
<td>0.0235830</td>
<td>0.0247781</td>
<td>0.0182951</td>
<td>0.0124088</td>
<td>0.0085789</td>
<td>0.0065210</td>
<td>0.0049502</td>
</tr>
<tr>
<td>8%</td>
<td>0.0266457</td>
<td>0.0283722</td>
<td>0.0228270</td>
<td>0.0157237</td>
<td>0.0113079</td>
<td>0.0086143</td>
<td>0.0065478</td>
</tr>
<tr>
<td>10%</td>
<td>0.0308361</td>
<td>0.0312542</td>
<td>0.0261154</td>
<td>0.0183544</td>
<td>0.0139230</td>
<td>0.0106368</td>
<td>0.0080999</td>
</tr>
<tr>
<td>12%</td>
<td>0.0337880</td>
<td>0.0317153</td>
<td>0.0279415</td>
<td>0.0211906</td>
<td>0.0163951</td>
<td>0.0125701</td>
<td>0.0095947</td>
</tr>
<tr>
<td>14%</td>
<td>0.0346552</td>
<td>0.0347544</td>
<td>0.0298910</td>
<td>0.0239985</td>
<td>0.0186973</td>
<td>0.0143945</td>
<td>0.0110199</td>
</tr>
<tr>
<td>16%</td>
<td>0.0327629</td>
<td>0.0365814</td>
<td>0.0324558</td>
<td>0.0264486</td>
<td>0.0207904</td>
<td>0.0160896</td>
<td>0.0123626</td>
</tr>
<tr>
<td>18%</td>
<td>0.0313320</td>
<td>0.0368880</td>
<td>0.0342397</td>
<td>0.0284717</td>
<td>0.0226367</td>
<td>0.0176335</td>
<td>0.0136094</td>
</tr>
<tr>
<td>20%</td>
<td>0.0353448</td>
<td>0.0353100</td>
<td>0.0350785</td>
<td>0.0299932</td>
<td>0.0241954</td>
<td>0.0190029</td>
<td>0.0147464</td>
</tr>
<tr>
<td>22%</td>
<td>0.0395809</td>
<td>0.0313788</td>
<td>0.0347811</td>
<td>0.0309469</td>
<td>0.0254214</td>
<td>0.0201310</td>
<td>0.0157589</td>
</tr>
</tbody>
</table>

Table 11.3 Radius of Gyration $\rho = 20 \cdot \sqrt{2}$

<table>
<thead>
<tr>
<th>$c = c_x$</th>
<th>$\gamma = 0.8$</th>
<th>$\gamma = 1.0$</th>
<th>$\gamma = 1.2$</th>
<th>$\gamma = 1.4$</th>
<th>$\gamma = 1.6$</th>
<th>$\gamma = 1.8$</th>
<th>$\gamma = 2.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>0.0071139</td>
<td>0.0083452</td>
<td>0.0033062</td>
<td>0.0032796</td>
<td>0.0021724</td>
<td>0.0016476</td>
<td>0.0012487</td>
</tr>
<tr>
<td>4%</td>
<td>0.0131003</td>
<td>0.0144393</td>
<td>0.0100926</td>
<td>0.0064258</td>
<td>0.0043242</td>
<td>0.0032823</td>
<td>0.0024890</td>
</tr>
<tr>
<td>6%</td>
<td>0.0176873</td>
<td>0.0185835</td>
<td>0.0137313</td>
<td>0.0093060</td>
<td>0.0064341</td>
<td>0.0048911</td>
<td>0.0037126</td>
</tr>
<tr>
<td>8%</td>
<td>0.0199843</td>
<td>0.0212791</td>
<td>0.0171203</td>
<td>0.0117928</td>
<td>0.0084809</td>
<td>0.0064607</td>
<td>0.0049108</td>
</tr>
<tr>
<td>10%</td>
<td>0.0231271</td>
<td>0.0234406</td>
<td>0.0196158</td>
<td>0.0137658</td>
<td>0.0104423</td>
<td>0.0079775</td>
<td>0.0060749</td>
</tr>
<tr>
<td>12%</td>
<td>0.0253410</td>
<td>0.0237865</td>
<td>0.0209561</td>
<td>0.0158929</td>
<td>0.0123963</td>
<td>0.0094275</td>
<td>0.0071960</td>
</tr>
<tr>
<td>14%</td>
<td>0.0259914</td>
<td>0.0260658</td>
<td>0.0224183</td>
<td>0.0179989</td>
<td>0.0140229</td>
<td>0.0107959</td>
<td>0.0082649</td>
</tr>
<tr>
<td>16%</td>
<td>0.0245722</td>
<td>0.0274361</td>
<td>0.0243418</td>
<td>0.0198364</td>
<td>0.0155928</td>
<td>0.0120672</td>
<td>0.0092719</td>
</tr>
<tr>
<td>18%</td>
<td>0.0234990</td>
<td>0.0276660</td>
<td>0.0256798</td>
<td>0.0221354</td>
<td>0.0169776</td>
<td>0.0132251</td>
<td>0.0102071</td>
</tr>
<tr>
<td>20%</td>
<td>0.0265086</td>
<td>0.0264825</td>
<td>0.0263089</td>
<td>0.0224949</td>
<td>0.0181465</td>
<td>0.0142522</td>
<td>0.0110598</td>
</tr>
<tr>
<td>22%</td>
<td>0.0296857</td>
<td>0.0235341</td>
<td>0.0260858</td>
<td>0.0232102</td>
<td>0.0190661</td>
<td>0.0151299</td>
<td>0.0118192</td>
</tr>
</tbody>
</table>
Table 11.4 Radius of Gyration $r = 25 \cdot \sqrt{2}$

<table>
<thead>
<tr>
<th>$e = e_x$</th>
<th>$\gamma = 0.8$</th>
<th>$\gamma = 1.0$</th>
<th>$\gamma = 1.2$</th>
<th>$\gamma = 1.4$</th>
<th>$\gamma = 1.6$</th>
<th>$\gamma = 1.8$</th>
<th>$\gamma = 2.0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2%</td>
<td>0.0056911</td>
<td>0.0066761</td>
<td>0.0042449</td>
<td>0.0026237</td>
<td>0.0017379</td>
<td>0.0013180</td>
<td>0.0009989</td>
</tr>
<tr>
<td>4%</td>
<td>0.0104802</td>
<td>0.0115514</td>
<td>0.0080421</td>
<td>0.0051407</td>
<td>0.0034594</td>
<td>0.0026259</td>
<td>0.0019913</td>
</tr>
<tr>
<td>6%</td>
<td>0.0141498</td>
<td>0.0148668</td>
<td>0.0109771</td>
<td>0.0074453</td>
<td>0.0051473</td>
<td>0.0039129</td>
<td>0.0029701</td>
</tr>
<tr>
<td>8%</td>
<td>0.0159874</td>
<td>0.0170233</td>
<td>0.0136962</td>
<td>0.0094342</td>
<td>0.0067847</td>
<td>0.0051686</td>
<td>0.0039287</td>
</tr>
<tr>
<td>10%</td>
<td>0.0185017</td>
<td>0.0187525</td>
<td>0.0156926</td>
<td>0.0110126</td>
<td>0.0083538</td>
<td>0.0063821</td>
<td>0.0048599</td>
</tr>
<tr>
<td>12%</td>
<td>0.0202728</td>
<td>0.0190292</td>
<td>0.0167649</td>
<td>0.0127143</td>
<td>0.0098370</td>
<td>0.0075420</td>
<td>0.0057569</td>
</tr>
<tr>
<td>14%</td>
<td>0.0207931</td>
<td>0.0208526</td>
<td>0.0171935</td>
<td>0.0143991</td>
<td>0.0112184</td>
<td>0.0086367</td>
<td>0.0066119</td>
</tr>
<tr>
<td>16%</td>
<td>0.0196577</td>
<td>0.0219489</td>
<td>0.0194735</td>
<td>0.0158691</td>
<td>0.0124742</td>
<td>0.0096538</td>
<td>0.0074175</td>
</tr>
<tr>
<td>18%</td>
<td>0.0187992</td>
<td>0.0221328</td>
<td>0.0205438</td>
<td>0.0170830</td>
<td>0.0135820</td>
<td>0.0105801</td>
<td>0.0081656</td>
</tr>
<tr>
<td>20%</td>
<td>0.0212069</td>
<td>0.0211860</td>
<td>0.0210471</td>
<td>0.0179959</td>
<td>0.0145172</td>
<td>0.0114018</td>
<td>0.0088479</td>
</tr>
<tr>
<td>22%</td>
<td>0.0237485</td>
<td>0.0188273</td>
<td>0.0208687</td>
<td>0.0185682</td>
<td>0.0152529</td>
<td>0.0121039</td>
<td>0.0094553</td>
</tr>
</tbody>
</table>
11.3 Model Construction and Testing Procedure

In the first part of this section the rationale for the scaled structural model (representative of a seismic isolated structure) and its characteristics are presented. The second part describes the procedure followed in performing the actual shake table test.

11.3.1 Structure Prototype

In order to construct a scaled model representative of a real seismic isolated structure, a prototype structure was selected. We opted for a five story building 20 meters by 20 meters in plan, and resting over 25 base isolators positioned 5 meters apart according to the modes of a square grid as shown in Fig. 11.32.

Considering a mass distribution of approximately 1.000 kg/m³ for each floor, the mass

![Diagram of the prototype building with a grid and isolators placed at 5 meters intervals.](image)

Figure 11.32 Plan view of the prototype building
and inertia characteristics of this structure are as follows:

- total mass \( m = 2 \cdot 10^6 \text{ kg} \):
- polar moment of inertia \( I = 1.3 \cdot 10^9 \text{ kg} \cdot \text{m}^2 \):
- radius of gyration \( \rho \equiv 8.16 \text{ m} \).

Considering a target prototype value of the uncoupled natural period of vibration \( T_L \equiv 2 \) seconds, and trying to match the usual setup of typical seismic isolated structures, the following characteristics could be deduced:

- total decoupled lateral stiffness (of all 25 isolators combined) \( k = 2 \cdot 10^7 \text{ N/m} \):
- total rotational stiffness \( k_{\theta\theta} = 2 \cdot 10^9 \text{ N} \cdot \text{m/ rad} \):
- ratio of decoupled longitudinal to rotational natural period \( \gamma \equiv 1.22 \):
- decoupled rotational natural period \( T_\theta \equiv 1.633 \text{ sec} \).

11.3.2 Model Structure

Rationale

To accommodate the characteristics of the Rice University shaking table, a time scale equal to 5. \( \lambda_T = \frac{T_0}{T_m} = 5 \), and a length scale of approximately 40. \( \lambda_l = \frac{L_0}{L_m} \equiv 40 \), were selected. Given the specific characteristics of the theory to be verified, we decided to construct a one-story linear elastic model with tunable distance between the center of mass and the center of rigidity (eccentricity), while keeping all of its other dynamic characteristics to fixed values. For this reason, the model had to satisfy the following requirements:
• maintain a linear elastic behavior (under lateral deformations).
• have a well defined and fixed position of center of rigidity.
• have a well defined and movable center of mass.
• have a well defined and constant value of the radius of gyration.
• have a constant value of uncoupled lateral natural period.
• have a constant value of uncoupled torsional natural period.

It was then decided to realize a model made of a light rigid top plate to which weights could be clamped in different positions (in order to obtain different positions of the center of the masses) and of a number of columns fixed at given positions.

After a lengthy investigation of different materials and construction techniques, a carbon fiber plate resting on nine plexiglas column rods fixed to a plexiglas base was built. The carbon fiber was selected for the construction of the top plate for its lightness, which maximized the range of the positions of the center of mass. Plexiglas was selected for its linear elastic behaviors, its low value of Young's modulus and its easy manufacturing.

*Design (target) characteristics of the model*

Given the chosen length and time scales, the model structure is supposed to have a natural frequency of approximately 2.5 Hz, a ratio of rotational to lateral uncoupled natural frequencies of approximately 1.22 and a radius of gyration of approximately 20.4 cm.

*Weights and dimensions*

As shown in Fig. 11.33, the carbon fiber top plate is squared 50 cm by 50 cm and weighs approximately 0.550 kg. The nine plexiglas column rods have \( \frac{1}{4} \) in (= 6.35 mm) diameter.
measure 22.5 cm in length and are set on the nodes of a 20 cm by 20 cm square grid. The dimension and the positioning of the rods were selected in order to obtain a model structure with an uncoupled lateral natural period of 0.40 sec. (2.5 Hz) for a model total mass of 8.00 kg assuming a Young's modulus for the plexiglas of 30,000 kg/cm². Particular attention was paid to the connection (accuracy and consistency of clamping condition) of these rods to both the top carbon fiber plate and the base plate, in order to locate the center of rigidity as close as possible to the geometrical center of the model.

![Figure 11.33 Plan view of the model](image)

Four weights of approximately 1.850 kg each (including the clamping bolts) can be attached to the carbon fiber plate. The positions at which the weights can be clamped to the top plate were carefully computed in order to precisely realize values of eccentricity equal to 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, and 20% of the equivalent diagonal $D_e$, while maintaining a constant value of the radius of gyration of the system. Mounts for the dis-
placement and acceleration transducers (having a total weight of approximately 1.100 kg) were attached to the carbon fiber plate. Their location and masses were taken into account in determining of the location of the camping of the weights. The total mass of the model (neglecting the weight of the rods) amounted 9.05 kgs.

11.3.3 Static Characteristics of the Model

Fig. 11.34 shows the Load/Displacement curve found for the reduced scale structural model built at Rice University. The plot indicates that the model behaves linearly under lateral deformations up to 0.6 inch, and has a lateral stiffness of approximately 10 lbs/in = 1.800 N/m. This value is close to the target value of 2.000 N/m selected during the design process (and obtained assuming the rods to be clamped at both ends and having a Young's modulus of 30,000 kg/cm²).

This overestimating of the value of the lateral rigidity of the model, together with an increase (+ 15%) of the total mass of the model (as compared to the design model) leads to a reduction of the natural frequency of the model of approximately 12% with respect to the target design value.

The model was statically tested under lateral loads up to 6.1 lbs = 27.13 Newtons. This maximum resisting force was selected to simulate the effect upon the scaled model of an absolute lateral acceleration response of 0.3 g which corresponds to a peak table acceleration of 0.10 g with a dynamic amplification factor of 3.
Figure 11.34 Load-deflection curve of the reduced scale model
11.3.4 Dynamic Characteristics of the Model

Fig. 11.35 shows the experimentally determined transfer function (magnitude and phase) for the lateral displacement response of the model. To obtain the uncoupled lateral natural frequency, the weights on the top plate were positioned such that the centers of mass and rigidity coincide. Inspection of Fig. 11.35 indicates that the lateral natural frequency of the system is approximately at 2.2 Hz. The damping ratio of the model structure can be estimated via the half-power-bandwidth method to approximately 6%.

