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Empirical Detection for Spread Spectrum and Code Division Multiple Access (CDMA) Communications

by

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A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

Doctor of Philosophy

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Empirical Detection for Spread Spectrum and Code Division Multiple Access (CDMA) Communications

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Abstract

In this thesis, the method of "classification with empirically observed statistics" – also known as empirical classification, empirical detection, universal classification, and type-based detection – is configured and applied to the despreading/detection receiver operation of a spread-spectrum (SS), code division multiple access (CDMA) communications system. In static and Rayleigh-fading environments, the empirical detector is capable of adapting to unknown noise environments in a superior manner than the linear matched-filter despreader/detector, and done with reasonable amounts of training. Compared to the optimum detector, when known, the empirical detector always approaches optimal performance, again, with reasonable amounts of training. In an interference-limited channel, we show that the single-user likelihood-ratio detector, which is the optimum single-user detector, can greatly outperform the matched filter in certain imperfect power-control situations. The near optimality of the empirical detector implies that it, too, will outperform the matched filter in these situations. Although the empirical detector has the added cost of requiring chip-based phase synchronization, its consistent and superior performance in all environments strongly suggests its application in lieu of the linear detector for SS/CDMA systems employing long, pseudo-random spreading. In order to apply empirical classification to digital communications, we derive the empirical forced-decision detector and show that it is asymptotically optimal over a large class of empirical classifiers.
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## Contents

Abstract .......................................................... ii
Acknowledgments .................................................. iii
List of Illustrations ............................................ ix
List of Tables ..................................................... xii

1 **Introduction** ................................................. 1
   1.1 Organization and Contribution of Thesis ............. 3

2 **Classification Using Empirically Observed Statistics** 6
   2.1 Classical Hypothesis Testing Revisited ............... 6
      2.1.1 Asymptotic Analysis ............................. 8
      2.1.2 Method of Types ............................... 11
      2.1.3 Quantifying Relative Performance ............... 13
   2.2 Classification based on Empirically Observed Statistics 15
      2.2.1 Preliminaries ................................ 17
      2.2.2 Null Hypothesis Test .......................... 20
      2.2.3 Test with Rejection ........................... 26
      2.2.4 Forced-Decision Test (no rejection) .......... 29
      2.2.5 Modification for Markov Sequences .......... 32
   2.3 Summary ................................................. 33

3 **Empirical Type-based Digital Receiver** ................. 35
   3.1 Empirical Type-based Detection System .............. 35
   3.2 Quantization for Detection ........................... 39
3.3 Simulation Study ................................................. 44
  3.3.1 Basic Properties ............................................. 44
  3.3.2 Quantization Effects ....................................... 47
  3.3.3 Symmetry Assumptions .................................... 55
3.4 Smoothing of Types ............................................. 60
3.5 Summary ......................................................... 66

4 Type-based Detector for Spread Spectrum and CDMA Communications  68
4.1 Introduction .................................................... 68
  4.1.1 Why Empirical Type Detector for Spread Spectrum? .... 69
4.2 Detector Topology .............................................. 71
  4.2.1 Detection after De-spreading .............................. 75
4.3 Basic Properties ............................................... 76
4.4 Single-Path Fading Channel ................................... 84
4.5 Multi-path and Diversity Combining .......................... 95
  4.5.1 Static Multi-path Channel ................................. 97
  4.5.2 Time Varying (Fading) Multi-path Channels .......... 99
  4.5.3 Simulations ................................................ 101
4.6 Comparison with Linear Detectors ........................... 108
4.7 Summary ......................................................... 110

5 Type-based Detector in Multiple Access Interference Channel  113
5.1 CDMA Background .............................................. 113
5.2 Analysis of LRT Detector in Multiple-Access Interference Channel 117
5.3 Local Behavior of LRT Detector with Equal-Powered Users ... 122
5.4 LRT Detector Performance with Imperfect Power Control ... 124
5.5 Simulation Results for Empirical Detector in MAI ............... 127
5.6 Summary .................................................................. 127

6 Conclusions ................................................................ 130

A Error Behavior Under Alternate Hypothesis ................. 133

B GLRT of Multiple Portioned Observations ...................... 136

C Important Sampling for Discrete-Valued Sequences ........ 137

D Chernoff Information with Equal-Powered Users .............. 141

Bibliography .................................................................. 143
Illustrations

3.1  Periodic training to probe properties of observations ........................................ 36
3.2  Empirical type-based detector .................................................................................. 37
3.3  Chernoff information for 1-5 bit uniform quantizers .............................................. 42
3.4  Quantizer Designed under KL versus Chernoff criterion ........................................ 43
3.5  Type detector performance in Gaussian noise ......................................................... 48
3.6  Predicted error probability bounds ........................................................................... 49
3.7  Type detector performance in Laplacian noise ......................................................... 50
3.8  Type detector performance in Hyperbolic Secant noise ........................................... 51
3.9  Type detector performance, 5 and 10 dB .............................................................. 52
3.10 Type detector performance: 2 – 4 bits .................................................................... 54
3.11 Symmetric v. baseline detector performance, Gaussian ........................................... 57
3.12 Symmetric v. baseline detector performance, Laplacian ......................................... 58
3.13 Symmetric v. baseline detector performance, 5 dB ................................................ 59
3.14 Smoothing of types ................................................................................................. 60
3.15 Type detector performance v. smoothing length, $n = 16$ ....................................... 63
3.16 Type detector performance v. smoothing length, $n = 64$ ....................................... 64
3.17 Type detector performance v. smoothing length, $n = 256$ ..................................... 65

4.1  Spread-spectrum transmit-receive topology ......................................................... 69
4.2  Spread-spectrum received signal model .................................................................... 72
4.3  Training of type detector for spread spectrum ....................................................... 73
4.4  Topology of empirical type detector for spread spectrum ....................................... 74
5.5 Comparison of LRT and MF with imperfect power control ..... 125
5.6 Comparison of 4-bit LRT and MF with imperfect power control ... 126
5.7 MAI rejection, simulation results . . . . . . . . . . . . . . . . . . 128
Tables

3.1 Summary of quantizer loss ............................................. 44
3.2 Summary of type detector performance at 2-5 quantizer bits .... 55
Chapter 1

Introduction

The classification problem based on empirically observed statistics lies at the heart of statistical decision making. In the classical problem of classification, or hypothesis testing, observations must be classified into one of many, known classes. Each of the classes is modeled by a probability measure that describes the statistics of observations from that class. In the empirically based classification problem, though, no models are available. Instead, training observations known to have emerged from each class are available. Empirical classification lies at the heart of all scientific inquiry – mathematical models are first motivated by data and become “theory” only with enough data verification. Thus, in any classification problem, any model used cannot be more accurate or contain more information than all of the data used for its motivation, derivation, and verification. With enough verification, a level of confidence is gained whereby using models can be more advantageous, due to parsimony, than using all collected data that describe it. But when the underlying classes change properties over time, or when the models to describe these classes are extremely complex, using training data may be more effective, especially if they can be captured by sufficient statistics that are of manageable complexity. As properties change, more data from known sources must be gathered to capture the most current properties.

The empirically based classification problem that is addressed in this thesis deals with discrete-valued, or categorical, data that are drawn from a finite alphabet. Finite-valued problems are familiar in information theory, where compression, capacity, rate-distortion, and hypothesis testing are typically addressed on finite sets [6]. Although data before measurement are typically continuous-valued, all digital signal process-
ing performed in digital hardware or computer operates on quantized, discrete data. After quantization, the most general model for the data is a probability mass function (rather than a density) and necessarily has a finite parameterization. Empirical detectors respect this generality and make no additional assumptions about the data.

The empirically based classification problem is sometimes called universal classification or classification by universal likelihood. This is because the solutions that have been investigated work equally well for all underlying sources – hence universal – and they are based on the probability of the observations – hence the term likelihood. (We have chosen the term type-based classification, which only describes the likelihood portion: For discrete-valued data, the type is the sufficient statistic for measuring probability. See Chapter 2.) The empirically based classification problem was first addressed by Ziv [62] and Gutman [14], and the resulting techniques were applied to numerous problems in inference and estimation [28, 59, 60]. The work of this thesis is primarily drawn and motivated by the powerful result derived by Ziv and Gutman: their classifier is optimal in the exponential rate of convergence of the probability of error among all empirically based classifiers, and that their empirically based classifier has the same error exponent as the optimum likelihood-ratio detector – which knows the true distributions – when the amount of training data grows faster than linearly with the amount of test data. It should be noted that work in universal compression is closely related to universal classification [9, 36, 53].

In digital communications, the receiver faces the classification problem in determining the transmitted information bit. Traditionally, the additive noise is assumed to be Gaussian, and the resulting detector* is linear. However, in many situations, the Gaussian assumption does not hold, and the linear detector may be far from optimal. In fact, we may question the utility of using any approach that attempts to model the underlying noise in environments that are time varying in unknown

*The term "detector" is used instead of classifier for historical reason. In radar systems, the problem is to detect the presence of a target.
ways. This motivates our use of empirically based classification techniques for digital communications.

In particular, the spread-spectrum, CDMA communications system offers a good application for empirical detection because of the high bandwidth expansion of the uncoded information symbols to the spread, chip observations. In an interference-limited environment, the statistics of the received observations can be very different from Gaussian, and therefore, the empirical detector may offer significant advantages over the traditional linear detector.

We apply empirical classifiers to the CDMA problem because of their simplicity and provable, asymptotically optimal properties. There may be other techniques, though, for continuous-valued data that have competitive or even superior performance. However, as we have noted, any digital signal processing approach must operate on quantized data, and using high-resolution quantization to emulate continuous-valued data is physically more costly. Using probability masses may be considered a more general parametric model than a specific parametric model, and for digital hardware or software, completely captures the statistics of the inputs.

1.1 Organization and Contribution of Thesis

Chapter 2 begins with a quick review of classical hypothesis testing. Then, the basics of empirical classification are presented, focusing on both the presentation of the results, how they are derived, and interpretation. Minor modifications to the basic results are provided. The derivation and analysis of the forced-decision detector follows, and its optimality is shown.

In Chapter 3, we investigate the basic properties of the empirical classifier through simulation studies of a basic digital communications system in a variety of different noise models. In particular, finite-sample properties of the empirical classifier are studied, and favorable operating conditions are learned. In Chapter 4, the empirical classifier is applied to the spread-spectrum, CDMA communications problem in
static, single-path fading, and multipath-fading channels in a variety of different additive noise models. In fading channels, we show that learning of the time-averaged properties of the observations is an effective way to combat fading, although more training data is needed to do so. In multipath-fading channels, we examine the effects of maximal ratio combining using an information-theoretic technique.

Chapter 5 is primarily a theoretical/numerical study of the optimum detector in an interference-limited, CDMA system and serves to promote the use of the empirical classifier. In particular, we show that the optimal detector can have far superior performance over the linear detector in certain situations. Our study of the empirically based classifier of the previous chapters have shown that optimality can be approached with reasonable amounts of training, and, therefore, the performance superiority of the optimum detector will imply the superior performance of the type detector.

The original contributions of this thesis are:

- Modification of Gutman’s basic result [14, Theorem 1] that quantifies the error probability under the alternate hypothesis. (Chapter 2)

- Derivation and analysis of optimality of the forced-decision detector. (Chapter 2)

- Important sampling for discrete-valued sequences is derived in Appendix C with provable asymptotic gains. The important sampling technique is used to estimate the performances of the optimal detectors that operate on quantized data. (Chapters 3 and 4)

- Extensive simulation studies of the empirical detector for digital communications. (Chapters 3 and 4)

- Application and simulation study of the empirical detector to spread spectrum, CDMA communications. The studies in this thesis are along the lines of our previously published results [19, 25] and related follow-up work [55, 56, 57, 58], but the current study is more extensive. (Chapter 4)
• Maximal-ratio combining for the empirical detector for time-varying multipath and diversity channels. (Chapter 4)

• Analysis of the optimum single-user detector in an interference-limited CDMA system and its comparison against the matched-filter detector. (Chapter 5)
Chapter 2

Classification Using Empirically Observed Statistics

In this chapter, we present basic results in universal classification, drawn especially from the works of Ziv [62] and Gutman [14]. We provide re-statements of Gutman's characterization of the error probability under the alternate hypothesis that appears to be incorrectly stated in [14, Theorem 1]. We begin with a review of hypothesis testing, in particular, of asymptotic analysis with the aid of the method of types, which are all well established concepts in hypothesis testing and information theory. We conclude with the derivation and analysis of the forced-decision detector, which are among the original contributions of this thesis.

2.1 Classical Hypothesis Testing Revisited

In the binary detection problem, one must choose which of two alternative models best describes the observations. A classic example is the radar problem, simply stated as: Is a target present? Mathematically, we can formulate this problem as follows: Given observations $x \in A^n$, where $A$ is a finite alphabet $\{a_1, \ldots, a_m\}$, which of the following two hypotheses is true?

\[ H_0 : x \sim P_0 \]
\[ H_1 : x \sim P_1. \quad (2.1) \]

For example, $A$ could be the output of an analog-to-digital (A/D) converter. $P_i$ is the $n^{th}$ order probability mass function (with support on $A^n$) that governs the statistics of $x$ when the $i^{th}$ hypothesis is true. Two types of errors are possible: we might say $H_1$ is true when $H_0$ is actually true, or $H_0$ when $H_1$ is true. Denote by $R_n \subset A^n$ the
set of \( x \in A^n \) for which we say that \( H_0 \) is true. That is, \( R_n \) is the acceptance region for \( H_0 \). The probabilities for the two errors are

\[
P(e|H_1) = P_1(R_n)
\]

\[
P(e|H_0) = P_0(R_n^c),
\]

where \( R_n^c \) is the complement of \( R_n \) and the acceptance region for \( H_1 \). For reasons related to the radar problem, \( P(e|H_1) \) is sometimes referred to as the "probability of miss," \( P_M \), and \( P(e|H_0) \) the "probability of false alarm," \( P_F \). Complementary to these error probabilities are the probabilities of correct detection: \( P(c|H_0) \) and \( P(c|H_1) \).

The error and detection probabilities defined here give rise to many performance criteria. One criterion that is relevant for communications systems is the Bayesian total probability of error. Assuming prior probability \( \pi_i \) that hypothesis \( i \) is true, the total probability of error \( P_e \) is

\[
P_e = \pi_0 P(e|H_0) + \pi_1 P(e|H_1).
\]

Here, the strategy is to classify \( x \) in a manner that minimizes \( P_e \). Another criterion is Neyman-Pearson, where the probability of error under one hypothesis is minimized while constraining the other error probability. That is, minimize \( P(e|H_1) \) while ensuring that \( P(e|H_0) \leq \epsilon \), or vice versa. The Neyman-Pearson criterion may be used when the prior probabilities \( \pi_i \) are unknown. The Bayesian total probability of error criterion can be said to be two-sided and symmetric, while Neyman-Pearson is one-sided and asymmetric.

One of the amazing facts in hypothesis testing is that the form of the optimum classifier, under all kinds of different performance criteria, is the likelihood ratio \( L(x) \) [42],

\[
L(x) = \frac{P_0(x)}{P_1(x)}.
\]

Simply stated, \( L(x) \) is the sufficient statistic for the classification problem. Denoting
\( T \in \mathbb{R} \) as the threshold, the likelihood-ratio test is defined as

\[
R_n = \{ x : L(x) > T \}, \quad R_n^c = \{ x : L(x) \leq T \},
\]

or more succinctly as

\[
\begin{align*}
&H_0 \\
L(x) \overset{\text{H}_1}{\gtrless} T.
\end{align*}
\]

(We assume that the set which yields \( L(x) = T \) is of measure zero.) When \( L(x) \) is known, the task of finding the optimum classifier is equivalent to setting the proper threshold \( T \). For example, it is well known that \( T = \pi_1/\pi_0 \) minimizes the total probability of error criterion, \( P_e \). In a Neyman-Pearson test, \( T \) is the value for which \( P(e|H_0) = \epsilon \). Detectors are often stated in terms of the log-likelihood ratio:

\[
\begin{align*}
&H_0 \\
h(x) \overset{\text{H}_1}{\gtrless} \lambda,
\end{align*}
\]

where \( h(x) = \frac{1}{n} \log L(x) \) and \( \lambda = \frac{\log T}{n} \). This equivalent formulation is used because \( \log L(x) \) simplifies for densities in the exponential family. The log-likelihood ratio is also convenient for asymptotic analysis.

When the underlying measures have unknown parameters, the generalized likelihood ratio test may be employed [42]. Here, the likelihood function of each hypothesis is maximized with respect to the unknown parameter:

\[
\text{GLRT} \equiv \frac{\max_{\theta} P_0(x|\theta)}{\max_{\theta} P_1(x|\theta)}.
\]

In essence, the maximum-likelihood estimate of the unknown parameter is used in the conditional likelihood function. Recently, it has been shown that the GLRT is optimal under a modified Neyman-Pearson criterion [60].

### 2.1.1 Asymptotic Analysis

It is of interest to know how well a detector works as more observations become available; i.e., with increasing sequence length \( n \). We need such results especially
in non-Gaussian cases because the exact probability of error is usually unavailable analytically. Luckily, asymptotic analysis is analytically tractable and may shed much insight when \( n \) is large. By asymptotics, here, we mean the exponential rate at which a particular probability converges to zero.

We begin by defining the well known Kullback-Leibler (KL) "distance" for univariate probability mass functions \( P \) and \( Q \) [24]

\[
D(P||Q) = \sum_{x \in A} P(x) \log \frac{P(x)}{Q(x)}.
\]  

(2.10)

The order in which \( P \) and \( Q \) appear in the definition is important because \( D(P||Q) \neq D(Q||P) \). \( D(P||Q) \) is also called the relative entropy between \( P \) and \( Q \) [6] or I-divergence [7]. In terms of hypothesis testing, the KL distance between \( P \) and \( Q \) is a measure of how well \( P \) and \( Q \) can be distinguished given the observation of independent, identically distributed (i.i.d.) \( x \). In particular, \( D(P||Q) \) is the largest exponential rate at which \( \Pr(e|Q \text{ true}) \) converges to zero when \( \Pr(e|P \text{ true}) \) is constrained to converge to zero. This summarizes Stein's Lemma.

**Theorem 1 (Stein's Lemma)** Let \( X_1, X_2, \ldots, X_n \) be i.i.d., to be classified between hypotheses \( H_0, H_1 \) which are represented by univariate measures \( P_0, P_1 \), respectively. Assume that \( D(P_0||P_1) < \infty \). Let

\[
\alpha_n = P_0^n(R_n^n) = P(e|H_0), \quad \beta_n = P_1^n(R_n) = P(e|H_1).
\]

(2.11)

For some \( \epsilon > 0 \) define

\[
\beta_n^\epsilon = \min_{R_n: \alpha_n < \epsilon} \beta_n.
\]

(2.12)

Then

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \beta_n^\epsilon = -D(P_0||P_1).
\]

(2.13)

Stein's Lemma tells us the asymptotic rate of error in the setting of a Neyman-Pearson criterion. When \( P(e|H_0) \) is constrained at \( \epsilon \), with \( \epsilon \) taken to zero arbitrarily slowly, \( P(e|H_1) \) has an exponential rate defined by the KL distance \( D(P_0||P_1) \). The
larger the KL distance, the faster the convergence, and the more "distinguishable" the hypotheses.

Since the asymptotic rate for the Bayesian total probability of error, \( P_e \) (2.4), is dominated by the slowest converging \( P(e|H_i) \), \( P_e \)'s rate is maximized when the individual rates are balanced. Let

\[
D^* = \lim_{n \to \infty} \min_{R_n} \frac{1}{n} \log P_e. \tag{2.14}
\]

**Theorem 2 (Chernoff)** The best achievable exponent in the Bayesian probability of error is \( D^* \), where

\[
D^* = D(P_{\alpha^*} || P_0) = D(P_{\alpha^*} || P_1), \tag{2.15}
\]

with

\[
P_{\alpha}(x) = \frac{P_0^\alpha(x)P_2^{1-\alpha}(x)}{\sum_{u \in A} P_0^\alpha(u)P_2^{1-\alpha}(u)} \tag{2.16}
\]

and \( \alpha^* \) the value of \( \alpha \) that satisfies

\[
D(P_{\alpha^*} || P_0) = D(P_{\alpha^*} || P_1). \tag{2.17}
\]

The Chernoff information is \( D^* \),

\[
C(P_0, P_1) \equiv D(P_{\alpha^*} || P_0) = D(P_{\alpha^*} || P_1), \tag{2.18}
\]

and it has an alternate formulation [6]:

\[
C(P_0, P_1) = -\min_{0 \leq \alpha \leq 1} \log \left( \sum_{u \in A} P_0^\alpha(u)P_1^{1-\alpha}(u) \right). \tag{2.19}
\]

The Chernoff information actually defines a strict exponentially decaying probability of error under each hypothesis [5, 6]:

\[
P(e|H_i) \leq 2^{-nC(P_0, P_1)}, \quad i = 0, 1. \tag{2.20}
\]
2.1.2 Method of Types

The method of types is a powerful set of tools used for asymptotic analysis, first refined in [8] and used in a variety of texts on information theory to prove theorems in capacity, hypothesis testing, and rate distortion theory [5, 6, 8]. A type is the empirical probability distribution (histogram) of a sequence of observations \( \mathbf{x} \) consisting of \( x_1, x_2, \ldots, x_n \). Recall from combinatorics that any sequences \( \mathbf{x} \) that result in the same histogram have the same probability. Our notation follows [6].

**Definition 1** The type \( q_\mathbf{x} \) (or empirical probability distribution) of \( \mathbf{x} \) is the relative proportion of occurrences of each symbol of \( \mathcal{A} \), that is, \( q_\mathbf{x}(a) = \frac{N(a|\mathbf{x})}{n} \) for all \( a \in \mathcal{A} \), where \( N(a|\mathbf{x}) \) is the number of times the symbol \( a \) occurs in the sequence \( \mathbf{x} \in \mathcal{A}^n \).

**Definition 2** Let \( \mathcal{P}_n \) denote the set of types with denominator \( n \), that is, types that result from a length-\( n \) observation \( \mathbf{x} \).

**Definition 3** If \( Q \in \mathcal{P}_n \), then the set of sequences of length \( n \) and type \( Q \) is called the type class of \( Q \), denoted \( T(Q) \), i.e.,

\[
T(Q) = \{ \mathbf{x} \in \mathcal{A}^n : q_\mathbf{x} = Q \}.
\] (2.21)

Major results regarding types are summarized in the following four relationships [6]:

\[
|\mathcal{P}_n| \leq (n + 1)^{|\mathcal{A}|}, \quad (2.22)
\]

\[
P^n(\mathbf{x}) = 2^{-n(D(q_\mathbf{x}||P) + H(q_\mathbf{x}))}, \quad (2.23)
\]

\[
|T(Q)| \approx 2^{nH(Q)} , \quad (2.24)
\]

\[
P^n(T(Q)) \approx 2^{-nD(Q||P)}. \quad (2.25)
\]

The relation \( \approx \) denotes operands that have the same exponential rate. The number of types is polynomial in \( n \) (2.22), while the number of sequences in each type class
is exponential in $n$ (2.24). The exact formula for the probability of any sequence is given in (2.23), and (2.25) gives an approximation for the probability of a type class. Note that (2.23) tells us that the maximum-likelihood estimate of the underlying pmf, $P$, is $q_\mathbf{x}$ since the KL is strictly non-negative and equals zero if and only if the two measures are identical.

Consider the one-sided null-hypothesis test defined in terms of $P_0$:

$$\Lambda_0 = \{ \mathbf{x} : D(q_\mathbf{x} \parallel P_0) \leq \lambda \}. \quad (2.26)$$

That is, accept the null hypothesis if the observations fall within a KL distance of $\lambda$ from $P_0$. The method of types allows us to easily determine the asymptotics of the error probability under $H_0$: Every type in $\Lambda_1 \equiv \Lambda_0^c$ has exponential probability of the order $\lambda$ (2.25), and there are at most a polynomial number of types in $\Lambda_1$; therefore

$$\lim_{n \to \infty} \frac{1}{n} \log P(\mathbf{e} | H_0) \leq -\lambda. \quad (2.27)$$

And consistent with Stein’s Lemma, we note that if $\lambda < D(P_1 \parallel P_0)$, then the error probability under $H_1$ is also exponentially decreasing:

$$\lim_{n \to \infty} \frac{1}{n} \log P(\mathbf{e} | H_1) \leq -e(\lambda) < 0, \quad (2.28)$$

where $e(\lambda)$ is the error exponent function [4, 5] defined by

$$e(\lambda) = \lim_{n \to \infty} \min_{q_\mathbf{x} \in \Lambda_0} D(q_\mathbf{x} \parallel P_1). \quad (2.29)$$

That the error probability under $H_1$ also converges exponentially can be easily shown using the method of types, and in particular, it is a direct consequence of Sanov’s Theorem [5, 6], which is stated below almost verbatim from [6].

**Theorem 3 (Sanov’s Theorem)** Let $X_1, \ldots, X_n$ be i.i.d. $\sim Q$. Let $E$ be a set of probability distributions. Then

$$\Pr(q_\mathbf{x} \in E) = Q^n(E) = Q^n(E \cap \mathcal{P}_n) \leq (n + 1)^{|\mathcal{A}|} 2^{-nD(P^{\ast} \parallel Q)}, \quad (2.30)$$
where

\[ P^* = \arg \min_{P \in E} D(P\|Q), \]

is the distribution in \( E \) that is closest to \( Q \) in relative entropy. If, in addition, the set \( E \) is the closure of its interior, then

\[ \lim_{n \to \infty} \frac{1}{n} \log Q^n(E) = -D(P^*\|Q). \]

### 2.1.3 Quantifying Relative Performance

Given two detectors or detection systems, how can their relative performances be quantified? It is typical in communications systems to compare performances on the decibel (dB) scale, and detector performances may be compared in the same way. The relative performances of the detectors may be captured by quantifying the amount of increased (decreased) SNR that is needed to make the performance of the worse (better) detector equal that of the other.

