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Optimal Design Problems for Quasidisks and Partially Clamped Drums: Existence, Symmetrization, and Numerical Methods

by

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Abstract

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It is shown that the class of quasidisks in the complex plane, with fixed quasicircle constant and area, is compact in both the Hausdorff metric and in the sense of Caratheodory convergence. Compactness for chord-arc domains with fixed chord-arc constant and area is shown as a result of the quasidisk compactness. Compactness is used to show that each eigenvalue of the Laplacian, subject to Dirichlet boundary conditions, attains its extrema over each of these classes.

The design problem of extremizing the fundamental tone of a drum fastened only on a fraction of the boundary is considered. The special case of minimizing the fundamental frequency of a circular drum is solved using symmetrization. The gradient of the tone of the drum with respect to the design is considered and approximated appropriately. This approximate gradient is then used to compute examples numerically.
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To the Lord Jesus Christ,
my loving wife Carmen,
Mom and Dad,
and everyone else
who has waited so long and
patiently for this work to finish.
Chapter 1

Quasidisks and Chord-Arc Domains

1.1 Introduction

For a bounded open subset $\Omega$ of the plane we denote by $\lambda_k(\Omega)$ the $k$th eigenvalue of $-\Delta$ on $H^1_0(\Omega)$. The question of finding extremal values of $\lambda_k(\Omega)$, where $\Omega$ is allowed to vary over a suitably restricted class of open sets, has a strong historical foundation. In general, the problem is to find the extrema of a given functional of a domain, over a given class of domains, and whenever possible describe the extremizing domains. The problem of extremizing more general domain functionals over certain restricted classes falls under the auspices of optimal shape design.

The story for Dirichlet eigenvalues begins with the class of open sets in the plane with a fixed area. Faber [12] and Krahn [21] independently proved a conjecture of Rayleigh that among all sets in this class, the smallest eigenvalue, $\Omega \mapsto \lambda_1(\Omega)$, achieves its minimum on the disk with the given fixed area. P. Szegö proved, and Pólya [26] published, the fact that $\Omega \mapsto \lambda_2(\Omega)$ attains its minimum over this class, and that the minimizer is the union of two disks of equal radius. More recently, Buttazzo and Dal Maso [4] have shown that in fact each $\lambda_k$ attains its minimum over this class.

Maximization of Dirichlet eigenvalues has also been considered. Pólya and Szegö [27] showed that the unit disk $D$ itself maximizes $\lambda_1$, in the class of all conformal images $\Omega = \mathcal{F}(D)$ of the unit disk, normalized by $\mathcal{F}'(0) = 1$. More recently, Cox and Ross [9] showed that each $\lambda_k$ attains its extrema over the class of starlike domains in the plane with fixed area, bounded boundary length, and uniform inradius about the
origin, and also over the class of convex domains with uniform inradius. These last results are the springboard for the work presented in this paper.

One of the key results of Garabedian and Schiffer [15] is the continuity of the Dirichlet spectrum over the class of Jordan domains; i.e. simply connected domains with boundary being a Jordan curve. They show that when the Riemann maps of a given sequence of domains and the corresponding areas are converging, then so is each $\lambda_k$. This continuity is shown by performing a conformal change of variables (via the Riemann maps) and moving all of the convergence back to the the unit disk.

The idea suggested by Cox for this work was to consider optimizing over the class of all quasiconformal images of the unit disk with proper constraints on area, diameter, and inradius. It so happens that once one has control of the dilation of a quasiconformal map and control of the area of its image, the inradius and diameter bounds come for free from a distortion theorem found in Gehring [16]. This makes the consideration of the class of quasidisks even more appealing. Since the boundary of a quasidisk satisfies the three-point property [16,29,2] or arc condition [22], the analytic machinery of quasiconformal maps may be removed from the problem. It turns out that the three-point property and area bounds are sufficient to provide the desired diameter and inradius bounds without need of the aforementioned distortion theorem. It is for this reason that the class of quasidisks with an area bound is considered a natural collection of domains over which to work.

The first section introduces the study of quasidisks with the definition of quasidisks and chord-arc domains and some immediate geometric consequences of these definitions. The class of MOA-quasidisks is then defined and is shown to be essentially all quasidisks.

The second section records some facts about the boundary behavior of Riemann maps onto quasidisks in the language of quasi-symmetric maps. It turns out that this
boundary behavior completely determines whether or not a Riemann map is mapping onto a quasidisk. The key result here is that any bounded uniformly quasisymmetric family of functions is a normal family.

In the third section, with this compactness of quasi-symmetric functions in hand, the continuity of area for convergent sequences of the corresponding conformal maps is shown. This is immediately applied to show compactness of MOA-quasidisks in the sense of Caratheodory convergence and the Hausdorff metric. With a little more work to show that arc-length is lower-semi continuous, one can then claim that the class of COA-chord-arc domains is also compact.

The topic of the fourth section focuses on the continuity of certain domain functionals, in particular the Dirichlet spectrum. This result for both quasidisks and chord-arc domains follows directly from the corresponding result from Cox and Ross [9]. Another functional of interest is the Cheeger constant thought of as a function of domain.

The fifth section lays out some less refined results and some possible directions for further investigation in this area. In particular, convexity constraints are considered as well as the relationship between the width of a convex set and its first eigenvalue. Other ideas that seem worth pursuing would include the understanding of how probability fits into this analysis, what can be extracted from looking at the polygonal and convex cases, and the omni-present question of the regularity of the extremizing domain.

1.2 Definitions and geometric properties

The study of quasidisks has a rich history founded in the theory of holomorphic functions on the complex plane. It is well known that the bi-holomorphisms of the extended complex plane are the Mobius transformations, and that these map circles
to circles. More generally, a holomorphic function is conformal, which means angle preserving, at each point where its complex derivative is non-vanishing. In this conformal case, general circles are not mapped to circles, rather, infinitesimal circles are mapped to infinitesimal circles; i.e. smaller and smaller circles about a point are mapped to sets which are closer and closer to being true circles.

Allowing a relaxation on the conformality condition so that infinitesimal circles are taken to infinitesimal ellipses leads to the study of quasi-conformal maps. This and other equivalent definitions are made precise in the book by Lehto and Virtanen called *Quasiconformal Mappings in the Plane* [23]. The study of quasiconformal maps is a non-trivial extension of the study of conformal maps since one no longer has the guarantee of regularity as in the conformal case [23]. It is this non-regularity and the fact that they are a generalization of conformal maps that gives quasiconformal maps their usefulness.

Conformal maps of the extended complex plane map disks to disks. Quasi-conformal maps of the extended complex plane map disks to quasidisks. An extensive description of quasidisks is given in Gehring's book *Characteristic Properties of Quasidisks* [16]. In particular, another analogy is drawn with the conformal case. Only circles in the extended complex plane admit a conformal reflection (defined on the extended plane) that reduces to the identity on the boundary. Ahlfors, in *Quasiconformal Reflections* [1], proved that a Jordan curve admits a quasiconformal reflection if and only if it bounds a quasidisk. Ahlfors also shows that a quasidisk is determined by the three-point property of its boundary [2]. It is from this characterization this work proceeds.

In all that follows we consider $\Omega$ to be a (simply connected) Jordan domain in the complex plane with boundary $\partial \Omega$. The following definition of quasidisk from Pommerenke [29] is equivalent to Ahlfors's three-point property.
Definition 1.1  We say $J$ is an $M$-quasicircle if there is a constant $M$ such that for any two points $a, b \in J$,

\[(M) \quad \text{diam} \, J(a, b) \leq M|a - b|,\]

where $J(a, b)$ is the arc of $J$ between $a$ and $b$ of smaller diameter. $\Omega$, the inner domain of $J$, is called a quasidisk, and the constant $M$ is referred to as the quasicircle constant.

It can be shown that any smooth Jordan curve is a quasicircle. However, there do exist quasicircles that are not rectifiable. An example of such is the Von-Koch snowflake curve [28]. One may note that the definition of quasicircle immediately excludes both inward and outward pointing cusps. A special sub-class of quasidisks is the set of chord-arc domains.

Definition 1.2  We say $\Omega$ is a $C$-chord-arc domain if there is a constant $C$ such that for any two points $a, b \in J = \partial \Omega$,

\[(C) \quad \text{length} \, J(a, b) \leq C|a - b|,\]

where $J(a, b)$ is the arc of $J$ between $a$ and $b$ of smaller length. The constant $C$ is referred to as the chord-arc constant.

That every chord-arc domain is a quasidisk follows immediately from the fact that $\text{diam} \, J(a, b) \leq \text{length} \, J(a, b)$. To see that these two definitions do not give the same class of domains, it suffices to point out that the chord-arc domain definition requires rectifiability of the boundary, and we have already seen an example of a quasidisk with a nonrectifiable boundary.

The following propositions list some geometric facts about quasidisks.
Proposition 1.1  If \( \Omega \) is an \( M \)-quasidisk, then there are two positive constants \( \hat{\rho} \) and \( \hat{\rho} \) each depending only on \( A \) and \( M \) such that

\[
(\rho) \quad D_{\hat{\rho}} \subset \Omega \subset D_{\hat{\rho}}
\]

for appropriate centers of the disks \( D_{\hat{\rho}} \) and \( D_{\hat{\rho}} \). The constants \( \hat{\rho} \) and \( \hat{\rho} \) are called the inradius and out-radius of the quasidisk \( \Omega \).

Proof  Let's set the following notation in addition to that already defined:

\[
B = (0, D) \times (-D, D)
\]

\[
L^\pm = [0, D] \times \{ \pm i \}
\]

Because \( \overline{\Omega} \) is compact, there are two points in \( J \) which attain the diameter \( D = \text{diam } \Omega \). The conclusion of the proposition is invariant under rotation and translation, so we may assume without loss of generality that these two points are 0 and \( D \) on the real axis. Thus we see that \( \Omega \subset B \). Furthermore, since \( \Omega \) connects 0 and \( D \), only intersects the boundary of \( B \) at 0 and \( D \), and is simply connected, we see that \( \Omega \) separates \( B \) into two components \( B^+ \) and \( B^- \) where \( \partial B^\pm \supset L^\pm \). We see further that \( J^\pm = J \cap B^\pm \) is a Jordan arc that separates \( \Omega \) from \( L^\pm \) in \( B \). Because \( L^\pm \subset \overline{B^\pm} \), \( \Omega \) separates \( L^+ \) and \( L^- \). Pictorially, this is shown in Figure 1.1.

To find \( \hat{\rho} \) we need only to show that each vertical slice of the domain \( \Omega \) has large length, then integrate using Fubini's Theorem and read off the upper bound for the diameter. To show that the vertical slices of \( \Omega \) have the correct length bound we make the following definitions. For each \( x \in (0, D) \) the \( x \)-slice of \( \Omega \) by

\[
\Omega_x = \overline{\Omega} \cap \{x\} \times (-D, D).
\]

Then for each \( x \)-slice we want to find an interval \( I_x \subset \Omega_x \) with an estimate on the length. To find the endpoints of such an interval we begin by defining

\[
K^\pm_x = J^\pm \cap \Omega_x.
\]
Now let
\[ y_x^- = \max\{y : (x, y) \in K_x^-\} \]
and
\[ y_x^+ = \min\{y : (x, y) \in K_x^+ \text{ and } y > y_x^-\} \].

The first point, \( y_x^- \), exists and is in \( J^- \) because \( K_x^- \) is a compact subset of \( J^- \). The second point, \( y_x^+ \), exists and is in \( J^+ \) because \( J^+ \) separates \( J^- \) and \( L^+ \), and so the minimum is over a nonempty compact subset of \( J^+ \). Furthermore, since \( \Omega \) separates \( J^+ \) and \( J^- \) we see that
\[ I_x = \{x\} \times (y_x^-, y_x^+) \subseteq \Omega_x, \]
where \((y_x^-, y_x^+)\) is the interval connecting \( y_x^- \) and \( y_x^+ \).

