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Optimal Combat Maneuvers of a
Next-Generation Jet Fighter Aircraft

by

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A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

Doctor of Philosophy

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Abstract

Optimal Combat Maneuvers of a Next-Generation Jet Fighter Aircraft

by

James B. Dabney

This thesis deals with the optimization of four classes of combat maneuvers for a next-generation jet fighter aircraft: climb maneuvers, fly-to-point maneuvers, pop-up attack maneuvers, and dive recovery maneuvers. For the first three classes of maneuvers, the optimization criterion is the minimization of the flight time, resulting in a Mayer-Bolza problem of optimal control; for the fourth class, the optimization criterion is the minimization of the maximum altitude loss during dive recovery, resulting in a Chebyshev problem of optimal control. Each class of problems is solved using the sequential gradient-restoration algorithm for optimal control.

Among the four classes of combat maneuvers studied, only dive recovery benefits from the ability of a next-generation fighter aircraft to maneuver at extremely high angles of attack. For the other three classes, relatively low angles of attack are required.

The optimal climb trajectories are characterized by three distinct segments: a central segment often flown with a load factor of nearly 1 and two terminal segments (dive or zoom) to and from the central segment. The central and final segments are nearly independent of the initial conditions, instead being dominated by the final conditions.
The optimal fly-to-point trajectories consist of three segments: turning, characterized by relatively high load factor; level acceleration at maximum thrust; and finally, resumption of steady-state cruising. The effects of the heading change magnitude and the load factor limit are discussed.

The optimal pop-up trajectories consist of three segments flown at maximum power: pitch-up, zoom, and pitch-down. The effects of using the afterburner, heading change magnitude, and dive angle magnitude are discussed.

The optimal dive recovery trajectories consist of one to three segments, depending on initial speed and flight path angle. All the optimal trajectories conclude with a pitch-up at the maximum available load factor. For very low initial speed, the pitch-up is preceded by a brief supermaneuver segment. For very low initial speed coupled with very high initial flight path angle, the supermaneuver segment is preceded by a dive initiation segment.

The optimal trajectories reported here serve two purposes. First, they can benefit aircraft designers by highlighting those flight characteristics that are most beneficial in combat. Second, they can benefit aircraft pilots as the basis for guidance trajectories that approximate the optimal trajectories.

**Key Words.** Combat climb maneuvers, fly-to-point maneuvers, pop-up attack maneuvers, dive recovery maneuvers, supermaneuvers, jet fighter aircraft, supermaneuvering jet fighter aircraft, flight mechanics, optimal trajectories, calculus of variations, optimal control.
Acknowledgments

I would like to express my sincere gratitude to Prof. Angelo Miele for his encouragement, advice, and guidance throughout my time at Rice University, and particularly throughout the period of thesis research. He has shared generously his time and his unique understanding of flight mechanics and optimal control.

I would also like to express my gratitude to the other members of my thesis committee: Prof. Pol Spanos and Prof. John Clark. I have enjoyed working with all the members of the Aero-Astronautics Group: Salvatore Mancuso, Chi Sun Chao, Tong Wang, and Julia Tang.
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1. Introduction

Over the past 25 years, there have been a number of studies of optimal climbing, turning, and evasive maneuvers of jet fighter aircraft. These studies have investigated time-optimal climbs and turns for both conventional and supermaneuvering aircraft (Refs. 1-2), as well as optimal evasive maneuvers (Refs. 2-7). Flight dynamics studies of optimal climbs and turns are important for two reasons. First, understanding the nature of the optimal climbing, turning, and evasive maneuvers provides useful insight to aircraft designers, allowing them to place greater emphasis on those aircraft characteristics that will prove most beneficial in combat. Second, understanding the nature of the optimal climbing, turning, and evasive maneuvers will benefit aircraft pilots, providing them advantages in combat.

This thesis deals with four classes of combat maneuvers for a next-generation fighter aircraft: climb maneuvers, fly-to-point maneuvers, pop-up attack maneuvers, and dive recovery maneuvers. For the first three classes of maneuvers, the optimization criterion is the minimization of the flight time, resulting in a Mayer-Bolza problem of optimal control; for the fourth class, the optimization criterion is the minimization of the maximum altitude loss during dive recovery, resulting in a Chebyshev problem of optimal control.

The first step which must be taken in this study is to develop a mathematical model of the jet fighter aircraft. The model should account for all factors necessary to fully characterize aircraft performance. The model used here extends the models used in Refs. 2, 8, 9, allowing flight studies to include speeds from low subsonic (Mach number less than
0.1) to moderate supersonic (Mach number approaching 2.0), altitudes from sea level to 82 kft, angles of attack from 0 to 90 deg, and parametrized wing loading and thrust-to-weight ratios.

A climb is a maneuver in which an aircraft flies to high altitude starting from low altitude. The aircraft must arrive at high altitude in level flight and at an airspeed such that level flight can be maintained. These maneuvers can be useful in intercepting high-altitude aircraft; they can also be useful in allowing an aircraft to climb above enemy surface defenses.

A fly-to-point is a maneuver performed when an aircraft reaches a turn point on a low-altitude, high-speed flight. This maneuver consists of a turn toward the next specified turn point, followed by flight to the next specified turn point. The objective is to arrive at the next turn point in minimum time without excessive deviation from the initial altitude and without exceeding a specified airspeed.

A pop-up attack is a maneuver in which an aircraft flying at low altitude gets ready to perform a dive-bombing attack against a surface target. The maneuver begins in level flight at low altitude, and ends with the aircraft diving toward the target at a specified altitude, airspeed, and dive angle. The objective is to complete the maneuver in minimum time.

A dive recovery is a maneuver in which an aircraft is piloted so as to return to level flight at an airspeed such that level flight can be maintained. The dive recovery maneuver can start with the aircraft in a steep climb or in level flight at an airspeed too low to maintain level flight, or in a dive at any airspeed. The objective of the dive recovery maneuver is to minimize the maximum altitude loss during recovery.
For each class of problem, the optimization is done using the sequential gradient-restoration algorithm for optimal control problems (SGRA, Refs. 10 - 11), developed and perfected by the Aero-Astronautics Group of Rice University. SGRA is a robust algorithm which has been employed with success to solve a wide variety of aerospace problems (Refs. 12 - 29) including interplanetary trajectories (Ref. 12), flight in a windshear (Refs. 13 - 16), aerospace plane trajectories (Refs. 17 - 19), aeroassisted orbital transfer trajectories (Refs. 20 - 26), and ascent trajectories for next-generation single-stage-to-orbit and two-stage-to-orbit spacecraft (Refs. 27 - 29).

This thesis is organized as follows. In Section 2, we present the system description including the differential system, aerodynamic model, atmospheric model, constraints, and supplementary data. In Section 3, we present the study of optimal climb maneuvers, and discuss the results in light of the classical theory of optimal climbs of jet aircraft. In Section 4, we present the study of optimal low altitude quasi-level fly-to-point maneuvers. In Section 5, we present the study of optimal pop-up attack maneuvers. In Section 6, we present the study of optimal dive recovery maneuvers. In Section 7, we present the conclusions and the implications for aircraft design and operation. Finally, in Section 8 (Appendix), we present a summary of the sequential gradient-restoration algorithm for optimal control.
2. System Description

The first step which must be taken in order to study optimal aircraft climbs and turns is to develop a mathematical model of the jet fighter aircraft. The model should account for all factors necessary to fully characterize the aircraft performance. This section describes a fighter aircraft flight dynamics model that extends the models used in Refs. 2, 8, 9, allowing flight studies to include speeds from low subsonic (Mach number less than 0.1) to moderate supersonic (Mach number approaching 2.0), altitudes from sea level to 82 kft, angles of attack from 0 to 90 deg, and parametrized wing loading and thrust-to-weight ratios.

We make the following assumptions: (A1) flight takes place at altitudes from sea level to 82 kft; (A2) flight takes place over a flat Earth; (A3) there is no wind; (A4) aircraft weight change during a maneuver is neglected; (A5) aircraft is controlled via angle of attack, angle of bank, and power setting; (A6) power setting can be adjusted continuously from zero thrust to maximum thrust; (A7) thrust is directed along the aircraft reference line (thrust vectoring is not considered); hence, thrust angle of attack is the same as aero-dynamic angle of attack.

2.1. Differential System. With the above assumptions, the motion of the aircraft is described by the following differential system for the along-track position $x$, cross-track position $y$, altitude $h$, velocity $V$, flight path angle $\gamma$, and velocity heading angle $\chi$:

\begin{align}
\dot{x} &= V \cos \gamma \cos \chi, \\
\dot{y} &= V \cos \gamma \sin \chi,
\end{align}

(1a) (1b)
\[ h = V \sin \gamma, \quad (1c) \]
\[ \dot{V} = (T \cos \alpha - D - W \sin \gamma) / m, \quad (1d) \]
\[ \dot{\gamma} = [(L + T \sin \alpha \cos \mu - W \cos \gamma) / m V, \quad (1e) \]
\[ \dot{\chi} = (L + T \sin \alpha \sin \mu / m V \cos \gamma, \quad (1f) \]
in which the dot denotes derivative with respect to time \( t \). Here, \( 0 \leq t \leq \tau \), where \( \tau \) is the final time. The quantities appearing on the right-hand side of (1) are the thrust \( T \), drag \( D \), lift \( L \), weight \( W \), mass \( m \), angle of attack \( \alpha \), and bank angle \( \mu \).

2.2. Functional Relations. In the system (1), the weight \( W \) is assumed to be constant,
\[ W = mg_e, \quad (2) \]
which is the product of the constant aircraft mass \( m \) and the sea level acceleration of gravity \( g_e \). Generally speaking, the aerodynamic forces can be represented by the functional relations
\[ D = D(h, V, \alpha), \quad (3a) \]
\[ L = L(h, V, \alpha), \quad (3b) \]
and the engine thrust is described by the functional relation
\[ T = T(h, V, \beta), \quad (4) \]
where \( \beta \) is the power setting.

2.3. Inequality Constraints. Inspection of the system (1) in light of (2) - (4) shows that the time history of the state \( x(t), y(t), h(t), V(t), \gamma(t), \chi(t) \) can be computed by
forward integration for given initial conditions, given controls \( \alpha(t), \mu(t), \beta(t) \), and given final time \( \tau \). In turn, the controls are subject to the two-sided inequality constraints

\[
0 \leq \alpha \leq \pi/2 \quad \text{or} \quad 0 \leq \alpha \leq \alpha_*,
\]

\[
0 \leq \beta \leq 1,
\]

which must be satisfied everywhere along the interval of integration. These inequality constraints can be accounted for via the trigonometric transformations

\[
\alpha = (\alpha_{\text{max}}/2)(1 + \sin u_1),
\]

\[
\beta = (1/2)(1 + \sin u_2),
\]

where \( u_1(t) \) and \( u_2(t) \) denote the auxiliary controls. In Eq. (6a), the upper bound \( \alpha_{\text{max}} \) is given by

\[
\alpha_{\text{max}} = \pi/2 \quad \text{or} \quad \alpha_{\text{max}} = \alpha_* = 0.436 \text{ rad.}
\]

In (6c), the first value applies if the aircraft is allowed to operate in the supermaneuver range; the second value applies if the aircraft is limited to angles of attack below that for maximum lift coefficient.

2.4. Atmospheric Model. The atmospheric temperature \( \Theta(h) \) and density \( \rho(h) \) are computed using approximations to the 1962 U.S. Standard Atmosphere (Ref. 30) as described in Ref. 8. These approximations are valid from the surface to 82.021 kft. The atmospheric temperature is in turn used to compute the speed of sound \( a(h) \).

In the troposphere \( (0 \leq h \leq 36.089 \text{ kft}) \), the atmospheric temperature is approximated as

\[
\Theta = \Theta_0 - Ch.
\]
where $\theta_0 = 518.69 ^\circ R$ is the sea level reference temperature and $C = 3.566 \, ^\circ R/\text{kft}$ is the rate of decrease of temperature with respect to altitude. The atmospheric density is approximated as

$$
\rho = \rho_0 (\theta/\theta_0)^{g_e/C R^{-1}} = \rho_0 (\theta/\theta_0)^{4.256},
$$

(8)

where $\rho_0 = 0.002377 \, \text{lb sec}^2/\text{ft}^4$ is the sea-level density, $g_e = 32.174 \, \text{ft/sec}^2$ is the sea-level acceleration of gravity, and $R = 1716.5 \, \text{ft}^2/\text{sec}^2 \, ^\circ R$ is the gas constant for air.

In the stratosphere ($36.089 \, \text{kft} \leq h \leq 82.021 \, \text{kft}$), the atmospheric temperature is assumed to be constant, $\theta = 389.99 \, ^\circ R$. The atmospheric density is approximated as

$$
\rho = \rho_* \exp[(h - h_*)/h_{\text{ref}}],
$$

(9)

where $\rho_* = 0.0007062 \, \text{lb sec}^2/\text{ft}^4$ is the density at the tropopause altitude $h_* = 36.089 \, \text{kft}$, and $h_{\text{ref}} = 21.0 \, \text{kft}$ is a reference altitude.

The speed of sound $a(h)$ is computed using the ideal gas approximation

$$
a = \sqrt{kR\theta},
$$

(10)

where $k = 1.4$ is the ratio of specific heats.

2.5. **Aerodynamic Forces.** The aerodynamic forces (3) are given by

$$
L = (1/2) C_L \rho(h) S V^2,
$$

(11a)

$$
D = (1/2) C_D \rho(h) S V^2,
$$

(11b)

where $C_L$ is the lift coefficient, $C_D$ the drag coefficient, and $S$ a reference surface area. The aerodynamic coefficients represent a hypothetical fighter aircraft based on the data of Ref. 2 and the typical fighter aircraft characteristics given in Ref. 9. The coefficients used here
account for flight at angles of attack up to 90 deg, and also take into account the dependence on Mach number of the maximum lift coefficient.

In the flight regimes of interest, the lift coefficient $C_L$ can be represented by the functional relation

$$C_L = C_L(\alpha, M),$$  

(12)

where $M$ is the Mach number. Here, the relation (12) is approximated as the product of two functions, one depending on angle of attack and one depending on Mach number,

$$C_L = F_{L1}(\alpha)F_{L2}(M).$$  

(13)

Here, $F_{L1}$ is defined by the polynomial approximation (Fig. 1)

$$F_{L1} = a_0 + a_1 \eta + a_2 \eta^2 + a_3 \eta^3 + a_4 \eta^4 + a_5 \eta^5 + a_6 \eta^6 + a_7 \eta^7, \quad \eta = 2\alpha / \pi. \quad (14)$$

$F_{L2}$ is defined to be the maximum lift coefficient at a given Mach number and the approximation used is (Fig. 2)

$$F_{L2} = b_1 \tanh[b_2(b_3 - M)] + b_4.$$  

(15)

Figure 3 shows the dependence of the lift coefficient computed via (13) - (15) on angle of attack and Mach number.

The drag coefficient $C_D$ can be represented by the functional relation

$$C_D = C_D(\alpha, M).$$  

(16)

To cover both conventional aircraft and supermaneuvering aircraft, the following particular form of Eq. (16) is assumed:

$$C_D = F_{D0}(M) + F_{D1}(\alpha)F_{D2}(M).$$  

(17)
where $F_{D0}(M)$ is the zero-lift drag coefficient and the product $F_{D1}(\alpha) F_{D2}(M)$ is the induced drag coefficient. $F_{D0}$ is taken to be (Fig. 4)

$$F_{D0} = d_1 \left[ \tanh d_2 (M + d_3) \right] + d_4 \tanh (d_5 (M + d_6)) + d_7. \quad (18)$$

$F_{D1}$ is the angle of attack component of the induced drag coefficient and is represented as the polynomial (Fig. 5)

$$F_{D1} = e_0 + e_1 \eta + e_2 \eta^2 + e_3 \eta^3 + e_4 \eta^4, \quad \eta = 2\alpha/\pi. \quad (19)$$

$F_{D2}$ is the Mach number component of the induced drag coefficient and is taken to be (Fig 5)

$$F_{D2} = f_0 + f_1 M + f_2 M^2. \quad (20)$$

Figure 7 shows the dependence of the drag coefficient computed via (17) - (20) on angle of attack and Mach number. The main comment on Eqs. (17) - (20) is that they produce reasonable values for the total drag coefficient for a supermaneuvering fighter aircraft throughout its flight envelope, while at the same time producing reasonable values of the total drag coefficient for a conventional fighter aircraft at angles of attack less than the angle of attack for maximum lift coefficient.