The signal-to-noise ratio is a function of the signal amplitude per observation and the number of observations \( n \). For a Gaussian system, SNR directly controls performance, but for a non-Gaussian system, performance depends explicitly on signal amplitude and \( n \). To measure the relative performances of two discrete-valued detectors, we may compare the required sample sizes at which the detectors achieve a certain, equal level of error probability. Because the sample length controls the relative amounts of energy (and SNR) input to the detectors, the SNR difference of the second detector from the first is:

\[ 10 \log_{10} \frac{n_2}{n_1} \text{dB}, \]

where \( n_1 \) and \( n_2 \) are the respective sample lengths that achieve a certain error probability. Depending on the particular system in question, this relative measure may be different for different levels of error probabilities. When the exact probability of error cannot be calculated, simulations may be used.
If we can capture the exponential rates of convergence of the competing detectors, say rates $C_1$ and $C_2$, they may be used to approximate the sample sizes where the errors are equal. Simple manipulations then yield

$$10 \log_{10} \frac{n_2}{n_1} = 10 \log_{10} \frac{C_1}{C_2}, \quad (2.34)$$

and performance does not depend explicitly on $n$ and error probability. The relative rate completely $C_1/C_2$ captures the asymptotic SNR difference.
2.2 Classification based on Empirically Observed Statistics

When classification is based on empirically observed statistics, we no longer have exact knowledge of $P_0$ and $P_1$, and we are instead provided with training sequences $t_0$ and $t_1$ known to have emerged from each hypothesis. This situation is often what practitioners face in the real world, when the system underlying each hypothesis is either very complicated to model or unknown altogether, for example, when the sources are non-Gaussian.* The observation space is more than the space of test vectors, $A^n$, and encompasses also the space of training vectors, $t_1 \in A^N$, $t_2 \in A^N$. The resulting observation space, $\{x, t_1, t_2\} \in A^n \times A^N \times A^N$, is a product space from which decision regions are determined. Because the decision regions rely only on the observations and hence independent of the true underlying measures $P_0$ and $P_1$, the label "universal" may be applied to such a problem. The universality of a particular classification strategy also imply that it works equally well for all underlying measures. The universal classification problem has been addressed recently by a number of works [14, 28, 52, 59, 62]. Our work is drawn primarily from Gutman[14].

While the results of Gutman [14] concern $M$-ary hypotheses, binary hypotheses can be discussed without loss of generality. In order to simplify notation, samples of the observed vectors are assumed to be i.i.d. Gutman's results [14] apply to finite-order Markov sequences, while Ziv's results [62] deal with a Markovian class that has diminishing dependence to the past. Dealing only with i.i.d. measures is not without loss of generality, but extension to the Markov processes is straightforward and presented in sequel. The methods that are used in proofs, and which also provide valuable intuition, can be readily learned with the i.i.d. system.

There are several variants in the universal classification problem that may be defined. In all cases, the performance measure of interest is the asymptotic exponential rate of convergence of the error probabilities.

*Here, we mean quantized data where the underlying continuous-valued density function is non-Gaussian.
Null Hypothesis Test

In the null hypothesis test, training vector \( t_0 \) is drawn from \( H_0 \). Based on the observation of the test sequence \( x \), the problem is to decide whether the test sequence \( x \) belongs to the null hypothesis \( H_0 \) based on the (empirical) decision regions \( (\Omega_0, \Omega_5) \) in \( \mathcal{A}^n \times \mathcal{A}^N \). The detector design problem is to maximize the exponential rate of convergence of \( P(e|H_0) \):

$$
\lim_{n \to \infty} \frac{1}{n} \log P(e|H_0),
$$

while allowing \( P(e|H_1) \nrightarrow 0 \). The null hypothesis, universal classification problem was addressed by Ziv [62] and Gutman [14].

Test with Rejection Region

Consider the classification problem where there are two training vectors, \( t_0 \sim P_0^N \) and \( t_1 \sim P_1^N \). Given the test sequence, \( x \), the classifier is allowed three possible decisions: \( H_0 \), \( H_1 \), or "no-decision" (called "reject" in [14]). This is the decision problem with rejection for two hypotheses, and the classification rule consists of a partition of the test- and training-data observation space, \( \mathcal{A}^n \times \mathcal{A}^N \times \mathcal{A}^N \), into three disjoint sets \( \{\Omega_0, \Omega_1, \Omega_R\} \), where \( \Omega_R \) is the rejection region. If \( x, t_0, t_1 \) falls in \( \Omega_i \), \( i = \{0, 1\} \), the \( i^{th} \) hypothesis is chosen. If \( x, t_0, t_1 \) falls in \( \Omega_R = (\Omega_0 \cup \Omega_1)^c \), no decision is made. Generally speaking, the rejection decision is chosen because the test sequence does not strongly resemble one training sequence over the other. A decision is considered erroneous only when the true model is \( j \) but hypothesis \( i \), \( i \neq j \), is chosen; a correct detection is made if hypothesis \( j \) is chosen. A rejection, regardless of the true hypothesis, results in neither an error nor correct detection. The probabilities of error, rejection, and correct detection, under hypothesis \( i \) are thus defined as

$$
P_\Omega(e|H_i) = \Pr(x, t_0, t_1 \in \Omega_j|H_i), \ j \neq i
$$

$$
P_\Omega(R|H_i) = \Pr(x, t_0, t_1 \in \Omega_R|H_i),
$$

(2.36) (2.37)
\[ P_{\Omega}(c|H_i) = 1 - P(e|H_i) - P(R|H_i), \] (2.38)

where the symbol \( \Omega \) denotes a particular partition of the observation space, or put another way, \( \Omega \) denotes a particular empirical decision rule. The test-with-rejection problem is conceptually similar to the sequential detection problem [30, 51] where the decision is to choose one hypothesis or to wait for more data. When inference does not clearly point to one of the hypotheses, the choice is to not make a decision. The test with rejection is characterized by Gutman [14].

**Forced-Decision Test (without rejection)**

In the classification problem without rejection, a hard decision pointing to one hypothesis must be made. This is the classic binary hypothesis test and is the most important one in digital communications where a candidate information bit must be selected. This forced-decision empirical test was addressed by neither Ziv nor Gutman, and its analysis in this thesis is an original contribution.

**2.2.1 Preliminaries**

This section contains preliminary material on empirical classification. The preliminary material should elucidate some of the mathematics to build intuition needed for later exploration. In the following sections, primary results are then presented. As well, re-statements to some of Gutman's results are included to correct minor points that do not appear to have been proven in [14], and which now do not appear provable. These "errors" were first reported by Warke [52] and Kelly [23]. We add several ways to re-state Gutman's main results. We also present Gutman's methods and results in a different order in such a way to demonstrate how results for the null-hypothesis test can be applied directly to the test with rejection. The i.i.d. assumption is made throughout these sections.

The first major result is that a type-based rule asymptotically performs as well or better than any other rule, to be defined below. Let \( \Omega = (\Omega_0, \Omega_1) \) be any decision rule
based on observations \(x, t_0, t_1\) and independent of true measures \(P_0, P_1\). From this rule, define a corresponding type-based rule, \(\Lambda = (\Lambda_0, \Lambda_1)\), based on the sufficient statistics (types) \(\{q_x, q_{t_0}, q_{t_1}\}\) as follows: Let the triplet set of types, \(\{q_x, q_{t_0}, q_{t_1}\}\), belong to \(\Lambda_0\) if at least half of its composite sequences belong to \(\Omega_0\); \(\Lambda_1\) is defined similarly. Then, Gutman shows that
\[
\lim_{n \to \infty} \frac{1}{n} \log P_\lambda(e|H_i) \leq \lim_{n \to \infty} \frac{1}{n} \log P_\Omega(e|H_i),
\]
for \(i = 0, 1\) and uniformly for all \(P_0\) and \(P_1\).

Equation (2.39) [14, Lemma 2] is both remarkable and evasive. It is remarkable that the error probabilities of a type-based rule are always better than those of a non-type-based rule, in the asymptotic sense. However, it is readily seen in the proof of Gutman’s Lemma 2 that equation 2.39 is derived from the following inequalities
\[
P(\Lambda_j|H_i) \leq P(\Omega_j|H_i)2^{n(\Theta_{logN} + O(\frac{logN}{n}))},
\]
for all \(i, j \in \{0, 1\}\), so that in the limit,
\[
\lim_{n \to \infty} \frac{1}{n} \log P(\Lambda_j|H_i) \leq \lim_{n \to \infty} \frac{1}{n} \log P(\Omega_j|H_i),
\]
for all \(i, j \in \{0, 1\}\). Equation (2.39) follows by noting that cases of (2.41) where \(i \neq j\) define error probabilities. The undesirable result, that the probabilities of correct detection (case \(i = j\)) for the type-based classifier are smaller, is omitted from the statement of [14, Lemma 2].

The preceding exercise demonstrates how polynomial terms may be conveniently ignored in asymptotic analysis. Because the construction of \(\Lambda\) from \(\Omega\) yielded probabilities that differ by a polynomial factor, the asymptotic rates of error and of correct-detection probabilities of the type-based test are unaffected. Perhaps a fairer interpretation of (2.40) is that the tests \(\Lambda\) and \(\Omega\) are asymptotically equivalent, and not that \(\Lambda\) has asymptotically smaller error regions than \(\Omega\). Equation (2.40) also plays an important role in the re-statements of Gutman’s theorems, which are addressed in sequel.
Gutman’s Classifier

Define $y_i$ to be the concatenation of $t_i$ with $x$, i.e.,

$$y_i(l) = \begin{cases} 
  t_i(l) & l \leq N \\
  x(l - N) & N < l \leq n + N.
\end{cases} \tag{2.42}$$

The type for $y_i$, $q_y$, is defined as those for $t_i$ and $x$ and can be shown to be:

$$q_y = \frac{n}{n + N} q_x + \frac{N}{n + N} q_t, \tag{2.43}$$

a linear combination of $q_t$ and $q_x$. Gutman defines the following test statistic for $H_i$:

$$h_i = h(x, t_i) = D(q_x || q_y) + \frac{N}{n} D(q_t || q_y) + \rho(n, N), \tag{2.44}$$

where $D(P || Q)$ is the Kullback-Leibler “distance” between measures $P$ and $Q$; $\rho(n, N)$ is a fixed, known term of order $O((\log n + 2 \log N)/n)$ that is used to “shape” the decision regions without affecting their convergence rates. The weighted sum of $D(q_x || q_y)$ and $D(q_t || q_y)$ may be seen to measure indirectly the divergence between $q_x$ and $q_t$ via an intermediary measure $q_y$. Statistic $h_i$ is a non-negative measure of dissimilarity between $x$ and the $i$th hypothesis as represented by $q_t$: It is large when $x$ and $t_i$ are distributed differently and small when they are similar.

Gutman’s discriminator is motivated from the generalized likelihood ratio test, or GLRT (Section 2.1), which has been shown to yield asymptotically optimal properties for a variety of decision problems [60]. Consider the problem of testing the hypothesis that $x$ and $t_i$ are differently distributed versus the hypothesis that they are identically distributed. Keeping in mind that the true distributions $P$ and $Q$ are unknown, the GLRT compares the individual likelihoods maximized over the unknown parameters, which in this case are the alphabet probabilities of the true measures:

$$\text{GLRT} \equiv \frac{1}{n} \log \frac{\sup_{P,Q} P(t_i)Q(x)}{\sup_P P(t_i)P(x)}. \tag{2.45}$$

Recognizing each supremum is reached by the type, straightforward manipulations yield:

$$\text{GLRT} \equiv \frac{1}{n} \log \frac{\sup_{P,Q} P(t_i)Q(x)}{\sup_P P(t_i)P(x)} = D(q_x || q_y) + \frac{N}{n} D(q_t || q_y). \tag{2.46}$$
Hence, from the GLRT perspective, it is evident that the statistic $h_i$ is large when $t_i$ and $x$ are differently distributed, since this hypothesis's generalized likelihood function is in the numerator. Likewise, statistic $h_i$ is small when the hypothesis whose generalized likelihood function in the denominator is true, as when $t_i$ and $x$ are similarly distributed. We note that the GLRT is made individually on each $i$. It is a simple exercise to show that (2.44) may be equivalently expressed as

$$h(x, t_i) = \frac{n + N}{n} H(q_y) - H(q_x) - \frac{N}{n} H(q_t) + \rho(n, N),$$

(2.47)

where $H(P) = \sum_{u \in A} P(u) \log P(u)$ is the entropy of a random variable distributed as $P$ [6].

2.2.2 Null Hypothesis Test

Having demonstrated that type-based universal tests are asymptotically as good as non-type-based tests, we next quantify the performance of Gutman's type-based discriminator (2.44) and compare it to non-type-based tests. The non-type-based test is symbolized by $\Omega = (\Omega_0, \Omega_1)$ and quantified to have a strictly exponential error rate of $\lambda$ under the null hypothesis, $H_0$:

$$P_\Omega(e|H_0) = \sum_{t_1} P_0(x) P_0(t_0) P_1(t_1) \leq 2^{-\lambda n}.$$  

(2.48)

For Gutman's test, symbolized by $\Lambda = (\Lambda_0, \Lambda_1)$ to denote a likelihood test, the acceptance region for $H_1, \Lambda_1$, is defined to be

$$\Lambda_1 = \{x, t_0, t_1 : h(x, t_0) \geq \lambda\},$$

(2.49)

for $\lambda > 0$. The acceptance region for $H_0, \Lambda_0$, is simply the complement of $\Lambda_1$:

$$\Lambda_0 \equiv \Lambda_1^c.$$  

(2.50)

Note, in particular, that the decision regions are defined independently of $t_1$. Gutman shows [14, Theorem 1] that

$$\lim_{n \to \infty} \frac{1}{n} \log P_\Lambda(e|H_0) \leq -\lambda, \; \forall P_0, P_1$$

(2.51)
with uniform convergence (uniform over \( P_0 \) and \( P_1 \)) and

\[
P_A(e|H_1) \leq P_{\Omega}(e|H_1), \quad \forall P_0, P_1.
\]  \( (2.52) \)

Equation (2.51) says that, using \( \lambda \) to define the decision regions, the type-based detector's probability of error under the null hypothesis \( H_0 \) achieves rate \( \lambda \), equaling that of the other test under comparison. Equation (2.52) says that the resulting probability of error under the alternate hypothesis \( H_1 \) is always smaller than the other test under comparison. It relies on Gutman's assertion that [14, eqn. (29)]

\[
\Omega_1 \subset \Lambda_1.
\]  \( (2.53) \)

Equations (2.51) and (2.52) effectively makes Gutman's detector asymptotically optimal among all detector's with null-hypothesis error rate greater than or equal to \( \lambda \).

Equation (2.51) can be easily shown, seen below using a slightly different procedure than in [14] to reveal the methodology using types. The probability of error under \( H_0 \),

\[
P_A(e|H_0) = \sum_{x,t_0,t_1 \in \Lambda_1} P_0(x) P_0(t_0) P_1(t_1),
\]  \( (2.54) \)
can be bounded without enumeration over \( t_1 \) because \( \Lambda_1 \) is defined independently of \( t_1 \),

\[
P_A(e|H_0) \leq \sum_{x,t_0 \in \Lambda_1} P_0(x) P_0(t_0).
\]  \( (2.55) \)

Because the type is the ML estimate of the true probability measure, \( P_0 \) can be replaced by \( q_{xt_0} \) to yield

\[
P_A(e|H_0) \leq \sum_{x,t_0 \in \Lambda_1} q_{xt_0}(x) q_{xt_0}(t_0).
\]  \( (2.56) \)

Now, re-enumerate the summation in terms of type classes of \( x \) and \( t_0 \), and use the upper bound of the probability of a type class (2.25):

\[
P_A(e|H_0) \leq \sum_{q_x q_{xt_0} \in \Lambda_1} 2^{-nD(q_x||q_{xt_0})} \cdot 2^{-nD(q_{xt_0}||q_{xt_0})}.
\]  \( (2.57) \)
As types in $\Lambda_1$ are defined by (2.49),

$$P_\Lambda(e|H_0) \leq 2^{-\lambda n} \cdot 2^{n\rho(n,N)} \sum_{\Lambda_1}(1).$$

(2.58)

The above summation can be replaced by the total number of types in $\mathcal{A}^n \times \mathcal{A}^N$: $(n + 1)^{|\mathcal{A}|}(N + 1)^{|\mathcal{A}|}$, yielding

$$P_\Lambda(e|H_0) \leq 2^{-\lambda n} \cdot 2^{|\mathcal{A}|(\log(n+1) + \log(N+1))} \cdot 2^{n\rho(n,N)},$$

(2.59)

which yields, when both sizes are operated by $\log(\cdot)$ and divided by $n$,

$$\frac{1}{n} \log P_\Lambda(e|H_0) \leq -\lambda + |\mathcal{A}| \left( \frac{\log(n+1)}{n} + \frac{\log(N+1)}{n} \right) + \rho(n,N).$$

(2.60)

Equation (2.51) is proven by requiring

$$\frac{\log(N+1)}{n} \xrightarrow{n \to \infty} 0$$

(2.61)

and $\rho(n,N) \xrightarrow{n \to \infty} 0$.

The sufficient condition $\frac{\log(N+1)}{n} \to 0$ is a curious one. It says that too much training data (in the mathematical sense defined) is not desirable, which is directly against intuition. It is instructive to briefly point out why this term is present. The $2^{\log(N+1)}$ term comes from counting the total number of types in $\mathcal{A}^N$. Because the asymptotic error rate with respect to $n$ is desired, the division by $n$ in the exponent of $2$ yields the $\frac{\log(N+1)}{n}$. When (2.61) is not satisfied, there are too many types in $\mathcal{A}^N$ to enumerate successfully. In standard analysis of classical hypothesis testing and rate-distortion theories (c.f. [6, 8]), the lack of the training-vector space precludes the comingling of the $n$ and $N$ terms, yielding only $\frac{\log(n+1)}{n}$, which always goes to zero, and the exponential rate is preserved without additional constraints. We believe this technicality to be unnecessarily burdensome. For one, we note that the upper bound for the total probability of error is extremely loose. Also, consider the likelihood statistic $h(x,t_0)$ as $C = N/n$ goes to $\infty$. It is easy to show that under $H_0$, $h(x,t_0)$ converges to zero, and under $H_1$, $h(x,t_0)$ converges to $D(P_1||P_0)$. Then, given any
\( \lambda > 0 \), the convergence point under \( H_0 \) is strictly inside \( \Lambda_0 \). Hence we believe an exponential error rate will result.

The second part of Gutman’s basic null-hypothesis results, as demonstrated in equation (2.52), compares the probabilities of error under the alternate hypothesis, \( H_1 \), for the Gutman test against an arbitrary empirically based test, which is denoted by \( \Omega \). Equation (2.52) comes from Gutman’s assertion that \( \Omega_1 \subset \Lambda_1 \), so that \( \Lambda_0 \subset \Omega_0 \), and hence (2.52). What Gutman actually shows is that \( \Omega_1 \subset \Lambda_1 \) for test \( \Omega \) that is type-based and satisfies (2.48). He then argues that, using (2.40) and (2.41) [14, Lemma 2], \( \Omega_1 \subset \Lambda_1 \) would hold for an arbitrary test \( \Omega \) satisfying (2.48). We are not able to verify this last logical step, although there are several ways to re-state Gutman’s Theorem 1. Possible re-statements are listed below; they are derived in Appendix A.

- Assume that \( \Omega \) is strictly a type-based test. Gutman has already shown as much. This is the assumption made by Kelly [23].

- Assume that \( \Omega \) is strictly a type-based test, but whose error probability under \( H_0 \) is not strictly less than \( 2^{-\lambda n} \) but relaxed to be less than \( 2^{-\lambda n + O(\log n) + O(\log N)} \), where the \( O(\cdot) \) terms are known.

- Claim that (2.52) is true only in the sense of the asymptotic exponential rate:

\[
\lim_{n \to \infty} \frac{1}{n} \log P_{\lambda}(e|H_1) \leq \lim_{n \to \infty} \frac{1}{n} \log P_{\Omega}(e|H_1), \quad \forall P_0, P_1. \tag{2.62}
\]

This is the approach taken by Warke [52].

We do not believe these re-statements substantially weaken Gutman’s results. One may even question the utility of using tests that are not based on types. It is against all established results in hypothesis testing (Neyman-Pearson, Bayesian) to classify differently two sequences that have the same probabilities under \( H_0 \) and \( H_1 \), that is,

\[
P_0(x_0) = P_0(x_1) \quad \text{and} \quad P_1(x_0) = P_1(x_1). \tag{2.63}
\]
Because all sequences in the same type class have the same probability under any measure, a non-type based test may classify $x_0$ and $x_1$ differently. The likelihood-ratio test, for example, always classifies an observation based on its probability of occurrence under each hypothesis and thus would always respect type-class boundaries. Here, we neglect discussion of randomized tests, which randomize decisions for observations that fall on the decision boundary, because randomization does not affect the asymptotics.

**Errors Under Alternate Hypothesis**

The type-based detector defined by (2.44) and (2.49) has exponential error rate $\lambda$, under the null hypothesis, $H_0$, regardless of the true measures $P_0$ and $P_1$. We examine in this section the exponential error rate under the alternate hypothesis, $H_1$ [14], as well as the conditions that cause the probability of error of $H_1$ to converge to 1 [14, 62].

Define

\[
C = \lim_{n \to \infty} \frac{N}{n} \tag{2.64}
\]

\[
P_{j,m} = \frac{n}{n + N} P_j + \frac{N}{n + N} P_m \tag{2.65}
\]

\[
\lambda_0(C, P_j, P_m) = \begin{cases} 
D(P_j \parallel P_m), & \text{if } C = \infty \\
D(P_j \parallel P_{j,m}) + CD(P_m \parallel P_{j,m}), & \text{if } 0 < C < \infty \\
0, & \text{if } C = 0 
\end{cases} \tag{2.66}
\]

Two conditions imply that $P(e|H_1)$ goes to 1:

\[
\lim_{n \to \infty} \frac{N}{n} = 0 \tag{2.67}
\]

and

\[
\lambda > \lambda_0(C, P_1, P_0). \tag{2.68}
\]

Conditions (2.67) and (2.68) are derived in [14, Theorem 3(1)]; condition (2.67) is also derived in [62, Theorem 3]. When the amount of training samples does not grow at least linearly with the amount of test data (2.67), the alternate-hypothesis acceptance
region $\Lambda_1$ (2.49) converges to $\emptyset$. This can be seen by noting that when $\lim_{n \to \infty} \frac{N}{n} = 0$, $q_{y_0} \to q_\infty$ and $h(x, t_0) \overset{\text{n}}{\to} 0$, which imply that $\Lambda_1 \overset{\text{n}}{\to} \emptyset$. The assertion of a constant error rate of $\lambda$ under $H_0$, even when training data is lacking, necessitates trading off the error probability under $H_1$ and causes $P(e|H_1) \overset{\text{n}}{\to} 1$.

The second condition, (2.68), generalizes Stein's Lemma for empirically based tests. Whereas for classical Neyman-Pearson tests, Stein's lemma shows that the largest effective\(^\dagger\) error rate under $H_0$ is $D(P_1||P_0)$, Gutman's results show that the largest effective error rate under $H_0$ is $D(P_1||P_{1,0}) + CD(P_0||P_{1,0})$, for finite $C > 0$. And when $C = \infty$, the empirically based test becomes equivalent to the clairvoyant classifier that knows $P_0$ exactly. To understand why $\lambda_0(C, P_1, P_0)$ is the limiting rate, note that under $H_1$, $q_\infty \overset{\text{n}}{\to} P_1$ and $q_{t_0} \overset{\text{n}}{\to} P_0$, and therefore

$$h(x, t_0) \overset{n}{\to} D(P_1||P_{1,0}) + CD(P_0||P_{1,0}) = \lambda_0(C, P_1, P_0). \quad (2.69)$$

Thus, if $\lambda > \lambda_0(C, P_1, P_0)$, the observed types $q_\infty$ and $q_{t_0}$ converge to points within $\Lambda_0$, and implies that $P(e|H_1) \to 1$.

Conditions (2.67) and (2.68) are applicable not only to Gutman's test, but to all empirically based tests that satisfy (2.48). The result for arbitrary $\Omega$ tests is derived from Gutman's test $\Lambda$ and noting the equivalent rate relationships between $\Lambda$ and arbitrary $\Omega$ tests. This gives an indication of the optimality of Gutman's classifier: it can achieve the largest allowable rate, $\lambda_0(C, P_1, P_0)$, under $H_0$, for all empirical classifiers.

Finally, given that good conditions

$$\lim_{n \to \infty} \frac{N}{n} > 0 \quad \text{and} \quad \lambda < \lambda_0(C, P_1, P_0) \quad (2.70)$$

are true, the error rate under $H_1$ is:

$$E_1 = \lim_{n \to \infty} \min_{(x, t_0) \in \{x, t_0 : h(x, t_0) < \lambda\}} \frac{D(q_x||P_1) + \frac{N}{n} D(q_{t_0}||P_0).}{n} \quad (2.71)$$

\(^\dagger\)By effective, we mean an error rate under $H_0$ such that the corresponding error rate under $H_1$ converges to 0.
Although not used here, $E_0$ can be defined similarly for a situation where the training samples come from $P_1$ and the decision regions are determined by $(t_1, x)$:

$$E_0 = \lim_{n \to \infty} \min_{(x, t_1) \in \{x, t_1 : h(x, t_1) < \lambda\}} D(q_x \| P_0) + \frac{N}{n} D(q_{t_1} \| P_1). \quad (2.72)$$

Equation (2.71) is a direct consequence of Sanov’s Theorem.