To obtain the necessary length estimate for \( I_x \) we define a function \( s \), which for each real value \( x \in [0, D] \) gives us the minimum distance between a point on \( J^\pm \cap \Omega_x \)
and any point on $J^+$ as restricted by the M-quasidisk restraint (M):

\[
(s) \quad s(x) = \begin{cases} 
  x/M & \text{if } x \leq D/2, \\
  (D - x)/M & \text{if } x \geq D/2.
\end{cases}
\]

Thus we have obtained the estimate that

\[
(sl) \quad |\Omega_x| \geq |I_x| \geq s(x).
\]

We now express $A$, the area of $\Omega$ as an area integral, apply Fubini’s Theorem, and use (sl) to estimate the $x$-integration and so obtain the area-diameter estimate

\[
A = \int_0^D dx |\Omega_x| \\
\geq \int_0^D s(x)dx \\
= \frac{D^2}{4M}
\]

This implies the diameter bound

\[
D \leq 2\sqrt{MA},
\]

which is what was needed for the right hand side of $(\rho)$ by choosing $\hat{\rho} = 2\sqrt{MA}$.

To get the inradius bound $(\hat{\rho})$ we consider

\[
x \in E = \left[\frac{DM}{2(M + 1)}, D - \frac{DM}{2(M + 1)}\right],
\]

which is a symmetric interval about $D/2$ with appropriate length for the upcoming estimate. On this interval, the function $s$, measuring the closeness of $J^+$ and $J^-$ allowed by (M), attains its minimum at both of the endpoints, with value

\[
d_0 = \frac{D}{2(M + 1)}.
\]

It follows that for each $x$ in $E$, the open ball centered at $(x, y_x^-)$ with radius $d_0$ does not intersect $J^+$. We will look for a ball satisfying $\hat{\rho}$ inside one of these $d_0$-balls.
Using the compactness of $J^-$ and $E$ we know that there is an $x_0 \in E$ such that for any other $x \in E$,

$$y_{x_0}^- \geq y_x^-,$$

where $y_{x_0}^-$ and $y_x^-$ are defined as for the diameter bound argument. For this choice of $x_0 = (x_0, y_{x_0}^-)$, the following set does not intersect either $J^+$ or $J^-$,

$$Q = \begin{cases} P_{left} & \text{if } x_0 \in [\frac{DM}{2(M+1)}, \frac{D}{2}], \\ P_{mid} & \text{if } x_0 = \frac{D}{2}, \\ P_{right} & \text{if } x_0 \in [\frac{D}{2}, D - \frac{DM}{2(M+1)}], \end{cases}$$

where

$$P_{left} = \left\{ z_0 + re^{i\theta} : r \in (0, d_0) \quad \text{and} \quad \theta \in (0, \frac{\pi}{2}) \right\},$$

$$P_{mid} = \left\{ z_0 + re^{i\theta} : r \in (0, d_0) \quad \text{and} \quad \theta \in (0, \pi) \right\},$$

$$P_{right} = \left\{ z_0 + re^{i\theta} : r \in (0, d_0) \quad \text{and} \quad \theta \in \left(\frac{\pi}{2}, \pi\right) \right\}.$$

These sets are pictured in Figure 1.2.

$Q$ does not intersect $J^+$ because of our choice of $d_0$, and it does not intersect $J^-$ because of our choice of $y_{x_0}^-$ as maximal. That $Q \subset \Omega$ follows from the fact that $Q$ does not intersect $J$, $I_{x_0}$ (as defined above) is contained in both $\Omega$ and the closure of $Q$, and $\Omega$ is open. We can now read-off the inradius bound from the fact that independent of the of $x_0 \in E$, there is a ball of radius $\frac{d_0}{1 + \sqrt{2}}$ contained in $Q$. Since $D$

![Figure 1.2](image-url)  

Figure 1.2 $P_{left}$, $P_{mid}$, and $P_{right}$, respectively.
depends only on \( A \) and \( M \), it follows that \( d_0 \) depends only on \( A \) and \( M \) and we may choose \( \hat{\rho} = \frac{d_0}{1+\sqrt{2}} \). Then \( \hat{\rho} \) satisfies the following lower bound,

\[
\hat{\rho} \geq \frac{D}{2(M+1)(1+\sqrt{2})} \geq \sqrt{\frac{2A}{\pi}} \frac{1}{2(M+1)(1+\sqrt{2})},
\]

by the isodiametric inequality. This proves the inradius bound and completes the proof of the proposition.

Now define

\[
\hat{\rho}_{M,A} = \sqrt{\frac{2A}{\pi}} \frac{1}{2(M+1)(1+\sqrt{2})}
\]

and

\[
\hat{\rho}_{M,A} = 2\sqrt{MA},
\]

which from the proposition bound the inradius and outradius respectively, of any \( M \)-quasidisk with area \( A \).

One also needs the following fact about the size of quasicircles. There are stronger results regarding the Hausdorff dimension of quasicircles, [17] [3] [13], which are far afield from the objectives of this exposition.

**Proposition 1.2** The area of a quasicircle is zero; that is, if \( \Omega \) is any \( M \)-quasidisk, then

\[
\text{area}(\partial \Omega) = 0.
\]

**Proof** To prove this, we will simply show that the two-dimensional Lebesgue density is strictly less than 1 everywhere in the plane; thus by Lebesgue’s density theorem, \( J = \partial \Omega \) must have area zero. To obtain the desired density result, we show that each sufficiently small concentric circle about a given point in the boundary contains an arc, of length linear in radius, that does not intersect \( J \).

Let \( z \in J \) be given and let \( d \) denote the radius of the smallest ball centered at \( z \) which contains \( \Omega \). Then there is a point \( w \in J \) with \( w \neq z \) such that the distance
between \( z \) and \( w \) is \( d \). \( J \) without both \( z \) and \( w \) has two disjoint connected components \( J^+ \) and \( J^- \) which connect \( w \) and \( z \). Now let \( r \) denote a real number in the interval \((0, \frac{\epsilon}{2})\) and consider the circle \( T_r \) centered at \( z \) with radius \( r \). \( T_r \) intersects \( J^+ \) and \( J^- \) because \( T_r \) separates \( w \) and \( z \). Let \( K_r^\pm \) denote the intersection of \( T_r \) with \( J^\pm \). These two sets are compact and so the distance between them is some positive number. This distance is bounded from below by \( s(r) \) where the function \( s \) is given in (s) from the proof of the previous proposition. If this distance was smaller than \( s(r) \) then we would fail to satisfy (M), the quasidisk constraint, because the diameter of the arc of \( J \) connecting the two points which achieve this distance would be greater than \( r \). Thus, there is an arc of \( T_r \), namely the shorter one connecting the two points achieving the distance between \( K_r^+ \) and \( K_r^- \), which has length greater than \( s(r) \) which does not intersect \( J \). So we have the estimate,

\[
H^1(T_r \cap J) \leq (2\pi r - s(x)) = \frac{r(2M\pi - 1)}{M}.
\]

If we do this for every \( r \in (0, \frac{\epsilon}{2}) \) and integrate using a polar form of the Fubini Theorem, we obtain for each \( \overline{r} \) in \((0, \frac{\epsilon}{2})\)

\[
\text{area } (B(z, \overline{r}) \cap J) = \int_0^{\overline{r}} |T_r \cap J| dr \leq \overline{r}^3 \frac{(2M\pi - 1)}{2M}.
\]

This estimate gives us that the two-dimensional density of \( J \) at an arbitrary point \( z \in J \) is bounded from above by \( \frac{(2M\pi - 1)}{2M\pi} < 1 \). Likewise, because \( J \) is compact, the density of \( J \) at points in its complement is zero. So we have that the density of \( J \) at every point in the plane is bounded away from 1. But the Lebesgue Density Theorem tells us that the density is 1 for almost every point in \( J \). Thus, we conclude that the area of \( J \) is zero.
**Definition 1.3** Define the set of MOA-quasidisks to be the collection of all pairs \( (\Omega, F_\Omega) \) where \( \Omega \) is an \( M \)-quasidisk with

\[
(A) \quad \text{area}(\Omega) = A,
\]

that contains a disk about the origin

\[
(O) \quad D_{\delta M, \lambda}(0) \subset \Omega,
\]

and \( F_\Omega \) is the unique Riemann map of the unit disk onto \( \Omega \) that satisfies

\[
(F) \quad F_\Omega(0) = 0 \quad \text{and} \quad F'_\Omega(0) > 0.
\]

Since \( F_\Omega \) is uniquely determined, it is reasonable to refer to an MOA-quasidisk \( (\Omega, F_\Omega) \) as just \( \Omega \). We likewise define the class of COA-chord-arc domains.

This definition is very general in that every \( M \)-quasidisk is an MOA-quasidisk up to scaling and translation. The restriction \( (O) \) only serves to mod out by translations since the inradius bound is guaranteed by \( (\rho) \). The \( (A) \) restriction is a natural control on scaling. \( (F) \) serves only to provide uniqueness in the Riemann mapping theorem.

### 1.3 Boundary behavior of Riemann maps

Since each MOA-quasidisk \( \Omega \) is a Jordan domain, the associated Riemann map \( F_\Omega \) extends to a homeomorphism \( F_{\overline{\Omega}} \) of \( \overline{D} \) onto \( \overline{\Omega} \). If we let \( \Sigma \) denote the boundary of the unit disk \( D \) and let \( h \) be the restriction of \( F_{\overline{\Omega}} \) to \( \Sigma \), then \( h \) will be a homeomorphism of \( \Sigma \) onto \( J = \partial \Omega \). Since this extension is unique, we will refer to \( h \) as the restriction of \( F_{\overline{\Omega}} \) to the boundary. We now turn our attention to this homeomorphism and its properties.
Definition 1.4 A map $h$ from the unit circle $\Sigma$ into the complex plane is called quasisymmetric if it is injective and there is a strictly increasing continuous function $\gamma : [0, \infty) \to \mathbb{R}$ with $\gamma(0) = 0$ such that
\[
(\gamma) \quad \frac{|h(\sigma_1) - h(\sigma_2)|}{|h(\sigma_2) - h(\sigma_3)|} \leq \gamma \left( \frac{\sigma_1 - \sigma_2}{\sigma_2 - \sigma_3} \right),
\]
for all $\sigma_i \in \Sigma$. We refer to $\gamma$ as the modulus of quasisymmetry. A sequence of quasisymmetric maps $\{h_n\}$ is called uniformly quasisymmetric if each $h_n$ is quasisymmetric with the same modulus of quasisymmetry $\gamma$ satisfying $(\gamma)$ for all $h_n$ simultaneously.

An immediate consequence of this definition is the following proposition from Pommerenke [29].

Proposition 1.3 If $h$ is a quasisymmetric map of $\Sigma$ into the plane, then $J = h(\Sigma)$ is a quasicircle.

The following theorem is a special case of a theorem by [28] and is a converse to the preceding proposition.

Theorem 1.1 If $(\Omega, F_\Omega)$ is an MOA-quasidisk, then $F_\Omega$ restricted to $\Sigma$ is quasisymmetric with $\gamma$ depending only on $M$ and $A$.

It is important to note that the conclusion that $\gamma$ depends only on $M$ and $A$ is false if we do not require the hypotheses (O) and (F) for our quasidisks. An example of this sort of failure is given by the sequence of Riemann maps $f_n$ from $D$ to itself defined by $f_n(0) = 1 - \frac{1}{n}$ and $f'_n(0) > 0$. It then suffices to choose $\sigma_1 = -1$, $\sigma_2 = 1$, and $\sigma_3 = i$. Then the argument of $\gamma$ in $(\gamma)$ is constant for all $n$, but the left-hand side grows without bound. This example clearly violates the assumption (O) in our definition of MOA-quasidisks since $f_n(0)$ is converging to 1 as $n$ goes to infinity so
that there is no uniform inradius bound. However, this example also violates (F), but only insofar as the origin is not mapped to the origin, and this can be fixed by a translation for each \( n \).

The following proposition is an exercise in Pommerenke.

**Proposition 1.4** If \( \{ h_n \} \) is a uniformly quasisymmetric sequence of maps from \( \Sigma \) into a fixed disk \( \overline{D_R(0)} \), then there is a subsequence \( \{ h_{n_k} \} \) which converges uniformly on \( \Sigma \) to either a quasisymmetric map or a constant.