2.6. Engine Model. The engine thrust approximation considers two regions: troposphere and stratosphere. In the troposphere, the engine thrust relation (4) is approximated as

$$T = \beta T_{ref} F_T(h, M), \quad (21)$$

where $\beta$ is the power setting, $T_{ref}$ is the maximum sea-level static thrust (a parameter of the problem), and $F_T(h, M)$ is a normalized thrust function. $F_T$ is computed using polyno-
mial approximations to the ratio $T/T_{\text{ref}}$ for the J85 turbojet engine. The curve fit is implemented in two stages. First, the normalized thrust function is computed using polynomials of degree three in the Mach number fit to the data from Ref. 31 for altitudes $h_0 = 0$, $h_1 = 15$ kft, $h_2 = 25$ kft, and $h_3 = 36$ kft (Fig. 8):

\begin{align*}
F_T(h_0, M) &= g_{00} + g_{01} M + g_{02} M^2 + g_{03} M^3, \quad (22a) \\
F_T(h_1, M) &= g_{10} + g_{11} M + g_{12} M^2 + g_{13} M^3, \quad (22b) \\
F_T(h_2, M) &= g_{20} + g_{21} M + g_{22} M^2 + g_{23} M^3, \quad (22c) \\
F_T(h_3, M) &= g_{30} + g_{31} M + g_{32} M^2 + g_{33} M^3. \quad (22d)
\end{align*}

Second, for a particular Mach number and starting from (21), the value of $F_T(h, M)$ is computed via Lagrange interpolation with respect to altitude; therefore,

\begin{align*}
F_T(h, M) &= F_T(h_0, M)(h - h_1)(h - h_2)(h - h_3)/(h_0 - h_1)(h_0 - h_2)(h_0 - h_3) \\
&\quad + F_T(h_1, M)(h - h_0)(h - h_2)(h - h_3)/(h_1 - h_0)(h_1 - h_2)(h_1 - h_3) \\
&\quad + F_T(h_2, M)(h - h_0)(h - h_1)(h - h_3)/(h_2 - h_0)(h_2 - h_1)(h_2 - h_3) \\
&\quad + F_T(h_3, M)(h - h_0)(h - h_1)(h - h_2)/(h_3 - h_0)(h_3 - h_1)(h_3 - h_2). \quad (23)
\end{align*}

In the stratosphere, Eq. (21) is retained and we use the approximation (Ref. 8)

\begin{equation}
F_T(h, M) = (\rho/\rho_*)F_T(h_*, M), \quad (24)
\end{equation}

where $\rho_* = 0.0007062 \text{ lb sec}^2/\text{ft}^4$, $h_* = 36.089$ kft; see Fig. 8, where $h_4 = 45$ kft and $h_5 = 55$ kft. Figure 9 shows the dependence of the normalized thrust function $F_T$ computed via (21) - (24) on Mach number and altitude.
2.7. **Path Constraint.** To avoid structural damage to the aircraft, the aerodynamic lift is constrained according to the relation

\[ L + T \sin \alpha \leq n_{\text{max}} W, \]  

(25)

where \( n_{\text{max}} \) is the so-called structural load factor limit. This constraint can be imposed either through the use of a penalty functional as in Ref. 27 or through a path inequality constraint via a slack variable transformation as described in Refs. 10 - 11. Here, we use the slack variable transformation

\[ n_{\text{max}} W - L - T \sin \alpha - u_3^2 = 0, \]  

(26)

with \( u_3(t) \) denoting an auxiliary control. We recall that the lift depends on angle of attack via the lift coefficient (12) and that the thrust depends on power setting via (21). Because the original controls \( \alpha(t) \) and \( \beta(t) \) are related to the auxiliary controls \( u_1(t) \) and \( u_2(t) \) via the trigonometric transformations (6), one must view Eq. (26) as a nondifferential relation between the auxiliary controls \( u_1(t), u_2(t), u_3(t) \). Among these, only two can be regarded as independent controls; the third one is automatically determined via (26).

2.8. **Boundary Conditions.** The initial conditions \( (t=0, \text{subscript i}) \) are inputs to the optimization problem. In all cases, the initial velocity heading angle is assumed to be \( \chi_i = 0 \). The final conditions \( (t = \tau, \text{subscript f}) \) are also inputs to the optimization problem. For more details, see Sections 3 - 6.

2.9. **Curve Fit Coefficients.** The numerical coefficients appearing in Eqs. (14) - (15), (18) - (20), and (22) - (23) have the following values:

\[ a_0 = 0.000, \quad a_1 = 4.371, \quad a_2 = 16.80, \quad a_3 = -115.7, \]  

(27a)
\[ a_4 = 172.6, \quad a_5 = -19.84, \quad a_6 = -125.4, \quad a_7 = 67.23; \]  
\[ b_1 = 0.300, \quad b_2 = 6.000, \quad b_3 = 0.800, \quad b_4 = 0.900; \]  
\[ d_1 = 0.004, \quad d_2 = 5.500, \quad d_3 = -1.000, \quad d_4 = 0.002, \]  
\[ d_5 = 3.500, \quad d_6 = -1.500, \quad d_7 = 0.007; \]  
\[ e_0 = 0.000, \quad e_1 = 1.023, \quad e_2 = 6.468, \quad e_3 = -11.36, \]  
\[ e_4 = 5.246; \]  
\[ f_0 = 0.416, \quad f_1 = 0.006, \quad f_2 = 0.020; \]  
\[ g_{00} = 0.983, \quad g_{01} = -0.096, \quad g_{02} = 0.424, \quad g_{03} = -0.143, \]  
\[ g_{10} = 0.615, \quad g_{11} = -0.139, \quad g_{12} = 0.596, \quad g_{13} = -0.192, \]  
\[ g_{20} = 0.424, \quad g_{21} = -0.248, \quad g_{22} = 0.703, \quad g_{23} = -0.201, \]  
\[ g_{30} = 0.317, \quad g_{31} = -0.392, \quad g_{32} = 0.693, \quad g_{33} = -0.177. \]  

2.10. Supplementary Data. The following data have been used in the numerical experiments:

- reference surface area \( S = 237.0 \text{ ft}^2 \),  
  \[ (33a) \]

- aircraft nominal weight \( W = 10 \text{ klb, 12 klb, 14 klb} \),  
  \[ (33b) \]

- load factor limit \( n_{\text{max}} = 6 \) or \( 9 \),  
  \[ (33c) \]

- maximum engine thrust with afterburning \( T_{\text{ref}} = 12.0 \text{ klb} \),  
  \[ (33d) \]

- maximum engine thrust without afterburning \( T_{\text{ref}} = 6.8 \text{ klb} \),  
  \[ (33e) \]

- sea level acceleration of gravity \( g_e = 32.174 \text{ ft/sec}^2 \).  
  \[ (33f) \]
3. Climb Maneuvers

Time-optimal climbs are important combat maneuvers for jet fighter aircraft. For example, the interception of a high-altitude aircraft requires the interceptor to climb to an appropriate altitude as rapidly as practical, and to arrive at that altitude in a state consistent with the capability to maintain level flight. As another example, it may be desirable for an aircraft, after expending ordnance against a ground target, to rapidly climb above enemy defenses, and to arrive at higher altitude with a speed such that the ability to maneuver evasively is preserved. As a consequence of the importance of time-optimal climbs of jet fighter aircraft, several studies of these trajectories have been reported over the last fifty years (Refs. 32 - 42).

This section describes a study of time-optimal climbs for a jet fighter aircraft capable of maneuvering at high angles of attack. The work described here extends the results of earlier studies in two ways. First, the maneuvers are studied for a hypothetical aircraft exhibiting flight characteristics similar to those expected of next-generation jet fighter aircraft, in particular the ability to fly at angles of attack greater than that for maximum lift coefficient, even angles of attack approaching 90 deg. Second, the maneuvers studied here represent feasible combat trajectories, whereas earlier studies did not require the aircraft to complete the maneuver in level flight and generally involved climbs to extremely high altitudes, much higher than those at which jet fighters typically operate.

The optimal trajectories computed here serve several purposes. First, they provide insight into the issue of the actual usefulness of the ability to maneuver at extreme angles of attack. Second, they can serve as the basis for guidance trajectories to approximate the
optimal climb. Finally, they illustrate the advantages accruing in certain situations via three-dimensional maneuvering, rather than via the purely vertical maneuvering studied in Refs. 32 - 42.

3.1. Optimization Problem. The minimum-time climb problem (Problem P1) can be formulated as follows:

$$\min I = \tau,$$  \hspace{1cm} (34)

where $\tau$ is the final time. The unknowns include the original state variables $x(t), y(t), h(t), V(t), \gamma(t), \chi(t)$, the original control variable $\mu(t)$, the auxiliary control variables $u_1(t), u_2(t), u_3(t)$, and the parameter $\tau$. With the solution known, the original control variables $\alpha(t), \beta(t)$ can be recovered from $u_1(t), u_2(t)$ via (6).

Time Normalization. In treating problem (34), it is convenient to normalize the dimensional time $t$ via the transformation

$$\theta = t/\tau,$$  \hspace{1cm} (35)

where $\theta$ is the dimensionless time and $\tau$ the final time. Clearly,

$$0 \leq \theta \leq 1, \text{ if } 0 \leq t \leq \tau.$$  \hspace{1cm} (36)

Approach. The optimization problem under consideration is of the Mayer type and can be solved via the sequential gradient-restoration algorithm for optimal control problems (SGRA, Refs. 10 - 11). The approach taken is to produce manually a nominal control history that results in a solution of the differential system (1), satisfying the specified initial conditions and the path constraint (26), and approximately satisfying the final conditions. This solution is used as the starting nominal trajectory for SGRA; this algo-
rithm produces a sequence of feasible trajectories, each characterized by a lower value of performance index $I$ than the previous feasible trajectory. The algorithm terminates when the mean square error in the optimality conditions becomes smaller than a preselected small positive number.

The trajectories studied fall into two groups: (a) three-dimensional trajectories and (b) two-dimensional trajectories flown in a vertical plane. In group (a), the nominal roll angle starts at a small positive value, and then increases smoothly to a value slightly less than 180 deg; subsequently, SGRA adjusts the roll angle while solving the optimization problem. In group (b), the nominal roll angle is set to zero throughout the trajectory; this results in the roll angle remaining at zero throughout the sequence of iterations.

3.2. **Computer Runs.** In this study, 54 optimum climb trajectories were computed (Trajectories A1 - A54), half of group (a), roll permitted (Trajectories A1 - A27), and half of group (b), roll excluded (Trajectories A28 - A54). For each group, 27 optimal climb trajectories were obtained by considering all possible combinations of the following parameter values:

$$W = 10, 12, 14 \text{ klb},$$

$$V(0) = 0.5, 1.0, 1.5 \text{ kft/sec},$$

$$V(\tau) = 0.5, 1.0, 1.5 \text{ kft/sec}.$$  \hspace{1cm} (37a) \hspace{1cm} (37b) \hspace{1cm} (37c)

For each run, the following initial conditions were assumed:

$$x(0) = 0 \text{ kft}, \quad y(0) = 0 \text{ kft}, \quad h(0) = 5 \text{ kft},$$

$$\gamma(0) = 0, \quad \chi(0) = 0.$$  \hspace{1cm} (38a) \hspace{1cm} (38b)

Together with the following final conditions:
\[ h(\tau) = 35 \text{ kft}, \quad \gamma(\tau) = 0, \quad \chi(\tau) = 0. \quad (39) \]

State variables which are not prescribed must be considered as free and must be determined from the optimization process together with the main unknown, the final time \( \tau \).

The computations via SGRA were done with an Intel Pentium II processor using the C language and double-precision arithmetic.

For convenience, Tables 1 - 2 present a summary of the trajectory parameters identifying Trajectories A1 - A54 [see (37)]. For the purpose of illustration, Trajectories A3, A10 - A18, A30, A40 - A42, and A52 are plotted in Figs. 10 - 24 along with a tabulation of the key parameters, data, and results.
Table 1. Optimal climb trajectories of group (a).

<table>
<thead>
<tr>
<th>Trajectory</th>
<th>$W$ [klb]</th>
<th>$V(0)$ [kft/sec]</th>
<th>$V(\tau)$ [kft/sec]</th>
<th>Remark</th>
<th>$\tau$ [sec]</th>
</tr>
</thead>
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Table 2. Optimal climb trajectories of group (b).

<table>
<thead>
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<th>Trajectory</th>
<th>$W$ [klb]</th>
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<th>$V(\tau)$ [kft/sec]</th>
<th>Remark</th>
<th>$\tau$ [sec]</th>
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</table>
3.3. **Analysis.** In this section, we discuss the computed optimal trajectories in light of the classical theory of optimal climb trajectories for turbojet aircraft (Refs. 32 - 36). Inspecting Trajectories A1 - A54, it is evident that each trajectory consists of three segments. The central segment is a relatively long duration climb. The initial segment consists of transferring the aircraft from the initial conditions to the central segment; the final segment is a low angle of attack zoom maneuver leading from the central segment to the specified final boundary conditions (level flight at a predetermined altitude and speed). In certain trajectories of group (a), the zoom maneuver is performed with sufficient excess kinetic energy so as to permit a final turn in a nearly-vertical plane, with relatively high load factor, leading to the desired level flight condition.

**Classical Theory of Optimal Climbs.** It is of interest to compare the trajectories produced in this study with the classical theory of optimal climbs of jet aircraft flying in a vertical plane (Refs. 32 - 36). As developed in Ref. 33, the classical theory is based on the assumptions that (i) the thrust is nearly tangent to the flight path and (ii) the drag can be computed approximately via the simplified assumption $L \equiv W$. For given power setting and weight, relations (3) and (4) simplify to

$$
T = T(h, V), \quad D = D(h, V),
$$

(40)

with the implication that the excess power per unit weight,

$$
E = V(T - D)/W
$$

(41)

is also a function of the form

$$
E = E(h, V).
$$

(42)

Hence, the so-called omega function (Ref. 33),
\[ \omega = \frac{\partial E}{\partial V} \frac{V \partial E}{g \partial h}, \]

(43)

has the form

\[ \omega = \omega(h, V). \]

(44)

In light of the above hypotheses, the flight path angle \( \gamma \) replaces the angle of attack \( \alpha \) as control and the minimum-time trajectory consists of the following segments:

\[ \omega = 0, \]

(45a)

\[ \gamma = -\pi/2, \]

(45b)

\[ \gamma = \pi/2. \]

(45c)

Equation (45a) constitutes the central segment of the optimal climbing trajectory. Equations (45b) and (45c) describe the initial and final segments. Depending on the boundary conditions, the initial segment can be a vertical dive or zoom; analogously, the final segment can be a vertical dive or zoom. With reference to the \((V, h)\) domain, the choice of (45b) vs. (45c) for the initial and final segments of the optimal trajectory depends on the location of the initial and final points with respect to the line \( \omega = 0 \).

**Central Segment of Optimal Trajectory.** Examining Trajectories A1 - A54, it is evident that many optimal climb trajectories involve a central segment flown with a load factor of approximately 1, thereby justifying the assumption \( L \equiv W \). In this section, we discuss the influence of the following factors on the nature of the central segment: initial speed, final speed, weight. With reference to the \((V, h)\) domain, we also compare the central segment with the trajectory \( \omega = 0 \) [Eq. (45a)] predicted by the classical theory of optimal climb.
Figures 25 - 30 illustrate optimal climb trajectories for the heavy weight case for initial speeds of 0.5, 1.0, 1.5 kft/sec and three final speeds: 0.5 kft/sec [Fig. 25 (roll permitted), Fig. 26 (vertical plane)], 1.0 kft/sec [(Fig. 27 (roll permitted), Fig. 28 (vertical plane)], 1.5 kft/sec [Fig. 29 (roll permitted), Fig. 30 (vertical plane)]. With respect to the central segment, these trajectories are typical of all of the cases studied. Each figure also shows the corresponding $\omega = 0$ trajectory. Examining these trajectories, it is clear that the influence of the initial speed on the central segment is slight. It is also evident that the influence of the final speed on the central segment is quite strong.

To compare the computed optimization results to the classical theory, consider again Figs. 25 - 30. The central segment of the trajectories featuring higher final speeds (1.0 and 1.5 kft/sec) follows closely the $\omega = 0$ trajectory. On the other hand, the central segment of the trajectories leading to a lower final speed (0.5 kft/sec) does not resemble the $\omega = 0$ trajectory.

Figures 31 - 32 illustrate the effect of the aircraft weight on the optimal trajectory. Examining these figures, it is apparent that the speed at which the central segment is flown increases as the aircraft weight increases.

**Initial Segment of Optimal Trajectory.** The trajectory comparisons illustrated in Figs. 33 - 38 show that there are two classes of initial segments: accelerating and decelerating. For low initial speed, the initial segment of the optimal trajectory consists of an acceleration at low angle of attack followed by a pitch-up maneuver with low load factor to intercept the central segment of the optimal trajectory. For high initial speed, the initial segment is an immediate pitch-up maneuver at a relatively high load factor so as to inter-
cept the central segment while simultaneously reducing speed. In particular, for high initial speed and a lightly loaded aircraft, the load factor in the initial pitch-up maneuver approaches the load factor limit; see, for example, Trajectory A52 of Fig. 24.

Examining Trajectories A1 - A54, it is evident that relatively steep dives and zooms are employed, albeit not the vertical dives and zooms predicted by the classical theory. For low initial speed, the acceleration maneuver results in a dive angle of approximately 20 deg; see for example, Trajectory A10 of Fig. 11. For high initial speed, the pitch-up maneuver tends to be quite steep, approaching 60 deg; see, for example, Trajectory A16 of Fig. 17.

**Final Segment of the Optimal Trajectory.** Trajectories A1 - A54 involve two types of final segments: those in which a roll is permitted (Trajectories A1 - A27) and those flown entirely in the vertical plane (Trajectories A28 - A54). Depending on the flight time \( \tau \), three cases must be considered: (i) rolling maneuver is superior, (ii) rolling maneuver and vertical plane maneuver are nearly the same, (iii) vertical plane maneuver is superior.

Case (i) occurs for low final speed, and more prominently for a lightly weighted aircraft. In this case, the zoom maneuver is performed with excess kinetic energy so that the rolling maneuver allows an abrupt terminal pitching maneuver simultaneously reducing the velocity to the specified value and the flight path angle to zero. The time saving via the rolling maneuver can be significant. Consider, for example, Trajectories A25 of Table 1 and A52 of Table 2: the flight time advantage via the rolling maneuver is 6.3 sec, or approximately 13%.
Also for case (i), it is interesting to note that there are a few instances in which the vertical plane trajectory includes a throttle reduction during the final segment. See, for example, Trajectory A52 of Fig. 24.