2.2.3 Test with Rejection

In the null-hypothesis universal test (Section 2.2.2), a training vector $t_0$ from $H_0$ defined the acceptance region of $H_1$, $A_1$, parameterized by a value, $\lambda$, that directly controls the exponential rate of $P(A_1 \mid H_0)$. Note especially that $t_i \sim P_i$ can play the same role for the error region of $H_1$. In particular, for $t_i$ and $H_i$:

$$\lim_{n \to \infty} \frac{1}{n} \log \Pr(h(x, t_i) \geq \lambda \mid x \sim P_i) \leq -\lambda, \quad (2.73)$$

where

$$\{x, t_0, t_1 : h(x, t_i) \geq \lambda\}, \quad (2.74)$$

defines the error region under $H_i$. Consider the sets based on both $t_0$ and $t_1$:

$$A_{0,i} \equiv \{x, t_0, t_1 : h(x, t_0) < \lambda, h(x, t_1) \geq \lambda\}$$

$$A_{0,1} \equiv \{x, t_0, t_1 : h(x, t_0) < \lambda, h(x, t_1) < \lambda\}$$

$$A_{0,1} \equiv \{x, t_0, t_1 : h(x, t_0) \geq \lambda, h(x, t_1) < \lambda\}$$

$$A_{0,1} \equiv \{x, t_0, t_1 : h(x, t_0) \geq \lambda, h(x, t_1) \geq \lambda\}, \quad (2.75)$$

where the subscript $i$ indicates that $h(x, t_i) < \lambda$ and $\overline{i}$ indicates that $h(x, t_i) \geq \lambda$.

Using (2.51), (2.71), and noting that

$$\Pr(A \cap B) \leq \Pr(A) \quad \text{and} \quad \Pr(A \cap B) \leq \Pr(B), \quad (2.76)$$
for measurable sets $A$ and $B$, we find the exponentially converging probabilities of sets $A_{i}$ to be:

\[
A_{0,i} : \lim_{n \to \infty} \frac{1}{n} \log P(A_{0,i}|H_1) \leq -\lambda \\
A_{0,1} : \lim_{n \to \infty} \frac{1}{n} \log P(A_{0,1}|H_0) \leq -E_0 \\
A_{0,i} : \lim_{n \to \infty} \frac{1}{n} \log P(A_{0,i}|H_1) \leq -E_1 \\
A_{0,i} : \lim_{n \to \infty} \frac{1}{n} \log P(A_{0,i}|H_0) \leq -\lambda \\
A_{0,i} : \lim_{n \to \infty} \frac{1}{n} \log P(A_{0,i}|H_1) \leq -\lambda.
\]  

(2.77)

Gutman uses these sets to construct the decision regions for the test with rejection:

\[
\Lambda_0 \equiv A_{0,i} \cup A_{0,i} \\
\Lambda_1 \equiv A_{0,i} \\
\Lambda_R \equiv A_{0,i}.
\]  

(2.78)

These decision regions yield symmetric error rates of $\lambda$ and a rejection rate of $\min(E_0, E_1)$. To see this, recall that an error only occurs under $H_i$ when the observation falls in $\Lambda_j$, $j \neq i$, and a rejection decision is never considered an error. Then, applying (2.77) and definition of $\Lambda$ (2.78),

\[
\lim_{n \to \infty} \frac{1}{n} \log P(e|H_i) = \lim_{n \to \infty} \frac{1}{n} \log P(\Lambda_j|H_i) \leq -\lambda, \quad i \neq j
\]  

(2.79)

and

\[
\lim_{n \to \infty} \frac{1}{n} \log P(R|H_i) = \lim_{n \to \infty} \frac{1}{n} \log P(\Lambda_R|H_i) \leq -E_i \leq -\min(E_0, E_1).
\]  

(2.80)

If the “bad” conditions

\[
\lim_{n \to \infty} \frac{N}{n} = 0 \quad \text{or} \quad \lambda > \min(\lambda_0(C, P_1, P_0), \lambda_0(C, P_0, P_1))
\]  

(2.81)

hold, then similar to the null-hypothesis case,

\[
P(R|H_i) \xrightarrow{\text{a.s.}} 1, \quad \text{for some } i.
\]  

(2.82)

Likewise, if the “good” conditions hold, then the rejection rate is strictly greater than 0:

\[
\lim_{n \to \infty} \frac{1}{n} \log P(R|H_i) \leq -E_i < 0.
\]  

(2.83)
Gutman's test with rejection can be compared to any other tests with rejection that satisfy dual error rates of $-\lambda$ using the same technique that made the comparisons for null-hypothesis tests. Thus, generally speaking, Gutman's test with rejection has the lowest probability of rejection, noting that the rejection region is comprised of the error regions of the alternate hypotheses. The strict statements regarding this comparison would require the re-statements we provided for the errors under the alternate hypothesis.

Intuitively, the test with rejection chooses $H_i$ if the received test vector is similar to $t_i$ and dissimilar to $t_j$, $j \neq i$. When $x$ is similar to both training vectors, a rejection decision is made. Gutman assigned the region, $A_{0,i}$, where $x$ is dissimilar to both training vectors to $H_0$, although this set is more appropriately considered a rejection region. That assignment was made for convenience of notation. Because the rate of $A_{0,i}$ under each hypothesis is $\lambda$, placing it in $\Lambda_R$ would result in the rate of rejection to be $\min(\lambda, E_i)$, which may not be as conveniently addressed. However, the following symmetric definition facilitates our subsequent analysis. We define the symmetric test with rejection, denoted by $\Lambda'$, to be:

$$
\begin{align*}
\Lambda'_0 &\equiv A_{0,i} \\
\Lambda'_1 &\equiv A_{0,i} \\
\Lambda'_R &\equiv A_{0,i} \cup A_{0,i},
\end{align*}
$$

(2.84)

noting that the corresponding rates are:

$$
\lim_{n \to \infty} \frac{1}{n} \log P_{\Lambda'}(e|H_i) \leq -\lambda
$$

(2.85)

$$
\lim_{n \to \infty} \frac{1}{n} \log P_{\Lambda'}(R|H_i) \leq -\min(\lambda, E_i)
$$

These results are used in Section 2.2.4.

The test with rejection is useful in situations where error rates under each hypothesis must be controlled and equal, with the rejection region minimized. If desired, the equality of the rates is simple to circumvent, and Warke [52] addressed using different rates $\lambda_i$ under each hypothesis. Possible applications of the test with rejection might
include communications problems with *erasures*, where a decision about the information bit only needs to be made if the observations strongly point to one hypothesis over the other.

### 2.2.4 Forced-Decision Test (no rejection)

In both the null-hypothesis test and the test with rejection, some of the error rates are controllable, but others are at the mercy of \( \lambda \) and the true underlying measures \( P_0 \) and \( P_1 \). While the lack of control may be acceptable in some situations, it is not in many others. Consider the test with rejection where it is desirable to maximize the rate of convergence of the probability of "not making a correct decision":

\[
1 - P(c|H_i) = P(e|H_i) + P(R|H_i),
\]

where "c" denotes "correct detection." Because the rate of the sum is dominated by the smallest rate term, we have

\[
\lim_{n \to \infty} \frac{1}{n} \log(1 - P(c|H_i)) \leq -\min(\lambda, E_i(\lambda)),
\]

where the dependence of \( E_i \) on \( \lambda \) is shown. Exponent \( E_i(\lambda) \) is a non-increasing function of \( \lambda \), since a larger \( \lambda \) always implies a larger set over which the minimization in (2.71) takes places. Hence, to maximize the rate in (2.87), it is desirable to use a \( \lambda^* \) that *balances* the error and rejection rates. But in the test with rejection, \( E_i(\lambda) \) (2.71) is a function of \( P_0 \) and \( P_1 \), thus balanced error rates cannot be made independent of \( P_0 \) and \( P_1 \), which are unknown.

For situations when a decision must be made, we derive the *forced-decision* detector and analyze its properties. We show in sequel that its error rate is the same as the balanced rate, \( \lambda^* \), described above. In addition, we show that among all other empirically based forced-decision tests that are derived from universal null-hypothesis tests, the forced-decision test based on Gutman's statistic has the best error rate. The analysis of this section is an original contribution.
Derivation

Consider the classification strategy that chooses $H_i$ if the likelihood statistic $h(x, t_i)$ is smaller than $h(x, t_j)$, $j \neq i$; i.e.,

$$\Lambda^F_0 \equiv \{x, t_0, t_1 : h(x, t_0) < h(x, t_1)\}$$
$$\Lambda^F_1 \equiv \{x, t_0, t_1 : h(x, t_0) \geq h(x, t_1)\}. \tag{2.88}$$

We call the resulting test, $\Lambda^F$, the forced-decision detector. Similar to the GLRT origins of $h(x, t_i)$, consider the two-sided GLRT that tests the hypothesis that $x$ and $t_0$ are identically distributed and $t_1$ is differently distributed against the hypothesis that $x$ and $t_1$ are identically distributed and $t_0$ is different:

$$x \sim t_0 \not\sim t_1 \text{ versus } x \sim t_1 \not\sim t_0. \tag{2.89}$$

The resulting GLRT and decision statistic, $h_{\text{GLRT}}^{\text{F}}(x, t_0, t_1)$, are expressed as

$$h_{\text{GLRT}}^{\text{F}}(x, t_0, t_1) \equiv \frac{1}{n} \log \frac{\sup_{P, Q} P(t_0) P(x) Q(t_1)}{\sup_{P, Q} P(t_1) P(x) Q(t_0)} \begin{array}{c} H_0 \\ H_1 \end{array} \geq 0. \tag{2.90}$$

Several observations can be made: the supremum over $P, Q$ is separable; the variables $P$ and $Q$ are dummy variables. Then, separating the supremums in (2.90), replacing $Q$ with $P$, and multiplying $\sup Q(x)$ in both the numerator and denominator, yields:

$$\frac{1}{n} \log \frac{\sup_{P, Q} P(t_0) P(x) Q(t_1)}{\sup_{P, Q} P(t_1) P(x) Q(t_0)} \equiv \frac{1}{n} \log \frac{(\sup_Q P(t_0) P(x))(\sup_Q P(t_1))}{(\sup_Q P(t_1) P(x))(\sup_Q P(t_0))} \begin{array}{c} \sup_Q Q(x) \\ \sup_Q Q(x) \end{array} \geq 0. \tag{2.91}$$

The second and third terms in the numerator and the first term in the denominator correspond to the GLRT for $H_1, h(x, t_1)$ (2.44), and the remaining terms correspond to $-h(x, t_0)$:

$$h_{\text{GLRT}}^{\text{F}}(x, t_0, t_1) = h(x, t_1) - h(x, t_0) \begin{array}{c} H_0 \\ H_1 \end{array} \geq 0. \tag{2.92}$$

Thus, the two-sided GLRT detector is equivalent to the forced-decision detector, defined by (2.88).
Optimality

Recall the symmetric test with rejection, $\Lambda'$, defined by the more intuitive rejection regions (2.84) that output "reject" if either $x$ is similar to $t_0$ and $t_1$ or $x$ is dissimilar to both $t_0$ and $t_1$. If $x, t_0, t_1 \in \Lambda'_0$, $h(x, t_0) < \lambda$ and $h(x, t_1) > \lambda$ by definition, that is, $h(x, t_0) < h(x, t_1)$. This observation implies that

$$\Lambda'_0 \subset \Lambda'_0^F \quad \text{and} \quad \Lambda'_1 \subset \Lambda'_1^F$$

is true for any value of $\lambda$. In particular, (2.93) implies that

$$(\Lambda'_i)^c = \Lambda'_j \subset (\Lambda'_j)^c = \Lambda'_R \cup \Lambda'_j, \quad j \neq i.$$  

(2.94)

Now consider the probability of error of the forced-decision test under $H_i$, noting that regions $\Lambda'_R$ and $\Lambda'_j$ are disjoint,

$$P_{\Lambda'}(c|H_i) = P(\Lambda'_j^c|H_i) \leq P((\Lambda'_i)^c|H_i) = P(\Lambda'_R|H_i) + P(\Lambda'_j|H_i), \quad j \neq i.$$  

(2.95)

Therefore, using the error and rejection rates of $\Lambda'$ (2.85), we can quantify the error rates of the forced-decision detector under both $H_0$ and $H_1$:

$$\lim_{n \to \infty} \frac{1}{n} \log P_{\Lambda'}(c|H_0) \leq -\min(\lambda, E_0) \quad \forall \lambda.$$

(2.96)

Note that the rate terms on the left side of the inequality are fixed given $P_0$ and $P_1$. The terms on the right side are functions of $\lambda$, $P_0$, and $P_1$. In particular, definitions (2.72) and (2.71) show $E_0$ and $E_1$ to be non-increasing functions of $\lambda$. Then, equation (2.96) implies that

$$\lim_{n \to \infty} \frac{1}{n} \log P_{\Lambda'}(c|H_0) \leq -\sup_\lambda \min(\lambda, E_0)$$

and

$$\lim_{n \to \infty} \frac{1}{n} \log P_{\Lambda'}(c|H_1) \leq -\sup_\lambda \min(\lambda, E_1).$$

(2.97)

Thus, under each hypothesis, the forced-decision detector achieves the optimum rate for the probability of not making a correct decision. It automatically balances the trade-off between $\lambda$ and $E_i$, which is something that the test with rejection cannot do.
*a priori* without knowledge of \( P_0 \) and \( P_1 \). Because the Gutman test with rejection is optimal in the rate sense among all empirical tests with rejection, the forced-decision detector has a better rate for the probability of not making a correct decision, for any \( P_0 \) and \( P_1 \), among all empirical tests *with rejection* that have dual error rates of \( \lambda \), for any \( \lambda \).

Recognizing that the interplay between \( \lambda \) and \( E_i \) is intrinsic of Gutman’s *null*-hypothesis test and that his test is optimal among all empirically based universal tests, we can state the forced-decision test’s optimality in another way. Consider a null-hypothesis universal test \( \Omega \) whose acceptance region is defined by some test statistic \( h_\Omega(x, t_0) \):

\[
\Omega_1 \equiv \{ x, t_0 : h_\Omega \geq \psi \}. \tag{2.98}
\]

This universal test statistic, \( h_\Omega \), leads to a forced-decision test:

\[
h_\Omega(x, t_1) \begin{cases} 
    H_0 & > \\
    H_1 & \leq \end{cases} h_\Omega(x, t_0). \tag{2.99}
\]

The forced-decision test based on Gutman’s \( h(x, t_i) \) is clearly better than this forced-decision test in the different ways that we have discussed (because of re-statement of Gutman’s analysis of the error under the alternate hypothesis).

### 2.2.5 Modification for Markov Sequences

All of the universal likelihood results presented in this chapter also hold for \( K \)-th order Markov processes. Here, \( K \)-th order conditional types are required to calculate test statistics. Conditional types are built by dividing the \((K + 1)\) order joint type by the \( K \)-order joint type

\[
q(x|s) = q(x, s)/q(s), \quad x \in A, \quad s \in A^K, \tag{2.100}
\]

and the \( K \) order type calculated by summing over \( q(x, s) \),

\[
q(s) = \sum_{x \in A} q(x, s). \tag{2.101}
\]
The \((K+1)\) order joint type for \(t_i\) and \(x\) are respectively

\[
q_{t_i}(x,s) = \frac{1}{N} \sum_{j=1}^{N} I(x, t_i(j)) I(s, (t_i(j-1), \ldots, t_i(j-K))) \quad (2.102)
\]

\[
q_{x}(x,s) = \frac{1}{n} \sum_{j=1}^{n} I(x, x(j)) I(s, (x(j-1), \ldots, x(j-K))). \quad (2.103)
\]

The type of the concatenated sequence \(y_i\) is similarly defined.

Entropy for a \(K^{th}\) order Markov process with \((K+1)\) order joint density \(P(x,s)\) is defined by

\[
H(P) = -\sum_{s \in A^K} P(s) \sum_{x \in A} P(x|s) \log P(x|s) \quad (2.104)
\]

Thus, \(H(q_{t_i}), H(q_{x}),\) and \(H(q_{y_i})\) are defined accordingly. Finally, the Kullback-Leibler distance \(D(P||Q)\) for \(P, Q\) representing \(K^{th}\) order Markov processes is defined as

\[
D(P||Q) = \sum_{s \in A^K} P(s) \sum_{x \in A} \log \frac{P(x|s)}{Q(x|s)}. \quad (2.105)
\]

The test statistic (2.44) is constructed using (2.105).

2.3 Summary

Major results presented in this chapter are summarized below. We note that all of the results presented are applicable to finite-order Markov chains and may be easily extended to \(M\)-ary hypothesis tests, following [14]. Kelly [23] extended some of the results of [14] to Markov chains of unknown orders using context trees, and we believe all of the results presented in this chapter may also be extended.

- The baseline null-hypothesis universal classifier is defined by

\[
\Lambda_1 = \{x, t_0 : h(x, t_0) \geq \lambda\},
\]

where \(h(x, t_0)\) (2.44) is a one-sided GLRT statistic testing if \(x\) and \(t_0\) are similarly distributed. Under \(H_0\), the error probability has rate \(\lambda\). If \(\lambda\) is taken too large (2.68) or \(N\) does not grow linearly with \(n\), the probability of error under
$H_1$ goes to one. However, if these bad conditions are avoided, then $P(e|H_1)$ converges exponentially at rate $E_1 > 0$ (2.71).

- Compared to other universal tests with $P(e|H_0)$ converging at rate $\lambda$, Gutman’s test has the best error rate under $H_1$. We provided re-statements to this comparison because Gutman’s result appears to have a gap.

- In the test with rejection, a third option – called rejection – may be chosen by the classifier. In this scenario, both $t_0$ and $t_1$ are used for classification. We showed how its properties translate directly from the null-hypothesis tests.

- We derived the forced-decision detector using a two-sided GLRT. We found that it has a close relationship to the test with rejection. Using this relationship, we showed that the forced-decision detector has the fastest convergence rate for $1 - P(c|H_i)$, which is the probability of not making a correct decision, among all tests with rejection. As well, we showed that the forced-decision detector based on the GLRT has better asymptotics than any other forced-decision detector that is based on the comparison of one-sided universal test statistics, as in

\[
h_{\Omega}(x, t_1) \stackrel{H_0}{>_{H_1}} h_{\Omega}(x, t_0),
\]

where $h_{\Omega}(x, t_i)$ defines a one-sided, universal test with exponential error rate under $H_i$. 

Chapter 3

Empirical Type-based Digital Receiver

In this chapter, we evaluate basic properties of the empirical type-based detector in a simple digital-communications system where the discrete-valued data are obtained via quantization of continuous-valued, analog data. As noted in the Introduction, the empirical detector may only be one of many possible solutions to the problem of detecting continuous-valued data in unknown environments. We use the type detector both for its simplicity and for its provable, asymptotically optimal properties, and, as shown in the remainder of this thesis, for its superior finite-sample properties as well.

3.1 Empirical Type-based Detection System

Because the type-based empirical classifier can be trained to function in unknown statistical environments, it is readily applicable to digital communication problems where channel statistics are either unknown or difficult to ascertain. We start with a simple digital communication system that uses binary phase shift keying (BPSK), where an information bit is sent/received every $T$ seconds. To transmit the binary information bit "0", analog signal $+s(t)$ is sent; for bit "1", $-s(t)$ is sent. For the moment, for additional simplicity, we take $s(t)$ to be constant over the bit period, i.e., $s(t)$ is a rectangular pulse that is zero outside the signaling interval $T$. The case for multi-valued signal $s(t)$ is addressed shortly. Finally, we assume perfect phase and frequency tracking so that we can consider the real-valued baseband signal, with the carrier of the original received signal removed. The received signal is represented as

$$r(t) = G(b \cdot s(t)) + w(t),$$  \hspace{1cm} (3.1)
where $G(\cdot)$ is a memoryless nonlinearity, representing the cumulative memoryless distortion to the signal; $b = \pm 1$ is the information bit; and $w(t)$ is the additive noise introduced by the channel and receiver electronics. A simple example of $G(u)$ is $Au$, which corresponds to an amplitude attenuation or gain; another example is the hard limiter. However, $G(\cdot)$ may be non-symmetric; for example, when induced by a non-symmetric nonlinearity of the receiver amplifier. The received signal is converted to discrete time by sampling every $T/n$ seconds, and quantized to a finite number of values. The sampled received signal is the “test” sequence and denoted $x$, following the definition and notation given in Chapter 2. The hypothesis testing problem confronting the detector is:

$$x \sim P_0$$

$$x \sim P_1,$$

where $P_0$ and $P_1$ are the $n^{th}$ order probability mass functions that are induced by the sampling and quantization of $G(b \cdot s(t)) + w(t)$ for $b$ equal to 1 and $-1$, respectively.

To give the receiver information about the characteristics of the modulated signal, channel, and noise, the transmitter sends a preamble consisting of repetitions of $\pm s(t)$ in an order known to the receiver (Figure 3.1). For example, the transmitter might send $s(t)$ 10 times followed by 10 instances of $-s(t)$. The preamble, after being distorted by the channel and the noise, contains information on how a “0” and “1” should appear at the receiver. In the parlance of empirical classification, the received
preamble bits are the training sequences, where the information governing hypothesis “0” is $t_0$, and $t_1$ represents hypothesis “1”.

Following (2.92), we use the forced-decision empirical detector to classify data signal $x$, choosing the smaller of generalized likelihood functions $h(x, t_0)$ and $h(x, t_1)$. Because the likelihood statistic can be written in terms of entropies (2.47),

$$h(x, t_i) = \frac{N + n}{n} H(q_{xt_i}) - \frac{N}{n} H(q_{t_i}) - H(q_x), \quad (3.3)$$

only the terms involving $q_x$ and $q_{xt_i}$ need to be computed “on the run;” the other parts can be computed beforehand and stored. Moreover, $H(q_x)$ does not have to be computed because it is common to both competing statistics. The type detector essentially stores the combined channel, noise, and signaling characteristics in types $(q_{t_0}, q_{t_1})$ and their entropies $H(q_{t_0}), H(q_{t_1})$. See Figure 3.2.

In real-world systems, $s(t)$ may not be constant over the bit period, as assumed in this section. The signal $s(t)$ is typically pulse-shaped, intentionally or not, by the digital and analog processing of the transmitter and receiver. In narrow-band TDMA systems, for example, the transmitted signal $s(t)$ is typically pulse-shaped to minimize inter-symbol interference (ISI) [33]. Multi-valued signals yield observations that are distributed differently from sample to sample because $G(s(t))$ changes with sample index, $t$. Thus, we need to modify the baseline type-based detector described previously.

Let the sampled version of $s(t)$ have $V$ different signal levels $\{s^1, \ldots, s^V\}$. Partition the training sequence $t_i$ into $V$ portions, $\{t_i^1, \ldots, t_i^V\}$, where $t_i^v$ contains the sam-
ples with the \( v^{th} \) signal level. Test sequence \( x \) is likewise partitioned into \( \{x^1, \ldots, x^V\} \).

Finally, the \( v^{th} \) portions of the test and training vectors are used to evaluate the \( v^{th} \) partial statistic \( h_i^v \), which are linearly combined to obtain the final likelihood value \( h_i \):

\[
h_i = \frac{1}{n} (n^1 h(t_i^1, x^1) + \cdots + n^V h(t_i^V, x^V)),
\]

where \( n^v \) is the number of samples in \( x \) that correspond to signal level \( s^v \). The derivation of (3.4) is straightforward and is found in Appendix B.

When the channel has dependent background noise or interference, joint statistics are needed for discrimination, and the observations with equal joint distributions are separated into one group. In the case of independent observations, the training and test sequences are split into \( V \) portions because \( s(t) \) has \( V \) values. In the case of \( p^{th} \)-order dependence, up to \( V^{p+1} \) groups must be formed to accommodate all possible consecutive, length-(\( p+1 \)) sequences with arbitrary values of \( \{s^1, \ldots, s^V\} \). Each "portion" of the observations is a sequence of length-(\( p+1 \)) vectors.

For example, let the signal have two values, \( s^1 \) and \( s^2 \), with the noise having first-order dependence (\( p=1 \)). The training and test vectors are each partitioned into \( V^{p+1} = 2^2 = 4 \) portions corresponding to pairwise signal values of \( (s^1, s^1) \), \( (s^1, s^2) \), \( (s^2, s^1) \), and \( (s^2, s^2) \). Each portion, say \( t_2^{1,1} \), is a sequence of ordered pairs. As in (3.4), the likelihood value of the \( i^{th} \) hypothesis, \( h_i \), is formed by linear combination

\[
h_i = \frac{1}{n} (n^{1,1} h(t_i^{1,1}, x^{1,1}) + n^{1,2} h(t_i^{1,2}, x^{1,2}) + n^{2,1} h(t_i^{2,1}, x^{2,1}) + n^{2,2} h(t_i^{2,2}, x^{2,2})).
\]

Multi-valued and dependent systems require longer preambles to achieve good training because the preamble is partitioned into \( V^{p+1} \) separate portions, and each has an average sample size of only \( \frac{N}{V^{p+1}} \). The dependent-observation case will not be pursued in this study, but similar effects arise from increasing the number of quantization bits (Section 3.3.2). For dependent systems, the baseline type-based implementation may be improved by one based on modeling the conditional pmf's with context trees [23, 53]. Chapter 4 examines in detail the special case \( V = 2 \),
which is applicable to spread-spectrum communications. A comparison with case $V = 1$ appears in Section 4.3.

### 3.2 Quantization for Detection

Quantization of continuous-valued observations degrades detection performance to some extent, but it must take place for digital processing. Typically, choosing the number of quantizer bits is a trade-off between detection performance and quantizer cost.