Fig. 11.36 represents the amplitude and phase spectra of the rotation response developed by the model in free vibration due to a given initial rotational deformation. These rotations were recorded with the weights still positioned for coincident centers of mass and rigidity. From these experimental results, the uncoupled rotational natural frequency is estimated to approximately 2.75 Hz, which gives a ratio of the uncoupled natural period to the rotational uncoupled natural period of \( \gamma = 1.25 \).

Fig. 11.37 gives the experimental transfer function between a lateral input and the rotation response developed by the model with the weights positioned in order to realize an eccentricity between the centers of mass and rigidity equal to 8% of \( D_e \). This experiment was performed in order to estimate the dynamic rotational characteristics of the model in the presence of eccentricity. It is observed that these experimental results are in agreement with the theory according to which the first and third modal frequencies bound closely the uncoupled lateral natural frequency.
Figure 11.35 Experimental transfer function of the scaled model for decoupled longitudinal vibrations (eccentricity = 0)
Figure 11.36 Amplitude and phase spectra of decoupled free rotation response of the scaled model (eccentricity = 0)
Figure 11.37 Experimental transfer function of the scaled model between longitudinal input and rotation response (eccentricity = 8 % of $D_e$)
11.3.5 Comparison Between Target and Actual Model Characteristics

As mentioned in the previous Section, the uncoupled lateral natural frequency of the model in working conditions is approximately 2.2 Hz. This value is about 12 % lower than the target value of 2.5 Hz and in agreement with the results expected considering that the actual value of the lateral stiffness is 12 % smaller than the target design value and the mass of the model is 10 % larger than the target design mass. Considering that the time scaling factor ($\lambda_T$) to be used in the dynamic testing is equal to 5, the actual reduced scale model is representative of a seismic isolated building having a natural frequency of 0.44 Hz (corresponding to a natural period of 2.27 sec). This value though it differs from the target one of 0.5 Hz, is still representative of common actual seismic isolated structures. The model retains its validity for the purpose of the investigation to be performed.

Fig. 11.36 has indicated that the experimentally determined decoupled rotational natural frequency of the model is approximately 2.75 Hz.

Thus, the experimental value of the ratio $\gamma$ between the decoupled rotational natural frequency and the decoupled lateral frequency is about 1.26, as compared to the target design value of 1.22.

Introducing in the theoretical expression of the parameter $\gamma$ ($\gamma = \sqrt{\frac{k_{99}}{\rho^3 \cdot k}}$):

1. the experimentally estimated value of $\gamma$.
2. the experimentally estimated value of the lateral stiffness $k$. and
3. the value of the torsional stiffness (see Section 11.2.15).
it is possible to obtain an estimate of the actual value of the radius of gyration \( \rho \) of 18.98 cm. This result gives a value of the length scale (relative to a prototype structure of 20 by 20 meters in plan with a radius of gyration of 8.16 m) of 43 \( \lambda_1 = \frac{L_P}{L_m} \equiv 43 \), compared to a target design value of 40.

11.3.6 Fundamental Considerations for Understanding the Experimental Test Procedure and Results

As mentioned in Section 11.2.12, the estimation of the maximum rotation response of a given eccentric structure using the proposed simplified "\( \alpha \)" method requires knowledge of the values of "\( \gamma \)", "\( \rho \)" (or the equivalent diagonal, \( D_e \)) and the damping ratio "\( \zeta \)" for the structure under consideration.

As previously mentioned, the experimental investigations performed on the model have shown that \( \gamma = 1.26 \), \( \rho = 18.98 \) cm and \( \zeta = 6 \% \). The actual model damping ratio "\( \zeta \)" is very close to the one of 5\% used in tabulating the values of the torsional factor "\( \alpha \)" and in determining Eqs. (11.20) a, b and (11.22) a, b. For this reason, the values of "\( \alpha \)" provided by these equations could be used for the validation of the proposed "\( \alpha \)" method. As the objective of this test was to validate the accuracy of the proposed simplified method for evaluation of the max. rotation response of actual eccentric real seismic isolated structure, it was decided to scale up the experimental results obtained from the model structure back to the displacement and rotation response values corresponding to the prototype structure. Thus, the measured displacement responses of the reduced scale model were multiplied by
the experimentally determined length scale factor of \( \lambda_1 = \frac{L_p}{L_m} \equiv 43 \) in order to estimate the corresponding displacement responses of the 20 m by 20 m prototype structure.

11.3.7 Testing Procedure

The reduced scale model was tested under seismic excitations corresponding to the following eight earthquake records: Parkfield 1966, El Centro 1940, Montenegro Bar 1979, Montenegro Petrovac 1979, Friuli Breginj 1976, Tolmezzo 1977, Brienza 1977, and a synthetic earthquake record from the Eurocode.

All of these earthquake records were scaled both in time and amplitude according to Table 11.5. Notice that various amplitude scalings were used for different earthquakes. These differences are due to the fact that we wanted to excite the model structure with a maximum base acceleration in the range between 0.1 g and 0.15 g for all earthquakes. This range of values of maximum ground (table) acceleration was selected in order to produce a displacement response of the model that is large enough to be detected accurately by the displacement transducers (linear potentiometers), but small enough to not threaten the structural integrity of the model.

The model was tested under eleven different weight configurations. Weights were positioned in order to increase progressively the transversal eccentricity between the centers of mass and rigidity from 0 % (non eccentric structure) to 20 % of \( D_e \) as reported in Table

---

1. As a last minute safety measure the amplitude of all of these earthquake records were further reduced by 40 %.
11.6.

Table 11.5 Time and Amplitude Scaling Factors for Earthquake Accelerograms

<table>
<thead>
<tr>
<th>Earthquake Record</th>
<th>Time Scale $\lambda_T$</th>
<th>Amplitude Scale $\lambda_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parkfield 1966</td>
<td>5</td>
<td>25.0</td>
</tr>
<tr>
<td>El Centro 1940</td>
<td>5</td>
<td>12.5</td>
</tr>
<tr>
<td>Montenegro Bar 1979</td>
<td>5</td>
<td>12.5</td>
</tr>
<tr>
<td>Montenegro Petrovac 1979</td>
<td>5</td>
<td>10.0</td>
</tr>
<tr>
<td>Friuli Breginj 1976</td>
<td>5</td>
<td>5.0</td>
</tr>
<tr>
<td>Synthetic earthquake from Eurocode</td>
<td>5</td>
<td>5.0</td>
</tr>
<tr>
<td>Tolmezzo 1977</td>
<td>5</td>
<td>5.0</td>
</tr>
<tr>
<td>Brienza 1977</td>
<td>5</td>
<td>5.0</td>
</tr>
</tbody>
</table>

Table 11.6 Transversal Eccentricities Used

<table>
<thead>
<tr>
<th>Transversal Eccentricity [% of $D_e$]</th>
<th>Actual distance between center of mass and center of rigidity [cm] (model scale)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>1.3</td>
</tr>
<tr>
<td>4</td>
<td>2.6</td>
</tr>
<tr>
<td>6</td>
<td>3.9</td>
</tr>
<tr>
<td>8</td>
<td>5.2</td>
</tr>
<tr>
<td>10</td>
<td>6.55</td>
</tr>
<tr>
<td>12</td>
<td>7.9</td>
</tr>
<tr>
<td>14</td>
<td>9.2</td>
</tr>
<tr>
<td>16</td>
<td>10.5</td>
</tr>
<tr>
<td>18</td>
<td>11.8</td>
</tr>
<tr>
<td>20</td>
<td>13.1</td>
</tr>
</tbody>
</table>
11.4 TEST RESULTS

As the objectives of this dynamic testing performed on the Rice shaking table were (1) to confirm the validity of the proposed “α” method for max. rotation response estimation (described in Section 11.2) and (2) to demonstrate the testing capability of the Rice Shaking table, this section will focus mainly on these two aspects.

11.4.1 Actual Maximum Rotations Observed in the Dynamic Testing and Corresponding Estimations Provided by the Proposed Simplified “α” Method

As extensively explained in Section 11.2, the “α” method provides an estimate, through Eq. (11.16), of the peak rotation response that a given eccentric structure develops under seismic excitation, given the value of the torsional factor “α” and the maximum deformation of the “equivalent SDOF oscillator” (as defined in Section 11.2.12).

The maximum deformation of the “equivalent SDOF oscillator” has been experimentally determined, for each earthquake excitation, by testing the structural model with the weight positioned on the top carbon fiber plate for coincident centers of mass and of rigidity. The value of the measured displacements developed by the model were then multiplied by the length scale factor (\( \lambda_1 = 43 \)) in order to obtain the corresponding displacement of the prototype structure (rotations remain constant in scaling). The value of the torsional factor “α” for the prototype structure (characterized by \( \gamma = 1.26 \) and \( D_e = 28 \text{ m} \)) has been estimated using both Eq. (11.20) a, b, and Eq. (11.22) a, b since in the present case of regularly spaced base isolators both expressions are valid. These two values of “α” lead to slightly different max. rotation estimations. In describing the results, the estimated maxi-
mum rotation response obtained using Eq. (11.20) a, b. and Eq. (11.22) a, b are referred to as Alpha1 and Alpha2 estimations, respectively.

Figs. 11.38 through Fig. 11.45 compare the Alpha1 and Alpha2 estimations of the maximum rotation response with the actual maximum rotation measured during the dynamic testing. The model structure was excited at the base with eight different earthquake records scaled according to Table 11.5.

All the experimental results obtained confirm the numerical results presented in Section 11.2.13: the maximum rotation responses estimated using the proposed "α" method are sufficiently close to the actual (measured) ones. Notice that in most cases the max rotation response estimated using the simplified "α" method provides an upper bound for the actual maximum rotation response. Furthermore, it is worth noting the high predictive capability of the "α" method in the cases of Parkfield 1966, El Centro 1940, Tolmezzo 1977 and Brienza 1977 earthquake excitations. As was already predicted by the numerical investigation (see Fig. 11.18), the actual maximum rotation responses developed under the seismic excitation relative to the Friuli Breginj earthquake record are much smaller than the ones predicted by the "α" method. Furthermore, note that in all cases, the Alpha1 and Alpha2 estimations are close.

These experimental results confirm both the validity of the proposed simplified "α" method for max rotation response estimation and the excellent testing capability of the Rice Shaking Table.
Figure 11.38 Actual Max. Rotation responses and their corresponding estimations by the “α” method for *El Centro 1940* excitation
Figure 11.39 Actual Max. Rotation responses and their corresponding estimations by the "α" method for Parkfield 1966 excitation
Figure 11.40 Actual Max. Rotation responses and their corresponding estimations by the "α" method for Montenegro Petrovac 1979 excitation.
Figure 11.41 Actual Max. Rotation responses and their corresponding estimations by the “α” method for *Montenegro Bar 1979* excitation
Figure 11.42 Actual Max. Rotation responses and their corresponding estimations by the “α” method for *Friuli Breginj 1976* excitation
Figure 11.43 Actual Max. Rotation responses and their corresponding estimations by the "α" method for *Tolmezzo 1977* excitation
Figure 11.44 Actual Max. Rotation responses and their corresponding estimations by the "α" method for a Synthetic Earthquake from Eurocode excitation.
Figure 11.45 Actual Max. Rotation responses and their corresponding estimations by the "α" method for Brienza 1977 excitation
11.4.2 Other Experimental Results

Fig. 11.46 shows the experimental values of the maximum longitudinal displacement response versus the model transversal eccentricity. Notice that, for a given base excitation, the maximum longitudinal displacement response developed by a given eccentric system does not change significantly for increasing values of transversal eccentricity. As for the numerical simulation results presented in Section 11.2.11, the variations in maximum longitudinal displacement do not exceed ±20% of the value obtained for the corresponding non-eccentric structure. $Y_{\text{max}-\text{ne}}$. Notice the similarities between the experimental results in Fig. 11.46 and their numerical counterparts in Fig. 11.7.

Fig. 11.47 compares the values of "α" provided by Eq. (11.20) a and b. and Eq. (11.22) a and b. with the corresponding values of $\frac{\theta_{\text{max}}}{Y_{\text{max}-\text{ne}}}$ obtained experimentally. $\theta_{\text{max}}$ is the maximum rotation response developed by the model structure during the test and $Y_{\text{max}-\text{ne}}$ is the maximum longitudinal displacement response developed by the model structure with no eccentricity converted to prototype dimensions. Notice that the values of "α" provided by Eq. (11.20) a and b. and Eq. (11.22) a and b represent, for most cases, an upper bound for the experimental values of $\frac{\theta_{\text{max}}}{Y_{\text{max}-\text{ne}}}$. Only the data relative to the Brienza 1977 earthquake record clearly significantly exceed the values of "α" provided by the analytical formulations, but only for large values of transversal eccentricity.

Figs. 11.48 a and b show the free vibration rotation response developed by the model structure when the weights are positioned for eccentricities between centers of mass and
stiffness equal to 2 and 4 % of $D_e$, respectively. Notice the beating behavior which was already predicted analytically in Section 11.2.4 (Eq. (11.5) c). Fig. 11.49 shows the free vibration longitudinal displacement response developed by the model structure with the same weight configuration described above for Fig 11.48. Notice the absence of any visible beating phenomenon. This fact is consistent with theoretical results of Eq. (11.5) a for structures having only a small transversal eccentricity (see Fig. 11.3 for the values of the \( \frac{\Omega_1 - 1}{\Omega_1 - \Omega_3} \) coefficient).

Fig. 11.50 shows the experimentally-determined free vibration time evolution of rotation versus longitudinal displacement responses. The weights are positioned in order to have a transversal eccentricity equal to 4 % of $D_e$. Notice that, as theoretically predicted, the maximum rotation response occurs always in the proximity of one of the two extremes of the longitudinal oscillation response. Furthermore, it is worth to mention the strong similarities between the experimental results represented in Fig. 11.50 and the numerical results shown in Fig. 11.10. These similarities further validate both the theory presented in Section 11.2 and the careful construction of the reduced scaled model built.