**Proof** First we find a uniformly convergent subsequence. To do this we show that any bounded uniformly quasisymmetric sequence of functions is equicontinuous and apply the Arzela-Ascoli Theorem. Uniform quasisymmetry means that for each \( n \)

\[
|h_n(\sigma_1) - h_n(\sigma_2)| \leq \gamma \left( \frac{|\sigma_1 - \sigma_2|}{|\sigma_2 - \sigma_3|} \right) |h_n(\sigma_2) - h_n(\sigma_3)|.
\]

Since the image of \( h_n \) is bounded, the triangle inequality gives

\[
\leq \gamma \left( \frac{|\sigma_1 - \sigma_2|}{|\sigma_2 - \sigma_3|} \right) 2R.
\]

Now the left-hand side is independent of \( \sigma_3 \) so for any choice of \( \sigma_1 \) and \( \sigma_2 \) in \( \Sigma \) we choose \( \sigma_3 \) so that \( |\sigma_2 - \sigma_3| = 1 \) and so obtain

\[
\leq \gamma(|\sigma_1 - \sigma_2|) \cdot 2R.
\]

Now given \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( \gamma(\delta) < \frac{\varepsilon}{2R} \) by the continuity of \( \gamma \). Thus for \( |\sigma_1 - \sigma_2| < \delta \) we have

\[
|h_n(\sigma_1) - h_n(\sigma_2)| < \varepsilon,
\]

where \( \delta \) depends only on \( \varepsilon \) and \( \gamma \). This shows that \( \{ h_n \} \) is an equicontinuous sequence of functions. Since the sequence is bounded, and \( \Sigma \) is compact, the Arzela-Ascoli Theorem provides the desired convergent subsequence \( \{ h_{n_k} \} \) which converges uniformly on \( \Sigma \) to a continuous function \( h : \Sigma \to \overline{D_R(0)} \).
We now show that the limit is either quasisymmetric or constant. We relabel our convergent subsequence \( \{ h_{n_k} \} \) to be \( \{ h_n \} \) and split this task into the following two cases:

**Case 1:** \( h \) is not injective. In this case there is a pair of distinct points \( \sigma_2 \) and \( \sigma_3 \) such that \( h(\sigma_2) = h(\sigma_3) \). Let \( K = \gamma(\frac{2}{|\sigma_2 - \sigma_3|}) \). (This is the largest that \( \gamma \) can be as \( \sigma_1 \) varies in \( \Sigma \) because of monotonicity.) By virtue of uniform convergence, given \( \varepsilon > 0 \), there exists an \( N_\varepsilon \geq 0 \) such that \( n \geq N_\varepsilon \) implies \( |h_n(\sigma_2) - h_n(\sigma_3)| < \frac{\varepsilon}{2K} \). Putting this together with 1.3 gives

\[
|h_n(\sigma_1) - h_n(\sigma_2)| < \frac{\varepsilon}{2},
\]

for every \( n \geq N_\varepsilon \) and every \( \sigma_1 \) in \( \Sigma \). Thus, \( \text{diam } h_n(\Sigma) < \varepsilon \) for every \( n \geq N_\varepsilon \), which shows \( \text{diam } h(\Sigma) = 0 \) by uniform convergence. Therefore \( h \) is constant in this case.

**Case 2:** \( h \) is injective. We need only show \((\gamma)\) holds for \( h \). Fix \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) in \( \Sigma \) with \( \sigma_2 \neq \sigma_3 \), and define

\[
q_n = \frac{|h_n(\sigma_1) - h_n(\sigma_2)|}{|h_n(\sigma_2) - h_n(\sigma_3)|}.
\]

This quotient converges to the left-hand side of \((\gamma)\) because \( h_n \) converges to \( h \) and the denominator does not vanish by the quasisymmetry of the sequence \( h_n \) and injectivity of the limit function \( h \). Furthermore, the uniform quasisymmetry of the sequence \( \{ h_n \} \) gives that \( q_n \) is bounded from above by the right-hand side of \((\gamma)\) for each \( n \geq 0 \). It follows that the limit of the \( q_n \)'s also satisfies this inequality, which is to say that \( h \) satisfies \((\gamma)\) with the same modulus of quasisymmetry \( \gamma \) as for the original sequence. Therefore, \( h \) is quasisymmetric in this case.

\[\square\]

The following proposition provides the necessary continuity of area statement for the compactness result of the next section.
Proposition 1.5 Suppose \( \{f_n\} \) is a sequence of conformal maps of the closed unit disk \( \overline{D} \) into the complex plane that converge uniformly on \( \overline{D} \) to the conformal map \( f \) of \( \overline{D} \) onto an \( M \)-quasidisk, then the areas of the images of the terms of the sequence converge to the area of the image of the limit function \( f \); i.e.

\[
\text{area } f_n(\overline{D}) \rightarrow \text{area } f(\overline{D}).
\]

Proof It suffices to show \( \chi_{f_n(\overline{D})} \rightarrow \chi_{f(\overline{D})} \) pointwise almost everywhere, since then, we may apply Lebesgue's Dominated Convergence Theorem and get convergence of the desired integrals. To check pointwise convergence at a point \( w \) in the plane, we can divide the task into three cases according to whether \( w \in f(D) \), \( w \in (f(\Sigma)) \), or \( w \in (f(\overline{D}))^c \).

Case 1 Here we assume that \( w \in f(D) \) with \( w = f(z_0) \). Then there is a \( \delta > 0 \) such that

\[
\text{dist } (w, f(\Sigma)) = \delta.
\]

Let \( \epsilon = \frac{\delta}{10} \), then the triangle inequality gives

\[
\text{dist } (B_\epsilon(f(\Sigma)), w) = \frac{9\delta}{10},
\]

where \( B_\epsilon(f(\Sigma)) \) is an \( \epsilon \)-neighborhood of \( f(\Sigma) \). Furthermore, there is an \( N \geq 0 \) such that \( n \geq N \) implies

\[
|f_n(z) - f(z)| < \epsilon
\]

for all \( z \in \overline{D} \) by the uniform convergence. In particular, for all \( n \geq N \) we have

\[
|f_n(z_0) - f(z_0)| < \epsilon
\]

and

\[
f_n(\Sigma) \subset B_\epsilon(f(\Sigma)).
\]
Therefore, the triangle inequality gives us
\[
\text{dist} (w, f_n(\Sigma)) > \frac{9\delta}{10} > \varepsilon.
\]
Thus \( f_n(\Sigma) \) does not separate \( f_n(z_0) \) and \( f(z_0) \); i.e. they are in the same component of \( (f_n(\Sigma))^c \). So, by the homeomorphism properties of \( f_n \) and \( f \), we have for all \( n \geq N \) that \( w \in f_n(\mathcal{D}) \). So on \( f(\mathcal{D}) \) the desired convergence is achieved.

**Case 2** In this case we assume that \( w \in f(\Sigma) \). But, by definition, \( f(\Sigma) \) is an \( M \)-quasicircle, so that area \( f(\Sigma) = 0 \) by proposition 1.2. Since we are only interested in showing convergence almost everywhere, we may ignore this case.

**Case 3** Finally, we assume \( w \in (f(\mathcal{D}))^c \). Then there is a \( \delta > 0 \) such that
\[
\text{dist} (w, f(\mathcal{D})) = \delta.
\]
So there is an \( N \geq 0 \) such that \( n \geq N \) implies
\[
|f_n(z) - f(z)| < \frac{\delta}{2}
\]
for all \( y \in \mathcal{D} \) by the uniform convergence. Thus for all \( n \geq N \) we have by the triangle inequality
\[
\text{dist} (w, f_n(\mathcal{D})) > \frac{\delta}{2},
\]
and so
\[
w \in (f_n(\mathcal{D}))^c.
\]
This proves convergence for all \( w \in \mathcal{D}^c \).

With convergence almost everywhere of the characteristic functions and the uniform bound on the images \( f_n(\mathcal{D}) \) afforded by the uniform convergence of the sequence \( \{f_n\} \) we can apply Lebesgue's Dominated Convergence Theorem and the conclusion follows immediately.
1.4 Convergence and compactness

We now define two types of convergence for simply connected Jordan domains. The first type is known as Caratheodory convergence and is analytic in nature insofar as Caratheodory convergence of domains implies local uniform convergence of the corresponding Riemann maps. This is spelled out for us in the following definition and theorem.

**Definition 1.5** Let \( \{ \Omega_n \} \) be a sequence of simply connected domains, each containing \( D_\rho \), the disk of radius \( \rho \) centered at the origin. The *kernel* of \( \{ \Omega_n \} \) is the largest domain \( \Omega_c \) containing \( D_\rho \) with the following property: every compact subset of \( \Omega_c \) lies in all but finitely many of the \( \Omega_n \). We say \( \{ \Omega_n \} \) converges to its kernel, \( \Omega_n \rightarrow \Omega_c \), if every subsequence of \( \{ \Omega_n \} \) has the same kernel \( \Omega_c \).

**Theorem 1.2** Let \( \{ F_n : D \rightarrow \Omega_n \} \) be a sequence of conformal diffeomorphisms with \( D_\rho \subset \Omega_n \) and with \( F_n(0) = 0 \) and \( F_n'(0) > 0 \). Let \( \Omega_c \), a proper subset of the plane, be the kernel of \( \{ \Omega_n \} \). Then \( F_n \) converges locally uniformly to a function \( F_c \) on \( D \) if and only if \( \Omega_n \rightarrow \Omega_c \). In the case of convergence, \( \Omega_c \) is simply connected, \( F_c \) is a conformal diffeomorphism onto \( \Omega_c \), and \( F_n^{-1} \rightarrow F_c^{-1} \) locally uniformly on \( \Omega_c \).

The second type of convergence is that of the convergence of compact sets in the Hausdorff metric. This is a more geometric approach to convergence of sets.

**Definition 1.6** Let \( \Gamma \) denote the collection of all compact subsets of the closed ball \( D_{R,M,A} \) and define a set function on pairs of elements of \( \Gamma \) by

\[
\text{dist}_H(K_1, K_2) = \inf \{ \varepsilon : K_1 \subset B_\varepsilon(K_2) \quad \text{and} \quad K_2 \subset B_\varepsilon(K_1) \}.
\]

This defines the *Hausdorff metric* on \( \Gamma \).
It is well known that \( (\Gamma, \text{dist}_H) \) is a compact metric space. We are now in a position to state the compactness result for MOA-quasidisks.

**Theorem 1.3** The collection of MOA-quasidisks is compact in the Hausdorff metric and in the sense of Caratheodory convergence.