Case (ii) occurs when the optimal trajectory ends with a zoom at zero load factor; see, for example, Trajectories A15 of Fig. 16 and A42 of Fig. 23, which are flown in the same time $\tau = 148.9$ sec. These trajectories are nearly identical until the zero load factor zoom begins. Afterward, the only difference is that there is a slow roll in the trajectories with roll permitted. This roll is a residue of the nominal trajectory used to start the SGRA algorithm; it remains in the converged solution because, at a load factor of zero, bank angle has no effect on aircraft performance.

Case (iii) occurs when the optimal trajectory ends with a final zoom at an angle of attack greater than zero; see for example Trajectories A3 of Fig. 10 and A30 of Fig. 20: the flight time advantage via maneuver in a vertical plane is 6.8 sec, or approximately 2.4%. In the trajectories of case (iii) for which roll is permitted, the nominal trajectory includes a roll maneuver to a bank angle of approximately 180 deg. During the optimization process, a relative minimum trajectory is located which ends with a zoom at zero load factor. The most prominent instances of case (iii) are those involving heavy weight and high final speed.

3.4. Climb Summary. Generally speaking, optimal climb trajectories for a jet fighter aircraft capable of maneuvering at high angles of attack do not require flight at high angles of attack. The optimal climbs tend to require low angles of attack and are characterized by three distinct segments: a central segment often flown with a load factor of
nearly 1 and two terminal segments (dive or zoom) to and from the central segment. The central and final segments are nearly independent of the initial conditions, instead being dominated by the final conditions. If roll is permitted, certain trajectories with excess speed at the end of the zoom maneuver conclude with a moderately high load factor maneuver to align the velocity vector with the horizon. In all cases, the initial segment of the maneuver consists of either a zero load factor dive or a pitch-up with high load factor so as to reach the central segment.
4. Fly-to-Point Maneuvers

Low altitude maneuvering to a surface target is an important flight regime for modern jet fighter aircraft. In many combat situations, it is desirable to penetrate enemy-controlled territory at low altitude to evade radar and surface-to-air weapons. Frequently, flight to a target is not direct, and can involve large course changes, characterized by a number of turns at designated turn points. It is beneficial in such situations to maneuver the aircraft so as to fly from one turn point to the next in minimum time, while at the same time conserving fuel. Level fly-to-point maneuvers for low-speed (0.15 to 0.5 kft/sec) aircraft have been studied previously (Ref. 43), but there appear to be no published studies of this type of maneuver for a jet fighter capable of supermaneuvers.

This section describes a study of minimum-time low altitude quasi-level fly-to-point maneuvers for a next-generation jet fighter aircraft capable of maneuvering at high angles of attack, even post-stall angles of attack. The optimal trajectories computed here serve two purposes. First, they can benefit aircraft designers by highlighting those flight characteristics that are most beneficial for this aspect of the surface attack mission. Second, they can benefit aircraft pilots as the basis for guidance trajectories that approximate the optimal fly-to-point maneuvers.

4.1. Additional Path Constraints. It is desirable for an aircraft flying at low altitude to turn in such a way as to remain near the selected altitude (quasi-level flight). Descending below the selected altitude could result in ground impact and climbing above the selected altitude could negate the tactical benefits of flying at low altitude. Therefore, altitude must be constrained to remain close to the initial altitude throughout the trajectory,
\[ h \equiv h_i. \] (46)

In this study, altitude is constrained via a penalty functional using the technique described in Ref. 27. The penalty functional used here is

\[ I_h = \int_0^x f_h(h, t) dt, \] (47)

where

\[ f_h(h, t) = k_h(h - h_i)^2 \] (48)

and \( k_h \) is a penalty factor that is held constant during an iteration, but which may change between iterations.

Typically, for low level flight, a pilot chooses a nominal speed that is a compromise between the benefits of flying at high speed (increased maneuverability and reduced exposure to threat) and the need to conserve fuel. Therefore, it is desirable while maneuvering at low altitude to remain close to the selected speed and to avoid extended periods of flight at speeds in excess of the selected speed. In this study, the aircraft velocity is subject to the inequality constraint

\[ V \leq V_i. \] (49)

In turn, Ineq. (49) is accounted for via the penalty functional

\[ I_V = \int_0^x f_V(h, t) dt \] (50)

where

\[ f_V(h, t) = \begin{cases} k_V[V_i - V(t)]^2, & V > V_i, \\ 0, & V \leq V_i, \end{cases} \] (51)
and $k_V$ is a penalty factor that is held constant during an iteration, but which may change between iterations.

4.2. Optimization Problem. In light of constraints (46) and (49), the minimum-time low altitude quasi-level fly-to-point problem (Problem P2) can be formulated as follows:

$$\min I = I_h + I_V + \tau,$$

where $I_h$ is computed via (47), $I_V$ is computed via (50), and $\tau$ is the final time. The unknowns include the original state variables $x(t), y(t), h(t), V(t), \gamma(t), \chi(t)$, the original control variable $\mu(t)$, the auxiliary control variables $u_1(t), u_2(t), u_3(t)$, and the parameter $\tau$.

With the solution known, the original control variables $\alpha(t), \beta(t)$ can be recovered from $u_1(t), u_2(t)$ via (6).

Time Normalization. In treating problem (52), it is convenient to normalize the dimensional time $t$ via the transformation

$$\theta = t/\tau,$$

where $\theta$ is the dimensionless time and $\tau$ the final time. Clearly,

$$0 \leq \theta \leq 1, \quad \text{if } 0 \leq t \leq \tau.$$

Approach. The optimization problem under consideration is of the Bolza type and can be solved via the sequential gradient-restoration algorithm for optimal control problems (SGRA, Refs. 10 - 11). The approach taken is to produce manually a nominal control history that results in a solution of the differential system (1), satisfying the specified initial conditions and the path constraint (25), and approximately satisfying the final
conditions. This solution is used as the starting nominal trajectory for SGRA; this algo-

rithm produces a sequence of feasible trajectories, each characterized by a lower value of

performance index $I$ than the previous feasible trajectory. The algorithm terminates when

the mean square error in the optimality conditions becomes smaller than a preselected

small positive number.

4.3. Computer Runs. In this study, 16 optimum fly-to-point trajectories

(Trajectories B1 - B16) were computed. First, for fixed $n_{\text{max}}$, 12 optimal fly-to-point tra-

jectories (Trajectories B1 - B12) were obtained by considering all possible combinations

of the following parameter values:

\[
\begin{align*}
    n_{\text{max}} &= 6, \\
    V(0) &= 0.7, 0.8, 0.9 \text{ kft/sec}, \\
    \psi(\tau) &= \pi/4, \pi/2, 3\pi/4, \pi, \\
\end{align*}
\]

(55a) (55b) (55c)

with

\[
\psi(\tau) = \arctan[y(\tau)/x(\tau)].
\]

(56)

Then, for fixed $V(0)$ and $\psi(\tau)$, 4 additional optimal trajectories (Trajectories B13 to B16)

were obtained by changing the load factor limit,

\[
\begin{align*}
    n_{\text{max}} &= 5, 4, 3, 2, \\
    V(0) &= 0.8 \text{ kft/sec}, \\
    \psi(\tau) &= \pi. \\
\end{align*}
\]

(57a) (57b) (57c)

For each run, the following initial conditions were assumed:

\[
\begin{align*}
    x(0) &= 0 \text{ kft}, \\
    y(0) &= 0 \text{ kft}, \\
    h(0) &= 5 \text{ kft}, \\
    \gamma(0) &= 0, \\
    \chi(0) &= 0, \\
\end{align*}
\]

(58a) (58b)
together with the following final conditions:

\[ x(\tau) = 50\cos\psi(\tau), \quad y(\tau) = 50\sin\psi(\tau), \quad h(\tau) = 5 \text{ kft}, \quad (59a) \]
\[ \gamma(\tau) = 0. \quad (59b) \]

State variables which are not prescribed must be considered as free and must be determined from the optimization process together with the main unknown, the final time \( \tau \).

The computations via SGRA were done with an Intel Pentium II processor using the C language and double-precision arithmetic.

For convenience, Table 3 presents a summary of the trajectory parameters identifying Trajectories B1 to B12 [see (55)] and Trajectories B13 to B16 [see (57)]. For the purpose of illustration, Trajectories B5 - B8 and B13 - B16 are plotted in Figs. 39 - 46, along with a tabulation of the key parameters, data, and results.
Table 3. Optimal fly-to-point trajectories.

<table>
<thead>
<tr>
<th>Trajectory</th>
<th>( V(0) ) [kft/sec]</th>
<th>( \psi(\tau) ) [deg]</th>
<th>( n_{\text{max}} )</th>
<th>( \tau ) [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1</td>
<td>0.7</td>
<td>45</td>
<td>6</td>
<td>72.9</td>
</tr>
<tr>
<td>B2</td>
<td>0.7</td>
<td>90</td>
<td>6</td>
<td>77.4</td>
</tr>
<tr>
<td>B3</td>
<td>0.7</td>
<td>135</td>
<td>6</td>
<td>85.7</td>
</tr>
<tr>
<td>B4</td>
<td>0.7</td>
<td>180</td>
<td>6</td>
<td>95.6</td>
</tr>
<tr>
<td>B5</td>
<td>0.8</td>
<td>45</td>
<td>6</td>
<td>64.1</td>
</tr>
<tr>
<td>B6</td>
<td>0.8</td>
<td>90</td>
<td>6</td>
<td>69.8</td>
</tr>
<tr>
<td>B7</td>
<td>0.8</td>
<td>135</td>
<td>6</td>
<td>78.5</td>
</tr>
<tr>
<td>B8</td>
<td>0.8</td>
<td>180</td>
<td>6</td>
<td>88.7</td>
</tr>
<tr>
<td>B9</td>
<td>0.9</td>
<td>45</td>
<td>6</td>
<td>57.9</td>
</tr>
<tr>
<td>B10</td>
<td>0.9</td>
<td>90</td>
<td>6</td>
<td>64.9</td>
</tr>
<tr>
<td>B11</td>
<td>0.9</td>
<td>135</td>
<td>6</td>
<td>74.8</td>
</tr>
<tr>
<td>B12</td>
<td>0.9</td>
<td>180</td>
<td>6</td>
<td>85.5</td>
</tr>
<tr>
<td>B13</td>
<td>0.8</td>
<td>180</td>
<td>5</td>
<td>90.2</td>
</tr>
<tr>
<td>B14</td>
<td>0.8</td>
<td>180</td>
<td>4</td>
<td>91.6</td>
</tr>
<tr>
<td>B15</td>
<td>0.8</td>
<td>180</td>
<td>3</td>
<td>96.0</td>
</tr>
<tr>
<td>B16</td>
<td>0.8</td>
<td>180</td>
<td>2</td>
<td>112.4</td>
</tr>
</tbody>
</table>

4.4. Analysis. Inspecting Trajectories B1 - B12 (\( n_{\text{max}} = 6 \)), it is evident that the optimal fly-to-point maneuvers do not exploit the high angle of attack capability of a next generation fighter. All of the trajectories consist of maneuvers flown at relatively low angles of attack. Each optimal trajectory consists of three segments: turning, accelerating, and cruising.

The turning segment begins at a relatively high load factor. For turns of 90 deg or more, the turn starts at the load factor limit, then decreases smoothly to a load factor of
nearly 1.0. It is worth noting that the angle of attack never approaches the value corresponding to maximum lift coefficient. Instead, it appears that the optimal turn is a compromise between minimizing the time to turn toward the destination point and conserving airspeed.

Examining the plots of the load factor \( n \) versus the heading angle-to-go \( \chi_f - \chi_s \), it appears that a good rule-of-thumb guidance trajectory would be to start the turn at the maximum available load factor and then uniformly decrease the load factor as the turn progresses, aiming toward a load factor \( n \equiv 1 \) at the time the aircraft velocity vector becomes aligned with the destination point. Thus, the guidance law

\[
 n = n_{\text{max}} (\chi_f - \chi_s) / (\chi_f - \chi_i) 
\]

(60)

will allow the pilot to fly approximately an optimal trajectory.

The second segment of the optimal trajectory is a level acceleration at maximum power. Due to the need to maintain nearly constant altitude, the load factor must remain at \( n \equiv 1 \) throughout the acceleration segment. Finally, the third segment of the optimal trajectory is a cruise at the selected airspeed, ending at the destination point.

Next, consider Trajectories B13 to B16, which are flown with maximum load factor \( n_{\text{max}} = 5, 4, 3, 2 \) lower than that of Trajectory B8 \( n_{\text{max}} = 6 \). The main observation is that, as the maximum load factor decreases from 6 to 2, the time necessary to complete the turn toward the destination point increases from 30 to 42 sec, with a corresponding increase in the time to complete the flight maneuver from 89 to 113 sec.

4.5. **Fly-to-Point Summary.** Generally speaking, minimum-time trajectories for low altitude quasi-level fly-to-point maneuvers do not involve high angles of attack.
The optimal trajectories consist of three segments: turning, accelerating, and cruising. The turning segment begins at a relatively high load factor. Then, the load factor decreases smoothly toward $n \equiv 1$, which is reached at the time the aircraft velocity vector points toward the destination point. The acceleration segment consists of a level acceleration at maximum thrust, which is followed in the final segment by resumption of steady-state cruising.
5. Pop-Up Attack Maneuvers

Low altitude maneuvering to a surface target is of considerable importance for modern jet fighter aircraft. In many combat situations, it is desirable to penetrate enemy-controlled territory at low altitude to evade radar and surface-to-air weapons. When the aircraft reaches the neighborhood of the target, it is necessary to maneuver so as to achieve values of heading, altitude, airspeed, and dive angle suitable for the initiation of a dive-bombing attack (pop-up maneuver). Typically, the pilot flies the aircraft to a preselected position, then begins the pop-up maneuver, frequently before the designated target is visually identified. During the climbing portion of the pop-up maneuver, the pilot identifies the target, and then adjusts the aircraft flight path so as to reach the bomb release point at suitable values of heading, altitude, airspeed, and dive angle. Since the pilot must visually identify the target, it is important that the aircraft be oriented so as to maximize the time in which the target is potentially visible to the pilot. Due to the obstruction to forward visibility presented by the aircraft nose, it is beneficial to begin the climb with the target displaced with respect to the aircraft centerline.

During the course of the pop-up maneuver and dive-bombing attack, the aircraft is quite vulnerable. It is reasonable to expect enemy surface-to-air defenses to be concentrated in the neighborhood of high-value targets, and the increased altitude the aircraft achieves during the pop-up maneuver and dive-bombing pass makes the aircraft more vulnerable. For these reasons, it is desirable to complete the pop-up maneuver in minimum time.
This section describes a study of optimal pop-up maneuvers for a next-generation jet fighter aircraft capable of maneuvering at high angles of attack. The optimal trajectories computed here serve two purposes. First, they can benefit aircraft designers by highlighting those flight characteristics that are most beneficial for this aspect of the surface attack mission. Second, they can benefit aircraft pilots by serving as the basis for guidance trajectories that approximate the optimal maneuvers.

5.1. Optimization Problem. The pop-up attack problem (Problem P3) can be formulated as follows:

$$\min I = \tau,$$

where $\tau$ is the final time. The unknowns include the original state variables $x(t), y(t), h(t), V(t), \gamma(t), \chi(t)$, the original control variable $\mu(t)$, the auxiliary control variables $u_1(t), u_2(t), u_3(t)$, and the parameter $\tau$. With the solution known, the original control variables $\alpha(t), \beta(t)$ can be recovered from $u_1(t), u_2(t)$ via (6).

Time Normalization. In treating problem (61), it is convenient to normalize the dimensional time $t$ via the transformation

$$\theta = t/\tau,$$

where $\theta$ is the dimensionless time and $\tau$ the final time. Clearly,

$$0 \leq \theta \leq 1, \quad \text{if} \quad 0 \leq t \leq \tau.$$

Approach. The optimization problem under consideration is of the Mayer type and can be solved via the sequential gradient-restoration algorithm for optimal control problems (SGRA, Refs. 10 - 11). The approach taken is to produce manually a nominal
control history that results in a solution of the differential system (1), satisfying the specified initial conditions and the path constraint (25), and approximately satisfying the final conditions. This solution is used as the starting nominal trajectory for SGRA; this algorithm produces a sequence of feasible trajectories, each characterized by a lower value of performance index $I$ than the previous feasible trajectory. The algorithm terminates when the mean square error in the optimality conditions becomes smaller than a preselected small positive number.

5.2. Computer Runs. In this study, 16 optimal pop-up trajectories (trajectories C1 to C16) were obtained by considering various combinations of the following parameter values:

\begin{align}
T_{\text{ref}} &= 6.8, 12.0 \text{ kib}, \\
h(\tau) &= 2, 4, 5, 7 \text{ kft}, \\
\gamma(\tau) &= -10, -20, -30, -45 \text{ deg}, \\
V(\tau) &= 0.8 \text{ kft/sec}, \\
\chi(\tau) &= 45, 90 \text{ deg}.
\end{align}

(64a) (64b) (64c) (64d) (64e)

For each run, the following initial conditions were assumed:

\begin{align}
x(0) &= 0 \text{ kft}, \\
y(0) &= 0 \text{ kft}, \\
h(0) &= 1 \text{ kft}, \\
V(0) &= 0.9 \text{ kft/sec}, \\
\gamma(0) &= 0, \\
\chi(0) &= 0.
\end{align}

(65a) (65b)

State variables which are not prescribed must be considered as free and must be determined from the optimization process together with the main unknown, the final time $\tau$. The computations via SGRA were done with an Intel Pentium II processor using the C language and double-precision arithmetic.
For convenience, Table 4 presents a summary of the trajectory parameters identifying Trajectories C1 to C16 [see (64)]. For the purpose of illustration, Trajectories C1, C4 - C5, C8 - C9, C12 - C13, and C16 are plotted in Figs. 47-54 along with a tabulation of the key parameters, data, and results.