Fig. 11.51 compares longitudinal and transversal vibrations developed by the model structure in free vibrations. Again, the weights are positioned for a transversal eccentricity of 4 % of $D_e$. Notice that, except for some small transversal displacements at the beginning of the free vibrations, the model does not develop any transversal motion. This is in perfect agreement with the theoretical predictions of Eq. (11.5) b in Section 11.2.4 for systems
Figure 11.46 Measured maximum longitudinal model displacement for increasing transversal eccentricity
Figure 11.47 Measured values of $\frac{\theta_{\text{max}}}{Y_{\text{max}} - y_e}$ and values of "α" for increasing values of transversal eccentricity
Figure 11.48 Free vibration rotation response of model structure: (a) system eccentricity = 2% of $D_e$ (b) system eccentricity = 4% of $D_e$
Figure 11.50 Time evolution of rotation versus longitudinal displacement responses for the model structure in free vibration (system eccentricity = 4 % of $D_e$) with transversal eccentricity only (as it is the case of the reduced scale model).
Figure 11.49 Free vibration longitudinal displacement response of model structure:
(a) system eccentricity = 2% of $D_e$; (b) system eccentricity = 4% of $D_e$
Figure 11.51 Longitudinal and transversal displacement responses developed by the model structure in free vibrations (system eccentricity = 4 % of $D_e$)
11.4.3 Accuracy in Earthquake Reproduction

Figs. 11.52 through Fig. 11.55 compare the earthquake ground displacement time histories to be reproduced on the Rice shaking table with the corresponding measured table displacement time histories. Notice the accuracy of Rice shaking table in reproducing actual earthquake ground displacement records.

Figs. 11.56 through Fig. 11.59 compare the earthquake ground acceleration time histories to be reproduced on the Rice shaking table with the corresponding measured table acceleration time histories. Notice that, also for the acceleration, the matching between target and measured time histories is very good. This is a further confirmation of the validity of the calibration performed for this shaking table and described extensively in Chapters 3 through Ch. 10 of this thesis.

All of the results illustrated in this section are relative to the scaled time histories used in the dynamic experimentation set to experimentally verify the validity of the simplified “α” method for max. rotational response estimation (see Table 11.5).
Figure 11.52 Earthquake ground displacement time history (solid line) and measured shaking table reproduction (dotted line)
Figure 11.53 Earthquake ground displacement time history (solid line) and measured shaking table reproduction (dotted line)
Figure 11.54 Earthquake ground displacement time history (solid line) and measured shaking table reproduction (dotted line)
Figure 11.55 Earthquake ground displacement time history (solid line) and measured shaking table reproduction (dotted line)
Figure 11.56 Earthquake ground acceleration time history (solid line) and measured shaking table reproduction (dashed-dotted line)
Figure 11.57 Earthquake ground acceleration time history (solid line) and measured shaking table reproduction (dashed-dotted line)
Figure 11.58 Earthquake ground acceleration time history (solid line) and measured shaking table reproduction (dashed-dotted line)
Figure 11.59 Earthquake ground acceleration time history (solid line) and measured shaking table reproduction (dashed-dotted line)
11.5 CONCLUSIONS

This chapter describes the shake table test performed with the Rice shaking table facility in order to provide support for a simplified theory that predicts the max. rotation response developed in eccentric seismic isolated buildings subjected to earthquake excitation. The experimental results confirm the validity of the simplified theory for max. rotation evaluation ("α" method) and are in very good agreement with numerical simulation results. The agreement between the experimental results and the results of the numerical investigation performed in developing the simplified theory confirmed also the capability of the Rice shaking table to perform accurate dynamic testing.
CHAPTER 12
CONCLUSIONS
12.1 SUMMARY OF WORK

The main objective of this research was to develop an analytical/experimental method to calibrate and optimize the performance of a servo-hydraulic shaking table facility, with particular emphasis to the Rice University shaking table. This objective was achieved through the following four phases. The first phase involved the development of a comprehensive analytical dynamic model of the shaking table. The second phase was concerned with the development and numerical validation of a procedure to accurately estimate the actual transfer function of the shaking table. The third phase includes the correlation between experimental and analytical results as well as an exhaustive study of the table sensitivities to control parameters and payload dynamic characteristics. In the fourth and final phase shaking table, set up for optimum performance is used to perform a dynamic test in order to validate the capabilities of the shaking table facility not only in reproducing accurately base excitation time histories, but also in acquiring reliably the dynamic response of the test structure. Furthermore, the structural dynamic test performed also served the purpose of verifying experimentally a simplified theory for predicting the maximum torsional response of eccentric seismic isolated structures subjected to base excitation.

The analytical model developed includes the proportional, integral, derivative, feed-forward, and differential pressure controls, and accounts for the time delay in the servovalve response, the compressibility of the actuator fluid, the flexibility of the foundation mass, and the dynamic characteristics of potential test structures.

In order to study the actual dynamic behavior of the shaking table, the transfer function between the commanded (target) and actual (measured) table acceleration had to be esti-
mated. The shaking table transfer function was estimated via spectrum estimation as the ratio of the estimated input-output cross spectrum to the estimated input spectrum, the spectral estimates being performed using Bartlett's procedure. A discrete-time stochastic simulation using ARMA models was conducted to understand the statistical relationship between estimated auto-/cross-power spectral density functions and their true underlying theoretical counterparts. This numerical simulation was used to determine the optimal number of non-overlapping time windows to be used in the Bartlett’s spectral estimation procedure and to select the appropriate anti-leakage window. Also, a numerical simulation of the adopted table transfer function estimation method was performed to assess the influence of various sources of errors on the final estimation of the table transfer function.

The estimated actual table acceleration transfer functions were correlated with their analytically predicted counterparts. This correlation study resulted in a thorough understanding of the actual shaking table dynamics and in an analytical mode thereof of high predictive capability.

A 1/40 scaled model representative of a five story, 20 m by 20 m in plane seismic isolated building structure was constructed and tested on the shaking table under 11 eccentricity conditions and 8 different seismic excitations. The earthquake accelerograms used correspond to the El Centro 1940, Parkfield 1966, Montenegro Petrovac 1979, Montenegro Bar 1979, Friuli Breginj 1976, Tolmezzo 1977, Brienza 1977, and a synthetic earthquake from the Eurocode.
12.2 SUMMARY OF FINDINGS

The development of an analytical, physically-based model of the Rice servo-hydraulic shaking table and the correlation between the analytically predicted and the experimentally determined table dynamic response provided a complete understanding of the shaking table dynamics and its sensitivities to both control and payload parameters. The correlation study showed that the introduction of the servovalve time delay in the analytical modeling of the shaking table was essential in order to achieve a high degree of correlation between analytical predictions and experimental results. This strong correlation is fundamental to confidently rely on the shaking table facility as a precise research tool. The exhaustive investigation of the dynamic behavior of the Rice shaking table indicated that the table, in its present condition, is able to reproduce accurately model earthquake acceleration time histories in the frequency bandwidth from 0 to 75 Hz. Furthermore, the analysis performed shows that the table acceleration reproduction accuracy is not affected by the presence of large (weight-wise) test models with a fundamental frequency up to 20 Hz. Test models having a higher fundamental frequency do affect the shaking table performance and require a modification of the optimal control gain setting that can be guided using the strong predictive ability of the analytical model of the shaking table facility.

The dynamic testing of a scaled model of an eccentric seismic isolated building structure successfully verified experimentally the predictive ability of a proposed simplified method for determining the torsional response of base isolated irregular structures. The consistency of these experimental results with those obtained through numerical simulation also established that the Rice shaking table facility can be used to perform advanced experi-
mental research in structural dynamics.

12.3 Future Research Work

Further developments of the research work reported here may include the following: (1) completion of a comprehensive stability analysis of the shaking table system; (2) improvement of the analytical modeling of the servo-valve transfer function, and (3) development and implementation of a batch table acceleration control algorithm which, based on the calibrated analytical model of the shaking table, is able to modify the input signal in order to obtain a table acceleration time history as close as possible to the target one. The actual implementation of such an algorithm requires an acute understanding of the table performance limits and of the potential dynamic interaction between test specimen and shaking table.
A.1 The Flexible Foundation

This Appendix reports about the analytical modeling and experimental investigation of the dynamic properties of the foundation, which was found to have a finite value of lateral stiffness and therefore to deform under lateral loads.

Fig. A.3 schematically represents the three concrete blocks used as foundation for the shaking table. These concrete blocks are connected to each other through six steel bars post tensioned at 2,000 lbs. The steel bars are then clamped to the top flange of an 8" by 6" I beam, and the bottom flange of the I-beam is connected to the strong floor of the laboratory through eight 3/4" bolts. In order to determine the dynamic characteristic of such a system, several analytical modelings were developed. Sections A.2 through A.4 present the analytical modeling which provided the best match between the foundation experimental dynamic behavior and its numerical counterpart. Section A.6 reports on the experimentally determined dynamic behavior of the foundation.

A.2 Lateral Stiffness

Fig. A.1 shows the MDOF analytical modeling which proved to be the best in capturing the flexible foundation dynamic behavior. This MDOF analytical model has the following characteristics:

- the 6 steel bars are considered to act as an elastic spring of given lateral stiffness, $K_{6b}$ = lateral stiffness of 6 steel bars;
- the elastic system composed by the two I-beam and by the eight bolts connecting the
beam to the strong floor has been considered to act as an elastic spring of lateral
stiffness $K_i$.

**A.2.1 Determination of the Lateral Stiffness of Steel Bars $K_{b6}$**

Fig. A.2 schematically represents the way the eight post-tensioned bars are embedded
within the three concrete blocks. It is known that:

- each steel bar has a diameter of 1" and sits in a 1' and 1/2' diameter hole;
- the corrugated surface of the concrete blocks gives a gap between the three concrete
blocks of about 1/8".

Given these physical and geometrical characteristics it was assumed that, under a lateral
load, the steel bars deform as if rigidly clamped at a distance equal to their diameter. This
results in a clamped-clamped lateral system separated by a 2 and 1/8" gap.

The lateral stiffness of such a system is known to be:

$$K_b = \frac{12 \cdot E \cdot I}{h^3} \quad (A.1)$$

where:

- $E$ is the elastic Modulus of Steel ($= 29,000,000$ psi);

- $I$ is the modulus of inertia of the steel bar. $I = \frac{\pi \cdot (radius)^4}{4} = 0.049 \quad [in^4]$;

- "h" is the gap between the two clamps, $2 \text{ and } 1/8" = 2,125 \text{ [in].}$

substituting the values for $E$, $I$ and "h" of the system in consideration in Eq. (A.1), leads to
the following value for the lateral stiffness, $K_b$, provided by each post-tensioned bar:

$$K_b = 1.777 \cdot 10^6 \text{ [lbs/in]}$$  \hspace{1cm} (A.2)

The lateral stiffness provided by the whole group of six bars is then obtained by multiplying the value expressed by Eq. (A.2) by 6:

$$K_{b6} = 10.66 \cdot 10^6 \text{ [lbs/in]}$$  \hspace{1cm} (A.3)

### A.2.2 Lateral Stiffness of the I-beam/bolts System

**Lateral stiffness of the I-beam**

The I-beam transfers the lateral forces applied to its upper flange by the concrete blocks which rest on its top, to the bolts connected to its lower flange. In transmitting such forces the I-beam deforms mainly in shear. Fig. A.4 shows how the shear deformation was accounted for. The lateral shear stiffness of the I-beam, $K_s$, was computed as follows:

$$K_s = \frac{G \cdot A}{L}$$  \hspace{1cm} (A.4)

where:

- $G$ is the tangential elastic modulus;
- $A$ is the cross section of the I beam subjected to shear. In this case the area $A$ is equal to $1/4''$ (the thickness of both flange and web) times $144''$ (the total length of the beam). This results in an area of 36 square inches;
- $L = 14$ inches ($3'' + 8'' + 3''$).

Substitution of these values in Eq. (A.4) leads to the following estimation of the lateral
stiffness, $K_s$, provided by the I beam:

$$K_s = 28.5 \cdot 10^6 \text{ [lbs/in]}$$  \hspace{1cm} (A.5)

Accounting for the fact that there are two beams in parallel the total stiffness is:

$$K_{st} = 57 \cdot 10^6 \text{ [lbs/in]}$$  \hspace{1cm} (A.6)

**Lateral stiffness of the bolts**

Fig. A.5 schematically represents the way the bottom flange of the I beam is connected to the laboratory strong floor through 1 and 1/2” diameter bolts. The gap between the head of the bolt and the top of the thread of the lab floor (which provides the bottom clamping for the bolt) is approximately 2”. Given these characteristics the lateral stiffness of each bolt, $K_b$, can be estimated as follows:

$$K_b = \frac{12 \cdot E \cdot I_b}{h_b^3}$$  \hspace{1cm} (A.7)

where:

- $E$ is the Young’s Modulus for Steel ($= 29,000,000$ psi);

- $I_b$ is the modulus of inertia of the bolt. $I_b = \frac{\pi \cdot (\text{radius})^4}{4} = 0.2485 \text{ [in}^4\text{]}$;

- $h_b$ is the distance between top and bottom clamps of the bolt. $h_b = 2.00 \text{ [in]}$.