**Proof** Let \( \{ (\Omega_n, F_n) \} \) be any sequence of MOA-quasidisks. Then Proposition 1.3 \((\rho)\) guarantees that this sequence is bounded in the Hausdorff metric. Theorem 1.5 gives us that the sequence \( \{ F_n \} \) extends to be uniformly quasisymmetric on the boundary. Applying Proposition 1.6 we can extract a subsequence, which we immediately relabel to be \( \{ F_n \} \), which has the property that its restriction to the boundary is converging uniformly to a new quasisymmetric map \( h \) with the same modulus of quasisymmetry as the original sequence. That this limit on the boundary is not constant follows from the \((O)\) assumption in the definition of MOA-quasidisks. The maximum principle for holomorphic functions then gives us that \( \{ F_n \} \) is converging uniformly on all of \( \overline{D} \) to a continuous function \( F \) on \( \overline{D} \). Because holomorphicity is preserved under uniform convergence, we have that \( F \) is holomorphic. It is also a bijection since each \( F_n \) is. Since \( F \) restricted to the boundary is \( h \) we see that \( \Omega = F(D) \) is a quasidisk. We need only show that this \( \Omega \) is an MOA-quasidisk. To show that \( \Omega \) is an \( M \)-quasidisk, we let \( a, b \in \partial \Omega \) be given. Then there is a pair of points \( z_a, z_b \) in the boundary of the unit disk which map to \( a \) and \( b \) respectively under \( F \). Define \( (a_n, b_n) = (F_n(z_a), F_n(z_b)) \). By convergence of the sequence \( F_n \) on the boundary, we get that \( |a_n - b_n| \to |a - b| \) as \( n \) goes to infinity. We do not however get that \( \text{diam} J(a_n, b_n) \to \text{diam} J(a, b) \) since there is no guarantee that \( J(a_n, b_n) \) converges to \( J(a, b) \). However, we can conclude that there is a subsequence of the arcs \( J(a_n, b_n) \) which will converge to one of the arcs of \( \partial \Omega \) which connect \( a \) and \( b \). For this arc, call it \( \tilde{J} \), and the subsequence, we have that \( \text{diam} J(a_n, b_n) \to \text{diam} \tilde{J} \). And so, for the
subsequence we have that
\[ \frac{\text{diam } J(a_n, b_n)}{|a_n - b_n|} \to \frac{\text{diam } \tilde{J}}{|a - b|} \]
and since each term in the sequence on the right is bounded by \( M \), we have
\[ \frac{\text{diam } J(a, b)}{|a - b|} \leq \frac{\text{diam } \tilde{J}}{|a - b|} \leq M. \]
Thus, \( \Omega \) is an \( M \)-quasidisk. We now apply Proposition 1.7 to conclude that area \( \Omega = A \). Finally, it is clear that \( F_n = F \) satisfies (O) and (F) from the definition of \( MOA \)-quasidisks by uniform convergence. This shows that \( (\Omega, F_n) \) is another \( MOA \) quasidisk. Uniform convergence on the closed disk guarantees that \( \Omega_n \) is in an \( \varepsilon \)-neighborhood of \( \overline{\Omega} \) for \( n \) large enough and vice versa. Therefore, \( \overline{\Omega} \) is the correct limit in the Hausdorff metric because the uniform convergence of Riemann maps implies Hausdorff metric convergence and the limit is unique. Likewise, the uniform convergence of Riemann maps is exactly what is required for Carathéodory convergence. \( \square \)

We are now in a position to read-off the corresponding theorem for \( COA \)-chord-arc domains. However, care is needed regarding rectifiability of the boundary, specifically, the lower semi-continuity of arc-length. To contend with this issue we record some definitions and theorems concerning boundary length that can be found in Duren’s book [11].

Suppose \( f \) is an analytic map of the unit disk \( D \) into the plane. We define
\[ M_1(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|d\theta. \]

**Definition 1.7** We define the Hardy space \( H^1 \) to be those analytic maps of the unit disk into a Jordan domain which satisfy
\[ ||f||_{H^1} = \sup_{0 \leq r < 1} M_1(r, f) < \infty. \]
Theorem 1.4 (Hardy's Convexity Theorem [11]). Suppose $f$ is an analytic map of the unit disk into the plane, then $M_1(r, f)$ is nondecreasing in $r$.

Theorem 1.5 A function $f$, analytic in $|z| < 1$, is continuous on $|z| \leq 1$ and absolutely continuous on $|z| = 1$ if and only if $f' \in H^1$.

Now the compactness theorem for chord-arc domains can be stated and proved.

Theorem 1.6 The collection of COA-chord-arc domains is compact in the Hausdorff metric and in the sense of Caratheodory convergence.

Proof Since each COA-chord-arc domain is a COA-quasidisk, we may apply the above theorem to conclude that every sequence of COA-chord-arc domains has a subsequence which converges to a COA-quasidisk. It remains to show that such a limit is also a COA-chord-arc domain. That (O) and (F) are satisfied follows from the quasidisk convergence. It remains to be shown that $\Omega$ is a $C$-chord-arc domain.

Let $(\Omega_n, F'_n)$ be the aforementioned subsequence with $F_n$ converging uniformly on $\overline{D}$ to $F$ as in the proof of 3.4. Since each $\Omega_n$ has rectifiable boundary, and the sequence $\Omega_n$ is uniformly $C$-chord-arc, we have

$$||F'_n||_{H^1} \leq 2C\hat{\rho}_{C,A}.$$ 

So, for each $r \in (0, 1)$ and index $n$ we have

$$M_1(r, F'_n) \leq 2C\hat{\rho}_{C,A}.$$ 

The uniform convergence of the $F_n$ sequence implies the uniform convergence of the $F'_n$ sequence on compact subsets of the disk. Thus,

$$M_1(r, F') \leq 2C\hat{\rho}_{C,A}.$$
for all \( r \in (0, 1) \). This tells us that \( F' \in H^1 \). Now Theorem 3.7 gives us that \( F \) extends to be absolutely continuous on \( \Sigma \) the boundary of the unit disk. Let \( a, b \in J = \partial \Omega \) be given. Then there are two points \( \sigma_a \) and \( \sigma_b \in \Sigma \) such that \( F(\sigma_a) = a \) and \( F(\sigma_b) = b \). Let \( \Sigma^+ \) and \( \Sigma^- \) be the two components (relatively open) of \( \Sigma \) without the points \( \sigma_a \) and \( \sigma_b \), and \( J^\pm \) the corresponding images \( F(\Sigma^\pm) \).

Then we need only show that

\[
\min_{\pm} \{ \text{length } J^\pm \} \leq C|a - b|.
\]

Since \( F \) is absolutely continuous on \( \Sigma \), we can express these lengths as

\[
\text{length } J^\pm = \int_{\Sigma^\pm} |F'(\sigma)|d\sigma
\]

\[
= \sup \left\{ \left| \int_{\Sigma^\pm} F'(\sigma)\phi(\sigma)d\sigma \right| : \phi \in C^1(\Sigma^\pm, \mathbb{R}^2) \text{ and } |\phi| \leq 1 \right\}
\]

\[
= \sup \left\{ \left| \int_{\Sigma^\pm} F(\sigma)\text{div}\phi(\sigma)d\sigma \right| : \phi \in C^1(\Sigma^\pm, \mathbb{R}^2) \text{ and } |\phi| \leq 1 \right\}
\]

where the last step was obtained by integration by parts. Now we are in a position to use the uniform convergence. So let \( \phi \in C^1_c(\Sigma^\pm, \mathbb{R}^2) \) and \( |\phi| \leq 1 \) be given. Then

\[
\int_{\Sigma^\pm} F(\sigma)\text{div}\phi(\sigma)d\sigma = \lim_{n \to \infty} \int_{\Sigma^\pm} F_n(\sigma)\text{div}\phi(\sigma)d\sigma,
\]

\[
= \lim_{n \to \infty} -\int_{\Sigma^\pm} F_n'(\sigma)\phi(\sigma)d\sigma
\]

by Lebesgue's Dominated Convergence theorem. Taking the absolute value of both sides and applying the triangle inequality,

\[
\left| \int_{\Sigma^\pm} F(\sigma)\text{div}\phi(\sigma)d\sigma \right| = \lim_{n \to \infty} \left| \int_{\Sigma^\pm} F_n'(\sigma)\phi(\sigma)d\sigma \right|,
\]

\[
\leq \lim_{n \to \infty} \int_{\Sigma^\pm} |F_n'(\sigma)|d\sigma,
\]

\[
= \lim_{n \to \infty} \text{length } J_n^\pm
\]

where \( J_n^\pm \) is the image of \( \Sigma^\pm \) under the map \( F_n \). Since the term on the left is controlled from above independent of \( \phi \), we take the supremum over \( \phi \) and obtain

\[
\text{length } J^\pm \leq \lim_{n \to \infty} \text{length } J_n^\pm.
\]
Furthermore, for each $n$, one of $J_n^\pm$ will satisfy (C), and so (C) will be satisfied for infinitely many $n$ in one of the sequences \{length $J_n^+$\} or \{length $J_n^-$\}, both of which converge. For this sequence, the limit will be bounded by $C|a - b|$. Thus the min will be controlled; i.e.

\[
\min_{\pm}\{\text{length } J_n^\pm\} \leq \min_{\pm}\{\lim_{n \to \infty} \text{length } J_n^\pm\} \\
\leq C|a - b|.
\]

This is the required inequality for $\Omega$ to satisfy (C).

\[\square\]

### 1.5 John domains and linearly connected domains

Natural questions at this point would be “Is such a compactness result true for a more general class of domains?” and “Why quasidisks?” The reason for choosing quasidisks is that they have a very nice geometric characterization in terms of (M) and are rather general. A fact from [29] is that every quasidisk is a linearly connected John domain. Each of these latter two conditions is given in terms of natural geometric conditions and are dual to each other in the plane; i.e. the complement of a John domain on the Riemann sphere is linearly connected and vice versa [29]. However, the following examples show what can go wrong in the Hausdorff limit of John domains and linearly connected domains.

**Definition 1.8** A bounded simply connected domain $\Omega$ is called an $M$-John domain if for every rectilinear crosscut $[a, b]$ of $\Omega$

\[
diam \Omega \leq M \cdot |a - b|.
\]

**Example 1.1** Each $\Omega_n$ pictured in Figure 1.3 is a 10-John domain. However, the Hausdorff limit of this sequence of compact sets is not a Jordan domain.
Definition 1.9 A bounded simply connected domain $\Omega$ is said to be $M$-linearly connected if any two points $w_1, w_2 \in \Omega$ can be connected by a curve $A \subset \Omega$ such that

$$\text{diam } A \leq M \cdot |a - b|.$$

Example 1.2 Each $\Omega_n$ pictured in Figure 1.4 is a 10-linearly connected domain. In this case, these sequence of compact sets converge to a set which has disconnected interior; i.e the domain has pinched into two pieces.

1.6 Continuity of domain functionals

We now want to consider the continuity of a certain functionals defined on our class of domains. The continuity of the Dirichlet Spectrum for a planar domain is shown first. After that, a related quantity, the Cheeger constant for a domain is shown to be upper semi-continuous with respect to domain.

With respect to an open bounded subset $\Omega$ of the plane we denote by $\lambda_k(\Omega)$ the $k$th (counting multiplicity) eigenvalue of $-\Delta$ on $H^1_0(\Omega)$. The continuity of $\Omega \mapsto \lambda_k(\Omega)$ for both the class of MOA-quasidisks as well as the class of $COA$-chord-arc domains with respect to Caratheodory convergence is now an easy consequence of the following theorem of Cox and Ross [9].

Theorem 1.7 Suppose $\{\Omega_n\}$ is a sequence of simply connected domains that satisfy $D_\rho \subset \Omega_n \subset D_{M\rho}$. If $\Omega_n \to \Omega_c$ and $|\Omega_n| \to |\Omega_c|$ then $\lambda_k(\Omega_n) \to \lambda_k(\Omega_c)$.

In particular, this shows:
Figure 1.3  A bad sequence of John domains.

Figure 1.4  A bad sequence of linearly connected domains.
**Corollary 1.1** The map $\Omega \mapsto \lambda_k(\Omega)$ achieves its extrema over both the class of MOA-quasidisks as well as the class of COA-chord-arc domains.

Another functional of interest is the **Cheeger constant** defined by the following for each bounded domain $\Omega$ in the plane

$$(h) \quad h(\Omega) = \inf_{G \subset \subset \Omega} \frac{\text{length} \partial G}{\text{area} G}.$$  

That $h(\Omega)$ is not $+\infty$ follows from the fact that $\Omega$ is open. That $h(\Omega)$ is not $0$ follows from the boundedness of $\Omega$. Cheeger’s constant is of interest in eigenvalue problems because of the following theorem known as Cheeger’s Inequality.

**Theorem 1.8** For each bounded domain $\Omega$ in the plane we have

$$\lambda_1(\Omega) \geq \frac{h^2(\Omega)}{4}.$$  

The proof [5] is Rayleigh’s quotient coupled with the co-area formula.

Thus Cheeger’s constant provides a natural lower bound for the first eigenvalue of a domain. We can now state how Cheeger’s constant varies with Caratheodory Convergence.

