INFORMATION TO USERS

This manuscript has been reproduced from the microfilm master. UMI films the text directly from the original or copy submitted. Thus, some thesis and dissertation copies are in typewriter face, while others may be from any type of computer printer.

The quality of this reproduction is dependent upon the quality of the copy submitted. Broken or indistinct print, colored or poor quality illustrations and photographs, print bleedthrough, substandard margins, and improper alignment can adversely affect reproduction.

In the unlikely event that the author did not send UMI a complete manuscript and there are missing pages, these will be noted. Also, if unauthorized copyright material had to be removed, a note will indicate the deletion.

Oversize materials (e.g., maps, drawings, charts) are reproduced by sectioning the original, beginning at the upper left-hand corner and continuing from left to right in equal sections with small overlaps. Each original is also photographed in one exposure and is included in reduced form at the back of the book.

Photographs included in the original manuscript have been reproduced xerographically in this copy. Higher quality 6" x 9" black and white photographic prints are available for any photographs or illustrations appearing in this copy for an additional charge. Contact UMI directly to order.

UMI
A Bell & Howell Information Company
300 North Zeeb Road, Ann Arbor MI 48106-1346 USA
313/761-4700 800/521-0600
RICE UNIVERSITY

Morse-Bott functions and the Witten Laplacian

by

Igor P. Prokhorenkov

A THESIS SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

Doctor of Philosophy

APPROVED, THESIS COMMITTEE:

Robin Forman, Chairman
Associate Professor of Mathematics

B. Frank Jones
Noah Harding Professor of Mathematics
Chair, Department of Mathematics

William W. Symes
Professor of Computational and Applied Mathematics
Chair, Department of Computational and Applied Mathematics

Houston, Texas

May, 1997
Morse-Bott functions and the Witten Laplacian

Igor P. Prokhor enkov

Abstract

Given a compact Riemannian manifold \((N, g)\), a flat vector bundle \(V\) over \(N\), and a Morse-Bott function \(h\), Witten considered the following one-parameter deformation of the differential \(d\) in the de Rham complex of \(V\)-valued differential forms on \(N\):

\[ d(\alpha) : \omega \mapsto e^{-\alpha h} d e^{\alpha h}. \]

In this thesis we study the asymptotics as \(\alpha \to \infty\) of the discrete spectrum of the Witten Laplacian

\[ L(\alpha) = d(\alpha) d^*(\alpha) + d^*(\alpha) d(\alpha). \]

Suppose \(g\) is a metric on \(N\), associated to a Morse-Bott function \(h\). The main result of the thesis states that as \(\alpha \to \infty\) the small eigenvalues of \(L(\alpha)\) approach all eigenvalues of the standard Laplacian \(\Delta\) on \(M\), twisted by the orientation bundle of the negative directions in the normal bundle to \(M\) in \(N\). We also prove the estimates on the rate of convergence as \(\alpha \to \infty\) of the small eigenvalues of \(L(\alpha)\).

The main idea of the proof is to use the adiabatic limit technique of Mazzeo-Melrose and Forman to analyze the spectrum of the Witten Laplacian on the tubular neighborhood of the critical submanifold \(M\) of \(h\).

As an application of our results we give a new Hodge theoretic proof of the Thom isomorphism. We obtain the localization of the trace of the heat kernel \(e^{-tL(\alpha)}\) along the critical submanifold \(M\) of the Morse-Bott function \(h\), and we prove the degenerate inequalities of Morse.
Acknowledgments

My greatest thanks go to my advisor Robin Forman for his incredible patience, guidance, numerous suggestions, and encouragement during the course of my graduate studies, without which the work would not have been possible. I would also like to mention his ever present sense of humor and positive outlook.

I am grateful to Frank Jones and William W. Symes for serving on my thesis committee.

I gratefully acknowledge the Mathematics Department at Rice University and the Department of Mathematics and Mechanics at Moscow State University for providing me with opportunities and generous support for my undergraduate and graduate studies.

During the course of my mathematical research, I have received many good ideas and suggestions from other mathematicians. Among these people, I would like to thank my undergraduate advisor Mikhail Shubin and Dan Burghelea.

I was privileged to be a part of the active mathematical community at Rice. I especially thank Zhiyong Gao, Bob Hardt, Frank Jones, Reese Harvey, John Polking and Mike Wolf for their personal interest in my work and welfare. Thanks to Maxine Turner and Janie McBane for their help above and beyond the call of duty, and to Sharon McDonough for being an absolutely incredible Math-Mum.

My fellow graduate students have been extremely supportive and taught me much. Hope McIlwain read this thesis and taught me many things about English. Nancy Cunningham and Ashley Ledbetter read parts of the thesis. Thanks especially for their warm friendship to Nancy Cunningham, Hope McIlwain, Ashley Ledbetter, David Handron, Paul Phillips, Chris Hawkins, Clayton Ward, ChangYou Wang, Paul Uhlig, Amy Lampazzi, Scott Berger and Lesley Ward.

I greatly benefited from discussions with my wonderful mathematical friends John Zweck, Ken Richardson, Efton Park and Robert Stingley. Special thanks go to Maxim Braverman, who found several gaps in the preliminary version of this thesis and suggested ways to fill them.
0. Introduction 1
1. Witten Laplacian 12
2. Deformation of the Witten Laplacian 19
3. Taylor analysis of zero eigenspaces and associated nested sequence of spaces 23
4. Isomorphism between cohomology of $E$ and $E^-$ 31
5. Hodge $*$-operator 36
6. Thom isomorphism 40
7. Asymptotics of the small eigenvalues of the Witten Laplacian 44
8. Witten Laplacian on compact manifolds 59
9. Asymptotics of the trace of the heat kernel 64
10. Morse-Bott inequalities 71

Appendix 1. Discreteness of the spectrum of the Witten Laplacian 76

Appendix 2. Bismut connection and bounds on the Witten Laplacian 79

Appendix 3. Space of rapidly decreasing forms 85

Appendix 4. Proof of Theorem 8.5 93

References 100
0. INTRODUCTION

0.0. **Summary.** Suppose $N$ is a compact manifold without boundary. Let $h : N \to \mathbb{R}$ be a smooth function. We assume that critical points of $h$ form a (disconnected) submanifold $M$ of $N$ and that the Hessian $D^2h$ of $h$ is a non-degenerate quadratic form on the normal bundle to $M$ in $N$. In this case the function $h$ is called a **Morse-Bott** function.

In [Wi1] E. Witten considered the following one-parameter deformation of the differential in the de Rham complex of $N$:

\begin{equation}
    d(\alpha) : \omega \mapsto e^{-\alpha h} de^\alpha = d\omega + \alpha dh \wedge \omega, \quad \alpha \in \mathbb{R}_{(\geq)}, \quad \omega \in \Omega^*(N).
\end{equation}

In this thesis we study the asymptotics as $\alpha \to \infty$ of the discrete spectrum of the **Witten Laplacian**

\begin{equation}
    L(\alpha) = d(\alpha)d^*(\alpha) + d^*(\alpha)d(\alpha).
\end{equation}

Here $d^*(\alpha)$ denotes the operator adjoint to $d(\alpha)$ with respect to a fixed Riemannian metric $g$ on $N$.

Suppose $g$ is a metric on $N$, associated to a Morse-Bott function $h$ (see Chapter 1). Then the spectrum of $L(\alpha)$ consists of small and large eigenvalues. As $\alpha \to \infty$, the small eigenvalues stay bounded and the large eigenvalues grow faster than $C\alpha$ for some constant $C > 0$. The main result of the thesis is Theorem 8.6 which states that the small eigenvalues of $L(\alpha)$ approach the eigenvalues of the standard Laplacian $\Delta$ on $M$ as $\alpha \to \infty$. The theorem also includes the estimates on the rate of the convergence of the small eigenvalues of $L(\alpha)$.

In Chapter 7 we apply the theorem to give a new analytic proof of the Thom isomorphism. In Chapter 10 we prove the degenerate inequalities of Morse. The estimates in Theorem 8.6 are further applied in Chapter 9 to obtain the localization of the trace of the heat kernel $e^{-tL(\alpha)}$ along the critical submanifold $M$ of the Morse-Bott function $h$.

The main idea of the proof is to use the adiabatic limit technique of Mazzeo-Melrose and Forman ([Ma-Me], [Fo]) to analyze the spectrum of the Witten Laplacian on the neighborhood of the critical submanifold $M$ of $h$. 

1
We now give a more detailed description of the results of the thesis.

0.1. A localization of the Witten Laplacian to a neighborhood of the critical submanifold. For a Morse-Bott function $h : N \to \mathbb{R}$ let $M_1, \ldots, M_A$ denote the connected components of the critical submanifold $M$ of $h$. According to the Generalized Morse Lemma (Lemma 8.3), each $M_i$ has a tubular neighborhood $E_i$, which is diffeomorphic to the normal bundle to $M_i$ in $N$. An easy calculation shows that

$$L(\alpha) = \Box + \alpha^2 |dh|^2 + \alpha A,$$

where $\Box$ is the usual Laplacian associated with the metric $g$ and $A$ is a bounded zeroth order operator. As $\alpha \to \infty$ the term $\alpha^2 |dh|^2$ becomes very large, except in the neighborhood of the critical submanifolds $M_i$, where $dh = 0$. Therefore, the eigenforms of $L(\alpha)$ corresponding to small eigenvalues are, for large $\alpha$, concentrated near the critical submanifolds of $h$. In Chapter 8 (Theorem 8.5) we show that the small eigenvalues of $L(\alpha)$ can be calculated by means of the restriction of $L(\alpha)$ on a tubular neighborhood of each connected component $M_i$ of $M$. We call this restriction $\Box(\alpha)$. Thus we are led to study the spectrum of $\Box(\alpha)$ on each $E_i \to M_i$.

0.2. Zero eigenvalues of the Witten Laplacian on the tubular neighborhood. We begin our study of the spectrum of $\Box(\alpha)$ with the examination of zero eigenvalues. Let $E \to M$ be a tubular neighborhood of a single connected component $M$ (dim $M = m$) of the critical submanifold of $h$. We will again denote by $h$ the pull-back of $h$ from $N$ to $E$ under the diffeomorphism between $E$ and the normal bundle to $M$ in $N$. Since the Hessian of $h$ is non-degenerate, the bundle $E$ splits into the Whitney sum of two subbundles $E = E^+ \oplus E^-$, such that the Hessian is strictly positive on $E^+$ and strictly negative on $E^-$. The dimension of the bundle $E^-$ is called the index of $M$ (as a critical submanifold of $h$) and is denoted $n^-$. Moreover,

$$h(y) = |y^+|^2 - |y^-|^2,$$
where $y = (y^+, y^-) \in E^+ \oplus E^-$ is the coordinate in the fiber.

We study the spectrum of $\square(\alpha)$ in a more general situation. Namely, we assume in addition that we are given a flat vector bundle $V \to E$. Then $\square(\alpha)$ is defined on the space $\Omega^*(E, V)$ of smooth differential forms on $E$ with values in $V$.

In chapters 1 through 5 we study the kernel of $\square(\alpha)$. Since the square of $d(\alpha)$ is zero, for each $\alpha$ we can define the de Rham cohomology $H^*(E, V, \alpha)$, associated to a differential complex $(\Omega^*(E, V, \alpha), d(\alpha))$. In Chapter 1 we show that the spectrum of $\square(\alpha)$ is discrete. Then by Hodge theory for $d(\alpha)$,

\begin{equation}
H^p(E, V, \alpha) \cong \ker \square^p(\alpha), \quad p = 0, 1, \ldots \dim E.
\end{equation}

As an essential step in the analytic proof of Morse-Bott inequalities, J. M. Bismut proved the following theorem about $H^p(E, V, \alpha)$ (without considering an additional bundle $V$):

**Theorem A** ([Bis, Section 2(h)]). *For $\alpha$ large enough and all $p$, $0 \leq p \leq \dim E$,

\begin{equation}
\dim H^p(E, \alpha) = \dim H^{p-n^-}(M, o(E^-)),
\end{equation}

where $o(E^-)$ denotes the orientation bundle of $E^-$.\)

In his proof Bismut uses the existence of a Thom form on $E^-$ and the retraction of $E^+$ on $M$ to construct local isomorphisms between $H^p(E|_U, \alpha)$ and $H^{m-n^-}(U)$ over open sets $U$. Then the Mayer-Vietoris argument finishes the proof.

M. Braverman and M. Farber in [Bra-Far, Section 3] proved Theorem B, which is the generalization of Theorem A, where in addition we have a flat vector bundle $V \to E$.

**Theorem B** (Theorem 5.3). *For large enough $\alpha$ and all $p$, $0 \leq p \leq \dim E$,

\begin{equation}
\dim H^p(E, V, \alpha) = \dim H^{p-n^-}(M, V|_M \otimes o(E^-)).
\end{equation}

Their proof is different from the proof of Bismut. They used the existence of a Thom form of the bundle $E^-$ to construct homotopy equivalences between the complexes $(\Omega^*(E, V, \alpha), d(\alpha))$ and $(\Omega(M, V \otimes o(E^-)), d)$.\)
In this thesis we take a new approach to prove Theorem B. It is motivated by the notion of the “adiabatic limit”, introduced in this sort of mathematical context by Witten in [Wi2]. Moreover, we use our approach to study the small eigenvalues of the Witten deformation of the Laplacian $\Box(\alpha)$ as the metric on $E$ is deformed.

0.3. The adiabatic limit. Suppose we chose a metric $g_E$ on $E$, compatible in the sense of Chapter 1 with the $(E^+ \oplus E^-, h)$-structure on $E$. We now describe the general set up for our approach. Let $A$ be a smooth distribution of $k$-planes, $A \subset TE$. Let $B$ be the orthogonal complement of $A$ in $TE$. Writing $g_E = g_A \oplus g_B$, for $0 < \delta \leq 1$ we define a 1-parameter family of metrics on $E$ by setting

\begin{equation}
(0.8) \quad g_\delta = g_A \oplus \delta^{-2} g_B.
\end{equation}

In addition, let $V \to E$ be a flat vector bundle. Then the limit of $(E, g_\delta)$ as $\delta \to 0$ is known as the adiabatic limit. The adiabatic limit was introduced in this form by Witten in [Wi2]. He considered the distribution $A$ consisting of the vertical vectors of a fibration

\begin{equation}
(0.9) \quad \mathcal{F} \hookrightarrow E \to M,
\end{equation}

where $\mathcal{F}$ is compact, $M = S^1$, and the metric $g$ makes (0.9) a Riemannian submersion. Witten investigated the limit of the eta-invariant of $E$ as $\delta \to 0$. We also refer to [Bis-Fr] and [Ch]. In [Bis-Ch] and [Dai] this investigation was extended to general base spaces $M$.

In [Ma-Me], R. Mazzeo and R. Melrose study the behavior of the space of harmonic forms on a compact manifold $E$ for the fibration (0.9) as $\delta \to 0$. They show that modulo a change of coordinates, the space of harmonic $p$-forms approaches a finite dimensional space, which can be identified from the Taylor series analysis. They use Melrose’s calculus of pseudodifferential operators on manifolds with corners to construct a parametrix for $\Box_\delta^p$, where $\Box_\delta^p$ denotes the Laplacian induced by the metric $g_\delta$ acting on $p$-forms on $E$ with values in $V$. This parametrix has
a uniform extension to the closed interval [0, 1]. This implies that in the case of a fibration (0.9) the eigenvalues and eigenvectors have well-defined asymptotic as \( \delta \to 0 \).

This thesis owes much to the ideas and techniques in [Fo]. In [Fo] R. Forman considers a more general situation than in [Ma-Me]. In particular, he does not require that the distribution \( A \subset TE \) arises from a fibration. He investigates a spectral sequence associated with \( A \) and \( B \) for the cohomology of \( E \) with values in \( V \). This spectral sequence arises naturally from a Taylor series analysis of the eigenvalues of \( \Box_\delta^p \) near \( \delta = 0 \). Moreover, in Chapter 5 of [Fo] he shows that the leading order asymptotics of the small eigenvalues of \( \Box_\delta^p \) and the corresponding eigenspaces are determined by the information contained in the spectral sequence.

We point out that our situation is different from the setting in [Ma-Me] or in [Fo] since the fibers of the fibration \( E \to M \) we study are not compact. As a result it is crucial in the analysis of Chapters 4 through 7 that for large enough \( \alpha \) the eigenforms of \( \Box(\delta) \) have rapid decay at \( \infty \). The proof of this result (Appendix 3) uses the notion of the Bismut connection on \( E \), introduced in [Bis]. This result is one of the main technical difficulties of the thesis.

Many results proved in chapters 1 through 7 of this thesis are similar to those proved in [Fo]. However, instead of the standard Laplacian on \( E \) we consider its Witten deformation.

### 0.4. The Thom isomorphism.

In Chapter 7 we prove the Thom isomorphism [Bott-Tu, Theorem 7.10] as an application of Theorem B. For \( E \to M \) a rank \( n \) smooth vector bundle over a compact connected manifold \( M \), we denote as \( H^p_{\delta}(E, V) \) the compactly supported de Rham cohomology of \( E \) with values in \( V \). Then our version of the Thom isomorphism is

**Theorem C. (Theorem 6.1)** For all \( p \), \( 0 \leq p \leq \dim E \),

(1) if \( E \) is orientable, then

\[
0.10 \quad \dim H^p_{\delta}(E, V) = \dim H^{p-n}(M, V_{|M});
\]
(2) if $E$ is not orientable, then

\[(0.11) \quad \dim H^p_c(E, V) = \dim H^{p-n}(M, V_M \otimes o(E)),\]

where $o(E)$ is the orientation bundle of $E$.

To prove Theorem C we choose $h(y) = -|y|^2$ as a Morse-Bott function on $E$. In this case $E = E^-$. Then Theorem C follows from Theorem B and the following equality, which is the main result of Section 7,

\[(0.12) \quad \dim H^p_c(E, V) = \dim \ker \Box^p(\alpha),\]

This equality holds for all $\alpha$ large enough and all $p$, $0 \leq p \leq \dim E$.

0.5. A simple example. Before we proceed further, let us consider a simple example which illustrates our results. In this example we can perform an explicit calculation of the eigenvalues and eigenforms of the deformed Laplacian. Let

$$E = M \times \left( \mathbb{R}^{n^+} \oplus \mathbb{R}^{n^-} \right),$$

and

$$g = g_M \oplus g_{\mathbb{R}^n},$$

where $g_{\mathbb{R}^n}$ is the standard Euclidean metric on $\mathbb{R}^n$. Let $y$ be the standard coordinate on $\mathbb{R}^n$, then

$$h(y) = \sum_{i=1}^{n^+} y_i^2 - \sum_{i=(n^+)+1}^{n} y_i^2.$$

Then we can perform the calculation as in [CFKS, Proposition 11.13] to get

$$\Box(\alpha) = H(\alpha) + \Delta_M,$$

where $\Delta_M$ is the Laplacian on $M$ and $H(\alpha) = \oplus_{k=1}^{n} H_k(\alpha)$ is the sum of harmonic oscillators. For all $\phi dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_\nu} \in \Omega^p(\mathbb{R}^n)$,

\[(0.13) \quad H_k(\phi dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_\nu}) = \left(-\frac{d^2 \phi}{dy_k^2} + 4\alpha^2 y_k^2 \phi\right) dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_\nu} \pm 2\alpha \phi B_k(dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_\nu}),\]
where we have (+) if \( k \in \{1, 2, \ldots, n^+\} \) and we have (−) otherwise. The operator \( B_k \) is a zeroth order operator defined by

\[
B_k(dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_p}) = \pm dy_{i_1} \wedge dy_{i_2} \wedge \cdots \wedge dy_{i_p},
\]

where we have (+) if \( k \in \{i_1, i_2, \ldots, i_p\} \) and we have (−) otherwise.

We let \( A = \mathbb{R}^n, B = TM \). Then the Witten Laplacian \( \square_\delta(\alpha) \), associated to the metric \( g_\delta \), equals to

\[
(0.14) \quad \square_\delta(\alpha) = H(\alpha) + \delta^2 \Delta_M.
\]

We can compute the eigenvalues and eigenforms of \( \square_\delta(\alpha) \) by separating variables. Let \( \omega \in \ker \square_\delta^p(\alpha) \) then we can assume that \( \omega = \gamma \otimes \beta \), where \( \gamma \in \ker H^i(\alpha) \) and \( \beta \in \ker \Delta_M^{p-i} \), where operator \( H^i \) denotes the restriction of the harmonic oscillator \( H \) to \( i \)-forms on \( \mathbb{R}^n \), and \( \Delta_M^{p-i} \) is the restriction of the Laplacian \( \Delta_M \) to \( (p-i) \)-forms on \( M \). It follows from (0.13) that if \( 0 \neq \gamma \in \ker H^i(\alpha) \), then \( i = n^− \) and

\[
(0.15) \quad \gamma = e^{-\alpha \|y\|^2} dy_{(n^+) + 1} \wedge \cdots \wedge dy_n.
\]

Thus if \( \omega \in \ker \square_\delta^p(\alpha) \), then \( \omega = \gamma^{n−} \otimes \beta \), where \( \gamma^{n−} \) is defined by (0.15). Since \( \dim H^p(E, \alpha) = \dim \ker \square_\delta^p(\alpha) \) and \( \dim H^{p-i}(M) = \dim \ker \square_\delta^{p-i} \), we demonstrated (for this example) the following theorem, which is a corollary of the Thom isomorphism (see Theorem C and Corollary 5.4).

**Theorem D.** For all \( p, 0 \leq p \leq \dim E \),

\[
\dim H^p(E, \alpha) = \dim H^{p-n^−}(M).
\]

Moreover from (0.14) we have

**Theorem E.** Let \( \lambda_i^p(\delta) \) be an eigenvalue of \( \square_\delta^p(\alpha) \) which is 0(\( \delta^2 \)) near \( \delta = 0 \), then

\[
\lambda_i^p(\delta, \alpha) = \delta^2 \mu_i^{(p-n^−)},
\]

where \( \mu_i^{p-n^−} \) is an eigenvalue of \( \Delta_M^{(p-n^−)} \). In particular, \( \lambda_i^p(\delta, \alpha) \) does not depend on \( \alpha \).

This theorem should be compared to Theorem 7.25.
0.6. A more detailed description of the content of chapters 1 through 7. In Chapter 1 we put a natural metric $g$ on $E$, compatible with the given decomposition $E = E^+ \oplus E^-$. Then we define a one parameter family of deformations $\Box(\alpha)$ of the Laplacian and describe the Hodge theory for $d(\alpha)$. We also define $\Omega^*(E, V)$, the space of rapidly decreasing forms on $E$, in terms of the Bismut connection. Finally, we observe that the cohomology $H^*(E, V, \alpha)$ of the complex $(\Omega^*(E, V), d(\alpha))$ is the same as cohomology $H^*_B(E, V, \alpha)$ of the complex $(\Omega^*_B(E, V), d(\alpha))$.

In Chapter 2 we consider a manifold $E$ as a fiber bundle over $E^-$. For reasons which will be clear in Chapter 4, we have to consider the bundle $E \to M$ as a filtration $E \to E^- \to M$. We denote as $A \subset TE$ the set of all vectors tangent to fibers of $E \to E^-$, and $B$ denotes the orthogonal complement of $A$ in $TE$ with respect to metric $g$. Then the family of metrics $g_\delta$ is defined as in (0.8). We also define a rescaling map $\rho_\delta$, which is an isometry

$$\rho_\delta : (\Omega^*(E, V), g_\delta) \to (\Omega(E, V), g).$$

We fix large enough $\alpha \geq 0$. We denote as $\hat{d}$ the differential $d(\alpha)$ for a fixed $\alpha$. Define

$$\hat{d}_\delta = \rho_\delta d(\alpha) \rho^{-1}_\delta, \quad \hat{d}^*_\delta = \rho_\delta d^*_\delta(\alpha) \rho^{-1}_\delta.$$

Then we show that $\dim \ker \Box^p(\alpha) = \dim \ker \hat{\Box}^p(\delta)$, where

$$\hat{\Box}(\delta) = \hat{d}_\delta \hat{d}^*_\delta + \hat{d}^*_\delta \hat{d}_\delta. \tag{0.16}$$

In chapters 3 and 4 we study the behavior of the space of the $\hat{d}_\delta$-harmonic forms on $M$ as $\delta \to 0$.

In Chapter 3 we define a nested sequence of spaces

$$E^p_0 \supseteq E^p_1 \supseteq E^p_2 \supseteq \ldots$$

by

$$E^p_k = \{ \omega \in \Omega^p_\delta(E, V) \mid \exists \omega_1, \ldots, \omega_{k-1} \text{ with}$$
\[ \hat{d}_\delta(\omega + \delta \omega_1 + \ldots + \delta^{k-1} \omega_{k-1}) \in O(\delta^k), \]
\[ \hat{d}_\delta^2(\omega + \delta \omega_1 + \ldots + \delta^{k-1} \omega_{k-1}) \in O(\delta^k). \]

By explicitly computing the spaces \( E_1^p \) and \( E_2^p \), we show that \( E_2^p \) is isomorphic to \( H_\alpha^p(E^-, V_{[E^-]}, \alpha) \), which denotes the cohomology of the complex \((\Omega^p_\alpha(E^-, V_{[E^-]}), d(\alpha))\), where a differential \( d(\alpha) \) on \( \Omega^p_\alpha(E^-, V_{[E^-]}) \) is defined by \( d(\alpha) = d - \alpha dh^- \wedge \).