Substitution of these values of $E$, $I_b$ and $h$ in Eq. (A.7) leads to:

$$K_b = 10.809 \cdot 10^6 \text{ [lbs/in]}$$  \hspace{1cm} (A.8)
Accounting for the fact that 8 bolts are used (in parallel) to connect the bottom flange of each I-beam to the strong floor of the laboratory, the total lateral stiffness provided by the bolts can be estimated to be:

$$K_{bt} = 86.48 \cdot 10^6 \text{ [lbs/in]}$$  \hspace{1cm} (A.9)

**Total stiffness of the beam-bolt system**

Accounting for the fact that each I-beam is connected in series with the eight bolts, the total stiffness $K_i$ of each I-beam-eight-bolts system can be computed using the following formulation:

$$\frac{1}{K_i} = \frac{1}{K_{bt}} + \frac{1}{K_s}$$  \hspace{1cm} (A.10)

Substitution of Eq. (A.5) and Eq. (A.9) in Eq. (A.10), leads to the following value for the lateral stiffness $K_i$ of the "I-Beam-bolts" system:

$$K_i = 34 \cdot 10^6 \text{ [lbs/in]}$$  \hspace{1cm} (A.11)
Figure A.1  MDOF Model of the Shaking Table Foundation

Figure A.2  Steel bar connecting two different concrete blocks
Figure A.3  Shaking Table Foundation
Figure A.4  I-beam deformation under shear
Figure A.5  Cross section of the bottom bolt
A.3 Masses

Mass of the concrete blocs

Each concrete block measures 12' by 12' by 1' which gives a volume of 144 cubic feet. Assuming the unit weight of the reinforced concrete to be equal to 150 lbs per cubic foot, the weight of each concrete block could be estimated to be equal to 21,600 lbs. This correspond to a mass of 55.95 lbs sec^2/in. The weight of the top block has to be increased with respect to that of the bottom two blocks, because it is rigidly loaded with the steel plate, the servo valve and the actuator equipments\(^{(1)}\). These equipments have the following estimated weights:

<table>
<thead>
<tr>
<th>Component</th>
<th>Weight</th>
<th>[lbs]</th>
</tr>
</thead>
<tbody>
<tr>
<td>steel plate</td>
<td>4050.0(^{(2)})</td>
<td></td>
</tr>
<tr>
<td>actuator</td>
<td>153.0(^{(3)})</td>
<td></td>
</tr>
<tr>
<td>accumulator/servo valve connector</td>
<td>27.0(^{(3)})</td>
<td></td>
</tr>
<tr>
<td>base swivel</td>
<td>66.0(^{(3)})</td>
<td></td>
</tr>
<tr>
<td>rod-end swivel</td>
<td>60.0(^{(3)})</td>
<td></td>
</tr>
<tr>
<td>servo valve</td>
<td>12.0(^{(2)})</td>
<td></td>
</tr>
<tr>
<td>accumulators</td>
<td>96.0(^{(3)})</td>
<td></td>
</tr>
<tr>
<td>slip table</td>
<td>1260</td>
<td></td>
</tr>
<tr>
<td>Other Mechanical Equipment(^{(4)})</td>
<td>180.0(^{(2)})</td>
<td></td>
</tr>
</tbody>
</table>

---

1. The element that concur to the mass of the top concrete block do not include the slip table mass. This is in accordance with what assumed in the analytical modeling of the shaking table described in Chapter 3.
2. Estimated value.
3. Information provided by the manufacturer of the Shaking Table: M.T.S. Mechanical Testing System.
4. This includes: rails, accumulator/servo valve connectors, structure for protection of the servovalve and actuator.
The total weight of these equipments, which are rigidly attached to the top concrete block, is equal to 4644 lbs.

This leads to a total weight of the top block of 26.244 lbs (21.600 + 4.644), which corresponds to a mass of 67.99 lbs sec^2/in.

### A.4 Modes of Vibration

Referring to the dynamic model represented Fig. A.1, the following masses and stiffness were assumed:

\[
\begin{align*}
m_1 &= 0.0 \quad [\text{lbs sec}^2/\text{in}]; \\
m_2 &= 56.0 \quad [\text{lbs sec}^2/\text{in}]; \\
m_3 &= 56.0 \quad [\text{lbs sec}^2/\text{in}]; \\
m_4 &= 68.0 \quad [\text{lbs sec}^2/\text{in}]; \\
K_1 = K_6 &= 34.00 \times 10^6 \quad [\text{lbf/in}]; \\
K_2 = K_3 = K_4 &= 10.66 \times 10^6 \quad [\text{lbf/in}].
\end{align*}
\]

These values led to the following modal frequencies and modal shapes:

- \( f_1 = 27.10 \) Hz, \( f_2 = 80.00 \) Hz, \( f_3 = 122.00 \) Hz

- \( \phi_1 = \begin{bmatrix} 0.09 \\ 0.36 \\ 0.58 \\ 0.71 \end{bmatrix} \) or, normalizing the maximum displacement to unity, \( \phi_1 = \begin{bmatrix} 0.12 \\ 0.50 \\ 0.81 \\ 1.00 \end{bmatrix} \)
\[ \Phi_2 = \begin{bmatrix} 0.18 \\ 0.75 \\ 0.32 \\ -0.538 \end{bmatrix} \text{ or. normalizing the maximum displacement to unity.} \quad \Phi_2 = \begin{bmatrix} 0.24 \\ 1.00 \\ 0.42 \\ -0.71 \end{bmatrix} \]

\[ \Phi_3 = \begin{bmatrix} 0.13 \\ 0.56 \\ -0.76 \\ 0.27 \end{bmatrix} \text{ or. normalizing the maximum displacement to unity.} \quad \Phi_3 = \begin{bmatrix} 0.17 \\ 0.73 \\ -1.00 \\ 0.35 \end{bmatrix} \]

A.5 Modal Masses

Through the analytical procedure described in Appendix E the following modal masses and modal weights were found:

<table>
<thead>
<tr>
<th>Modal mass [lbs sec^2/s/in]</th>
<th>Modal Weight [lbs]</th>
<th>Percentage of the total mass [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m_1 = 64.966 )</td>
<td>( w_1 = 168.3 )</td>
<td>( p_1 = 93.71 )</td>
</tr>
<tr>
<td>( m_2 = 3.940 )</td>
<td>( w_2 = 10.2 )</td>
<td>( p_2 = 5.68 )</td>
</tr>
<tr>
<td>( m_3 = 419 )</td>
<td>( w_3 = 1.1 )</td>
<td>( p_3 = 0.61 )</td>
</tr>
</tbody>
</table>
A.6 Dynamic Properties of the Shaking Table Foundation: Experimental Results

This Section presents the experimental results obtained exciting the base foundation through the shaking of the slip table. The transfer function of the flexible foundation has been obtained from the accelerations time histories recorded by accelerometers rigidly connected to the foundation mass.

A.6.1 Modes of Vibration

Figs. A.8 and A.9 show the magnitude of the transfer function of the shaking table foundation at the center-point of the first, second, third concrete bloc, and at the top of the I-Beam flange. The magnitude of the transfer function at the center-point of each concrete block has been derived by averaging the transfer functions obtained at the top and the bottom of each block. As it will be explained in the next sections all the three concrete blocks behaved almost rigidly (the accelerations recorded at the top and at the bottom of each concrete block are almost the same). This indicates that each concrete block slips rigidly on the top of the other and is in agreement with the assumptions used for the development of the analytical modeling.

The experimentally determined shaking table base transfer function plotted in Fig. A.8 indicates that the first mode of vibration of the table base occurs at approximately 25 - 26 Hz which is in very good agreement with the analytical prediction obtained from the analytical modeling, developed in Sections A.2 through A.5. Fig. A.8 also indicates that the second a mode of vibration occurs at approximately 35 - 40 Hz. This frequency is much lower than the one predicted by the analytical model. A careful inspection of the transfer function in correspondence of this frequency shows that the bottom concrete block and the
I-Beam web have the same amplitude of acceleration, behaving almost as a single degree of freedom. Furthermore, the top and middle concrete blocks have similar amplitudes of acceleration. This indicates that at a frequency of about 35–40 Hz the shaking table foundation behaves like a two degrees of freedom system with one degree of freedom in correspondence of the bottom concrete block and the I beam, and the other degree of freedom in correspondence of the top and middle concrete block. This 2 DOF behavior is very different from that predicted by the analytical modeling, as it is the frequency at which this second mode of vibration occurs (about 30–40 Hz instead of approximately 80 Hz). This suggests that in the second mode of vibration of the flexible foundation some unexpected phenomena which have not been predicted by the analytical modeling occur. Nonetheless, these not-modeled phenomena have an amplitude which is much smaller than that of the first mode of vibration which is perfectly captured by the analytical modeling. For this reason the analytical model described in Sections A.2 through A.4 was considered to be accurate enough.
Figure A.6  Magnitude of the Transfer Function at the center-points of the concrete blocs and at the top flange of the I-beam of the shaking table foundation
A.6.2 First Mode of Vibration

Fig. A.9 shows the experimentally-determined table foundation transfer function in correspondence of the first mode of vibration natural frequency. This frequency of about 25.5 Hz is in good agreement with the 27.1 Hz predicted by the analytical modeling. The magnitude of the transfer function at for the center-point of the top, middle, bottom concrete bloc, and for the I-Beam flange are respectively:

0.640, 0.360, 0.20, 0.065.

This gives an experimentally-estimated first mode of vibration equal to\(^1\):

\[
\Phi_{1\text{experimental}} = \begin{bmatrix} 0.10 \\ 0.31 \\ 0.56 \\ 1.00 \end{bmatrix}
\]

which is in good accordance with that predicted by the analytical model:

\[
\Phi_{1\text{analytical model}} = \begin{bmatrix} 0.12 \\ 0.50 \\ 0.81 \\ 1.00 \end{bmatrix}
\]

---

1. The magnitude normalization was performed in order to have the maximum displacement equal to one. The coordinate system is equal to that used in the analytical modeling, namely: the first DOF corresponds to the motion of the I-Beam flange, the second DOF corresponds to the motion of the bottom concrete bloc, the third DOF corresponds to the motion of the middle concrete bloc, and the fourth DOF corresponds to the motion of the top concrete bloc.
Figure A.7  Magnitude of the Transfer Function at the center-points of the concrete blocks and at the top flange of the I-beam of the shaking table foundation: First Mode of Vibration
A.6.3 Vibrations Within each Concrete Block

Figs. A.8 through A.10 compare the magnitude of the transfer function obtained from the acceleration time histories recorded at the top and at the bottom of each concrete block. The top block shows a detectable difference between the amplitude of the two transfer functions in correspondence of the first mode of vibration resonant peaks. The amplitude of motion at the bottom of the top concrete block can be estimated to be approximately 30% smaller than that recorded at the top of the block. The center and bottom blocks do not show any appreciable difference between the amplitude of the motion at the top and at bottom of each block at the frequency corresponding to the first mode of vibration.

Fig. A.11 compares the amplitude of the transfer function obtained for the top flange of the I-Beam with the amplitude of the transfer function obtained for the center-point of the web of the I-Beam. The amplitude of the motion recorded at the top flange of the I-Beam can be estimated to be approximately the double of the motion recorded in the web. This suggests the I-Beam deforms mainly in shear.
Figure A.8  Magnitude of the Transfer Function of third (Top) concrete block of the Shaking Table Foundation (Base)
Figure A.9  Magnitude of the Transfer Function of second (Middle) concrete block of the Shaking Table Foundation (Base)
Figure A.10 Magnitude of the Transfer Function of the first (Bottom) concrete block of the Shaking Table Foundation (Base)
Figure A.11 Magnitude of the Transfer Function of the I-Beam that supports the concrete blocks of the Shaking Table Foundation
A.6.4 Modal Damping

Fig. A.12 shows the magnitude of the acceleration transfer function obtained for the center-point of the top concrete block. This transfer function has been used to experimentally estimate the damping of the shaking table foundation through the half-power-bandwidth method. According to this estimation technique, the damping coefficient, $\xi$, of a structure can be estimated as:

$$\xi = \frac{f_2 - f_1}{f_2 + f_1} \quad \text{(A.12)}$$

where $f_1$ and $f_2$ are the frequencies at which the power spectrum of the structure response has an amplitude which is half that of the resonant peak.

Fig. A.12 plots the amplitude spectrum of the response of the structure. $f_1$ and $f_2$ are the frequencies at which the amplitude of the response is $\frac{1}{\sqrt{2}}$ the amplitude of the response at the resonant peak.

Tracing a horizontal line at an amplitude of 0.452 (corresponding to $\frac{1}{\sqrt{2}}$ the amplitude of the response at the resonant peak of 0.639), $f_1$ and $f_2$ could be estimated to be equal to 24.2 and 27 Hz, respectively.

Substituting these values into Eq. (A.12) leads to:

$$\xi = \frac{27 - 24.2}{27 + 24.2} = 0.0547 \approx 5.5 \%$$
Figure A.12 Magnitude of the transfer function of the center-point of top concrete block of the shaking table foundation (first mode of vibration) - estimation of the modal damping through half-power-bandwidth method
B.1 The Three Story Steel Model Built at Rice University

Figure B.1 shows the three story model built at Rice University. It is a steel frame model made of members having a “T” shaped cross section (2” by 2” and 0.25” thick). In order to simulate the mass of a real building structure, several concrete blocks, each one weighting approximately 150 lbs, were attached to the floors of the steel structure of the model using high strength steel bolts (7/16” in diameter). The model was tested under two different weight configuration:

- **"450 lbs Model"**:  
  This configuration had only one concrete block attached to each floor of the steel structure.

- **"900 lbs Model"**:  
  This configuration had two concrete blocks attached to each floor of the steel structure.

Each column of the model was clamped to the top surface of the aluminum slip table through 4 high strength steel bolts of 7/16” diameter.
Figure B.1  Structure of the three story steel model built at Rice University
B.2 Analytical Modeling of the Steel Structure as a Shear Building

This very simple analytical model was developed considering a shear behavior of the steel structure, as represented in Fig. B.2. The lateral rigidity $k_1$ of each floor was computed using the following formula:

$$k_1 = N \cdot 12 \cdot \frac{E \cdot I}{L^3} \quad (B.1)$$

where

- $N$ is the total number of columns attached to each floor;
- $E$ is the elastic (Young’s) modulus of steel;
- $I$ is the moment of inertia of the cross section of the steel member in the columns computed with respect to the axis orthogonal to the direction of deformation;
- $L$ is the height of each story (column length).

In order to account for the not perfectly clamping of the columns, the lateral rigidity given by Eq. (B.1) was reduced as follows:

- lateral rigidity of the bottom floor - 30 %;
- lateral rigidity of the intermediate floor - 40 %;
- lateral rigidity of the top floor - 50 %.

These reductions led to the following lateral stiffness matrix:

$$K = \begin{bmatrix} 11,323 & -5,226 & 0 \\ -5,226 & 9,581 & -4,355 \\ 0 & -4,355 & 4,355 \end{bmatrix} \text{[lbs/in]} \quad (B.2)$$
B.2.1 Dynamic Characteristics of the "450 lbs Model"

Considering the steel structure loaded with three concrete blocks of 150 lbs each (one per floor), the mass matrix has the following value:

\[
M = \begin{bmatrix}
0.3882 & 0 & 0 \\
0 & 0.3882 & 0 \\
0 & 0 & 0.3882
\end{bmatrix} \text{ [lbs sec}^2/\text{in]} \quad (B.3)
\]

Solving the eigenvalue problem for the mass and stiffness matrices \( M \) and \( K \) given by Eqs. (B.3) and (B.2), the following dynamic characteristics were obtained:

*Modal frequencies of vibration*

\( f_1 = 8.45 \text{ Hz}, f_2 = 22.42 \text{ Hz}, f_3 = 32.77 \text{ Hz} \).