**Theorem 1.9** If $\Omega_n$ is a sequence of simply connected domains all containing $D_\rho$ that is converging in the sense of Caratheodory, then the Cheeger constant is upper semi-continuous; i.e.

$$(L) \quad \limsup_{n \to \infty} h(\Omega_n) \leq h(\Omega).$$

**Proof** Let $G \subset \subset \Omega$ be given. Then, by virtue of Caratheodory convergence there is an $N$ such that $n \geq N$ implies that $G \subset \subset \Omega_n$. Thus

$$\frac{\text{length} \partial G}{\text{area} G} \geq h(\Omega_n)$$
for all \( n \geq N \). This gives us that

\[
\frac{\text{length} \partial G}{\text{area} \, G} \geq \limsup_{n \to \infty} h(\Omega_n).
\]

Since \( G \) was an arbitrary contender and the right hand side of the equation is independent of \( G \), we take the infimum over all such \( G \) and obtain (L).

Upper semi-continuity, although not as strong as continuity, suffices in the context of maximizing the first eigenvalue. This is because Cheeger's constant is a lower bound for the \( \lambda_1 \) and upper-semi-continuity says that the lower bound will only get better in the limit.

1.7 Other directions and ideas

In all that precedes, existence of extremizers has been demonstrated without construction. This deficiency is not easily remedied because there is no closed formula for the first eigenvalue of a general planar domain. Numerical experiments can provide "good" candidates, but the proof of maximality is not within reach currently, at least for quasidisks.

The story for convex sets with area and length bounds is more promising. By identifying a convex set (containing the origin) with its radial parameterization \( f \), it can be shown that \( \lambda_k \) is Lipschitz over this class of domains (with respect to the \( L^\infty \) norm on the radial parameterizations) [9]; i.e.

\[
\lambda_k(\hat{f}) \leq (3/\rho)\lambda_k(D_\rho) \sup_{[0,2\pi]} |\hat{f} - f| + \lambda_k(f)
\]

where \( \hat{f} \) is the maximizer and \( f \) is any other convex domain. The choice of \( f \) that minimizes the right hand side is not obvious. We can, however, bound the sup in terms of the diameter and inradius alone, and then choose \( f \) to be the disk of radius
\[ \sqrt{A/(2\pi)} \]. This is a candidate upper bound for all convex domains. Now the problem is reduced to "finding" a domain that achieves this bound, or finding a better bound.

Another candidate approach for finding a maximizer is the following. For a convex set \( \Omega \) define the width of \( \Omega, W(\Omega) \), to be the smallest distance between any two parallel planes that bound \( \Omega \). It then seems reasonable to conjecture:

\[
W(\Omega_1) \leq W(\Omega_2) \Rightarrow \lambda_1(\Omega_1) \geq \lambda_1(\Omega_2) \quad (W1)
\]

where each \( \Omega_i \) has area= \( A \). However, this is not the case. To see this, we note an easy corollary of (W1):

\[
W(\Omega_1) = W(\Omega_2) \Rightarrow \lambda_1(\Omega_1) = \lambda_1(\Omega_2) \quad (W1C)
\]

which follows by symmetry. This assertion is false in general. To see this, we employ an example of Daniel Joseph from his paper *Parameter and Domain Dependence of Eigenvalues of Elliptic Partial Differential Equations* [18]. The example of reference. Joseph considers the first eigenvalue of a parallelogram, undergoing shear, as a function of shear. Joseph shows that this is an analytic function and computes the power series for \( \lambda_1 \) in terms of the shear. He then shows that although the first derivative is zero (by symmetry), the second derivative is positive for a parallelogram which has height greater than its width divided by the square root of three. For the problem at hand, if the pure shear is vertical and the height of the parallelogram is greater than its width then, its width \( W \) is constant. But from the fact that the second derivative of \( \lambda_1 \) is positive, \( \lambda_1 \) must increase with the shear.

The previous example is not too surprising. However, it would be disappointing if (W1) didn't hold in some sense. Fortunately, there is a weaker version of (W1) which follows from domain monotonicity which is now stated as a proposition on width.
Proposition 1.6  For any convex domain $\Omega$ in the plane

$$\frac{\pi^2}{W(\Omega)^2} \leq \lambda_1(\Omega) \leq \frac{48\lambda_1(D_0)}{W(\Omega)^2}. \quad (W2)$$

Proof  The lower bound is achieved by an infinite rectangular strip of width $W(\Omega)$. The upper bound is given by the first eigenvalue of the disk of radius $\frac{W(\Omega)}{4\sqrt{3}}$ since every convex domain of width $W(\Omega)$ contains a disk of that radius. This follows from

$$W(\Omega)^2 \leq \sqrt{3}A$$

in [32] and the following corollary of Bonnesen's inequality [25,9] relating inradius($\rho$), area($A$) and perimeter($P$)

$$\rho \geq \frac{A}{L}.$$ 

For the problem at hand, the smaller the set $\Omega$, the better. So it is reasonable to assume that the "maximizing" set of width $W(\Omega)$ is contained in a square of side length $W(\Omega)$. With this assumption, an upper bound for the perimeter is obtained as $P \leq 4W(\Omega)$. This, plus the two previous inequalities give that any domain of width $W(\Omega)$ will have an inner disk of radius

$$\rho \geq \frac{W(\Omega)}{4\sqrt{3}}.$$ 

This proves the desired upper bound by the inverse monotonicity of $\lambda_1$ with respect to set inclusion.

A direction of interest along these lines is whether or not (W1) holds under more stringent conditions on $\Omega$. The most generic conditions are simply a perimeter bound and fixed area. A generic overview of historical as well as related results can be found in Extremal Eigenvalue Problems for Starlike Planar Domains by Cox and Ross [9].

Also of current interest for convex sets is shape gradients. Dealing with the variations of $\lambda_1(\Omega)$ with respect to perturbation of $\partial \Omega$ has been extensively studied
[6, 14, 19]. More recently, the generalized gradient of $\lambda_1(\Omega)$ for starlike domains has been considered by Cox and Ross [9] and used to obtain regularity results for extremizers. We've already alluded to the work of D. Joseph [18] in which he obtains a power series expansion for the first eigenvalue of a parallelogram experiencing pure shear. This leads one to question the regularity of the map $\Omega \mapsto \lambda_1(\Omega)$. When restricted to the space of convex domains with uniform inradius, this map is known to be Lipschitz [9]. If we consider the space of $n$-gons as parameterized by vertex coordinates, the concern is how to compute the $2n$-dimensional gradients. However, Lee Segal in his paper *Application of Conformal Mapping to Boundary Perturbation Problems for the Membrane Equation* [31], computes exact solutions to the membrane problem for an isosceles right triangle, mentions that this is also possible for rectangles, regular hexagons and equilateral triangles, and finally suggests that these solutions could possibly be perturbed. It is easy to hope that good gradient estimates for the polygonal case could be transmitted through limit (of domain convergence) and so to more general domains.

A slightly different direction of pursuit, relating back to continuity of domain functionals with respect to quasidisk convergence, is that of continuity of other functionals: for example, Korn's constant or Fredholm Eigenvalues. *An Extremal Problem for the Fredholm Eigenvalues* is a paper by Schiffer and Schober [30] which already addresses this issue to some extent. Kondratiev and Oleinik [20] have addressed the continuity of Korn's constant with respect to "parameters characterizing the geometry of the region".

A considerably different direction of pursuit is the study of the spectrum of vibrating plates. A good historical account of this area of study can be found in [6]. Recently, Nadirashvili published results on the minimization of the fundamental frequency for the vibrating plate [24]. The optimization of the first nonzero eigenvalue
becomes interesting in this case, because one no longer has a guarantee that its multiplicity will remain 1. To handle this, in general, one must usually apply the tools of non-smooth analysis. In particular, the approach used in [8] could possibly serve as a template for the analysis of this problem.
Chapter 2

The Partially Clamped Drum Problem

2.1 Problem summary and preliminary results

The basis for this section is the paper "Where best to hold a drum fast" [10] written by my advisor, Steven J. Cox and myself. That paper contains the complete story and details behind the following discussion. As such, only the relevant facts from that paper will be included in the following discussion.

The object of study is, once again, the first eigenvalue of the Laplacian on the domain $\Omega$ subject to mixed boundary conditions on $\partial \Omega$,

$$-\Delta u = \lambda u, \quad x \in \Omega, \quad (2.1)$$

$$1_{\Gamma}u + (1 - 1_{\Gamma})\partial u/\partial n = 0, \quad x \in \Gamma. \quad (2.2)$$

Denoting this least eigenvalue by $\lambda_1(\Gamma)$, the problem is to determine the extremes of $\lambda_1(\Gamma)$ as $\Gamma$ varies over subsets of $\partial \Omega$ of prescribed measure. Furthermore, in addition to the extremes, a description of the optimizing Dirichlet sets $\Gamma$ is also desired. In general, finding an optimizer is very difficult, since, as in the Dirichlet case from the previous section, there is no closed formula for $\lambda_1(\Gamma)$. However, there are well developed computational techniques that provide feasible means by which to construct numerical solutions. The next sections describe the complete answer for the disc in the plane, issues that have arisen in attempt to apply numerical methods to construct approximations to optimizing Dirichlet sets, and numerical results in some special cases.
As done in the paper [10], the original boundary condition is replaced by the following relaxed regularized version,

$$\frac{1}{\varepsilon} \theta u + \partial u / \partial n = 0.$$ 

where \( \theta \) is an element of the following collection of admissible Robin coefficients denoted by,

$$ad_* \equiv \{ \theta : 0 \leq \theta(x) \leq 1, \int_{\partial \Omega} \theta(x) \, dx = \gamma|\partial \Omega| \}.$$ 

and \( \gamma, \varepsilon > 0 \) are fixed. Denote by \( \lambda_1(\theta) \) the corresponding first eigenvalue of 2.1 with the relaxed regularized boundary condition. The following theorem is a conglomeration of some relevant theorems from [10].

**Theorem 2.1**

1) \( \lambda_1(\theta) = \inf_{u \in H_0^1(\Omega)} \int_{\Omega} |\nabla u|^2 \, dx + \frac{1}{\varepsilon} \int_{\partial \Omega} \theta u^2 \, dx \)

2) \( \theta \mapsto \lambda_1(\theta) \) is weak* continuous.

3) \( \theta \mapsto \lambda_1(\theta) \) is concave on \( ad_* \).

4) \( \theta \mapsto \lambda_1(\theta) \) attains its minimum at an extreme point of \( ad_* \). The set of extreme points of \( ad_* \) is given by \( ad_* \equiv \{ 1\Gamma : \Gamma \subset \partial \Omega, |\Gamma| = \gamma|\partial \Omega| \} \).

5) \( \theta \mapsto \lambda_1(\theta) \) is smooth and \( \langle \partial \lambda_1(\theta), \psi \rangle = \frac{1}{\varepsilon} \int_{\partial \Omega} \psi u^2 \, dx \) where \( u \) is the nonnegative eigenfunction with \( L^2 \)-norm 1 associated with \( \theta \).

6) \( \theta \) is unique.

The weak* continuity of \( \lambda_1(\theta) \) guarantees that the extrema will be obtained over \( ad_* \) which is compact in the weak* topology on \( L^\infty \). Although finding the optimizing \( \theta \) in most cases is very hard, which is the motivation for the numerical work, the case of \( \Omega \) being a disk is completely solvable. The next section describes this result.
2.2 Minimizer for the disk via rearrangements

The proof of the following theorem is the motivation and goal of this section.

**Theorem 2.2** If $\Omega$ is the disc, then the maximizer is constant $\hat{\theta} = \gamma$
and the support of minimizer $\hat{\theta}$ is connected.

The idea for the proof of this theorem is two-fold. First, the maximizer is unique
and has the symmetry of the domain $\Omega$. The only kind of function that has the
symmetries of the disk is a constant function. This is spelled out by Cox in [10].
Second, to show that the minimizer has connected support, a re-arrangement theorem
of Kawohl [19] is applied. This re-arrangement reduces $\lambda(\theta)$ (energy) without
changing the $L^2$ norm which when coupled with the variational formulation gives the
desired result.