<table>
<thead>
<tr>
<th>Trajectory</th>
<th>$T_{\text{ref}}$ [kib]</th>
<th>$h_{f}$ [kft]</th>
<th>$\gamma_{f}$ [deg]</th>
<th>$\chi_{f}$ [deg]</th>
<th>$\tau$ [sec]</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>6.8</td>
<td>2</td>
<td>-10</td>
<td>45</td>
<td>13.8</td>
</tr>
<tr>
<td>C2</td>
<td>6.8</td>
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<td>-20</td>
<td>45</td>
<td>19.8</td>
</tr>
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<td>C3</td>
<td>6.8</td>
<td>5</td>
<td>-30</td>
<td>45</td>
<td>28.7</td>
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<td>6.8</td>
<td>7</td>
<td>-45</td>
<td>45</td>
<td>44.8</td>
</tr>
<tr>
<td>C5</td>
<td>6.8</td>
<td>2</td>
<td>-10</td>
<td>90</td>
<td>23.3</td>
</tr>
<tr>
<td>C6</td>
<td>6.8</td>
<td>4</td>
<td>-20</td>
<td>90</td>
<td>27.0</td>
</tr>
<tr>
<td>C7</td>
<td>6.8</td>
<td>5</td>
<td>-30</td>
<td>90</td>
<td>35.4</td>
</tr>
<tr>
<td>C8</td>
<td>6.8</td>
<td>7</td>
<td>-45</td>
<td>90</td>
<td>51.0</td>
</tr>
<tr>
<td>C9</td>
<td>12</td>
<td>2</td>
<td>-10</td>
<td>45</td>
<td>9.2</td>
</tr>
<tr>
<td>C10</td>
<td>12</td>
<td>4</td>
<td>-20</td>
<td>45</td>
<td>13.0</td>
</tr>
<tr>
<td>C11</td>
<td>12</td>
<td>5</td>
<td>-30</td>
<td>45</td>
<td>18.1</td>
</tr>
<tr>
<td>C12</td>
<td>12</td>
<td>7</td>
<td>-45</td>
<td>45</td>
<td>25.2</td>
</tr>
<tr>
<td>C13</td>
<td>12</td>
<td>2</td>
<td>-10</td>
<td>90</td>
<td>13.2</td>
</tr>
<tr>
<td>C14</td>
<td>12</td>
<td>4</td>
<td>-20</td>
<td>90</td>
<td>16.4</td>
</tr>
<tr>
<td>C15</td>
<td>12</td>
<td>5</td>
<td>-30</td>
<td>90</td>
<td>21.0</td>
</tr>
<tr>
<td>C16</td>
<td>12</td>
<td>7</td>
<td>-45</td>
<td>90</td>
<td>27.8</td>
</tr>
</tbody>
</table>

5.3. **Analysis.** Inspecting Trajectories C1 - C16, it is evident that each optimal pop-up maneuver consists of three segments: pitch-up, zoom, pitch-down.
The pitch-up segment begins at the maximum allowable load factor. In some of the trajectories in which afterburner is permitted, the load factor remains at the limit $n_{\text{max}} = 6$ briefly (see, for example, Trajectories C13 - C16 of Figs. 53 - 54). The load factor decreases smoothly as the pitch attitude increases.

The zoom segment is performed at low angle of attack. In the runs involving a smaller heading change (45 deg), the zoom is performed in most cases at $\alpha \equiv 0$ deg; in the runs involving a larger heading change (90 deg), the zoom is performed at a low angle of attack. In the runs involving a smaller heading change, all of the heading change occurs during the initial pitch-up segment and final pitch-down segment; in the runs involving a larger heading change, turning occurs throughout the zoom maneuver.

The final segment of the optimal trajectories consists of a pitch-down in which the flight path angle is reduced to the prescribed final value. In most cases, the pitch down is performed at a relatively low load factor. In the cases of large heading changes (90 deg) without afterburning, the pitch-down segment is performed at $\alpha \equiv 0$ deg.

None of the optimal trajectories involves flight at angles of attack in excess of that for maximum lift coefficient. For all the optimal trajectories, the power setting remains at its maximum value $\beta = 1$.

It is interesting to compare the trajectories with and without afterburning. Even in the case of short flight time (small heading change, small dive angle), the benefit of afterburner use is large. Compare for example Trajectories C1 and C9 of Table 4 (45 deg heading change, 10 deg dive angle). Employing the afterburner reduces the final time from
13.8 to 9.2 sec, a 33% reduction. Similar reductions in flight time are evident when comparing the remaining runs with and without afterburning.

It is also interesting to compare the effect of increasing the heading change during the maneuver from 45 to 90 deg. In the runs involving shallow dive angles (10 and 20 deg), the effect on final time of increasing the heading change is large. For example, compare the runs involving a dive angle of 10 deg with afterburner excluded, namely Trajectories C1 (45 deg heading change) and C5 (90 deg heading change) of Table 4. Increasing the heading change from 45 to 90 deg increases the flight time from 13.8 to 23.3 sec, a 69% increase. On the other hand, in the runs involving steep dive angles (30 and 45 deg), the effect on final time of increasing the heading change is smaller. For example, compare the runs involving 45 deg dive angles with afterburner excluded, namely Trajectories C4 (45 deg heading change) and C8 (90 deg heading change) of Table 4. In this case, increasing the heading change from 45 to 90 deg increases the final time from 44.8 to 51.0 sec, a 14% increase. Thus, in situations requiring steep dive angles, the increased ease of target acquisition that accrues from the larger heading change might be worthwhile, whereas in situations in which a shallow dive angle is acceptable, the additional exposure time that would result from larger heading changes may not be justifiable.

Finally, comparing the trajectories from the perspective of the dive angle, it is clear that the effect on final time of increasing the dive angle is quite large. For example, compare the runs involving 45 deg heading change with afterburner excluded, namely Trajectories C1 (dive angle 10 deg) and Trajectory C4 (dive angle 45 deg) of Table 4. In this case, increasing the dive angle from 10 deg to 45 deg increases the flight time from 13.8 to 44.8 sec, a 225% increase.
5.4. Pop-Up Attack Summary. Generally speaking, the optimal trajectories for pop-up attack maneuvers do not involve flight at high angles of attack. The optimal trajectories consist of three segments flown at maximum power: pitch-up, zoom, and pitch-down. The use of afterburner during the pop-up maneuver is quite beneficial. The increase in flight time resulting from increasing heading change from 45 to 90 deg is large for shallow dive angles, and is small for steep dive angles. Also, the increase in flight time resulting from increasing the dive angle is quite large.
6. Dive Recovery Maneuvers

This section describes a study of optimal dive recovery maneuvers for a next-generation jet fighter aircraft capable of maneuvering at high angles of attack. In a dive recovery maneuver, the pilot attempts to return the aircraft to level flight at an airspeed such that level flight can be maintained afterward. The optimization criterion is the minimization of the maximum loss of altitude; hence, the optimization problem is a minimax problem (Chebyshev problem) of optimal control.

The need for a dive recovery maneuver can arise (i) intentionally or (ii) unintentionally. As an example of type (i), in performing a dive bombing attack, the pilot intentionally places the aircraft in a dive; knowledge of the optimal dive recovery trajectory will assist the pilot in planning the attack and subsequent recovery.

As an example of type (ii), in the course of air-to-air combat, maneuvering relative to an opponent in an effort to force the opponent's flight path to overshoot the pilot's own flight path can result inadvertently in extremely high pitch attitudes and vanishing airspeed; in such a situation, the pilot must place the aircraft in a dive so as to accelerate the aircraft to an airspeed at which level flight is possible. As another example of type (ii), an aircraft can enter a dive as the result of the pilot's momentary distraction or spatial disorientation.

The optimal trajectories computed in this section serve two purposes. First, they can benefit aircraft designers by highlighting those flight characteristics that are most beneficial for this aspect of the flight of a next-generation aircraft. Second, they can benefit
aircraft pilots by serving as the basis for guidance trajectories that approximate the optimal trajectories.

6.1. Optimization Problem. The dive recovery problem (Problem P4) can be formulated as follows:

$$\min I, \quad I = \max_t [h(0) - h(t)].$$  \hspace{1cm} (66)

This is a minimax problem of optimal control, in which one minimizes with respect to the controls the maximum value with respect to time of the altitude loss. The unknowns include the original state variables $x(t), y(t), h(t), V(t), \gamma(t), \chi(t)$, the original control variable $\mu(t)$, the auxiliary control variables $u_1(t), u_2(t), u_3(t)$, and the parameter $\tau$. With the solution known, the original control variables $\alpha(t), \beta(t)$ can be recovered from $u_1(t), u_2(t)$ via (6).

Time Normalization. In treating problem (66), it is convenient to normalize the dimensional time $t$ via the transformation

$$\theta = t/\tau,$$  \hspace{1cm} (67)

where $\theta$ is the dimensionless time and $\tau$ the final time. Clearly,

$$0 \leq \theta \leq 1, \quad \text{if } 0 \leq t \leq \tau.$$  \hspace{1cm} (68)

Approach. The optimization problem (66) is of the Chebyshev type (Problem P4). Using the technique of Refs. 44 - 45, the Chebyshev problem can be reformulated as a Lagrange problem of optimal control (Problem P5) by replacing (66) with
min \( J, \quad J = \int_0^T [h(0) - h(t)]^q dt, \) \hspace{1cm} (69)

where \( q \) is a large, positive, even exponent (for example, \( q = 20 \)). This replacement is based on the observation that (Ref. 45)

\[
\lim_{q \to \infty} J^{1/q} = \max_t [h(0) - h(t)]. \quad (70)
\]

This problem can be solved via the sequential gradient-restoration algorithm for optimal control problems (SGRA, Refs. 10 - 11). The approach taken is to produce manually a nominal control history that results in a solution of the differential system (1), satisfying the specified initial conditions and the path constraint (25), and approximately satisfying the final conditions. This solution is used as the starting nominal trajectory for SGRA; this algorithm produces a sequence of feasible trajectories, each characterized by a lower value of performance index \( J \) than the previous feasible trajectory. The algorithm terminates when the mean square error in the optimality conditions becomes smaller than a preselected small positive number.

**6.2. Computer Runs.** In this study, 20 optimal dive recovery trajectories (Trajectories D1 - D20) were obtained by considering various combinations of the following parameter values:

\[
\alpha_{\text{max}} = \pi/2 \quad \text{rad} = 90 \text{ deg,} \quad (71a)
\]

\[
T_{\text{ref}} = 6.8, 12.0 \text{ klb,} \quad (71b)
\]

\[
V(0) = 0.05, 0.3, 1.0, 1.5 \text{ kft/sec,} \quad (71c)
\]

\[
\gamma(0) = 85, 0, -45, -85 \text{ deg.} \quad (71d)
\]
If $\gamma(0) = 85$ or 0 deg, the aircraft is able to maintain level flight at the initial airspeeds $V(0) = 0.3, 1.0, 1.5$ kft/sec; therefore, no dive recovery is necessary. Then, for $V(0) = 0.05$ kft/sec, 8 optimal supplementary dive recovery trajectories (Trajectories D21 - D28) were computed by limiting the angle of attack to the value $\alpha_*$ corresponding to maximum lift coefficient. Hence, the parameter values (71) were replaced by

$$\alpha_{\text{max}} = \alpha_* = 0.436 \text{ rad} = 25 \text{ deg},$$

(72a)

$$T_{\text{ref}} = 6.8, 12.0 \text{ klb},$$

(72b)

$$V(0) = 0.05 \text{ kft/sec},$$

(72c)

$$\gamma(0) = 85, 0, -45, -85 \text{ deg}.$$

(72d)

For each run, the following initial conditions were assumed:

$$x(0) = 0 \text{ kft}, \quad y(0) = 0 \text{ kft}, \quad h(0) = 10 \text{ kft},$$

(73a)

$$\chi(0) = 0,$$

(73b)

together with the following final condition:

$$\gamma(\tau) = 0.$$  

(74)

State variables which are not prescribed must be considered as free and must be determined from the optimization process together with the final time $\tau$ and the maximum value of the altitude loss $h(0) - h(\tau)$. The computations via SGRA were done with an Intel Pentium II processor using the C language and double-precision arithmetic.

Table 5 presents a summary of the trajectory parameters identifying Trajectories D1 - D20 [see (71)]. Table 6 presents a summary of the trajectory parameters identifying Trajectories D21 - D28 [see (72)]. For the purpose of illustration, Trajectories D1, D9,
D11 - D15, D18, D20, and D25 - D28 are plotted in Figs. 55 - 67 along with a tabulation of the key parameters, data, and results.

<table>
<thead>
<tr>
<th>Trajectory</th>
<th>$T_{ref}$ [klb]</th>
<th>$V(0)$ [kft/sec]</th>
<th>$\gamma(0)$ [deg]</th>
<th>$h(\tau)$ [kft]</th>
<th>$h(0) - h(\tau)$ [kft]</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1</td>
<td>6.8</td>
<td>0.05</td>
<td>85</td>
<td>7.93</td>
<td>2.07</td>
</tr>
<tr>
<td>D2</td>
<td>6.8</td>
<td>0.05</td>
<td>0</td>
<td>8.19</td>
<td>1.81</td>
</tr>
<tr>
<td>D3</td>
<td>6.8</td>
<td>0.05</td>
<td>-45</td>
<td>7.97</td>
<td>2.03</td>
</tr>
<tr>
<td>D4</td>
<td>6.8</td>
<td>0.05</td>
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<td>-85</td>
<td>7.77</td>
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<td>-45</td>
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</tr>
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<td>85</td>
<td>8.83</td>
<td>1.17</td>
</tr>
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<tr>
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<td>9.23</td>
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<td>0.84</td>
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<td>-85</td>
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Table 6. Optimal supplementary dive recovery trajectories.

<table>
<thead>
<tr>
<th>Trajectory</th>
<th>$T_{ref}$ [klb]</th>
<th>$V(0)$ [kft/sec]</th>
<th>$\gamma(0)$ [deg]</th>
<th>$h(\tau)$ [kft]</th>
<th>$h(0) - h(\tau)$ [kft]</th>
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<tr>
<td>D21</td>
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<td>7.90</td>
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</tr>
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<td>0.05</td>
<td>-85</td>
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<td>2.33</td>
</tr>
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<td>85</td>
<td>8.78</td>
<td>1.22</td>
</tr>
<tr>
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<td>12</td>
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<td>0</td>
<td>8.90</td>
<td>1.10</td>
</tr>
<tr>
<td>D27</td>
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<td>0.05</td>
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<td>8.50</td>
<td>1.50</td>
</tr>
<tr>
<td>D28</td>
<td>12</td>
<td>0.05</td>
<td>-85</td>
<td>8.10</td>
<td>1.90</td>
</tr>
</tbody>
</table>

6.3. Comments on Load Factor. If the load factor $n$ is defined as

$$n = \frac{(L + T \sin \alpha)}{W},$$  \hspace{1cm} (75)

the path constraint (25) can be rewritten as

$$n \leq n_{\text{max}}.$$  \hspace{1cm} (76)

On the other hand, $n$ given by (75) has the functional form (see Appendix, Section 6.6.)

$$n = n(h, V, \alpha, \beta),$$  \hspace{1cm} (77)

which exhibits a maximum with respect to $\alpha$ for given values of $h, V, \beta$. This maximum value is given by (see Appendix, Section 6.6.)

$$n_* = \max_{\alpha} n(h, V, \alpha, \beta) = n(h, V, \alpha_*, \beta),$$  \hspace{1cm} (78)

with the implication that

$$n \leq n_*.$$  \hspace{1cm} (79)
Upon comparing (76) with (79), we conclude that

\[ n \leq \tilde{n}, \]  

where

\[ \tilde{n} = \min(n_*, n_{max}) \]  

is the maximum available load factor (see Appendix, Section 6.6.). To sum up, \( \tilde{n} \) is equal to either \( n_* \) or \( n_{max} \), depending on the aircraft velocity. The particular velocity at which

\[ \tilde{n} = n_* = n_{max} \]  

is called the corner velocity \( V_c \) (Ref. 1). As a consequence,

\[ \tilde{n} = n_*, \quad \text{if } V \leq V_c, \]  

\[ \tilde{n} = n_{max}, \quad \text{if } V \geq V_c. \]

6.4. Analysis. Inspecting Trajectories D1 - D20 of Table 5, it is evident that the optimal dive recovery trajectories consist of one to three segments, depending on initial speed and flight path angle. Common to the optimal trajectories is a pitch-up maneuver flown at approximately the maximum available load factor \( \tilde{n} \); we refer to this segment as the \( \tilde{n} \)-segment. For trajectories starting with very low initial speed (0.05 kft/sec), the \( \tilde{n} \)-segment is preceded by a brief supermaneuver segment. For trajectories starting with very low initial speed and very high flight path angle, the supermaneuver segment is preceded by a brief segment in which dive initiation occurs.
One-Segment Optimal Trajectories. Inspecting Trajectories D5 - D10 and D15 - D20 of Table 5 (see, for example, Figs. 56 and 61 - 63), it is apparent that, in the runs in which the initial speed is 0.3 kft/sec or greater, the optimal dive recovery trajectory consists of a single segment: a pitch-up at the maximum available load factor \( \tilde{n} \) [see (81)]. At speeds below the corner velocity, \( \tilde{n} = n_* \) [see (83a)]; at speeds above the corner velocity, \( \tilde{n} = n_{\text{max}} \) [see (83b)].