*Mode shapes of vibration*

\[
\Phi_1 = \begin{bmatrix} 0.2928 \\ 0.5731 \\ 0.7654 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 0.6605 \\ 0.4576 \\ -0.5953 \end{bmatrix}, \quad \Phi_3 = \begin{bmatrix} 0.6914 \\ -0.6799 \\ 0.2446 \end{bmatrix}.
\]

B.2.2 Dynamic Characteristics of the "900 lbs Model"

Considering the steel structure loaded with six concrete blocks\(^1\) of 150 lbs each, the mass matrix has the following value:

\[
M = \begin{bmatrix}
0.7764 & 0 & 0 \\
0 & 0.7764 & 0 \\
0 & 0 & 0.7764
\end{bmatrix} \text{ [lbs sec}^2/\text{in]} \quad (B.4)
\]

Solving the eigenvalue problem for the mass and stiffness matrices \( M \) and \( K \) given by Eqs. (B.3) and (B.2), the following dynamic characteristics were obtained:

*Modal frequencies of vibration*

\( f_1 = 8.45 \text{ Hz}, f_2 = 22.42 \text{ Hz}, f_3 = 32.77 \text{ Hz} \).

*Mode shapes of vibration*

\[
\Phi_1 = \begin{bmatrix} 0.2928 \\ 0.5731 \\ 0.7654 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 0.6605 \\ 0.4576 \\ -0.5953 \end{bmatrix}, \quad \Phi_3 = \begin{bmatrix} 0.6914 \\ -0.6799 \\ 0.2446 \end{bmatrix}.
\]

---

1. Two blocks per floor.
(B.4) and (B.2), the following dynamic characteristics were obtained:

**Modal frequencies of vibration**

\[ f_1 = 5.97 \text{ Hz}, \ f_2 = 15.85 \text{ Hz}, \ f_3 = 23.17 \text{ Hz} \]

**Mode shapes of vibration**

\[
\begin{align*}
\Phi_1 &= \begin{bmatrix} 0.2928 \\ 0.5731 \\ 0.7654 \end{bmatrix}, \\
\Phi_2 &= \begin{bmatrix} 0.6605 \\ 0.4576 \\ -0.5953 \end{bmatrix}, \\
\Phi_3 &= \begin{bmatrix} 0.6914 \\ -0.6799 \\ 0.2446 \end{bmatrix}
\end{align*}
\]

Figure B.2 The degrees of freedom selected for the Shear Type modeling of the steel frame built at Rice University
B.3 Analytical Modeling Obtained Through the CAL90 Finite Element Program

A more refined analytical modeling for the steel structure was obtained introducing the rotational flexibility in correspondence of the joints between the vertical and horizontal members. This analytical modeling was performed using the finite element program CAL90. Column beam elements which neglect axial deformation were used. Fig. B.3 shows the degrees of freedom considered for this modeling.

Using static condensation to eliminate the rotational degree of freedom, the following lateral stiffness matrix was obtained:

\[
K = \begin{bmatrix}
16,198 & -8,640 & 1,008 \\
-8,640 & 14,884 & -7,346 \\
1,008 & -7,346 & 6424
\end{bmatrix} \text{ [lbs/in]} \quad (B.5)
\]
Figure B.3  Degrees of freedom selected in the modeling for the steel structure through the CAL90 finite element program
B.3.1 Dynamic Characteristics of the “450 lbs Model”

Considering the steel structure loaded with three concrete blocks of 150 lbs each, and taking into consideration the mass of each steel member, the following mass matrix could be estimated:

\[
M = \begin{bmatrix}
0.5782 & 0 & 0 \\
0 & 0.5782 & 0 \\
0 & 0 & 0.5782
\end{bmatrix} \text{ [lbs sec}^2/\text{in]} \tag{B.6}
\]

The eigenvalue problem for the mass and stiffness matrices \(M\) and \(K\) given by Eqs. (B.6) and (B.5) gave the following modal frequencies of vibration

\[
f_1 = 7.2194 \text{ Hz, } f_2 = 21.584 \text{ Hz, } f_3 = 33.859 \text{ Hz}
\]

B.3.2 Dynamic Characteristics of the “900 lbs Model”

Considering the steel structure loaded with six concrete blocks\(^1\) of 150 lbs each, and taking into consideration the mass of each steel member, the following mass matrix could be estimated:

\[
M = \begin{bmatrix}
0.96668 & 0 & 0 \\
0 & 0.96668 & 0 \\
0 & 0 & 0.96668
\end{bmatrix} \text{ [lbs sec}^2/\text{in]} \tag{B.7}
\]

The eigenvalue problem for the mass and stiffness matrices \(M\) and \(K\) given by Eqs. (B.7) and (B.5) gave the following modal frequencies of vibration:

\[
f_1 = 5.53 \text{ Hz, } f_2 = 16.605 \text{ Hz, } f_3 = 26.147 \text{ Hz}
\]

---

1. Two blocks per floor.
B.4 Three Dimensional Analytical Modeling

A third, and even more refined analytical model for the steel structure was developed using the finite element program SAP90. This modeling accounted for the spatial (three-dimensional) behavior of the frame. Fig. B.3 shows the node numbering used in the modeling of the steel frame. Each node (except node 1 through 4) was considered to have 6 degrees of freedom, thus leading to a system with 126 degrees of freedom. The masses of the concrete blocks were positioned in correspondence of nodes 18, 21 and 24.
Figure B.4  Node numbering of the steel frame and coordinate system used for the Three Dimensional Analytical Modeling
B.4.1 Dynamic Characteristics of the “450 lbs” Model

The dynamic analysis performed with SAP90 provided the following values for the natural frequencies of the modes of vibration of the “450lbs” Model:

<table>
<thead>
<tr>
<th>Type of Mode</th>
<th>Frequency [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>First lateral mode in y direction</td>
<td>6.74</td>
</tr>
<tr>
<td>Second lateral mode in y direction</td>
<td>15.91</td>
</tr>
<tr>
<td>Third lateral mode in y direction</td>
<td>19.28</td>
</tr>
<tr>
<td>First lateral mode in x direction</td>
<td>7.07</td>
</tr>
<tr>
<td>Second lateral mode in x direction</td>
<td>20.25</td>
</tr>
<tr>
<td>First torsional mode</td>
<td>12.42</td>
</tr>
</tbody>
</table>
### B.4.2 Dynamic Characteristics of the "900 lbs" Model

The dynamic analysis performed with SAP90 provided the following values for the natural frequencies of the modes of vibration of the "900lbs" Model:

<table>
<thead>
<tr>
<th>Type of Mode</th>
<th>Frequency [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>First lateral mode in y direction</td>
<td>5.15</td>
</tr>
<tr>
<td>Second lateral mode in y direction</td>
<td>11.8</td>
</tr>
<tr>
<td>Third lateral mode in y direction</td>
<td>14.14</td>
</tr>
<tr>
<td>First lateral mode in x direction</td>
<td>5.41</td>
</tr>
<tr>
<td>Second lateral mode in x direction</td>
<td>14.92</td>
</tr>
<tr>
<td>First torsional mode</td>
<td>9.48</td>
</tr>
<tr>
<td>Second torsional mode</td>
<td>17.71</td>
</tr>
<tr>
<td>Third torsional mode</td>
<td>20.94</td>
</tr>
</tbody>
</table>
B.4.3 Reduced System for Table Transfer Function Simulation

Given that a) the analytical model had to simulate the dynamic response of the steel frame along only one direction (the y-direction, according to the adopted coordinate system), and b) the structure is basically symmetric, all the degrees of freedom other than that on the y-direction could be eliminated through static condensation. A further reduction in the number of the degrees of freedom of the system was obtained considering the steel member to be axially not extensible. This resulted in a dynamic system which allowed only the horizontal displacement of nodes 5, 7, 9, 10, 13, 14, 17, 18, 19 (according to the node numbering illustrated in Fig. B.4). This reduction in the number of the degrees of freedom led to the following 9 by 9 lateral stiffness matrix:

\[
K = \begin{bmatrix}
1.0584 & 0.1854 & -0.4374 & 0.0097 & 0.0499 & 0.0025 & -0.4488 & 0.0002 & 0.0001 \\
0.1854 & 1.0578 & 0.0097 & -0.4371 & 0.0025 & 0.0499 & -0.4446 & 0.0002 & 0.0001 \\
-0.4374 & 0.0097 & 0.9824 & 0.1932 & -0.3729 & 0.0138 & 0.0003 & -0.4446 & 0.0003 \\
0.0097 & -0.4371 & 0.1932 & 0.9823 & 0.0138 & -0.3729 & 0.0002 & -0.4446 & 0.0003 \\
0.0499 & 0.0025 & -0.3729 & 0.0138 & 0.5487 & 0.2053 & 0.0000 & 0.0003 & -0.4444 \\
0.0025 & 0.0499 & 0.0138 & -0.3729 & 0.2053 & 0.5487 & 0.0001 & 0.0003 & -0.4444 \\
-0.4488 & -0.4446 & 0.0003 & 0.0002 & 0.0005 & 0.0001 & 0.8907 & -0.0012 & 0.0005 \\
-0.4446 & -0.4446 & 0.0003 & 0.0003 & -0.0012 & 0.8906 & -0.0012 & 0.8894 \\
0.0001 & 0.0001 & 0.0003 & -0.4444 & -0.4444 & 0.0005 & -0.0012 & 0.8894 \\
\end{bmatrix} \text{[lbs/in]} \quad (B.8)
B.4.4 Dynamic Characteristics of the Reduced “450 lbs” Model

Considering the steel frame loaded with only one concrete bloc per floor, the following masses could be estimated:

- \( m_5 = m_6 = m_9 = m_{10} = 0.0758 \text{ [lbs sec}^2\text{/in]} \):
- \( m_{13} = m_{14} = 0.0554 \text{ [lbs sec}^2\text{/in]} \):
- \( m_{17} = m_{20} = m_{23} = 0.0528 + 150/386.4 = 0.4410 \text{ [lbs sec}^2\text{/in]} \):

These values lead to the following mass matrix:

\[
M = \begin{bmatrix}
0.0758 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.0758 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0758 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.0758 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.0554 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0554 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.4410 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4410 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.4410 \\
\end{bmatrix}
\text{ [lbs sec}^2\text{/in]} \quad (B.9)
\]

The eigenvalue problem for the mass and stiffness matrices \( M \) and \( K \) given by Eqs. (B.9) and (B.8), provided the following dynamic characteristics for the system:

---

1. The subscript of the masses refer to the node at which they correspond - See Fig. B.4.
<table>
<thead>
<tr>
<th>Mode of Vibration</th>
<th>Frequency [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>First lateral mode in y direction</td>
<td>6.7457</td>
</tr>
<tr>
<td>Second lateral mode in y direction</td>
<td>15.91</td>
</tr>
<tr>
<td>Third lateral mode in y direction</td>
<td>19.28</td>
</tr>
<tr>
<td>Fourth mode in y direction</td>
<td>48.833</td>
</tr>
<tr>
<td>Fifth mode in y direction</td>
<td>62.174</td>
</tr>
<tr>
<td>Sixth mode in y direction</td>
<td>78.384</td>
</tr>
<tr>
<td>First torsional mode</td>
<td>16.25</td>
</tr>
<tr>
<td>Second torsional mode</td>
<td>46.046</td>
</tr>
<tr>
<td>Third torsional mode</td>
<td>68.843</td>
</tr>
</tbody>
</table>
B.4.5 Dynamic Characteristics of the Reduced “900 lbs” Model

Considering the steel frame loaded with two concrete blocs per floor, the following masses could be estimated:

- \( m_5 = m_6 = m_9 = m_{10} = 0.0758 \) [lbs sec^2/in]:
- \( m_{13} = m_{14} = 0.0554 \) [lbs sec^2/in]:
- \( m_{17} = m_{20} = m_{23} = 0.0528 + 300/386.4 = 0 \) [lbs sec^2/in]:

these values lead to the following mass matrix:

\[
M = \begin{bmatrix}
0.0758 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.0758 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0758 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.0758 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.0554 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.0554 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \text{[lbs sec}^2\text{/in]} \quad (B.10)
\]

The eigenvalue problem for the mass and stiffness matrices \( M \) and \( K \) given by Eqs (B.10) and (B.8), provided the following dynamic characteristics for the system:

---

1. The subscript of the masses refer to the node at which they correspond - See Fig. B.4.
<table>
<thead>
<tr>
<th>Mode of Vibration</th>
<th>Frequency [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>First lateral mode in y direction</td>
<td>5.1575</td>
</tr>
<tr>
<td>Second lateral mode in y direction</td>
<td>11.802</td>
</tr>
<tr>
<td>Third lateral mode in y direction</td>
<td>14.141</td>
</tr>
<tr>
<td>Fourth mode in y direction</td>
<td>46.5802</td>
</tr>
<tr>
<td>Fifth mode in y direction</td>
<td>61.1297</td>
</tr>
<tr>
<td>Sixth mode in y direction</td>
<td>77.9386</td>
</tr>
<tr>
<td>First torsional mode</td>
<td>16.25</td>
</tr>
<tr>
<td>Second torsional mode</td>
<td>46.046</td>
</tr>
<tr>
<td>Third torsional mode</td>
<td>68.843</td>
</tr>
</tbody>
</table>
B.5 Experimental Dynamic Characteristics of the "450 lbs" Model

Figs. B.5 and B.6 show the magnitude and the phase of the transfer function between the accelerations recorded at the base of the steel structure and those recorded in correspondence of the three floors of the steel structure for the steel structure loaded with one concrete block per floor("450lbs" Model).

Notice that Fig. B.5 indicates the presence in the steel structure transfer function of a group of resonant peaks in the frequency range between 5 and 25 Hz. This results is in accordance with what predicted by the various analytical modeling developed in Sections B.2 through B.4. These peaks clearly correspond to the first few modes of vibration, namely, lateral (along the y direction) and torsional. Most likely, the torsional modes were excited because of non perfect centering of the concrete blocks.

Fig. B.5 shows also a group of high peaks in the frequency range between 50 and 60 Hz. These peaks perhaps are related with the second and third torsional modes of vibration, as predicted by the three dimensional analytical modeling described in Section B.4.
Figure B.5  Experimentally-Determined Magnitude of the Transfer Function for the 450 lbs Model
Figure B.6  Experimentally-Determined Phase of the Transfer Function for the 450 lbs Model
B.5.1 First Lateral Mode of Vibration

The experimentally estimated value of the first natural frequency of vibration of the steel structure is 6.4 Hz. This value is slightly smaller than the corresponding ones predicted by the analytical modelings developed for the steel structure (found to be between 6.7 and 8.45 Hz). This finding suggests that the actual behavior of the steel structure is slightly more flexible than that predicted by the analytical models.