This result only requires consideration of rearrangements in dimensions 1 and 2.
For dimension 1, let $D \in [-\pi, \pi]$ be compact and denote the Lebesgue measure of $D$
by $m_1(D)$. Define the symmetric decreasing rearrangement $D^*$ of $D$ by

$$D^* = \begin{cases} 
[-\frac{1}{2}m_1(D), \frac{1}{2}m_1(D)] & D \neq \emptyset \\
\emptyset & D = \emptyset
\end{cases}$$

and the symmetric increasing rearrangement $^*D$ of $D$ by

$$^*D = \begin{cases} 
[-\pi + \frac{1}{2}m_1(D), \pi - \frac{1}{2}m_1(D)] & D \neq \emptyset \\
\emptyset & D = \emptyset
\end{cases}$$

With these in hand, rearrangements for functions can be defined as well. This is
done by means of rearranging the level sets of the given function. Let $f : [-\pi, \pi] \to \mathbb{R}$
be Lipschitz and for each $c \in \mathbb{R}$ let $\Omega_c = \{x \in [-\pi, \pi] | f(x) \geq c\}$. Define the
symmetric decreasing rearrangement $f^*$ of $f$ by

$$f^*(x) = \sup\{c \in \mathbb{R} | x \in \Omega_c^*\}.$$
Define the symmetric increasing rearrangement \( ^*f \) of \( f \) by

\[
^*f(x) = \sup \{ c \in \mathbb{R} | x \in ^* \Omega_c \}.
\]

For dimension 2, let \( B = [0, 1] \times [-\pi, \pi] \) which will heretofore be thought of as \( \rho \phi \)
space. For each compact \( D \subset B \) and every \( \rho \in [0, 1] \), denote and define the \( \rho \)-slice of \( D \) by

\[
D(\rho) = D \cap (\{\rho\} \times [-\pi, \pi]).
\]

For this kind of set, the Steiner symmetrization with respect to \( \{\phi = 0\} \) is considered. It is convenient to abuse notation and define

\[
D^* = \cup_{\rho \in [0, 1]} D^*(\rho),
\]

where \( D^*(\rho) \) is the symmetric decreasing rearrangement of \( D(\rho) \) (considered as a segment in the direction of \( \phi \))

\[
D^*(\rho) = [D(\rho)]^*.
\]

For \( u : B \to \mathbb{R} \), the Steiner symmetrization \( u^* \) of \( u \) with respect to the line \( \{ \phi = 0 \} \) (also known as the symmetrically decreasing rearrangement in the variable \( \phi \)) is defined by

\[
u^*(\rho, \phi) = \sup \{ c \in \mathbb{R} | (\rho, \phi) \in \Omega_c^* \}.
\]

Let \( D \) denote the unit disk \( \{(x, y) | x^2 + y^2 \leq 1 \} \). For \( u : D \to \mathbb{R} \), define the circular symmetrization with respect to \( \{ \phi = 0 \} \) by

\[
u^*(x, y) = [u(\rho \cos \phi, \rho \sin \phi)]^*.
\]

That is, the circular symmetrization of a function on \( D \) is the Steiner symmetrization of its pull-back to \( B \).
**Lemma 2.1** If \( u : D \to \mathbb{R} \) is in \( L^2 \), then \( u^* : D \to \mathbb{R} \) is also in \( L^2 \) and
\[
||u^*|| = ||u||.
\]

**Proof** \( u^* \) and \( u \) are equimeasurable [Kawohl]. \( \square \)

A theorem of Kawohl is now recalled regarding Steiner symmetrization with respect to \( \{ \phi = 0 \} \). Let \( I \subset \mathbb{R} \) be a bounded interval and set \( \Omega = I \times (-\pi, \pi) \). The following two assumptions will be needed in the rearrangement theorem.

**Assumption 2.2.1** \( u : \bar{\Omega} \to \mathbb{R} \) is in \( C^1(\bar{\Omega}) \) and is periodic in \( \phi \); i.e. \( u(\rho, \pi) = u(\rho, -\pi) \).

**Assumption 2.2.2** There is a closed subset \( N \subset I \) with \( m_1(N) = 0 \) so that for every \( \rho \in I \times N \) and every \( c \in (\min_{\Omega} u, \max_{\Omega} u) \) the following sets are finite:
\[
W = \{(\rho, \phi) \in \Omega | u(\rho, \phi) = c\}
\]
\[
V = \{(\rho, \phi) \in \Omega | \frac{\partial u}{\partial \phi}(\rho, \phi) = 0\}.
\]

**Theorem 2.3** (Kawohl). Let \( u : \bar{\Omega} \to \mathbb{R} \) satisfy the two assumptions. Let \( x_k : I \to \mathbb{R}^+_0 \) for \( k = 1, 2 \) be nonnegative and continuous and let \( G : \mathbb{R}^+_0 \to \mathbb{R} \) be monotone, nondecreasing and convex. Then
\[
\int_\Omega G \left( \left\{ x_1(\rho) \left| \frac{\partial u}{\partial \rho} \right|^2 + x_2(\rho) \left| \frac{\partial u}{\partial \phi} \right|^2 \right\} \right) \frac{1}{2} \ d\rho d\phi \geq \\
\geq \int_\Omega G \left( \left\{ x_1(\rho) \left| \frac{\partial u^*}{\partial \rho} \right|^2 + x_2(\rho) \left| \frac{\partial u^*}{\partial \phi} \right|^2 \right\} \right) \frac{1}{2} \ d\rho d\phi.
\]
If in addition, the \( x_k(\rho) \) are positive and \( G \) is monotone increasing and strictly convex, then equality holds if and only if \( u = u^* \) modulo translation in the \( \phi \)-direction (with respect to periodicity).
Denote by \( A(r_1, r_2) \) the annulus with inner radius \( r_1 > 0 \) and outer radius \( r_2 < 1 \) and \( B(r_1, r_2) = (r_1, r_2) \times [-\pi, \pi] \).

**Corollary 2.1** Suppose \( u : A(r_1, r_2) \to \mathbb{R} \) and that the pull-back of \( u \) to \( B(r_1, r_2) \) satisfies conditions (1) and (2). Then circular symmetrization reduces the Dirichlet energy of \( u \). That is

\[
\int_{A(r_1, r_2)} |\nabla u|^2 \, dx \, dy \leq \int_{A(r_1, r_2)} |\nabla u|^2 \, dx \, dy,
\]

with equality if and only if \( u = u^\ast \) modulo rotation.

**Proof** We apply the theorem to the pull-back of \( u \) to \( B(r_1, r_2) \). In these coordinates, the desired Dirichlet energy becomes

\[
\int_{r_1}^{r_2} \int_{-\pi}^{\pi} \left\{ \rho \left| \frac{\partial u}{\partial \rho} \right|^2 + \frac{1}{\rho} \left| \frac{\partial u}{\partial \phi} \right|^2 \right\} \, d\phi \, d\rho.
\]

So long as \( r_1 \) and \( r_2 \) are positive, we can apply the theorem with \( x_1 = \rho \), \( x_2 = \frac{1}{\rho} \) and \( G(t) = t^2 \). Thus

\[
\int_{r_1}^{r_2} \int_{-\pi}^{\pi} \left\{ \rho \left| \frac{\partial u}{\partial \rho} \right|^2 + \frac{1}{\rho} \left| \frac{\partial u}{\partial \phi} \right|^2 \right\} \, d\phi \, d\rho \geq \int_{r_1}^{r_2} \int_{-\pi}^{\pi} \left\{ \rho \left| \frac{\partial u^\ast}{\partial \rho} \right|^2 + \frac{1}{\rho} \left| \frac{\partial u^\ast}{\partial \phi} \right|^2 \right\} \, d\phi \, d\rho.
\]

Finally, the equality statement follows from the fact that rotation of \( u \) on the annulus corresponds exactly to translation of the pull-back of \( u \) in the \( \phi \) direction on the square (with respect to periodicity). \( \square \)

**Lemma 2.2** Suppose \( v, w : S^1 \to \mathbb{R} \) are measurable, then

\[
\int_{S^1} v w \, d\sigma \geq \int_{S^1} v^\ast w^\ast \, d\sigma.
\]

This lemma is a special case of a more general theorem of Hardy, Littlewood and Pólya that can be found stated in [27]. It is also necessary to know that rearrangement commutes with composition by certain functions. The following lemma covers this in the special case that is needed.
Lemma 2.3 Suppose $w : [-\pi, \pi] \to \mathbb{R}$ is a nonnegative and continuous. Then $[w^2]^* = [w^*]^2$.

Proof This follows immediately from a more general theorem of Cox [7] on the commutativity of composition and rearrangement.

The idea to come is that the preceding lemmas will be applied to the rearrangement of the first eigenfunction corresponding to some $\theta$. It is not obvious that such an eigenfunction will obtain continuous boundary values. However, it is known that the first eigenfunction is in $H^1$ (Sobolev space), which implies its rearrangement is also in $H^1$, and so both of these have at worst measurable extensions to the boundary. So, to take care of the boundary term in the expression for $\lambda(\theta)$, the lemma will be applied on subdisks where the eigenfunction is continuous up to the boundary, then a limit will be taken as the subdisk approaches the original. This is done in the following corollary.

Corollary 2.2 Suppose $u : D \to \mathbb{R}$ satisfies 2.1 and $w : S^1 \to \mathbb{R}$ is measurable, then $u^* \in H^1$ and

$$\int_{S^1} w u d\sigma \geq \int_{S^1} w u^* d\sigma$$

Proof That $u^*$ is in $H^1$ (and is Lipschitz) follows from the following corollary on rearrangements. For each $\rho \in (0, 1)$ let $u_\rho$ and $u_\rho^*$ denote the restriction of the first eigenfunction and its rearrangement to the boundary of the disk of radius $\rho$. For each such $\rho$ the preceding lemmas apply and give that

$$\int_{S^1} w u_\rho d\sigma \geq \int_{S^1} w u_\rho^* d\sigma.$$  

Fortunately, $u$ and $u^*$ are both uniformly bounded on the the entire disk, and $u_\rho$ and $u_\rho^*$ are converging pointwise a.e. This means that Lebesgue's dominated convergence
Theorem applies and allows the limit to be pulled into the integrals. The inequality persists and gives the desired conclusion. □

**Corollary 2.3** Suppose \( u : D \to \mathbb{R} \) satisfies 2.1, Then

\[
\int_D |\nabla u^*|^2 \, dx \, dy \leq \int_D |\nabla u|^2 \, dx \, dy,
\]

with equality if and only if \( u = u^* \) modulo rotation.

**Proof** If it is assumed that the pull-back of \( u \) in polar coordinates satisfies (1) and (2), then on each proper sub-annulus \( A(r_1, r_2) \subseteq D \), we have

\[
\int_{A(r_1, r_2)} |\nabla u^*|^2 \, dx \, dy \leq \int_{A(r_1, r_2)} |\nabla u|^2 \, dx \, dy \leq \int_D |\nabla u|^2 \, dx \, dy.
\]

Since the right hand side is independent of \( r_1 \) and \( r_2 \), it follows that

\[
\int_D |\nabla u^*|^2 \, dx \, dy \leq \int_D |\nabla u|^2 \, dx \, dy
\]

For the equality statement, it is enough to notice that if there is strict inequality on some sub-annulus, then it persists, since as the radii of the annuli approach their limits, this inequality can only worsen; i.e. if the strict inequality were to improve at some radius, it would be possible to show that rearrangement increased energy on some small annulus with corresponding radius.

To show that the conditions (1) and (2) are satisfied it will suffice to use the fact that \( u \in C^\omega(A(r_1, r_2)) \). It follows that the pull-back of \( u \) to \( \rho, \phi \) space is also real analytic and periodic in the \( \phi \) direction. Thus, condition (1) is satisfied. For each \( \rho \in (r_1, r_2) \), it turns out that the sets \( V \) and \( W \) are finite. This follows from the fact that if either of these sets were infinite, then \( u(\rho, \bullet) \) would be constant for all \( \phi \in [-\pi, \pi] \). This would imply that \( u \) is a radial function; i.e. radially symmetric on all of \( D \). This cannot happen since the boundary conditions are not radially
symmetric. Thus neither of these sets is infinite, and so condition (2) is satisfied with \( N = \emptyset \). This proves the corollary.