In Chapter 4 we show that the nested sequence define in Chapter 3 stabilizes at \( E_2^p \):
\[ E_2^p = E_3^p = \ldots = E_\infty^p. \]

Then we describe an isomorphism between \( H_\alpha^p(E, V, \alpha) \) and \( E_\infty^p \). We note that we can prove this fact directly for the fibration \( E \to E^- \), but not for the fibration \( E \to M \). Together with the results of Chapter 2, we have an isomorphism
\[ H_\alpha^p(E, V, \alpha) \cong H_\alpha^p(E^-, V_{[E^-]}, \alpha). \]

In Chapter 5 we use the Hodge theoretic \(*\)-operator as a convenient tool to deduce the following equality
\[ \dim H_\alpha^p(E^-, V_{[E^-]}, \alpha) = \dim H^{m+n-p}(M, V_{[M]} \otimes O(E^-)), \]

Together with (0.17) and the Poincare duality on \( M \) this equality proves Theorem B of Braverman and Farber. We note that equality (0.18) of Chapter 5 could be independently proved by the methods of chapters 3 and 4.

In Chapter 6 we present a Hodge theoretic version of the de Rham cohomology of \( E \). We use this version and (0.18) to prove the Thom isomorphism.

In Chapter 7 we study the asymptotics of the small eigenvalues of \( \square(\alpha) \) as \( \alpha \to \infty \). Our observation is that if we put
\[ \delta = \alpha^{-1/2} \]
then the operators \( \square(\alpha) \) and \( \delta^{-2} \hat{\square}(\delta) \) are isospectral (we assume that in the definition of \( \hat{\square}(\delta) \) the parameter \( \alpha \) equals to one). This implies that if \( \lambda_\alpha^p(\alpha) \)
denotes the $j$-th eigenvalues of $\Box^p(\alpha)$ and $\lambda_{j}^p(\delta)$ of $\Box^p(\delta)$, then for any $p, 0 \leq p \leq \dim E$, $\alpha \geq 0$, and $j = 1, 2, \ldots \dim E$, we have

\begin{equation}
\lambda_{j}^p(\alpha) = \delta^{-2}\lambda_{j}^p(\delta).
\end{equation}

To investigate the asymptotics of the small spectra of $\Box(\delta)$ we use the Taylor analysis of the eigenspaces of the Witten Laplacian in the spirit of Section 5 of [Fo]. The main result of Chapter 7 is Theorem 7.20, which states that the small eigenvalues of $\Box(\alpha)$ approach the eigenvalues of the Laplacian

$$\Delta : \Omega^*(M, V_{|M} \otimes o(E^-)) \to \Omega^*(M, V_{|M} \otimes o(E^-)).$$

**0.7. The main result.** In Chapter 8 observe that the small eigenvalues of $L(\alpha) : \Omega^*(N, V) \to \Omega^*(N, V)$ can be calculated by means of the restriction of $L(\alpha)$ onto a tubular neighborhoods $E_{i}$ of connected components $M_{i}$ of the critical submanifold $M$ and then applying Theorem 7.20. We prove Theorem 8.6, which is the main result of the thesis. This theorem states that the small eigenvalues of $L(\alpha)$ on $N$ approach the eigenvalues of the standard Laplacian $\Delta$ on $M$, twisted by orientation bundles $o(E^-)$ as $\alpha \to \infty$. The theorem also contains estimates on the rate of convergence of the eigenvalues of the Witten Laplacian $L(\alpha)$ on $N$.

**0.8. The trace of the heat kernel.** In Chapter 9 we apply the estimates in Theorem 8.6 to obtain the localization of the trace of the heat kernel $e^{-tL(\alpha)}$ along the critical submanifolds of $h$. This result is used in the proof of Morse-Bott inequalities and in the forthcoming work of the author on the localization formula for the Witten deformation of the analytic torsion.

**0.9 The Morse-Bott inequalities.** In 1954, R. Bott [Bott] generalized the Morse inequalities to the case when the critical points of the function $h$ form a submanifold $M$, satisfying some conditions of non-degeneration (see Section 8). Let $M = \bigoplus_{i=1}^{\lambda} M_{i}$, where $M_{i}$ denotes the connected component of $M$ of index $n_{i}$. For each $i$, consider the following twisted Poincare polynomial of $M_{i}$

$$P_{i}^{-}(t) = \sum_{p} t^{p} \dim H^{p}(M_{i}, V_{|M_{i}} \otimes o(E_{i}^-)),$$
and the following Poincaré polynomial of $N$

$$P(t) = \sum t^p \dim H^p(N, V).$$

Then the Morse-Bott inequalities say that there exists a polynomial $Q(t)$ given by $Q(t) = Q_0 + Q_1 t + \ldots$ with all non-negative coefficients such that

$$0.20 \sum_{i=1}^\Lambda t^{n_i} P_i^- (t) - P(t) = Q(t)(1 + t).$$

The idea of applying the Witten deformation to prove the Morse-Bott inequalities (0.20) was suggested by Witten in [Wi1].

J.-M. Bismut [Bis] introduced a slight modification of the Witten deformation (using two parameters), such that the study of the corresponding family of operators leads to a family of operators on $E \to M$. Bismut applied a probabilistic technique to study the eigenvalues which approach 0 of the deformed Laplacian on $E \to M$.

M. Braverman and M. Farber [Bra-Far] used essentially the same modification of the Witten deformation as Bismut. However, they excluded probability considerations and used instead an explicit estimate of the number of the eigenvalues of the deformed Laplacian approaching 0. They proved the existence of the spectral gap which separates the eigenvalues that approach 0 from the rest of the spectrum. Moreover, Braverman and Farber also proved the twisted degenerate Novikov inequalities.

In [H-S] Helffer and Sjöstrand gave an analytic proof of the Morse-Bott inequalities. Although they also used the ideas of [Wi1], their method is completely different from the method in [Bis] and [Bra-Far].

In our case the Morse-Bott inequalities easily follow from more general theorems of Chapters 8 and 9. In particular, the twisted degenerate Novikov inequalities of Braverman and Farber [Bra-Far] also can be easily recovered from those theorems.
1. WITTEN LAPLACIAN

1.0. Introduction. In this chapter we put a natural metric on $E$, compatible with the given decomposition $E = E^+ \oplus E^-$ and the Morse-Bott function $h$. Then we define the family

$$\Box (\alpha) = d(\alpha)d(\alpha)^* + d(\alpha)^*d(\alpha),$$

of the Witten deformations of the Laplacian and describe its properties. We prove the Hodge decomposition for $\Box (\alpha)$, and we define the space of rapidly decreasing forms.

1.1. Description of the data. The following set up naturally arises when we consider tubular neighborhoods of connected components of the critical submanifold for a Morse-Bott function. (see Section 0.2 and Chapter 8). Let $E = E^+ \oplus E^-$ be the $\mathbb{Z}_2$ graded finite-dimensional vector bundle of rank $n$, not necessarily orientable, over a compact connected Riemannian manifold $(M, g_M)$ of dimension $m$. Let $p : E \to M$ be the projection. Dimensions of $E^+$ and $E^-$ are $n^+$ and $n^-$, where $n = n^+ + n^-$. 

Suppose that $V$ is a flat vector bundle over $E$ with a choice of flat connection on $V$. Then $\Omega^p(E, V)$ denotes the space of smooth differential $p$-forms on $E$ with values in $V$. Let $\partial : \Omega^p(E, V) \to \Omega^{p+1}(E, V)$ be the standard differential on $\Omega^*(E, V)$, corresponding to our choice of a flat connection on $V$.

1.2. The Morse-Bott function on $E$. We choose Euclidean metrics on each fiber of $E^+$ and $E^-$ which vary smoothly with the fibers. We define the metric on $E$ to be the direct sum of the metrics on $E^+$ and $E^-$. With such a metric $E$ becomes a Euclidean vector bundle.

For a vector $y \in E^+ \oplus E^-$ we denote by $|y|$ its Euclidean norm with respect to the metric on $E$.

$$|y| = |y^+| + |y^-|. \quad (1.1)$$

Let $h : E \to \mathbb{R}$ be the function defined on a vector $y = (y^+, y^-) \in E^+ \oplus E^-$ by
the formula
\[(1.2) \quad h(y) = |y^+|^2 - |y^-|^2 = h^+(y) - h^-(y).\]

Then \(h(y)\) is a Morse-Bott function on \(E\). Since the 1-form \(dh\) degenerates on \(TM \subset TE\), the function \(h\) has \(M\) as its critical submanifold. Moreover, the Hessian of \(h\) is positive on \(E^+\) and is negative on \(E^-\). The index of \(M\) is \(n^-\).

In order to study the Witten Laplacian we need metrics on \(E\) and \(V\).

1.3. A Riemannian metric on \(E\). It is convenient to choose a Riemannian metric on \(E\) to be compatible with a given Morse-Bott function \(h\) in the following sense.

We denote by \(T^\text{ver}E\) the space of all vertical vectors in \(TE\). For each \(y \in E\), we have the canonical identification of \(E\) and \(T^\text{ver}E\) (identification of fibers and vectors tangent to fibers). Via this identification the metric on \(E\) induces the metric \(g^\text{ver}\) on \(T^\text{ver}E\).

We choose a Euclidean connection \(\nabla^{E^+}\) on \(E^+\) and \(\nabla^{E^-}\) on \(E^-\). Then we define a connection on \(E\) as the direct sum of connections on \(E^+\) and on \(E^-\):

\[\nabla^E = \nabla^{E^+} \oplus \nabla^{E^-}.\]

We recall that a connection is called Euclidean if it is compatible with the given Euclidean metric on \(E\). We note that the splitting \(E = E^+ \oplus E^-\) is parallel for our choice of \(\nabla^E\).

The choice of a connection defines the horizontal subspace \(T^\text{hor}E \subset TE\) (see Section 1 of [Roe]), and thus the splitting of \(TE\) into complimentary vertical and horizontal subspaces \(TE = T^\text{ver}E \oplus T^\text{hor}E\). Note that for each \(y \in E\) we have an identification

\[p_* : T^\text{hor}_y \rightarrow T^p(y)M.\]

Thus we can lift \(g_M\) to the metric \(g^\text{hor}\) on \(T^\text{hor}E\).

We define the metric \(g\) on \(TE\) to be

\[g = g^\text{ver} \oplus g^\text{hor} = g^\text{ver} \oplus p^*g_M.\]
1.4. The Bismut connection and a choice of basis on \( TE \). In order to do computations we need to choose a basis and a connection on \( TE \). As it will be explained in Appendix 2, the most computationally convenient choice of a connection is the Bismut connection. The Bismut connection \( \tilde{\nabla} \) on \( TE \) [Bis, Section 2] can be defined as a direct sum of two connections \( \nabla^E \) and \( \nabla^M \):

\[
\tilde{\nabla} = \nabla^E \oplus \nabla^M,
\]

where \( \nabla^E \) is the connection on \( T^{\text{ver}}E \) (identified with \( E \)), and \( \nabla^M \) is the connection on \( T^{\text{hor}}E \) (identified with \( TM \)). The properties of this connection are discussed in more detail in Appendix 2.

In order to choose a basis on \( TE \), we first choose a basis on \( TM \oplus E \) and then, we lift this basis to \( TE \). Take \( x \in M \). Let \( \{a_i\}_{i=1,...,n}, \{b_j\}_{j=1,...,m} \) be orthogonal bases of \( E_x, T_xM \). Let \( \{a^i\}_{i=1,...,n}, \{b^j\}_{j=1,...,m} \) be the corresponding dual bases. Take \( y \in E_x \). We can lift \( \{a_i\}_{i=1,...,n}, \{b_j\}_{j=1,...,m} \) to \( TE \). Since there is no risk of confusion we can assume as well that \( \{a_i\}_{i=1,...,n} \) is the basis of \( A_y \) and \( \{b_j\}_{j=1,...,m} \) is the basis of \( B_y \). In addition we can choose vertical basis to be parallel in the horizontal direction.

1.5. A Euclidean metric on \( V \). In order to choose a basis on \( V \), we identify the manifold \( M \) with the zero section of \( E \). Let \( V|_M \) denote the restriction of \( V \) on \( M \). Fix an arbitrary Euclidean metric \( q \) on \( V|_M \), compatible with the flat connection on \( V \). The flat connection on \( V \) defines a trivialization of \( V \) along the fibers of \( E \) and, therefore, gives a natural extension of \( q \) to an Euclidean metric \( q_V \) on \( V \) which is flat along the fibers of \( E \).

1.6. The Witten differential and the Witten Laplacian. We define after Witten [Wi1] a one-parameter family of differentials \( d(\alpha), \alpha \geq 0 \), by the formula

\[
d(\alpha) = e^{-\alpha h}d e^{\alpha h} = d + \alpha dh \wedge .
\]

It is easy to see that \( d(\alpha) \circ d(\alpha) = e^{-\alpha h}(d \circ d)e^{\alpha h} = 0 \).

The metric \( g \) on \( TE \) induces the metric on \( \Lambda^*T^*E \) and, together with \( q_V \), leads to an \( L^2 \)-metric on \( \Omega^*(E, V) \).
Let $\Omega^*_\epsilon(E, V)$ be the space of square integrable forms on $E$ with values in $V$. If the bundle $E \to M$ is not orientable then we let $\Omega^*_\epsilon(E, V)$ denote the space of square integrable densities.

The Witten Laplacian $\Box(\alpha)$ on the bundle $E$, associated to the metrics $g$ and $q_V$, is defined by the formula:

$$\Box(\alpha) = d(\alpha)d^*(\alpha) + d^*(\alpha)d(\alpha),$$

where $d^*(\alpha)$ denotes the formal adjoint of $d(\alpha)$ with respect to the $L^2$-metric on $\Omega^*_\epsilon(E, V)$. We denote by $\Box^p(\alpha)$ the restriction of $\Box(\alpha)$ to the space of $p$-forms.

1.7. The spectrum of the Witten Laplacian. In this section we formulate several results about the spectrum of $\Box(\alpha)$.

First we need to define a self-adjoint extension of the Witten Laplacian. We restrict $\Box(\alpha)$ to the space $\Omega^*_\epsilon(E, V)$ of smooth differential forms on $E$ with compact support. A simple calculation ([CFKS, Proposition 11.13] or [Bis, Proposition 2.6]) shows that

$$\Box(\alpha) = \Box + \alpha^2|dh|^2 + \alpha A,$$

where $\Box$ is the usual Laplacian associated to the metrics $g$ and $q_V$, and $A$ is a zeroth order operator. The operator $A$, as expressed in the basis of Section 1.4, is bounded [Bis, Proposition 2.6]. Because the manifold $E$ is non-compact, the operator $A$ can be unbounded in a different basis.

Since for any fixed $\alpha \geq 0$ zeroth order operator $\alpha^2|dh|^2 + \alpha A$ is symmetric and bounded below by a constant (see Appendix 1), it follows that $\Box(\alpha) : \Omega^*_\epsilon(E, V) \to \Omega^*_\epsilon(E, V)$ is essentially self-adjoint ([Bra, Theorem 1] or [Cher, Section 3]). Thus it has a unique self-adjoint extension [R-S, Chapter viii], which we will denote again as $\Box(\alpha)$. Moreover, it follows from [Cher, Section 3] that the powers of $\Box(\alpha)$, restricted to $\Omega^*_\epsilon(E, V)$, are also essentially self-adjoint.

Theorem 1.1. For any $\alpha > 0$ and any $p$, $0 \leq p \leq \dim E$, the spectrum of $\Box^p(\alpha)$ is discrete. In particular ker $\Box^p(\alpha)$ is finite-dimensional. Moreover,
eigenforms of $\Box(\alpha)$ form a basis for $\Omega^\bullet_{(2)}(E, V)$ in the $L^2$-topology, associated to the metric $g$ on the tangent space $TE$.

The proof of the theorem can be found in the Appendix 1.

1.8. The Hodge theory for the Witten Laplacian. In this section we introduce the complex whose cohomology will be our main interest in the coming chapters. We then show that even though the underlying manifold is non-compact, by using the deformation introduced in section 1.5 this cohomology can be studied via Hodge theory.

We denote by $\Omega^\bullet(E, V, \alpha)$ the space of smooth ($C^\infty$)-square integrable forms $\omega \in \Omega^\bullet_{(2)}(E, V)$ which have the property that $d(\alpha)\omega \in \Omega^\bullet_{(2)}(E, V)$ and $d(\alpha)^\ast \omega \in \Omega^\bullet_{(2)}(E, V)$. In Chapters 1 through 6, we study $H^\bullet(E, V, \alpha)$, the cohomology of the complex

$$0 \to \Omega^0(E, V, \alpha) \to \Omega^1(E, V, \alpha) \to \cdots \to \Omega^{m+n}(E, V, \alpha) \to 0,$$

associated to the differential $d(\alpha)$. We prove the following Hodge decomposition theorem for $d(\alpha)$.

**Theorem 1.2.** For any $\alpha > 0$ we have an orthogonal decomposition

$$\Omega^\bullet(E, V, \alpha) = \text{image } d(\alpha) \oplus \text{image } d^\ast(\alpha) \oplus \ker \Box(\alpha).$$

**Proof.** Since by Theorem 1.1 $\dim \ker \Box(\alpha) < \infty$, for any $\omega \in \Omega^\bullet(E, V)$ we have the decomposition $\omega = \beta + h$, where $\beta \in (\ker \Box(\alpha))^\perp$ and $h \in \ker \Box(\alpha)$.

Writing

$$\beta = ((\alpha)d^\ast(\alpha) + d^\ast(\alpha)d(\alpha))(\Box(\alpha))^{-1} \beta$$

$$= d(\alpha)(d^\ast(\alpha)(\Box(\alpha))^{-1} \beta) + d^\ast(\alpha)(d(\alpha)(\Box(\alpha))^{-1} \beta),$$

we have $\beta = \beta_1 + \beta_2$, where $\beta_1 \in \text{image } d(\alpha)$ and $\beta_2 \in \text{image } d^\ast(\alpha)$. Moreover, since $(d(\alpha))^2 = 0$, $\beta_1$ and $\beta_2$ are orthogonal. Thus for any $\omega \in \Omega^\bullet(E, V)$ we have an orthogonal decomposition $\omega = h + \beta_1 + \beta_2$. ■

We also have the following immediate corollary:
Corollary 1.3.

\[ H^\bullet(E, V, \alpha) \cong \ker \Box(\alpha) = \ker d(\alpha) \cap \ker d^*(\alpha). \]

1.9. The space of rapidly decreasing forms. Until now, we have worked with the space of square integrable forms, but it will be convenient to work instead with the space of rapidly decreasing forms. The purpose of this section is to show that restricting to this smaller space of forms does not change the cohomology of the complex defined in Section 1.7.

It is convenient to define the space of rapidly decreasing forms on \( E \) in terms of the Bismut connection on \( TE \). In order to simplify the notation, let

\[ \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_m\} = \{e_1, \ldots, e_{n+m}\} \]

be the basis, chosen in Section 1.4.

We now introduce the spaces \( \Omega^p_s(E, V) \) of smooth rapidly decreasing \( p \)-forms. We say that \( \omega \in \Omega^p_s(E, V) \) if \( |y|^l \nabla^\kappa \omega \in \Omega^p_{(2)}(E, V) \) for any \( l \geq 0 \) and any multi-index \( \kappa, |\kappa| = 0, 1, \ldots \). Here

\[ \nabla^\kappa = \nabla_{e_{i_1}} \circ \cdots \circ \nabla_{e_{i_k}}, \ \kappa = \{i_1, \ldots, i_k\}. \]

If \( E = M \times \mathbb{R}^n \), then \( \Omega^p_s(E, V) \) becomes the space of smooth Schwartz \( p \)-forms on \( E \).

Since we can express \( d \) in terms of the Bismut connection, we conclude (see Theorem A.3.1) that \( \Omega^p_s(E, V) \) is invariant under \( d(\alpha) \). Thus we can define \( H^p_s(E, V, \alpha) \), the \( p \)-th cohomology of \( (\Omega^p_s(E, V), d(\alpha)) \).

Clearly, for any \( \alpha \geq 0 \), \( \Omega^p_s(E, V) \subseteq \Omega^p(E, V, \alpha) \). Thus, we have an induced decomposition of \( \Omega^p_s(E, V) \):

\[(1.7) \quad \Omega^p_s(E, V) = \text{image } d(\alpha) \oplus \text{image } d^*(\alpha) \oplus \ker \Box(\alpha). \]

Theorem 1.4. For any large enough \( \alpha \) and any \( \omega \in \Omega^\bullet(E, V, \alpha) \), such that \( \Box(\alpha)\omega = \lambda(\alpha)\omega \), we have \( \omega \in \Omega^p_s(E, V) \); i.e. the eigenforms of \( \Box(\alpha) \) are rapidly decreasing forms.
**Corollary 1.5.** For every $\alpha > 0$,

$$H^*_s(E, V, \alpha) = H^*(E, V, \alpha).$$

**Remark 1.6.** M. Shubin in Appendix to [Sh2] proved a theorem about the decay of eigenfunctions of matrix-valued differential operators on $\mathbb{R}^n$ similar to Theorem 1.
2. DEFORMATION OF THE WITTEN LAPLACIAN

2.0. Introduction. In this chapter we fix the parameter $\alpha > 0$ to be large enough to ensure the conclusions of Theorem 1.4. For the sake of simplicity the notation $\tilde{d}$ and $\tilde{\Box}$ will indicate that the differential and the Witten Laplacian depend on a fixed $\alpha$. Thus $d(\alpha) = \tilde{d}$ and $\Box(\alpha) = \tilde{\Box}$.

In Section 2.1 we introduce for all $0 < \delta \leq 1$ an adiabatic deformation $g_\delta$ of metric $g$ by expanding the metric in the directions orthogonal to fibers of $E \to E^\perp$.

For a fixed $\alpha > 0$ this deformation leads to a corresponding deformation $\tilde{\Box}_\delta$ of the Witten Laplacian. To simplify the situation we remove the dependence of the metric on the parameter by introducing in Section 2.4 a new family $\tilde{\Box}(\delta)$ of operators. This will be done in such a way that for all $\delta > 0$ the operators $\tilde{\Box}_\delta$ and $\tilde{\Box}(\delta)$ will be isospectral.

In sections 2.2 and 2.3 we conclude that there is a natural bigrading on the space of differential forms on $E$, which is associated to an orthogonal decomposition of $TE$ into horizontal and vertical vectors. This bigrading leads to a corresponding bigrading of the differential $d$.

2.1. A one-parameter deformation of the metric on $E$. Now we consider $E$ as a vector bundle over a non-compact manifold $E^\perp$. Let $\pi : E \to E^\perp$ be the projection. The tangent space $TE$ has an orthogonal decomposition

$$TE = A \oplus B$$

where $A$ is the set of all vectors tangent to fibers of $E \to E^\perp$ and

$$B = T^\text{hor} E \oplus \{\text{vectors, tangent to fibers of } E^\perp \to M\}.$$

Note that

$$g = g_A \oplus g_B,$$

where $g_A$ and $g_B$ are the restrictions of $g$ to $A$ and $B$.

We define a one-parameter family of metrics on $TE$ by setting

$$g_\delta = g_A \oplus \delta^{-2} g_B.$$ (2.1)
2.2. A bigrading of the space of forms. The decomposition $TE = A \oplus B$ leads to the corresponding decomposition of the dual space

$$T^*E = A^* \oplus B^*.$$

This decomposition in turn induces a bigrading on $\Omega^p(E, V)$ by

$$\Omega^p(E, V) = \bigotimes_{i=0}^p \Omega^{i,p-i}(E, V),$$

where

$$\Omega^{i,p-i}(E, V) = \Gamma(A^i A^* \oplus A^j B^* \oplus V).$$

2.3. A bigrading of the Witten differential. Similarly, all operators on forms inherit a corresponding decomposition. In particular, the $d$-operator inherits a bigrading

$$d = d^{2,-1} + d^{1,0} + d^{0,1} + d^{-1,2},$$

where $d^{a,b}: \Omega^{i,j}(E, V) \to \Omega^{i+a,j+b}(E, V)$. Note that $d^{1,0}$ and $d^{0,1}$ are first order differential operators and $d^{2,-1}$ and $d^{-1,2}$ are zeroth order.

Observe that our distribution $A$ is integrable. It follows from the computation of $d$ in terms of the Bismut connection that $d^{2,-1} = 0$ (see Corollary A.2.3).

For the operator $\tilde{d} = d + \alpha dh \wedge$ we have a similar decomposition

$$\tilde{d} = \tilde{d}^{1,0} + \tilde{d}^{0,1} + \tilde{d}^{-1,2}.$$

Since $dh |_{T^{nor} E} = 0$ and the metric $g$ is chosen so that $T^{ver} E^+$ is perpendicular to $T^{ver} E^-$, then

$$dh = dh^+ - dh^-,$$

where $dh^+ \wedge = d^{1,0} h^+ \wedge$ is a $(1,0)$-operator and $dh^- \wedge = d^{1,0} h^- \wedge$ is a $(0,1)$-operator. Therefore,

$$\tilde{d}^{1,0} = d^{1,0} + \alpha dh^+ \wedge,$$

$$\tilde{d}^{0,1} = d^{0,1} - \alpha dh^- \wedge,$$
and

\[ \hat{d}^{-1.2} = d^{-1.2} \]

The identity \((\hat{d}^2) = 0\) yields the identities

\[ 0 = (\hat{d}_p^2)^{2.0} = (\hat{d}_p^{1.0})^2, \]

\[ 0 = (\hat{d}_p^{1.1}) = \hat{d}_p^{1.0} \hat{d}_p^{0.1} + \hat{d}_p^{0.1} \hat{d}_p^{1.0}, \]

\[ 0 = (\hat{d}_p^{0.2}) = (\hat{d}_p^{2.0})^2 + d^{-1.2} \hat{d}_p^{1.0} + \hat{d}_p^{1.0} d^{-1.2}, \]

\[ 0 = (\hat{d}_p^{0.4}) = (\hat{d}_p^{1.2})^2. \]

2.4. A one-parameter deformation of the Witten Laplacian. For each \( p \) and \( \delta \) we have an induced Laplacian

\[ \hat{\Box}_p^\delta = \hat{d}_p^{*} \hat{d}_p + \hat{d}_p \hat{d}_p^{*} : \Omega_p^p(E, V) \to \Omega_p^p(E, V), \]

where \( \hat{d}_p^{*} \) is the adjoint of \( \hat{d} \) with respect to the \( L^2 \)-metric on \( \Lambda^* T^* E \otimes V \) induced by the metrics \( g_\delta \) and \( q_\nu \).