There is a large body of work on the design of optimal quantizers under various criteria; a small, representative sample listed with corresponding criteria includes: distortion (mean-squared error) [11, 27], efficacy and local power slope [21], minimum probability of error [15], and Ali-Silvey distance, or $f$-divergence [30, 31]. A survey of quantization for detection is found in [22]. A short discussion on these cited works follows.

Ideally, we would like to design a quantizer that minimizes the probability of error (Bayesian or Neyman-Pearson). However, due to the nonlinear operation of the quantizer, the probability of error cannot typically yield tractable solutions and practical design procedures. Consider calculating the exact minimum probability of error given some quantizer (the minimum error probability is achieved by using the likelihood-ratio test). For each $P_0$ and $P_1$ induced by a quantizer, calculating probabilities requires the enumeration of every event in the observation space. Although events are completely captured by types, there are still of the order $(n + 1)^{|A|}$ types (2.22). For even a modest number of quantizer bins and test length $n$, the total number of events can be very large. Strictly speaking, the optimum quantizer is a function of observation length $n$, and a different optimum quantizer exists for each $n$, even though the per-sample observations are identically distributed. This is essentially the approach taken in [15], wherein an approximation of $P_e$ is made in order to make $P_e$ continuous with respect to the probability of each event (type). Sample
quantizers are given for $|A| = 4$ and $n$ up to 20. For manageable complexity, we would prefer a design criterion that is invariant with $n$ and depends only on the per-sample statistics.

In many works, a "weak" signal assumption is made wherein the competing hypotheses are assumed to be asymptotically equal, and the solutions correspond to detection problems with very small per-sample signals. The weak signal justification is made in response to many classical detection problems – such as the detection of targets using radar – where quantizer design may not be so crucial for signals with large SNR. However, in a communications system, any degradation contributes to reduced quality of service and reduced system capacity, so the weak signal assumption may not be as appropriate. It is interesting that the optimal quantizer for weak signals in Gaussian noise is identical to the quantizer found by minimizing the mean-squared error, as pointed out in [1], even though the mean-squared error has no direct relevance to detection.

Because the Chernoff information, $C$, bounds the exponential rate of the Bayesian $P_e$ (Section 2.1.1) and is only a function of the per-sample observations’ statistical properties, it is a ready candidate for a quantizer’s criterion. The relevant works of Poor and Thomas [31] and Poor [30] use various Ali-Silvey distances [2] or so-called $f$-divergences as criteria for quantizer design. Writing the Chernoff information as

$$C = -\log \left( \sum_{x \in A} P_0(x) \left( \frac{P_0(x)}{P_1(x)} \right)^{\alpha} \right) = -\log E_{P_0} L^\alpha,$$

for some specific $\alpha$, $0 \leq \alpha \leq 1$, and $L$ the log-likelihood ratio, the Chernoff’s membership as an Ali-Silvey measure is revealed. Poor and Thomas derive necessary local conditions (in particular the gradient) for optimizing an Ali-Silvey distance, which can be used with conventional gradient-descent techniques. Using the analysis of Section 2.1.3, the ratio of Chernoff informations readily conveys a tangible comparison between quantizers and may be used, for example, to determine a sensible stopping criterion in the numerical search for the optimal quantizer.
In this thesis, we consider only the uniform quantizer that is symmetric about zero, which has a very simple structure parameterized by the number of bins and the maximum (or minimum) bin boundary. Let \( M \) be the number of quantizer bins and \( x_Q > 0 \) be the quantizer set-point, then the \( M + 1 \) break points are:

\[
s_0 = -\infty, \quad s_1 = -x_Q + \frac{2x_Q}{M}, \ldots, s_{M-1} = x_Q - \frac{2x_Q}{M}, \quad s_M = \infty. \tag{3.7}
\]

In general, the optimization of the quantizer is done given the number of desired bins \( M \), so the only parameter to optimize is \( x_Q \). We choose the symmetric, uniform quantizer for its simplicity and practicality despite its sub-optimality. In additive noise having a smooth and unimodal amplitude distribution, as in the examples in the remainder of this chapter, the degradation from the optimum is small. We consider that degradation below.

Finding the Chernoff information (3.6, 2.19) requires minimizing over \( \alpha \), which poses an additional hurdle. We make the additional assumption that the noise distributions, \( p_w \), are symmetric about zero. In this case, the Chernoff information corresponds to \( \alpha = 1/2 \), where the Ali-Silvey distance is the so-called Matsusita-Hellinger-Bhattacharyya coefficient [30].

In Figure 3.3, the Chernoff information as a function of the quantizer's set-point is shown for 1-5 bits of quantization for Gaussian and Laplacian noise. These figures show that the optimum Chernoff information increases as more bits are used, with the increase saturating quickly at 4 or 5 bits. Having more bits, however, offer more protection against using sub-optimal quantizer set-points. The systems in these figures have the same signal-to-noise ratio and RMS values, but their optimum set-points are drastically different. This implies that a gain-control strategy cannot simply use second-order statistics, as commonly featured in gain-control algorithms. The resultant Chernoff information of each quantizer is measured against the unquantized Chernoff information, in the manner discussed in Section 2.1.3, and shown in the lower plots of Figure 3.3.

Figure 3.4 shows the optimum set-points found using the Kullback-Leibler (KL)
Figure 3.3: Chernoff information for symmetric, uniform quantizers with 1-5 bits of resolution is plotted as a function of the quantizer's set-point, for the cases of zero-mean, unit-variance Gaussian (left) and Laplacian (right) additive noise. The underlying signal has an amplitude of 0.5. The optimum set-points are very different, suggesting that an energy-based quantizer-design methodology will be inadequate. The corresponding dB losses relative to the continuous-valued Chernoff information are also shown.
Figure 3.4: Quantizer Designed under KL versus Chernoff criterion. In the left-hand plot, the optimum quantizer set-point—found using both KL and Chernoff criteria in a Gaussian system—is shown against the signal amplitude. For large signal amplitudes, the resulting quantizers are very different. The right-hand plot quantifies the performance loss of the KL-designed quantizer relative to the Chernoff-designed quantizer. The comparison is made using the resultant Chernoff informations of the two quantizers.

measure and the Chernoff information as design criteria at different signal amplitudes. As the signal amplitude increases, the set-points under each criterion become drastically different. Figure 3.4 also shows the loss in SNR of the quantizers that use the set-points of the optimal KL measures instead of the optimal Chernoff information. The loss becomes significant when the signal amplitude is large. This re-enforces the fact that quantizer design must be made with respect to the Chernoff information and not to the KL measure.

Table 3.1 summarizes the quantizer losses as a function of the number of bits for Gaussian, Laplacian, Hyperbolic Secant, and Cauchy noises. These losses are computed at the optimum quantizer set-points at a signal amplitude of 0.2223 (which corresponds to bit SNR of 5 dB when $n$ is 64).
<table>
<thead>
<tr>
<th>number of bits</th>
<th>Gaussian</th>
<th>Laplacian</th>
<th>Hyp. Secant</th>
<th>Cauchy</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.96</td>
<td>0.36</td>
<td>0.93</td>
<td>0.91</td>
</tr>
<tr>
<td>2</td>
<td>0.54</td>
<td>0.05</td>
<td>0.19</td>
<td>0.61</td>
</tr>
<tr>
<td>3</td>
<td>0.16</td>
<td>&lt; 0.01</td>
<td>0.05</td>
<td>0.40</td>
</tr>
<tr>
<td>4</td>
<td>0.05</td>
<td>&lt; 0.002</td>
<td>0.01</td>
<td>0.16</td>
</tr>
<tr>
<td>5</td>
<td>0.01</td>
<td>&lt; 0.001</td>
<td>&lt; 0.01</td>
<td>0.07</td>
</tr>
</tbody>
</table>

Table 3.1: Summary of quantizer loss. Losses in dB quantify the differences of the Chernoff informations for the corresponding $x$-bit quantizer and the continuous-valued distribution. The losses are computed at the optimum quantizer set-points at a signal amplitude of 0.2223 (which corresponds to bit SNR of 5 dB when $n$ is 64). These comparisons are made according to the discussion in Section 2.1.3.

### 3.3 Simulation Study

Through simulations, the effects of parameters $n$, $N$, $C$, and the amount of quantization are examined. The examinations look at the BPSK communication scenario laid out in the previous sections and not at arbitrary discrete-valued hypotheses tests. The simulations address only the case of constant signaling and i.i.d. noise.

A simulation study cannot be comprehensive but serves an exploratory purpose by looking at several relevant examples of noise distributions, signal amplitudes, etc. General trends may be learned to guide the practitioner in the application of the type detector. In particular, finite sample behaviors may emerge to complement the theoretical, asymptotic results.

#### 3.3.1 Basic Properties

In Figure 3.5, for a system with Gaussian noise and three bits of quantization, the probability of error is plotted against increasing $n$ at a fixed signal amplitude. Shown are the error probabilities corresponding to the type receiver with different amounts of training, as indicated by the parameter $C = N/n$. For each $C$, the errors are ap-
proximately linear on the semi-log scale and behave exponentially. For increasing $C$, not only do the error rates become larger, but the actual (finite-sample) error probabilities become smaller. Here, the observed trends are only partially corroborated by the theoretical results presented in Chapter 2. Judging from the null-hypothesis result where $\lambda_0(C, P_1, P_0)$ increases with increasing $C$ and converges to $D(P_1 \parallel P_0)$, we may expect that the rate of the forced-decision detector to also increase with $C$, and converging to $C(P_0, P_1)$, the Chernoff information. Unfortunately, for finite $C$, we don't know the exact error rate of the forced-decision detector. Let us speculate for the moment that the error rate is $\frac{1}{4}\lambda_0(C, P_1, P_0)$ (equation 2.66), which is true for Gaussian noise when $C \to \infty$. Using (2.59) to bound the error probabilities, the predicted probabilities are shown in Figure 3.6, where the curves shown in the top plot include the polynomial terms of $n$ and $N$ that come from the total number of types in $\mathcal{P}_n$. This upper bound is clearly much too loose to be useful, as the probabilities approach $10^{50}$ and are not yet decreasing for $n$ of 256. In the lower plot of Figure 3.6, the polynomial terms are omitted, and the resulting bounds are more reasonable, about one order of magnitude away from the simulated errors. Thus, the theoretical predictions seem accurate for error rates, but the upper bounds are too loose. Tightening the bounds by omitting the polynomial terms better captures the errors and implies both increased rate and decrease error probability for increasing $C$.

Also shown in Figure 3.5 are the optimal error probabilities given the quantized observations, that is, as obtained by the likelihood-ratio detector operating on discrete-valued inputs. These values are estimated using the quick simulation technique called important sampling [29] applied to the current situation of testing discrete-valued sequences. This estimator is derived in Appendix C, where its gain over the Monte-Carlo simulation method is shown to increase exponentially with $n$ as well as inversely with the true probability of error.

For fixed $C$, although the (log) error probabilities are approximately linear in $n$,
the loss relative to the optimum is greater at small values of \( n \). This behavior is also outside the scope of asymptotic theory. Since \( N = Cn \), for very small \( n \) the amount of training is quite small compared to the same \( C \) at a larger \( n \). Intuition would suggest that the performance will be relatively worse, although it is not revealed by theory. The loss of the empirical detector from optimum is shown in the table in Figure 3.5, which is measured in the manner discussed in Section 2.1.3: the ratio of the type detector’s \( n \) that is required to achieve a certain \( P_e \) and the likelihood-ratio detector’s \( n \) needed to achieve the same error.

The table in Figure 3.5 reveals another trend, that at a fixed \( N \), the performance relative to the optimum is better at small rather than large values of \( n \). At a fixed \( N \), a small \( n \) corresponds to a larger \( C \) than a large value of \( n \). Thus, as \( n \) increases and \( C \) decreases, the instantaneous rate (at \( n \)) will decrease. If we model the error probabilities as strictly exponentially decreasing (neglecting the polynomial terms) at a rate controlled by \( C \), we can predict that the resulting error (at fixed \( N \)) will continue to intersect the error curves corresponding to smaller and smaller \( C \) values as \( n \) increases. This means that the relative dB loss increases with \( n \), as observed. However, if we include the polynomial terms, we may, as before, arrive at indeterminate bounds for finite \( n \).

Performance results for the Laplacian noise and Hyperbolic Secant noise (at the same amplitude as the Gaussian case) are displayed and summarized in Figure 3.7 and Figure 3.8, respectively, and the tables therein. All of the type detector’s trends discovered for the Gaussian case are present in the Laplacian and Cauchy cases. We also tested the type detector at signal amplitudes of approximately 0.1 to 0.4 (not shown) and found that the same trends govern their behaviors, although the actual probability errors and relative dB losses, for example, were not identical. We note that the Laplacian losses, shown by the table in Figure 3.7, present inconsistencies at \( C \) of 128 and 256, where the relative performance at \( n = 4 \) is better than all of the other \( n \) values. We believe this is a statistical “blip,” and it is not found in the other
Laplacian (nor the other noises) simulations we have conducted (not shown).

Overall, we tested operating ranges with error probabilities down to only about $10^{-5}$ due to simulation-time constraints, but this error level is sufficiently low for wide-ranging communications applications, especially if we consider this bit error rate to be the rate before channel decoding, which means the decoded error level would be much lower [33]. Testing at amplitudes larger than about 0.5 would have resulted in very low error probabilities for relatively small values of $n$.

In our final trend exploration of the basic properties of the type detector, we fix SNR and vary $n$. As we will see in Chapter 4, varying $n$ with fixed SNR is a system design issue for spread-spectrum communications systems. Although the different amplitudes so far tested (0.1 - 0.4; only results for 0.2223 are shown) range in SNR by up to 10 dB, the amplitude differences are still fairly small. This is in contrast to the fixed SNR case (with varying $n$), where the amplitudes of the signal vary greatly over $n$. For example, at $n$'s of (1, 2, 16, 64, 256), the respective amplitudes for a fixed SNR of 5 dB are (1.7783, 1.2574, 0.4446, 0.2223, 0.1111); an even greater range is encountered for fixed SNR of 10 dB. To accommodate these different amplitudes, the quantizer at each amplitude has a different set-point, found by methods discussed in Section 3.2.

We test the Gaussian system at 5 and 10 dB, at observation lengths, $n$, ranging from 2 to 512. Results appear in Figure 3.9. At fixed $C$, small $n$ again yields slightly worse performance than large $n$. The contrast is greater at 10 dB than at 5 dB. Performance differences diminish, though, with increasing $C$. We note that the quantizer loss is about the same for varying amplitudes, seen from the equal performance of the optimal detector across different values of $n$.

### 3.3.2 Quantization Effects

Increasing the quantizer resolution increases the optimum performance but, due to a larger number of probability masses that must be estimated, is done at the cost
Figure 3.5: **Type detector performance in Gaussian noise.** System parameters: signal amplitude 0.2223, 3-bit quantizer, set-point 2.3. The table summarizes the performance loss, in dB, of the type detector relative to the likelihood-ratio detector operating on the same quantized data. The values in bold show trends for fixed $C$ (horizontal) and fixed $N$ (diagonal).
Figure 3.6: Predicted error probability bounds. Polynomial terms are included in the top plot, showing the looseness of the upper bounds. In the lower plot, the polynomial terms are omitted. Note that the curves in the lower plot are still about an order of magnitude larger than the actual simulated probabilities, shown in Figure 3.5.
Figure 3.7: **Type detector performance in Laplacian noise.** System parameters: signal amplitude 0.2223, 3-bit quantizer, set-point 0.25. The table summarizes the performance loss, in dB, of the type detector relative to the likelihood-ratio detector operating on the same quantized data. The values in bold show trends for fixed $C$ (horizontal) and fixed $N$ (diagonal).
Figure 3.8: Type detector performance in Hyperbolic Secant noise. System parameters: signal amplitude 0.2223, 3-bit quantizer, set-point 0.95. The table summarizes the performance loss, in dB, of the type detector relative to the likelihood-ratio detector operating on the same quantized data. The values in bold show trends for fixed $C$ (horizontal) and fixed $N$ (diagonal).
Figure 3.9: Type detector performance, 5 and 10 dB. The probability of error is plotted against varying $n$ with a fixed SNR of 5 and 10 dB, respectively, in the top and bottom plots. The additive noise is Gaussian, and a 3-bit quantizer at varying set-points are used.
of increasing the amount of training. These offsetting effects are examined in this section.

In Figure 3.10, the performances of the type detector for 2 - 4 bits are shown at selected $C$ values. Gaussian noise is assumed. We have already looked at this quantizer’s performance back in Section 3.2, in particular in Table 3.1, where the losses from optimum are 0.54, 0.16, 0.05, and 0.01 dB for 2-5 bits, respectively. With small amounts of training, for example at $C \leq 16$, the 2-bit quantizer performed the best at $n$’s of 16 and 64. For $C$ larger than 32, the 3-bit quantizer is better at all values of $n$ greater than 16. Quantizer resolution and training size interact in the way we would expect. In the comparison of 3 versus 4 bits, we see that at $C = 32$, the 4-bit quantizer is better only after $n > 192$, where the actual number of training samples $N$ would be sufficiently large. At smaller $n$ and smaller $C$, the 3-bit quantizer wins out. The 5-bit quantizer (not shown) only beats the 3-bit one at $C = 32$ and $n = 256$. Tabulated results for these tests are shown in Table 3.2 at selected values of $n$.

While these tests provide guidelines governing the behavior of the type detector for varying $C$, $n$, and the number of quantization bits, actual results, such as $P_e$, may vary for other noise distributions and system settings. Examination at specific system operating ranges must be done to determine the best quantizer resolution.
Figure 3.10: **Type detector performance: 2 – 4 bits.** In the top plot, shown are 2 and 3 bits; in the bottom, 3 and 4 bits. System parameters: Gaussian noise, signal amplitude 0.2223, set-point 2.0 for all quantizers.
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
 & \multicolumn{4}{|c|}{n = 16} & \multicolumn{4}{|c|}{n = 64} \\
\hline
$C$ & 2 bits & 3 bits & 4 bits & 5 bits & 2 bits & 3 bits & 4 bits & 5 bits \\
\hline
4 & -2.94  & -5.07  & -6.44 & -8.23 & -1.72  & -1.67  & -3.31  & -4.62  \\
8 & -1.77  & -2.70  & -4.11 & -6.10 & -1.17  & -1.25  & -1.51  & -2.81  \\
16 & -0.94  & -1.22  & -2.12 & -4.31 & -1.01  & -0.61  & -0.82  & -1.42  \\
32 & -0.94  & -0.82  & -1.50 & -2.51 & -0.62  & -0.42  & -0.60  & -0.85  \\
64 & -0.66  & -0.33  & -0.71 & -1.24 & -0.63  & -0.27  & -0.45  & -0.28  \\
128 & -0.31  & -0.15  & -0.60 & -0.95 & -0.59  & -0.20  & -0.12  & -0.02  \\
256 & -0.58  & -0.38  & -0.01 & -0.62 & -0.61  & -0.00  & -0.20  & -0.24  \\
\hline
\end{tabular}
\caption{Summary of type detector performance at 2-5 quantizer bits. Shown are the dB losses of the type detector at 2-5 bits of quantization relative to the likelihood-ratio detector operating on \textit{continuous-valued} data. Shown at $n$ of 16 and 64. Gaussian noise.}
\end{table}

3.3.3 Symmetry Assumptions

Assuming that channel distortion $G(\cdot)$ is symmetric for positive and negative-valued signals and that the noise distribution is symmetric, the training vectors under each hypothesis can be combined to yield a larger training set. Here, the training vectors $[t_0, -t_1]$ can be combined to form the augmented training vector under $H_0$: $t_0^*$. Similarly, the augmented training vector $H_1, t_1^*$, is $[-t_0, t_1]$. An immediate consequence is that the effective training length is doubled, so the required amount of training is halved.

Figures 3.11 (for Gaussian) and 3.12 (for Laplacian) show the relative performances of the baseline detector with the "symmetric" detector for a fixed signal amplitude. Performance summaries are provided in tables included with Figures 3.11 and 3.12. Additional comparisons are provided in Figures 3.13, where the detectors are compared at a fixed SNR. These latter plots provide a better look at the trends.

From these plots, it is evident that the training requirement is reduced, but ap-
parently by a factor greater than two. This effect may be explained by comparing
the total amount of "randomness," as captured by the total number of types, in the
baseline detector at $2C$ versus the symmetric detector at $C$. Recall that many of
the probability upper-bounds described in Chapter 2 (c.f. 2.59) was in the form of
the total number of types multiplied by the largest type probability. At $2C$ for the
baseline detector, there are up to a total of $2^{|A|2\log N} = 2^{|A|2\log 2Cn}$ types making up
$q_{t_0}$ and $q_{t_1}$, while the symmetric detector at $C$ has only $2^{|A|2\log 2Cn}$ types making up
$q_{t_5}$. Thus, the upper bound to the error probability is larger for the baseline detector
at $2C$ than the symmetric detector at $C$. The implication here is that having $N$
observations from $P_0$ and an exact model relating $P_0$ and $P_1$ provides a better look
at the detection problem than having $N$ observations from each of $P_0$ and $P_1$ without
assuming a coupled relationship between $P_0$ and $P_1$. Asymptotically, we would expect
their rates to be identical because for each likelihood statistic $h_{i}$, the ratios $N/n$ are
the same, at $2C$.

Trade-off analysis must be made to determine if the loss (if any) from the assump-
tion of symmetry is offset by the performance gain. When the symmetry assumption
is reasonable, it should be made. A reduction of training length from 32 to 16 bits or
from 16 to 8 bits can be significant.
Figure 3.11: Symmetric v. baseline detector performance, Gaussian. System parameters: 0.2223 signal amplitude, 3-bit quantization, set-point 2.3. The table summarizes the performance loss, in dB, of the symmetric type detector relative to the likelihood-ratio detector operating on the same quantized data. Compare baseline detector’s losses, shown in Figure 3.5.
Figure 3.12: Symmetric v. baseline detector performance, Laplacian. System parameters: 0.2223 signal amplitude, 3-bit quantization, set-point 0.25. The table summarizes the performance loss, in dB, of the symmetric type detector relative to the likelihood-ratio detector operating on the same quantized data. Compare with baseline detector's losses, shown in Figure 3.7.
Figure 3.13: Symmetric v. baseline detector performance, 5 dB. Top plot shows Gaussian results; bottom plot shows Laplacian. Quantizer has 3 bits, set at different set-points depending on signal amplitude; Gaussian: 2.3 for all amplitudes; Laplacian: 0.36 0.25 0.18 0.125 for increasing $n$. 
Figure 3.14: **Smoothing of types.** The input histogram is smoothed by a filter—we used a Gaussian window—of varying length, and the output has a larger number of letters and is no longer a type. The smoothed "types" for training and test are then used in the type detector exactly as the original, un-smoothed, types.

### 3.4 Smoothing of Types

It is of the utmost importance to reduce training requirements, since more training necessarily reduces the throughput* of real information. Thus far, we found that using the symmetric type detector can reduce the amount of training by more than two-fold. As well, reducing the resolution of the quantizer reduces training requirements but is done at the cost of degraded performance.

The discrete-valued data in our communications scenario arise from quantizing continuous-valued data. For the optimal detector (likelihood-ratio test) — which knows the true properties of the quantized data — the reduction in performance due to quantization is fairly small, e.g., 0.2 dB for 3-bit quantizer in Gaussian noise, 0.02 dB for Laplacian. For the type detector operating on quantized data, though, our simulations have shown that a large number of bits $C > 32$ typically are required to come near the optimal detector. The implication here is that when the histogram estimator has enough training samples to accurately estimate the underlying probability mass function, the type detector performs as well as the optimal detector.

---

*In a communications system whose properties change over time in unknown ways, training sequences must be transmitted periodically. For a fixed time interval between training sequences, increasing the amount of training decreases the throughput of useful information.*
From the perspective of processing analog data, the histogram is a crude estimator of the true probability density function. It is well known that a more general kernel density estimator yields a more accurate estimate [37]. In this way, given continuous-valued data, a kernel density estimator could be used to pre-process data, with the resulting density estimate mapped into a discrete-valued pmf that is used as the training type. The problem here is one of complexity and cost. In order to form a kernel estimate, the received data must be sampled at a much higher resolution, which is costly in practice. The added computations may render it impractical.

It is reported in [18] that linear filtering of the training and test histograms can dramatically improve the performance of the type detector. Here, the so-called smoothing process is post-quantization (Figure 3.14). The smoothed type has a larger number of elements than the original that was obtained from quantizing data, due to the length of the convolution filter. The smoothed types, for training and test, are then used in the type detector in the same way as the original un-smoothed types.

To demonstrate the effect of smoothing on the type detector, we use a Gaussian smoothing filter, $g$, of length $l_F$:

$$g(i) = \exp \left( -\frac{(\frac{l_F-1}{2} + i)^2}{2\sigma^2} \right), i = 0, \ldots, l_F - 1. \quad (3.8)$$

In Figures 3.15, 3.16, and 3.17, for a Gaussian system, we show resulting error probabilities as a function of the length of the smoothing filter, for $n$ of 16, 64, and 256, respectively. In each figure, the performances of the symmetric and baseline type detectors are both displayed to demonstrate and contrast the effects of smoothing on each. With increasing smoothing length $l_F$, we keep $\sigma$ constant at 1.

For the baseline type detector, smoothing reduces the amount of training by a factor less than 4 when $n$ is 16, by just over 2 when $n$ is 64, and by less than 2 when $n$ is 256. For the symmetric type detector, the reduction factor is, respectively, less than 4, less than 4, and greater than 4. Because the symmetric type detector already has a training reduction factor of greater than 2 over the baseline, un-smoothed detector,
the training requirement of the baseline detector are reduced by a factor of about 8!