For example, consider Trajectory D15 \( [T_{\text{ref}} = 12.0 \text{ klb}, V(0) = 0.3 \text{ kft/sec, } \gamma(0) = -45.0 \text{ deg}] \), which is flown at speeds below the corner velocity. For this trajectory, Fig. 70 compares the time histories of the actual load factor \( n \) and the maximum available load factor \( \tilde{n} = n_* \). Clearly, the plots of the functions \( n(t) \) and \( \tilde{n}(t) \) are quite close to one another.

As another example, consider Trajectory D18 \( [T_{\text{ref}} = 12.0 \text{ klb}, V(0) = 1.0 \text{ kft/sec, } \gamma(0) = -85.0 \text{ deg}] \), which is flown at speeds partly above and partly below the corner velocity. For this trajectory, Fig. 71 compares the time histories of the actual load factor \( n \) and the maximum available load factor, with \( \tilde{n} = n_{\text{max}} \) above the corner velocity and \( \tilde{n} = n_* \) below the corner velocity. Clearly the plots of the functions \( n(t) \) and \( \tilde{n}(t) \) are so close to one another that they are indistinguishable in the scale of Fig. 71.

Two-Segment Optimal Trajectories. Inspecting Trajectories D2 - D4 and D12 - D14 of Table 5 (see, for example, Figs. 58 - 60), it is apparent that, in the runs with initial speed 0.05 kft/sec and flight path angle between 0 and -85 deg, the optimal dive
recovery trajectory consists of two segments. The first segment is a supermaneuver flown at extremely high angles of attack (see in particular Trajectories D13 - D14); the supermaneuver lasts approximately 2.5 sec and ends as the speed increases through approximately 0.2 kft/sec. The second segment is a pitch-up at the maximum available load factor $ar{n} = n_*$ [sec (83a)].

To determine the magnitude of the benefits accrued via supermaneuvers, the supplementary Trajectories D21 - D28 of Table 6 were computed. In the trajectories of Table 5, the inequality $\alpha \leq \pi/2$ is enforced (supermaneuver allowed); in the trajectories of Table 6, the inequality $\alpha \leq \alpha_*$ is enforced (supermaneuver not allowed).

Trajectories D21 - D24 correspond to Trajectories D1 - D4, albeit with $\alpha_{\text{max}}$ changed from $\pi/2$ in Trajectories D1 - D4 to $\alpha_*$ in Trajectories D21 - D24. Analogously, Trajectories D25 - D28 correspond to Trajectories D11 - D14, albeit with $\alpha_{\text{max}}$ changed from $\pi/2$ in Trajectories D11 - D14 to $\alpha_*$ in Trajectories D25 - D28. In Trajectories D21 - D28, the supermaneuver of Trajectories D1 - D4 and D11 - D14 is replaced by a segment flown with $\alpha = \alpha_*$. For each pair of corresponding trajectories, one with supermaneuver allowed and one with supermaneuver not allowed, limiting the angle of attack to $\pi/2$ results in a reduced loss of altitude with respect to limiting the angle of attack to $\alpha_*$. The largest benefit due to supermaneuver can be seen by comparing Trajectory D14 (supermaneuver allowed) and Trajectory D28 (supermaneuver not allowed): the altitude loss reduces from
1.90 kft in Trajectory D28 to 1.74 kft in Trajectory D14, a relative decrease in altitude loss of 8 percent due to supermaneuver.

**Three-Segment Optimal Trajectories.** Inspecting Trajectories D1 and D11 of Table 5 (see Figs. 55 and 57), it is apparent that, in the runs with very low initial speed (0.05 kft/sec) and very high flight path angle (85.0 deg), the optimal dive recovery trajectory consists of three segments. The first segment is a dive initiation flown with \( \alpha = 0 \), ending as the flight path angle decreases through zero. The second segment is a supermaneuver flown at large angles of attack, lasting about 2.5 sec and ending as the speed increases through approximately 0.2 kft/sec. The third segment is a pitch-up at the maximum available load factor \( \bar{n} = n_* \) [see (83a)].

**Effect of Afterburner Usage.** Comparing Trajectories D1 - D10 (afterburner excluded) and Trajectories D11 - D20 (afterburner permitted), it appears that the magnitude of the benefit due to afterburner usage depends on the initial speed. For very low initial speed, afterburner usage has a pronounced effect; compare for example Trajectories D1 and D11 of Table 1; the benefit of afterburner usage is a reduction in altitude loss of from 2.07 to 1.17 kft, a 43% reduction. On the other hand, for higher initial speeds, the benefit of afterburner usage is negligible; compare for example Trajectories D9 and D19 of Table 5; here, there is no benefit due to afterburner usage.

**6.5. Dive Recovery Summary.** Optimal dive recovery trajectories consist of one to three segments. All the optimal trajectories conclude with a pitch-up at the maximum available load factor \( \bar{n} \), which is the smaller between the maximum aerodynamically
available load factor $n_*$ and the load factor limit $n_{\text{max}}$. In optimal trajectories starting with very low speed, the $\tilde{n}$-segment is preceded by a brief supermaneuver segment. In the special case of optimal trajectories that start at very low speed and very high flight path angle, the supermaneuver segment is preceded by a dive initiation segment.

6.6. Appendix: Maximum Available Load Factor. The load factor $n$ is defined as

$$n = (L + T \sin \alpha)/W,$$  \hspace{1cm} (84)

with lift functionally given by [see (10), (11a), (12)]

$$L = L(h, V, \alpha),$$ \hspace{1cm} (85)

and thrust functionally given by [see (10), (21)]

$$T = T(h, V, \beta).$$ \hspace{1cm} (86)

Hence, the load factor has the functional form

$$n = n(h, V, \alpha, \beta).$$ \hspace{1cm} (87)

For given altitude $h = 10$ kft and power setting $\beta = 1$, the function (87) is plotted in Fig. 68 versus the angle of attack $\alpha$ for three values of the velocity $V$. As Fig. 68 shows, the load factor $n$ has a maximum value $n_*$ for $\alpha_* = 0.436$ rad $\approx 25$ deg. Note that, for

$V = 0.90$ kft/sec, $n_*$ exceeds $n_{\text{max}}$; for $V = 0.75$ kft/sec, $n_*$ equals $n_{\text{max}}$; for $V = 0.30$ kft/sec, $n_*$ is less than $n_{\text{max}}$.

Let

$$\tilde{n} = \min(n_*, n_{\text{max}})$$ \hspace{1cm} (88)

denote the maximum available load factor. With reference to Fig. 68, we have that
\[ \tilde{n} = 9.0, \quad \text{if } V = 0.90 \text{ kft/sec}, \quad (89a) \]
\[ \tilde{n} = 9.0, \quad \text{if } V = 0.75 \text{ kft/sec}, \quad (89b) \]
\[ \tilde{n} = 1.8, \quad \text{if } V = 0.30 \text{ kft/sec}. \quad (89c) \]

For \( \beta = 1 \), the maximum available load factor \( \tilde{n} \) is shown in Fig. 69 versus the velocity \( V \) for three values of the altitude. The particular velocity at which \( \tilde{n} = n_\ast = n_{\text{max}} \) is called the corner velocity \( V_c \) (Ref. 1). With reference to Fig. 69, we have that

\[ V_c = 0.62 \text{ kft/sec}, \quad \text{if } h = 0 \text{ kft}, \quad (90a) \]
\[ V_c = 0.74 \text{ kft/sec}, \quad \text{if } h = 10 \text{ kft}, \quad (90b) \]
\[ V_c = 1.17 \text{ kft/sec}, \quad \text{if } h = 20 \text{ kft}. \quad (90c) \]

Figure 70 refers to Trajectory D15 of Table 5 and compares the time histories of the maximum available load factor \( \tilde{n} \) and the actual load factor \( n \). Clearly, the load factor \( n \) of the optimal trajectory is very close to the maximum available load factor \( \tilde{n} \).

Figure 71 refers to Trajectory D18 of Table 5 and compares the time histories of the maximum available load factor \( \tilde{n} \) and the actual load factor \( n \). Clearly, the load factor \( n \) of the optimal trajectory is so close to the maximum available load factor \( \tilde{n} \) that the two time histories are indistinguishable from one another.
7. Summary and Conclusions

In this thesis, four classes of optimal combat maneuvers for a next-generation fighter aircraft are considered: climb maneuvers, fly-to-point maneuvers, pop-up attack maneuvers, and dive recovery maneuvers. For the first three classes of maneuvers, the optimization criterion is the minimization of the flight time, resulting in a Mayer-Bolza problem of optimal control; for the fourth class, the optimization criterion is the minimization of the maximum altitude loss during dive recovery, resulting in a Chebyshev problem of optimal control. Each class of problems is solved using the sequential gradient-restoration algorithm for optimal control.

The studies were performed using a mathematical model of the jet fighter aircraft which accounts for all factors necessary to fully characterize aircraft performance in the flight regimes of interest. In particular, the model allows flight studies to include speeds from low subsonic (Mach number less than 0.1) to moderate supersonic (Mach number approaching 2.0), altitudes from sea level to 82 kft, angles of attack from 0 to 90 deg, and parametrized wing loading and thrust-to-weight ratios.

Among the four classes of optimal trajectories, only dive recovery benefits from the ability of a next-generation fighter aircraft to perform a supermaneuver, that is, to fly at extremely high angles of attack, even angles of attack approaching 90 deg. For the other three classes, relatively low angles of attack are required.

With the above studies completed, the following main conclusions emerge:

(i) Optimal climbs tend to require low angles of attack and are characterized by three distinct segments: a central segment often flown with a load factor of nearly 1 and
two terminal segments (dive or zoom) to and from the central segment. The central and final segments are nearly independent of the initial conditions, instead being dominated by the final conditions. If roll is permitted, certain trajectories with excess speed at the end of the zoom conclude with a moderately high load factor to align the velocity vector with the horizon. In all cases, the initial segment consists of either a zero load factor dive or a pitch-up with high load factor so as to reach the central segment.

(ii) Optimal fly-to-point trajectories consist of three segments: turning, accelerating, and cruising. The turning segment begins at a relatively high load factor; then, the load factor decreases smoothly toward \( n = 1 \), which is reached at the time the aircraft velocity vector points toward the destination point. The acceleration segment consists of a level acceleration at maximum thrust, which is followed in the final segment by resumption of steady-state cruising.

(iii) Optimal pop-up attack trajectories consist of three segments flown at maximum power: pitch-up, zoom, and pitch-down. The use of afterburner during the pop-up maneuver is quite beneficial, particularly in cases involving steep dive angles. The increase in flight time of the optimal pop-up maneuver resulting from increasing the heading change from 45 to 90 deg is large for shallow dive angles, and is small for steep dive angles. The increase in flight time of the optimal pop-up maneuver resulting from increasing the dive angle is quite large.

(iv) Optimal dive recovery trajectories consist of one to three segments. All the optimal trajectories conclude with a pitch-up at the maximum available load factor \( \tilde{n} \), which is the smaller between the maximum aerodynamically available load factor \( n_* \) and
the load factor limit \( n_{\text{max}} \). In optimal dive recovery trajectories starting with very low speed, the \( \tilde{n} \)-segment is preceded by a brief supermaneuver segment. In the special case of optimal dive recovery trajectories that start at very low speed coupled with very high flight path angle, the supermaneuver segment is preceded by a dive initiation segment.

The optimal trajectories reported here serve two purposes. First, they can benefit aircraft designers by highlighting those flight characteristics that are most beneficial in combat. Second, they can benefit aircraft pilots as the basis for guidance trajectories that approximate the optimal trajectories.

Future work will concern guidance schemes that allow the aircraft pilot to approximate the optimal trajectories in real time using information available in the aircraft. For fly-to-point and dive recovery maneuvers, simple candidate guidance schemes emerge from the optimal trajectories; these guidance schemes should be tested to verify that they approximate the optimal trajectories and that they are robust in the presence of dispersion effects. More difficult work will be required to devise guidance schemes for combat climb and pop-up attack maneuvers.
Appendix: Sequential Gradient-Restoration Algorithm

The problems of Sections 3 - 6 were solved using the sequential gradient-restoration algorithm for optimal control. SGRA consists of a sequence of cycles, each including a gradient phase and a restoration phase. In the gradient phase, the value of the augmented functional $J$ [see (93)] is reduced. In the restoration phase, the constraint error $P$ [see (97)] is reduced until a specified tolerance is achieved. The algorithm produces a sequence of feasible trajectories, each characterized by a lower value of the functional $I$ [see (91)] than that of the previous feasible trajectory.

The optimization problems under consideration are of either the Mayer type, Lagrange type, or Bolza type. Because the Mayer and Lagrange problems can be viewed as special cases of the Bolza problem, we give a summary of SGRA for the Bolza problem. For a more detailed description of the algorithm, see Refs. 10 - 11.

8.1. Optimization Problem. The optimization problem can be formulated as follows. Minimize the functional

$$ I = \int_{0}^{1} f(x, u, \pi, t) dt + [g(x, \pi)]_{1}, \quad (91) $$

with respect to the state $x(t)$, control $u(t)$, and parameter $\pi$, subject to the constraints

$$ \dot{x} = \phi(x, u, \pi, t), \quad 0 \leq t \leq 1, \quad (92a) $$

$$ S(x, u, \pi, t) = 0, \quad 0 \leq t \leq 1, \quad (92b) $$

$$ x(0) = \text{given}, \quad (92c) $$

$$ [\psi(x, \pi)]_{1} = 0. \quad (92d) $$
In the above equations, the independent variable is the time \( t \) (a scalar), \( 0 \leq t \leq 1 \); the quantities \( I, f, g \) are scalar, \( x \) and \( \phi \) are \( n \)-vectors, \( u \) is an \( m \)-vector, \( \pi \) is a \( p \)-vector, \( \psi \) is a \( q \)-vector, and \( S \) is a \( k \)-vector, \( k \leq m \). A trajectory satisfying (92) is called a feasible trajectory.

**First-Order Conditions.** From calculus of variations, the above problem can be recast as the minimization of the augmented functional

\[
J = I + L,
\]

subject to (92), with \( L \) is defined as

\[
L = \int_0^1 \left[ \lambda^T (\dot{x} - \phi) + \rho^T S \right] dt + (\mu^T \psi)_1.
\]

(94)

here, the \( n \)-vector \( \lambda(t) \) and \( k \)-vector \( \rho(t) \) are time-varying Lagrange multipliers and the \( q \)-vector \( \mu \) is a constant Lagrange multiplier. The first-order optimality conditions for the above problem are

\[
\dot{\lambda} = f_x - \phi_x \lambda + S_x \rho,
\]

\[
0 \leq t \leq 1,
\]

(95a)

\[
f_u - \phi_u \lambda + S_u \rho = 0,
\]

\[
0 \leq t \leq 1,
\]

(95b)

\[
\int_0^1 (f_\pi - \phi_\pi \lambda + S_\pi \rho) dt + (g_\pi + \psi_\pi \mu)_1 = 0,
\]

(95c)

\[
(\lambda + g_x + \psi_x \mu)_1 = 0.
\]

(95d)

To summarize, we seek the variables \( x(t), u(t), \pi \) and multipliers \( \lambda(t), \rho(t), \mu \) that satisfy the feasibility conditions (92) and optimality conditions (95). A trajectory satisfying (92) and (95) is called an optimal trajectory.
8.2. **Performance Measures.** Since the system (92) and (95) is generally non-linear, approximate methods must be used to seek a solution iteratively. Regardless of the technique employed, let the norm squared of a vector \( y \) be denoted by

\[
N(y) = y^T y.
\] (96)

Then, the constraint error \( P \) can be written as

\[
P = \int_0^1 N(\dot{x} - \phi) dt + \int_0^1 N(S) dt + N(\psi)_1,
\] (97)

and the error in the optimality conditions \( Q \) can be written as

\[
Q = \int_0^1 N(\lambda - f_x + \phi_x \lambda - S_x \rho) dt + \int_0^1 N(f_u - \phi_u \lambda + S_u \rho) dt
+ N\left[ \int_0^1 (f_\pi - \phi_\pi \lambda + S_\pi \rho) dt + (g_\pi + \psi_\pi \mu)_1 \right] + N(\lambda + g_\lambda + \psi_\lambda)_1.
\] (98)

For an exact optimal solution, \( P = 0 \) and \( Q = 0 \). For an approximation to the optimal solution, the following relations must be satisfied:

\[
P \leq \varepsilon_1,
\] (99a)

\[
Q \leq \varepsilon_2,
\] (99b)

where \( \varepsilon_1 \) and \( \varepsilon_2 \) are small, preselected positive numbers (for example, \( \varepsilon_1 = 10^{-20} \),

\( \varepsilon_2 = 10^{-4} \)).