The operator \( \hat{\Box}_p^\delta \) depends on \( \delta \) through the metric \( g_\delta \), which varies with \( \delta \). To simplify the situation we introduce an isometry

\[ \rho_\delta : (\Omega_p^p(E, V), g_\delta) \to (\Omega_p^p(E, V), g), \]

where for \( \omega \in \Omega_{i,j} \),

\[ \rho_\delta \omega = \delta^i \omega. \]

We define \( \hat{\Box}_p^\delta \) by the formula

\[ \hat{\Box}_p^\delta = \rho_\delta^{-1} \hat{\Box}_p^\delta \rho_\delta. \]

Then \( \hat{\Box}_p^\delta \) and \( \hat{\Box}_p^\delta \) are isospectral. We also observe that \( \hat{d}_p^{*} = \rho_\delta^{-1} \hat{d}^{*} \rho_\delta \) and hence, for all \( p \) and \( \delta \)

\[ \text{ker} \hat{\Box}_p^\delta = \text{ker} \hat{\Box}_p^\delta = \text{ker} \hat{\Box}_p^\delta. \]

Moreover, we have the following lemma, which is a simple corollary of the min-max description of the eigenvalues.
Lemma 2.1. ([Fo, Section 1]) For any $0 \leq p \leq \dim E$

\[ \hat{\Box}^p(\delta) = \hat{d}_\delta \hat{d}_\delta^* + \hat{d}_\delta^* \hat{d}_\delta, \]

where

\[ \hat{d}_\delta = \hat{d}^{1,0} + \delta \hat{d}^{0,1} + \delta^2 \hat{d}^{-1,2} \]

and

\[ \hat{d}_\delta^* = (\hat{d}^{1,0})^* + \delta (\hat{d}^{0,1})^* + \delta^2 (\hat{d}^{-1,2})^* \]

is the adjoint, where all adjoints are taken with respect to metrics $g$ and $q_\nu$. 
3. TAYLOR ANALYSIS OF ZERO EIGENSPACES AND ASSOCIATED NESTED SEQUENCE OF SPACES

3.0. Introduction. In this chapter we start a Taylor analysis of the kernel of the operator \( \hat{\delta} \). We observe that \( \omega \in \ker \hat{\delta} \) if and only if \( \hat{d} \omega = 0 \) and \( \hat{d}^* \omega = 0 \). This description of the kernel motivates an introduction in Section 3.1 of the nested sequence of spaces \( \{ E^p_k \} \) of “approximate solutions” to the equations above. We also define the appropriate differentials \( \pi_k d_k \pi_k : E^p_k \to E^p_k \). Those spaces and differentials were first introduced in this form by R. Forman in [Fo]. Similar spaces were studied by Mazzeo and Melrose in [Ma-Me].

In this chapter we compute the spaces \( E^p_1 \) and \( E^p_2 \). In particular, in Section 3.3 we find that the forms from \( E^p_1 \) restrict to \( \hat{\delta}^{0,1} \)-harmonic forms on fibers of \( E \to E^- \). Furthermore, there is an isomorphism between \( E^p_2 \) and \( H^p_\delta(E^-, V|_{E^-}, \alpha) \), where \( H^p_\delta(E^-, V|_{E^-}, \alpha) \) denotes the de Rham cohomology associated to the differential complex \( (\Omega^p_\delta(E^-, V|_{E^-}), d(\alpha) = d - adh^{-1}) \).

3.1. A nested sequence of spaces. For each \( p \) we define a nested sequence of spaces

\[
E^p_0 \supseteq E^p_1 \supseteq E^p_2 \supseteq \ldots,
\]

where \( E^p_0 = \Omega^p_\delta(E, V) \), by

\[
E^p_k = \{ \omega \in \Omega^p_\delta(E, V) | \exists \omega_1, \ldots, \omega_{k-1} \text{ with}
\]

\[
\hat{d}_\delta(\omega + \delta \omega_1 + \cdots + \delta^{k-1} \omega_{k-1}) \in O(\delta^k),
\]

\[
\hat{d}^*_\delta(\omega + \delta \omega_1 + \cdots + \delta^{k-1} \omega_{k-1}) \in O(\delta^k), \quad k = 1, 2, \ldots.
\]

We denote by \( \pi_k \) the orthogonal projection onto \( E^p_k \):

\[
\pi_k : \Omega^p_\delta(E, V) \to E^p_k.
\]

We also define an operator \( \hat{d}_k \) on \( E^p_k \) by setting, for \( \omega \in E^p_k \)

\[
\hat{d}^k \omega = \lim_{\delta \to 0} \delta^{-k} \hat{d}_\delta(\omega + \delta \omega_1 + \cdots + \delta^{k-1} \omega_{k-1}), \quad k = 0, 1, 2, \ldots
\]
We observe that generally the map $\hat{d}_k$ depends on $\omega_i$'s, however in the next section we are going to see that $\pi_k \hat{d}_k \pi_k$ does not. For example it is clear that $\pi_0 \hat{d}_0 \pi_0 \omega = \hat{d}_0 \omega = \hat{d}^{1,0} \omega$. In particular, we will show that $(\pi_k \hat{d}_k \pi_k)^2 = 0$ and $\pi_k \hat{d}_k \pi_k = 0$ for $k \geq 2$.

3.2. Computation of $E^p_1$.

By definition

$$E^p_1 = \{ \omega \in \Omega^p_s(E, V) | \hat{d}_0 \omega \in 0(\delta), \hat{d}_0^*(\omega) \in 0(\delta) \}.$$ 

Since

$$\hat{d}_0(\omega) = (\hat{d}^{1,0} + \delta \hat{d}^{0,1} + \delta^2 \hat{d}^{-1,2})(\omega) = \hat{d}^{1,0} \omega + 0(\delta)$$

and

$$\hat{d}_0^*(\omega) = (\hat{d}^{1,0})^* \omega + 0(\delta)$$

we have that

$$E^p_1 = \{ \omega \in \Omega^p_s(E, V) | \hat{d}^{1,0} \omega = 0, (\hat{d}^{1,0})^* \omega = 0 \}$$

$$= \{ \omega \in \Omega^p_s(E, V) | \pi_0 \hat{d}_0 \pi_0 \omega = 0, (\pi_0 \hat{d}_0 \pi_0)^* \omega = 0 \}. \quad (3.3)$$

Therefore,

$$E^p_1 = E^0_0 \cap \ker(\pi_0 \hat{d}_0 \pi_0) \cap \ker(\pi_0 \hat{d}_0 \pi_0)^*. \quad (3.4)$$

3.3. Some further properties of $E^p_1$. Recall that we chose a metric $g$, so that the vertical space $A$ consists of the fibers of the Riemannian submersion.

Let $\omega \in E^p_1$. Since the differential $\hat{d}^{1,0}$ preserves the bigrading, we may assume that $\omega \in \Omega^{i,j}_s(E, V)$ for some $i, j$, $i + j = p$. Then $\omega$ can be considered as a $j$-form on $E^-$ with the values in an infinite-dimensional bundle

$$\pi : \Omega^j_s(A, V|_A) \rightarrow E^-$$

(where $\Omega^j_s(A, V|_A)$ denotes the bundle whose fiber at $(x, y^-) \in E^-$ is $\Omega^j_s(\pi^{-1}(x, y^-), V|_{\pi^{-1}(x, y^-)})$.}
We write $\omega$ as

$$\omega = \gamma \otimes \pi^* \beta,$$

where $\gamma \in \Omega^i_s(A, \mathcal{V}_A)$, $\beta \in \Omega^i_s(E^-, \mathcal{V}_{|E^-})$. Then we have

$$\partial^{1,0} \omega = (\partial \omega)^{i+1,j}$$

$$= (\partial \gamma \otimes \pi^* \beta)^{i+1,j} + (\gamma \otimes \partial \pi^* \beta)^{i+1,j}$$

$$= \partial \gamma \otimes \pi^* \beta = \partial_A \gamma \otimes \pi^* \beta.$$

Similarly,

$$(\partial^{1,0})^* \omega = \partial_A^* \gamma \otimes \pi^* \beta,$$

and finally,

$$\tilde{\nabla}^{1,0} \omega = (\tilde{\nabla}_A \gamma) \otimes \pi^* \beta,$$

where $\partial_A$, $\partial_A^*$ and $\tilde{\nabla}_A$ are respectively the differential, codifferential and Laplacian on $\Omega^i_s(\pi^{-1}(x,y^{-}), \mathcal{V}_{|\pi^{-1}(x,y^-)})$.

Thus

$$\ker \tilde{\nabla}^{1,0} = \Gamma \left( E^-, \Lambda^* E^* \otimes \mathcal{H}(A, \mathcal{V}_A) \right),$$

where $\mathcal{H}(A, V)$ denotes the vector bundle over $E^-$ whose fibers at $(x, y^-)$ are the $\tilde{\nabla}_A$-harmonic forms (i.e. elements of $\ker \tilde{\nabla}_A$) on $\pi^{-1}(x, y^-)$ with values in $V$.

The following lemma computes the kernel of $\tilde{\nabla}_A$ in the fiber.

**Lemma 3.1.**

(1) Let $\tilde{\nabla}_A$ denote the restriction of the operator $\tilde{\nabla}_A$ to the space $\Omega^i_s(\pi^{-1}(x,y^-), \mathcal{V}_{|\pi^{-1}(x,y^-)})$. Then for every point $(x, y^-) \in E^-$, $\dim \mathcal{V} \ker \tilde{\nabla}_A^{(i)} = 1$, if $i = 0$; $\dim \mathcal{V} \ker \tilde{\nabla}_A^{(i)} = 0$, otherwise.

(2) If $\omega \in \ker \tilde{\nabla}_A^{(i)}$, then $\omega = \gamma \otimes \pi^* \beta$, where after an orthogonal change of coordinates in the fiber over $(x, y^-) \in E^-$,

$$\gamma|_{\Omega^i_s(\pi^{-1}(x,y^-), \mathcal{V}_{|\pi^{-1}(x,y^-)})} = e^{-\alpha |Dy^+|^2} \otimes v,$$

where $D$ is some diagonal matrix.

(3) The bundle $\mathcal{H}(A, V)$ is trivial.
Proof. Recall that we chose $g$, so that $g_E$ and, hence $g_{E^+}$ is a smooth fiberwise Euclidean metric. Therefore, for each $(x, y^-) \in E^-$ we can perform a calculation similar to [CFKS, Proposition 11.13] in the fiber over $(x, y^-)$ to get

$$\hat{\Delta}^i_A \gamma = (\Delta + \alpha^2 |d\mathbf{h}^+|^2 + \alpha B) \gamma,$$

where $\Delta$ is the Euclidean Laplacian in the fiber and $B$ is a zeroth order operator. Locally $\gamma = \phi \otimes v$, where $\phi \in \Omega^i_s(\pi^{-1}(x, y^-))$ and $v \in V|_{\pi^{-1}(x, y^-)}$. Moreover, after an orthogonal change of the coordinates in the fiber, the variables separate and $\hat{\Delta}^i_A$ acts on the $i$-form $\gamma$ by

$$\hat{\Delta}^i_A \gamma = (\oplus_{k=1}^{n^+} H_k(\alpha) \phi) \otimes v.$$

The harmonic oscillators $H_k(\alpha)$ in the formula above can be expressed as

$$L_k \phi = (\Delta_k + \lambda_k(4\alpha^2 y_k^2 + 2\alpha B_k)) \phi,$$

where for each $k$, $\Delta_k$ is the one-dimensional non-negative Laplacian, $\{\lambda_k\}$ are some positive numbers, $B_k \phi = \phi$ if $\phi$ contains $d\mathbf{y}_k$, and $B_k \phi = -\phi$ otherwise. In particular, if $\gamma \in \ker \hat{\Delta}^i_A$, then $\phi \in \ker H_k(\alpha)$ for all $k = 1, \ldots, n^+$; that is $\phi$ does not contain $d\mathbf{y}_k$, $k = 1, \ldots, n^+$. So $\dim \ker \hat{\Delta}^i_A$ is not zero if and only if $i = 0$. In this case there is a unique (up to multiplication) $\hat{\Delta}^i_A$-harmonic 0-form in the fiber. This form can be represented as $\phi \otimes v$, where $\phi = e^{-\alpha|d\mathbf{y}^+|^2}$. ■

The following very useful corollary follows from part (2) of Theorem 3.2.

**Corollary 3.2.** For any $p$, $0 \leq p \leq \dim E$,

$$E^p_i \subset \Omega^0_s(E, V).$$

Now with the help of the above corollary, we will get a Hodge decomposition for the differential $\hat{d}^{1,0}$. Namely, as in Section 1.7 we have the orthogonal (Hodge) decomposition in each fiber $F$, associated with the operator $\hat{d}_A$:

$$\Omega^*_{s}(F, V|_{F}) = \text{image } \hat{d}_A \oplus \text{image } (\hat{d}_A)^* \oplus \ker \hat{\Delta}_A.$$
It induces a decomposition of $\Omega^*_s(E, V)$.

$$\Omega^*_s(E, V) = \text{image } \hat d^{1,0} \oplus \text{image } (\hat d^{1,0})^* \oplus \ker \hat d^{1,0}$$

(3.5) 

$$= \text{image } \hat d^{1,0} \oplus \text{image } (\hat d^{1,0})^* \oplus \left( \ker \hat d^{1,0} \cap \ker (\hat d^{1,0})^* \right).$$

In particular

$$\Omega^0_s(E, V) = \text{image } (\hat d^{1,0})^* \oplus \left( \ker \hat d^{1,0} \cap \ker (\hat d^{1,0})^* \right) = \text{image } (\hat d^{1,0})^* \oplus E^0_2.$$

3.4. Some further properties of $E^p_2$. Our next goal is to relate $E^p_2$ to the cohomology $H^p_s(E^-, V, \alpha)$ of the differential complex $(\Omega^*_s(E^-, V), d(\alpha))$. To this extent we prove the following theorem:

**Theorem 3.3.** For any $p$, $0 \leq p \leq \dim E$,

$$\dim E^p_2 = \dim H^p_s(E^-, V, \alpha).$$

Moreover, if $n^- = 0$ (i.e. $E = E^+$), then

$$\dim E^p_2 = \dim H^p(M, V).$$

This theorem has an important corollary.

**Corollary 3.4.** The spaces $E^p_2$ are finite-dimensional.

We prove Theorem 3.3 by a sequence of lemmas. The first lemma deals with the computation of $E^p_2$.

**Lemma 3.5.** For any $p$, $0 \leq p \leq \dim E$,

$$E^p_2 = \{ \omega \in E^p_1 \mid \pi_1 \hat d^{0,1} \pi_1 \omega = 0, \ (\pi_1 \hat d^{0,1} \pi_1)^* \omega = 0 \}.$$ 

**Proof.** By (3.1)

$$E^p_2 = \{ \omega \in \Omega^p_s(E, V) \mid \exists \omega_1 \text{ such that }$$

$$\hat d_\delta(\omega + \delta \omega_1) = 0(\delta^2) \text{ and } (\hat d_\delta)^*(\omega + \delta \omega_1) = 0(\delta^2) \}. $$
Since
\[ \hat{d}_\delta(\omega + \delta\omega_1) = \hat{d}^{1.0}\omega + \delta(\hat{d}^{1.0}\omega_1 + \hat{d}^{0.1}\omega) + O(\delta^2). \]
and
\[ (\hat{d}_\delta)^*(\omega + \delta\omega_1) = (\hat{d}^{1.0})^*\omega + \delta((\hat{d}^{1.0})^*\omega_1 + (\hat{d}^{0.1})^*\omega) + O(\delta^2). \]
we have
\[ E^p_2 = \{ \omega \in E^p_1 \mid \exists \omega_1 \text{ such that } \hat{d}^{1.0}\omega_1 + \hat{d}^{0.1}\omega = 0 \text{ and } (\hat{d}^{1.0})^*\omega_1 + (\hat{d}^{0.1})^*\omega = 0 \}. \]

Let \( \omega \in E^p_1 \), then the equation \( \hat{d}^{1.0}\omega_1 + \hat{d}^{0.1}\omega = 0 \) can be solved if and only if
\[ \hat{d}^{0.1}\omega \in \text{image } \hat{d}^{1.0}. \]

Since by (2.7) \( \hat{d}^{1.0}\hat{d}^{0.1} + \hat{d}^{0.1}\hat{d}^{1.0} = 0 \), we have
\[ \hat{d}^{1.0}\hat{d}^{0.1}\omega = -\hat{d}^{0.1}\hat{d}^{1.0}\omega = 0. \]

The last equality uses the fact that for \( \omega \in E^p_1 \) we have \( \hat{d}^{1.0}\omega = 0 \). Therefore, it follows from the decomposition (3.5) that \( \hat{d}^{0.1}\omega \in \text{image } \hat{d}^{1.0} \) if and only if the harmonic component of \( \hat{d}^{0.1}\omega \) is 0, that is if and only if \( \pi_1\hat{d}^{0.1}\pi_1\omega = \pi_1\hat{d}^{0.1}\omega = 0 \).

Similarly, the equation \( (\hat{d}^{1.0})^*\omega_1 + (\hat{d}^{0.1})^*\omega = 0 \) can be solved if and only if
\[ (\hat{d}^{0.1})^*\omega \in \text{image } (\hat{d}^{1.0})^*. \]

Now since \( (\hat{d}^{1.0})^*(\hat{d}^{0.1})^*\omega = 0 \) it follows from the Hodge decomposition (3.5) of \( \Omega^{0,p}_s \) that \( (\hat{d}^{0.1})^*\omega \in \text{image } (\hat{d}^{1.0})^* \) if and only if
\[ \pi_1(\hat{d}^{0.1})^*\omega = \pi_1(\hat{d}^{0.1})^*\pi_1\omega = 0. \]

\[ \textbf{Lemma 3.6.} \] Suppose \( \gamma \) is any locally constant section of the bundle \( \mathcal{H}(A,V|A) \), then the following diagram is commutative:

\[ \begin{array}{ccc}
E^p_1 & \xrightarrow{\pi_1\hat{d}^{0.1}\pi_1} & E^{p+1}_1 \\
\uparrow_{\gamma \otimes \pi^*} & & \uparrow_{\gamma \otimes \pi^*} \\
\Omega^{p}_s(E^-, V|E^-) & \xrightarrow{\hat{d}} & \Omega^{p+1}_s(E^-, V|E^-),
\end{array} \]
where \( \hat{d} = d(\alpha) = d - \alpha dh^- \land \) is a differential on the bundle \( E^- \to M \), and an isomorphism
\[
\gamma \otimes \pi^* : \Omega^p_\pi(E^-, V_{E^-}) \to E^p_1
\]
is defined by
\[
\beta \mapsto \gamma \otimes \pi^* \beta.
\]

**Proof.** Let \( \beta \in \Omega^p_\pi(E^-, V_{E^-}) \). We want to check that
\[
(\pi_1 d^{0,1} \pi_1) \circ (\gamma \otimes \pi^*) \beta = (\gamma \otimes \pi^*) \circ (\hat{d}) \beta.
\]
Indeed, we have equalities:
\[
\begin{align*}
\pi_1 d^{0,1} \pi_1 (\gamma \otimes \pi^* \beta) &= \pi_1 d^{0,1} (\gamma \otimes \pi^* \beta) \\
&= \pi_1 (d^{0,1} - \alpha dh^- \land)(\gamma \otimes \pi^* \beta) \\
&= \pi_1 d^{0,1} (\gamma \otimes \pi^* \beta) - \pi_1 \gamma \otimes (\alpha dh^- \land \pi^* \beta) \\
&= \pi_1 \gamma \otimes d^{0,1} \pi^* \beta - \gamma \otimes \pi^* (\alpha dh^- \land \beta) \\
&= (\gamma \otimes \pi^* d\beta - \gamma \otimes \pi^* (\alpha dh^- \land \pi^* \beta))^{0,p+1} \\
&= \gamma \otimes \pi^* \hat{d} \beta,
\end{align*}
\]
where the fourth equality is satisfied since \( \gamma \) is locally constant and since for a vertical form \( dh^- \) we have \( \pi^* dh^- \land = dh^- \land \pi^* \). ■

The following lemma completes the proof of Theorem 3.3.

**Lemma 3.7.** We have:
\[
\dim \ker(\pi_1 d^{0,1} \pi_1) \cap \ker(\pi_1 d^{0,1} \pi_1)^* = \dim(\ker \hat{d} \cap \ker \hat{d}^*).
\]

**Proof.** After taking the adjoints, we have a commutative diagram similar to (3.7):
\[
\begin{array}{ccc}
E^p_1 & \xleftarrow{\pi_1 d^{0,1} \pi_1} & E^{p+1}_1 \\
\downarrow{\gamma \otimes \pi^*} & & \downarrow{\gamma \otimes \pi^*} \\
\Omega^p_\pi(E^-, V_{E^-}) & \xleftarrow{(\hat{d})^*} & \Omega^{p+1}_\pi(E^-, V_{E^-})
\end{array}
\]
It follows from (3.7) and (3.8) that $\ker \pi_1 \tilde{d}^{0,1} \pi_1 \cong \ker \tilde{d}$ and $\ker (\pi_1 \tilde{d}^{0,1} \pi_1)^* \cong \ker \tilde{d}^*$. Thus we have equality in the lemma. ■

Now Theorem 3.3 follows from Lemma 3.6 and from the description of $H^*_s(E^-, V_{|E^-}, \alpha)$ as

$$H^*_s(E^-, V_{|E^-}, \alpha) = \ker d(\alpha) \cap \ker d^*(\alpha).$$
4. ISOMORPHISM BETWEEN COHOMOLOGY OF E AND E^-

4.0. Introduction. The goal of this chapter is to relate for a fixed large $\alpha$ the cohomology $H_*^p(E, V, \alpha)$ of the complex $(\Omega^*(E, V), d(\alpha))$ to the cohomology $H_*^p(E^-, V_{|E^-}, \alpha)$ of the complex $(\Omega^*(E^-, V_{|E^-}), d(\alpha))$. The main result of this chapter is the following theorem:

**Theorem 4.1.** For any $p$, $0 \leq p \leq \dim E$, and for large enough $\alpha$

$$\dim H_*^p(E, V, \alpha) = \dim H_*^p(E^-, V_{|E^-}, \alpha).$$

This theorem has an important corollary.

**Corollary 4.2.** If $n^\gamma = 0$ (i.e. $E = E^+$ and $E^- = M$), then

$$\dim H_*^p(E, V, \alpha) = \dim H^p(M, V_M).$$

To prove Theorem 4.1 we show first that the nested sequence of spaces $E^p_0 \supset E^p_1 \supset E^p_2 \supset ...$ stabilizes at $E^p_2$, i.e. $E^p_2 = E^p_3 = ... = E^p_\infty$. Then the arguments from [Fo, p. 60], recalled in Section 4.4., lead to an isomorphism between $E^p_\infty$ and $H_*^p(E, V, \alpha)$. This isomorphism together with Theorem 3.3 provides us with the chain of equalities

$$\dim H_*^p(E, V, \alpha) = \dim E^p_\infty = \dim E^p_2 = \dim H_*^p(E^-, V_{|E^-}, \alpha).$$

Thus we have the conclusion of Theorem 4.1.

4.1. A preliminary result. In this section we will prove a preliminary result.

Namely we will show that in the definition of spaces $E^p_k$ we can choose $\omega_1, \omega_{k-1}$ to be from the space of rapidly decreasing forms. This result will allow us to do all computations completely in the space of rapidly decreasing forms.

**Lemma 4.3.** Let $\omega_0 \in E^p_k$, and $\omega_1, ..., \omega_{k-1}$ be such that $\hat{d}_8(\omega_0 + \delta \omega_1 + ... + \delta^{k-1} \omega_{k-1}) = O(\delta^k)$ and $\hat{d}_8(\omega_0 + \delta \omega_1 + ... + \delta^{k-1} \omega_{k-1}) = O(\delta^k)$. Then $\omega_j \in \Omega^* E, V)$ for $j = 0, 1, ..., k-1$.

**Proof.** We use induction on $k$.

$k=1$. If $\omega_0 \in E^p_1$ then $\omega_0 \in \Omega^* E, V)$ by definition of $E^p_1$. 

31
$k=i-1$. Suppose the statement of Lemma 4.3 is true for $k=i-1$.