Also shown are the optimum un-quantized error probabilities at 0.0, 0.2, 0.5, and 1 dB from the optimum. Keeping in mind that a 3-bit quantizer for Gaussian noise has about a 0.2 dB loss, we see in all of the tests that the smoothed, symmetric type detector is practically performing optimally at $C = 8$. Its performance is especially dramatic for $n$ of 256, where $C = 1$ almost has optimal performance. The effect of smoothing on the baseline type detector is significant as well, but just not as dramatic. We note that in all tests, the symmetric detector’s error probabilities overlapped for some values of $C$, which implies that more training (a larger $C$) does not always yield better results.

Clearly, smoothing works because simple binning of data does not adequately capture the properties of the smooth, underlying probability density function. Unfortunately, theoretical analysis of the resulting smoothed detector is difficult because the space of smoothed types is not strictly a space of types with a larger alphabet. For one, a smoothed type is not a type because it doesn’t have counts. As well, because the smoothed type is a function of an un-smoothed type, the number of possible smoothed types must be smaller than the number of original types, even though the smoothed type has a larger alphabet. These properties render invalid the conventional methods of using types to enumerate set probabilities.

Smoothing should be used only with some knowledge that the pre-quantized data has underlying smoothness. Additionally, as pointed out in [18], smoothing seems to work only when the hypotheses are separated by some mean, as in the case of BPSK communications.
Figure 3.15: Type detector performance v. smoothing length, n = 16. Top plot: baseline type detector; bottom plot: symmetric type detector. System parameters: Gaussian, 0.2223 signal amplitude, 3 bits quantization, n = 16, set-point 2.25. The dB lines indicate the performance of the optimum detector (operating on continuous-valued data) that have the indicated levels of SNR loss, as compared to that in the simulated examples.
Figure 3.16: Type detector performance v. smoothing length, \( n = 64 \). Top plot: baseline type detector; bottom plot: symmetric type detector. System parameters: Gaussian, 0.2223 signal amplitude, 3 bits quantization, \( n = 64 \), set-point 2.25. The dB lines indicate the performance of the optimum detector (operating on continuous-valued data) that have the indicated levels of SNR loss, as compared to that in the simulated examples.
Figure 3.17: Type detector performance v. smoothing length, n = 256. Top plot: baseline type detector; bottom plot: symmetric type detector. System parameters: Gaussian, 0.2223 signal amplitude, 3 bits quantization, n = 256, setpoint 2.25. The dB lines indicate the performance of the optimum detector (operating on continuous-valued data) that have the indicated levels of SNR loss, as compared to that in the simulated examples.
3.5 Summary

The simulation tests in this chapter yielded many guidelines in the use of the empirical type detector and its variants. The first observations below pertain to the baseline, un-smoothed type detector.

• When per-sample amplitude and \( n \) are fixed, a larger \( C \) always improve performance. This result is somewhat corroborated by theory. Although the theory predicts a larger error rate for increasing \( C \), it also predicts a larger probability. In particular, the polynomial terms that enumerate the number of types cause the probability bounds to be too loose. Omitting them yielded tighter bounds. We speculated on the forced-decision detector’s error exponents, which the theory does not yet provide.

• When per-sample amplitude and \( C \) are fixed, a larger \( n \) performs better than a smaller \( n \), when both are compared to the corresponding performances of the optimal detector. This is a finite-sample effect that is not predicted by theory.

• When per-sample amplitude and \( N \) are fixed, a smaller \( n \) is better than a larger \( n \) when both are compared to the corresponding performances of the optimal detector.

• For SNR and \( C \) fixed, a larger \( n \) results in smaller \( P_e \).

We make the following observations about the symmetric type detector and the effects of smoothing:

• Making the symmetry assumption reduces training requirements by a factor strictly greater than 2. We believe this is true in all situations where the observations have symmetric distributions.

• Smoothing reduces training by a factor of about 2 in the baseline detector for small to moderate values of \( n \). For large \( n \), the reduction is smaller than 2.
- Smoothing reduces training requirements by an additional factor of about 4 in the symmetric type detector. This gain appears to be less sensitive to \( n \) than in the baseline type detector.

- For the symmetric type detector, the smoothed performance of a larger \( C \) may be worse than the smoothed performance of a smaller \( C \).

The following observations are made regarding quantization.

- For all of the systems examined, a 3-bit quantizer sufficed for reasonable amounts of training \( C \leq 32 \). Only when training vectors are overly abundant should a 4- or 5-bit quantizer be considered. A simple, symmetric, uniform quantizer was used in all tests.

- How to find the quantizer set-point at run time is an open issue that we did not address. In all of our simulations, a good quantizer was chosen \textit{a priori}. One possible strategy is to use finely quantized training data to allow the computation of the Chernoff information at different set-points of a cruder quantizer. This is essentially the approach taken in [54]. Another is to use the probability of error, computed using quantized training data, to slowly vary the quantizer’s set-point.

Remaining observations include:

- The type detector is most suitable for systems with a small number of signal values and i.i.d per-sample observations.

Many of these observations and guidelines are directly relevant to the application of the type detector to spread-spectrum, code division multiple access (CDMA) communications. This is the subject of Chapter 4.
Chapter 4

Type-based Detector for Spread Spectrum and CDMA Communications

4.1 Introduction

In a communications system, source symbols at $R_s$ symbols/sec are transmitted to the receiver. In most cases, these "symbols" are just bit values of $\{0,1\}$, resulting from source coders. These $R_s$-rate symbols are then channel coded into $R_b$-rate "bits," for example, via convolutional coding. The code "bits" have values of $\{0,1\}$. By design, $R_b > R_s$ so that there is an effective rate expansion that provides redundancy in the coded stream. For terrestrial communications systems, this expansion is typically small, for example, 2 or 3.

In a narrow-band, non-spread-spectrum system such as time-division multiple-access (TDMA), the coded bits are sent directly by the transmitter to the receiver via a channel that has a bandwidth of at least $\frac{R_b}{2}$ [33]. (To simplify discussion, we ignore analog filtering and pulse-shaping of the digital bit signal.) The receiver digitally samples the received signal (once or twice per bit), sends it to a detector, then performs channel decoding.

In a wide-band, spread-spectrum or CDMA system, the channel-coded bits are additionally "spread" in bandwidth to rate $R_c$ "chips" using a pseudo-random $\{0,1\}$ spreading sequence. See Figure 4.1. For example, in the forward link (base station to mobile station) of the IS-95 CDMA cellular system, the spreading factor $\frac{R_b}{R_b}$ is 64 [39]. Spreading has many beneficial effects, which include diminished susceptibility to channel fading [33, 35, 61], covert and anti-jamming operations [38, 61], and the efficient use of channel bandwidth for capacity [12, 13].
Figure 4.1: **Spread-spectrum transmit-receive topology.** $R_s$ is the information symbol rate, $R_b$ is the channel-coded bit rate, and $R_c$ is the spread-spectrum chip rate. $R_s \leq R_b \leq R_c$. $\frac{R_s}{R_b}$ is the total bandwidth expansion, with $\frac{R_s}{R_b}$ being expansion due to channel coding and $\frac{R_c}{R_b}$ the expansion due to spreading.

The total bandwidth expansion factor from “symbols” to “chips” is $\frac{R_c}{R_s}$. Allocating the total bandwidth expansion between channel coding and pseudo-noise spreading is an active research topic [44]. As a precursor of what’s to come, we note that the type-based receiver is applicable to the spread spectrum system regardless of the amount of bandwidth expansion due to spreading or to channel coding. However, it is important that the total bandwidth expansion, $\frac{R_c}{R_s}$, be relatively large, so that the per-sample SNR is relatively small.

### 4.1.1 Why Empirical Type Detector for Spread Spectrum?

The type-based detector is well suited for spread spectrum, serving as the “despreading/detection” block in the receiver diagram shown in Figure 4.1. One reason is the typically large spreading length, which is $n$ in the detection terminology we
use. (In spread-spectrum terminology, $10 \log_{10} \eta$ is the “spreading gain.”) One of the so-called “third-generation” CDMA system currently undergoing standardization has spreading lengths ranging from 4 to 128 [3]. The Global Positioning System, or GPS, has a spreading length of more than $10^4$ [20]. As we have seen, the type-based detector works best at large values of $n$. This means that in a spread-spectrum system, the required amount of bits used for training, which is $2C$, will be smaller than a non-spread system (to achieve the same $P_e$). A small $C$ is good for two reasons: First, the overhead of training is reduced, hence, the effective system throughput, $\hat{R}_e^*$, is improved. Second, a small $C$ is important in systems where channel, interference, and noise characteristics change over time, and the training duration is restricted to be less than the system coherence time. Finding exactly how large $C$ needs to be for reasonable, real-world operating ranges is one of the goals of this chapter.

In systems where channel coding consumes most of the bandwidth expansion and spreading length $n$ is small, the type-based detector is even more relevant. This operating range yields higher system capacity in some cases [44] and occurs, for example, on the reverse link (mobile station to base station) in IS-95, where $n = 4$ [39]. The simulation study in Chapter 3 found that for fixed $N$, the performance of the type detector is better at small, rather than large $n$, when compared to the respective optimal detectors. Note that the number of samples in the training sequence, $N$, directly controls throughput, $\hat{R}_e$. Therefore, at a fixed throughput, the type detector will perform better in the spread-spectrum system with more channel coding than the system with more spreading when the two systems have equal symbol and chip rates and equal chip energies.

In a spread-spectrum system, digital sampling of the received observations is typically done once per chip, following chip-matched filtering using analog components [34]. In this way, the noiseless signal only takes on the spreading signal’s values, which

---

*The effective throughput, $\hat{R}_e$, is the amount of real information “symbols” transmitted per second.*
are ±1 with an unknown gain. (As we will see, using the "symmetry" technique described in Section 3.3.3 effectively makes \( V = 1 \) and doubles the training size.) The small \( V \) inherent in spread-spectrum CDMA systems provides another attraction for the type-based detector.

Another attractive feature for a spread spectrum system is the pseudo-randomness of the spreading signal. Pseudo-noise spreading effectively diminishes the dependence introduced by the channel (i.e., inter-symbol interference, or ISI) which means that the interference caused by a delayed version of the desired spreading signal can be treated as independent noise. The same holds for the interference due to other users' codes. Pseudo-randomness and the wide-band additive noise means that SS systems have the desirable feature that the received samples may be considered i.i.d., or \( p = 0 \).

Finally, spread-spectrum systems require quantizers with only a few bits. One reason is that having wide-band signaling diminishes the susceptibility of the signal to channel fading [33, 35, 48], which reduces the fade margin built into the receiver, and which in turn has the effect of reducing the number of required quantizer bits. A larger bandwidth expansion factor also yields sample-by-sample SNR's that are smaller, which means that fewer bits are required to capture the sampled signal. Examples that illustrate these two factors are: the IS-54 [41] TDMA (narrow-band signaling) mobile receiver, where the bits are oversampled only by a factor of 2 and some 10-13 bits are needed in the quantizer [40, refer to "ARCTIC"]; IS-95 CDMA (wide-band) mobile receivers (\( n \) is 64) may use only 4 bits [34, Section 4 on "BBA2"]; GPS receivers (\( n > 1023 \)) may use 2 bits [10, 20].

4.2 Detector Topology

Consider a spread-spectrum system employing binary-phase-shift keying (BPSK), with spreading signal \( s(\cdot) \), operating at \( n \) chips per bit. Following carrier down-conversion and analog chip-matched filtering, the receiver samples the received signal once per chip period, resulting in a discrete-time signal of \( r(t) \), where \( t \) is the chip, or
Figure 4.2: Spread-spectrum received signal model. \( b \) is the transmitted information bit. The spreading signal, \( s(t) \), has discrete-time index, \( t \), and is made of pseudo-random "chips." (\( b \) and \( s(t) \) are shown to take \( \pm 1 \) values (instead of \{0,1\}) for simplicity of illustration.) Function \( G(\cdot) \) is a memoryless nonlinearity that represents the combined effects on the signal from the transmitter to the receiver. Additive noise \( w(t) \) is assumed i.i.d. with unknown distribution.

discrete-time, index. To start, we assume that the receiver sees only one signal path in additive noise \( w(t) \). The resulting baseband waveform in discrete-time is real and represented as:

\[
r(t) = G(b(t)s(t)) + w(t),
\]

where \( b(t) \) is the information bit (\(+1\) or \(-1\)) sequence (which may change value once per \( n \) increments of \( t \)). \( G(\cdot) \) is an unknown nonlinear mapping and includes the effects of attenuation, \( s(t) \) is the spreading code of \(+1\) and \(-1\) values. The additive noise \( w(t) \) is assumed to be i.i.d. but with unknown distribution. This received signal model, shown in Figure 4.2, assumes perfect frequency synchronization and chip-level phase recovery. (The extension of type-based detection to separate in-phase (I) and quadrature (Q) samples is addressed briefly in Section 4.7.) Additionally, we assume a very slowly fading channel, so that \( G(\cdot) \) is effectively time invariant. The extension to more realistic rates of fading is addressed in sequel in Section 4.4.

The spreading \( s(t) \) alternates between \(+1\) and \(-1\), so the observations need to be separated into two portions, as discussed in Section 3.1. The two portions will be denoted by superscripts "+" and "−". The actual distortion \( G(\cdot) \) is unknown but irrelevant, because all of the samples of \( r(t) \) where \( b(t)s(t) \) is \(+1\) (or \(-1\)) are
Figure 4.3: **Training of type detector for spread spectrum.** Training observations are separated into two portions, "+" and "−", depending on the sign of $b \cdot s(t)$. The collected portions result in separate types $q_{t+}$ and $q_{t−}$.

distributed as $p_w(\cdot - G(+1))$ (or $p_w(\cdot - G(-1))$), where $p_w$ is the probability density function of $w(t)$. Thus, all of the samples in the "+" portions are similarly distributed, as are all of the samples in the "−" portions. This is true regardless of the properties of $G(\cdot)$, and applies even if it inverts the signal.

The receiver acquires training data from a preamble $t$, which is a known bit sequence transmitted periodically to probe the channel. For consistency with the constant-signal communications system addressed in Chapter 3, we let training data $t$ contain $2C$ bits, or $2Cn$ samples. As the preamble and the spreading sequence $s(t)$ have known patterns, the receiver can distinguish which samples of $r(t)$ belong to each of the "+" and "−" portions of the preamble. The collected training sequences are denoted as $t^+$ and $t^−$ and made into types $(q_{t^+}, q_{t^−})$. Figure 4.3 depicts the training operation. The same type of separation is made on the test vector $x$, resulting in $(x^+, x^-)$ and $(q_{x^+}, q_{x^-})$. When separating the test vector, the convention is to assume that $+s(t)$ (bit "0" or +1) is sent.

As in Section 3.1, the likelihood function $h_i$ is composed of the partial likelihood statistics of the matching "portions" in the training and test vectors. When the transmitted bit is +1 (corresponding to hypothesis $H_0$), $t^+$ and $x^+$ are governed by
Figure 4.4: Topology of empirical type detector for spread spectrum. During the test period, the received signal \( r(t) \) is quantized into \( x \). Samples of \( x^+ \) are collected when \(+s(t) > 0\) and likewise for the samples of \( x^- \). Likelihood statistics \( h_0 \) and \( h_1 \) are computed according to (4.2).

The same measure, and likewise for the pair \( t^- \) and \( x^- \). To compute statistic \( h_0 \), we add partial likelihood statistics \( h(t^+, x^+) \) and \( h(t^-, x^-) \) together, similar to (3.4). When the transmitted bit is \(-1\) (corresponding to hypothesis \( 1 \)), \( x^- \) and \( x^+ \) become the positive and negative portions, respectively, of \( x \). Then, to compute \( h_1 \), we add together \( h(t^+, x^-) \) and \( h(t^-, x^+) \). In summary, the type-based spread-spectrum detector computes the following

\[
\begin{align*}
    h_0 &= \frac{1}{n}( n^+ h(t^+, x^+) + n^- h(t^-, x^-) ) \\
    h_1 &= \frac{1}{n}( n^+ h(t^-, x^+) + n^- h(t^+, x^-) ),
\end{align*}
\]  

(4.2)

where \( n^+ \) and \( n^- \) are the numbers of observed samples in \( x \) in the \( "+" \) and \( "-" \) portions, respectively. The smaller of the \( h_i \) results in choosing the \( i^{th} \) hypothesis. Figure 4.4 illustrates this detector topology.

Although (4.2) is similar to the detector for the multi-valued signal case, as described by (3.4), the spread-spectrum type detector uses all of the training samples to compute each test statistic \( h_i \). The training types are enumerated in the order \( (+, -) \) when used to compute \( h_0 \) and in the order \( (-, +) \) for compute \( h_1 \), as shown in (4.2). Hence, there is an inherent degree symmetry in spread-spectrum signaling: The \( "+" \) portion during training comes as the result of either \(+1\) training bit and \(+1\) spreading chip or \(-1\) training bit and \(-1\) spreading chip. The \( "-" \) portion during training results similarly.
Figure 4.5: **Training procedure for de-spread type detector.** After de-spreading, the training sequences have a positive mean and are used to build \( q_{t_0} \). If the quantizer is symmetric about the origin, \( q_{t_1} \) is simply the “mirror” image of \( q_{t_0} \), build by reversing the probability masses of \( q_{t_0} \).

### 4.2.1 Detection after De-spreading

In the previous section, the “basic” type-based detector for spread spectrum trains and detects each individual “+” and “−” portions separately because their probability measures may not be symmetric about zero. This may occur, for example, when the channel distortion function \( G(\cdot) \) is not an odd function or if the background noise, \( w \), is not symmetric.

If the symmetry assumption can be made, as often done in practice, the type-based detector can operate on the sufficient statistics after de-spreading the received samples. De-spreading is the operation of toggling received chip values by +1 or −1 depending on the spreading code of the desired user. The preamble after de-spreading is always distributed as the “+” portion and becomes \( t_0 \). Training vector \( t_1 \) is the “negative” of \( t_0 \). The test-vector type is also formed after de-spreading. The test statistic \( h_i \) is simply \( h(x, t_i) \), and the smaller \( h_i \) results in bit \( i \) chosen. If the quantizer is symmetric about zero, which is typically true, only one training type, \( q_{t_0} \), needs to be built. The alternate training type, \( q_{t_1} \), is simply \( q_{t_0} \) with its probability masses reversed. The training procedure is shown in Figure 4.5, and
Figure 4.6: **Topology of de-spread type detector.** The quantized received signal $x$ is first de-spread, then its type is formed. For a symmetric quantizer, $Q$, only one training and one test type are built.

The corresponding detector is shown in Figure 4.6. The resulting "de-spread" type-based detector is identical to the "symmetric" type detector described in Section 3.3.3, where training requirements are reduced by a factor of at least 2 over the baseline type detector. The performance of the "de-spread" detector is compared to the baseline spread-spectrum detector next. Its superior training requirements will make it our primary spread-spectrum detector.

### 4.3 Basic Properties

In this chapter, simulation studies of the type-based detector are made in the context of a spread-spectrum communications system, as opposed to the studies made in Chapter 3, where the examination was closely tied to the theory and concept of empirical classification. The basic examination here looks at bit error probability versus signal-to-noise ratio, taking into account the effects of $N$ and $n$. We examine the spread-spectrum type detector at parameter ranges of SNR and spreading length $n$ near operating regions of today's CDMA cellular systems. Performance in static-channel conditions is examined first. Single and multi-path fading channel conditions are addressed subsequently.

To start, we make basic comparisons between the spread-spectrum type detectors and the constant-signal type detector (of Chapter 3). There are three variations of these detectors: baseline type detector for constant signals, baseline type detector for
spread spectrum, and "de-spread" detector for spread spectrum. The "symmetric" type detector introduced in Section 3.3.3 is exactly the "de-spread" type detector for spread spectrum. In both, all of the $2C$ bits of training are used to form the training type under $H_0$, which automatically generates the training type under $H_1$ via mirroring about zero, and all samples of the test sequence are used to form one test type. (Recall that for consistency reasons, an empirical detector characterized by $C = C_0$ uses $2C_0$ bits of training.)

Figure 4.7 shows the performances of these detectors for Gaussian and Laplacian noise, at a fixed bit SNR of 5 dB. For any given amount of training (fixed $C$), the baseline SS type detector is always better than the baseline constant-signal detector, but it is never better than the de-spread type detector. Also, the baseline SS type detector at $C = C_0$ is never better than the constant-signal detector at $C = 2C_0$. At $C_0$, the baseline SS type detector actually uses all of the $2C_0$ training bits to test each hypothesis, using both training types $q_+^t$ and $q_-^t$ to compute each detector statistic $h_i$, as shown in (4.2). In contrast, the constant-signal type detector at $C = C_0$ only uses one training type (formed from $C_0$ bits) to test each hypothesis, using either $q_+^t$ or $q_-^t$ for $H_0$ or $H_1$, respectively. For the constant-signal detector at $C = 2C_0$, computing each likelihood statistic requires one training type built from $2C_0$ bits. This is compared to the baseline SS detector at $C = C_0$, where testing each hypothesis involves two types, each from $C_0$ bits. Hence, we would expect the larger variance in the training types of the baseline SS detector to result in more errors. The Laplacian results shown in Figure 4.7 have the same trends described above. The de-spread detector at $C = C_0$ is always better than the baseline SS detector at $C = C_0$, but never better than the same detector at $C = 2C_0$. The relationships can be understood by examining the amount of training required, as characterized by $C$, in computing each likelihood statistic, $h_i$.

The remaining simulations in this section focus on the "de-spread" type detector, given its reduced training requirements over the baseline SS type detector, and on
Figure 4.7: Comparison of SS baseline, de-spread, and constant-signal baseline. Gaussian results are shown in the top plot, and Laplacian in the bottom plot. System parameters: bit SNR 5 dB; 3-bit quantizer set at various levels depending on signal amplitude.
the additional benefits of smoothing the training and test histograms, as introduced in Section 3.4. Simulation results are obtained for Gaussian, Laplacian, and Cauchy additive noises, and are shown in Figures 4.8, 4.9, and 4.10, respectively. In each figure, results at spreading lengths, $n$, of 32 and 256 are shown.

In the Gaussian noise case (Fig. 4.8), we note that the smoothed detector for $C = 1$ (2-bit preamble) is nearly as good as the un-smoothed for $C = 4$ (8 bits). Even better is the performance at $C = 2$ (4-bit preamble), where the smoothed detector is better than the un-smoothed at $C = 8$, and falls within 0.1 dB from the optimal detector (for 3-bit quantizer). The un-smoothed type detector falls within 0.2 dB from the optimum when $C = 8$. As found previously, the type detector performs better, though just marginally for Gaussian noise, at spreading length, $n$, of 256 than at 32.

For the Laplacian noise case (Fig. 4.9), the type detector shows very strong performance. For $C = 1, 2$, the smoothed detector is indistinguishable from the optimal detector, especially at the larger value of $n$. This is compared to the 0.1 to 0.3 dB range of the un-smoothed type detector at $C = 8$. Laplacian noise displays a larger performance disparity for different $n$ values than the Gaussian. The optimum detector is almost 1 dB better at $n = 256$ than at 32. The implication here is that for Laplacian noise, the SNR does not capture enough information about performance. The sample signal amplitudes, noise variance, and $n$ all contribute to performance in a way not captured by the SNR. For Laplacian noise, more spreading is better†. More spreading is also good for type detector in a relative sense. For any $C$, the type detector is closer to the optimum at $n$ of 256. The dB differences are more striking for smaller values of $C$.

In Cauchy noise (Fig. 4.10), results are encouraging, but they are not as good as with Gaussian and Laplacian. The un-smoothed type detector at $C = 8$ performs at

†For Laplacian, the best distribution of energy is to put everything in one sample, which is also reached for the case of $n = 1$. More spreading is good for Laplacian if we assume $n > 1$ and signal energy is equally distributed to all $n$ samples.
approximately 0.3 dB from the optimum; smaller $C$ values have unacceptably large degradation. The benefits of smoothing are minimized for Cauchy. For $C = 1, 2$ (shown), smoothing is beneficial, but the smoothed detector at $C = 2$ is almost 1.0 dB from optimum. For larger $C$ (not shown), smoothing does not produce any significant benefits. Any SNR degradations are enhanced in the Cauchy case due to the relative insensitivity of performance to SNR. A small difference in error probability may translate to a large difference in SNR.

We also simulated each of these systems at $n = 64$ and 128. These performance trends fell within those for $n$ of 32 and 256 and are not shown. Smoothing was performed at lengths 1,5,10,15, and 20, but the displayed results are taken at a smoothing length of 10. The majority of the optimum smoothing lengths fell at 5 and 10, except for Cauchy, wherein smoothing lengths of 1 and 5 were superior..
Figure 4.8: Performance of smoothed, de-spread type detector in static Gaussian channel. Top plot: $n = 32$; bottom: $n = 256$. Smoothing length is 10. The smoothed, de-spread type detector approaches the optimum detector starting at $C = 2$. The un-smoothed detector at $C = 8$ is about 0.2 dB from optimum. Performance at $n = 256$ is just marginally better that at $n = 32$. 
Figure 4.9: Performance of smoothed, de-spread type detector in static Laplacian channel. For $C = 2$, the smoothed, de-spread type detector is practically identical to the optimal value. The performances of the optimal and type detectors at $n = 256$ are both significantly better than at 32. The type detector also compares more favorably to the optimum at $n = 256$ than at 32. System parameters are as in Figure 4.8.
Figure 4.10: Performance of smoothed, de-spread type detector in static Cauchy channel 1. For Cauchy noise, smoothing (length 10) yields only minor benefits. The un-smoothed empirical detector is about 0.2 dB from optimum at $C = 16$. System parameters are as in Figure 4.8.
4.4 Single-Path Fading Channel

In all of the simulations thus far, we have assumed a static system where the transmitted signal does not experience time-varying attenuation. In a land-based wireless system, such as a cellular phone system, the transmitted signal undergoes amplitude distortions due to the movement of the mobile receiver that results in time-varying multi-paths. This time-varying attenuation is actually a spatial phenomenon, since the envelope of the electro-magnetic (EM) waves have crests and valleys at different locations. As the mobile receiver moves through the EM field the received signal amplitude varies, and the mobile receiver experiences temporal variations, or fading [17, 33, 35]. The same principles hold for the base-station receiver when the mobile transmitter is in motion. In this section, the performance and operating ranges of type-based detector in the single-path fading channel are examined. The effects of the multi-path fading channel are examined in Section 4.5.