Recall that if \( \theta \in \text{ad}_\tau \) and \( u_\theta \) satisfies 2.1 then

\[
\lambda(\theta) = \int_\Omega |\nabla u_\theta|^2 \, dx + \frac{1}{\varepsilon} \int_{\partial \Omega} \theta u_\theta^2 \, d\Sigma
\]

**Proposition 2.1** If \( \theta \in \text{ad}_\tau \), then \( \lambda^* \leq \lambda_\theta \) with strict inequality if \( ^*\theta \neq \theta \) modulo rotation.

**Proof** Let \( u_\theta \) be the eigenvector associated to \( \lambda_\theta \) with \( ||u_\theta||_2 = 1 \). Then \( ||u^*_\theta||_2 = 1 \) and so by the corollaries

\[
\lambda_\theta = \int_\Omega |\nabla u_\theta|^2 \, dx + \frac{1}{\varepsilon} \int_{\partial \Omega} \theta u_\theta^2 \, d\Sigma \geq \int_\Omega |\nabla u^*_\theta|^2 \, dx + \frac{1}{\varepsilon} \int_{\partial \Omega} ^*\theta[u^*_\theta]^2 \, d\Sigma \geq \lambda^*.
\]

The last inequality follows from the fact that \( u^*_\theta \) is only one contender in the infimum that defines \( \lambda^* \). To show strict inequality as predicted, it is enough to show that if the Dirichlet energy is unchanged by the symmetrization that the boundary condition is unchanged by symmetrization. So suppose that the Dirichlet energy is unchanged. This means, by the corollary that \( u_\theta = u^*_\theta \) modulo rotation. This implies (by uniqueness of the first eigenfunction as normalized) that \( \theta = \theta^* = ^*\theta \) modulo rotation. This proves the contrapositive of the strict inequality statement.

**Lemma 2.4** For all \( \theta \in \text{ad}_\tau \),

\[
^*\theta(\phi) = \begin{cases} 1 & |\phi| \geq (1 - \frac{\tau}{2})\pi \\ 0 & \text{otherwise.} \end{cases}
\]

This lemma, whose proof follows immediately from the definitions, tells us that the support of \(^*\theta \) on the boundary of the unit disk is connected, and is independent of choice of \( \theta \in \text{ad}_\tau \). Thus we define \( ^c\theta = ^*\theta \) for any \( \theta \in \text{ad}_\tau \).
Proposition 2.2 If $\Omega$ is a ball, then $\theta$ is the minimizing choice of boundary condition (modulo rotation) for the given design problem. That is, the minimizing choice of Dirichlet set is connected.

Proof This follows immediately from the proposition. \qed

2.3 Discretization of the design problem

With computation as the goal, it will be assumed that the given domain $\Omega$ is a planar polygon and that the designs $\theta$ are constant on each edge of the polygon. Since the boundary conditions are constant on the edges of the given polygon, it may be necessary for the polygon to have trivial vertices (i.e. vertices along straight edges) so that these boundary conditions which are constant on each of these smaller edges can combine to give a better approximation to a more general boundary condition along the original longer edge.

Project the design, $\theta$, and the eigenfunction, $u$, to finite-dimensional function spaces and so arrive at a discrete optimization problem intended to be solved by a computer. It shall be convenient to suppose that $\Omega$ is a planar polygon. Write $\partial \Omega$ as the closure of the disjoint union of $m$ open edges, $\{\partial \Omega_j\}_{j=1}^m$ and then restrict $\theta$ to

$$\theta(x) = \sum_{j=1}^m \Theta_j 1_{\partial \Omega_j}(x),$$

where $\Theta \in \mathbb{R}^m$ satisfies the pointwise constraints

$$0 \leq \Theta_j \leq 1, \quad i = j, \ldots, m$$

and the integral constraint

$$\sum_{j=1}^m \Theta_j |\partial \Omega_j| = \gamma |\partial \Omega|.$$
In order to compute \( \lambda \) at such a \( \theta \) we restrict our search to eigenfunctions of the form

\[
u(x) = \sum_{i=1}^{p} U_i \phi_i(x)\]

where \( p < \infty \) and the \( \phi_i \) compose a so-called Galerkin basis. So an eigenfunction \( u \) can be identified with the vector \( U \in \mathbb{R}^p \) via 2.3. On substituting this expansion into the weak form of 2.1 with \( v \) running through the \( \phi_i \) we arrive at the \( p \times p \) eigensystem

\[
K(\Theta)U = \lambda MU.
\]

Let us denote the least eigenvalue of 2.3 by \( \tilde{\lambda}(\Theta) \). For well chosen basis functions, e.g., piecewise linear hats, it can be shown that \( \tilde{\lambda}(\Theta) \to \lambda(\theta) \) as \( m \) and \( p \) approach \( \infty \).

In attempting to extremize \( \Theta \mapsto \tilde{\lambda}(\Theta) \) one should expect to be queried for its gradient. The finite-dimensional analog of Theorem 2.1(5) reads

\[
\frac{\partial \tilde{\lambda}(\Theta)}{\partial \Theta_j} = \left\langle \frac{\partial K(\Theta)}{\partial \Theta_j} \tilde{U}, \tilde{U} \right\rangle,
\]

where the associated eigenvector, \( \tilde{U} \), is normalized according to

\[
\langle M\tilde{U}, \tilde{U} \rangle = 1.
\]

Though simple in appearance, the implementation of 2.3, in particular the application of \( \partial K(\Theta)/\partial \Theta_i \), requires intimate knowledge of how the stiffness matrix, \( K(\Theta) \), is assembled. As such information is not typically available to users of modern finite-element codes we settle for an approximation. Namely, given that \( \tilde{\lambda}(\Theta) \) is a good approximation to \( \lambda(\Theta) \) we may expect that

\[
\frac{\partial \tilde{\lambda}(\Theta)}{\partial \Theta_i} \approx \frac{\partial \lambda(\Theta)}{\partial \Theta_i}.
\]

The advantage of this line of reasoning is that Proposition 3.1 allows us to compute

\[
\frac{\partial \lambda(\Theta)}{\partial \Theta_j} = \langle \partial \lambda(\Theta), 1_{\partial \Omega_j} \rangle = \varepsilon^{-1} \int_{\partial \Omega_j} |u|^2 \, dx \approx \varepsilon^{-1} \sum_{k \in T_j} \langle \tilde{U} \rangle_k |\omega_k|,
\]
where $I_j$ is the set of indices of mesh edges $\omega_k$ contained in $\partial \Omega_j$ and

$$
\langle \tilde{U} \rangle_k = \frac{1}{3} [\tilde{U}^2_{\omega_k^+} + [\tilde{U}]_{\omega_k^+} [\tilde{U}]_{\omega_k^-} + \frac{1}{3} [\tilde{U}^2_{\omega_k^-}],
$$

where $\omega_k^\pm$ are the endpoints of $\omega_k$. This calculation requires only knowledge of $\tilde{U}$ at vertices on $\partial \Omega$. So define

$$
\frac{\partial \bar{\lambda}}{\partial \theta_j}(\Theta) = \sum_{k \in I_j} \langle \tilde{U} \rangle_k |\omega_k|.
$$

With this notation, the following theorem will addresses how appropriate the given approximation is. Before stating the theorem, recall that there are three different gradients available. Namely the distributional gradient of $\lambda$ given in Theorem 2.1.5, the approximation of this gradient given in 2.3, and the (finite-dimensional) gradient of the first eigenvalue $\bar{\lambda}$ (computed by the finite-element method). The distributional gradient is exactly the right gradient in terms of performing theoretical analysis. However, to deal with the problem numerically, one has to restrict consideration to approximate designs. For these approximate designs, it is still sensible to consider the "real" generalized gradient, but it is not practical because in general, it is very hard to find a closed formula for the first eigenfunction in most cases. However, it is possible to find an approximate first eigenfunction by applying some numerical theory like the finite-element method. By substituting the approximate eigenfunction for the real one, the approximate distributional gradient is obtained. At this point it is tempting to think that the optimization is step is ready for execution. However, a careful examination of what is about to be done reveals that something more needs to be checked. The optimization package is not going to optimize $\lambda$. Rather, the function optimized will be $\bar{\lambda}$ as computed by the finite-element method. The gradient provided to the optimization package should be the gradient of the function being optimized.
namely the gradient of $\bar{\lambda}$. This is the third gradient in the list. The following theorem says that everything checks out just fine.

**Theorem 2.4** The gradient of the approximate eigenvalue is the approximate (distributional) gradient of the eigenvalue; i.e.

$$\frac{\partial \bar{\lambda}}{\partial \Theta_i} (\Theta) = \frac{\partial \lambda}{\partial \Theta_i} (\Theta).$$

**Proof** It is clear from the previous discussion that the structure of $K(\Theta)$ will play a pivotal role in this proof. Thus it ill be shown how the finite-element method creates $K$. The finite-element system is created in three steps.

Step1: Describe the domain $\Omega$ and boundary conditions $\Theta$.

Step2: Create a triangular mesh on the domain. The description of this triangular mesh is stored in three parts; i.e. $P$, $E$ and $T$. These variables are the point list, edge list and triangle list respectively.

Step3: Discretize the PDE and boundary conditions. This is the guts of the matter for the computation that is needed.

For each point $x_j$ in the point list, associate a basis function $\phi_j$ which satisfies the following properties:

1) $\phi_j(x_i) = \delta_{ij}$.

2) $\phi_j$ is piecewise linear on each closed triangle.

We now project the weak formulation of the PDE onto $\text{span}\{\phi_j\}$ by considering as above,

$$u(x) = \sum_{i=1}^{p} \hat{U}_i \phi_i(x)$$

To obtain the weak formulation, in general, we multiply by an arbitrary test function $v$ and integrate

$$\int_{\Omega} -\nabla u \cdot v \, dx = \int_{\Omega} \lambda uv \, dx.$$
Apply Green's Formula
\[ \int_{\Omega} \nabla(u) \nabla(v) \, dx - \int_{\partial \Omega} (\mathbf{n} \cdot \nabla(u)) v \, d\sigma = \lambda \int_{\Omega} uv \, dx. \]

Substituting in the boundary condition gives
\[ \int_{\Omega} \nabla(u) \nabla(v) \, dx - \int_{\partial \Omega} \frac{\Theta}{\varepsilon} uv \, d\sigma = \int_{\Omega} uv \, dx. \]

If this holds for all test functions \( v \), \( u \) is called a weak solution. For the FEM solution, the weak formulation is required to hold for all test functions \( v \in \text{span}\{\phi_i\} \); i.e.
\[ \sum_j \left[ \int_{\Omega} \nabla(\phi_i) \nabla(\phi_j) \, dx + \int_{\partial \Omega} \frac{\Theta}{\varepsilon} \phi_i \phi_j \, d\sigma \right] U_j = \tilde{\lambda} \sum_j \int_{\Omega} \phi_i \phi_j \, dx U_j \quad \text{for all } i. \]

This can be expressed in matrix form as
\[ [K + Q(\Theta)] \hat{U} = \tilde{\lambda} M \hat{U}(\Theta) \]

where
\[
K = (k_{ij}) \quad \text{with} \quad k_{ij} = \int_{\Omega} \nabla(\phi_i) \nabla(\phi_j) \, dx \\
Q = (q_{ij}) \quad \text{with} \quad q_{ij} = \int_{\partial \Omega} \frac{\Theta}{\varepsilon} \phi_i \phi_j \, d\sigma \\
M = (m_{ij}) \quad \text{with} \quad m_{ij} = \int_{\Omega} \phi_i \phi_j \, dx
\]

The FEML system can now be solved for the first eigenvalue \( \tilde{\lambda}(\Theta) \) and corresponding eigenvector \( \hat{U}(\Theta) \). Once again, this is considering a design that is piecewise constant on the sides of \( \Omega \); i.e. considering \( \lambda(\Theta) \) as a function of \( n \) variables. Thus, there is sense in computing \( \frac{\partial \lambda(\Theta)}{\partial \Theta_k} \) in the classical fashion.