$k=i$. Let $\omega_0 \in E_r^p$ and let $\omega_1, \ldots, \omega_{i-1}$ be such that

\begin{equation}
\hat{d}_\delta (\omega_0 + \delta \omega_1 + \cdots + \delta^{i-1} \omega_{i-1}) = 0(\delta^i)
\end{equation}

and

\begin{equation}
\hat{d}_\delta^* (\omega_0 + \delta \omega_1 + \cdots + \delta^{i-1} \omega_{i-1}) = 0(\delta^i).
\end{equation}

Then $\omega_1, \ldots, \omega_{i-2}$ are such that $\hat{d}_\delta (\omega_0 + \delta \omega_1 + \cdots + \delta^{i-2} \omega_{i-2}) = 0(\delta^{i-1})$ and $\hat{d}_\delta^* (\omega_0 + \delta \omega_1 + \cdots + \delta^{i-2} \omega_{i-2}) = 0(\delta^{i-1})$. By the previous step of the induction, $\omega_j \in \Omega_\ast (E, V)$, $j = 0, \ldots, i-2$. Computing coefficients in front of $\delta^{k-1}$ in (4.1) and (4.2), we see that $\omega_{i-1}$ satisfies two equations:

\[
d^{1,0}_{\ast} \omega_{i-1} + d^{0,1}_{\ast} \omega_{i-2} + d^{-1,2}_{\ast} \omega_{i-3} = 0,
\]

\[
(d^{1,0}_{\ast})^* \omega_{i-1} + (d^{0,1}_{\ast})^* \omega_{i-2} + (d^{-1,2}_{\ast})^* \omega_{i-3} = 0.
\]

Hence, $\omega_{i-1}$ is a solution of

\[
(d^{1,0}_{\ast})^* (d^{1,0}_{\ast} \omega_{i-1} + d^{0,1}_{\ast} \omega_{i-2} + d^{-1,2}_{\ast} \omega_{i-3})
\]

\[
+ d^{1,0}_{\ast} ((d^{1,0}_{\ast})^* \omega_{i-1} + (d^{0,1}_{\ast})^* \omega_{i-2} + (d^{-1,2}_{\ast})^* \omega_{i-3}) = 0.
\]

We rewrite this equation as

\begin{equation}
\hat{\Box}^{1,0} \omega_{i-1} = \chi,
\end{equation}

where

\[
\chi = -(d^{1,0}_{\ast})^* (d^{0,1}_{\ast} \omega_{i-2} + d^{-1,2}_{\ast} \omega_{i-3}) - d^{1,0}_{\ast} ((d^{0,1}_{\ast})^* \omega_{i-2} + (d^{-1,2}_{\ast})^* \omega_{i-3}).
\]

We note that $\chi \in (\ker \hat{\Box}^{1,0})^\perp$. Since both $\omega_{i-2}$ and $\omega_{i-3}$ are in $\Omega_\ast (E, V)$, and $\Omega_\ast (E, V)$ is invariant under $\hat{d}^{1,0}_{\ast}$ and $\hat{d}^{0,1}_{\ast}$ (see Theorem A.3.1), then $\chi \in \Omega_\ast (E, V)$.

From (4.3) we can represent $\omega_{i-1}$ as

\[
\omega_{i-1} = (\hat{\Box}^{1,0})^{-1} \chi + h,
\]

where $h \in \ker \hat{\Box}^{1,0}$. Thus $\omega_{i-1} \in \Omega_\ast (E, V)$, since $(\hat{\Box}^{1,0})^{-1} \chi \in \Omega_\ast (E, V)$, $h \in \Omega_\ast (E, V)$, and $(\hat{\Box}^{1,0})^{-1}$ leaves $\Omega_\ast (E, V)$ invariant. $\blacksquare$
4.2. The spaces $E_k^p$ for $k > 1$. In Section 3 we defined the differentials $\pi_k \hat{d}_k \pi_k : E_k^p \to E_{k+1}^p$ and $(\pi_k \hat{d}_k \pi_k)^* : E_k^p \to E_{k-1}^p$. Here we will show that the differentials equal zero for $k \geq 2$. This will lead (via the result of R. Forman [Fo, Theorem 2.5]) to the equality $E_2^p = E_{\infty}^p$. In other words, our nested sequence of spaces stabilizes at $k = 2$.

We start with an important lemma in the proof of which the dimension considerations of Corollary 3.2 are crucial.

**Lemma 4.4.** For any $k \geq 2$, $\pi_k \hat{d}_k \pi_k = (\pi_k \hat{d}_k \pi_k)^* = 0$.

**Proof.** We prove that for any $k \geq 2$, $(\pi_k \hat{d}_k \pi_k)^* = 0$. Then, after taking the adjoints, the identities $\pi_k \hat{d}_k \pi_k = 0$ will follow.

Let $\omega \in E_k^p$, then

$$(\pi_k \hat{d}_k \pi_k)^* \omega = \pi_k \hat{d}_k \pi_k \omega = \pi_k \hat{d}_k \pi_k \omega.$$

Since $\omega \in E_k^p$, $\exists \omega_1, \ldots, \omega_{k-1}$, such that

$$\hat{d}_s(\omega + \delta \omega_1 + \cdots + \delta^{k-1} \omega_{k-1}) \in O(\delta^k).$$

Then

$$\hat{d}_s(\omega + \delta \omega_1 + \cdots + \delta^{k-1} \omega_{k-1}) = \delta^k \left( (\hat{d}^{0,1})^* \omega_{k-1} + (d^{-1,2})^* \omega_{k-2} \right) + O(\delta^{k+1}).$$

Thus

$$\hat{d}_s^* \omega = \lim_{\delta \to 0} \delta^{-k} \hat{d}_s^*(\omega + \delta \omega_1 + \cdots + \delta^{k-1} \omega_{k-1}) = (d^{0,1})^* \omega_{k-1} + (d^{-1,2})^* \omega_{k-2}.$$

Therefore

$$\pi_k \hat{d}_k^* \pi_k \omega = \pi_k (d^{0,1})^* \omega_{k-1} + \pi_k (d^{-1,2})^* \omega_{k-2}.$$

We consider each term in the right-hand side separately.

Since $E_k^p \subseteq E_1^p$, $\pi_k (d^{-1,2})^* \omega_{k-1}$ is a $d^{1,0}$-harmonic form. Then, according to Corollary 3.2, the restriction of $\pi_k (d^{-1,2})^* \omega_{k-1}$ to the fiber must be a 0-form.
However, the 0-degree component of $(d^{-1,2})^\ast \omega_{k-1}$ in a fiber must be 0, since $(d^{-1,2})^\ast \omega_{k-1}$ is a form in a fiber of degree at least 1. Thus

$$\pi_k (d^{-1,2})^\ast \omega_{k-1} = 0.$$  

Similarly, when $\pi_k (d^{0,1})^\ast \omega_{k-1}$ is restricted to a fiber, it is a 0-form. Therefore,

$$\pi_k (d^{0,1})^\ast \omega_{k-1} = \pi_k (d^{0,1})^\ast \omega_{k-1}^{(0)},$$

where $\omega_{k-1}^{(0)}$ denotes the component of $\omega_{k-1}$ which, when restricted to a fiber, has degree 0.

Since by Lemma 4.3 $\omega_{k-1}^{(0)} \in \Omega^0_s(E, V)$, it follows from (3.6) that $\omega_{k-1}^{(0)}$ can be decomposed into $\tilde{d}^{1,0}$-harmonic and $\tilde{d}^{1,0}$-coexact components,

$$\omega_{k-1}^{(0)} = \pi_1 + (\tilde{d}^{1,0})^\ast h_2.$$  

Then

$$\pi_k (d^{0,1})^\ast \omega_{k-1}^{(0)} = \pi_k (d^{0,1})^\ast h_1 + \pi_k (d^{0,1})^\ast (\tilde{d}^{1,0})^\ast h_2.$$  

Now,

$$\pi_k (d^{0,1})^\ast h_1 = \pi_k \pi_2 \pi_1 (d^{0,1})^\ast \pi_1 h_1 = 0.$$  

The equality to zero follows from the description of $E_2^p$ in Lemma 3.5. Also, from (2.7) we have

$$\pi_k (d^{1,0})^\ast (d^{0,1})^\ast h_2 = \pi_k \pi_1 (d^{1,0})^\ast (d^{0,1})^\ast h_2 = 0,$$

where the last equality on the right is a consequence of the orthogonal Hodge decomposition (3.5).

In Section 3 we proved that $\dim E_2^p < \infty$. Thus there exists $N$, such that $E_0^p \supseteq E_1^p \supseteq \cdots \supseteq E_N^p = E_{N+1}^p = \cdots$. We denote such $E_N^p$ as $E_{\infty}^p$. Now we will show that the nested sequence of spaces stabilizes at $k = 2$.

We have the same situation as in [Fo].
Theorem 4.5. [Fo, Theorem 2.5] For all \( k \geq 0 \)

(i) \((\pi_k \hat{d}_k \pi_k)^2 = 0\).

(ii) The kernel of

\[
\hat{\Delta}_k = (\pi_k \hat{d}_k \pi_k)(\pi_k \hat{d}_k \pi_k)^* + (\pi_k \hat{d}_k \pi_k)^*(\pi_k \hat{d}_k \pi_k) : E^p_k \to E^p_k
\]

is precisely \( E^p_{k+1} \).

Note that we proved this theorem for \( k = 0 \) and \( k = 1 \) in Section 3.

Corollary 4.6. \( E^p_\infty = E^p_2 \).

4.3. An isomorphism between \( E^p_\infty \) and \( H^p_\ast(E, V, \alpha) \). We outline the steps of the argument taken from [Fo, p. 60]. The proofs are given in [Fo, Sec. 3].

(i) For every \( \omega \in E^p_\infty \), there is a formal power series

\[
\omega_\delta = \omega + \delta \omega_1 + \delta^2 \omega_2 + ...
\]

such that, formally,

\[
\hat{d}_\delta \omega_\delta = \hat{d}^*_\delta \omega_\delta = 0.
\]

(ii) The \( \omega_\delta \)'s arising in (i) form a basis, modulo the action of \( T \) (the ring of formal real Taylor series), for the cohomology of the complex \((T[\Omega^p_\ast], \hat{d}_\delta)\).

Here \( T[\Omega^p_\ast] \) denotes the space of formal Taylor series with coefficients in \( \Omega_\ast^p(E, V) \).

(iii) The operator \( \rho_\delta \) provides an isomorphism between \((T[\Omega^p_\ast], \hat{d}_\delta)\) and \((T[\Omega^p_\ast], d(\alpha))\).

(iv) The cohomology of \((T[\Omega^p_\ast], d(\alpha))\) is canonically isomorphic to \( T[H^p_\ast(E, V, \alpha)] \)

and hence, modulo \( T \), \( H^p_\ast(E, V, \alpha) \) provides a basis.

Observations (i)-(iv) allow us to conclude, in particular, that for all \( p \),

\[
\dim E^p_\infty = \dim H^p_\ast(E, V, \alpha).
\]

This fact completes the proof of Theorem 4.1.

Remark 4.6. In part (ii) of the argument above we used that all \( \omega_i \)'s arising in the expansion of \( \omega_\delta \) belong to \( \Omega_\ast^p(E, V) \).
5. Hodge $*$-Operator

5.0. Introduction. In this chapter for a fixed $\alpha$ we use the Hodge $*$-operator as a convenient tool to study the cohomology $H^*_*(E^-, V_{E^-}, \alpha) \equiv H^*_*(E^-, V_{E^-}, d(\alpha))$ associated to the differential complex $(\Omega^*_*(E^-, V_{E^-}), d(\alpha))$. This study allows us to compare $H^*_*(E^-, V_{E^-}, d(\alpha))$ with $H^*(M, V_M)$, the de Rham cohomology of $M$ with values in $V$.

We note that Corollary 5.2 in this section can also be deduced without the help of the Hodge $*$-operator by methods of the previous sections. Note also that the notation in this chapter differs slightly from the notations of previous chapters.

5.1. The set up. First we want to compare $H^*_*(E, V, d(\alpha))$ to $H^*_*(E, V, d(-\alpha))$, the cohomology of the differential complex $(\Omega^*_*(E, V), d(-\alpha))$, where

$$d(-\alpha) = e^{\alpha h} de^{-\alpha h} = d - \alpha dh \wedge.$$

We consider $d(-\alpha)$ as the differential, associated to a new Morse-Bott function $\tilde{h}$, where $\tilde{h} = -h$, $\tilde{h}^+ = h^-$, $\tilde{h}^- = h^+$.

This new Morse-Bott function $\tilde{h}$ on $E$ leads to a new decomposition of $E$:

$$E = \tilde{E}^+ \oplus \tilde{E}^-,$$

where $\tilde{E}^+ \cong E^-$, $\tilde{E}^- \cong E^+$.

5.2. The main result. In this section we develop Poincare duality for the above situation.

Theorem 5.1.

(1) If $E \to M$ is orientable, then

$$\dim H^*_*(E, V, d(\alpha)) = \dim H^{m+n-p}_*(E, V, d(-\alpha)).$$

(2) If $E \to M$ is not orientable, then

$$\dim H^*_*(E, V, d(\alpha)) = \dim H^{m+n-p}_*(E, V \otimes o(E), d(-\alpha)),$$

where $H^*_*(E, V \otimes o(E), d(-\alpha))$ is the cohomology, twisted by the orientation bundle of $E$ with values in $V$.  

36
Before we prove this theorem we derive some important corollaries.

5.3 Some applications of Theorem 5.1. The main application of Theorem 5.1 is the proof of the theorem of Braverman and Farber [Bra-Far, Section 3]. Although, the first application of Theorem 5.1 is the following corollary

Corollary 5.2. Let \( n^+ = 0 \), that is we have \( E = E^- \), \( n = n^- \), and \( H^s(E, V, \alpha) = H^s_s(E^-, V|_{E^-}, \alpha) \). Then

1. if \( E^- \to M \) is orientable, then

\[
\dim H^s_s(E^-, V|_{E^-}, \alpha) = \dim H^{m+n^- - p}(M, V|_M);
\]

2. if \( E^- \to M \) is not orientable, then

\[
\dim H^p_s(E^-, V|_{E^-}, \alpha) = \dim H^{m+n^- - p}(M, V|_M \otimes o(E^-)).
\]

Proof. We will prove (2). By Theorem 5.1, part (ii),

\[
\dim H^p_s(E^-, V|_{E^-}, \alpha) = \dim H^{m+n^- - p}(E^-, V|_{E^-} \otimes o(E^-), d(-\alpha)) = \dim H^{m+n^- - p}(\tilde{E}^+, V|_{E^-} \otimes o(E^-), \alpha).
\]

Then we can apply Corollary 4.2 to get an equality

\[
\dim H^{m+n^- - p}(\tilde{E}^+, V|_{E^-} \otimes o(E^-), \alpha) = \dim H^{m+n^- - p}(M, V|_M \otimes o(E^-)). \blacksquare
\]

Now we can prove

Theorem 5.3. Let \( E = E^+ \oplus E^- \to M \), then

1. if the vector bundle \( E^- \to M \) is orientable, then for all \( p \geq 0 \)

\[
\dim H^p(E, V, \alpha) = \dim H^{m+n^- - p}(M, V|_M);
\]

2. if the vector bundle \( E^- \to M \) is not orientable, then for all \( p \geq 0 \)

\[
\dim H^p(E, V, \alpha) = \dim H^{m+n^- - p}(M, V|_M \otimes o(E^-)),
\]

where \( H^*(M, V|_M \otimes o(E^-)) \) is the de Rham cohomology of \( M \) twisted by the orientation bundle of \( E^- \to M \).
Proof. We have the following sequence of equalities

\[ \dim H^p(E, V, \alpha) = \dim H^p_\ast(E, V, \alpha) = \dim H^p_\ast(E^-, V_{|E^-}, \alpha) = \dim H^{m+n-n}_\ast(M, V_{|M} \otimes o(E^-)), \]

where the first equality is the conclusion of Theorem 1.4, the second follows from Theorem 4.1 and the third is part (2) of the Corollary 5.2.  ■

The following corollary easily follows from Theorem 5.3 and Poincare duality on \( M \).

Corollary 5.4. Let \( E = E^+ \oplus E^- \to M \), then

1. if the vector bundle \( E^- \to M \) is orientable, then for any \( 0 \leq p \leq \dim E \)

\[ \dim H^p(E, V, \alpha) = \dim H^{p-n}_\ast(M, V_{|M}); \]

2. if the vector bundle \( E^- \to M \) is not orientable, then

\[ \dim H^p(E, V, \alpha) = \dim H^{p-n}_\ast(M, V_{|M} \otimes o(E^-)). \]

We will use Theorem 5.3 in the next section to give an analytic proof of the Thom isomorphism.

5.4 Proof of Theorem 5.1. We want to define the Hodge \( \ast \)-operator on \( E \). If \( E \) is orientable, we choose an orientation on \( E \) by choosing a volume form on \( TE \). Then we define the \( \ast \)-operator as in [CFKS, Proposition 11.9]. If \( E \) is not orientable, then instead of the volume form on \( TE \) we use the volume density.

Then we have

Lemma 5.5. If \( \omega \in \Omega^p_\ast(E, V) \), then

\[ d^\ast \omega = (-1)^{(m+n)(p+1)+1} \ast [d(\ast \omega)]. \]

Proof of Lemma 5.5 is the same as in [CFKS, Theorem 11.10 ].
A proof of Theorem 5.1. (ii) From Corollary 1.3 for all $p \geq 0$,

$$\dim H^p_s(E, V, d(\alpha)) = \dim \{ \omega \in \Omega^p_s(E, V) \mid d(\alpha)\omega = 0, d(\alpha)^*\omega = 0 \}$$

Similarly,

$$\dim H^{m+n-p}_s(E, V \otimes o(E), d(-\alpha))$$

$$\dim \{ \phi \in \Omega^{m+n-p}_s(E, V \otimes o(E)) \mid d(-\alpha)\phi = 0, d(-\alpha)^*\phi = 0 \}$$

To finish the proof we only need to show that $\omega \in \Omega^p_s(E, V)$ is $d(-\alpha)$-harmonic if and only if $*\omega \in \Omega^{m+n-p}_s(E, V \otimes o(E))$ is $d(\alpha)$-harmonic. Indeed if

$$d(\alpha)\omega = e^{-\alpha h}de^{\alpha h}\omega = 0,$$

then

$$0 = *e^{-\alpha h}de^{\alpha h} *\omega = e^{-\alpha h} * d * e^{\alpha h}(*\omega)$$

$$= e^{-\alpha h} d^* e^{\alpha h}(*\omega) = d^*(-\alpha)(*\omega).$$

Thus $d(\alpha)\omega = 0$ if and only if $d^*(-\alpha)(*\omega) = 0$. Similarly, $d^*(\alpha)\omega = 0$ if and only if $d(-\alpha)(*\omega) = 0$. ■
6. THOM ISOMORPHISM

6.0. Introduction. In this chapter we complete an analytic proof of the Thom isomorphism. We recall that the Thom isomorphism (see [Bott-Tu, Chapter 1.6]) relates compactly supported de Rham cohomology of the bundle to the de Rham cohomology of the base.

The final step in the proof of the Thom isomorphism relates (Theorem 6.2) the compactly supported de Rham cohomology of $E$ to the cohomology associated to the differential complex $(\Omega_c^*(E, V, \alpha), d(\alpha))$. Then our version of Thom isomorphism (for cohomology with values in a flat bundle $V$) will follow from Theorem 5.3.

6.1. The statement of Thom isomorphism. Let $E \to M$ be a vector bundle of rank $n$ over $M$. Let $V \to E$ be a flat vector bundle over $E$. We denote the $V$-valued cohomology of $E$ with compact support in the vertical direction as $H_c^*(E, V)$. Thus by definition $H^*_c(E, V)$ is the cohomology associated to the differential complex $(\Omega_c^*(E, V), d)$.

Theorem 6.1. (Thom isomorphism.)

(i) if $E \to M$ is orientable then

$$\dim H^p_c(E, V) = \dim H^{p-n}(M, V|_M);$$

(ii) if $E \to M$ is not orientable then

$$\dim H^p_c(E, V) = \dim H^{p-n}(M, V|_M \otimes o(E)).$$

6.2. Proof of Thom isomorphism. As in Chapter 1 we put a metric $g$ on $E$, a Morse-Bott function $h(y) = -h^-(y) = -|y|^2$ on $E$ and a metric $q$ on $V$. Thus $E = E^-$. In Section 6.3 we prove the following theorem:

Theorem 6.2. For any large enough $\alpha$ and any $p$, $0 \leq p \leq \dim E$,

$$\dim H^p_c(E, V) = \dim H^p(E, V, \alpha).$$
Theorem 6.2 together with Theorem 5.3 proves the Thom isomorphism via the following sequence of equalities:

$$\dim H^p_c(E, V) = \dim H^p(E, V, \alpha) = \dim H^{m+n-p}(M, |V| \times o(E)).$$

and the observation that by Poincare duality on $M$

$$\dim H^{m+n-p}(M, |V| \times o(E)) = \dim H^{p-n}(M, |V| \times o(E)).$$

6.3. Proof of Theorem 6.2. We prove Theorem 6.2 by using a sequence of lemmas.

We denote the space of $\{e^{\alpha h} \omega | \omega \in \Omega^*(E, V, \alpha)\}$ as $\tilde{\Omega}^*(E, V, \alpha)$. Then the following lemma is an easy corollary of definition (1.3) of $d(\alpha)$.

**Lemma 6.3.** For any large enough $\alpha$ and any $p, 0 \leq p \leq \dim E$, the following diagram is commutative:

$$
\begin{array}{ccc}
\tilde{\Omega}^p(E, V, \alpha) & \xrightarrow{d} & \tilde{\Omega}^{p+1}(E, V, \alpha) \\
\uparrow_{e^{-\alpha h}} & & \uparrow_{e^{-\alpha h}} \\
\Omega^p(E, V, \alpha) & \xrightarrow{d(\alpha)} & \Omega^{p+1}(E, V, \alpha),
\end{array}
$$

where the vertical arrows are isomorphisms.

We introduce a diffeomorphism $f: \mathbb{R} \to (0, 2)$, where

$$f(t) = \frac{2}{\pi} \arctan \left( \frac{t}{2} \right) + 1.$$

For all $(x, y) \in (M, \pi^{-1}(x))$ we also define a diffeomorphism $\psi$ between manifolds $E$ and $F_2$ by the formula

$$\psi(x, y) = \left( x, \frac{y}{|y|} f(|y|) \right).$$

The manifold $F_2$ is defined as the image of $E$ under $\psi$. Observe that $F_2$ is a disc bundle over $M$ with the fibers being open discs $\{y | |y| < 2\}$. 

The diffeomorphism $\psi^{-1}$ induces the map $(\psi^{-1})^*$ on the spaces of differential forms on $E$ with values in $V$:

$$(\psi^{-1})^* : \Omega^p_c(E, V) \to \Omega^p_c(F_2, W),$$

$$(\psi^{-1})^* : \tilde{\Omega}^p(E, V, \alpha) \to (\psi^{-1})^*(\tilde{\Omega}^p(E, V, \alpha)),$$

where $W = (\psi^{-1})^* V$ is an induced flat bundle over $F_2$. We denoted $(\psi^{-1})^*(\tilde{\Omega}^p(E, V, \alpha))$ as $\tilde{\Omega}^p(F_2, W, \alpha)$.

Since differential $d$ and an isomorphism $(\psi^{-1})^*$ commute, we have the following lemma:

**Lemma 6.4.** For any large enough $\alpha$ and any $p$, $0 \leq p \leq \dim E$,

$${\tilde{\mathcal{H}}}^p(E, V, \alpha) \cong \tilde{\mathcal{H}}^p(F_2, W, \alpha),$$

$${\mathcal{H}}^p_c(E, V) \cong \mathcal{H}^p_c(F_2, W).$$

Let $F_1 \subset F_2$ be a subbundle of $F_2$, where each fiber of $F_1 \to M$ is an open disc $\{y \mid |y| < 1\}$. Let $k : F_1 \to F_2$ be a diffeomorphism which is the multiplication by 2 map in each fiber.

For every $p \geq 0$ we have the following maps

$$i : \Omega^p_c(F_1, W) \to \tilde{\Omega}^p(F_1, W, \alpha), \quad i = 1, 2,$$

$$j : \tilde{\Omega}^p(F_1, W, \alpha) \to \Omega^p_c(F_2, W),$$

$$k^* : \Omega^p_c(F_2, W) \to \Omega^p_c(F_1, W), \quad \text{and}$$

$$k^* : \tilde{\Omega}^p(F_2, W) \to \tilde{\Omega}^p(F_1, W),$$

where $i$ is the inclusion, $j$ is the extension maps and $k^*$ is the map induced by the diffeomorphism $k$.

We now describe the map $j$. Let $\theta \in \tilde{\Omega}^p(F_1, W, \alpha)$, then $\theta = (\psi^{-1})^* e^{-\alpha |y|^2} \omega$, for some $\omega \in \Omega^p_c(E, V)$. Clearly $\theta$ and all its derivatives are zero on $\partial F_1$, since the form $\omega$ and all its derivatives decay rapidly at infinity. Thus, $\theta$ can be extended by zero to a form in $\Omega^p_c(F_2, W)$.

Since differential $d$ and the maps $i$, $j$ and $k^*$ commute, they induce the maps on the corresponding cohomology.
Lemma 6.5. For large enough $\alpha$ and any $p$, $0 \leq p \leq \dim E$,

$$k^* \circ j \circ i : H^p_c(F_1, W) \to H^p_c(F_1, W)$$

and

$$k^* \circ i \circ j : \check{H}^p(F_1, W, \alpha) \to \check{H}^p(F_1, W, \alpha)$$

are isomorphisms.

Moreover, since $k^*$ is an isomorphism on the cohomology, so is $i$ and $j$, and we have a corollary:

Corollary 6.6. For any large enough $\alpha$ and any $p$, $0 \leq p \leq \dim E$,

$$H^p_c(F_2, W) \cong \check{H}^p(F_2, W).$$

This corollary together with Lemma 6.4. proves Theorem 6.2.
7. ASYMPTOTICS OF THE SMALL EIGENVALUES OF THE WITTMAN LAPLACIAN

7.0. Introduction. For each \( \alpha > 0 \) and \( p = 1, \ldots, m + n \), let \( 0 \leq \lambda_1^p(\alpha) \leq \lambda_2^p(\alpha) \leq \cdots \) be the eigenvalues of \( \Box^p(\alpha) \). The goal of this chapter is to investigate the asymptotics of the eigenvalues of \( \Box^p(\alpha) \) as \( \alpha \to \infty \).

The main result of this chapter is Theorem 7.20. This theorem states that the small eigenvalues of \( \Box(\alpha) \) approach the eigenvalues of the standard Laplacian

\[
\Delta : \Omega^*(M, V|_M \otimes o(E^-)) \to \Omega^*(M, V|_M \otimes o(E^-))
\]

on the space of \( V \)-valued differential forms on \( M \), which are twisted by the orientation bundle of \( E^- \).