Generalizing the static model of (4.1), the received BPSK signal model in a single-path fading channel is described by the following model, assuming linear distortion:

\[ r(t) = a(t) \cdot b(t)s(t) + w(t), \] (4.3)

where the channel function \( G(\cdot) \) in (4.1) is now a time-varying amplitude gain or attenuation \( a(t) \). Once again, this model assumes chip-based phase and frequency correction. This assumption means that \( a(t) \) is strictly positive, representing the magnitude of the complex-valued channel distortion. For a slow fading channel, changes are infrequent, and the received model may be assumed to be static in each bit interval [33, 35].

The rate of variation in the fading channel is captured by the Doppler frequency, \( f_d \), which is a function of the mobile’s speed and the carrier frequency. The spatial envelope of the transmitted signal has crests and valleys that are of the order one-half wavelength of the EM wave, which is \( \frac{c}{f_c} \), where \( c \) is the speed of light, and \( f_c \) is the carrier frequency. The vehicle speed, \( v \), divided by the wavelength gives the number
of peaks traversed by the mobile receiver per time unit and results in the Doppler frequency:

$$f_d = v \frac{f_c}{c}.$$  \hspace{1cm} (4.4)

For example, in a system with carrier frequency of 1 GHz, a vehicle at velocity 30 m/s (~67 mph) experiences a maximum Doppler frequency of 100 Hz. Doppler frequencies of 100 to 200 Hz are the maximum typical values experienced by systems at 1 and 2 GHz. A slow-fading system is one whose bit rate $1/T$ is an order of magnitude greater than $f_d$, so that the channel changes little inside a bit period. To capture the amount of fading against the bit rate, the normalized Doppler “frequency,” $\bar{f}_d$, is defined as

$$\bar{f}_d = f_d T.$$  \hspace{1cm} (4.5)

A slow-fading system has $\bar{f}_d \ll 1$. In the above example, if the bit rate is at 20 KHz, the normalized Doppler is 0.005. For a fixed Doppler frequency, very high bit-rate systems experience very slow fading.

While the received statistics changes little over one bit period, they can vary greatly over many bit periods. A detector, such as one based on the type-based detector, typically must change over time to accommodate these variations. A linear receiver, on the other hand, is sub-optimal except in additive Gaussian noise, but it is amplitude invariant and does not need to adapt to channel attenuation.

Figure 4.11 shows the normalized magnitude of a sample path of the Rayleigh-fading channel, at $\bar{f}_d = .005$, produced according to Jakes model [17]. The magnitude $a(t)$, over time, has a Rayleigh distribution:

$$a(t) \sim p(x) = \frac{2x}{A^2} \exp\left(-\frac{x^2}{A^2}\right), \quad x \geq 0,$$  \hspace{1cm} (4.6)

where $A$ is the per-chip amplitude of the transmitted signal. The upper plot shows a sample path lasting 5000 bits (0.25 seconds at 20KHz bit rate), and the amount of change is evident. The zoomed-in lower plot provides a good look at the amount of change that might occur during the training period and between training periods: in
Figure 4.11: Magnitude of fading envelope. A fading envelope is shown, where the normalized Doppler frequency, $f_d$, is 0.005. The top plot reveals long-term behavior, while the bottom plot zooms in to reveal local variations. The top plot is shown in dB scale and the bottom in linear scale.

A matter of 10’s of bits, the channel magnitude greatly changes. The rate of change at other Doppler frequencies is directly scalable from the plots shown.

By having new training sequences that are periodically transmitted, the type detector can adapt to the changing environment. In the static system, more training is always better, but the gain is minimal after a certain threshold, which was found for a 3-bit quantizer to be about 4 bits in Gaussian and Laplacian noise (when smoothing is performed) but around 64 bits in Cauchy (Section 4.3). In a fading channel, due to changing conditions during the training period, increased training may actually degrade performance. Another aspect of interest is the frequency training. Training should take place often enough for current behavior to be captured, but not too
frequently as to significantly reduce the communication throughput.

In Figure 4.12, for several training lengths characterized by $C$, performance is shown as a function of the amount of test bits following each training period. The periodicity of training is simply the number of training bits ($2C$) added to the number of test bits. In these simulations, which address various Doppler frequencies with Gaussian and Laplacian noise, the spreading length $n$ is 64 and the bit SNR is 12 dB. Surprisingly, for all of the Doppler frequencies tested, performance appears to be relatively insensitive to the frequency of training, and improves monotonically with additional training.

Some of the results shown in Figure 4.12 do appear to have slight upward or downward trends as a function of the number of test bits. We observed that the error estimates in many cases were noisy, even when the error estimate is based on greater than $10^6$ bit decisions. Trends that appeared in one simulation did not return in the next. This was especially true in slow-fading situations, where the randomness of the channel only appears with the passage of many bit periods. Consequently, to sample enough randomness in the channel requires prohibitively long simulations. We examined the variability of the error estimates using bootstrap and found that the bootstrapped variance estimate was higher than what would be expected at the prescribed error probability. Nevertheless, we do believe that slight trends are present, but they are difficult to discern, for reasons discussed below.

Additional simulation results are shown in Figures 4.13, 4.14, and 4.15 for Gaussian, Laplacian, and Cauchy noises, respectively. In these simulations, for every $2C$ training bits, $10C$ test bits are transmitted, so the effective information throughput is $10/12$, approximately 85%. We observe again that regardless of the Doppler frequency, more training always yields better results.

Recall that the fading channel experiences peaks approximately every $(2\hat{f}_d)^{-1}$ bits (Figure 4.11), which means that the channel undergoes great changes during this time. In fact, the fading envelope loses approximately half of its correlation in $T_{3.5} = (6\hat{f}_d)^{-1}$
bits [35]. At the Doppler frequencies tested, $T_{0.5}$ ranges from 250 bits ($\tilde{f}_d = 0.0005$) to 12 bits ($\tilde{f}_d = 0.01$). Hence, we see that in simulations at $\tilde{f}_d = 0.0005$, training takes place in relatively static conditions, and re-training intervals longer than 250 bits would not sample the channel fast enough. At $\tilde{f}_d = 0.01$, the channel changes greatly for $C \geq 8$ (16 training bits), and most of the re-training intervals are at a slower rate than channel variations.

We believe the reason that performance is insensitive to the frequency of training and improves with more training is due to the small per-chip signal value relative to the background noise. At spreading length, $n$, of 64 and bit SNR of 12 dB, the transmitted signal has an amplitude of approximately 0.5, and the background noise has variance one. Even though the channel changes greatly, the mismatch in the received statistics only varies slightly. Hence, performance would be insensitive to the frequency of training. The same reasoning implies that the time-averaged property of the observations, which is what the type detector captures when using a large amount of training ($2C > T_{0.5}$), is not far from the properties at any instance in time and hence can be used effectively for classification. In Figure 4.16, we show that the time-averaged densities for Gaussian and Laplacian noise appear very similar to their counterparts at the mean values. More striking are the similarities between the log-likelihood-ratio detectors of the time-averaged properties and the actual optimum detectors, which are a linear detector and a hard-limiter for Gaussian and Laplacian, respectively.

The small per-sample signal property is intrinsic to systems with large amounts of bandwidth expansion, like in spread-spectrum. In such systems, the empirical type detector only needs to learn of the time-averaged properties of the observations and need not be concerned with variations caused by channel fading. Training does not need to take place at the order of the channel-fading rate but depends only on other factors that may cause the received signal properties to change outside the bounds of Rayleigh fading and current noise properties.
Although not shown in these simulations, smoothing of the training and test types is quite beneficial for moderate amounts of training. For Gaussian noise, at $C = 8$, the smoothed detector performs nearly optimally. Its gains are not so significant at larger $C$ values, and, in fact, degraded performance sometimes occurs. Because smoothing convolves a smooth function (Gaussian window) with the observed types, its effect may be said to be similar to the effect of capturing the time-averaged properties, which are captured by the distribution of the background noise convolved with the channel’s amplitude distribution. Because training does not have to take place frequently, we advocate using long training sequences to improve performance, without the need to perform smoothing.
Figure 4.12: Performance versus training frequency. These plots show Gaussian test results at $f_d$ of 0.001 (top left) and 0.005 (top right), with bit SNR of 12 dB, and $n$ of 64. Laplacian results are shown at $f_d$ of 0.0005 (lower left) and 0.001 (lower right) at the same SNR and $n$. The type detector appears insensitive to the frequency of training, regardless of the amount of training. As well, more training benefits performance, even though the channel may change significantly during training. We believe both of these effects are due to the small per-chip amplitude value of a spread-spectrum system; see text. No smoothing is performed in these simulations.
Figure 4.13: Performance in single-path fading channel, Gaussian. Four fading rates, $f_d$, are shown: 0.0005 (top left), 0.001 (top right), 0.005 (bottom left), 0.01 (bottom right). Larger amounts of training result in better performance. Although the gain in $P_e$ value is small, the flatness of the detector performance versus SNR translates small $P_e$ gains into considerable dB gains. The fading rates shown here correspond to 7 - 140 mph at carrier frequency 1 GHz. In all of the simulations, 2C bits of training are followed by 10C bits of information, and so on.
Figure 4.14: Performance in single-path fading channel, Laplacian. Four fading rates, $\dot{f}_d$, are shown: 0.0005 (top left), 0.001 (top right), 0.005 (bottom left), 0.01 (bottom right). Larger amounts of training result in better performance. Although the gain in $P_e$ value is small, the flatness of the detector performance versus SNR translates small $P_e$ gains into considerable dB gains. The fading rates shown here correspond to 7 - 140 mph at carrier frequency 1 GHz. In all of the simulations, 2C bits of training are followed by 10C bits of information, and so on.
Figure 4.15: Performance in single-path fading channel, Cauchy. Four fading rates, $\tilde{f}_d$, are shown: 0.0005 (top left), 0.001 (top right), 0.005 (bottom left), 0.01 (bottom right). Larger amounts of training result in better performance. Although the gain in $P_e$ value is small, the flatness of the detector performance versus SNR translates small $P_e$ gains into considerable dB gains. The fading rates shown here correspond to 7 - 140 mph at carrier frequency 1 GHz. In all of the simulations, $2C$ bits of training are followed by $10C$ bits of information, and so on.
Figure 4.16: Mismatch due to long training. Shown on the left are the per-chip instantaneous (dashed) and time-averaged (solid) densities, and shown on the right are the resulting log-likelihood ratios for Gaussian (top) and Laplacian (bottom). (The densities shown in solid lines are offset in the y-axis by 0.5 to facilitate visual comparisons with the dashed curves.) The Gaussian and Laplacian have a mean at \( \approx 0.5 \) and variance of 1, which for \( n = 64 \) corresponds to a bit SNR of 12 dB. Because the signal amplitude is fairly small with respect to the additive noise, the time-averaged densities (averaged over a Rayleigh random variable) resemble the original densities. When the type detector trains itself to the time-averaged properties of the observations, its processing resembles the LRT detector that assumes the time-averaged densities. As shown in the right plots, we see that the mismatch between the time-averaged LRT and the actual LRT is small.
4.5 Multi-path and Diversity Combining

One of the major wins of a spread-spectrum or CDMA system is that wide-band signaling allows some multi-paths to be resolvable, where the same multi-paths in a narrow-band system would have either contributed to the Rayleigh fading or induced inter-symbol interference. The mechanisms behind Rayleigh fading and multi-path fading will not be discussed here but can readily be found in established texts [33, 35].

As we have seen, single-path Rayleigh fading causes significant performance degradation compared to the static channel (compare, for example, Figures 4.8 and 4.13; comparisons are also shown below in Figures 4.25, 4.26, and 4.27). Intuitively, this is due to the fact that Rayleigh fading causes the instantaneous signal-to-noise ratio to vary substantially from the expected SNR and that the probability of error is a decreasing, convex function of SNR. Analytic results can be found in [33]. With a multi-path channel, when the detector statistics from all of the paths are correctly incorporated — to be discussed below — the effect of having multiple paths is to reduce the variance in the instantaneous SNR, thereby reducing the degradation in performance. Asymptotically, as the number of paths go to infinity, it is known that performance converges to the static case when the additive noise is Gaussian [33]. Receivers used for SS/CDMA in terrain (such as terrestrial) that induces multiple paths always have multiple “demodulator” elements, individually called “arms” or “fingers” and collectively a “rake” receiver, where each detector element is applied to each resolvable path of sufficient strength [33, 35, 48, 61].

Having multiple fingers allows a cellular CDMA system to have “make-before-break,” or “soft,” handoffs as a mobile station traverses from the coverage area of one base station to another. When located between the coverage areas of two or three base stations, the mobile station may receive transmission from multiple base stations. Then, as the mobile station moves within the main coverage area of a targeted base station, the link to former base stations can be severed. A similar situation holds on the base station side, where multiple base stations can receive the
independently faded transmission from a desired mobile station. Diversity of this kind has the benefit of reducing dropped calls during handoff and is said to increase the coverage and capacity of the system [12, 50]. In contrast, a TDMA receiver has one demodulator element and must sever its connection with the former base station before linking with the new target base station, thus the term "hard" handoff.

Multiple paths at the receiver can be modeled by

\[ r(t) = \sum_{l=1}^{L} a_l(t) \cdot b(t - \tau_l) \cdot s(t - \tau_l) + w(t), \tag{4.7} \]

where \( L \) is the number of paths, \( a_l(t) \) is the independent fading amplitude of the \( l^{th} \) path, and \( \tau_l \) is the delay of the \( l^{th} \) path, in integral number of chips. We assume, as before, perfect phase recovery takes place on each Rake finger, and the phase-corrected signal at each finger is projected from the complex I/Q plane to the real line. The delays are assumed to be at least one chip from each other. Then, due to the pseudo-noise property of the spreading sequence, the received samples synchronized to one path may be considered to be independent of the interference coming from the other paths. We also assume that the multiple paths are chip-synchronous, so that \( t \) and \( \tau_l \) are integers. The delays are known to the receiver using readily available synchronization methods [33, 38]. For soft-handoff situations, the spreading sequences of the different base stations are different and thus independent, so that the delays between the received paths may be arbitrary without violating the independence assumption. The model of the received observations for soft handoff is simply (4.7) with the spreading sequence \( s(t - \tau_l) \) replaced by \( s_l(t - \tau_l) \), which represents a different spreading sequence.

The classic linear rake receiver is depicted in Figure 4.17. Each finger is a linear detector. The output of each finger is combined, with a weight depending on the path's current SNR (actually, \( a_l(t)/\sigma_l^2 \)), to form a centralized detector statistic [33, 38, 48, 61].
Figure 4.17: Classic rake receiver. The received signal is passed to \( L \) independent demodulator elements, or "fingers," each focused on a different path characterized by delay \( \tau_i \). Each matched-filter output is weighted by a strength parameter (see text) before being combined with the other weighted finger outputs. This weighting and combining process is called maximal-ratio combining (MRC).

4.5.1 Static Multi-path Channel

To start, we first assume that the multi-path channel is static, so that the \( a_i(t) \)'s are constant but may be different. In any test bit period \( i \), \( n + \tau_{\text{diff}} \) received samples contain the observations for all of the paths for bit \( i \), where \( \tau_{\text{diff}} \) is the maximum delay between any two paths, in integer chip index. Given the assumption that the paths are independent, these \( n + \tau_{\text{diff}} \) samples are equivalently treated as \( L n \) samples, \( \{x_i^l\}_{i=1,\ldots,L} \), where \( x_i^l \) is length-\( n \) and used as the test sequence for the \( l^{th} \) path. The same concept is applied to training, so that there are effectively \( L \) sets of length-\( N \) observations under each hypothesis, \( \{t_0^l, t_1^l\}_{i=1,\ldots,L} \). Because the observations from each path are independent, the joint probability mass function of the training sequences (under \( H_i \) and the test sequences can be written as the product of the individual probability mass functions \( P_l \):

\[
P(\{t_i^l\}_{i=1,\ldots,L}, \{x_i^l\}_{i=1,\ldots,L}) = \prod_{i=1,\ldots,L} P_l(t_0^l, x^l).
\]  

(4.8)

Thus, the Gutman GLRT statistic for \( H_i \) is:

\[
h_i = \frac{1}{Ln} \left( \log \frac{\sup P_l \ P_l(x^l) P_l(t_1^l)}{\sup P_l, Q_l \ P_l(x^l) Q_l(t_1^l)} + \cdots + \log \frac{\sup P_L \ P_L(x^L) P_L(t_1^L)}{\sup P_L, Q_L \ P_L(x^L) Q_L(t_1^L)} \right),
\]

(4.9)
Figure 4.18: Empirical detector for static multi-path channels. Similar to the classic, linear rake receiver (Figure 4.17), the received signal is sent to separate "fingers," where each is made of a separate type-based detector. The output of each type detector, however, are combined with equal weights, in contrast to the linear rake receiver. The equal weighting reflects the intrinsic weighting of each type detector due to its corresponding training vectors \((t_0^l, t_1^l)\). The configuration shown here is for the de-spread detector. The topology of the baseline detector follows straightforwardly.

which simplifies to

\[
h_i = \frac{1}{L}(h(x^1, t_0^1) + \cdots + h(x^L, t_0^L)).
\]

The Gutman statistics for all of the observations is simply the sum of the individual Gutman statistics, with equal weighting. The emphasis (or de-emphasis) given to the stronger (or weaker) paths is intrinsic to the individual likelihood functions and captured by the individual training sequences \(\{t_0^l, t_1^l\}_{l=1,\ldots,L}\). In forced-decision mode, the empirical detector of the \(l^{th}\) path computes \(h(x^l, t_0^l) - h(x^l, t_0^l)\), and the differences are combined equally to arrive at the final decision statistic. This equal-weighting topology is depicted in Figure 4.18, where the de-spread type detector is shown. The topology of the baseline spread-spectrum type detector follows a similar form and may be found by direct application of equations (4.8), (4.9), and (4.10), which assume de-spreading.

In the classic linear rake receiver, maximal-ratio combining (MRC) is nothing more than the matched filtering of the the finger (matched filter) outputs, which is
well known to be optimal for Gaussian noise. In thinking of the rake receiver, we may, confusingly, consider the finger operation (linear detector) as separate from the weighting. A better way to conceptualize MRC is to consider the weighting as part of the finger detector (weighting before or after a linear detector amounts to the same thing), so that each finger detector has built into itself the statistics of the path, in the same way as the type detector, as succinctly expressed in (4.10).

4.5.2 Time Varying (Fading) Multi-path Channels

The analysis of the static multi-path type detector does not always apply when the multi-path channel is dynamic, time varying. In a very slowly fading channel, when the properties of the training set and the test bits are nearly equal, the channel may be considered static. When the fading rate is such that the training and testing periods experience different statistics, the static analysis does not apply.

In the GLRT expression for the static multi-path channel (4.9), each of the numerator terms expresses the generalized likelihood that the test and training are equally distributed, and hence the supremum is taken over the same $P_t$ for $X_l$ and $T_l$. The same expression cannot strictly be used for a fading system where the training and test vectors are differently distributed. Although it is clear that the supremum must be taken over separate pmf's for the training and test sequences, it is not clear how the relationship of the two pmf's can be modeled. The type-based detector topology for the multi-path fading channel cannot readily be formulated in the spirit of the "universal" classifier. In fact, the same is true for the type detector in the single-path fading channel, but that case is simplified somewhat by not having multiple detector statistics to contend with.

Similar to the single-path fading scenario, the difficulties lie in the different statistical characteristics experienced by the training and test sequences. Consider the following scenario: In one path, the channel was strong during training but is weak during testing. In the other path, the opposite occurs. Adding the individual detec-
Figure 4.19: Empirical detector for *fading* multi-path channels. Unlike the type detector for static multi-path channels, shown in Figure 4.18, the outputs from the individual type detectors are weighted differently. The output of each type detector is first normalized to offset the intrinsic weighting of the training sequence, then is weighted according to the current "strength" of the path. The normalization term is the square-root of the Chernoff information, computed from the corresponding training sequences. The weighting term, \( W(x') \), corresponds to the current path strength and may be computed in a number of ways, as discussed in the text. The Chernoff information may be considered a generalized measure of SNR; they are identical for Gaussian systems.

Detector statistics with equal weight would put more emphasis on the path that doesn’t deserve it, and vice versa. Weighting by the current path strength, such as in MRC, places more emphasis on the currently stronger path and should do better than equal weighting. But the training sequence of the currently weaker path (due to its former strength) intrinsically places a larger emphasis, incorrectly, on the detector statistic.

Thus, we seek a method that would first "normalize" the output of the type detector according to conditions during training, then weights the output according to current conditions. We propose that the \( i^{th} \) path statistic, for the \( i^{th} \) hypothesis, be computed as follows,

\[
h_i^l = h(x^l, t_i^l) \frac{1}{\sqrt{C(t_i^l, t_i^l)}} W(x^l),
\]

and the combined detector statistic to be

\[
h_i = \frac{1}{L}(h_1^l + \cdots + h_L^l).
\]
See Figure 4.19. The normalization factor $C(t_0^l, t_1^l)$ is the empirical Chernoff information, computed using types $q_{t_0^l}$ and $q_{t_1^l}$. Its square-root may be considered a generic measure of signal-to-noise ratio, motivated by the fact that the Chernoff information and the SNR are identical when the additive noise is Gaussian. The normalization attempts to remove the intrinsic weighting applied to the type detector due to $t_0^l$ and $t_1^l$. The weighting term, $W(\{x^l\})$, accounts for the current strength of the path. Its exact form is not given here because of various implementation considerations. Ideally, the current Chernoff information would be used. However, the test type, $q_{x^l}$, is likely to yield a noisy estimate of the Chernoff information because only $n$ samples are used. (This is compared to $2Cn$ samples for estimating the Chernoff information using training sequences.) One possibility is to average the Chernoff informations for a window of the nearby bit periods. Another would be to use decision-feedback to first guess the information bit, then build one type based on adjacent, detected bits, which could then be used to compute the Chernoff information. The brace delimiter in the weighting expression, $W(\{x^l\})$, means to convey that the weight may depend on the test vectors of nearby bit periods. We do not resolve this issue in this thesis and leave it for future consideration. In all of our simulations, we make $W(\{x^l\}) \propto$ be the true value of $a_l/\sigma_l^2$, which is the ideal weighting statistics used in linear MRC.

4.5.3 Simulations

We begin by comparing the Chernoff-based MRC detector, the equal-weighting detector, and the SNR-weighted detector that only accounts for the current path strength, in three different two-path fading scenarios: equal powered paths at $\bar{f}_d = 0.005$, equal powered paths at $\bar{f}_d = 0$, and unequal powered paths at $(0.8, 0.2)$ at $\bar{f}_d = 0$. Simulations are conducted for Gaussian and Laplacian additive noise, and the results are shown in Figures 4.20 and 4.21, respectively. In the non-zero Doppler situations, it is clear that the Chernoff-based MRC type detector performs best, typically by a significant margin (over 1 dB) over the equal-weighting type detector and a non-trivial
margin (0.0 to 0.5 dB) over the SNR-weighted type detector. In the zero-Doppler test of equal-powered paths, all of the detectors performed equally, as expected, since no normalization is needed due to unequal strengths during training. In the zero Doppler, unequal-powered-paths test, the equal-weighting detector is expected to be the best due to static conditions, and simulation results are consistent with this prediction. In the Laplacian test, the Chernoff-weighting detector seems to perform better than the equal-weighting detector, but we believe this variability is within the variability inherent in the estimates of the error probabilities.

These simulations show clearly that normalizing the Gutman statistics by $\sqrt{C(t_0^r, t_1^r)}$ rightfully weights the individual statistics. We tried other normalizations factors (not shown): normalizing by the Chernoff (not the square-root) is inferior; normalizing by the square-root of SNR is almost as good; here, we used the true, not estimated, SNR, as opposed to using the estimated $C$ for the Chernoff-based normalization. In all of the tests, the true SNR was used, regardless of the weighting due to training or testing.