To begin, it is natural to assume that \( \hat{U}^T(\Theta) M \hat{U}(\Theta) = 1 \) so that the formula for \( \tilde{\lambda} \) is explicit as
\[ \tilde{\lambda}(\Theta) = \hat{U}^T(\Theta) [K + Q(\Theta)] \hat{U}(\Theta). \]
Now compute the derivative with respect to $\Theta_k$ and obtain the following expression

$$\frac{\partial \tilde{\lambda}(\Theta)}{\partial \Theta_k} = \tilde{U}^T(\Theta) \left[ \frac{\partial}{\partial \Theta_k} Q(\Theta) \right] \tilde{U}(\Theta).$$

It is now convenient to assume $k = 1$. Once the right hand side is simplified it will be clear that the choice of $k = 1$ was unimportant and the corresponding calculation for any other choice of $k$ would be analogous.

Define $F = (f_{ij})$ by

$$f_{ij} = \int_{\Omega_1} \phi_i \phi_j \, d\sigma.$$ 

By a sufficient change of notation, it may be assumed that

**Assumption 2.3.1**

1) $\{x_1, \ldots, x_n\} \subset \Omega_1$, 
2) $x_j \in \Omega_1$ for all $j > n$, 
3) $[x_1, x_n] = \Omega_1$, 
4) $U_i^{n-1}[x_i, x_{i+1}] = \Omega_1$ and the $\{[x_i, x_{i+1}]\}$ have pairwise disjoint interiors.

Geometrically this means that the $\{x_i\}$ partition the edge $\Omega_i$.

With this notation,

$$Q(\Theta) = \begin{bmatrix} \Theta F + \ast & \ast \\ \ast & \ast \end{bmatrix},$$

where $F$ is the $n \times n$ tridiagonal matrix

$$F = \begin{bmatrix} f_{1,1} & f_{1,2} & 0 & 0 & 0 & \cdots & 0 \\ f_{2,1} & f_{2,2} & f_{2,3} & 0 & 0 & \cdots & 0 \\ 0 & f_{3,2} & f_{3,3} & f_{3,4} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & f_{n-1,n} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & f_{n,n} \end{bmatrix}$$
and all of the * matrices are independent of $\Theta_1$. After differentiating with respect to $\Theta_1$, all terms leave except $F$ and so

$$\frac{\partial}{\partial \Theta_k} Q(\Theta) = \frac{1}{\epsilon} \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix}$$

The actual values of $F$ are now needed and are easily computed to be

$$f_{i,i} = \begin{cases} \frac{1}{3}(|x_{i-1} - x_i| + |x_i - x_{i+1}|) & \text{for } 1 < i < n \\ \frac{1}{3}|x_1 - x_2| & \text{for } i = 1 \\ \frac{1}{3}|x_{n-1} - x_n| & \text{for } i = n \end{cases}$$

and

$$f_{i,j} = \frac{1}{6}|x_i - x_j| \quad \text{for } i - j = \pm 1.$$  

With this in hand, this computation is reduced to multiplying all terms out and collecting in a careful way.

$$\bar{U}^T(\Theta) \left[ \frac{\partial}{\partial \Theta_k} Q(\Theta) \right] \bar{U}(\Theta) = \left[ \bar{U}_1 \ldots \bar{U}_n \ldots \right] \frac{1}{\epsilon} \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{U}_1 \\ \vdots \\ \bar{U}_n \\ \vdots \end{bmatrix}$$

$$= \frac{1}{\epsilon} \left[ \bar{U}_1, \ldots, \bar{U}_n \right] F \begin{bmatrix} \bar{U}_1 \\ \vdots \\ \bar{U}_n \end{bmatrix}$$

$$= \frac{1}{\epsilon} (\bar{U}_1^2 f_{1,1} + \bar{U}_1 \bar{U}_2 f_{1,2} + \bar{U}_1 \bar{U}_2 f_{2,1} + \bar{U}_2^2 f_{2,2} + \bar{U}_2 \bar{U}_3 f_{3,2,3} + \ldots + \bar{U}_{j-1} \bar{U}_j f_{j,j-1} + \bar{U}_j^2 f_{j,j} + \bar{U}_j \bar{U}_{j+1} f_{j,j+1} + \ldots + \bar{U}_{n-1} \bar{U}_n F_{n,n-1} + \bar{U}_n^2 F_{n,n} ).
In this sum, split the \( f_{ii} \) terms into their \((i - 1, i)\) and \((i, i + 1)\) parts and collect terms edgewise to obtain

\[
\frac{\partial \hat{\lambda}(\Theta)}{\partial \Theta_1} = \frac{1}{\varepsilon} \left[ \frac{1}{3} \left( \tilde{U}_1^2 |x_1 - x_2| + \tilde{U}_1 \tilde{U}_2 |x_1 - x_2| + \tilde{U}_2^2 |x_1 - x_2| \right) 
+ \frac{1}{3} \left( \tilde{U}_2^2 |x_2 - x_3| + \tilde{U}_2 \tilde{U}_3 |x_2 - x_3| + \tilde{U}_3^2 |x_2 - x_3| \right) 
+ \ldots 
+ \frac{1}{3} \left( \tilde{U}_{n-1}^2 |x_{n-1} - x_n| + \tilde{U}_{n-1} \tilde{U}_n |x_{n-1} - x_n| + \tilde{U}_n^2 |x_{n-1} - x_n| \right) \right] 
= \frac{1}{\varepsilon} \left[ \frac{1}{3} \left( \tilde{U}_1^2 \ + \tilde{U}_1 \tilde{U}_2 + \tilde{U}_2^2 \right) |x_1 - x_2| 
+ \frac{1}{3} \left( \tilde{U}_2^2 + \tilde{U}_2 \tilde{U}_3 + \tilde{U}_3^2 \right) |x_2 - x_3| 
+ \ldots 
+ \frac{1}{3} \left( \tilde{U}_{n-1}^2 + \tilde{U}_{n-1} \tilde{U}_n + \tilde{U}_n^2 \right) |x_{n-1} - x_n| \right].
\]

If the relevant mesh edges are named

\[\omega_k = [x_k, x_{k+1}] \quad \text{for} \quad k = 1, \ldots, n - 1,\]

then it is clear that the preceding sum is equal to the desired approximation 2.3 of the gradient. This completes the proof. \(\square\)

### 2.4 Computational method and results

MatLab is an environment in which one finds, in a integrated fashion, the tools of optimization, partial differential equations, and visualization. For the constrained extremization of \(\lambda_1\) we used the CONSTR function found in MatLab’s Optimization Toolbox. The assembly of 2.3 and the computation of \(\lambda_1\) and \(U^*_f\) was carried out by the PDEEIG function found in MatLab’s PDE toolbox. Given \(U^*_f\) we coded the
gradient computation, 2.3, ourselves. The details of our implementation are spelled out in [U]. We present here only the results of our computations.

Given that $\theta \mapsto \lambda_1(\theta)$ is a concave function with a unique maximizer we may expect our numerical method to perform better on the maximization of $\lambda_1$ than on the minimization. Let us begin then with our results for

$$\max_{\theta \in AD_\gamma^\ast} \lambda_1(\Theta),$$

where $AD_\gamma^\ast$ denotes those vectors satisfying 2.3 and 2.3. In Figures 2.1 through 2.5 we depict the best $\Theta$ for 5 distinct choices of $\Omega$. In each case we chose $\gamma = 1/2$, $\varepsilon = 10^{-6}$, and performed 3 mesh refinements beyond the default and so arrived at stiffness and mass matrices on the order of $10^4 \times 10^4$. In each figure is plotted the underlying polygon $\Omega$, the best $\Theta$, and contours of the corresponding eigenvector, $U_1^\ast$.

In Figure 2.1 the underlying polygon $\Omega$ is a unit area equilateral triangle. Each of the original sides of the triangle were partitioned into 24 edges for a total of 72 edges in the refined triangle, and so $\Theta \in \mathbb{R}^{72}$. The computed maximum eigenvalue for this design is $\lambda_1 = 22.8$.

In Figure 2.2 we have the unit square. With each of its 4 original sides partitioned into 24 edges we find that $\Theta$ varies over $\mathbb{R}^{96}$. The computed maximum eigenvalue for this design is $\lambda_1 = 19.74$.

In Figure 2.3 we find a regular 25–gon. We partitioned each side into 4 edges and so considered $\Theta \in \mathbb{R}^{100}$. The computed maximum eigenvalue for this design is $\lambda_1 = 18.17$.

It is not hard to a discern a definite pattern in these three figures. Namely, the more acute the angle the more miserly one can be with regard to distributing $\Theta$. Clearly, as we approach a disk by increasing the number of sides, we arrive at a
Figure 2.1 Maximizing $\theta$ for the equilateral triangle.

Figure 2.2 Maximizing $\theta$ for the square.
uniform distribution of material, i.e., $\Theta \rightarrow \gamma$. In the next two figures we treat the maximum problem over a pair of nonconvex domains.

In Figure 2.4 we find the chevron, a non-convex quadrilateral with one axis of symmetry. We introduced 8 edges per side and so searched over $\mathbb{R}^{32}$. The computed maximum eigenvalue for this design is $\lambda_1 = 24.17$.

In Figure 2.5 we see the plus-sign. With 12 sides and 8 edges per side we searched over $\Theta \in \mathbb{R}^{96}$. The computed maximum eigenvalue for this design is $\lambda_1 = 18.3$.

Both figures demonstrate the dramatic difference between inward and outward pointing corners. By placing Dirichlet data at the inward pointing corners one effectively reduces a domain to its largest inscribed disk.

We now turn to the minimization problem

$$\min_{\Theta \in AD} \lambda_1(\Theta),$$
Figure 2.4  Maximizing $\theta$ for a chevron.

Figure 2.5  Maximizing $\theta$ for a plus–sign.
with, as before, \( \gamma = 1/2, \varepsilon = 10^{-6} \) and \( p \) on the order of \( 10^4 \). Given the apparent abundance of global minima we've no reason to believe there are not an equal number of local minima. As such we can expect our algorithm to converge to nothing better than a local minimum. In fact, without some modification, we did not even observe convergence. In particular, we found that as the iterates approached a classical bang–bang design the problem becomes ill-conditioned and the numerical errors in the gradient approximation, 2.3 and 2.3, grow to unacceptable levels. We therefore abandoned the gradient calculation and let CONSTR approximate it via finite differences. Though this increased the execution time considerably it nonetheless produced convergence.

Figure 2.6 depicts the application of this procedure to the equilateral triangle. As predicted by Corollary 2.5, the minimizer is a characteristic function; i.e. taking values 0 and 1 alone. Another feature of this example is that it is "dual" to the corresponding maximizer insofar as the minimization process here has pushed all of the Dirichlet set to the corners of the triangle. This symmetric solution is beautiful and seems intuitively correct as a "dual" to the maximization problem. The computed minimum eigenvalue for this picture is \( \lambda_1 = 6.86 \).

In keeping with the result that the minimizing Dirichlet set for the disk is connected, it is worth considering what a connectedness assumption might say about other domains. In the following figure 2.7, we see the behavior of the first eigenvalue as a function of the placement of a connected Dirichlet set.

For this picture, the underlying polygon is an equilateral triangle with unit area. Each side of the triangle was subdivided into 32 edges of equal length for a total of 96 edges. The numbers along the y-axis represent displacement (number of (trivial) edges) from the midpoint of the Dirichlet set to a fixed vertex of the original equilateral triangle. The x-axis corresponds to the Dirichlet fraction. For a fixed Dirichlet
fraction, the eigenvalue, as a function of the displacement of the midpoint of the Dirichlet set, is a periodic function with period equal to one edge of the underlying equilateral triangle. The symmetries of this triangle immediately show that the corners and midpoints of the sides of the original triangle are local extrema for each \( \gamma \) (as seen in the picture). However, for \( \gamma \) large enough, other local extrema appear also.
Figure 2.7 First eigenvalue as function of connected Dirichlet set.
Bibliography