In Section 7.1 we relate the spectrum of \( \Box(\alpha) \) to the spectrum of the adiabatic deformation \( \hat{\Box}(\delta) \) of the Witten differential \( \Box = \Box(\alpha) \) (for \( \alpha = 1 \)). We observe that if \( \delta = \alpha^{-1/2} \) then the operators \( \Box(\alpha) \) and \( \delta^{-2} \hat{\Box}(\delta) \) are isospectral. We assumed that in the definition of \( \hat{\Box}(\delta) \) the parameter \( \alpha \) equals to one. The isospectrality means that if \( \{ \lambda_j^p(\delta) \} \) denote the eigenvalues of \( \hat{\Box}^p(\delta) \), then for any \( p, 0 \leq p \leq \dim E \) and \( \alpha \geq 0 \) we have

\[
\lambda_j^p(\alpha) = \delta^{-2} \lambda_j^p(\delta), \quad j = 1, 2, \ldots .
\]

In Section 7.2 we study the kernel of \( \hat{\Box}(\delta) \). We conclude that for large enough \( \delta \) the dimension of \( \ker \hat{\Box}(\delta) \) does not depend on \( \delta \) and is equal to the dimension of \( H^{p-n^-}(M, V|_M \otimes o(E^-)) \). Moreover, by the Hodge theory on \( M \) we have

\[
\dim H^{p-n^-}(M, V|_M \otimes o(E^-)) = \dim \ker \Delta^{p-n^-}.
\]

In Section 7.3 we start the Taylor analysis of the small spectrum of \( \hat{\Box}(\delta) \) by proving several useful preliminary results.

In sections 7.4 and 7.5 we introduce a model operator \( \Delta \) and prove the main result of this chapter. We use the classical variational approach as in [Du-Sc] to compare the small spectrum of the Witten Laplacian to the spectrum of the model operator.
7.1. A rescaling of the eigenvalues. For a bigrading \( E = E^+ \oplus E^- \) of a vector bundle \( E \to M \) let \( A = T^{\text{ver}}E, \ B = T^{\text{hor}}E \). Then \( TE = A \oplus B \). As in Section 1.3 we have chosen the metric \( g \) on \( TE \) to be the sum of the metrics on \( A \) and \( B \):

\[ g = g_A \oplus g_B. \]

As in Section 2.2 we define a deformation \( g_\delta (0 < \delta \leq 1) \) of the metric \( g \) on \( TE \) by the formula

\[ g_\delta = g_A \oplus \delta^{-2}g_B. \]  

We also define the operators \( \mathring{d} \) and \( \mathring{d}_\delta \) by

\[ \mathring{d} = e^{-h}de^h = d + dh \wedge \quad \text{and} \quad \mathring{d}_\delta = d^{1,0} + \delta d^{0,1} + \delta^2 d^{-1,2} + dh \wedge. \]

We recall that the horizontal space \( B \) was chosen so that \( dh|_B = 0 \). Then \( dh \wedge \) is a \((1,0)\)-operator. Therefore,

\[ \mathring{d}_\delta = \mathring{d}^{1,0} + \delta d^{0,1} + \delta^2 d^{-1,2}, \quad \text{where} \quad \mathring{d}^{1,0} = d^{1,0} + dh \wedge. \]  

As in Section 2.4

\[ \square^p(\delta) = \mathring{d}_{\delta}^* \mathring{d}_{\delta} + \mathring{d}_{\delta}^* \mathring{d}_{\delta} \]

denotes the adiabatic deformation of the Witten Laplacian \( \square \).

We have the following theorem:

**Theorem 7.1.** Let \( \delta = \alpha^{-1/2}, \) then for all \( p, \ 0 \leq p \leq \dim E, \) and \( j = 1, 2, \ldots \)

\[ \lambda_j^p(\alpha) = \delta^{-2} \lambda_j^p(\delta). \]

In other words the operators \( \square(\alpha) \) and \( \delta^{-2} \square(\delta) \) are isospectral.

**Proof.** We prove the theorem by a simple rescaling argument. Let \( m \in M \) be any point. Let \( \pi^{-1}(m) \) be the fiber of the fibration \( \pi : E \to M \) over point \( m \), and let \( y_1, \ldots, y_n \) be the coordinates in the fiber. If we introduce a new (rescaled) coordinates \( z_i = \alpha^{-1/2} y_i \). It is equivalent to choosing a new metric

\[ g_\alpha = (\alpha^{-1}g_A) \oplus g_B = \alpha^{-1}(g_A \oplus \alpha g_B) \]

\[ = \delta^2 (g_A \oplus \delta^{-2} g_B) = \delta^2 g_\delta, \]
where \( g_\delta \) is given by (7.1). We have
\[
\left( \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right)_g = \left( \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j} \right) g_\alpha.
\]

In the new coordinates
\[
\alpha h(y) = \alpha(|y^+|^2 - |y^-|^2) = h(z).
\]

On the other hand we use the rescaling property of the Laplacian and the equality \( \delta = \alpha^{-1/2} \) to conclude that

(7.6) \[\hat{\Delta}^p(\alpha) = \delta^{-2} \hat{\Delta}_\delta^p,\]

where \( \hat{\Delta}_\delta^p \) is the Witten Laplacian associated to the metric \( g_\delta \). The statement of the theorem follows from (7.6) after we recall from Section 2.4 that for all \( p \) the operator \( \hat{\Delta}_\delta^p \) is isospectral to the operator \( \hat{\Delta}^p(\delta) \).

**7.2. Zero eigenvalues of \( \hat{\Delta}(\delta) \).** In this section we reproduce the results of chapters 3 and 4 in the setting of this chapter. We do not give proofs, but only indications of the necessary changes. The main result of this section states that for all \( p \), \( \dim \ker \hat{\Delta}^p(\delta) \) does not depend on \( \delta \) and is equal to \( \dim E^p_2 = \dim H^{p-n^{-}}(M, V|_M \otimes o(E^-)) \).

As in Chapter 3 for each \( p \) we can define a nested sequence of spaces
\[
E^p_0 \supseteq E^p_1 \supseteq E^p_2 \supseteq \ldots
\]

by
\[
E^p_k = \{ \omega \in \Omega^p(E, V) \mid \exists \omega_1, \ldots, \omega_{k-1} \text{ with } \\
\hat{d}_\delta(\omega + \delta \omega_1 + \cdots + \delta^{k-1} \omega_{k-1}) \in O(\delta^k), \\
\hat{d}_\delta^*(\omega + \delta \omega_1 + \cdots + \delta^{k-1} \omega_{k-1}) \in O(\delta^k) \}, \quad k = 1, 2, \ldots.
\]

Our computations of \( E^p_1 \) and \( E^p_2 \) work as before and we have
\[
E^p_1 = \{ \omega \in \Omega^p(E, V) \mid \hat{d}^{1,0} \omega = 0, (\hat{d}^{1,0})^* \omega = 0 \},
\]
\begin{equation}
E_2^p = \{ \omega \in E_1^p \mid \pi_1 d^{0,1} \pi_1 \omega = 0, (\pi_1 d^{0,1} \pi_1)^* \omega = 0 \}.
\end{equation}

Let \( \omega \in E_1^p \). Since the differential \( \hat{d}^{1,0} \) preserves the bigrading, we may assume that \( \omega \in \Omega^i_j(E, V) \) for some \( i, j \), \( i + j = p \).

We write \( \omega \) as

\[ \omega = \gamma \otimes \pi^* \beta, \]

where \( \gamma \in \Omega^i_A(A, V_A), \beta \in \Omega^j(M, V_M) \). Then we have

\begin{equation}
\hat{\Box}^{1,0} \omega = (\hat{\Box}_A \gamma) \otimes \pi^* \beta,
\end{equation}

where \( \hat{\Box}_A \) is the Laplacian on \( \Omega^i_A(\pi^{-1}(m), V_{\pi^{-1}(m)}) \).

Thus

\begin{equation}
\ker \hat{\Box}^{1,0} = \Gamma(M, \Lambda^* T^* M \otimes \mathcal{H}(A, V_A)),
\end{equation}

where \( \mathcal{H}(A, V) \) denotes the vector bundle over \( M \) whose fibers at \( m \) are the \( \hat{\Box}_A \)-harmonic forms (i.e. elements of \( \ker \hat{\Box}_A \)) on \( \pi^{-1}(m) \) with values in \( V \). Now we have the following lemma:

**Lemma 7.2.**

1. Let \( \hat{\Box}^i_A \) denote the restriction of the operator \( \hat{\Box}_A \) to the space \( \Omega^i_A(\pi^{-1}(m), V_{\pi^{-1}(m)}) \).

   Then, for every \( m \in M \), \( \dim_V \ker \hat{\Box}^i_A = 1 \), if \( i = n^- \); \( \dim_V \ker \hat{\Box}^i_A = 0 \), otherwise.

2. If \( \omega \in \ker \hat{\Box}^i_A \), then \( \omega = \gamma \otimes \pi^* \beta \). After an orthogonal change of coordinates in the fiber \( \pi^{-1}(m) \),

\[ \gamma(\Omega^i_A(\pi^{-1}(m), V_{\pi^{-1}(m)})) = e^{-a |Dy|} dy_{n+1} \wedge \cdots \wedge dy_n \otimes v, \]

where \( D \) is some diagonal matrix.

We note that \( \mathcal{H}(A, V_A) \) is a rank-one bundle of the \( n^- \)-forms. If \( E^- \to M \) is not orientable, then the line bundle \( \mathcal{H}(A, V_A) \to \Omega^{p-n^-}(M, V_M) \) is not trivial. On the other hand, the line bundle \( \mathcal{H}(A, V_A) \to \Omega^{p-n^-}(M, V_M \otimes o(E^-)) \) is trivial.

Now we have the following theorem:
Theorem 7.3. For any \( p, 0 \leq p \leq \dim E \),

\[
(7.10) \quad \dim E^p_2 = \dim H^{p-n^-}(M, V|M \otimes o(E^-)).
\]

In a similar way as Theorem 3.2, this theorem follows from the following lemma:

Lemma 7.4. Suppose \( \gamma \) is any locally constant section of the bundle \( H(A, V_A) \), then the following diagram is commutative:

\[
\begin{array}{ccc}
E^p_1 & \xrightarrow{\pi_1 d^{p,1} \pi_1} & E^{p+1}_1 \\
\uparrow_{\gamma \otimes \pi^*} & & \uparrow_{\gamma \otimes \pi^*} \\
\Omega^{p-n^-}(M, V|M \otimes o(E^-)) & \xrightarrow{d} & \Omega^{p+1-n^-}(M, V|M \otimes o(E^-))
\end{array}
\]

where an isomorphism

\[\gamma \otimes \pi^* : \Omega^{p-n^-}(M, V|M \otimes o(E^-)) \to E^p_1\]

is defined by

\[\beta \mapsto \gamma \otimes \pi^* \beta.\]

Now we can prove our next theorem:

Theorem 7.5. For any small enough \( \delta \) and any \( p, 0 \leq p \leq \dim E \),

\[
\dim \ker \hat{\Delta}^p(\delta) = \dim E^p_2 = \dim H^{p-n^-}(M, V|M \otimes o(E^-)).
\]

Proof. By Theorem 7.3,

\[
(7.11) \quad \dim E^p_2 = \dim H^{p-n^-}(M, V|M \otimes o(E^-)).
\]

Then by Corollary 5.4,

\[
(7.12) \quad \dim H^{p-n^-}(M, V|M \otimes o(E^-)) = \dim H^p_\epsilon(E, V, \alpha) = \dim \ker \hat{\Delta}^p(\delta),
\]

where the last equality is Corollary 1.3. The statement of the theorem then follows from (7.11) and (7.12). \( \blacksquare \)
Corollary 7.6. For any \( p \), \( 0 \leq p \leq \dim E \),

\[
E^p_2 = E^p_3 = \ldots = E^p_\infty.
\]

Proof. From (7.12) we have that \( \dim E^p_2 = \dim H^p(E, V, \alpha) \). Moreover, by applying the arguments from Section 4.3 to our setting we have \( \dim E^p_\infty = \dim H^p(E, V, \alpha) \). Combining these two equalities we conclude that \( \dim E^p_2 = \dim E^p_\infty \). ■

7.3. Asymptotics of the small eigenvalues of \( \hat{\Delta}^p(\delta) \). Preliminary results. We recall that \( \pi_1 : \Omega^p(E, V) \rightarrow E^p_1 \) denotes the orthogonal projection. We also denote as \( \tilde{\pi}_0 \) the orthogonal projection onto \( (E^p_1)^\perp \). Then \( \tilde{\pi}_0 + \pi_1 = 1 \).

Lemma 7.7. There exists a constant \( c > 0 \), so that for all \( \omega \in \Omega^p_2(E, V) \),

\[
(\hat{\Delta}^{1,0}\omega, \omega) \geq c|\tilde{\pi}_0 \omega|^2.
\]

Proof. It follows from (7.8) that

\[
\inf\{\lambda \in \text{spec}(\hat{\Delta}^{1,0}) | \lambda > 0\}
\]

\[
= \inf_{m \in M} \left\{ \inf\{\lambda \in \text{spec} \left( \hat{\Delta}_A \circ \Omega^p(\pi^{-1}(m), V_{\pi^{-1}(m)} \rightarrow \Omega^p(\pi^{-1}(m), V_{\pi^{-1}(m)}) \right) | \lambda \geq 0\} \right\}.
\]

The spectrum of \( \hat{\Delta}_A \) varies continuously over \( M \), and the multiplicity of 0 is constant. Thus the smallest positive eigenvalue is a continuous function on \( M \) and therefore achieves a positive minimum. This implies that the infimum above is positive, which is precisely the statement of Lemma 7.7 ■

Theorem 7.8. There is \( C > 0 \) such that for all \( \delta \) small enough

\[
\hat{\Delta}(\delta) \geq \frac{1}{2} \hat{\Delta}^{1,0} + \delta^2 (\square^{0,1} - C) \geq \delta^2 (\hat{\Delta}^{1,0} + \square^{0,1} - C).
\]

The proof of this theorem is given in the Appendix 2 (see Theorem A.2.7).

The proof of the following corollary is a simple application of Lemma 7.7 and Theorem 7.8.
Corollary 7.9. There exist \( c_1 > 0 \) and \( c_2 > 0 \), so that

\[
\| \hat{\pi}_0 \omega \| \leq c_1 \| \square (\delta) \omega \| + c_2 \| \omega \|.
\]

Lemma 7.10.

\[
(7.15) \quad \hat{\pi}_0 (d^{0,1})^\ast \pi_1 = \pi_1 (d^{0,1})^\ast \pi_0 = \pi_1 (d^{0,1})^\ast \hat{\pi}_0 = 0.
\]

Proof. We will prove that \( \hat{\pi}_0 d^{0,1} \pi_1 = 0 \). It is enough to show that for any \( \omega \in E_1^p \), \( d^{0,1} \omega \) is \( \hat{d}^{1,0} \)-harmonic. By the commutativity relation (2.7) we have

\[
d^{2,0} d^{0,1} \omega = -d^{0,1} \hat{d}^{1,0} \omega = 0.
\]

By another commutativity relation in the statement of Lemma A.2.5 part (1) we have

\[
(\hat{d}^{1,0})^\ast d^{0,1} \omega = -d^{0,1} (\hat{d}^{1,0})^\ast \omega = 0.
\]

Thus \( d^{0,1} \omega \in E_1^{p+1} \Rightarrow \hat{\pi}_0 d^{0,1} \omega = 0. \]

7.4. The model operator. A definition of \( \delta \)-small eigenvalues. We denote as

\[
(7.16) \quad \Delta_M^p : \Omega^p (M, V|_M \otimes o(E^-)) \rightarrow \Omega^p (M, V|_M \otimes o(E^-))
\]

the standard Laplacian on \( M \) acting on \( (V|_M \otimes o(E^-)) \)-valued \( p \)-forms.

Theorem 7.11. Operators \( \pi_1 \square_p^{0,1} \pi_1 : E_1^p \rightarrow E_1^p \) and \( \Delta_M^{p-n} \) are isospectral.

Proof. The statement of the theorem follows from Lemma 7.4 and (7.15) \( \blacksquare \)

Let

\[
0 \leq \mu_1^{p-n} \leq \mu_2^{p-n} \leq \cdots \quad \text{and} \quad u_1, u_2, \cdots \in E_1^p
\]

denote the eigenvalues (counting multiplicities) of \( \pi_1 \square_p^{0,1} \pi_1 \) and the corresponding orthonormal eigenforms. By Theorem 7.11 the eigenvalues above equal to the eigenvalues of \( \Delta_M^{p-n} \).
It follows from Theorem 7.3 that

\[ \dim E_2^p = \dim \ker(\pi_1 \Box_p^1 \pi_1) = \dim \ker(\hat{\Delta}^p(\delta)). \]

Therefore for small enough \( \delta \geq 0 \) the number of zero eigenvalues of \( \Delta^p_M \) equals to the number of zero eigenvalues of \( \hat{\Delta}^p(\delta) \).

Fix \( 0 < \varepsilon < 1/3 \). For every \( 0 \leq p \leq \dim E \) and for each \( 0 < \delta < 1 \) we define the space \( \widetilde{W}^p(\delta) \) to be the span of the eigenforms \( \{ \omega^p_j(\delta) \}_{j=1}^{\hat{k}_p(\delta)} \), satisfying

\[ \hat{\Delta}^p(\delta) \omega^p_j(\delta) = \lambda^p_j(\delta) \omega^p_j(\delta) \quad \text{with} \quad \lambda^p_j(\delta) < \delta^{2-\varepsilon}, \quad j = 1, \ldots, \hat{k}_p(\delta). \]  

Since \( \hat{\Delta}^p(\delta) \) has discrete spectrum, \( \hat{k}_p(\delta) = \dim \widetilde{W}^p(\delta) < \infty \).

Similarly, we define the space \( W^p(\delta) \subset E_1^p \) to be the span of the eigenforms \( \{ u^p_j \}_{j=1}^{k_p(\delta)} \) satisfying

\[ \pi_1 \Box_p^1 \pi_1 u^p_j = \mu^p_j \ u^p_j \quad \text{with} \quad \mu^p_j < \delta^{-\varepsilon}, \quad j = 1, \ldots, k_p(\delta). \]

**Definition 7.12.** We call an eigenvalue \( \lambda^p_j(\delta) \) (\( \mu^p_j \)) \( \delta \)-small if it satisfies the inequality

\[ \lambda^p_j(\delta) < \delta^{2-\varepsilon} \quad (\mu^p_j < \delta^{-\varepsilon}) \]

for small enough \( \delta \).

Our goal is to compare the small eigenvalues of \( \hat{\Delta}^p(\delta) \) to the small eigenvalues of \( \Delta^{p-n} \). The following theorem estimates the eigenvalues of \( \hat{\Delta}^p(\delta) \) in terms of the eigenvalues of \( \Delta^{p-n} \).

**Theorem 7.13.** There exists a constant \( C > 0 \) such that for all small enough \( \delta \)

\[ \lambda^p_j \leq \delta^2 \mu^p_j + C \delta^{3-\varepsilon}, \quad 1 \leq j \leq k_p(\delta). \]

**Proof.** Let \( P(\delta) \) denote the orthogonal projection on \( W^p(\delta) \). Then, since

\[ \pi_1 \tilde{d}^{1,0} = \tilde{d}^{1,0} \pi_1 = \pi_1 (d^{1,0}^*) = (d^{1,0}^* \pi_1)^*, \]
it follows from an explicit calculation of $\hat{\Box}^p(\delta)$ in (A.2.13) that
\begin{equation}
\hat{P}(\delta) \hat{\Box}^p(\delta) \Box P(\delta) = P(\delta) \pi_1 \hat{\Box}^p(\delta) \pi_1 P(\delta) \\
= \delta^2 P(\delta) \pi_1 \Box_p^{0,1} \pi_1 P(\delta) + \delta^3 P(\delta) K_3 P(\delta) + \delta^4 P(\delta) \Box^{-1,2} P(\delta).
\end{equation}  
(7.22)

We now will estimate the second and the third terms in the right-hand side of (7.22). From an estimate
\begin{equation}
K_3 \leq \Box^{0,1} + \Box^{-1,2}
\end{equation}  
(7.23)
in (A.2.21) we conclude that
\begin{equation}
\delta^3 \| P(\delta) K_3 P(\delta) \| \leq \delta^3 \| P(\delta) \Box_p^{0,1} P(\delta) \| + \delta^3 \| P(\delta) \Box^{-1,2} P(\delta) \|.
\end{equation}  
(7.24)

Moreover, by the definition of $W^p(\delta)$,
\begin{equation}
\| P(\delta) \Box_p^{0,1} P(\delta) \| \leq \delta^{-\epsilon}.
\end{equation}  
(7.25)

To estimate $\| P(\delta) \Box^{-1,2} P(\delta) \|$ we recall that (see A.2.17)
\begin{equation}
\Box^{-1,2} \leq c_1 \Box^{1,0} + c_2.
\end{equation}  
(7.26)

Therefore, since $P(\delta) \Box^{1,0} P(\delta) = P(\delta) \pi_1 \Box^{1,0} \pi_1 P(\delta) 0$ we conclude that
\begin{equation}
\| P(\delta) \Box^{-1,2} P(\delta) \| \leq \| P(\delta) (c_1 \Box^{1,0} + c_2) P(\delta) \| \leq c_2.
\end{equation}  
(7.27)

Now we use (7.22), (7.24), (7.26), and (7.27) to get the following inequality
\begin{equation}
\| P(\delta) \hat{\Box}^p(\delta) \Box P(\delta) - \delta^2 P(\delta) \pi_1 \Box_p^{0,1} \pi_1 P(\delta) \| \leq \delta^3 \| P(\delta) \Box_p^{0,1} P(\delta) \| + 2\delta^3 \| P(\delta) \Box^{-1,2} P(\delta) \| \\
\leq \delta^3 - \epsilon + 2c_1 \delta^3 \leq C\delta^{3 - \epsilon}.
\end{equation}  
(7.28)

Both operators $P(\delta) \hat{\Box}^p(\delta) P(\delta)$ and $\delta^2 P(\delta) \pi_1 \Box_p^{0,1} \pi_1 P(\delta)$ are matrices of the same rank. The operator $\delta^2 P(\delta) \pi_1 \Box_p^{0,1} \pi_1 P(\delta)$ has exactly $k_p(\delta)$ eigenvalues. Therefore from (7.28) we can conclude that $P(\delta) \hat{\Box}^p(\delta) P(\delta)$ also has exactly $k_p(\delta)$ eigenvalues which we denote as $0 \leq \lambda_1^p(\delta) \leq \cdots \leq \lambda_{k_p(\delta)}^p(\delta)$. Then as a simple application of the classical min-max principle [Du-Sc] and (7.28) we conclude that
\begin{equation}
\lambda_j^p(\delta) \leq \hat{\lambda}_j^p(\delta) \leq \delta^2 \mu_j^{p-\eta} + C\delta^{3 - \epsilon},
\end{equation}
which is the statement of the theorem. ■
Corollary 7.14. For all small enough $\delta \geq 0$ we have

$$\tag{7.29} k_p(\delta) \leq \hat{k}_p(\delta).$$

Our next goal is to prove the following theorem which estimates the small eigenvalues of $\Delta^{p-n^-}$ in terms of the small eigenvalues of $\hat{\Delta}^p(\delta)$.

Theorem 7.15. There exists a constant $C > 0$ such that for any small enough $\delta > 0$ and any $p$, $0 \leq p \leq \dim E$, we have the following inequality:

$$\tag{7.30} \delta^2 \mu_j^{p-n^-} \leq \lambda_j^p + C\delta^{3-2\epsilon}, \quad 1 \leq j \leq \hat{k}_p(\delta).$$

Let $\hat{P}(\delta)$ denote the orthogonal projection on $\hat{W}(\delta)$. In order to prove Theorem 7.15 we need the following lemma:

Lemma 7.16. There exists $C > 0$, such that for all $\delta$ small enough we have the following estimates:

$$\tag{7.31} \|\hat{P}(\delta)\bar{\eta}_0 \hat{P}(\delta)\| \leq C\delta^{1-\epsilon/2};$$

$$\tag{7.32} \|\hat{d}^{1,0} \hat{P}(\delta)\| \leq C\delta^{1-\epsilon/2}, \quad \|\varphi^{1,0} \hat{P}(\delta)\| \leq C\delta^{1-\epsilon/2};$$

$$\tag{7.33} \|d^{0,1} \hat{P}(\delta)\| \leq C\delta^{1-\epsilon/2}, \quad \|(d^{0,1})^* \hat{P}(\delta)\| \leq C\delta^{1-\epsilon/2};$$

$$\tag{7.34} \|d^{0,1} \pi_1 \hat{P}(\delta)\| \leq C\delta^{1-\epsilon/2}, \quad \|(d^{0,1})^* \pi_1 \hat{P}(\delta)\| \leq C\delta^{1-\epsilon/2};$$

$$\tag{7.35} \|d^{-1,2} \hat{P}(\delta)\| \leq C, \quad \|(d^{-1,2})^* \hat{P}(\delta)\| \leq C;$$
\[(7.36) \quad \| d^{-1,2} \pi_1 \hat{P}(\delta) \| \leq C, \quad \| (d^{-1,2})^* \pi_1 \hat{P}(\delta) \| \leq C. \]

**Proof.** Let \( \omega \) be any form of norm one from \( \hat{W}^p(\delta) \).

(7.31): It follows from Theorem 7.8 and Lemma 7.7 that
\[
2 \delta^{2-\epsilon} \geq 2 \langle \tilde{\Omega}^p(\delta) \omega, \omega \rangle \geq \langle \tilde{\Omega}^{1,0} \omega, \omega \rangle - c_1 \delta^2
\]
\[(7.37) \quad \geq c_2 \| \tilde{\pi}_0 \omega \|^2 - c_1 \delta^2. \]

Inequality (7.31) easily follows from (7.37).