8.3. **Description of Algorithm.** SGRA consists of a sequence of two-phase cycles, each containing a gradient phase and a restoration phase. The gradient phase is started when Ineq. (99a) is satisfied; it involves a single iteration and is designed to reduce the value of the augmented functional \( J \) while satisfying the constraints (92) to first order.
The restoration phase is initiated only if Ineq. (99a) is violated; it involves a number of iterations, each designed to reduce the constraint error $P$ while producing the least square change in the variations of the control $u(t)$ and parameter $\pi$. The restoration phase terminates whenever Ineq. (99a) is satisfied.

A complete gradient-restoration cycle begins with Ineq. (99a) satisfied; it ends with Ineq. (99a) satisfied and a decrease in the functional $I$. The algorithm terminates when Ineqs. (99) are both satisfied.

Let $x(t), u(t), \pi$ denote the nominal functions; let $\tilde{x}(t), \tilde{u}(t), \tilde{\pi}$ denote the varied functions; let $\Delta x(t), \Delta u(t), \Delta \pi$ denote the variations leading from the nominal functions to the varied functions; let $A(t), B(t), C$ denote the variations per unit stepsize $\eta$. Then, the following relations hold:

\begin{align}
\tilde{x}(t) &= x(t) + \Delta x(t) = x(t) + \eta A(t), \\
\tilde{u}(t) &= u(t) + \Delta u(t) = u(t) + \eta B(t), \\
\tilde{\pi} &= \pi + \Delta \pi = \pi + \eta C.
\end{align}

(100a) (100b) (100c)

Therefore, each iteration of the gradient phase and the restoration phase includes two operations: (i) determination of the variations per unit stepsize $A(t), B(t), C$; and (ii) determination of the stepsize $\eta$.

**Linear Two-Point Boundary-Value Problem.** The computation of the variations per unit stepsize for both the gradient phase and the restoration phase involves the solution of a linear two-point boundary-value problem which can be stated in general form. The linearized constraint equations are
\[ \dot{A} - \phi_x^T A - \phi_u^T B - \phi_n^T C + K_1 (\dot{x} - \phi) = 0, \quad 0 \leq t \leq 1, \tag{101a} \]
\[ S_x^T A + S_u^T B + S_n^T C + K_1 S = 0, \quad 0 \leq t \leq 1, \tag{101b} \]
\[ \left( \psi_x^T A + \psi_n^T C + K_1 \psi \right) = 0, \tag{101c} \]

and the linearized optimality equations are
\[ \dot{\lambda} - K_2 f_x + \phi_x \lambda - S_x \rho = 0, \quad 0 \leq t \leq 1, \tag{102a} \]
\[ B + K_2 f_u - \phi_u \lambda + S_u \rho = 0, \quad 0 \leq t \leq 1, \tag{102b} \]
\[ C + \int_0^1 (K_2 f_\pi - \phi_\pi \lambda + S_\pi \rho) dt + (K_2 g_\pi + \psi_\pi \mu)_1 = 0, \tag{102c} \]
\[ (\lambda + K_2 g_x + \psi_x \mu)_1 = 0, \tag{102d} \]

where \( K_1 = 0, K_2 = 1 \) for the gradient phase and \( K_1 = 1, K_2 = 0 \) for the restoration phase.

The linear two-point boundary-value problem (101) - (102) is solved with the method of particular solutions (MPS, Ref. 46). The MPS solution requires \( n + p + 1 \) independent sweeps of the differential system (101) - (102), each characterized by a different value of the \((n + p)\)-vector \( w \), with
\[ w = \left[ \lambda^T(0), \ C^T \right]^T. \tag{103} \]

For further details, see Ref. 46.

**Stepsize Search.** Once \( A(t), B(t), C \) and \( \lambda(t), \rho(t), \mu \) are known, one can form the one-parameter family of varied functions (100). For this one-parameter family, the functionals (91), (93), (97) depend only on the stepsize,
\[ I = \tilde{I}(\eta), \quad J = \tilde{J}(\eta), \quad P = \tilde{P}(\eta). \] (104)

For the gradient phase, the stepsize \( \eta \) is selected so that the following relations are satisfied:
\[ \tilde{J}(\eta) < \tilde{J}(0), \quad \tilde{P}(\eta) < P_* . \] (105)

Here, \( P_* \) is a preselected number, not necessarily small, which limits the constraint violation at the end of the gradient phase. Satisfaction of (105) is guaranteed by the descent property of the gradient phase and is enforced via a bisection process on \( \eta \), starting from a reference stepsize \( \eta_0 \). In turn, \( \eta_0 \) is obtained via a scanning process followed by cubic interpolation.

For the restoration phase, the stepsize \( \eta \) is selected so that
\[ \tilde{P}(\eta) < \tilde{P}(0). \] (106)

Satisfaction of (106) is guaranteed by the descent property of the restoration phase and is enforced via a bisection process on \( \eta \), starting from the reference stepsize \( \eta_0 = 1 \).

8.4. Summary of Algorithm. SGRA can be summarized as follows:

Step 1. Starting with nominal functions \( x(t), u(t), \pi \) that satisfy the initial conditions (92c), compute the constraint error via (97). If \( P \leq \varepsilon_1 \), proceed to Step 2. If \( P > \varepsilon_1 \), proceed to Step 3.

Step 2. For the gradient phase, solve the linear two-point boundary-value problem
\[ (101) - (102) \] with \( K_1 = 0, K_2 = 1 \). Check the error in the optimality conditions \( Q \) via (98). If \( Q \leq \varepsilon_2 \), the algorithm has converged; stop. If \( Q > \varepsilon_2 \), choose the step-
size $\eta$ such that Ineqs. (105) are both satisfied. Then, check the constraint error $P$.

If $P > \varepsilon_1$, proceed to Step 3. If $P \leq \varepsilon_1$, proceed to Step 4.

Step 3. For the restoration phase, solve the linear two-point boundary-value problem

$$(101) - (102)$$

with $K_1 = 1$, $K_2 = 0$. Determine the stepsize $\eta$ such that Ineq. (106) is satisfied. Check the constraint error $P$. If $P > \varepsilon_1$, repeat Step 3. If $P \leq \varepsilon_1$, proceed to Step 4.

Step 4. At the end of a gradient-restoration cycle, compute the value of the functional $I$ and compare it to that at the beginning of the cycle. If $I$ has decreased, return to Step 2. If $I$ has increased, bisect the gradient stepsize, and return to Step 3.
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Fig. 10C. Optimal climb trajectory A3: parameters, data, and results.

| Parameters | $W = 14$ klb, | $T_{\text{ref}} = 12$ klb, | $n_{\text{max}} = 9$. |
| Initial conditions | $x(0) = 0$ kft, | $y(0) = 0$ kft, | $h(0) = 5$ kft, |
| | $V(0) = 0.5$ kft/sec, | $\gamma(0) = 0$ deg, | $\chi(0) = 0$ deg. |
| Final conditions | $h(\tau) = 35$ kft, | $V(\tau) = 1.5$ kft/sec, | $\gamma(\tau) = 0$ deg, |
| | $\chi(\tau) = 0$ deg. |
| Results | $\tau = 283.8$ sec | $x(\tau) = 381.9$ kft | $y(\tau) = -0.3$ kft. |
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Fig. 11B. Optimal climb trajectory A10: medium weight, low initial speed, low final speed, roll permitted.

Fig. 11C. Optimal climb trajectory A10: parameters, data, and results.

<table>
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<th>Parameters</th>
<th>$W = 12$ klb, $T_{ref} = 12$ klb, $n_{max} = 9$.</th>
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<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft, $y(0) = 0$ kft, $h(0) = 5$ kft, $\gamma(0) = 0$ deg, $\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 35$ kft, $V(\tau) = 0.5$ kft/sec, $\gamma(\tau) = 0$ deg, $\chi(\tau) = 0$ deg.</td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 101.4$ sec $x(\tau) = 73.3$ kft $y(\tau) = -0.4$ kft.</td>
</tr>
</tbody>
</table>
Fig. 12A. Optimal climb trajectory A11: medium weight, low initial speed, medium final speed, roll permitted.
Fig. 12B. Optimal climb trajectory A11: medium weight, low initial speed, medium final speed, roll permitted.

Fig. 12C. Optimal climb trajectory A11: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( W = 12 \text{ klb} )</th>
<th>( T_{\text{ref}} = 12 \text{ klb} )</th>
<th>( n_{\text{max}} = 9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>( x(0) = 0 \text{ kft} ), ( y(0) = 0 \text{ kft} ), ( V(0) = 0.5 \text{ kft/sec} ), ( \gamma(0) = 0 \text{ deg} ), ( \chi(0) = 0 \text{ deg} )</td>
<td>( h(0) = 5 \text{ kft} )</td>
<td></td>
</tr>
<tr>
<td>Final conditions</td>
<td>( h(\tau) = 35 \text{ kft} ), ( V(\tau) = 1.0 \text{ kft/sec} ), ( \gamma(\tau) = 0 \text{ deg} )</td>
<td>( \chi(\tau) = 0 \text{ deg} )</td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>( \tau = 135.9 \text{ sec} )</td>
<td>( x(\tau) = 138.6 \text{ kft} )</td>
<td>( y(\tau) = -0.1 \text{ kft} )</td>
</tr>
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</table>
Fig. 13A. Optimal climb trajectory A12: medium weight, low initial speed, high final speed, roll permitted.
Fig. 13B. Optimal climb trajectory A12: medium weight, low initial speed, high final speed, roll permitted.

Fig. 13C. Optimal climb trajectory A12: parameters, data, and results.

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<th>$T_{\text{ref}} = 12$ klb,</th>
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<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 5$ kft,</td>
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<tr>
<td></td>
<td>$V(0) = 0.5$ kft/sec,</td>
<td>$\gamma(0) = 0$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 35$ kft,</td>
<td>$V(\tau) = 1.5$ kft/sec,</td>
<td>$\gamma(\tau) = 0$ deg,</td>
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<tr>
<td></td>
<td>$\chi(\tau) = 0$ deg.</td>
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<td></td>
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<tr>
<td>Results</td>
<td>$\tau = 170.8$ sec</td>
<td>$x(\tau) = 212.4$ kft</td>
<td>$y(\tau) = -0.2$ kft.</td>
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Fig. 14A. Optimal climb trajectory A13: medium weight, medium initial speed, low final speed, roll permitted.
Fig. 14B. Optimal climb trajectory A13: medium weight, medium initial speed, low final speed, roll permitted.

Fig. 14C. Optimal climb trajectory A13: parameters, data, and results.

<table>
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<tr>
<th>Parameters</th>
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<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
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</tr>
<tr>
<td></td>
<td>$V(0) = 1.0$ kft/sec,</td>
<td>$\gamma(0) = 0$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 35$ kft,</td>
<td>$V(\tau) = 0.5$ kft/sec,</td>
<td>$\gamma(\tau) = 0$ deg,</td>
</tr>
<tr>
<td></td>
<td>$\chi(\tau) = 0$ deg.</td>
<td></td>
<td></td>
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<tr>
<td>Results</td>
<td>$\tau = 79.3$ sec</td>
<td>$x(\tau) = 57.2$ kft</td>
<td>$y(\tau) = 0.0$ kft.</td>
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</tbody>
</table>
Fig. 15A. Optimal climb trajectory A14: medium weight, medium initial speed, medium final speed, roll permitted.
Fig. 15B. Optimal climb trajectory A14: medium weight, medium initial speed, medium final speed, roll permitted.

Fig. 15C. Optimal climb trajectory A14: parameters, data, and results.

<table>
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<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 5$ kft,</td>
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<tr>
<td></td>
<td>$V(0) = 1.0$ kft/sec,</td>
<td>$\gamma(0) = 0$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 35$ kft,</td>
<td>$V(\tau) = 1.0$ kft/sec,</td>
<td>$\gamma(\tau) = 0$ deg,</td>
</tr>
<tr>
<td></td>
<td>$\chi(\tau) = 0$ deg.</td>
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<tr>
<td>Results</td>
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<td>$x(\tau) = 122.2$ kft</td>
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Fig. 16A. Optimal climb trajectory A15: medium weight, medium initial speed, high final speed, roll permitted.
Fig. 16B. Optimal climb trajectory A15: medium weight, medium initial speed, high final speed, roll permitted.

Fig. 16C. Optimal climb trajectory A15: parameters, data, and results.

<table>
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</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 35 \text{ kft}$, $V(\tau) = 1.5 \text{ kft/sec}$, $\gamma(\tau) = 0 \text{ deg}$.</td>
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<tr>
<td>Results</td>
<td>$\tau = 148.9 \text{ sec}$, $x(\tau) = 196.0 \text{ kft}$, $y(\tau) = -0.1 \text{ kft}$.</td>
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Fig. 17A. Optimal climb trajectory A16: medium weight, high initial speed, low final speed, roll permitted.
Fig. 17B. Optimal climb trajectory A16: medium weight, high initial speed, low final speed, roll permitted.

Fig. 17C. Optimal climb trajectory A16: parameters, data, and results.

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<td></td>
<td>$V(0) = 1.5$ kft/sec,</td>
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<td>Final conditions</td>
<td>$h(\tau) = 35$ kft,</td>
<td>$V(\tau) = 0.5$ kft/sec,</td>
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<tr>
<td></td>
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<tr>
<td>Results</td>
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<td>$x(\tau) = 39.1$ kft</td>
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Fig. 18A. Optimal climb trajectory A17: medium weight, high initial speed, medium final speed, roll permitted.
Fig. 18B. Optimal climb trajectory A17: medium weight, high initial speed, medium final speed, roll permitted.

Fig. 18C. Optimal climb trajectory A17: parameters, data, and results.

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<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
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<tr>
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<td>$V(0) = 1.5$ kft/sec,</td>
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<td>$\chi(0) = 0$ deg.</td>
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<tr>
<td>Final conditions</td>
<td>$h(\tau) = 35$ kft,</td>
<td>$V(\tau) = 1.0$ kft/sec,</td>
<td>$\gamma(\tau) = 0$ deg,</td>
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<tr>
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<td>$\chi(\tau) = 0$ deg.</td>
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<td>Results</td>
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<td>$x(\tau) = 78.1$ kft</td>
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Fig. 19A. Optimal climb trajectory A18: medium weight, high initial speed, high final speed, roll permitted.
Fig. 19B. Optimal climb trajectory A18: medium weight, high initial speed, high final speed, roll permitted.

Fig. 19C. Optimal climb trajectory A18: parameters, data, and results.

<table>
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<tr>
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<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$\gamma(0) = 0$ kft,</td>
<td>$h(0) = 5$ kft,</td>
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<tr>
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<td>$V(0) = 1.5$ kft/sec,</td>
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<td>$\chi(0) = 0$ deg.</td>
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<tr>
<td>Final conditions</td>
<td>$h(\tau) = 35$ kft,</td>
<td>$V(\tau) = 1.5$ kft/sec,</td>
<td>$\gamma(\tau) = 0$ deg,</td>
</tr>
<tr>
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<td>$\chi(\tau) = 0$ deg.</td>
<td></td>
<td></td>
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<tr>
<td>Results</td>
<td>$\tau = 108.1$ sec</td>
<td>$x(\tau) = 150.4$ kft</td>
<td>$\gamma(\tau) = 0.0$ kft.</td>
</tr>
</tbody>
</table>
Fig. 20A. Optimal climb trajectory A30: heavy weight, low initial speed, high final speed, vertical plane.
Fig. 20B. Optimal climb trajectory A30: heavy weight, low initial speed, high final speed, vertical plane.

Fig. 20C. Optimal climb trajectory A30: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb,</th>
<th>$T_{ref} = 12$ klb,</th>
<th>$n_{\text{max}} = 9$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 5$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.5$ kft/sec,</td>
<td>$\gamma(0) = 0$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 35$ kft,</td>
<td>$V(\tau) = 1.5$ kft/sec,</td>
<td>$\gamma(\tau) = 0$ deg,</td>
</tr>
<tr>
<td></td>
<td>$\chi(\tau) = 0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 276.9$ sec</td>
<td>$x(\tau) = 367.9$ kft</td>
<td>$y(\tau) = 0.0$ kft.</td>
</tr>
</tbody>
</table>
Fig. 21A. Optimal climb trajectory A40: medium weight, medium initial speed, low final speed, vertical plane.
Fig. 21B. Optimal climb trajectory A40: medium weight, medium initial speed, low final speed, vertical plane.

Fig. 21C. Optimal climb trajectory A40: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 12$ klb, $T_{ref} = 12$ klb, $n_{max} = 9$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft, $y(0) = 0$ kft, $h(0) = 5$ kft, $\gamma(0) = 0$ deg, $\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 35$ kft, $V(\tau) = 0.5$ kft/sec, $\gamma(\tau) = 0$ deg, $\chi(\tau) = 0$ deg.</td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 80.1$ sec, $x(\tau) = 55.5$ kft, $y(\tau) = 0.0$ kft.</td>
</tr>
</tbody>
</table>
Fig. 22A. Optimal climb trajectory A41: medium weight, medium initial speed, medium final speed, vertical plane.
Fig. 22B. Optimal climb trajectory A41: medium weight, medium initial speed, medium final speed, vertical plane.

Fig. 22C. Optimal climb trajectory A41: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 12$ klb,</th>
<th>$T_{\text{ref}} = 12$ klb,</th>
<th>$n_{\max} = 9$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 5$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 1.0$ kft/sec,</td>
<td>$\gamma(0) = 0$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 35$ kft,</td>
<td>$V(\tau) = 1.0$ kft/sec,</td>
<td>$\gamma(\tau) = 0$ deg,</td>
</tr>
<tr>
<td></td>
<td>$\chi(\tau) = 0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 113.9$ sec</td>
<td>$x(\tau) = 121.8$ kft</td>
<td>$y(\tau) = 0.0$ kft.</td>
</tr>
</tbody>
</table>
Fig. 23A. Optimal climb trajectory A42: medium weight, medium initial speed, high final speed, vertical plane.
Fig. 23B. Optimal climb trajectory A42: medium weight, medium initial speed, high final speed, vertical plane.