More thorough two-path simulations are made for the same fading conditions as those in the single-path fading tests. All of the single-path trends hold for the present two-path scenario. Results for the Gaussian, Laplacian, and Cauchy cases are shown in Figures 4.22, 4.23, and 4.24. Here, diversity from multiple paths greatly increases performance over the single-path situation. (Compare with Figures 4.13, 4.14, 4.15.)
Figure 4.20: Comparison of multi-path weighting strategies, Gaussian. The Chernoff-weighting detector, the equal-weighting detector, and the SNR-weighting detector that accounts only for the current path strength are compared. Comparisons are made at various two-path fading situations: equal-powered paths, $\tilde{f}_d = 0.005$ (top row); equal-powered paths, $\tilde{f}_d = 0$ (middle row); unequal-powered paths $(0.8, 0.2)$, $\tilde{f}_d = 0$ (bottom row). The figures on the left side represent the un-smoothed detector, and the right side shows a smoothing length of 8; the de-spread detector is used for all tests. In all cases, the Chernoff-based weighting strategy is superior to the others.
Figure 4.21: Comparison of multi-path weighting strategies, Laplacian. The same tests as those in Figure 4.20, but with Laplacian noise, are applied here. In the unequal-powered static channel, with results shown in the bottom plots, the Chernoff-weighting detector seems to outperform the equal-weighting detector. We believe this is due to the statistical variability of the error-probability estimates, and in this case, the culprit is the error-probability estimates at 7.5 dB.
Figure 4.22: Performance in 2-path fading channel, Gaussian. Four fading rates are shown: $\bar{f}_d$ of 0.0005 (top left), 0.001 (top right), 0.005 (bottom left), 0.01 (bottom right). The spreading length is 64. Tests shown have constant throughput: For every $2C$ bits of training, $10C$ information bits are transmitted. More training typically means better performances, as in the one-path fading case, although in $\bar{f}_d$ of 0.0005, there appear to be some inconsistencies. We caution that simulations at small $\bar{f}_d$’s yield noisier results. Nevertheless, the $C = 32$ training case yields consistent performance compared to the others. No smoothing is performed. A uniform 3-bit quantizer is used in all tests.
Figure 4.23: Performance in 2-path fading channel, Laplacian. Four fading rates are shown: $\tilde{f}_d$ of 0.0005 (top left), 0.001 (top right), 0.005 (bottom left), 0.01 (bottom right). The spreading length is 64. Tests shown have constant throughput: For every $2C$ bits of training, 10$C$ information bits are transmitted. More training typically means better performances, as in the one-path fading case, although in $\tilde{f}_d$ of 0.0005 and 0.001, there appear to be some inconsistencies. Nevertheless, the $C = 32$ training case yields consistent performance compared to the others. No smoothing is performed. A uniform 3-bit quantizer is used in all tests.
Figure 4.24: Performance in 2-path fading channel, Cauchy. Four fading rates are shown: $f_d$ of 0.0005 (top left), 0.001 (top right), 0.005 (bottom left), 0.01 (bottom right). The spreading length is 64. Tests shown have constant throughput: For every $2C$ bits of training, $10C$ information bits are transmitted. More training almost always means better performances, especially for the faster fading rates of 0.005 and 0.01, The $C = 32$ training case yields consistent performances throughout. No smoothing is performed. A uniform 3-bit quantizer is used in all tests.
4.6 Comparison with Linear Detectors

Finally, we compare the type detector against the matched-filter detector (MF) and the sign detector (SIGN). The matched-filter is the predominantly used detector for spread-spectrum, CDMA receivers, and it is optimum for Gaussian noise. The sign detector is a linear detector operating on one-bit quantized data. It is the locally optimal detector for Laplacian noise [30]. Simulation tests are conducted in static, one-path fading, and two-path fading channels, for Gaussian, Laplacian, and Cauchy noise. Corresponding results are shown in Figures 4.25, 4.26, and 4.27. In all cases, the type detector is close to the best detector, if it isn’t already the best, and its superiority over the worst-case detector, which varied between the MF and SIGN detectors, is always significant.
Figure 4.25: Comparison with linear detectors, Gaussian. The performance of the type detector is summarized here and compared against the matched-filter detector (MF) and the sign detector (SIGN). The type detector shown is the despread detector operating at $C = 32$. In Gaussian noise, the MF is optimal, and outperforms the others. The type detector, though, performs almost identically. We note that the MF shown operates on continuous-valued data, which means that it has an intrinsic advantage of about 0.15 dB. Without this margin, the type detector would perform identically. Maximal-ratio combining (MRC) is used with all detectors in the two-path fading channel. The spreading length, $n$, is 64. A 3-bit uniform quantizer is used.
Figure 4.26: Comparison with linear detectors, Laplacian. In small-signal situations, the sign detector (SIGN) is optimal in Laplacian noise. Although the per-chip SNR at most of the bit SNR shown is small, the type detector almost always performs better. The matched filter (MF) does not fare well in Laplacian noise and significantly lags the others.

4.7 Summary

Through simulations, we examined finite-sample performance of the type-based detector applied to the spread spectrum problem. Tests were conducted in static channel conditions, single-path Rayleigh fading, and multi-path Raleigh fading. We summarize our findings below.

- Spread-spectrum, CDMA communications is a good application of the type detector because of the large bandwidth-expansion factor from the information symbols to the transmitted samples. A large bandwidth expansion results in small per-chip SNR's, more training samples, and less quantizer resolution, all of which are favorable conditions to the type detector.
Figure 4.27: Comparison with linear detectors, Cauchy. The type detector performs best in most cases, with the sign detector (SIGN) matching its performance in the two-path fading case. The heavy tail of Cauchy noise renders the matched filter (MF) ineffective.

- The baseline type-based spread-spectrum detector separated training and test sequences depending on whether the chips of the spreading signal were positive or negative. The de-spread detector made symmetry assumptions on the noise and signal, resulting in only one type for training and test and effectively reduced training requirements by less than 2.

- In the static channel, the de-spread type detector - with smoothing - required only a small number of training bits (4) to approach within 0.1 dB of optimum. With Cauchy noise, more training was required to achieve near optimality.

- When the channel undergoes Rayleigh fading, more training is better at almost all fading rates. The long training period implies that the type detector captures
the time-averaged properties of the channel, which we showed is similar to the instantaneous, local properties when the per-chip SNR is small. We noted that the bandwidth expansion of spread-spectrum systems results in small per-chip SNR’s for reasonable symbol SNR’s.

- For multi-path, diversity channels, one type detector is applied to each path. In the static, multi-path channel, the individual likelihood statistics from each path are equally added to form the final decision statistic. In a time-varying channel, we normalized each path’s likelihood statistic by the Chernoff information in its training sequence and weighed it by the current path strength. Good performance was shown.

While we showed superior performance of the empirical type detector over the linear and sign detectors in a variety of static and fading channels, we always made the assumption that sample-based phase and frequency synchronization were in effect. This assumption effectively makes the empirical detector structure more costly, since sample-based correction methods require operations at the chip rate. A type made of I,Q data when frequency offset or channel fading is present must be updated extremely fast, since changes in the phase will greatly distort the type. Thus, a direct implementation of the type detector to I,Q data may be practical only in extremely slow-fading channels, much slower than those we have simulated. Parameterization of the I,Q type, by phase, if available, could be used to rotate types as needed, but this may be just as costly as chip-based correction of phase. Having guaranteed, superior performance in unknown environments, however, may very well be worth the added cost of sample-based synchronization.
Chapter 5

Type-based Detector in Multiple Access Interference Channel

This chapter analyzes the single-user likelihood-ratio-test (LRT) detector in a multiple-access environment, where the dominant source of "noise" is due to the transmission of other users. Its performance is compared to the conventional matched-filter (MF) detector at various system operation regions: near-far, noise dominated, interference-limited, perfect power control, imperfect power control. We show that in certain, common situations, the single-user LRT significantly outperforms the matched filter. Because of the near-optimality of the type detector, found in the performance studies of the previous chapters, we rely on the theoretical analysis of the LRT detector's effectiveness to convey the applicability of the type detector in multiple-access situations. The superior performance of the optimal detector directly implies the superior performance of the empirical type detector, especially when the performance margin is large. Therefore, other than including a small number of simulation results that are in our published works [19, 25], the focus of this section is primarily theoretical, with help from numerical examples.

We note that related topics are currently pursued in another thesis development [54], where the type detector is compared to the MMSE detector [16] and in some cases, implemented as a multi-user detector.

5.1 CDMA Background

In addition to protection against intentional and unintentional narrow-band interference (jamming), a spread-spectrum system can also reject interference from other
friendly spread-spectrum systems. This concept is the basis for the spread-spectrum based multiple access communications system called Code Division Multiple Access (CDMA) system, which is designed for multiple users to simultaneously operate in the same frequency spectrum. Users are assigned unique codes so that the desired user's signal can be distilled from the aggregate signal coming from multiple transmissions. If \( K \) users are transmitting, assuming a single-path from each user, the baseband model of the received signal is (assuming chip synchrony):

\[
r(t) = A_1 b_1 s_1(t - \tau_1) + \sum_{k=2}^{K} A_k b_k s_k(t - \tau_k) + w(t),
\]

where \( A_k, b_k, s_k(t) \) are, respectively, the amplitude, information bit, and spreading signal of the \( k^{th} \) user. The chip-synchrony assumption means that \( t \) and \( \tau_k \) are integers and represent discrete-time "time" and "delay" indices, respectively. For our analysis here, static conditions are assumed, so that the signal magnitude \( A_k \) does not vary with time.

Consider the case when \( s_i(t) \) and \( s_j(t) \) have a small correlation. Then, if a matched filter (which performs a linear correlation of the received signal with the desired user's spreading code) is used for the detection of a desired user, the contribution from other codes - called multiple access interference (MAI) - will be small, and reception for the desired user can take place with high performance, assuming that the strengths of the other users are reasonable. We consider CDMA systems with random-code spreading, so that the codes are effectively statistically independent from chip to chip of each user and between users. The correlation in any one bit period between any two codes, which is the linear interference they give to each other, is therefore random and noise like. In particular, the correlation is binomially distributed composed of \( n \) Bernoulli, with \( p = \frac{1}{2} \), taking values of \( \pm 1 \). The random code implemented is typically a very long deterministic code - such as an m-sequence or Gold code - long enough for the randomness assumption to hold. For example, the Long PN Code in the IS-95 CDMA system has a period of \( 2^{42} - 1 \), which is approximately 40 days, at a 1.3 MHz chip
rate, and it is produced from a 42-bit m-sequence [39].

The most widely used detector is the conventional linear detector, or matched filter. In the single-user case, the linear detector has an SNR, per bit, of

$$\text{SNR} = \frac{nA^2}{\sigma^2}, \quad (5.2)$$

where $A$ is the per-chip amplitude, and $\sigma^2$ is the per-chip variance of the additive noise. The spreading length, $n$, increases the SNR proportionally; hence the term “spreading gain.” The SNR is the measure of interest because in additive Gaussian noise, the probability of error is $Q(\sqrt{\text{SNR}})$. In a multi-user channel of $K$ users, assuming random codes, the signal-to-interference ratio (SIR) is:

$$\text{SIR} = \frac{nA^2}{\sum_{k=2}^{K} A_k^2 + \sigma^2}. \quad (5.3)$$

That is, each of the other users appear as independent noise with per-chip variance equaling the square of its amplitude. For large $Kn$, the MAI may be assumed Gaussian at the output of the MF detector due to the Central Limit Theorem (CLT). The resulting probability of error is $Q(\sqrt{\text{SIR}})$. Equation (5.3) reveals many of the limitations with the linear detector. Any increase in the signal strength of one user necessarily means that the interference to the other users increases. When some users’ energies are much larger than the rest, the small-signaled users won’t be able to maintain effective communications. In CDMA communications, this is called the “near-far” problem: Transmitters that are near-by can drown out the signals from transmitters far away. To address this problem, current day systems, such as the reverse-link of IS-95, use power-control strategies, whereby the transmitters are commanded by the receiver to increase or decrease their powers in a way that the received powers are balanced in a desirable way [39, 45, 49, 50].

A variety of multi-user detectors for interference-cancelation have been proposed to address the detection of multiple users, including the near-far problem. A survey may be found in [46]. Two linear multi-user detectors, the MMSE detector [16] and the decorrelating detector [26], are generalizations of the single-user matched filter. In
all of these methods, detection is made for all users – sequentially or simultaneously – and knowledge of all user’s spreading code is required. Currently, there is intense debate on both the theoretical and practical applicability of these methods [45, 46]. Proponents on both sides have made good points, but the debate is unresolved and on-going. Our analysis in this chapter is relevant only in the single-user detection scenario, although some comparison to the linear multi-user detectors will be made. As we have pointed out, the type detector is analyzed in the context of multi-user detection in [54].

The type-based detector is a single-user detector with a simple implementation. It has the same “philosophy” as the likelihood-ratio detector that assumes the received observations, at the chip-level, come from either \( P_0 (H_0) \) or \( P_1 (H_1) \). All “noise” – whether due to multi-path interference, additive noise, and as we will see, multiple-access interference as well – only combine to change the pmf’s \( P_0 \) and \( P_1 \). The type detector does not use the linear structure that may exist in the users’ spreading codes for interference cancelation like the linear multiuser detectors (MMSE and decorrelator).

In what follows, we analyze the LRT detector’s performance under the assumption that the spreading codes of the other users are completely random and not known a priori. We quantify the performance of the LRT (hence, the type detector as well) over that of the MF detector in a variety of operating conditions – noise limited, interference limited, near-far, perfect power control, and imperfect power control.

*Here we assume that the received observations have already been de-spread, so that all observations in a bit period come from either only the “+” portion \((H_0)\) or only the “−” portion \((H_1)\).
Figure 5.1: Distributions of received samples in MAI. Three users are present (two interferers) are present. The top plot shows the continuous-valued densities under $H_0$ and $H_1$. The lower plots show the distribution under each hypothesis using a 4-bit uniform quantizer.

5.2 Analysis of LRT Detector in Multiple-Access Interference Channel

The asymptotic performance of the LRT detector is captured completely by the Chernoff information between $P_0$ and $P_1$. In this section, we quantify this Chernoff information for an interference-limited channel.

After de-spreading, the per-chip probability density function when the transmitted bit is "0" (or "+1") is

$$p_0(x) = \frac{1}{2^{K-1}} \sum_{c_2=\pm 1, \ldots, c_K=\pm 1} p_w \left( x - A_1 + \sum_{k=2}^K c_k A_k \right),$$  \hspace{1cm} (5.4)

where $K$ is the number of total users, user 1 is the desired user, $A_k$ is the signal
amplitude of the $k^{th}$ user, and $p_w(\cdot)$ is the pdf of the additive noise. The lack of knowledge of the other users’ spreading codes and their bits results in the $c_k$’s terms. The factor $1/2^{K-1}$ comes from the fair-coin modeling of the interference coming from the $K-1$ other users. The model we want assumes that the interference from each of the other users is i.i.d. and random per chip. In particular, the other users must not have the same spreading code as the desired user. We emphasize this point because the randomness (the $c_k$ terms) in (5.4) could have arisen from the random interference due to the unknown bit information and when the interfering users all have the same spreading code as the desired user. In this degenerate case, performance is strictly
dependent on only the signal amplitudes and information bits of the interfering users and not on the spreading length $n$. This point is noted because it is easy to mistakenly consider the performance of the LRT detector in an MAI channel to depend only on the interfering power levels, and not on the randomness in the code. Another way to look at (5.4) is as a train of impulses convolved with $p_w(\cdot)$:

$$p_0(x) = \frac{1}{2^{K-1}} \sum_{c_2=\pm1,\ldots,c_K=\pm1} \delta \left(x - A_1 + \sum_{k=2}^{K} c_k A_k \right) * p_w(x).\quad (5.5)$$

Figure 5.1 shows an example of $p_0$ and $p_1$, for the case of 3 total users (2 interferers), and both the continuous-valued and quantized distributions are shown. Visualizing the arbitrary case of $K$ users readily follows.

Figure 5.2 looks at the LRT's performance in near-far situations and shows the Chernoff information for the desired user when the other users' combined power increases without bound, for the cases of 2–8 users. The desired user's amplitude is 0.5 in the top figure and 3.0 in the bottom. An interesting effect emerges: with large interference, the performance of the LRT detector is identical to the no-interference case. This level of interference rejection isn't even reached by any of the linear multi-user (decorrelating and MMSE) receivers, let alone by the MF. For the MF, the degradation in performance can be seen from (5.3) and is shown in Figure 5.2 for the case of 2 and 8 users; for the linear multi-user detectors, performance for large interference levels never gets as good as with no interference [32, 43]. The worst-case operating range occurs when each user's amplitude is equal to the desired user's, when the desired user's amplitude is more than about 0.2 plus the standard deviation of $p_w$, which is $N(0,1)$. When the desired user's amplitude is less than 1.2, the worst case point occurs at 1.2. We will return to explore the LRT detector's performance at small deviations away from the equal-power point. While the LRT has very good near-far properties, a large region of moderate interference levels exists in which its performance is substantially compromised.

We next examine the LRT detector's performance when all users' powers are
Figure 5.3: Chernoff information for equal-powered users. The LRT detector's Chernoff information of the desired user is shown as a function of each user's amplitude, including that of the desired user. Chernoff information does not continue to increase with increasing user amplitudes. The background noise is $\mathcal{N}(0,1)$.

balanced so that the mutual interference among all users are equal. This is the ideal operating condition for the MF and linear multi-user detectors (for capacity reasons) [12, 43, 44, 50]. Figure 5.3 shows the Chernoff information for 2-8 equal-powered users as the powers increase. Similar to the MF detector, increasing power does not continue to increase performance after some threshold. We show in Appendix D that for $K$ total users, the limiting Chernoff for each user, independent of noise density $p_w$, is

$$C(p_0, p_1) = -\log_2 \left( \sum_{i=1}^{K-1} \sqrt{\binom{K-1}{i-1} \binom{K-1}{i}} \right) + K - 1, \quad (5.6)$$

where $\binom{n}{k}$ denotes $\frac{n!}{(n-k)!k!}$. The limiting Chernoff information is shown against the
Figure 5.4: Limiting Chernoff information, LRT v. MF. The limiting Chernoff information, achieved at large, equal user amplitudes, is shown in the upper plot as a function of the number of total users. In the lower plot, this Chernoff information is compared to the Chernoff information at the output of the matched-filter detector (shown in dB scale).

number of users in upper plot of Figure 5.4. The displayed values have been verified against values obtained via numerical integration.

While the Chernoff information is a useful measure for asymptotic analysis and may be used as a comparative tool, our experience has found that it only provides a loose bound, $2^{-C_n}$, on the actual probability of error. As such, the error bound should not be compared to a true error probability such as that derived for a Gaussian system using the Q function. Besides using simulations, we would like to compare the LRT performance, given in Chernoff information, to that of the matched filter. The LRT performs better, since it is the optimal single-user detector that assumes a random
MAI, which is the same assumption made by the matched filter, but we would like to know by how much. For the case of equal-powered users, as the number of users increase, the MAI – at the chip level – converges to Gaussian by virtue of the Central Limit Theorem. Therefore, the MF converges to the optimum detector for growing $K$, but it would be useful to know when the MF’s performance approaches the LRT. We quantify the MF’s performance by using the LRT detector’s Chernoff information in a static Gaussian channel that has the same SNR as the MAI channel’s SIR. This makes sense when the output of the MF becomes Gaussian, which occurs when both $K$ and $n$ are large. If the spreading length $n$ is large, the output of the MF may be close to Gaussian even when its input, which are the received chip observations, do not. Then, we can measure the Chernoff information at the output of the MF, assuming the output has a certain SIR level. The Chernoff information for the matched filter in Figure 5.2 is computed in this way.

The lower half of Figure 5.4 compares the Chernoff information of the LRT versus the MF, shown in dB scale. As the number of users increase, the performance margin of the LRT over the MF decreases. At 12 users, the margin is about 1%, or 0.05 dB; at 8 users, the margin is about 5%, or 0.2 dB, which is a modest gain. These observations are consistent with the LRT’s optimality and the Central Limit Theorem.

5.3 Local Behavior of LRT Detector with Equal-Powered Users

The point where all users have equal magnitude is the optimal set point for MF in the sense that the minimum performance amongst all users is maximized, given any power constraint. That it is a local solution can be seen by noting that the gradient of SIR – derived from (5.3) – is zero when $A_j = A_i$ for all $i, j$, and the Hessian is less than zero in a direction where user 1’s amplitude decreases relative to all of the other users. Intuition concurs: Any increase in the magnitude of one user necessarily implies the decrease of another user’s SIR. The LRT’s local behavior at the equal-
powered point is different and shall be analyzed in this section. The surprising story here is that the equal-powered point is the \textit{worst local operating point} for the LRT detector when the interference environment is \textit{extreme}, that is, when the per-chip user signals are large.

A glimpse of the Chernoff's local minimality at the equal-powered point was seen in the lower plot of Figure 5.2, where the desired user's Chernoff information reached a minimum as each interfering user's magnitude equaled the desired user's. Here, although the desired user's Chernoff information increases with a particular magnitude perturbation,

\[ \delta A = \pm 1[011...1], \]  

(5.7)

the behavior of the other users' Chernoff information is not revealed, nor are the behaviors of all users' Chernoff information with an arbitrary perturbation $\delta A$. Another glimpse was revealed in Figure 5.3, where the Chernoff information saturates for large signal magnitudes, and implying that the gradient and Hessian of any user's Chernoff information are both zero in the direction,

\[ \delta A = \pm 1[111...1], \]  

(5.8)

when $A$ is large.

Although we cannot analytically prove at this time, we believe that at the large, equal-power operating point, the Chernoff information of each user is locally convex, so that any perturbation in the power of any combination of users will increase every user's Chernoff information (except when the perturbation is in a direction of unison increase or decrease of all user amplitudes, as described above). That is, whether user powers increase or decrease, the net result is that every user's Chernoff information increases. These are based on empirical observations of simulation results, which we discuss in detail in the following section.

It may be interesting to search for the allocation of user powers that would optimize some function of the Chernoff information of all users, given a power constraint,
e.g.,
\[
\max_{A_i \leq A_0} f(A),
\]
for some objective function \( f \). An example objective is the minimum Chernoff information among all users, and another is the average Chernoff. The unconstrained optimum may be one where user powers increase in an exponential fashion:
\[
A_1 = A, \ A_2 = 2A, \ A_k = 2^{k-1} A, \ldots, \ A_K = 2^{K-1} A,
\]
which was studied in [47] for a different purpose. Clearly, this power setting will be impractical for moderately large values of \( A \) and \( K \). We will not explore the unconstrained nor constrained power-allocation issues in this thesis. Instead, we examine next the LRT detector's performance in an imperfect and random power-control environment, which is a realistic operating point of the matched-filter detector [49].

5.4 LRT Detector Performance with Imperfect Power Control

It is of interest to study the LRT detector's behavior when random fluctuations in user powers exist about the equal-power operating point. This scenario is not unlike that encountered in the reverse-link of IS-95 [39], where the reported variations are 1–2.5 dB of variance due to log-normal shadowing and delay in closed-loop power control [49], with the user powers modeled as log-normal. The occurrence of random fluctuations can be seen as an intended (but random) power-allocation strategy that enhances the performance of the LRT detector.

Figure 5.5 shows the gain of the LRT detector over the MF detector as a function of the target magnitude of each user, simulated at 0 to 4 dB of log-normal variance. Eight (8) users are present. The gain over the MF increases without bound as a function of increasing target magnitude. The gain is greatest for 1 dB variance and decreases monotonically with increasing log-normal variance, implying that the behavior of the
Figure 5.5 : Comparison of LRT and MF with imperfect power control. Eight (8) users. As the user amplitudes rise above the span of the background noise, \( N(0,1) \), the LRT detector begins to outperform the MF detector in significant ways. The local sensitivity of the LRT's Chernoff information is evident, as the performance is best with 1 dB of log-normal variation in the user powers. Because the MF detector's performance saturates with increasing user amplitude (as discussed in text), the continued superiority of the LRT detector with increasing user amplitude implies that the LRT detector's performance continues to increase with increasing user amplitude. Results shown are actually obtained with an 11-bit uniform quantizer; results are almost identical with a 6-bit quantizer (not shown).

LRT detector's Chernoff information is very sensitive (in a desirable way) at the large, equal-power operating point. Its performance is much better than the matched filter near (1 dB variation) the large, equal-power operating point, but this gain becomes less dramatic (but still significant) when the power variation increases. At the equal-power operating point, with no random fluctuation, the gain over the matched filter is minimal. Another measure of system performance is the average minimum Chernoff
Figure 5.6: Comparison of 4-bit LRT and MF with imperfect power control. Using a 4-bit uniform quantizer can still yield significant gains over the MF detector. Qualitatively, the behavior of the LRT operating on 4-bit data is identical to the LRT operating on continuous-valued data, as shown in Figure 5.5. Eight (8) users are present.

information, with respect to all users, and its behavior is similar to the average Chernoff information (not shown). From a “quality-of-service” perspective, it would be important to maintain the worst-case user’s Chernoff information at a reasonable level.

Figure 5.6 shows the comparison between the 4-bit quantized LRT and unquantized MF. For small user magnitudes, when the Gaussian assumption holds well, the quantization loss of the LRT results in a loss, albeit small, compared to the MF. For larger values of user magnitude, though, the gain of the quantized LRT over the MF is significant.
The simulation of Figure 5.5 is based on 11 bits of quantization, which we believe well approximates the underlying continuous-valued densities. At 1 dB of power variation, for user amplitudes larger than 6, every simulated random power perturbation yielded Chernoff information values (numerically computed) – for every user – that are strictly larger than the corresponding Chernoff information at the equal-powered operating point. When A is large, we have never witnessed a small power perturbation to yield a smaller Chernoff information. These observations imply the local convexity of the Chernoff information.

5.5 Simulation Results for Empirical Detector in MAI

Figure 5.7A shows simulation results for a “near-far” environment, where another user’s power increases relative to that of the intended user. This situation is in addition interference-limited, because the MAI dominates the additive receiver noise, which is $N(0,1)$. As expected, the type detector rejects the interference once it becomes very large.

Shown in Figure 5.7B are the performances of the type-based detector and the matched filter, in the same MAI channel as above, but as a function of the signal to additive-noise ratio. The beneficial effect of increasing SNR is much more pronounced in the type-based detector because the distributions under the two hypotheses become more distinct. The matched filter’s performance saturates to a level that is a function of the signal-to-interference ratio (SIR), which becomes constant.