(7.32): We have an equality
\[(7.38) \quad \| d^{1,0} \|^2 + \| (d^{1,0})^* \|^2 = \langle \tilde{\Omega}^{1,0} \omega, \omega \rangle, \]

From Theorem 7.8 we can estimate \( \tilde{\Omega}^{1,0} \). This estimate implies the following inequality:
\[(7.39) \quad \| d^{1,0} \|^2 + \| (d^{1,0})^* \|^2 \leq c_1 \langle \tilde{\Omega}^p(\delta) \omega, \omega \rangle + c_2 \delta^2 \leq c \delta^{2-\epsilon}. \]

(7.33): Similarly to the proof of (7.32) we have
\[
\delta^2 \| d^{0,1} \|^2 + \delta^2 \| (d^{0,1})^* \|^2 \leq \delta^2 \langle \tilde{\Omega}^{0,1} \omega, \omega \rangle
\]
\[
\leq \langle \tilde{\Omega}^p(\delta) \omega, \omega \rangle + c_2 \delta^2 \leq c \delta^{2-\epsilon}. \]

Inequalities (7.33) follow from (7.40) after dividing both sides of 7.40 by \( \delta^2 \).

(7.34): Since by Lemma 7.10 \( \pi_1 \square^{0,1} \pi_1 + \tilde{\pi}_0 \square^{0,1} \tilde{\pi}_0 = \square^{0,1} \), we have
\[(7.41) \quad \pi_1 \square^{0,1} \pi_1 \leq \pi_1 \square^{0,1} \pi_1 + \tilde{\pi}_0 \square^{0,1} \tilde{\pi}_0 = \square^{0,1}. \]

Now the inequality (7.34) follows from (7.40).

(7.35): We have an equality
\[(7.42) \quad \| d^{-1,2} \omega \|^2 + \| (d^{-1,2})^* \omega \|^2 = \langle \square^{-1,2} \omega, \omega \rangle. \]
It follows from (7.26) that

\[ \|d^{-1,2}\omega\|^2 + \|\langle d^{-1,2}\omega, \omega \rangle + c_1 \leq C. \] (7.43)

Therefore both \(\|d^{-1,2}\omega\|\) and \(\|\langle d^{-1,2}\omega, \omega \rangle\|\) are bounded.

(7.36): The proof is similar to the proof of the inequality (7.35).

With the help of the estimates of Lemma 7.16 we can now prove Theorem 7.15.

**Proof of Theorem 7.15.** Since

\[ 1 = \pi_1 + \tilde{\pi}_0, \] (7.44)

we have

\[ \hat{P}(\delta)^{\Box p}(\delta)\hat{P}(\delta) = \hat{P}(\delta)\pi_1 \hat{\Box} p(\delta)\hat{P}(\delta) + \hat{P}(\delta)\tilde{\pi}_0 \hat{\Box} p(\delta)\hat{P}(\delta). \] (7.45)

From (7.31) we have \(\hat{P}(\delta)\tilde{\pi}_0\hat{P}(\delta) \leq C \delta^{1+\epsilon/2}\). Moreover, since \(\hat{P}(\delta)\) is an orthogonal projector on the space \(\hat{W}(\delta)\) of \(\delta\)-small forms, we have \(\|\hat{\Box} p(\delta)\hat{P}(\delta)\| \leq \delta^{2-\epsilon}\). Therefore,

\[ \|\hat{P}(\delta)\hat{\Box} p(\delta)\hat{P}(\delta) - \hat{P}(\delta)\pi_1 \hat{\Box} p(\delta)\hat{P}(\delta)\| = \|\hat{P}(\delta)\tilde{\pi}_0 \hat{\Box} p(\delta)\hat{P}(\delta)\| \]
\[ \leq \|\hat{P}(\delta)\tilde{\pi}_0\hat{P}(\delta)\|\|\hat{\Box} p(\delta)\hat{P}(\delta)\| \]
\[ \leq C \delta^{1-\epsilon/2}\delta^{2-\epsilon} \leq C \delta^{3-3\epsilon/2}. \] (7.46)

From an explicit calculation of \(\hat{\Box}(\delta)\) in (A.2.13) and from vanishing in (7.21) we conclude that

\[ \hat{P}(\delta)\pi_1 \hat{\Box} p(\delta)\hat{P}(\delta) = \delta^2 P(\delta)\pi_1 \Box^{0,1} \hat{P}(\delta) + \delta^2 P(\delta)\pi_1 K_2 \pi_1 K_2 \hat{P}(\delta) \]
\[ + \delta^3 P(\delta)\pi_1 K_3 \hat{P}(\delta) + \delta^4 P(\delta)\pi_1 \Box^{-1,2} \hat{P}(\delta). \] (7.47)

We now estimate each term in the right-hand side of (7.47). From Lemma 7.10 we have an equality for the first term:

\[ \pi_1 \Box^{0,1} = \pi_1 \Box^{0,1} \pi_1. \] (7.48)
To estimate the next term we write

\[(7.49) \quad \delta^2 \pi_1 K_2 = \delta^2 \pi_1 d^{-1,2}(d^{0,1})^* + \delta^2 \pi_1 (d^{-1,2})^* d^{0,1}. \]

For any norm one form \( \omega \in \tilde{W}^p(\delta) \) we have

\[(7.50) \quad |(\delta^2 \pi_1 K_2 \omega, \omega)| \leq \delta^2 |(d^{1,0})^* \omega, (d^{-1,2})^* \pi_1 \omega| + \delta^2 |(d^{1,0} \omega, d^{-1,2} \pi_1 \omega)|
\leq \delta^2 (\|d^{1,0}\| \omega \|d^{-1,2}\| \pi_1 \omega\| + \|d^{1,0} \omega\| \|d^{-1,2} \pi_1 \omega\|)
\leq C \delta^3 \delta^{1-\epsilon/2} \delta^{-\epsilon/2} \leq C \delta^{3-\epsilon}.\]

We used (7.32) and (7.36) to get to the last line in the formula above. Therefore,

\[(7.51) \quad \delta^2 \|\hat{P}(\delta) \pi_1 K_2 \hat{P}(\delta)\| \leq C \delta^{3-\epsilon}. \]

To estimate \( \hat{P}(\delta) \pi_1 K_3 \hat{P}(\delta) \) we write

\[(7.52) \quad \langle \pi_1 K_3 \omega, \omega \rangle = \langle d^{0,1} \omega, d^{-1,2} \pi_1 \omega \rangle + \langle (d^{0,1})^* \omega, (d^{-1,2})^* \pi_1 \omega \rangle
+ \langle d^{-1,2} \omega, d^{0,1} \pi_1 \omega \rangle + \langle (d^{-1,2})^* \omega, (d^{0,1})^* \pi_1 \omega \rangle.\]

It follows from (7.33), (7.34), (7.35), and (7.36) that

\[|\langle \pi_1 K_3 \omega, \omega \rangle| \leq C \delta^{-\epsilon/2}. \]

Therefore,

\[(7.53) \quad \delta^3 \|\hat{P}(\delta) \pi_1 K_3 \hat{P}(\delta)\| \leq C \delta^{3-\epsilon/2}. \]

Similarly, we use (7.35) and (7.36) to deduce an estimate:

\[(7.54) \quad \delta^4 \|\hat{P}(\delta) \pi_1 \square^{-1,2} \hat{P}(\delta)\| \leq C \delta^4. \]

Finally, by combining (7.46), (7.47), (7.48), (7.51), (7.53), and (7.54), we get

\[(7.55) \quad \|\hat{P}(\delta) \square^p(\delta) \hat{P}(\delta) - \hat{P}(\delta) \pi_1 \square^{0,1} \pi_1 \hat{P}(\delta)\| \leq C \delta^{3-2\epsilon}. \]

Operators \( \hat{P}(\delta) \square^p(\delta) \hat{P}(\delta) \) and \( \hat{P}(\delta) \pi_1 \square^{0,1} \pi_1 \hat{P}(\delta) \) are matrices of finite rank. The operator \( \hat{P}(\delta) \square^p(\delta) \hat{P}(\delta) \) has exactly \( \hat{k}(\delta) \) eigenvalues. Therefore, it follows
from (7.55) that \( \hat{P}(\delta) \pi_1 \hat{\Sigma} \pi_1 \hat{P}(\delta) \) also has exactly \( \hat{k}(\delta) \) eigenvalues which we will call \( \hat{\mu}_1^p(\delta) \leq \cdots \leq \hat{\mu}_{k_p(\delta)}^p(\delta) \). Moreover, it follows from (7.55) that

\[
\delta^2 \hat{\mu}_i^p(\delta) \leq \lambda_i^p(\delta) + C\delta^{3-2\epsilon}, \quad 1 \leq i \leq \hat{k}_p(\delta).
\]

From the classical min-max principle we have

\[
\delta^2 \mu_i^{p-n^-} \leq \delta^2 \hat{\mu}_i^p(\delta) \leq \lambda_i^p(\delta) + C\delta^{3-2\epsilon}, \quad 1 \leq i \leq \hat{k}_p(\delta),
\]

which is the statement we want to prove. ■

**Corollary 7.17.** For any small enough \( \delta > 0 \) and any \( p, 0 \leq p \leq \dim E \), we have

\[
\hat{k}_p(\delta) = k_p(\delta).
\]

Now we combine Theorem 7.13, Corollary 7.14, Theorem 7.16, and Corollary 7.17 into a single theorem.

**Theorem 7.18.**

1. For small enough \( \delta \) and for any \( p, n^- \leq p \leq \dim E \), the number \( \hat{k}_p(\delta) \) of \( \delta \)-small eigenvalues of \( \hat{\Delta}^p(\delta) \) is equal to the number \( k_p(\delta) \) of \( \delta \)-small eigenvalues of \( \Delta^{p-n^-} \).

2. There exists \( C \) such that for \( \delta \) small enough and for \( 1 \leq j \leq k_p(\delta) \) we have

\[
|\lambda_j^p(\delta) - \delta^2 \mu_j^{p-n^-}| \leq C\delta^{3-2\epsilon}.
\]

3. If \( p \leq n^- \), \( \hat{\Delta}^p(\delta) \) does not have \( \delta \)-small eigenvalues.

**7.6. The main result about the spectrum of \( \Box(\alpha) \).** To reformulate the main result of this section, Theorem 7.23, for the eigenvalues of \( \Box^p(\alpha) \), we recall that \( \delta = \alpha^{-1/2} \) and \( \lambda^p(\delta) = \alpha^{-1} \lambda^p(\alpha) \). Then we have the following definition, equivalent to Definition 7.12:

**Definition 7.19.** We call an eigenvalue \( \lambda_j^p(\alpha) \) (\( \mu_j^{p-n^-} \)) \( \alpha \)-small if it satisfies the inequality

\[
\lambda_j^p(\alpha) < \alpha^{-\epsilon/2} \quad (\mu_j^{p-n^-} < \alpha^{-\epsilon/2})
\]

for large enough \( \alpha \).

Theorem 7.18 can be reformulated as:
Theorem 7.20.

(1) For large enough $\alpha$ and for any $p$, $n^- \leq p \leq \dim E$, the number $k^p(\alpha)$ of $\alpha$-small eigenvalues of $\Box^p(\alpha)$ is equal to the number $k^p(\alpha)$ of $\alpha$-small eigenvalues of $\triangle^{p-n^-}$.

(2) There exists a constant $C > 0$ such that for large enough $\alpha$ and for $1 \leq j \leq k^p(\alpha)$ we have

\begin{equation}
|\lambda_j^p(\alpha) - \mu_j^{p-n^-}| \leq C\alpha^{-1/2+\epsilon}.
\end{equation}

(3) If $p \leq n^-$, $\Box^p(\alpha)$ does not have $\alpha$-small eigenvalues.
8. WITTEN LAPLACIAN ON COMPACT MANIFOLDS

8.0. Introduction. In this chapter we observe that the small eigenvalues of the Witten Laplacian

\[ L(\alpha) : \Omega^*(N, V) \to \Omega^*(N, V) \]

can be calculated by restricting the operator \( L(\alpha) \) onto tubular neighborhoods \( \{ E_i \} \) of connected components \( \{ M_i \} \) of the critical submanifold \( M \) and then applying Theorem 7.20.

In Section 8.1 we recall the definition of a Morse-Bott function and the statement of the Generalized Morse Lemma about the structure of a tubular neighborhood of a connected component of the critical submanifold.

In Section 8.2 we define a metric on \( N \), which in the neighborhood of the critical submanifold comes from the metric on the tubular neighborhood defined in Section 1.3. In that section we also define the Witten Laplacian on \( N \).

In Section 8.3 we prove Theorem 8.6. This theorem states that as \( \alpha \to \infty \) the small eigenvalues of \( L(\alpha) \) on \( N \) approach the eigenvalues of the standard Laplacian \( \Delta \) on \( M \), twisted by the orientation bundles \( o(E^-) \). The theorem also contains the estimate of the rate of convergence of the eigenvalues of the Witten Laplacian \( L(\alpha) \) on \( N \).

8.1. The Generalized Morse Lemma. We start by giving a definition of a Morse-Bott function.

Let \( N \) be a compact smooth manifold without boundary. Let \( h : N \to \mathbb{R} \) be a \( C^\infty \)-function. We call a point \( m \in M \) a nondegenerate critical point of index \( k \) if for any (or, equivalently, for some) submanifold \( W \subset N \), which is transverse to \( M \) at \( m \), the point \( m \) is a nondegenerate critical point of \( h|_W \) of index \( k \). A smooth submanifold \( M \) of \( N \) is called critical if every point of \( M \) is critical. A critical submanifold \( M \) is called nondegenerate of index \( k \), if each point of \( M \) is nondegenerate of index \( k \).

Definition 8.1. A \( C^\infty \)-function \( h : N \to \mathbb{R} \) is called a Morse-Bott function if all critical submanifolds of \( h \) are nondegenerate.
Definition 8.2. A tubular neighborhood of a submanifold \( M \subset N \) is a pair \((f, E)\), where \( E \rightarrow M \) is a vector bundle over \( M \) and \( f : E \rightarrow N \) is an embedding such that

1. \( f|_M = \text{id}|_M \), where \( M \) is identified with the zero section of \( E \);
2. \( f(E) \) is an open neighborhood of \( M \) in \( N \).

Lemma 8.3 (Generalized Morse Lemma) [Hir]. Let \( h : N \rightarrow \mathbb{R} \) be a Morse-Bott function, \( M \) be a critical submanifold of \( h \) of index \( n^- \). If \( M \) is connected, then there is a \( C^\infty \) tubular neighborhood \((f, E = E^+ \oplus E^-)\), \( \dim E^- = n^- \) and a Euclidean structure on \( E^+ \oplus E^- \) such that the composition \( h \circ f : E^+ \oplus E^- \rightarrow \mathbb{R} \) is given by

\[
(y^+, y^-) \mapsto |y^+|^2 - |y^-|^2 + C
\]

for all \((y^+, y^-) \in E^+_m \oplus E^-_m, m \in M, \) and where \( C = h(M) \).

8.2. The Witten Laplacian on \( N \). Let \( N \) be a compact Riemannian manifold with the metric \( g_0 \). Let \( h : N \rightarrow \mathbb{R} \) be a Morse-Bott function. We denote as \( M_1, \ldots, M_\Lambda \) disjoint connected components of the critical submanifold \( M \) of \( h \).

For all \( j \), \( \text{ind}(M_j) = n^-_j \).

From Lemma 8.3 each \( M_j \) has a tubular neighborhood \((f_j, (p_j, E_j))\). Since submanifolds \( M_1, \ldots, M_\Lambda \) are disjoint we can always assume that neighborhoods \( U_1 = f_1(E_1), \ldots, U_\Lambda = f_\Lambda(E_\Lambda) \) are also disjoint.

On each \( E_j \) we put a metric \( g_j \), chosen as in Section 1.3. Let \( \tilde{E}_j \subset E_j \) be the subset of all vectors in \( E_j \) with the norm less then 1 and \( f_j(\tilde{E}_j) = \tilde{U}_j \). We choose a smooth non-negative partition of unity \( \{\chi_j\}_{j=0, \ldots, \Lambda} \), such that \( \text{supp}(\chi_j) \subset U_j \) and \( \chi_j = 1 \) on \( \tilde{U}_j, j = 1, \ldots \Lambda \). We define

\[
\chi_0 = 1 - \sum_{j=1}^\Lambda \chi_j = 1 - \chi.
\]

Finally, we put a new metric \( g \) on \( N \), defined by

\[
g = \chi_0 g_0 + \sum_{j=1}^\Lambda \chi_j(f_j^{-1})^* g_j.
\]
Let
\[ L(\alpha) : \Omega^*(N, V) \to \Omega^*(N, V) \]
be the Witten deformation of the Laplacian on \( N \) associated to \( h \) and \( g \). That is
\[ L(\alpha) = d(\alpha)d^*(\alpha) + d^*(\alpha)d(\alpha), \]
where
\[ d(\alpha) = e^{-\alpha h}d e^{\alpha h}, \]
and \( d^*(\alpha) \) is an adjoint of \( d(\alpha) \) with respect to the metric \( g \) on \( N \) and the metric \( q_V \) on \( V \).

Since for any \( p, 0 \leq p \leq \dim N \), and for any \( \alpha \) the self-adjoint extension \( L^p(\alpha) \) to \( L^2 \)-integrable forms on \( N \) is an elliptic self-adjoint differential operator on compact manifold. Then the spectrum of \( L^p(\alpha) \) is discrete. We denote as \( 0 \leq \nu_1^p(\alpha) \leq \nu_2^p(\alpha) \ldots \) and \( \phi_1^p(\alpha), \phi_2^p(\alpha), \ldots \) the eigenvalues (counting multiplicity) and the corresponding orthonormal eigenforms of \( L^p(\alpha) \).

Our goal is to find the asymptotics of the small eigenvalues of \( L(\alpha) \) as \( \alpha \to \infty \).

**8.3. The asymptotics of the small eigenvalues of \( L(\alpha) \).** We start by introducing some new notation. Let
\[ \Box_f^p(\alpha) : \Omega^p(E_j, V) \to \Omega^p(E_j, V), \]
be the Witten Laplacian on \( E_j \) defined by
\[ \Box_f^p(\alpha) = d(\alpha)d^*(\alpha) + d^*(\alpha)d(\alpha), \]
where
\[ d(\alpha) = e^{-\alpha h_j}d e^{\alpha h_j}, \quad h_j = f_j \circ h, \]
and \( d^*(\alpha) \) is an adjoint of \( d(\alpha) \) with respect to the metric \( g_j \) on \( E_j \) and the metric \( q_V \) on the flat vector bundle \( V \) over \( E_j \) (Chapter 1). Here we do not distinguish
between the vector bundle $V$ on $M$ and its push forward under $f$ to a bundle over $E$.

We denote the disjoint union of $E_j$'s as $E$:

$$E = E_1 \cup \cdots \cup E_\Lambda.$$

Then

$$\Box^p(\alpha) = \oplus_{j=1}^\Lambda \Box^p_j(\alpha) : \Omega^p_\ast(E, V) \to \Omega^p_\ast(E, V),$$

where

$$\Omega^p_\ast(E, V) = \oplus_{j=1}^\Lambda \Omega^p_j(E, V).$$

We denote as $0 \leq \lambda^p_1(\alpha) \leq \lambda^p_2(\alpha) \ldots$ the eigenvalues of $\Box^p(\alpha)$ (counting multiplicities). Let $\omega^p_1(\alpha), \omega^p_2(\alpha), \ldots$ be the corresponding orthonormal eigenforms. Then

$$\lambda^p_i(\alpha) = \sum_{j=1}^\Lambda \lambda^p_{l(i), j}(\alpha),$$

where $\lambda^p_{l(i), j}(\alpha)$ denotes the $l(i)$'s eigenvalue in the spectrum of $\Box^p_j(\alpha)$.

Fix $0 \leq \epsilon < 1$. As in Section 7.6 we give

**Definition 8.4.** We say that an eigenvalue $\nu^p_j(\alpha) (\lambda^p_j(\alpha))$ is $\alpha$-small if for $\alpha$ large enough $\nu^p_j(\alpha) < \alpha^\epsilon$ ($\lambda^p_j(\alpha) < \alpha^\epsilon$).

The following theorem is the main result of this chapter. It shows that as $\alpha \to \infty$ the small eigenvalues of the Witten Laplacian $L(\alpha)$ on $N$ localize to the small eigenvalues of the Witten Laplacian $\Box(\alpha)$ on the disjoint union of tubular neighborhoods of the critical submanifolds.

Fix $\epsilon$, where $0 \leq \epsilon \leq 1$.

**Theorem 8.5.** For any large enough $\alpha > 0$ and any $p$, $0 \leq p \leq \dim N$, the number of $\alpha$-small eigenvalues of $L^p(\alpha)$ is equal to the number $\hat{k}^p(\alpha)$ of $\alpha$-small eigenvalues of $\Box^p(\alpha)$. 
Moreover, there exists $C > 0$, such that for $\alpha$ large enough and for all $1 \leq j \leq \hat{k}^p(\alpha)$ we have

\begin{equation}
|\lambda_j^p(\alpha) - \nu_j^p(\alpha)| \leq C \alpha^{-1/2}.
\end{equation}

The proof of this theorem is given in Appendix 4.

Now fix $\varepsilon$ with $0 \leq \varepsilon < 1/2$. We now apply Theorem 7.20 to the setting of this chapter. To do so we need to introduce some new notation. For each critical submanifold $M_j$ of $\text{ind}(M_j) = n_j^-$ and $\dim M_j = m_j$ we denote by $\sigma_p(M_j)$ the spectrum of the standard Laplacian $\Delta^{p-n_j^-}$ on $\Omega^{p-n_j^-}(M_j, V_{M_j} \otimes o(E_j^-))$. This spectrum is only defined for $n_j^- \leq p \leq m_j + n_j^-$. We define $\sigma_p(M_j)$ to be empty otherwise. We denote as $\sigma_p(M)$ the union of $\sigma_p(M_j)$ over all critical submanifolds $M_j$ for which inequality $n_j^- \leq p \leq m_j + n_j^-$ is satisfied. We arrange all numbers from the set $\sigma_p(M)$ in non-decreasing order $0 \leq \mu_1^p \leq \mu_2^p \leq \ldots$. In other words each $\mu_i^p$ is an eigenvalue of the standard Laplacian on $\Omega^{p-n_j^-}(M_j, V_{M_j} \otimes o(E_j^-))$ for some $1 \leq j \leq \Lambda$.

Fix $\alpha > 0$. Let $\{\lambda_i^p(\alpha)\}$ for $1 \leq i \leq \hat{k}^p(\alpha)$ be the set of all $\alpha$-small eigenvalues. By applying Theorem 7.20 to the Witten Laplacian on $E = E_1 \cup \cdots \cup E_\Lambda$ we see that there is a constant $C$, such that

\begin{equation}
|\lambda_i^p(\alpha) - \mu_i^p| \leq C \alpha^{-1/2+\varepsilon},
\end{equation}

for all $1 \leq i \leq \hat{k}^p(\alpha)$.

Inequalities (8.7) and (8.8) prove the following theorem:

**Theorem 8.6.** For any large enough $\alpha \geq 0$ and any $0 \leq p \leq \dim N$ the number $\hat{k}^p(\alpha)$ of $\alpha$-small eigenvalues of $L^p(\alpha)$ is equal to the number $k^p(\alpha)$ of $\alpha$-small eigenvalues from $\sigma_p(M)$.

Moreover, there exists $C > 0$, such that for $\alpha$ large enough and for all $1 \leq j \leq k^p(\alpha)$ we have

\begin{equation}
|\nu_j^p(\alpha) - \mu_j^p| \leq C \alpha^{-1/2+\varepsilon}
\end{equation}
9. ASYMPTOTICS OF THE TRACE OF THE HEAT KERNEL

9.0. Introduction. In this chapter we use Theorem 8.6 and the estimates on the spectral counting function of the Witten Laplacian $L(\alpha)$ to obtain a localization as $\alpha \to \infty$ of the trace of the heat kernel of $L(\alpha)$ along the critical submanifold $M$ of the Morse-Bott function $h$ (Theorem 9.3). The conclusion of Theorem 9.3 is used in the next chapter to prove the Morse-Bott inequalities.

In Section 9.1 we define the counting function for the spectrum of $L(\alpha)$.

In Section 9.2 we separate the trace of the heat kernel into large and small traces, where the large trace is associated to the large eigenvalues of $L(\alpha)$ and the small trace is associated to the small eigenvalues of $L(\alpha)$. Theorem 9.3 follows from estimates on the large trace and the small trace given in lemmas 9.1 and 9.2.

In Section 9.3 we start the proof of Lemma 9.1 by estimating from below the counting function for the spectrum of $L(\alpha)$.

In Section 9.4 we complete the proofs of the lemmas.

9.1. The trace and the counting function. For any fixed $\alpha \geq 0$ the Witten Laplacian

\begin{equation}
L_p(\alpha) = d^*d + dd^* + \alpha^2 |dh|^2 + \alpha A
\end{equation}

is a Laplace type operator on a compact manifold $N$ of dim $N = n$. We recall that a Laplace type operator is a second order partial differential operator on $\Omega^*(N, V)$ with the leading symbol given by the metric tensor [Gilkey, 4.1]. It follows from the discussion of such operators in [Roe, Section 7] that for any $p$, $0 \leq p \leq n = \dim N$, there exists a constant $c_p$ such that asymptotically

\begin{equation}
\nu_j^p(\alpha) \sim c_p j^{2/n} \text{ as } j \to \infty.
\end{equation}

Since the eigenvalues grow at infinity as a positive power of $j$, we can define the trace of the heat kernel associated to $L_p(\alpha)$ by

\begin{equation}
\text{Tr} \left( e^{-tL_p(\alpha)} \right) = \sum_j e^{-t\nu_j^p(\alpha)},
\end{equation}

64
where the sum on the right converges equicontinuously for \( t \geq T > 0 \).