Fig. 23C. Optimal climb trajectory A42: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 12$ klb,</th>
<th>$T_{ref} = 12$ klb,</th>
<th>$n_{max} = 9$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 5$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 1.0$ kft/sec,</td>
<td>$\gamma(0) = 0$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 35$ kft,</td>
<td>$V(\tau) = 1.5$ kft/sec,</td>
<td>$\gamma(\tau) = 0$ deg,</td>
</tr>
<tr>
<td></td>
<td>$\chi(\tau) = 0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 148.9$ sec</td>
<td>$x(\tau) = 195.8$ kft</td>
<td>$y(\tau) = 0.0$ kft.</td>
</tr>
</tbody>
</table>
Fig. 24A. Optimal climb trajectory A52: light weight, high initial speed, low final speed, vertical plane.
Fig. 24B. Optimal climb trajectory A52: light weight, high initial speed, low final speed, vertical plane.

Fig. 24C. Optimal climb trajectory A52: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 10$ klb,</th>
<th>$T_{\text{ref}} = 12$ klb,</th>
<th>$n_{\text{max}} = 9$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 5$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 1.5$ kft/sec,</td>
<td>$\gamma(0) = 0$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 35$ kft,</td>
<td>$V(\tau) = 0.5$ kft/sec,</td>
<td>$\gamma(\tau) = 0$ deg,</td>
</tr>
<tr>
<td></td>
<td>$\chi(\tau) = 0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 53.6$ sec</td>
<td>$x(\tau) = 28.2$ kft</td>
<td>$y(\tau) = 0.0$ kft.</td>
</tr>
</tbody>
</table>
Fig. 25.  Optimal climb trajectories A1, A4, A7: heavy weight, low final speed, roll permitted.

Fig. 26.  Optimal climb trajectories A28, A31, A34: heavy weight, low final speed, vertical plane.
Fig. 27. Optimal climb trajectories A2, A5, A8: heavy weight, medium final speed, roll permitted.

Fig. 28. Optimal climb trajectories A29, A32, A35: heavy weight, medium final speed, vertical plane.
Fig. 29. Optimal climb trajectories A3, A6, A9: heavy weight, high final speed, roll permitted.

Fig. 30. Optimal climb trajectories A30, A33, A36: heavy weight, high final speed, vertical plane.
Fig. 31. Optimal climb trajectories A28, A37, A46: low initial speed, low final speed, vertical plane.

Fig. 32. Optimal climb trajectories A30, A39, A48: low initial speed, high final speed, vertical plane.
Fig. 33. Optimal climb trajectories A1, A2, A3: heavy weight, low initial speed, roll permitted.

Fig. 34. Optimal climb trajectories A28, A29, A30: heavy weight, low initial speed, vertical plane.
Fig. 35. Optimal climb trajectories A4, A5, A6: heavy weight, medium initial speed, roll permitted.

Fig. 36. Optimal climb trajectories A31, A32, A33: heavy weight, medium initial speed, vertical plane.
Fig. 37. Optimal climb trajectories A7, A8, A9: heavy weight, high initial speed, roll permitted.

Fig. 38. Optimal climb trajectories A34, A35, A36: heavy weight, high initial speed, vertical plane.
Fig. 39A. Optimal fly-to-point trajectory B5: medium initial speed, final position vector direction 45 deg.
Fig. 39B. Optimal fly-to-point trajectory B5: medium initial speed, final position vector direction 45 deg.

Fig. 39C. Optimal fly-to-point trajectory B5: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ kib,</th>
<th>$T_{ref} = 6.8$ kib,</th>
<th>$n_{max} = 6$,</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\psi(\tau) = 45$ deg.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 5$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.8$ kft/sec,</td>
<td>$y(0) = 0$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$x(\tau) = 35.4$ kft</td>
<td>$y(\tau) = 35.4$ kft,</td>
<td>$h(\tau) = 5$ kft,</td>
</tr>
<tr>
<td></td>
<td>$\gamma(\tau) = 0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 64.1$ sec,</td>
<td>$V(\tau) = 0.8$ kft/sec,</td>
<td>$\chi(\tau) = 54$ deg.</td>
</tr>
</tbody>
</table>
Fig. 40A. Optimal fly-to-point trajectory B6: medium initial speed, final position vector direction 90 deg.
**Fig. 40B.** Optimal fly-to-point trajectory B6: medium initial speed, final position vector direction 90 deg.

**Fig. 40C.** Optimal fly-to-point trajectory B6: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klf,</th>
<th>$T_{ref} = 6.8$ klf,</th>
<th>$n_{max} = 6$,</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\psi(\tau) = 90$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ klf,</td>
<td>$y(0) = 0$ klf,</td>
<td>$h(0) = 5$ klf,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.8$ klf/sec,</td>
<td>$\gamma(0) = 0$ deg.</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$x(\tau) = 0.0$ klf</td>
<td>$y(\tau) = 50.0$ klf,</td>
<td>$h(\tau) = 5$ klf,</td>
</tr>
<tr>
<td></td>
<td>$\gamma(\tau) = 0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 69.8$ sec,</td>
<td>$V(\tau) = 0.8$ klf/sec,</td>
<td>$\chi(\tau) = 103$ deg.</td>
</tr>
</tbody>
</table>
Fig. 41A. Optimal fly-to-point trajectory B7: medium initial speed, final position vector direction 135 deg.
Fig. 41B. Optimal fly-to-point trajectory B7: medium initial speed, final position vector direction 135 deg.

Fig. 41C. Optimal fly-to-point trajectory B7: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb,</th>
<th>$T_{\text{ref}} = 6.8$ klb,</th>
<th>$n_{\text{max}} = 6$,</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\psi(\tau) = 135$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 5$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.8$ kft/sec,</td>
<td>$\gamma(0) = 0$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$x(\tau) = -35.4$ kft</td>
<td>$y(\tau) = 35.4$ kft,</td>
<td>$h(\tau) = 5$ kft,</td>
</tr>
<tr>
<td></td>
<td>$\gamma(\tau) = 0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 78.5$ sec,</td>
<td>$V(\tau) = 0.8$ kft/sec,</td>
<td>$\chi(\tau) = 149$ deg.</td>
</tr>
</tbody>
</table>
Fig. 42A. Optimal fly-to-point trajectory B8: medium initial speed, final position vector direction 180 deg.
Fig. 42B. Optimal fly-to-point trajectory B8: medium initial speed, final position vector direction 180 deg.

Fig. 42C. Optimal fly-to-point trajectory B8: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14 \text{k lb}$, $T_{ref} = 6.8 \text{k lb}$, $n_{max} = 6$, $\psi(\tau) = 180 \text{ deg}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0 \text{k ft}$, $y(0) = 0 \text{k ft}$, $h(0) = 5 \text{k ft}$, $\gamma(0) = 0 \text{ deg}$, $\chi(0) = 0 \text{ deg}$</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$x(\tau) = -50.0 \text{k ft}$, $y(\tau) = 0.0 \text{k ft}$, $h(\tau) = 5 \text{k ft}$, $\gamma(\tau) = 0 \text{ deg}$</td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 88.7 \text{ sec}$, $V(\tau) = 0.8 \text{k ft/sec}$, $\chi(\tau) = 193 \text{ deg}$</td>
</tr>
</tbody>
</table>
Fig. 43A. Optimal fly-to-point trajectory B13: medium initial speed, final position vector direction 180 deg.
Fig. 43B. Optimal fly-to-point trajectory B13: medium initial speed, final position vector direction 180 deg.

Fig. 43C. Optimal fly-to-point trajectory B13: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb, $\psi(\tau) = 180$ deg.</th>
<th>$T_{\text{ref}} = 6.8$ klb, $n_{\text{max}} = 5$,</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft, $V(0) = 0.8$ kft/sec,</td>
<td>$y(0) = 0$ kft, $\gamma(0) = 0$ deg, $h(0) = 5$ kft,</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$x(\tau) = -50.0$ kft, $\gamma(\tau) = 0$ deg.</td>
<td>$y(\tau) = 0.0$ kft, $\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 90.2$ sec, $V(\tau) = 0.8$ kft/sec,</td>
<td>$\chi(\tau) = 194$ deg.</td>
</tr>
</tbody>
</table>
Fig. 44A. Optimal fly-to-point trajectory B14: medium initial speed, final position vector direction 180 deg.
Fig. 44B. Optimal fly-to-point trajectory B14: medium initial speed, final position vector direction 180 deg.

Fig. 44C. Optimal fly-to-point trajectory B14: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb, $T_{\text{ref}} = 6.8$ klb, $n_{\text{max}} = 4$, $\psi(\tau) = 180$ deg.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft, $y(0) = 0$ kft, $h(0) = 5$ kft, $\gamma(0) = 0$ deg, $\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$x(\tau) = -50.0$ kft, $y(\tau) = 0.0$ kft, $h(\tau) = 5$ kft, $\gamma(\tau) = 0$ deg.</td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 91.6$ sec, $V(\tau) = 0.8$ kft/sec, $\chi(\tau) = 195$ deg.</td>
</tr>
</tbody>
</table>
Fig. 45A. Optimal fly-to-point trajectory B15: medium initial speed, final position vector direction 180 deg.
Fig. 45B. Optimal fly-to-point trajectory B15: medium initial speed, final position vector direction 180 deg.

Fig. 45C. Optimal fly-to-point trajectory B15: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb,</th>
<th>$T_{ref} = 6.8$ klb,</th>
<th>$n_{max} = 3$,</th>
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<tbody>
<tr>
<td></td>
<td>$\psi(\tau) = 180$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 5$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.8$ kft/sec,</td>
<td>$y(0) = 0$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$x(\tau) = -50.0$ kft</td>
<td>$y(\tau) = 0.0$ kft,</td>
<td>$h(\tau) = 5$ kft,</td>
</tr>
<tr>
<td></td>
<td>$\gamma(\tau) = 0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 96.0$ sec,</td>
<td>$V(\tau) = 0.8$ kft/sec,</td>
<td>$\chi(\tau) = 197$ deg.</td>
</tr>
</tbody>
</table>
Fig. 46A. Optimal fly-to-point trajectory B16: medium initial speed, final position vector direction 180 deg.
Fig. 46B. Optimal fly-to-point trajectory B16: medium initial speed, final position vector direction 180 deg.

Fig. 46C. Optimal fly-to-point trajectory B16: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14 \text{ klb,}$</th>
<th>$T_{\text{ref}} = 6.8 \text{ klb,}$</th>
<th>$n_{\text{max}} = 2,$</th>
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</thead>
<tbody>
<tr>
<td>$\psi(\tau) = 180 \text{ deg.}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0 \text{ kft,}$</td>
<td>$y(0) = 0 \text{ kft,}$</td>
<td>$h(0) = 5 \text{ kft,}$</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.8 \text{ kft/sec,}$</td>
<td>$\gamma(0) = 0 \text{ deg,}$</td>
<td>$\chi(0) = 0 \text{ deg.}$</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$x(\tau) = -50.0 \text{ kft}$</td>
<td>$y(\tau) = 0.0 \text{ kft,}$</td>
<td>$h(\tau) = 5 \text{ kft,}$</td>
</tr>
<tr>
<td></td>
<td>$\gamma(\tau) = 0 \text{ deg.}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 112.4 \text{ sec,}$</td>
<td>$V(\tau) = 0.8 \text{ kft/sec,}$</td>
<td>$\chi(\tau) = 204 \text{ deg.}$</td>
</tr>
</tbody>
</table>
Fig. 47A. Optimal pop-up trajectory C1: 10 deg dive angle, 45 deg heading change, afterburner excluded.
Fig. 47B. Optimal pop-up trajectory C1: 10 deg. dive angle, 45 deg. heading change, afterburner excluded.

Fig. 47C. Optimal pop-up trajectory C1: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( W = 14 \text{ klb} )</th>
<th>( T_{ref} = 6.8 \text{ klb} )</th>
<th>( n_{max} = 6 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>( x(0) = 0 \text{ kft} )</td>
<td>( y(0) = 0 \text{ kft} )</td>
<td>( h(0) = 1 \text{ kft} )</td>
</tr>
<tr>
<td></td>
<td>( V(0) = 0.9 \text{ kft/sec} )</td>
<td>( \gamma(0) = 0 \text{ deg} )</td>
<td>( \chi(0) = 0 \text{ deg} )</td>
</tr>
<tr>
<td>Final conditions</td>
<td>( h(\tau) = 2 \text{ kft} )</td>
<td>( V(\tau) = 0.8 \text{ kft/sec} )</td>
<td>( \gamma(\tau) = -10 \text{ deg} )</td>
</tr>
<tr>
<td></td>
<td>( \chi(\tau) = 45 \text{ deg} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>( \tau = 13.8 \text{ sec} )</td>
<td>( x(\tau) = 9.1 \text{ kft} )</td>
<td>( y(\tau) = 5.2 \text{ kft} )</td>
</tr>
</tbody>
</table>
Fig. 48A. Optimal pop-up trajectory C4: 45 deg dive angle, 45 deg heading change, afterburner excluded.
Fig. 48B. Optimal pop-up trajectory C4: 45 deg. dive angle, 45 deg. heading change, afterburner excluded.

Fig. 48C. Optimal pop-up trajectory C4: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14, \text{kib}$,</th>
<th>$T_{\text{ref}} = 6.8, \text{kib}$,</th>
<th>$n_{\text{max}} = 6$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0, \text{kib}$,</td>
<td>$y(0) = 0, \text{kib}$,</td>
<td>$h(0) = 1, \text{kib}$,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.9, \text{kib/\text{sec}}$,</td>
<td>$\gamma(0) = 0, \text{deg}$,</td>
<td>$\chi(0) = 0, \text{deg}$.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 7, \text{kib}$,</td>
<td>$V(\tau) = 0.8, \text{kib/\text{sec}}$,</td>
<td>$\gamma(\tau) = -45, \text{deg},$</td>
</tr>
<tr>
<td></td>
<td>$\chi(\tau) = 45, \text{deg}$.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 44.8, \text{sec}$,</td>
<td>$x(\tau) = 25.1, \text{kib}$,</td>
<td>$y(\tau) = 11.3, \text{kib}$.</td>
</tr>
</tbody>
</table>
Fig. 49A. Optimal pop-up trajectory C5: 10 deg dive angle, 90 deg heading change, afterburner excluded.
Fig. 49B. Optimal pop-up trajectory C5: 10 deg. dive angle, 90 deg. heading change, afterburner excluded.

Fig. 49C. Optimal pop-up trajectory C5: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb,</th>
<th>$T_{\text{ref}} = 6.8$ klb,</th>
<th>$n_{\text{max}} = 6$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 1$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.9$ kft/sec,</td>
<td>$\gamma(0) = 0$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 2$ kft,</td>
<td>$V(\tau) = 0.8$ kft/sec,</td>
<td>$\gamma(\tau) = -10$ deg,</td>
</tr>
<tr>
<td></td>
<td>$\chi(\tau) = 90$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 23.3$ sec,</td>
<td>$x(\tau) = 7.5$ kft,</td>
<td>$y(\tau) = 13.6$ kft.</td>
</tr>
</tbody>
</table>
Fig. 50A. Optimal pop-up trajectory C8: 45 deg dive angle, 90 deg heading change, afterburner excluded.
**Fig. 50B.** Optimal pop-up trajectory C8: 45 deg. dive angle, 90 deg. heading change, afterburner excluded.

**Fig. 50C.** Optimal pop-up trajectory C8: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ kib,</th>
<th>$T_{ref} = 6.8$ kib,</th>
<th>$n_{max} = 6$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 1$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.9$ kft/sec,</td>
<td>$\gamma(0) = 0$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 7$ kft,</td>
<td>$V(\tau) = 0.8$ kft/sec,</td>
<td>$\gamma(\tau) = -45$ deg.</td>
</tr>
<tr>
<td></td>
<td>$\chi(\tau) = 90$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 51.0$ sec,</td>
<td>$x(\tau) = 19.3$ kft,</td>
<td>$y(\tau) = 18.0$ kft.</td>
</tr>
</tbody>
</table>
Fig. 51A. Optimal pop-up trajectory C9: 10 deg dive angle, 45 deg heading change, afterburner permitted.
Fig. 51B. Optimal pop-up trajectory C9: 10 deg. dive angle, 45 deg. heading change, afterburner permitted.

Fig. 51C. Optimal pop-up trajectory C9: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14 \text{ klb}$, $T_{\text{ref}} = 12.0 \text{ klb}$, $n_{\text{max}} = 6$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0 \text{ kft}$, $y(0) = 0 \text{ kft}$, $h(0) = 1 \text{ kft}$, $\gamma(0) = 0 \text{ deg}$, $\chi(0) = 0 \text{ deg}$.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 2 \text{ kft}$, $V(\tau) = 0.8 \text{ kft/sec}$, $\gamma(\tau) = -10 \text{ deg}$, $\chi(\tau) = 45 \text{ deg}$.</td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 9.2 \text{ sec}$, $x(\tau) = 6.6 \text{ kft}$, $y(\tau) = 3.0 \text{ kft}$</td>
</tr>
</tbody>
</table>
Fig. 52A. Optimal pop-up trajectory C12: 45 deg dive angle, 45 deg heading change, afterburner permitted.
Fig. 52B. Optimal pop-up trajectory C12: 45 deg. dive angle, 45 deg. heading change, afterburner permitted.