Magnitude of the Transfer Function of the Steel Structure (M.T.F.S.S.)

Figure B.7 shows that, at frequencies corresponding to the first lateral mode of vibration, the magnitude of the transfer function increases almost linearly passing from the first to the second to the third floor. This suggests a simple lateral sway mode.

Phase of the Transfer Function of the Steel Structure (P.T.F.S.S.)

Figure B.8 shows a perfect synchronization (all modes in perfect phase) at a frequency of 6.6 Hz. Notice that this frequency is slightly higher than the 6.4 Hz identified from the analysis of the M.T.F.S.S. as the one corresponding to the first mode of vibration and slightly closer to the first natural frequency of vibration predicted by the analytical modelings.
Figure B.7  Magnitude of the Transfer Function of the 450 lbs Model in correspondence of the frequency of the first mode of vibration
Figure B.8  Phase of the Transfer Function of the 450 lbs Model in correspondence of the frequency of the first mode of vibration
B.5.2 Second Lateral Mode of Vibration

The experimentally-estimated value for the second natural frequency of vibration is 16.8 Hz. This value falls within the range of values predicted by the analytical modelings of the steel structure which spans between 15.9 and 22.4 Hz. This indicates that, as far as the frequency of the second lateral mode of vibration is concerned, the actual behavior of the structure is very well captured by the analytical models.

Magnitude of the Transfer Function of the Steel Structure (M.T.F.S.S.)

Figure B.9 show that, at frequencies corresponding to the second lateral mode of vibration, the magnitude of the transfer function increases from the first to the second to the third floor. However, the difference between the amplitude of the transfer function at the first and second floor is not very large and suggests that the second mode of vibration has one inversion between the first and second floor (as analytically predicted).

Phase of the Transfer Function of the Steel Structure (P.T.F.S.S.)

Figure B.10 does not give a clear indications. In the frequencies around the one identified as corresponding to the second lateral mode of vibration, it is not possible to identify any case in which the second and third floor are in phase together and out of phase with the first floor (as predicted by the analytically determined mode shapes).
Figure B.9  Magnitude of the Transfer Function of the 450 lbs Model in correspondence of the frequency of the second mode of vibration
Figure B.10 Phase of the Transfer Function of the 450 lbs Model in correspondence of the frequency of the second mode of vibration.
B.5.3 Third Lateral Mode of Vibration

The experimentally estimated value of the third natural frequency of vibration is 22.2 Hz. This value falls within the range of values predicted by the analytical models developed for the steel structure which spans between 19.28 and 32.77 Hz. This indicates that, as far as the frequency of the third lateral mode of vibration is concerned, the actual behavior of the structure is very well captured by the analytical models.

Magnitude of the Transfer Function of the Steel Structure (M.T.F.S.S.)

Figure B.11 shows that, at frequencies corresponding to the third lateral mode of vibration, the largest magnitude of the model transfer function is that relative to the second floor motions. The magnitudes of the transfer function relative to the third floor is larger than that of the first one. This suggests a mode third mode of vibration characterized by one inversion between the first and second floor and one inversion between the second and third floor.

Phase of the Transfer Function of the Steel Structure (P.T.F.S.S.)

Figure B.12 shows that, at a frequency of 22.2 Hz. the response of the first and third floor are in phase together. The response of the second floor is out of phase with respect to the phase of the first and third floor. This confirms that this mode shapes has two inversions: a first one between the first and second floor and a second one between the second and the third floor.
Figure B.11 Magnitude of the Transfer Function of the 450 lbs Model in correspondence of the frequency of the third mode of vibration
Figure B.12  Phase of the Transfer Function of the 450 lbs Model in correspondence of the frequency of the third mode of vibration
B.5.4 Correlation Between Experimental Results and Analytical Modeling, for the “450 lbs” Model

Table B.1 compares the experimentally determined natural frequencies of vibration of the steel frame loaded with one concrete block per floor (450 lbs model) with the corresponding frequencies predicted by the analytical models.

<table>
<thead>
<tr>
<th>Mode Type</th>
<th>Experimental natural frequency (Hz)</th>
<th>Shear Model natural frequency (Hz)</th>
<th>CAL90 Model natural frequency (Hz)</th>
<th>Three Dimensional Model natural frequency (Hz)</th>
<th>Reduced Three Dimensional Model natural frequency (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1\textsuperscript{st} lateral in the y direction</td>
<td>6.4</td>
<td>8.45</td>
<td>7.219</td>
<td>6.74</td>
<td>6.74</td>
</tr>
<tr>
<td>2\textsuperscript{nd} lateral in the y direction</td>
<td>16.82</td>
<td>22.42</td>
<td>21.584</td>
<td>15.91</td>
<td>15.91</td>
</tr>
<tr>
<td>3\textsuperscript{rd} lateral in the y direction</td>
<td>22.2</td>
<td>32.77</td>
<td>33.859</td>
<td>19.28</td>
<td>19.28</td>
</tr>
<tr>
<td>4\textsuperscript{th} lateral in the y direction</td>
<td>38.5</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>48.83</td>
</tr>
<tr>
<td>5\textsuperscript{th} lateral in the y direction</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>62.174</td>
</tr>
<tr>
<td>6\textsuperscript{th} lateral in the y direction</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>78.38</td>
</tr>
<tr>
<td>1\textsuperscript{st} torsional</td>
<td>12.5</td>
<td>-</td>
<td>-</td>
<td>12.42</td>
<td>16.25</td>
</tr>
<tr>
<td>2\textsuperscript{nd} torsional</td>
<td>19.5</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>46.05</td>
</tr>
<tr>
<td>3\textsuperscript{rd} torsional</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>68.84</td>
</tr>
</tbody>
</table>
B.6 Experimental Dynamic Characteristics of the “900 lbs” Model

Figs. B.13 and B.14 show the magnitude and the phase of the experimentally determined transfer function between the acceleration at the base of the steel structure and the acceleration recorded in correspondence of the three floors of the steel structure loaded with two concrete blocks per floor ("900lbs" Model).

Notice that Fig. B.13 indicates the presence of a group of resonant peaks in the frequency range between 5 and 25 Hz. As predicted by the analytical modelings described in Sections B.2 through B.4.3, these peaks correspond to the first few modes of vibration (lateral along the y direction and torsional). As for the “450lbs” Model, although the excitation of the structure was along the y direction, some torsional modes were excited probably due to the not perfect centering of the concrete blocks.
Figure B.13  Magnitude of the Transfer Function for the 900 lbs Model
Figure B.14 Phase of the Transfer Function for the 900 lbs Model
B.6.1 First Mode of Vibration

The experimentally-estimated value of the first natural frequency of vibration is 5.15 Hz. This value bounded within the range of values predicted by the analytical models developed in Sections B.2 through B.4 and perfectly matches the value predicted by the three dimensional analytical models\(^1\).

Magnitude of the Transfer Function of the Steel Structure (M.T.F.S.S.)

Figure B.15 shows that, at the frequencies corresponding to the first lateral mode of vibration, the amplitude of the transfer function increases almost linearly passing from the first to the second to the third floor. More precisely, the magnitude of the transfer function for the first floor is much smaller (it has a value of approximately 1) than that relative to the second (approximate value of 4.0) and the third (approximate value of 8.5). This suggest a lateral mode which is similar to that found for structures that deform in bending.

Phase of the Transfer Function of the Steel Structure (P.T.F.S.S.)

Figure B.16 does not give a clear indication of a frequency at which all the responses are in phase. This can be probably due to the incorrect polarity of the response provided by accelerometer positioned in correspondence of the second floor. In fact by reversing the value of the phase of the second floor, all three responses are in phase at the frequency of 5.15 Hz.

---

\(^1\) The three dimensional analytical modeling is the more accurate of the modeling carried out.
Figure B.15 Magnitude of the Transfer Function of the 900 lbs Model in correspondence of the frequency of the first mode of vibration
Figure B.16 Phase of the Transfer Function of the 900 lbs Model in correspondence of the frequency of the first mode of vibration.
B.6.2 Second Mode of Vibration

The experimentally-estimated value of the second natural frequency of vibration is 13.82 Hz. This value falls within the range of values (between 13.8 and 15.8 Hz) which were predicted by the analytical models developed for the steel structure. This suggests that, as far as the natural frequency of the second lateral mode of vibration is concerned, the actual behavior of the structure is well captured by all the analytical models.

Magnitude of the Transfer Function of the Steel Structure (M.T.F.S.S.)

Figure B.17 shows that, at frequencies corresponding to the second lateral mode of vibration, the amplitude of the transfer function for the first and the second floor are almost equal (at a value of approximately 2). The amplitude of the transfer function for the third floor has a much larger value (equal to 4). This suggests that the second mode shape has one inversion between the first and second floor.

Phase of the Transfer Function of the Steel Structure (P.T.F.S.S.)

As for the first mode of vibration, the plot of the phase of the transfer function, represented in Figure B.18, does not provide clear indications. It is not possible to identify a precise frequency at which the second and third floor are in phase together and out of phase with the first floor. Again, as it was the case for the first mode of vibration, this can be explained taking into account a possible polarity inversion for the accelerometer placed in correspondence of the second floor. In fact, by inverting the phase of the response of the second floor, it is possible to identify that at a frequency of about 13.6 - 13.8 (close to the
frequency identified to be that of the second mode of vibration) the third and the second floor are in phase together and out of phase with the first as analytically predicted.
Figure B.17  Magnitude of the Transfer Function of the 900 lbs Model in correspondence of the frequency of the second mode of vibration
Figure B.18  Phase of the Transfer Function of the 900 lbs Model in correspondence of the frequency of the second mode of vibration
B.6.3 Third Mode of Vibration

The experimentally-determined value of the third natural frequency of vibration is 20.05 Hz. This value falls within the range of values (14.14 to 23.17 Hz) predicted by the analytical models developed for the steel structure. This result suggests that as far as the frequency of the third lateral mode of vibration is concerned, the actual behavior of the structure is very well captured by the analytical models.

Magnitude of the Transfer Function of the Steel Structure (M.T.F.S.S.)

Figure B.19 shows that, at frequencies corresponding to those of the third lateral mode of vibration, the amplitude of the transfer functions of the first second and third floor are all very similar. This clearly suggests a mode shape with two inversion: one between the first and second floor and one between the second and third.

Phase of the Transfer Function of the Steel Structure (P.T.F.S.S.)

Figure B.20 shows that, at a frequency of 20.05 Hz, the response of the first and third floor are in phase together. The response of the second floor is out of phase, compared with the one of the first and third. This confirms that the mode shapes has two inversions.
Figure B.19  Magnitude of the Transfer Function of the 900 lbs Model in correspondence of the frequency of the third mode of vibration
Figure B.20  Phase of the Transfer Function of the 900 lbs Model in correspondence of the frequency of the third mode of vibration
B.6.4 Correlation Between Experimental Results and Analytical Modeling, for the "900 lbs" Structure

Table B.1 compares the experimentally-determined natural frequencies of vibration of the steel frame loaded with two concrete blocks per floor with the corresponding frequency of vibration predicted by the analytical models prepared.

<table>
<thead>
<tr>
<th>Mode Type</th>
<th>Experimental natural frequency [Hz]</th>
<th>Shear Model natural frequency [Hz]</th>
<th>CAL90 Model natural frequency [Hz]</th>
<th>Three Dimensional Model natural frequency [Hz]</th>
<th>Reduced Three Dimensional Model natural frequency [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt; lateral in the y direction</td>
<td>5.15</td>
<td>5.97</td>
<td>5.53</td>
<td>5.15</td>
<td>5.15</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt; lateral in the y direction</td>
<td>13.82</td>
<td>15.85</td>
<td>16.605</td>
<td>11.8</td>
<td>11.802</td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt; lateral in the y direction</td>
<td>20.05</td>
<td>23.17</td>
<td>26.147</td>
<td>14.14</td>
<td>14.14</td>
</tr>
<tr>
<td>4&lt;sup&gt;th&lt;/sup&gt; lateral in the y direction</td>
<td>46.3</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>46.58</td>
</tr>
<tr>
<td>5&lt;sup&gt;th&lt;/sup&gt; lateral in the y direction</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>61.12</td>
</tr>
<tr>
<td>6&lt;sup&gt;th&lt;/sup&gt; lateral in the y direction</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>77.93</td>
</tr>
<tr>
<td>1&lt;sup&gt;st&lt;/sup&gt; torsional</td>
<td>10.5</td>
<td>-</td>
<td>-</td>
<td>9.48</td>
<td>16.25</td>
</tr>
<tr>
<td>2&lt;sup&gt;nd&lt;/sup&gt; torsional</td>
<td>22.5</td>
<td>-</td>
<td>-</td>
<td>17.71</td>
<td>46.046</td>
</tr>
<tr>
<td>3&lt;sup&gt;rd&lt;/sup&gt; torsional</td>
<td>51.0</td>
<td>-</td>
<td>-</td>
<td>20.94</td>
<td>68.843</td>
</tr>
</tbody>
</table>
B.6.5 Coherency

Fig. B.21 shows the Coherency Spectrum relative to the acceleration records obtained at the third and first floor of the steel structure. The coherency spectrum shows a constant value equal to unity. This result confirms that the steel structure behaved elastically throughout the whole dynamic testing and is in accordance with the theoretical predictions.
Figure B.21  Coherency Spectrum between different degrees of freedom of the 900 lbs Model
APPENDIX C

C.1 The "Suny Buffalo" Analytical Model

The "Suny Buffalo Structural Model" is a scaled down model structure which was carefully calibrated and tested in several laboratories (Stanford, Berkley) in order to be effectively representative of a real building.