Now we define the counting function \( N^p_\alpha(r) \) for the spectrum of \( L^p(\alpha) \) by

\[
N^p_\alpha(r) = \sum_{\nu^p_\alpha(\alpha) < r} 1.
\]

Since \( L^p(\alpha) \) is a non-negative operator we have \( N^p_\alpha(0) = 0 \). With the help of the counting function we can rewrite (9.3) as an integral with respect to the measure \( dN^p_\alpha(r) \):

\[
\text{Tr} \left( e^{-tL^p(\alpha)} \right) = \int_0^\infty e^{-tr} dN^p_\alpha(r),
\]

where the integration should be understood in the sense of Lebesgue-Stieltjes.

**9.2. The small and the large traces. The main result.** Fix \( \varepsilon > 0 \). Our goal now is to find the asymptotics of \( \text{Tr} \left( e^{-tL^p(\alpha)} \right) \) as \( \alpha \to \infty \). To do so we separate this trace into large and small parts. We compute the asymptotics of each part separately.

We write \( \text{Tr} \left( e^{-tL^p(\alpha)} \right) \) as the sum of small and large traces:

\[
(9.5) \quad \text{Tr} \left( e^{-tL^p(\alpha)} \right) = \text{Tr}_{\text{sm}} \left( e^{-tL^p(\alpha)} \right) + \text{Tr}_{\text{l}} \left( e^{-tL^p(\alpha)} \right), \quad \text{where}
\]

\[
(9.6) \quad \text{Tr}_{\text{sm}} \left( e^{-tL^p(\alpha)} \right) = \sum_{\nu^p_\alpha(\alpha) < \alpha^*} e^{-tr^p_\alpha(\alpha)}, \quad \text{and}
\]

\[
(9.7) \quad \text{Tr}_{\text{l}} \left( e^{-tL^p(\alpha)} \right) = \sum_{\nu^p_\alpha(\alpha) \geq \alpha^*} e^{-tr^p_\alpha(\alpha)} = \int_{\alpha^*}^\infty e^{-tr} dN^p_\alpha(r).
\]

We have the following two lemmas. The first lemma estimates the large trace and the second lemma estimates the small trace:

**Lemma 9.1.** There exists \( C_1 \) such that for any large enough \( \alpha \), any \( \varepsilon > 0 \), any \( p, 0 \leq p \leq \dim N \), and all \( t \geq T > 0 \),

\[
(9.8) \quad \text{Tr}_{\text{l}} \left( e^{-tL^p(\alpha)} \right) \leq C_1 e^{-T\alpha^*/2} \left( 1 + \left( \frac{\alpha}{T} \right)^{n/2} \right).
\]

We combine statements of Lemma 9.1 and Lemma 9.2 into one theorem.
Lemma 9.2. There exist constants $C_2 > 0$, $d$, such that $1 \leq d \leq n$, such that for any large enough $\alpha$, $\epsilon$ such that $0 < \epsilon < 1/4$, any $p$ such that $0 \leq p \leq \dim N$, and all $t$ in the interval $\alpha^{1/4} \geq t \geq T > 0$,

$$|\text{Tr}_{\text{sm}} (e^{-tL^p(\alpha)}) - \sum_{M_j, \quad n_j^+ \leq p \leq m_j + n_j^+} \text{Tr} \left( e^{-t\Delta_j^{p-n_j^-}} \right) | \leq C_2 \left( \alpha^{-1/4+3\epsilon} + e^{-(T\alpha^{s/2})/2}(1 + T^{-d/2}) \right).$$

(9.9)

Theorem 9.3 easily follows from lemmas 9.1 and 9.2.

Theorem 9.3. There exist constants $C > 0$ and $d > 2$ such that for any large enough $\alpha$, any $\epsilon$ such that $0 < \epsilon < 1/4$, any $p$ such that $0 \leq p \leq \dim N$, and all $t$ such that $\alpha^{1/4} \geq t \geq T > 0$ we have

$$|\text{Tr} (e^{-tL^p(\alpha)}) - \sum_{M_j, \quad n_j^+ \leq p \leq m_j + n_j^+} \text{Tr} \left( e^{-t\Delta_j^{p-n_j^-}} \right) | \leq C_2 \left( \alpha^{-1/4+\epsilon} + e^{-(T\alpha^{s/2})/2}(1 + T^{-d/2}) \right) + C_1 e^{-T\alpha^{s/2}} \left( 1 + (\alpha/T)^n/2 \right).$$

In particular, $\text{Tr} (e^{-tL^p(\alpha)})$ converges uniformly for all $t$ such that $T_1 \geq t \geq T > 0$ to

$$\sum_{M_j, \quad n_j^+ \leq p \leq m_j + n_j^+} \text{Tr} \left( e^{-t\Delta_j^{p-n_j^-}} \right)$$

as $\alpha \to \infty$.

9.3. An upper bound on the counting function for the Witten Laplacian. Before we can prove lemmas 9.1 and 9.2 we need to find an upper bound for $N_{\alpha}^p(r)$. Since $A$ is a bounded operator and $\alpha^2|dh|^2$ is non-negative, we can estimate $L^p(\alpha)$ from below. If we choose $C_3 \geq ||A||$, then

$$L^p(\alpha) \geq d^*d + dd^* - \alpha C_3.$$
Let $N^p(r)$ be the counting function for the spectrum of the Laplace type operator $d^* d + dd^*$. Then the counting function for the spectrum of $d^* d + dd^* - \alpha C_3$ is $N^p(r + \alpha C_3)$. From (9.11) we conclude that

$$N^p_\alpha(r) \leq N^p(r + \alpha C_3) \leq N^p(2\alpha r),$$

where the last inequality is valid for $r \geq C_3$ and $\alpha \geq 1$. It follows from the discussion in [Roe, Section 7] that there exists a constant $C_p$ such that

$$N^p(\xi) \sim C(p)\xi^{n/2}, \text{ as } \xi \to \infty.$$  

Therefore

$$N^p(2\alpha r) \sim C(p)(2\alpha r)^{n/2}, \text{ as } r \to \infty.$$  

Moreover, there exists a constant $C_4$ such that $N^p(2\alpha r) \leq C_4(1 + (2\alpha r)^{n/2})$, for any $p, 0 \leq p \leq \dim N$ and all $r \geq 0$. Therefore, we proved the following result:

**Lemma 9.4.** There exists $C_4 > 0$, such that for any $p$ such that $0 \leq p \leq \dim N$, all $r \geq 0$ and $\alpha > 1$ we have

$$N^p_\alpha(r) \leq C_4(1 + (2\alpha r)^{n/2}).$$

**9.4. Proof of Lemmas 9.1 and 9.2.** Lemma 9.4 allows us to estimate the rate of decay of the large part of the trace of $e^{-t L^p(\alpha)}$ as $\alpha \to \infty$.

**Proof of Lemma 9.1.** Writing $e^{-tr}$ as $e^{-tr/2}e^{-tr/2}$, and estimating $e^{-tr/2}$ from below by $e^{-t\alpha^\epsilon/2}$ for $r \geq \alpha^\epsilon$, we have

$$\mathcal{T}_\alpha \left(e^{-t L^p(\alpha)}\right) \leq e^{-t\alpha^\epsilon/2} \int_{\alpha^\epsilon}^{\infty} e^{-tr/2}dN^p_\alpha(r) \leq e^{-t\alpha^\epsilon/2} \int_{0}^{\infty} e^{-tr/2}dN^p_\alpha(r).$$

To estimate $I = \int_{0}^{\infty} e^{-tr/2}dN^p_\alpha(r)$ we integrate by parts, using the fact that $N^p_\alpha(0) = 0$:

$$I = \lim_{r \to \infty} e^{-tr/2}N^p_\alpha(r) - \frac{t}{2} \int_{0}^{\infty} e^{-tr/2}N^p_\alpha(r)dr = -\frac{t}{2} \int_{0}^{\infty} e^{-tr/2}N^p_\alpha(r)dr.$$
After estimating $N^p_{\alpha}(r)$ by $C_4(1 + (2\alpha r)^{n/2})$ inside the integral and changing variables $\xi = tr/2$ we get

\[
I \leq -\frac{t}{2} \int_0^\infty e^{-\xi^2/2} C_4(1 + (2\alpha r)^{n/2}) dr \\
\leq C_4 \left( 1 - 2^n(\alpha/t)^{n/2} \int_0^\infty e^{-\xi^2} d\xi \right) \\
\leq C_4 \left( 1 + 2^n(\alpha/t)^{n/2} \text{const} \right) \\
\leq C_1 \left( 1 + (\alpha/t)^{n/2} \right).
\]

(9.17)

Therefore,

\[
\text{Tr}_a \left( e^{-tL^p_{\alpha}} \right) \leq C_1 e^{-t\alpha^{n/2}} \left( 1 + (\alpha/t)^{n/2} \right).
\]

(9.18)

Since the right hand side of (9.18) is a decreasing function of $t$, we have for $t \geq T > 0$,

\[
\text{Tr}_a \left( e^{-tL^p_{\alpha}} \right) \leq C_1 e^{-T\alpha^{n/2}} \left( 1 + \left( \frac{\alpha}{T} \right)^{n/2} \right),
\]

(9.19)

which is the statement of the lemma.

Proof of Lemma 9.2. We observe that

\[
\text{Tr}_{sm} \left( e^{-tL^p_{\alpha}} \right) = \sum_{i=1}^{k_p(\alpha)} e^{-t\nu^p_i(\alpha)}.
\]

On the other hand by definition of $\{\mu^p_i\}$'s (see Section 8.3):

\[
\sum_{\mathcal{M}_j, n_i^- \leq p \leq m_j + n_i^+} \text{Tr} \left( e^{-t\Delta^p_j - n_i^-} \right) = \sum_{i=1}^\infty e^{-t\mu^p_i}.
\]

(9.20)

We write the difference of two traces as the sum of two terms:

\[
I = \sum_{i=1}^{k_p(\alpha)} e^{-t\nu^p_i(\alpha)} - \sum_{i=1}^\infty e^{-t\mu^p_i} \\
= \sum_{i=1}^{k_p(\alpha)} \left( e^{-t\nu^p_i(\alpha)} - e^{-t\mu^p_i} \right) + \sum_{i=k_p(\alpha)+1}^\infty e^{-t\mu^p_i} \\
= I_1 + I_2.
\]

(9.21)
Since \( t \leq \alpha^{1/4} \) it follows from the inequality (8.9) that for some \( C_3 \geq 0 \),
\[
(9.22) \quad |t(\mu_i^p(\alpha) - \mu_i^p)| \leq C_3 \alpha^{-1/4+\epsilon}, \quad 1 \leq i \leq k_p(\alpha).
\]
Therefore, for any large enough \( \alpha > 0 \)
\[
(9.23) \quad |e^{-t(\mu_i^p(\alpha) - \mu_i^p)} - 1| \leq 2|t(\mu_i^p(\alpha) - \mu_i^p)| \leq 2C_3 \alpha^{-1/4+\epsilon}.
\]

Therefore,
\[
|I_1| = \sum_{i=1}^{k} e^{-t\mu_i^p} \left( e^{-t(\mu_i^p(\alpha) - \mu_i^p)} - 1 \right)
\leq 2C_3 \alpha^{-1/4+\epsilon} \sum_{i=1}^{k_p(\alpha)} e^{-t\mu_i^p}
\leq 2C_3 \alpha^{-1/4+\epsilon} \sum_{i=1}^{\infty} e^{-t\mu_i^p}
\leq 2C_3 C_4 \alpha^{-1/4+\epsilon},
\]
where \( C_4 = \sum_{i=1}^{\infty} e^{-t\mu_i^p} \).

To estimate \( |I_2| \) we note that for large \( \alpha \), \( \alpha^{2\epsilon} \geq \nu_{k_p(\alpha)+1}(\alpha) \geq \alpha^{\epsilon} \). If we change \( \epsilon \) by \( 2\epsilon \) in the inequality (8.9) we get
\[
(9.25) \quad |\nu_{k_p(\alpha)+1}^p(\alpha) - \mu_{k_p(\alpha)+1}^p| \leq C_3 \alpha^{-1/2+2\epsilon}.
\]

From (9.25) it follows that
\[
\mu_{k_p(\alpha)+1}^p \geq \nu_{k_p(\alpha)+1}^p(\alpha) - c \alpha^{-1/2+2\epsilon} \geq \alpha^{\epsilon/2}.
\]

Thus if \( N_{\sigma_p}(r) \) is the counting function for the spectrum \( \sigma_p(M) \) defined in Section 8.3, then
\[
(9.27) \quad I_2 = \sum_{i=k_p(\alpha)+1}^{\infty} e^{-t\mu_i^p} \leq \int_{\alpha^{\epsilon/2}}^{\infty} e^{-tr} dN_{\sigma_p}(r).
\]

Since \( N_{\sigma_p}(r) \) is the counting function for the spectrum of the sum of standard Laplacians on forms, there exist constants \( 1 \leq d \leq n \), \( C > 0 \), such that for all \( r > 0 \) we have an estimate
\[
N_{\sigma_p}(r) \leq C(1 + r^{d/2}).
\]
Finally, the same estimates as in the proof of Lemma 9.1 show that for \( t \geq T > 0 \)

\[
I_2 \leq Ce^{-\left(T\alpha^{-1/2}\right)/2} \left(1 + T^{-d/2}\right).
\]

Therefore, for large enough \( \alpha \)

\[
I \leq 2C_3C_4\alpha^{-1/4+\epsilon} + Ce^{-\left(T\alpha^{-1/2}\right)/2} \left(1 + T^{-d/2}\right).
\]

The statement of the lemma easily follows from this inequality. ■
10. MORSE-BOTT INEQUALITIES

10.0. Introduction. In this chapter we prove the Morse-Bott inequalities as an application of the results of Chapter 9. Morse-Bott inequalities (or degenerate Morse inequalities of R. Bott) [Bott] relate the Betti numbers of a compact manifold $N$ to the Betti numbers of the connected components of the critical submanifold $M$ of a Morse-Bott function $h$ on $N$.

The non-degenerate Morse inequalities are a particular case of the Morse-Bott inequalities when the critical submanifold $M$ is a union of a finite number of critical points.

Our proof repeats the heat equation argument of Bismut in [Bis]. Instead of complicated probability considerations in [Bis] we use the estimates of Chapter 9 on the trace of the heat kernel of the Witten Laplacian $L(\alpha)$ on $N$.

In fact, in order to prove the Morse-Bott inequalities we only need the estimates on the number of zero eigenvalues of $L(\alpha)$ and the existence of the spectral gap separating zero eigenvalues of $L(\alpha)$ from the rest of the spectrum ([CFKS], [Bra-Far]).

10.1. The basic inequalities. In this section we prove the trace inequalities of Bismut, which are an adequate substitution for the well-known trace equality in the heat equation method for the proof of the index theorem in Atiyah, Bott and Patodi [A-B-P].

For any $p$ let

\begin{equation}
B_p = \dim H^p(N, V)
\end{equation}

denote the $p$-th Betti number of the $V$-valued cohomology of $N$. Moreover, for any $0 \leq p \leq n$ and $t > 0$ let

\[ K_p(t, \alpha) = \text{Tr} \, e^{-tL^p(\alpha)}. \]

Then we have the following theorem:
Theorem 10.1. [Bis, Theorem 1.3]. For any $\alpha > 0$, $t > 0$ and $p$ such that $0 \leq p \leq n$, we have

\[
K_p(t, \alpha) - K_{p-1}(t, \alpha) + \cdots + (-1)^p K_0(t, \alpha) \\
\geq B_p - B_{p-1} + \cdots + (-1)^p B_0,
\]

where equality holds for $p = n$.

Proof. It follows from definition (8.2) of $d(\alpha)$ that the map

\[
e^{-\alpha h} : \Omega^*(N, V) \rightarrow \Omega^*(N, V)
\]

induces an isomorphism between $H^*(N, V, \alpha)$, the cohomology of $(\Omega^*(N, V), d(\alpha))$, and $H^*(N, V)$. Therefore, for any $0 \leq p \leq n$,

\[
\dim L^p(\alpha) = B_p.
\]

Let $\nu$ be a positive eigenvalue of $L(\alpha) = \bigoplus_{p=0}^n L^p(\alpha)$. Let $F_\nu(\alpha)$ be the corresponding eigenspace in the set of smooth forms. Then

\[
F_\nu(\alpha) = \bigoplus_{p=0}^n F^p_\nu(\alpha),
\]

where $F^p_\nu(\alpha)$ is the corresponding eigenspace in the space of smooth $p$-forms which are eigenforms for the eigenvalue $\nu$.

We observe that $d(\alpha)$ commutes with $L(\alpha)$ and thus

\[
d(\alpha)F^p_\nu(\alpha) \subseteq F^{p+1}_\nu(\alpha)
\]

Therefore, we have the sequence:

\[
0 \rightarrow F^0_\nu(\alpha) \rightarrow F^1_\nu(\alpha) \rightarrow \cdots \rightarrow F^n_\nu(\alpha) \rightarrow 0.
\]

To see that the sequence (10.6) is exact, we observe that we have the following orthogonal Hodge decomposition:

\[
\Omega^*(N, V) = \text{image } d(\alpha) \oplus \text{image } d^*(\alpha) \oplus \ker L(\alpha).
\]
Since both \( d(\alpha) \) and \( d^*(\alpha) \) commute with \( L(\alpha) \), it follows from (10.7) that for any \( 0 \leq p \leq n \)
\[
F^p_\nu(\alpha) = d(\alpha)F^{p-1}_\nu(\alpha) \oplus d^*(\alpha)F^{p+1}_\nu(\alpha).
\]

Therefore,
\[
(10.8) \quad \ker(d(\alpha) : F^p_\nu(\alpha) \to F^{p+1}_\nu(\alpha)) = \text{image} \ (d^*(\alpha) : F^{p-1}_\nu(\alpha) \to F^p_\nu(\alpha)).
\]

Equality (10.8) proves the exactness of (10.7).

To finish the proof of the lemma, we define \( R^p_\nu(\alpha) \) by
\[
(10.9) \quad R^p_\nu(\alpha) = \dim F^p_\nu(\alpha) - \dim F^{p-1}_\nu(\alpha) + \cdots + (-1)^p \dim F^0_\nu(\alpha).
\]

Since (10.7) is exact, we have \( R^p_\nu(\alpha) \geq 0 \) and \( R^0_\nu(\alpha) = 0 \). Now the left-hand side of (10.9) is given by
\[
B_p - B_{p-1} + \cdots + (-1)^p B_0 + \sum_{\nu \neq 0} e^{-\nu t} R^p_\nu(\alpha).
\]

\[
10.2. \text{The Morse-Bott inequalities.} \text{ To deduce the Morse-Bott inequalities, we would like to take a double limit as } t \to \infty \text{ and } \alpha \to \infty \text{ in the left-hand side of 10.2. It follows from Theorem 9.3 that for any } p, 0 \leq p \leq \dim N, \text{ and any } t \geq 0,
\]
\[
(10.10) \quad \lim_{\alpha \to \infty} K_p(\alpha, t) = \sum_{M_j, \nu_j \leq p \leq m_j + n_j} \text{Tr} \left( e^{-t\Delta_j^{p-n_j^-}} \right).
\]

For every \( j, 0 \leq j \leq \Lambda \), we define the \textit{twisted Betti numbers} \( \tilde{b}_{k,j} \) of the critical submanifold \( M_j \) by
\[
(10.11) \quad b_{k,j} = \dim \ker \left( \partial_j^k : \Omega^k (M_j, V|M_j \otimes \mathcal{O}(E_j^-)) \to \Omega^k (M_j, V|M_j \otimes \mathcal{O}(E_j^-)) \right).
\]

In particular, \( b_{k,j} = 0 \) for \( k > \dim M_j \).
Lemma 10.2. For any \( j, 1 \leq j \leq \Lambda \), and every \( k, 0 \leq k \leq \dim M_j \), we have

\[
\lim_{t \to \infty} \text{Tr} \left( e^{-t \Delta_j^k} \right) = b_{k,j}^-.
\]

(10.12)

Proof. We can rewrite \( \text{Tr} \left( e^{-t \Delta_j^k} \right) \) as

\[
\begin{align*}
\text{Tr} \left( e^{-t \Delta_j^k} \right) &= \sum_{\nu \in \sigma(\Delta_j^k), \nu = 0} e^{-t \nu} + \sum_{\nu \in \sigma(\Delta_j^k), \nu \neq 0} e^{-t \nu} \\
&= b_{k,j}^- + \sum_{\nu \in \sigma(\Delta_j^k), \nu > \tilde{\nu}} e^{-t \nu},
\end{align*}
\]

where \( \tilde{\nu} \) is the first non-zero eigenvalue of \( \Delta_j^k \). The same estimates as in the proof of Lemma 9.2 work to show that

\[
\lim_{t \to \infty} \sum_{\nu \in \sigma(\Delta_j^k), \nu > \tilde{\nu}} e^{-t \nu} = 0. \blackslug
\]

The following theorem is an easy consequence of (10.2), (10.10) and (10.12).

Theorem 10.3. For any \( p \) with \( 1 \leq p \leq \dim N \), the following inequality holds:

\[
\sum_{i=1}^{\Lambda} \left[ b_{p-n_i^-}^{i} - b_{p-n_i^- - 1}^{i} - \cdots - (-1)^{p-n_i^-} b_{0}^{i} \right] \\
\geq B_p - B_{p-1} + \cdots + (-1)^p B_0.
\]

(10.13)

For \( p = n \) equality holds in (10.13).

Let \( P(t) \) be the Poincaré polynomial for \( H^\bullet(N.V) \). For each \( i \), let \( 1 \leq i \leq \Lambda \), \( P_i^-(t) \) be the Poincaré polynomial for \( H^\bullet(M_i, V|_{M_i} \otimes o(E_i^-)) \), the cohomology of \( M_i \) with values in the bundle \( V|_{M_i} \) twisted by the orientation bundle of \( E_i^- \). The Morse-Bott inequalities [Bott] say that there exists a polynomial \( Q(t) \) given by \( Q(t) = Q_0 + Q_1 t + \ldots \) with all non-negative coefficients, such that

\[
\sum_{i=1}^{\Lambda} t^{n_i^-} P_i^-(t) - P(t) = Q(t)(1 + t).
\]

(10.14)
It is an easy observation that (10.13) and (10.14) are equivalent.

In order to recover the non-degenerate Morse inequalities, we assume that all $M_i$’s are critical points and $V$ is a bundle of rank one. Let $m_k$ denote the number of critical points of index $k$. Then, since $b_{p-n_i}^{-} = 1$ if $p = n_i^{-}$ and $b_{p-n_i}^{i-} = 0$ if $p \neq n_i^{-}$, the inequality (10.13) becomes

\begin{equation}
(10.14) \quad m_p - m_{p-1} + \cdots + (-1)^p m_0 \geq B_p - B_{p-1} + \cdots + (-1)^p B_0.
\end{equation}

Inequalities (10.14) are called non-degenerate Morse inequalities ([BFKS]).
APPENDIX 1
DISCRETENESS OF THE SPECTRUM
OF THE WITTEN LAPLACIAN

In this appendix we want to prove the following theorem:

**Theorem 1.1.** For every $\alpha > 0$ and every $p$, $0 \leq p \leq \dim E$, the spectrum of $\Box^p(\alpha)$ is discrete. In particular $\ker \Box^p(\alpha)$ is finite-dimensional. Moreover, the eigenforms of $\Box^p(\alpha)$ form a basis for $\Omega^p_{(2)}(E, V)$ in the $L^2$-topology, associated to the metric $g$ on the tangent bundle $TE$.

The idea of the prove is to show that for some $C > 0$ the operator $\Box(\alpha) + \alpha C$ has compact inverse.

It is useful to recall (see formula (1.5)) that

$$\Box(\alpha) = \Box + \alpha^2 |dh|^2 + \alpha A,$$

where $A$ is a bounded zeroth order operator in the basis chosen in Section 1.3.

We start the proof of the theorem with the following lemma which establishes the existence of the inverse:

**Lemma A.1.1.** Choose a constant $C$ so that $\|A\| < C$. Then the operator $\Box^p(\alpha) + \alpha C$ is positive on the domain $D(\Box^p(\alpha))$ of $\Box^p(\alpha)$. Thus it has a bounded self-adjoint inverse $(\Box^p(\alpha) + \alpha C)^{-1}$.

**Proof of Lemma A.1.1.** Let $\omega \in D(\Box^p(\alpha))$, then $d\omega \in \Omega^p_2(E, V)$ and $d^*\omega \in \Omega^p_2(E, V)$, (see [Bra, Lemma 1]).

Let $\phi(s) : \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function defined by $\phi(s) = 1$ for $0 \leq s \leq 1$, $\phi(s) = 0$ for $s > 2$. We define $J_t(y) : E \to \mathbb{R}$ by $J_t(y) = \phi(|y|/t)$, where $|y|$ is the Euclidean norm of $y \in E$.

We want to show that for any $\omega \in D(\Box^p(\alpha))$, $\langle \Box \omega, \omega \rangle \geq 0$. We write

$$\langle \Box \omega, \omega \rangle = \langle J_t \Box \omega, \omega \rangle + \langle (1 - J_t) \Box \omega, \omega \rangle.$$  \hspace{2cm} (A.2.1)

We estimate

$$|\langle (1 - J_t) \Box \omega, \omega \rangle| \leq \|\Box \omega\| \|(1 - J_t) \omega\| < \text{const} \|\Box \omega\| \|(1 - J_t) \omega\|.$$  \hspace{2cm} (A.2.2)
Since $\omega \in \Omega_2^r(E, V)$, we have $|((1 - J_t) \Box \omega, \omega)| \to 0$ as $t \to \infty$.