Fig. 52C. Optimal pop-up trajectory C12: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb,</th>
<th>$T_{ref} = 12.0$ klb,</th>
<th>$n_{max} = 6$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 1$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.9$ kft/sec,</td>
<td>$y(0) = 0$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 7$ kft,</td>
<td>$V(\tau) = 0.8$ kft/sec,</td>
<td>$\gamma(\tau) = -45$ deg,</td>
</tr>
<tr>
<td></td>
<td>$\chi(\tau) = 45$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 25.2$ sec,</td>
<td>$x(\tau) = 15.9$ kft,</td>
<td>$y(\tau) = 6.7$ kft.</td>
</tr>
</tbody>
</table>
Fig. 53A. Optimal pop-up trajectory C13: 10 deg dive angle, 90 deg heading change, afterburner permitted.
Fig. 53B. Optimal pop-up trajectory C13: 10 deg. dive angle, 90 deg. heading change, afterburner permitted.

Fig. 53C. Optimal pop-up trajectory C13: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb, $T_{ref} = 12.0$ klb, $n_{max} = 6$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft, $y(0) = 0$ kft, $h(0) = 1$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.9$ kft/sec, $\gamma(0) = 0$ deg, $\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 2$ kft, $V(\tau) = 0.8$ kft/sec, $\gamma(\tau) = -10$ deg,</td>
</tr>
<tr>
<td></td>
<td>$\chi(\tau) = 90$ deg.</td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 13.2$ sec, $x(\tau) = 5.0$ kft, $y(\tau) = 7.2$ kft.</td>
</tr>
</tbody>
</table>
Fig. 54A. Optimal pop-up trajectory C16: 45 deg dive angle, 90 deg heading change, afterburner permitted.
Fig. 54B. Optimal pop-up trajectory C16: 45 deg. dive angle, 90 deg. heading change, afterburner permitted.

Fig. 54C. Optimal pop-up trajectory C16: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb,</th>
<th>$T_{ref} = 12.0$ klb,</th>
<th>$n_{max} = 6$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 1$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.9$ kft/sec,</td>
<td>$\gamma(0) = 0$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$h(\tau) = 7$ kft,</td>
<td>$V(\tau) = 0.8$ kft/sec,</td>
<td>$\gamma(\tau) = -45$ deg,</td>
</tr>
<tr>
<td></td>
<td>$\chi(\tau) = 90$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 27.8$ sec,</td>
<td>$x(\tau) = 11.7$ kft,</td>
<td>$y(\tau) = 12.2$ kft.</td>
</tr>
</tbody>
</table>
Fig. 55A. Optimal dive recovery trajectory D1: very low initial speed, climbing initial flight path angle, afterburner excluded.
Fig. 55B.  Optimal dive recovery trajectory D1: very low initial speed, climbing initial flight path angle, afterburner excluded.

Fig. 55C.  Optimal dive recovery trajectory D1: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb,</th>
<th>$T_{ref} = 6.8$ klb,</th>
<th>$n_{max} = 9$.</th>
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</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 10$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.05$ kft/sec,</td>
<td>$\gamma(0) = 85$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$\gamma(\tau) = 0.0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 21.1$ sec,</td>
<td>$x(\tau) = 3.3$ kft,</td>
<td>$y(\tau) = 0.0$ kft.</td>
</tr>
<tr>
<td></td>
<td>$h(\tau) = 7.93$ kft,</td>
<td>$V(\tau) = 0.3$ kft/sec,</td>
<td>$\chi(\tau) = 0$ deg.</td>
</tr>
</tbody>
</table>
Fig. 56A. Optimal dive recovery trajectory D9: high initial speed, diving initial flight path angle, afterburner excluded.
Fig. 56B. Optimal dive recovery trajectory D9: high initial speed, diving initial flight path angle, afterburner excluded.

Fig. 56C. Optimal dive recovery trajectory D9: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb,</th>
<th>$T_{\text{ref}} = 6.8$ klb,</th>
<th>$n_{\text{max}} = 9$.</th>
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</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 10$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 1.50$ kft/sec,</td>
<td>$\gamma(0) = -45$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$\gamma(\tau) = 0.0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 3.6$ sec,</td>
<td>$x(\tau) = 3.8$ kft,</td>
<td>$y(\tau) = 0.0$ kft.</td>
</tr>
<tr>
<td></td>
<td>$h(\tau) = 8.16$ kft,</td>
<td>$V(\tau) = 0.9$ kft/sec,</td>
<td>$\chi(\tau) = 0$ deg.</td>
</tr>
</tbody>
</table>
Fig. 57A. Optimal dive recovery trajectory D11: very low initial speed, climbing initial flight path angle, afterburner permitted.
Fig. 57B. Optimal dive recovery trajectory D11: very low initial speed, climbing initial flight path angle, afterburner permitted.

Fig. 57C. Optimal dive recovery trajectory D11: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14 \text{ klb}$,</th>
<th>$T_{\text{ref}} = 12.0 \text{ klb}$,</th>
<th>$n_{\text{max}} = 9$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0 \text{ kft}$,</td>
<td>$y(0) = 0 \text{ kft}$,</td>
<td>$h(0) = 10 \text{ kft}$,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.05 \text{ kft/sec}$,</td>
<td>$\gamma(0) = 85 \text{ deg}$,</td>
<td>$\chi(0) = 0 \text{ deg}$,</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$\gamma(\tau) = 0.0 \text{ deg}$.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 18.6 \text{ sec}$,</td>
<td>$x(\tau) = 2.8 \text{ kft}$,</td>
<td>$\gamma(\tau) = 0.0 \text{ kft}$.</td>
</tr>
</tbody>
</table>
| | $h(\tau) = 8.83 \text{ kft}$, | $V(\tau) = 0.3 \text{ kft/sec}$, | $\chi(\tau) = 0 \text{ deg}$.
Fig. 58A. Optimal dive recovery trajectory D12: very low initial speed, level initial flight path angle, afterburner permitted.
Fig. 58B. Optimal dive recovery trajectory D12: very low initial speed, level initial flight path angle, afterburner permitted.

Fig. 58C. Optimal dive recovery trajectory D12: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( W = 14 \text{ klb} )</th>
<th>( T_{ref} = 12.0 \text{ klb} )</th>
<th>( n_{max} = 9 ).</th>
</tr>
</thead>
</table>
| Initial conditions | \( x(0) = 0 \text{ kft} \), \( y(0) = 0 \text{ kft} \), \( V(0) = 0.05 \text{ kft/sec} \), \( \gamma(0) = 0 \text{ deg} \), \( \chi(0) = 0 \text{ deg} \) | \( h(0) = 10 \text{ kft} \), \( \gamma(0) = 0 \text{ kft} \), \( \chi(0) = 0 \text{ deg} \) |\( 
| Final conditions  | \( \gamma(\tau) = 0.0 \text{ deg} \) |\( 
| Results          | \( \tau = 14.5 \text{ sec} \), \( x(\tau) = 2.7 \text{ kft} \), \( \gamma(\tau) = 0.0 \text{ kft} \), \( h(\tau) = 8.92 \text{ kft} \), \( V(\tau) = 0.3 \text{ kft/sec} \), \( \chi(\tau) = 0 \text{ deg} \) |\( 

Fig. 59A. Optimal dive recovery trajectory D13: very low initial speed, diving initial flight path angle, afterburner permitted.
Fig. 59B. Optimal dive recovery trajectory D13: very low initial speed, diving initial flight path angle, afterburner permitted.

Fig. 59C. Optimal dive recovery trajectory D13: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb,</th>
<th>$T_{ref} = 12.0$ klb,</th>
<th>$n_{max} = 9$.</th>
</tr>
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<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 10$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.05$ kft/sec,</td>
<td>$\gamma(0) = -45$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$\gamma(\tau) = 0.0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 14.2$ sec,</td>
<td>$x(\tau) = 2.7$ kft,</td>
<td>$y(\tau) = 0.0$ kft.</td>
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<td></td>
<td>$h(\tau) = 8.57$ kft,</td>
<td>$V(\tau) = 0.3$ kft/sec,</td>
<td>$\chi(\tau) = 0$ deg.</td>
</tr>
</tbody>
</table>
Fig. 60A. Optimal dive recovery trajectory D14: very low initial speed, steep diving initial flight path angle, afterburner permitted.
Fig. 60B. Optimal dive recovery trajectory D14: very low initial speed, steep diving initial flight path angle, afterburner permitted.

Fig. 60C. Optimal dive recovery trajectory D14: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
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<th>$T_{ref} = 12.0$ klb,</th>
<th>$n_{max} = 9$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 10$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.05$ kft/sec,</td>
<td>$\gamma(0) = -85$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$\gamma(\tau) = 0.0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 14.3$ sec,</td>
<td>$x(\tau) = 2.6$ kft,</td>
<td>$y(\tau) = 0.0$ kft.</td>
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<tr>
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<td>$h(\tau) = 8.26$ kft,</td>
<td>$V(\tau) = 0.3$ kft/sec,</td>
<td>$\chi(\tau) = 0$ deg.</td>
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</tbody>
</table>
Fig. 61A. Optimal dive recovery trajectory D15: low initial speed, diving initial flight path angle, afterburner permitted.
Fig. 61B. Optimal dive recovery trajectory D15: low initial speed, diving initial flight path angle, afterburner permitted.

Fig. 61C. Optimal dive recovery trajectory D15: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
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<th>$T_{ref} = 12.0$ klbf,</th>
<th>$n_{max} = 9$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 10$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.30$ kft/sec,</td>
<td>$\gamma(0) = -45$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$\gamma(\tau) = 0.0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 6.8$ sec,</td>
<td>$x(\tau) = 1.9$ kft,</td>
<td>$y(\tau) = 0.0$ kft.</td>
</tr>
<tr>
<td></td>
<td>$h(\tau) = 9.23$ kft,</td>
<td>$V(\tau) = 0.3$ kft/sec,</td>
<td>$\chi(\tau) = 0$ deg.</td>
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</tbody>
</table>
Fig. 62A. Optimal dive recovery trajectory D18: medium initial speed, steep diving initial flight path angle, afterburner permitted.
Fig. 62B. Optimal dive recovery trajectory D18: medium initial speed, steep diving initial flight path angle, afterburner permitted.

Fig. 62C. Optimal dive recovery trajectory D18: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb,</th>
<th>$T_{ref} = 12.0$ klb,</th>
<th>$n_{max} = 9$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 10$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 1.00$ kft/sec,</td>
<td>$g(0) = -85$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$\gamma(\tau) = 0.0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 4.6$ sec,</td>
<td>$x(\tau) = 2.1$ kft,</td>
<td>$y(\tau) = 0.0$ kft.</td>
</tr>
<tr>
<td></td>
<td>$h(\tau) = 7.79$ kft,</td>
<td>$V(\tau) = 0.5$ kft/sec,</td>
<td>$\chi(\tau) = 0$ deg.</td>
</tr>
</tbody>
</table>
Fig. 63A. Optimal dive recovery trajectory D20: high initial speed, steep diving initial flight path angle, afterburner permitted.
Fig. 63B. Optimal dive recovery trajectory D20: high initial speed, steep diving initial flight path angle, afterburner permitted.

Fig. 63C. Optimal dive recovery trajectory D20: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>( W = 14 \text{ klb} )</th>
<th>( T_{\text{ref}} = 12.0 \text{ klb} )</th>
<th>( n_{\text{max}} = 9 ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>( x(0) = 0 \text{ kft} )</td>
<td>( y(0) = 0 \text{ kft} )</td>
<td>( h(0) = 10 \text{ kft} )</td>
</tr>
<tr>
<td></td>
<td>( V(0) = 1.50 \text{ kft/sec} )</td>
<td>( \gamma(0) = -85 \text{ deg} )</td>
<td>( \chi(0) = 0 \text{ deg} )</td>
</tr>
<tr>
<td>Final conditions</td>
<td>( \gamma(\tau) = 0.0 \text{ deg} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>( \tau = 5.7 \text{ sec} )</td>
<td>( x(\tau) = 3.4 \text{ kft} )</td>
<td>( y(\tau) = 0.0 \text{ kft} )</td>
</tr>
<tr>
<td></td>
<td>( h(\tau) = 5.51 \text{ kft} )</td>
<td>( V(\tau) = 0.7 \text{ kft/sec} )</td>
<td>( \chi(\tau) = 0 \text{ deg} )</td>
</tr>
</tbody>
</table>
Fig. 64A. Optimal supplementary dive recovery trajectory D25: very low initial speed, climbing initial flight path angle, afterburner permitted.
Fig. 64B. Optimal supplementary dive recovery trajectory D25: very low initial, speed, climbing initial flight path angle, afterburner permitted.

Fig. 64C. Optimal supplementary dive recovery trajectory D25: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb, $T_{ref} = 12.0$ klb, $n_{max} = 9.$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft, $y(0) = 0$ kft, $h(0) = 10$ kft, $V(0) = 0.05$ kft/sec, $\gamma(0) = 85$ deg, $\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$\gamma(\tau) = 0.0$ deg.</td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 17.7$ sec, $x(\tau) = 2.7$ kft, $y(\tau) = 0.0$ kft, $h(\tau) = 8.78$ kft, $V(\tau) = 0.3$ kft/sec, $\chi(\tau) = 0$ deg.</td>
</tr>
</tbody>
</table>
Fig. 65A. Optimal supplementary dive recovery trajectory D26: very low initial speed, level initial flight path angle, afterburner permitted.
Fig. 65B. Optimal supplementary dive recovery trajectory D26: very low initial, speed, level initial flight path angle, afterburner permitted.

Fig. 65C. Optimal supplementary dive recovery trajectory D26: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb, $T_{ref} = 12.0$ klb, $n_{\text{max}} = 9$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft, $y(0) = 0$ kft, $h(0) = 10$ kft, $\gamma(0) = 0$ deg, $\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$\gamma(\tau) = 0.0$ deg.</td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 14.1$ sec, $x(\tau) = 2.7$ kft, $y(\tau) = 0.0$ kft.</td>
</tr>
<tr>
<td></td>
<td>$h(\tau) = 8.90$ kft, $V(\tau) = 0.3$ kft/sec, $\chi(\tau) = 0$ deg.</td>
</tr>
</tbody>
</table>
Fig. 66A. Optimal supplementary dive recovery trajectory D27: very low initial speed, diving initial flight path angle, afterburner permitted.
Fig. 66B. Optimal supplementary dive recovery trajectory D27: very low initial speed, diving initial flight path angle, afterburner permitted.

Fig. 66C. Optimal supplementary dive recovery trajectory D27: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ klb,</th>
<th>$T_{ref} = 12.0$ klb,</th>
<th>$n_{max} = 9$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 10$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.05$ kft/sec,</td>
<td>$\gamma(0) = -45$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$\gamma(\tau) = 0.0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 13.4$ sec,</td>
<td>$x(\tau) = 2.6$ kft,</td>
<td>$\gamma(\tau) = 0.0$ kft.</td>
</tr>
<tr>
<td></td>
<td>$h(\tau) = 8.50$ kft,</td>
<td>$V(\tau) = 0.3$ kft/sec,</td>
<td>$\chi(\tau) = 0$ deg.</td>
</tr>
</tbody>
</table>
Fig. 67A. Optimal supplementary dive recovery trajectory D28: very low initial speed, steep diving initial flight path angle, afterburner permitted.
Fig. 67B. Optimal supplementary dive recovery trajectory D28: very low initial, speed, steep diving initial flight path angle, afterburner permitted.

Fig. 67C. Optimal supplementary dive recovery trajectory D28: parameters, data, and results.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$W = 14$ kllb,</th>
<th>$T_{ref} = 12.0$ kllb,</th>
<th>$n_{max} = 9$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial conditions</td>
<td>$x(0) = 0$ kft,</td>
<td>$y(0) = 0$ kft,</td>
<td>$h(0) = 10$ kft,</td>
</tr>
<tr>
<td></td>
<td>$V(0) = 0.05$ kft/sec,</td>
<td>$\gamma(0) = -85$ deg,</td>
<td>$\chi(0) = 0$ deg.</td>
</tr>
<tr>
<td>Final conditions</td>
<td>$\gamma(\tau) = 0.0$ deg.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Results</td>
<td>$\tau = 13.5$ sec,</td>
<td>$x(\tau) = 2.5$ kft,</td>
<td>$y(\tau) = 0.0$ kft.</td>
</tr>
<tr>
<td></td>
<td>$h(\tau) = 8.10$ kft,</td>
<td>$V(\tau) = 0.3$ kft/sec,</td>
<td>$\chi(\tau) = 0$ deg.</td>
</tr>
</tbody>
</table>
Fig. 68. Load factor $n$ vs angle of attack $\alpha$ for $h = 10$ kft, $\beta = 1$, and three values of velocity.

Fig. 69. Maximum available load factor $\bar{n}$ vs velocity $V$ for $\beta = 1$ and three values of altitude.
Fig. 70. Comparison of maximum available load factor $\bar{n}$ and optimal load factor $n$ for Trajectory D15.

Fig. 71. Comparison of maximum available load factor $\bar{n}$ and optimal load factor $n$ for Trajectory D18.
IMAGE EVALUATION
TEST TARGET (QA-3)

1.0  1.1  1.25  1.4  1.6

2.0  1.8

2.5  2.2

2.8  2.6

1.0  1.1  1.25  1.4  1.6

150mm

6"

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