An analytical model for this particular model structure, developed by the method of mass simulation\(^1\), has the following mass, stiffness and damping matrices:

\[
M = \begin{bmatrix}
5.6 & 0 & 0 \\
0 & 5.6 & 0 \\
0 & 0 & 5.6 \\
\end{bmatrix} \quad \text{[lbs s}^2\text{/in]}:
\]

\[
K = \begin{bmatrix}
15649 & -9370 & 2107 \\
-9370 & 17250 & -9274 \\
2107 & -9274 & 7612 \\
\end{bmatrix} \quad \text{[lb/in]}:
\]

\[
C = \begin{bmatrix}
2.185 & -0.327 & 0.352 \\
-0.327 & 2.608 & -0.015 \\
0.352 & -0.015 & 2.497 \\
\end{bmatrix} \quad \text{[lbs s/in]}
\]

which lead to the following dynamic characteristics:

\[
Modal \, Frequencies: \quad f = \begin{bmatrix}
2.24 \\
6.83 \\
11.53 \\
\end{bmatrix} \quad \text{[Hz]}
\]

---

1. "For proper modeling of dynamic behavior, mass similitude of the model must be satisfied. Using the constant acceleration scaling and same material for the model, an additional mass must be applied to the model in order to compensate for the difference in the required and provided material densities" [J.M. Bracci].
Modal Damping Ratios: \[ \tilde{\xi} = \begin{bmatrix} 1.62 \\ 0.39 \\ 0.36 \end{bmatrix} \text{[\%].} \]

C.2 The so-called "Reduced Suny Buffalo Model"

The weights and dimensions of the "Suny Buffalo Structural Model" were further reduced in order to simulate a reduced scale model, still representative of a real building structure, but able to satisfy the limitations (in terms of maximum weight and maximum dimensions) of the Rice University Shaking Table\(^1\). For the above reasons the "Suny Buffalo Structural Model" was reduced according to the following scale factors:

- Young's Modulus ratio \( \lambda_E = 1 \):
- Density ratio \( \lambda_\rho = 1 \):
- Mass ratio \( \lambda_{\text{mass}} = 5.404(1) \):
- Length ratio \( \lambda_L = 1.755 \):
- Stiffness ratio \( \lambda_K = 1.755 \).

This reduction gave the following values for the mass, stiffness and damping matrices:

Mass matrix: \[ M_{\text{Rice}} = \begin{bmatrix} 1.036 & 0 & 0 \\ 0 & 1.036 & 0 \\ 0 & 0 & 1.036 \end{bmatrix} \text{[lbs s}^2/\text{in}].\]

---

1. Maximum Payload weight = 1,500 lbs.
Stiffness matrix: \( K_{\text{Rice}} = \begin{bmatrix} 8917 & -5339 & 1201 \\ -5339 & 9829 & -5284 \\ 1201 & -5284 & 4337 \end{bmatrix} \) [lbs/in].

According to these reduced characteristics the following dynamic behavior could be computed:

Modal Frequencies: \( f_{\text{Rice}} = \begin{bmatrix} 3.93 \\ 11.94 \\ 20.16 \end{bmatrix} \) [Hz];

Modal Damping Ratios: \( \xi_{\text{Rice}} = \begin{bmatrix} 1.62 \\ 0.39 \\ 0.36 \end{bmatrix} \) [%].

Figs. 3.1 a and b show the magnitude and the phase of the analytically-determined shaking table transfer function, when the table is loaded with a MDOF payload equal to the Rice Steel Model and to the Suny Buffalo Model. The differences between the two Table Transfer Functions are very limited. This indicates that the Rice Model is suitable to simulate the effects that a scaled down model structure representative of a real building may produce upon the dynamic behavior of the Rice University shaking table.
Figure 3.1  Effect of a MDOF payload upon the table transfer function: Rice model
Vs. scaled down Suny Buffalo Model:
(a) Magnitude  (b) Phase (Radians)
APPENDIX D

D.1 Laplace Transform

Given a function \( f(t) \), its Laplace Transform \( F(s) \) or \( L\{f(t)\} \), where \( L\{\ldots\} \) represents the Laplace Transform operator, is defined as:

\[
F(s) = L\{f(t)\} = \int_{-\infty}^{\infty} e^{-s \cdot t} \cdot f(t) dt
\]  

(D.1)

For functions \( f(t) \) defined only for \( t > 0 \), the Laplace Transform becomes

\[
F(s) = \int_{0}^{\infty} e^{-s \cdot t} \cdot f(t) dt
\]

(D.2)

D.2 Differentiation

The Laplace Transform of the derivative of a signal, \( L\left\{ \frac{d}{dt} f(t) \right\} \), and the Laplace transform of the signal itself, \( L\{f(t)\} \), are related as follows:

\[
L\left\{ \frac{d}{dt} f(t) \right\} = \int_{0}^{\infty} e^{-s \cdot t} \cdot \frac{d}{dt} f(t) dt = e^{-s \cdot t} \cdot f(t) \bigg|_{0}^{\infty} + \int_{0}^{\infty} e^{-s \cdot t} \cdot f(t) dt
\]

(D.3)

Since the Laplace Transform of \( f(t) \) can be indicated as \( F(s) \) and

\[ e^{-s \cdot t} \cdot f(t) \rightarrow 0 \]

as \( t \rightarrow \infty \),

it is possible to obtain:
\[
\text{L}\left\{ \frac{d}{dt}f(t) \right\} = -f(0^-) + s \cdot F(s)
\]

Then, if the function \( f(t) \) is defined only for \( t > 0 \) we have:

\[
\text{L}\left\{ \frac{d}{dt}f(t) \right\} = s \cdot F(s)
\]

(D.4)

### D.3 Integration

Let us assume that we want to determine the Laplace Transform of the integral of a time function \( f(t) \), that is, to find:

\[
F_1(s) = \text{L}\left\{ \int_0^t f(\zeta) \cdot d\zeta \right\} = \int_0^s \int_0^t f(\zeta) \cdot d\zeta \cdot e^{-s \cdot t} \cdot dt
\]

Through integration by parts, where:

\[
u = \int_0^t f(\zeta) \cdot d\zeta \quad \text{and} \quad dv = e^{-s \cdot t} \cdot dt
\]

it is possible to obtain:

\[
F_1(s) = \left[ -\frac{1}{s} \cdot e^{-s \cdot t} \cdot \left( \int_0^t f(\zeta) \cdot d\zeta \right) \right]_{t=0}^{t=\infty} - \int_0^t \frac{1}{s} \cdot f(t) \cdot dt = \frac{1}{s} \cdot F(s)
\]

### D.4 Time Delay

Given the Laplace Transform, \( F(s) = \text{L}\{ f(t) \} \), of a function \( f(t) \); the Laplace Transform, \( F_2(s) = \text{L}\{ f(t-\tau) \} \), of the function \( f(t) \) time-delayed of a constant time lag \( \tau \) is given by the following expression:
\[ F_2(s) = L\{ f(t - \tau) \} = \int_0^\infty e^{-st} \cdot f(t - \tau) \cdot dt \quad (D.5) \]

where, as previously noticed, \( L\{ \ldots \} \) represents the Laplace Transform operator.

If a new variable \( y(t) \) such that \( y(t) = t - \tau \) is defined, then, since the time lag \( \tau \) is constant it is true that

\[ dy = dt \]

and therefore, substituting \( y(t) = t - \tau \) in Eq. (D.5) it is possible to obtain

\[ F_2(s) = \int_{-\tau}^{\infty} e^{-st} \cdot (y + \tau) \cdot f(y) \cdot dy \quad (D.6) \]

If the function \( f(t) \) is defined only for \( t > 0 \), \( f(t) = 0 \) for \( t < 0 \) and Eq. (D.6) becomes:

\[ F_2(s) = \int_0^{\infty} e^{-st} \cdot (y + \tau) \cdot f(y) \cdot dy \quad (D.7) \]

Then, recalling that the time delay \( \tau \) is constant, \( e^{-s \cdot \tau} \) is independent from the integrating variable \( y \) and can extracted from the integral. This leads to the following expression for \( F_2(s) = L\{ f(t - \tau) \} \):

\[ F_2(s) = e^{-s \cdot \tau} \cdot \int_0^{\infty} e^{-s \cdot y} \cdot f(y) \cdot dy = e^{-s \cdot \tau} \cdot F(s) \quad (D.8) \]
APPENDIX E

E.1 Modal Mass

For a given MDOF system characterized by the following mass, damping and stiffness matrices:

- mass matrix \( [m_p] \);
- damping matrix \( [c_p] \);
- lateral stiffness matrix \( [k_p] \);

and by the following corresponding modal parameters:

- \( n \)-th mode shape \( \{ \phi_n \} = \begin{bmatrix} \phi_{n,1} \\ \phi_{n,2} \\ \vdots \\ \phi_{n,N} \end{bmatrix} \);

The \( n \)-th modal mass \( m_n^m \) can be computed as follows\(^1\):

\[
  m_n^m = \frac{\left( \sum_{i=1}^{N} m_i \cdot \phi_{n,i} \right)^2}{\sum_{i=1}^{N} m_i \cdot \phi_{n,i}^2} \quad (E.1)
\]

where:

- \( \phi_{n,i} \) is the \( i \)-th component of the \( n \)-th mode shape vector, and
- \( m_i \) is the \( n \)-th element of the mass matrix \( [m_p] \).

---

\(^1\) As Defined by A. Chopra, 1995.
E.2 Modal Mass of the Flexible Table Foundation (Base)

The shaking table base has been modeled in Chapter 10 and Appendix A, as a three degrees of freedom system having the following characteristics:

- mass matrix: \[ \mathbf{m}_p = \begin{bmatrix} 56.9948 & 0 & 0 \\ 0 & 56.9948 & 0 \\ 0 & 0 & 56.9948 \end{bmatrix} \text{ [lbs sec}^2\text{/in]} \]

  (which corresponds to a weight of each concrete block of 22,000 lbs. and to a total weight of 66,000 lbs)

- stiffness matrix: \[ \mathbf{k}_p = \begin{bmatrix} 46 & -23 & 0 \\ -23 & 46 & -23 \\ 0 & -23 & 23 \end{bmatrix} \cdot 10^6 \text{ [lbs/in]} \]

According to these characteristics the following modal masses can be computed using the procedure described in Section E.1:

- \( m_1^m = 156.29 \) [lbs sec\(^2\)/in].
- \( m_2^m = 12.8 \) [lbs sec\(^2\)/in].
- \( m_3^m = 1.88 \) [lbs sec\(^2\)/in].

These values correspond to the following modal weights:

- \( w_1^m = 60,329 \) [lbs].
- \( w_2^m = 4,942 \) [lbs].
• \( w_3^m = 729 \) [lbs].

The modal weights (or masses) represent the following percentage of the total weight (or mass) of the foundation:

• first modal mass = \( w_1^m = 91.4 \% \) of total foundation weight;

• second modal mass = \( w_2^m = 7.5 \% \) of total foundation weight;

• third modal mass = \( w_3^m = 1.1 \% \) of total foundation weight.

E.3 Shaking Table Transfer Function Sensitivity to Base Mass

This Section reports the comparison between the table transfer function obtained using the modal mass of the foundation and the table transfer function obtained using the estimated foundation weight.

In the development of the analytical model for the shaking table transfer function described in Chapter 3, the flexibility of the foundation mass was modeled using a single degree of freedom which had a weight of 90.000 lbs (corresponding to a mass of 288.16 lbs sec\(^2\)/in). The computation of the first modal mass for the shaking table base carried out in this appendix\(^1\) gave a mass of 60.329 lbs (corresponding to a mass of 156.29 lbs sec\(^2\)/in).

Figs. E.1(a) and (b) compare the analytical shaking table transfer functions obtained using as weight for the single degree of freedom representing the flexible shaking table base the

---

1. In Chapter number 10 it has been shown how the motion of the flexible foundation can be correctly approximated with using just the motion of its first mode of vibration.
following values:

- 90,000 lbs. corresponding to the total foundation mass used in the analytical model developed in Chapter 3:

- 60,000 lbs. corresponding to the (more appropriate) first modal mass derived in this appendix.

Inspection of the shaking table transfer functions represented in Fig. E.1 shows that the differences between the two transfer functions are very limited. Notice that only one peak in the magnitude of the shaking table transfer function (and the corresponding notch in the phase of the shaking table transfer function) obtained using the first modal mass is slightly larger than the corresponding one obtained using the total foundation mass.
Figure E.1  Table Transfer Function with Flexible Foundation - foundation mass equal to the total foundation mass or the first modal mass
(a) Magnitude  (b) Phase (Radians)
F.1 The “Rigid” Payload

A “Rigid” payload was used to test the sensitivity of the shaking table response to very stiff payloads. This payload consisted of a series of concrete blocks clamped to the slip table. Each concrete block measures 29” by 14” by 4 1/2” and weights approximately 150 lbs. Fig. F.1 shows the way in which each concrete block was clamped to the slip table. Each concrete block was fixed to the slip table in two points. At each point the same type of clamping system (composed by one iron block, one iron bar and one 7/16” bolt with nut) was used. The iron bar was positioned horizontally above concrete block and the iron block. The iron block is a specific device with variable height, which could be adjusted in order to level its upper surface with upper face of the concrete block. The 7/16” bolt was passed through the iron bar and screwed in the slip table, thus providing a vertical pressure on the concrete block.
F.2 Analytical Modeling of the Lateral Stiffness of the “Rigid” Payload

This clamping system provided a horizontal restrain to the lateral motion of the concrete blocks. The lateral restraining forces are provided mainly by the 7/16” bolt. Fig. F.2 shows how the bolt is screwed to the slip table and post tensioned by a nut positioned at the top of the iron bar.

The lateral rigidity provided by each bolt, $K_{lb}$, can be estimated as follows:

$$K_{lb} = \frac{12 \cdot E \cdot I}{L^3} \quad \text{(F.1)}$$

where:

- $E$ is the elastic modulus of steel:
- $I$ is the moment of inertia of the cross section of the bolt:
- $L$ is the distance between the top and bottom clamps of the bolt.

The elastic modulus of steel, $E$, is $29 \cdot 10^6$ psi, the moment of inertia $I$ of the cross section of the bolt is estimated to be equal to 0.0018 in$^4$, and the length $L$ can be estimated to be equal to 5 and $\frac{5}{8}$ in ($= 4 \frac{1}{2} + 1 + \frac{1}{8}$). These values give the following lateral stiffness:

$$K_{lb} = \frac{12 \cdot 29 \cdot 10^6 \cdot 0.0018}{(5.625)^3} = 3519.87 \text{ [lbs/in]} \quad \text{(F.2)}$$

1. The moment of inertia of a circular cross section is equal to $\frac{\pi \cdot r^4}{4}$. where $r$ is the radius of the circle.
Considering the fact that each concrete block was clamped with two of these systems, the total lateral rigidity, \(K_2\), restraining each concrete block is

\[
K_2 = 2 \cdot K \approx 7040 \text{ [lbs/in]}
\] (F.3)

### F.2.1 Dynamic Characteristics of the “Rigid” Payload

Inspection of the experimental results obtained with the “Rigid” payload clamped on the top of the slip table suggested that, despite all our efforts to create a rigid connection, this clamping system was quite flexible. In fact, the concrete blocks behaved to dynamically almost as single degree of freedom.