5.6 Summary

In this chapter, we have shown through various means the gain of the single-user, likelihood-ratio test (LRT) detector over the matched-filter (MF) detector. The underlying assumption of this chapter is that the performance superiority of the optimal detector (LRT) directly implies the performance superiority of the empirical type detector, since the simulation studies of the previous chapters have shown that the
Figure 5.7: MAI rejection, simulation results. The type-based detector has good "near-far" properties in multiple-access environments. In A, system parameters are: 3 users, SNR/chip = 20dB, n = 16, C = 64. The second user's power increases relative to the intended user; the third user has the same power as the intended user. The type-based detector learns the interference and the background noise using training data and does not assume a Gaussian noise structure like the matched filter. It is interesting to note the two local maxima that appear in A. The first occurs near 0 dB because the intended user and the interferer can be difficult to distinguish at equal powers. The second maximum appears at about 6 dB when the interferer has twice the amplitude of the intended user. We believe it occurs due to the nonlinearity of the 3-bit quantizer used, which was not uniform. In B, the signal to additive-noise ratio, per chip, of the intended user is varied while the interference level is fixed, with all 3 users having equal power. As the additive SNR increases, detection performance improves.

type detector approaches optimal performance with reasonable amounts of training. Hence, this chapter focused on the analysis of the optimal detector's performance.

In particular, we found that the disparity in performance of the LRT versus the MF is a function of the chip observations' resemblance to Gaussian. With a combination of large user powers and moderate number of users, the CLT does not hold well, and the LRT detector's gain can be large. The assumption of large user powers relative to the background noise may not be practical for a cellular communications system where maximizing capacity is desired. An often quoted value for out-of-cell interference (which contributes to background noise) is 1/2 of the total in-cell power
[48]. This would lead to a per-chip signal amplitude that is considerably smaller than the values where the LRT detector outperforms the MF. Intuitively, cranking up in-cell user powers to be much greater than the background noise cannot occur in all cells. The increased in-cell power contributes readily to the noise floor of an outside cell, which would then require its users to increase power, which would then increase the in-cell background noise, and so on. However, it may be possible in off-peak hours to arrive at LRT detector’s favorable settings when capacity is not an issue.

Thus, the LRT detector would best be implemented in an isolated-cell system, both for its optimality and implementation simplicity (via a type-based detector). While it is true that in such a system the linear multi-user detector can have better performance, the implementation simplicity of the LRT detector may still give it competitive advantages.
Chapter 6

Conclusions

Classification based on empirically observed statistics, first developed by Ziv [62] and Gutman [14] and further developed in this thesis, is a powerful tool in statistical decision making. The resulting universal classifier, which operates on discrete data, has provable asymptotically optimal properties for any given data distributions $P_0$ and $P_1$: it has the fastest exponentially converging error probability among all empirically based classifiers, and it has the same error rate as the likelihood-ratio detector – which has full knowledge of the underlying sources and is optimal – when the amount of training data grows faster than linearly with the amount of test data. The forced-decision universal detector, derived and analyzed in this thesis, is the two-sided extension to Gutman’s null-hypothesis classifier. We showed in this thesis its asymptotically optimal properties among a large class of universal tests.

The application of the universal classifier to wireless digital communications is motivated by the difficult channel conditions that are often encountered in wireless communications systems. The hostile environments arise from unpredictable channel distortions – due to weather, to the motion of transmitters and receivers, and to the surrounding terrain, from noise and nonlinearities of receiver electronics, and from other users who share the same frequency bandwidth. In particular, we applied the universal detector to the spread-spectrum, CDMA communications system because it offers operating conditions that are favorable to the universal detector – large number of samples per bit, low quantizer resolution, independence of observations even when channel has dependence, and the possibility of abundant training – all of which arise from the SS-CDMA system’s large bandwidth expansion and pseudo-random
spreading codes. Using simulations, we showed that the empirical detector has superior performance in real-world, finite-sample situations. Compared to the optimum detector's performance, when known, we showed that its performance was nearly identical using reasonable amounts of training; compared to well-established linear detectors (matched filter and sign detector), the empirical detector's consistent performance in all environments was in stark contrast to those of the linear detectors, which only perform well in favorable conditions that match their assumptions. Although the empirical detector requires sample-based synchronization, which is more costly than bit-based synchronization, we believe this is justified by its performance. We successfully showed the empirical detector's superior, consistent performance in static conditions, single- and multi-path fading, and interference-limited situations. Its ability to adapt to unknown, time-varying environments, through training, makes it superior over detectors designed for specific environments.

The symmetric uniform quantizer that was used throughout this thesis is simple as well as powerful, but a run-time bin-width determination strategy was outside the scope of this thesis. In [58], the quantizer is found using the continuous-valued training data, and results are promising. We proposed, without study, in Chapter 3 that instead of the continuous-valued data, quantized data with higher resolutions may be used, and work to study the required level of resolution, i.e., the number of quantizer bits, needs to be made. Another method we proposed without study is quantizer adaptation based on a run-time LMS-like method that updates from one training period to another, depending on whether the change in the quantizer's bin width increased or decreased decision making on the training data.

The interested reader might ask why the empirical likelihood-ratio test was not used, mainly

\[
\text{ELRT} \equiv D(q_x|q_{e_1}) - D(q_x|q_{e_0}) \begin{cases} H_0 & \geq T. \\ H_1 & < T. \end{cases}
\]  

(6.1)

A facetious answer is, "Why?" Although this structure imitates the LRT and intuitive
arguments may suggest its application, we don’t have the right tools to analyze it. In all of our simulations, we found the ELRT’s performance (not shown) to be very close to the Gutman detector, which is based on the GLRT. But the strong theoretical results for the Gutman detector provide us many reasons to use it and reasons for caution when using other detectors. If the ELRT detector is used, we have to accept the possibility that there will be situations in which its performance plummets.

While we have the satisfaction of showing the universality and optimality of the forced-decision GLRT detector, stronger results have escaped us. Although we can show the optimality of the forced-decision detector over any other forced-decision detector constructed from a one-sided universal test, we cannot make comparison with others, e.g., the ELRT. In addition, the geometry that almost certainly must be behind empirically based classification – as it is in classical hypothesis testing with such concepts as the geodesic of tilted measures and the minimizing properties of the KL distance – is outside our grasp. These extensions would significantly add to the current body of knowledge.
Appendix A

Error Behavior Under Alternate Hypothesis

Let $\Omega = (\Omega_0, \Omega_1)$ be a decision rule based solely on types, that is, determined by sufficient statistics $q_X, q_{t_0},$ and $q_{t_1}$. Let this test possess a strictly exponential error rate of $\lambda$ under $H_0$ for any $P_0$ and $P_1$, as defined in (2.48). Then,

$$2^{-n\lambda} \geq \sum_{x, t_0, t_1 \in \Omega_1} P_0(x) P_0(t_0) P_1(t_1)$$

$$= \sum_{q_X, q_{t_0}, q_{t_1} \subset \Omega_1} P_0(q_X) P_0(q_{t_0}) P_1(q_{t_1})$$

$$\geq \sum_{q_X, q_{t_0}, q_{t_1} \subset \Omega_1} (n+1)^{-|A|2^{-nD(q_X\|P_0)}} \cdot (N+1)^{-2|A|2^{-nD(q_{t_0}\|P_0)}} \cdot 2^{-ND(q_{t_1}\|P_1)})$$

(A.1)

where the last inequality is due to the lower bound on the probability of a type class:

$$Q^n(q_X) \geq (n+1)^{-|A|2^{-nD(q_X\|Q)}},$$

(A.2)

for some measure $Q$. For any $x, t_0, t_1 \in \Omega_1$, therefore, the inequality (A.1) implies

$$2^{-n\lambda} \geq (n+1)^{-|A|2^{-nD(q_X\|P_0)}} \cdot (N+1)^{-|A|2^{-nD(q_{t_0}\|P_0)}} \cdot (N+1)^{-|A|2^{-nD(q_{t_1}\|P_1)}},$$

(A.3)

for any $P_0$ and $P_1$. In particular, this relationship holds for

$$P_0 = q_{A,t_0} \quad \text{and} \quad P_1 = q_{t_1}.$$  \hspace{1cm} (A.4)

Hence,

$$2^{-n\lambda} \geq (n+1)^{-|A|2^{-nD(q_X\|q_{A,t_0})}} \cdot (N+1)^{-|A|2^{-nD(q_{t_0}\|q_X,t_0)}} \cdot (N+1)^{-|A|},$$

(A.5)

which implies

$$\lambda \leq D(q_X\|q_{A,t_0}) + \frac{N}{n} D(q_{t_0}\|q_{A,t_0}) + \frac{|A|}{n} (\log(n+1) + 2 \log(N+1)),$$

(A.6)
for any \( x, t_0, t_1 \in \Omega_1 \). Recall that Gutman’s acceptance region, \( \Lambda_1 \), is defined by

\[
\Lambda_1 = \{ x, t_0, t_1 : D(q_x\|q_{y_0}) + \frac{N}{n} D(q_{t_0}\|q_{y_2}) + \rho(n, N) \geq \lambda \}. \tag{A.7}
\]

If we make the polynomial term \( \rho(n, N) \) to be large, such that

\[
\rho(n, N) \geq \frac{|A|}{n} (\log(n + 1) + 2 \log(N + 1)), \tag{A.8}
\]

then,

\[
\Omega_1 \subset \Lambda_1, \tag{A.9}
\]

which means that \( \Omega_0 \supset \Lambda_0 \) and, in particular,

\[
P_{\Lambda}(e|H_1) \leq P_\Omega(e|H_1). \tag{A.10}
\]

Hence, the acceptance region for \( H_0 \) of any type-based test that satisfies the strictly exponential error rate of \( \lambda \) under \( H_0 \) (2.48) must be strictly larger than the \( H_0 \) acceptance region of Gutman’s test, provided that \( \rho(n, N) \) is taken large enough, as required by (A.8). This is what Gutman has shown [14, proof, Thm. 1].

Recall from (2.60) that under \( H_0 \), the Gutman test’s error probability rate at finite sample sizes is:

\[
\frac{1}{n} \log P_{\Lambda}(e|H_0) \leq -\lambda + \frac{|A|}{n} (\log(n + 1) + \log(N + 1)) + \rho(n, N). \tag{A.11}
\]

Hence, we see how the term \( \rho(n, N) \) can “shape” the sizes of the decision regions. For example, if we make \( \rho(n, N) \) small,

\[
\rho(n, N) \leq -\frac{|A|}{n} (\log(n + 1) + \log(N + 1)), \tag{A.12}
\]

then the error rate of Gutman’s test under \( H_0 \) would satisfy the strictly exponential relationship defined in (2.48). However, with a small \( \rho(n, N) \), the relationships (A.9) and (A.10) would no longer hold. The \( H_1 \) error rate for Gutman’s test can nevertheless be shown to be the same as \( \Omega \) by noting that as \( \rho(n, N) \) converges to zero, \( \Omega_1 \) converges to \( \Lambda_1 \).
We now derive the second re-statement of Gutman's Theorem 1. Let \( \Omega \) be a type-based test such that

\[
\frac{1}{n} \log P_{\Omega}(e|H_0) \leq -\lambda + \xi(n, N),
\]

where \( \xi(n, N) \) is known and has size

\[
\xi(n, N) = O\left(\frac{\log(n+1)}{n} + \frac{\log(N+1)}{n}\right).
\]

Using the same methods as before, we can show for all \( x, t_0, t_1 \in \Omega_1 \),

\[
\lambda \leq D(q_x||q_{x,t_0}) + \frac{N}{n} D(q_{t_0}||q_{x,t_0}) + \frac{|A|}{n} (\log(n+1) + 2 \log(N+1)) + \xi(n, N),
\]

which is identical in spirit to (A.6). Then, by taking \( \rho(n, N) \) large enough, relationships (A.9) and (A.10) once again hold.

In the last re-statement of Gutman's Theorem 1, we look at the relationship between an arbitrary test, \( \Omega \), not necessarily type-based, and Gutman's test. From \( \Omega \), a corresponding type-based rule \( \Omega' \) can be defined by the following rule, which was first introduced in Section 2.2.1: Let the triplet set of types, \( \{q_x, q_{t_0}, q_{t_1}\} \), belong to \( \Omega'_0 \) if at least half of the sequences in the type class of \( x, t_0, t_1 \) belong to \( \Omega_0 \); \( \Omega'_1 \) is defined similarly. In [14, proof, Lemma 2], it is shown that

\[
P_{\Omega'}(e|H_1) \leq P_{\Omega}(e|H_1) 2^{n(O(\frac{\log n}{n}) + O(\frac{\log N}{n}))},
\]

which was discussed in Section 2.2.1. The \( O(\frac{\log n}{n}) + O(\frac{\log N}{n}) \) terms are known. We have already shown that a type-based test with a known polynomial term in addition to the exponential rate satisfies the relationships (A.9) and (A.10). Since the error probabilities of \( \Omega' \) have at most a polynomial relationship to the error probabilities of \( \Omega \), the strict containment of \( \Lambda_0 \) in \( \Omega' \) implies that the error probability of \( \Lambda \) under \( H_1 \) has at most a polynomial relationship to that of \( \Omega \). Hence,

\[
\lim_{n \to \infty} \frac{1}{n} \log P_{\Lambda}(e|H_1) \leq \lim_{n \to \infty} \frac{1}{n} \log P_{\Omega}(e|H_1),
\]

and the last statement characterizing \( P_{\Lambda}(e|H_1) \) is complete.
Appendix B

GLRT of Multiple Portioned Observations

For simplicity, the case of two "portions," \( V = 2 \) is examined. The extension to general \( V \) is straightforward.

The generalized likelihood statistic testing \( x \) and \( t_i \) is:

\[
\frac{1}{n} \log \frac{\sup_P P(x)P(t_i)}{\sup_{P,Q} P(x)Q(t_i)}. \tag{B.1}
\]

To denote the different distributions of the two portions, the observations \( x, t_i \) are separated into \( x^1, x^2 \), and so on. Similarly, the measures \( P \) and \( Q \) need to be denoted by the separate measures:

\[
P(x) = P^1(x^1)P^2(x^2), \quad P(t_i) = P^1(t_i^1)P^2(t_i^2), \tag{B.2}
\]

and \( Q \) is defined similarly. Substituting the expressions (B.2) into (B.1) and separating the suprema, the generalized likelihood statistics can then be shown to be

\[
\frac{1}{n} \left( \log \frac{\sup_P P(x^1)P(t_i^1)}{\sup_{P,Q} P(x^1)Q(t_i^1)} + \log \frac{\sup_P P(x^2)P(t_i^2)}{\sup_{P,Q} P(x^2)Q(t_i^2)} \right), \tag{B.3}
\]

which equals

\[
\frac{1}{n} \left( n^1 h(x^1, t_i^1) + n^2 h(x^2, t_i^2) \right), \tag{B.4}
\]

where \( n^j \) is the number of samples in \( x \) that belong to portion \( j \). The extension to \( V \)-valued portions follows the same steps shown here and is straightforward.
Appendix C

Important Sampling for Discrete-Valued Sequences

For the biasing probability measure, $P^\alpha$, we use the exponential tilt probability measure, $P^\alpha$:

$$P^\alpha = \frac{P_0^\alpha P_1^{1-\alpha}}{\tilde{P}}, \quad \alpha \in [0, 1]$$  \hspace{1cm} (C.1)

where $\tilde{P}$ is the normalizing factor,

$$\tilde{P} = \sum_{u \in \mathcal{A}} P_0^\alpha(u)P_1^{1-\alpha}(u).$$  \hspace{1cm} (C.2)

The exponential tilt is chosen because for the likelihood-ratio test detector (LRT), it is the nearest probability measure in the acceptance region $\Lambda_j$ to probability measure $P_i$, $i \neq j$. This is due to the fact that the exponential tile $P^\alpha$ traverses the geodesic from $P_0$ to $P_1$ as $\alpha$ varies from 0 to 1 [6, Ch. 12].

In our BPSK communications scenario, we assume that the additive noise is symmetric about the origin and that the signaling is symmetric, i.e., $(+A, -A)$, where $A$ is the amplitude of the signal. If, in addition, we assume a quantizer that is symmetric about zero, then the induced probability measures $P_0$ and $P_1$ are “symmetric,” or mirror images of each other. Consider the induced alphabet to be enumerated from 1 to $|\mathcal{A}|$ in the order of the quantizer bins beginning from the most negative to the most positive: 1 corresponds to the most negative quantizer bin, 2 corresponds to the bin just to the right of bin 1, and $|\mathcal{A}|$ corresponds to the most positive quantizer bin. The symmetry assumptions on the noise density and signaling amplitude imply that

$$P_0(u) = P_1(|\mathcal{A}| + 1 - u), \quad u \in [1, \ldots, |\mathcal{A}|].$$  \hspace{1cm} (C.3)

For the likelihood-ratio detector optimizing the Bayesian total probability of error, the error exponent under each hypothesis must be equal, and the exponentially tilted
measure is one that equalsizes
\[
D(P^{\alpha*}\|P_0) = D(P^{\alpha*}\|P_1). \tag{C.4}
\]
As we discussed in Section 2.1.1, this KL distance is called the Chernoff information \(C(P_0, P_1)\), which can be defined as
\[
C(P_0, P_1) = -\log \sum_{u \in \mathcal{A}} P_0^\alpha(u) P_1^{1-\alpha}(u). \tag{C.5}
\]
The symmetry of \(P_0\) and \(P_1\) (C.3) implies that the KL distance of the exponential-tilt measure is symmetric, that is,
\[
D(P^\alpha\|P_0) = D(P^{1-\alpha}\|P_1). \tag{C.6}
\]
Since \(D(P^\alpha\|P_0)\) is strictly decreasing for increasing \(\alpha\), equation (C.6) means that the tilt satisfying (C.4) occurs at \(\alpha^* = 0.5\), and this tilted measure is used for the important-sampling biasing measure \(P^*\):
\[
P^* = \frac{P_0^{1/2} P_1^{1/2}}{\hat{P}}, \tag{C.7}
\]
where \(\hat{P}\) is defined in (C.2) with \(\alpha = \frac{1}{2}\).

The detector that we want test is the likelihood ratio detector optimizing the Bayesian error assuming the prior probability of \(H_0\) and \(H_1\) are equally likely:
\[
L(x) \equiv \frac{P_0(x)}{P_1(x)} \frac{H_0}{H_1} > 1, \quad x \in \mathcal{A}^n \tag{C.8}
\]
where \(x\) is a sequence of discrete-valued observations that we want to classify as \(H_0\) or \(H_1\).

Without loss of generality, we assume that the error probability under \(H_0\) is desired. The important sampling estimator requires a weight term \(W(x)\) to “normalize” the error estimator:
\[
\hat{p}_{IS} = \frac{1}{M_{IS}} \sum_{x(m) \in \Lambda_1} I(x(m) \in \Lambda_1) W(x(m)), \tag{C.9}
\]
where $M_{IS}$ is the number of "trials" and $x(m)$ is the observations from the $m$-th trial. It is well established that for $\hat{p}_{IS}$ to be an unbiased estimator of $p = \Pr(e|H_0)$, $W(x)$ is
\[ W(x) \equiv \frac{P_0^n(x)}{P^\ast_n(x)} = \prod_{i=1}^{n} \frac{P_0(x_i)}{P^\ast(x_i)}, \] (C.10)
where the superscript $n$ on the measures indicates that the observations are i.i.d. and the $n$-th order joint measure is simply a product measure. By using the exponentially tilted measure as $P^\ast$, we have
\[ W(x) = \prod_{i=1}^{n} \frac{P_0(x_i)}{\bar{P}^{1/2}P_0^{1/2}(x_i)\bar{P}^{1/2}(x_i)} = \bar{P}^{n/2} \prod_{i=1}^{n} \left( \frac{P_0(x_i)}{P_1(x_i)} \right)^{1/2} = \bar{P}^{n/2}(x). \] (C.11)
The variance of $\hat{p}_{IS}$ is
\[ \sigma^2_{IS} = E[\hat{p}^2_{IS}] - (E[\hat{p}_{IS}])^2 = E[W^2(x)I(x \in \Lambda_1)|x \sim P^\ast] - p^2. \] (C.12)
We start with the mean-squared term:
\[ E[W^2(x)I(x \in \Lambda_1)|x \sim P^\ast] = \sum_{x \in \Lambda_1}(\bar{P}^{n/2}(x))^2 P^\ast_n(x) = \bar{P}^{n} \sum_{x \in \Lambda_1} L^{1/2}(x)P_0(x), \] (C.13)
where the first equality results from the definition of $W(x)$ (C.11), and the second equality comes from the definition of $P^\ast$ (C.7). Noting that the likelihood-ratio $L(x) \leq 1$ when $x \in \Lambda_1$ yields
\[ E[W^2(x)I(x \in \Lambda_1)|x \sim P^\ast] \leq \bar{P}^{n} \sum_{x \in \Lambda_1} P_0(x) = \bar{P}^{n} \cdot \Pr(e|H_0) = \bar{P}^{n}p. \] (C.14)
Hence, the variance of the important-sampling estimator, $\sigma^2_{IS}$, is
\[ \sigma^2_{IS} \leq \bar{P}^{n}p - p^2 = p(\bar{P}^{n} - p). \] (C.15)
The gain, $\Gamma$, of the important-sampling estimator compared to the Monte-Carlo estimator is found by taking the ratio of $\sigma^2_{MC}$ and $\sigma^2_{IS}$:
\[ \Gamma = \frac{\sigma^2_{MC}}{\sigma^2_{IS}} \geq \frac{p(1-p)}{p(\bar{P}^{n} - p)} = \frac{1-p}{\bar{P}^{n} - p}, \] (C.16)
noting that $\sigma_{MC}^2 = p - p^2$. Now we make a few observations. First, by the definition of Chernoff information (C.5),

$$\bar{P}^n = \left( \sum_{u \in A} p_0^{\frac{1}{2}} p_1^{\frac{1}{2}} \right)^n = (2^{-\mathcal{C}(P_0, P_1)})^n = 2^{-n\mathcal{C}}.$$  \hspace{1cm} (C.17)

Second, according to (2.20), the probability of error, $p$, is upper-bounded by

$$p \leq 2^{-n\mathcal{C}} = \bar{P}^n,$$ \hspace{1cm} (C.18)

but that it has the same rate (Sanov’s Theorem),

$$p \doteq \bar{P}^n.$$ \hspace{1cm} (C.19)

Making the approximation that $1 - p \approx 1$ for small $p$, we write the important sampling gain as

$$\Gamma > \frac{1 - p}{\bar{P}^n - p} \approx \frac{1}{\bar{P}^n - p} > \frac{1}{\bar{P}^n} = 2^{n\mathcal{C}} \doteq p^{-1}.$$ \hspace{1cm} (C.20)

As a function of the number of observations, $n$, the important sampling gain increases exponentially at the rate $\mathcal{C}$. With respect to the error probability, $p$, which includes the effects of $n$ and $\mathcal{C}$, $\Gamma$ increases inversely with $p$. 
Appendix D

Chernoff Information with Equal-Powered Users

Define $p_0(x)$ as:

$$p_0(x) = \frac{1}{2^{K-1}} \sum_{c_2=\pm 1, \ldots, c_K=\pm 1} p_w \left( x - A_1 + \sum_{k=2}^{K} c_k A_k \right), \quad (D.1)$$

with $p_w(x)$ symmetric about zero. The alternate measure, $p_1(x)$, is then $p_1(x) = p_0(-x)$. This symmetry means that the Chernoff information is

$$C = -\log \int \sqrt{p_0(x)p_1(x)} \ dx .$$

Substituting for $p_0(x)$ and $p_1(x)$, we have:

$$C = -\log \int \frac{1}{2^{K-1}} \sqrt{\sum_{c_2=\pm 1, \ldots, c_K=\pm 1} p_w \left( x - A_1 + \sum_{k=2}^{K} c_k A_k \right)}$$

$$\cdot \sqrt{\sum_{c_2=\pm 1, \ldots, c_K=\pm 1} p_w \left( x + A_1 + \sum_{k=2}^{K} c_k A_k \right)} \ dx . \quad (D.3)$$

Note that the shift of each $p_w(\cdot)$, by $\sum c_k A_k$, is a binomial-like term controlled by $c_2, \ldots, c_K$. Recall that $A_i = A$, for all $i$. Define

$$J = (\sum c_k + K - 1)/2 = \{0, \ldots, K - 1\} \quad (D.4)$$

to parameterize the binomial-like term in the first summation. Similarly define $J'$ to parameterize the second summation. Rewrite (D.3), re-enumerating the summations over $c_2, \ldots, c_K$ by $J$ and $J'$:

$$C = -\log \int \frac{1}{2^{K-1}} \sqrt{\sum_{J=0, \ldots, K-1} \left( \begin{array}{c} K - 1 \\ J \end{array} \right) p_w(x + A(2J - K))}$$
\[ \sqrt{\sum_{J'=0,\ldots,K-1} \binom{K-1}{J'} p_w(x + A(2J' - K + 2)) \, dx}, \quad (D.5) \]

where we have used the fact that \( A_i = A \), for all \( i \). When \( A \) is very large, larger than the effective support of \( p_w(x) \), the shifted \( p_w(x) \) terms become very distinct at the shifted locations. In this case, the square root may be brought inside the summations because the terms inside each summation have essentially no overlap. In addition, the product of the two summations, when integrated, has non-zero values only when

\[ J' = J - 1 = \{1, 2, \ldots, K - 1\}, \quad (D.6) \]

because for other \( J' \) and \( J \), the shifted densities have no overlap. Because \( p_w(x) \) is symmetric, the integrated value is actually 1, i.e.,

\[ \int \sqrt{p_w(x + A(2J - K)) \cdot p_w(x + A(2J' - K + 2))} \, dx = 1, \quad (D.7) \]

for \( J' \) and \( J \) satisfying (D.6). Hence, we can rewrite (D.5) as

\[ \mathcal{C} = -\log \frac{1}{2K-1} \sum_{J'=1,\ldots,K-1} \sqrt{\binom{K-1}{J'-1} \binom{K-1}{J'}}. \quad (D.8) \]

Separating the log of the product then results in equation (5.6), as claimed in Section 5.2.
Bibliography