To understand the dynamic behavior of the “Rigid Payload” an analytical model for the dynamic behavior of the concrete blocks clamping system was developed.

This dynamic analytical modeling is schematically represented in Fig. F.3 and assumes that the concrete blocks have a double behavior: their bottom part acts as if rigidly connected to the slip table, and their top part acts as a single degree of freedom restrained by the lateral rigidity provided by the bolts of the clamping system.

A least square fit between the experimental and analytical shaking table transfer function indicated that the concrete block mass can be split as follows:

- the bottom five-sixth of the concrete bloc mass behaves as part of the slip table

\[
m_{b} = \frac{5}{6} \cdot \frac{150}{386.4} = 0.3235 \text{ [lbs s}^2/\text{in]}:
\]
- the top one-sixth of the concrete bloc mass behaves as a flexible SDOF

\[ m_t = \frac{1}{6} \cdot \frac{150}{386.4} = 0.065 \text{ [lbs s}^2\text{/in].} \]

Using the value of \( m_t \) indicated above, the following natural frequency could be computed

for the SDOF representative of the dynamic behavior of the "Rigid" Payload:

\[ f_{SDOF} = \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{K_2}{m_t}} = \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{7040}{0.065}} = 52 \text{ Hz} \]

These values closely matches the experimentally determined natural frequency of vibration of the concrete blocks of about 54 Hz.
Figure F.1 System used to clamp the concrete blocks to the Slip Table
Figure F.2  Lateral Stiffness of the System that clamps the Concrete Blocks to the Slip Table
Figure F.3  Dynamic Modeling of the Concrete Blocks clamping
ABREVIATIONS AND SYMBOLS

A&S1 Symbols Valid for Shaking Table Characteristics - all Chapters

\( A \) effective piston area

\( A(s) \) feedback transfer function \( A(s) = \frac{x_a(s)}{x_t(s)} \)

\( B(s) \) Base transfer function: \( B(s) = \frac{x_b(s)}{x_t(s)} \) (no Payload on the table)

\( B'(s) \) Base transfer function (SDOF Flexible Payload on the table)

\( B''(s) \) Base transfer function (MDOF Flexible Payload on the table)

\( \beta \) bulk modulus of the fluid

\([C]\) generalized damping matrix

\( c_p \) damping coefficient of the SDOF payload

\([c_p]\) damping matrix of the MDOF payload

\( \Delta P(t) \) pressure drop across the actuator piston

\( \Delta P_p(t) \) differential pressure in the pilot spool

\( e(t) \) DC error signal

\( \varepsilon(t) \) electric servovalve command component due to the PID Gains

\( F_a(t) \) force in the actuator

\( \dot{F}_a(t) \) time derivative of the force in the actuator
\( F_s(t) \) base shear transmitted from the payload to the shaking table

\([\Phi]\) matrix of mode shapes

\(\{\phi_n\}\) mode shape vector

\(\phi_{n1}\) first component (1st degree of freedom) of the \(n\)-th mode shape

\(H(s)\) system transfer function: transfer fn. between external command (input) and motion of the table with respect to the base (output): \(H(s) = \frac{x_i(s)}{x_c(s)}\) (Fixed Base. no Payload on the table)

\(H'(s)\) system transfer function (Flexible Base. no Payload on the table)

\(H''(s)\) system transfer function (Flexible Base. SDOF payload on the table)

\(H'''(s)\) system transfer function (Flexible Base. MDOF payload on the table)

\(H_{F1}(s)\) operator: \(H_{F1}(s) = 1 + B(s)\)

\(H_{F2}(s)\) operator: \(H_{F2}(s) = 1 + B'(s) + \frac{m_p}{m} \cdot \left[ 1 + H_p(s) \right] \cdot \left[ 1 + B'(s) \right]\)

\(H_{F3}(s)\) operator: \(H_{F3}(s) = 1 + B'(s) - \frac{k_p}{s^2 \cdot m} \cdot H_p(s) \cdot \left[ 1 + B'(s) \right]\)

\(H_{F4}(s)\) operator: \(H_{F4}(s) = 1 + B''(s) - \frac{K_{11}}{s^2 \cdot m} \cdot H_{pm1}(s) \cdot \left[ 1 + B''(s) \right]\)

\(H_p(s)\) SDOF payload transfer function: \(H_p(s) = \frac{x_p}{x_{ta}}\)

\(\{H_{pm}\}\) MDOF payload transfer function (vector)
$H_{pm1}(s)$ is an operator that allows to express the shear at the base (scalar) of a of the MDOF payload in terms of the MDOF payload transfer function

$H_c(s)$ is the three stage servovalve Transfer Function $H_c(s) = \frac{q_s(s)}{x_c(s)}$

$\zeta_b$ is the critical damping coefficient for the flexible foundation

$\zeta_n$ is the critical damping coefficient of the nth mode of vibration (MDOF)

$\zeta_p$ is the critical damping coefficient of the flexible payload (SDOF)

$[K]$ is the generalized stiffness matrix

$K_{cell}$ is the conversion factor for pressure to voltage

$K_{der}$ is the derivative gain (in practical experimentation referred as D-gain)

$K_{dp}$ is the Delta Pressure gain (in practical experimentation referred as dP-gain)

$K_{ff}$ is the feed-forward gain (in practical experimentation referred as FF-gain)

$K_{int}$ is the integral gain (in practical experimentation referred as I-gain)

$k_{le}$ is the force-flow coefficient expressing the linear relationship between the force in the actuator and the flow of leakage fluid. (to be determined experimentally)

$k_p$ is the lateral stiffness of the payload (SDOF)

$[k_p]$ is the lateral stiffness matrix of the payload (MDOF)

$K_{pro}$ is the proportional gain (in practical experimentation referred as P-gain)

$k_t$ is the table gain factor (to be experimentally determined)
$K_{11}$  stiffness coefficient of the first degree of freedom of the MDOF payload (shear type)

$[M]$  generalized mass matrix

$m_p$  mass of the payload (SDOF)

$[m_p]$  mass matrix of the payload (MDOF)

$m_t$  mass of the table

$m_T$  system total mass: $m_T = m_t + m_b$

$\omega_b$  circular natural frequency of the flexible foundation (base)

$\omega_n$  circular natural frequency of vibration of the $n$th mode of vibration (MDOF)

$\omega_p$  circular natural frequency of the flexible payload (SDOF)

$P_{DP}(s)$  operator: $P_{DP}(s) = \frac{s^2 \cdot K_{dp} \cdot m_t}{A}$

$P_{PID}(s)$  operator: $P_{PID}(s) = K_{pro} + \frac{1}{s} K_{int} + s K_{der}$

$P_{FF}(s)$  operator: $P_{FF}(s) = s \cdot K_{ff}$

$q_{am}(s)$  is the component of flow due to actuator motion

$q_{com}(s)$  is the component of flow due to the fluid compression

$q_{le}(s)$  is the component of flow due to leakage

$q_s(t)$  high pressure hydraulic fluid flow that ports from the servo valve to the actuator
\( S(s) \)  
Servo valve transfer function: transfer fcn. between servo valve command (input) and motion of the table relative to the base (output): 
\[
S(s) = \frac{x_r(s)}{x_c(s)} 
\]
(Rigid Base. no Payload on the table)

\( S'(s) \)  
Servo valve transfer function (Flexible Base. no Payload on the table)

\( S''(s) \)  
Servo valve transfer function (Flexible Base. SDOF payload on the table)

\( S'''(s) \)  
Servo valve transfer function (Flexible Base. MDOF payload on the table)

\( T(s) \)  
Table transfer function: transfer fcn. between external command (input) and absolute motion of the table (output) 
\[
T(s) = \frac{x_{ta}(s)}{x_c(s)} = H'(s) \cdot (B(s) + 1) 
\]
(Flexible Base. no Payload on the table)

\( T'(s) \)  
Table transfer function: 
\[
T'(s) = H''(s) \cdot (B'(s) + 1) 
\]
(Flexible Base. SDOF Payload on the table)

\( T''(s) \)  
Table transfer function: 
\[
T''(s) = H'''(s) \cdot (B''(s) + 1) 
\]
(Flexible Base. MDOF Payload on the table)

\( V \)  
Volume of the actuator load chamber

\( x_a(t) \)  
Displacement feedback signal

\( x_b(t) \)  
Motion of the flexible base with respect to the ground

\( x_c(t) \)  
Servo valve signal

\( \bar{x}_c(t) \)  
Command signal
$x_{cell}(t)$  electric signal given by the Delta Pressure Load Cell

$x_d(t)$  electric servovalve command component due to the Delta Pressure Gain

$x_f(t)$  electric servovalve command component due to the Feed Forward Gain

$x_p(t)$  relative displacement of the payload with respect to the table

$\{x_p\}$  vector of payload relative displacements (MDOF)

$x_{pa}(t)$  absolute displacement of the payload

$x_t(t)$  actual extension of the actuator (motion of the table with respect to the base)

$x_{ta}(t)$  absolute motion of the table: $x_{ta}(t) = x_t(t) + x_b(t)$

$x_1(s)$  displacement of the 1st degree of freedom (MDOF payload)

$\{1\}$  vector of ones

**A&S2 Special Symbols Used in Spectral Estimation - Sections 5.2 and 5.3**

$\Delta t$  sampling time interval

$E[\ldots]$  expectation or ensemble-average operator

$f(t)$  deterministic input excitation time history

$f(t)$  input random process

$F(\omega)$  frequency spectrum of the deterministic input $f(t)$

$F_0$  deterministic complex input vector (characteristics in terms of amplitude and phase of the sinusoidal input)
\( H(\omega) \)  System Transfer Function

\( \Phi_{ff}(\omega) \)  power spectral density of the input r.p. \( f(t) \)

\( \Phi_{fx}(\omega) \)  Cross spectral density between the input and output r.p.'s

\( \Phi_{rr}(\omega) \)  power spectrum of the process \( r(t) \)

\( \Phi_{xx}(\omega) \)  power spectral density of the output r.p. \( x(t) \)

\( \hat{\Phi}_{rr}(\omega) \)  estimation of the power spectrum (often referred also as "Periodogram") of a stochastic process

\( \hat{\Phi}_{rr}^{B}(\omega, k) \)  Bartlett's estimation of the power spectrum (k refers to the number of process records - segments - used in the estimation)

\( \hat{\Phi}_{rr}^{(k)}(\omega) = \frac{1}{M} \left| R_{M}^{(k)}(\omega) \right|^2 \)

\( K \)  number of process observation (segments) of equal time duration

\( K_r(m) \)  co-variance function of the zero mean, stationary process \( r(t) \)

\( \hat{K}_r(m) \)  estimate of the co-variance function \( K_r(m) \)

\( i \)  the imaginary unit

\( m \)  the time lag

\( M \)  time duration of process observations

\( \omega \)  circular frequency of the deterministic sinusoidal excitation

\( r(t) \)  zero mean, stationary stochastic process

\( R_L(\omega) \)  Fourier Transform of the windowed signal of the stochastic process \( r(t) \)
$R_{M}^{(k)}(\omega)$  Fourier Transform of the k-th segment of the stochastic process $r(t)$

$\sigma_{\phi r}(\omega)$  standard deviation of the Bartlett's estimation of the power spectrum

$w_{L}^{R}(l)$  rectangular or "boxcar" window applied to the data in the time domain

$w_{L}(l)$  window function in the time domain (general window shape)

$\tilde{w}_{2L-1}(m)$  window function in the lag domain, corresponding to $w_{L}(1)$

$\tilde{\psi}_{2L-1}(\omega)$  Fourier Transform of the window function in the lag domain, sometimes referred also as kernel

$\tilde{w}_{2L-1}^{\tau}(m) = 1 - \frac{|m|}{L}$  triangular window referred in literature as a triangular Bartlett Window. It is the window function $w_{L}^{g}(l)$ in the lag domain

$\tilde{\psi}_{2L-1}^{T}(\omega)$  Fourier Transform of the Bartlett window, also known as the Fejer kernel:

$\tilde{\psi}_{2L-1}^{T}(\omega) = \frac{1}{L} \cdot \left( \frac{\sin(\omega \cdot L)}{\sin(\omega)} \right)^{2} = (\text{sinc}(\omega))^{2}$

$x(t)$  deterministic output response time history

$X(t)$  output random process

$X(\omega)$  frequency spectrum of the deterministic output $x(t)$

$X_{0}$  deterministic complex output vector (characteristics in terms of amplitude and phase of the sinusoidal output)
A&S3 Special Symbols Used for ARMA-Models - Section 5.4

\( a_k = a(k \cdot \Delta t) \) discrete realization of the random process

\( C \) dashpot damping coefficient

\( C_d \) proportion of the input displacement applied to the dashpot

\( C_s \) proportion of the input displacement applied to the linear spring

\( \zeta_g \) coefficient of critical damping of the co-variance-equivalent SDOF system

\( \Delta t \) sampling time interval

\( \{e_k\} \) zero-mean discrete Gaussian white-noise of variance \( \sigma^2_e \)

\( f_{Nyq} \) Nyquist frequency

\( \Phi_i \) auto-regressive coefficients

\( \gamma_n = E[(a_k - \mu_a) \cdot (a_{k+n} - \mu_a)] \) Auto-covariance function

\( K \) linear spring stiffness

\( \omega_d \) damped natural frequency of the co-variance-equivalent SDOF system

(underlying physical system)

\( \omega_g \) natural frequency of the co-variance-equivalent SDOF system

\( p(f) \) One-sided spectrum

\( \mathcal{R}_n = E[a_k \cdot a_{k+n}] \) auto-correlation function

\( \rho_n \) auto-correlation coefficient function

\( \theta_i \) moving average coefficients
\( X(t) \)  
input displacement

**A&S4 Special Symbols Used for the Simulation of the Errors in Table Transfer Function Estimation - Section 5.5**

\( A_i \)  
maximum amplitude of the white noise that simulates the error in the input line

\( A_o \)  
maximum amplitude of the white noise that simulates the error in the output line

\( \zeta \)  
coefficient of critical damping

\( L(t) \)  
noiseless line signal (unaffected by line noise)

\( L_a(t) \)  
line signal affected by the line noise

\( \omega_n \)  
natural circular frequency of the dynamic system

\( \dot{X}_a(t) \)  
output of the accelerometer (acceleration, affected by the accelerometer error

\( \ddot{X}_i(t) \)  
actual acceleration "felt" by the accelerometer

\( W(t) \)  
white noise uniformly distributed between -1 and +1