To estimate $\langle J_t \Box \omega, \omega \rangle$ we integrate by parts to get

$$
\langle J_t \Box \omega, \omega \rangle = \int_E J_t \Box \omega \wedge *\omega
$$

$$
= \int_E J_t (d\omega \wedge *d\omega + d^* \omega \wedge *d^* \omega) + \int_E dJ_t (\omega \wedge *d\omega + d^* \omega \wedge *\omega)
$$

$$
= \langle J_t d\omega, d\omega \rangle + \langle dJ_t d^* \omega, d^* \omega \rangle + \langle dJ_t \omega, d\omega \rangle - \langle dJ_t \omega, d^* \omega \rangle.
$$

Since $J_t \geq 0$, both $\langle J_t d\omega, d\omega \rangle$ and $\langle J_t d^* \omega, d^* \omega \rangle$ are non-negative. Moreover

$$
|\langle dJ_t \omega, d\omega \rangle| \leq \|dJ_t \omega\| d\omega \leq (\text{const}) t^{-1}
$$

and similarly

$$
|\langle dJ_t \omega, d^* \omega \rangle| \leq (\text{const}) t^{-1}.
$$

Therefore

$$
\langle J_t \Box \omega, \omega \rangle \geq -2(\text{const}) t^{-1}

$$

(A.2.3)

Thus combining (A.2.1), (A.2.2), and (A.2.3) together we get

$$
\langle \Box \omega, \omega \rangle \geq -\langle (1 - J_t) \Box \omega, \omega \rangle - (\text{const}) t^{-1}.
$$

Taking the limit as $t \to \infty$, we will have $\langle \Box \omega, \omega \rangle \geq 0$. Now we get the statement of the lemma if we choose $C$ so that $\alpha A + \alpha C > 0$. ■

We have the following general result of E. Bueler [Bu, Theorem 4.5] which applies to our situation.

**Theorem A.1.2 (Bueler).** Let $x \to W(x)$ be a continuous map on a complete Riemannian manifold $N$ with the image $W(x)$ a symmetric (zeroth order) operator on $\Lambda^* T^*_x \otimes V$. Assume that

$$
H = \Box + W
$$

is non-negative and essentially self-adjoint with core $\Omega_2^r(E, V)$. Let $\nu(x)$ be the smallest eigenvalue of $W(x)$ on $\Lambda^* T^*_x \otimes V$. Assume that $\nu(x) \to \infty$, i. e. for each $N \geq 0$, there exists $R \geq 0$ such that $\nu(x) \geq N$ if $x \in N/B(R)$. Then $H$ has compact resolvent.
Corollary A.1.3.

\((\Box^p(\alpha) + \alpha C)^{-1} : \Omega_2^p(E; V) \to \Omega_2^p(E; V)\)

is compact.

The statement of Theorem 1.1 now follows from this corollary and from general properties of compact operators.
APPENDIX 2
BISMUT CONNECTION AND BOUNDS
ON THE WITTEN LAPLACIAN

A.2.0. Introduction. In this appendix we define and study the Bismut connection on the tangent space $TE$ of a vector bundle $E$, which we consider as a non-compact manifold. This connection is defined as a direct sum of a chosen Euclidean connection on $E$ and the Levi-Civita connection on the base $M$. The Bismut connection is a more natural choice of a connection for our purposes than the Levi-Civita connection on $TE$ because (as it will be seen in Section A.2.1) the Bismut connection preserves the decomposition of $TE$ into horizontal and vertical subspaces. The drawback of the Bismut connection is that it has a non-trivial torsion.

The Bismut connection was introduced in [Bis]. It is studied in more detail in [B-G-V].

In Section A.2.1 we define the Bismut connection and compute its torsion and curvature tensors.

In Section A.2.2 we choose a basis on $TE$ and express the differential $d$ in terms of the Bismut connection. In order to simplify proofs we assume in sections A.2.2 and A.2.3 that the bundle $V$ is a one-dimensional trivial bundle.

In Section A.2.3 we give estimates on the Witten Laplacian $\Box(\alpha)$ by computing $\Box(\alpha)$ in terms of the Bismut connection. These estimates are used in Chapter 7.

A.2.1. The Bismut connection. The curvature and the torsion of the Bismut connection. Let $\nabla^E$ be a Euclidean connection on $E$ chosen in Section 1.3, and let $\nabla^M$ be the Levi–Civita connection on $TM$. Then we define the Bismut connection $\tilde{\nabla}$ on $TE$ by

$$\tilde{\nabla} = \nabla^E \oplus \nabla^M.$$  

(A.2.1)

For any $y \in E$, in order to parallel translate vector $X \in T_y E$ along the path $\gamma : [0,1] \to E$ with $\gamma(0) = y$, we identify $X$ with the pair $(X^{\text{ver}}, X^{\text{hor}}) \in A \oplus B$,  

79
where $A \oplus B$ is identified in turn with $E \oplus T_{p(y)}M$. Then the result of the parallel translation will be the pair $(\tilde{X}^\text{ver}, \tilde{X}^\text{hor}) \in E \oplus T_{p(y)}M$, where $\tilde{X}^\text{ver}$ is the result of the parallel translation of $X^\text{ver}$ along $p\gamma$ with respect to $\nabla^E$ and $\tilde{X}^\text{hor}$ is the result of the parallel translation of $X^\text{hor}$ along $p\gamma$ with respect to $\nabla^M$.

For $X, Y \in TE$ we denote as $\bar{T}(X, Y)$ the value of the torsion tensor $\bar{T}$ of $\bar{\nabla}$ on the vectors $X, Y$. Similarly, we denote as $\bar{L}(X, Y)$ the value of the curvature tensor $\bar{L}$ of $\bar{\nabla}$ on vectors $X, Y$. By definition

\begin{align}
\bar{T}(X, Y) &= \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y], \\
\bar{L}(X, Y) &= \bar{\nabla}_X \bar{\nabla}_Y Y - \bar{\nabla}_Y \bar{\nabla}_X X - \bar{\nabla}_{[X, Y]}.
\end{align}

We denote as $L$ and $R$ respectively the curvature tensors associated to $\nabla^E$ and $\nabla^M$. That is, $R$ is just the Levi-Civita curvature of $TM$ and $L \in \Omega^2(M, \text{Hom}(E, E))$ is defined for $X \in TM, Y \in TM$ by

\begin{align}
L(X, Y) &= \nabla^E_X \nabla^E_Y \bar{\nabla}^E_Y \nabla^E_X - \nabla^E_{[X, Y]}.
\end{align}

Then we have the following lemma.

**Lemma A.2.1** ([Bis, Theorem 2.1]. The metric $g$ on $TE$ is parallel for $\bar{\nabla}$. Moreover if $X, Y, Z \in T_yE$, then

\begin{align}
\bar{T}(X, Y) &= [L(X^\text{hor}, Y^\text{hor})y] = -[X, Y]^\text{ver}, \\
\bar{L}(X, Y) &= [L(X^\text{hor}, Y^\text{hor})Z^\text{ver}] + [R(X^\text{hor}, Y^\text{hor})Z^\text{hor}].
\end{align}

**A.2.2. A choice of basis. The expression of a differential in terms of the Bismut connection.** Take $x \in M$. Let $\{a_i\}_{i=1, \ldots, n}, \{b_j\}_{j=1, \ldots, m}$ be orthogonal bases of $E_x, T_xM$. Let $\{a^i\}_{i=1, \ldots, n}, \{b^j\}_{j=1, \ldots, m}$ be the corresponding dual bases. Take $y \in E_x$. We can lift $\{a_i\}_{i=1, \ldots, n}, \{b_j\}_{j=1, \ldots, m}$ to $TE$. Since there is no risk of
confusion we can assume as well that \{a_i\}_{i=1,\ldots,n} is the basis of \(A_y\) and \{b_j\}_{j=1,\ldots,m}

is the basis of \(B_y\).

From the definition of the Bismut connection it follows that for all \(i\) and \(j\)

\[(A.2.7) \quad \tilde{\nabla}_{a_i}a_j = 0 \text{ and } \tilde{\nabla}_{a_i}b_j = 0.\]

In addition, we can choose the \(a\) vertical basis to be parallel in the horizontal direction. Then for all \(i\) and \(j\)

\[(A.2.8) \quad \tilde{\nabla}_{b_i}a_j = 0.\]

Now we describe the formulas for the operators \(d\) and \(d^*\) in terms of the Bismut connection. In order to simplify the computations we assume that the bundle \(V\) is a trivial one-dimensional bundle over \(N\).

We denote by \(i(v)\) the operator of interior multiplication by the vector \(v\). Then we have the following lemma:

**Lemma A.2.2** ([Bis, Proposition 2.2.]). If \(x \in M\) and if \(y \in E_x\), we have

\[(A.2.8) \quad d = a^j \tilde{\nabla}_{a_j} + b^j \tilde{\nabla}_{b_j} + \frac{1}{2} b^k \wedge b^l i([L(b_k, b_l)y]),\]

\[(A.2.9) \quad d^* = -a^j \nabla_{a_j} - b^j \nabla_{b_j} - \frac{1}{2} [L(b_k, b_l)y]i(b_k)i(b_l).\]

**Corollary A.2.3.**

\[(A.2.10) \quad d^{1,0} = a^j \nabla_{a_j}, \quad (d^{1,0})^* = -i(a_j)\nabla_{a_j},\]

\[(A.2.11) \quad d^{0,1} = b^j \nabla_{b_j}, \quad (d^{0,1})^* = -i(b_j)\nabla_{b_j},\]

\[(A.2.12) \quad d^{-1,2} = \frac{1}{2} b^k \wedge b^l i([L(b_k, b_l)y]^{ver}), (d^{-1,2})^* = -\frac{1}{2} [L(b_k, b_l)y]i(b_k)i(b_l).\]
Corollary A.2.4.

\[ d^{2,-1} = (d^{2,-1})^* = 0. \]

A.2.3. Some estimates for the Witten Laplacian. In order to estimate the adiabatic deformation of the Witten Laplacian we explicitly compute \( \hat{\Box}(\delta) \). Let \( \Box^{a,b} \) denote \( d^{a,b}(d^{a,b})^* + (d^{a,b})^* d^{a,b} \), then

\[ \hat{\Box}(\delta) = \hat{\Box}^{1,0} + \delta^2 \Box^{0,1} + \delta^4 \Box^{-1,2} \]

\[ + \delta((d^{1,0}(d^{0,1})^* + (d^{0,1})^* d^{1,0} + (d^{1,0})^* d^{0,1} + d^{0,1}(d^{1,0})^*)) + \]

\[ + \delta^2(d^{1,0}(d^{-1,2})^* + (d^{-1,2})^* d^{1,0} + (d^{1,0})^* d^{-1,2} + d^{-1,2}(d^{1,0})^*) + \]

\[ + \delta^3(d^{-1,2}(d^{-1,2})^* + (d^{-1,2})^* d^{0,1} + (d^{0,1})^* d^{-1,2} + d^{-1,2}(d^{0,1})^*) , \]

where \( \hat{\Box}^{1,0} = \Box^{1,0} + |dh|^2 + A \). In our computation we used the fact that the multiplication by \( e^{-h} \) commutes with \( d^{0,1} \) and \( d^{-1,2} \). To simplify notation we will write

\[ \hat{\Box}(\delta) = \hat{\Box}^{1,0} + \delta^2 \Box^{0,1} + \delta^4 \Box^{-1,2} + \delta K_1 + \delta^2 K_2 + \delta^3 K_3, \]

Lemma A.2.5.

(1) \( d^{1,0}(d^{0,1})^* + (d^{0,1})^* d^{1,0} = 0, (d^{1,0})^* d^{0,1} + d^{0,1}(d^{1,0})^* = 0. \)

(2) The operator \( K_2 \) is bounded zeroth order.

(3) The operator \( \Box^{-1,2} \) is a zeroth order operator. Moreover for any \( \omega \in \Omega_s^*(E, V) \), \( |\Box^{-1,2} \omega| \leq C |y|^2 \).

Proof. To prove part (1) it is enough to show that \( (d^{1,0})^* d^{0,1} + d^{0,1}(d^{1,0})^* = 0. \) We use Corollary A.2.3 to write

\[ (d^{1,0})^* d^{0,1} + d^{0,1}(d^{1,0})^* = -i(a_j) \bar{\nabla}_{a_j} b^k \bar{\nabla}_{b_k} - b^k \bar{\nabla}_{b_k} i(a_j) \bar{\nabla}_{a_j} \]

\[ = -i(a_j) b^k \bar{\nabla}_{a_j} \bar{\nabla}_{b_k} - b^k i(a_j) \bar{\nabla}_{b_k} \bar{\nabla}_{a_j} \]

\[ = -(i(a_j) b^k + b^k i(a_j)) \bar{\nabla}_{b_k} \bar{\nabla}_{a_j} = 0. \]

To get from the second to the third line in the formula above we use (A.2.7) and (A.2.8) To get the last line in the formula we use that \( \bar{L}(a_j, b_k) = 0 \). Therefore, \( \bar{\nabla}_{a_j} \bar{\nabla}_{b_k} = \bar{\nabla}_{b_k} \bar{\nabla}_{a_j} \). Moreover, \( i(a_j) b^k - b^k i(a_j) = 0. \)
To prove part (2) we explicitly compute $K_2$ in terms of the Bismut connection. From [Bis, Proposition 2.6] we have

$$\begin{align*}
  2 &= a^j \land [L(b_k, b_l)a_j]i(b_k)i(b_l) + i(a_j)i([L(b_k, b_l)a_j])b_k \land b_l.
\end{align*}$$

(A.2.15)

The statement of part (2) of the lemma then follows from the fact that $[L(b_k, b_l)a_j]$ is bounded.

To prove part (3) we explicitly compute $\Box^{-1,2}$ in terms of the Bismut connection. From [Bis, Proposition 2.6] we have

$$\begin{align*}
  \Box^{-1,2} &= -\frac{1}{4} (b^k \land b^l i([L(b_k, b_l)y]) [L(b_k', b_l') y] i(b_k') i(b_l'))
  
  + [L(b_k', b_l') y] i(b_k') i(b_l') b^k \land b^l i([L(b_k, b_l)y])).
\end{align*}$$

(A.2.16)

The statement then follows from the fact that $[L(b_k, b_l)y]$ is linear in $y$. Therefore, we have an estimate $|[L(b_k, b_l)y]| \leq c|y|$ which becomes an estimate on $\Box^{-1,2}$. □

**Corollary A.2.6.** There exist constants $c_1$ and $c_2$ such that

$$\begin{align*}
  \Box^{-1,2} \leq c_1 \Box^{1,0} + c_2.
\end{align*}$$

(A.2.17)

**Proof.** The proof of the corollary is the following string of inequalities:

$$\begin{align*}
  \Box^{-1,2} \leq c|y|^2 \leq c_1 |dh|^2 \leq c_1 \Box^{1,0} + c_2., \quad □
\end{align*}$$

Next we estimate the operator $\Box(\delta)$ from below in the following theorem:

**Theorem A.2.7.** There exists a constant $C > 0$ such that for all $\delta$ small enough

$$\begin{align*}
  \Box(\delta) \geq \frac{1}{2} \Box^{1,0} + \delta^2 (\Box^{0,1} - C) \geq \delta^2 (\Box^{1,0} + \Box^{0,1} - C).
\end{align*}$$

(A.2.18)

**Proof.** We observe that $\Box^{-1,2}$ is non-negative. Moreover, by Lemma A.2.5 the operator $K_1$ equals to zero, and $K_2$ is a bounded zeroth order operator. Therefore, there exists $c$, so that

$$\begin{align*}
  \Box(\delta) \geq \Box^{1,0} + \delta^2 \Box^{0,1} + \delta^3 K_3 - c\delta^2.
\end{align*}$$

(A.2.19)
To estimate $K_3$ we observe that for any $\omega \in \Omega_*(E, V)$ we have the inequality:

$$\langle \delta^3 K_3 \omega, \omega \rangle = 2\langle \delta^{3/2} d^{0.1} \omega, \delta^{3/2} d^{-1.2} \omega \rangle + 2\langle \delta^{3/2} (d^{0.1})^* \omega, \delta^{3/2} (d^{-1.2})^* \omega \rangle,$$

so that

(A.2.20) \[ |\langle \delta^3 K_3 \omega, \omega \rangle| \leq \delta^3 \langle \square^{0.1} \omega, \omega \rangle + \delta^3 \langle \square^{-1.2} \omega, \omega \rangle > \]

Finally, by Corollary A.2.6

(A.2.21) \[ \langle \square^{-1.2} \omega, \omega \rangle \leq c_1 \langle \square^{0.1} \omega, \omega \rangle + c_2 \langle \omega, \omega \rangle. \]

Thus, for $\delta \leq 1/2$

(A.2.22) \[ \hat{\square}(\delta) \geq \hat{\square}^{1.0} + \delta^2 \hat{\square}^{0.1} - c_3 \delta^3 \hat{\square}^{1.0} - C \delta^2 \]

\[ \geq 1/2 \hat{\square}^{1.0} + \delta^2 (\hat{\square}^{0.1} - c). \]

Inequality (A.2.22) is the statement of the theorem. ■
APPENDIX 3
SPACE OF RAPIDLY DECREASING FORMS

A.3.0. Introduction. It is useful to recall in a slightly different form the definition of the space of rapidly decreasing forms from Section 1.8. For any choice of numbers $a \geq 0$ and $l, l = 0, 1, 2, \ldots$, we define the space

$$S_{a,l} = \{ \omega \in \Omega^\bullet(E, V) \mid \| (1 + |y|)^a (\tilde{\nabla})^\kappa \omega \| \leq \infty, \ |\kappa| = 0, 1, 2, \ldots, l \},$$

where $\tilde{\nabla}$ is the Bismut connection, defined in the Section 1.4, and

$$(\tilde{\nabla})^\kappa = \tilde{\nabla}i_{i_1} \circ \cdots \circ \tilde{\nabla}i_{i_{|\kappa|}},$$

where $\kappa = \{i_1, \ldots, i_{|\kappa|}\}$ is a multi-index. Here $\{e_1, \ldots, e_{m+n}\} = \{a_1, \ldots, a_n\} \cup \{b_1, \ldots, b_m\}$ is the basis of $TE$.

We say that $\omega$ belongs to the space of rapidly decreasing forms $\Omega^\bullet_s(E, V)$ if $\omega \in S_{a,l}$ for all $a \geq 0$ and $l \geq 0$. We denote as $S_{\infty,l}$ an intersection of $S_{a,l}$ for all $a \geq 0$. This is the same definition as in the Section 1.9. In this notation $\Omega^\bullet_s(E, V) = S_{\infty,\infty}$.

The space of rapidly decreasing forms is very convenient to work with since, on one hand, for large $\alpha > 0$ the eigenforms of $\Box(\alpha)$ are rapidly decreasing forms. On the other hand, $\Omega^\bullet_s(E, V)$ is invariant under operators $d^{1,0}, d^{0,1}, d^{-1,2}$ and under exterior multiplication by $dh$. In particular, $\Omega^\bullet_s(E, V)$ is also invariant under $d(\alpha)$ and $d^*(\alpha)$.

The goal of this appendix is to prove the following two theorems:

**Theorem 1.4.** For any large enough $\alpha$ and any $\omega \in \Omega^\bullet(E, V, \alpha)$, such that $\Box(\alpha)\omega = \lambda(\alpha)\omega$, we have $\omega \in \Omega^\bullet_s(E, V)$; i.e. the eigenforms of $\Box(\alpha)$ are rapidly decreasing forms.

**Theorem A.3.1.** The space of rapidly decreasing forms is invariant under operators $d^{1,0}, d^{0,1}, d^{-1,2}, dh\wedge$, and their adjoints. In particular, $\Omega^\bullet_s(E, V)$ is invariant under $d(\alpha)$ and $d^*(\alpha)$.

We will prove Theorem A.3.1 in Section A.3.1. The proof of Theorem 1.4 is contained in Section A.3.2.

85
A.3.1. Proof of Theorem A.3.1. To simplify the computations we assume that the vector bundle $V$ is a trivial one-dimensional vector bundle.

We recall from Appendix 2 (Corollary A.2.3) the expression of the components of the differential $d$ in terms of the Bismut connection.

(A.3.1) \[ d^{1,0} = a^j \nabla_{a_j}, \quad (d^{1,0})^* = -i(a_j) \nabla_{a_j}, \]

(A.3.2) \[ d^{0,1} = b^j \nabla_{b_j}, \quad (d^{0,1})^* = -i(b_j) \nabla_{b_j}, \]

(A.3.3) \[ d^{-1,2} = \frac{1}{2} b^k \wedge b^l i([L(b_k, b_l) y] \omega^r), \]

(A.3.4) \[ (d^{-1,2})^* = -\frac{1}{2} [L(b_k, b_l) y] i(b_k) i(b_l). \]

By definition, the space of rapidly decreasing forms is invariant under $\nabla^i$. Therefore, we only need to show that $\Omega^*_i(E, V)$ is invariant under exterior multiplications by the basis elements $\{e^j\}$ of $T^*E$, by $dh$, by $L(b_k, b_l) y$, and by all the adjoints of those operators.

Let $\omega \in \Omega^*_i(E, V)$. As an example we show that $e^j \wedge \omega \in \Omega^*_i(E, V)$. Since $\|e^j \wedge \omega\| = \|\omega\|$, and the operator $e^j \wedge$ commutes with the multiplication by $(1 - |y|)^a$, we conclude that $e^j \wedge \omega \in S_{\infty,0}$ for all $j = 1, \ldots, m + n$.

To show that $e^j \wedge \omega \in S_{\infty, \infty}$ we will use the simultaneous (for all $j$) induction in $l$. The case $l = 0$ is settled above.

Assume that $e^j \wedge \omega \in S_{\infty,l}$ for all $j = 1, \ldots, m + n$.

We want to show that $e^j \wedge \omega \in S_{\infty, l+1}$ for all $j = 1, \ldots, m + n$. It is equivalent to show that for any multi-index $\kappa$, $|\kappa| = l + 1$, we have $\nabla^\kappa(e^j \wedge \omega) \in S_{\infty,0}$. We write

(A.3.5) \[ \nabla^\kappa(e^j \wedge \omega) = e^j \wedge \nabla^\kappa \omega + [\nabla^\kappa, e^j] \wedge \omega, \]

where $[\nabla^\kappa, e^j] = \nabla^\kappa e^j - e^j \nabla^\kappa$ is the commutator. Since $\nabla^\kappa \omega \in \Omega^*_i(E, V)$, it follows that $e^j \wedge \nabla^\kappa \omega \in S_{\infty,0}$.

The crucial step is to show that $[\nabla^\kappa, e^j] \wedge \omega \in S_{\infty,0}$. This step easily follows from the following lemma.
Lemma A.3.2. The commutator $[\bar{\nabla}^\kappa, e^j]$ can be represented as

\[(A.3.6) \quad [\bar{\nabla}^\kappa, e^j] = e^k \wedge P_{\kappa,j,k}(\bar{\nabla}, y),\]

where each $P_{\kappa,j,k}(\bar{\nabla}, y)$ is a polynomial in $y$ and $\bar{\nabla}$ of the form

\[\sum_{\beta, \gamma, |\gamma| \leq l} C_{\beta, \gamma}(x) y^\beta \bar{\nabla}^\gamma\]

Here $\beta$ and $\gamma$ are multi-indexes, and coefficients $C_{\beta, \gamma}(x)$ depend only on the coordinate on the base.

The statement of the lemma above can be concluded from the calculation of the commutator $[\bar{\nabla}^\kappa, e^j]$ in local coordinates. Local expressions for the Bismut connection and the basis elements are all linear in $y$. Moreover, the transition functions between the coordinate neighborhoods $U \times \mathbb{R}^n$ which cover the bundle $E$ are also linear in $y$.

A.3.2. Proof of Theorem 1.4. We start by observing that the local elliptic estimates imply that for any $\alpha$ all eigenforms of the Witten Laplacian $\Box(\alpha)$ are smooth.

The idea of the proof of Theorem 1.4 is to show by means of elliptic estimates that for large $\alpha$ the powers of the Witten Laplacian control the powers of $|y|$ and the powers of covariant derivatives with respect to the Bismut connection.

The following lemma is the first step in the proof of Theorem 1.4.

Lemma A.3.3. For any large enough $\alpha > 0$ and any $\omega \in \Omega^*(E, V, \alpha)$, such that $\Box(\alpha)\omega = \lambda(\alpha)\omega$, we have $\omega \in S_{\infty,0}$, $d(\alpha)\omega \in S_{\infty,0}$, and $d^*(\alpha)\omega \in S_{\infty,0}$.

Proof. We will start the proof by recalling the definition of the family $\{J_t \mid t \geq 0\}$ of cut-off functions on $E$. This family was used first in Appendix 1.

Let $\phi(s) : \mathbb{R} \to \mathbb{R}$ be a smooth cut-off function defined by $\phi(s) = 1$ for $0 \leq s \leq 1$, $\phi(s) = 0$ for $s > 2$. We define the family $J_t(y) : E \to \mathbb{R}$ by $J_t(y) = \phi(|y|/t)$, where $|y|$ is the Euclidean norm of $y \in E$. Then for any $t > 0$ and any form $\omega \in \Omega^*(E, V, \alpha)$ of norm one, such that $\Box(\alpha)\omega = \lambda(\alpha)\omega$, we have

\[(A.3.7) \quad \lambda(\alpha) = \langle J_t|y|^{2\alpha} \Box(\alpha)\omega, \omega \rangle.\]
After substituting $\Box(\alpha) = \Box + \alpha^2 |d\Omega|^2 + \alpha A$ into (A.3.7), we have

$$\lambda(\alpha) = \langle J_t | y|^{2a} \Box \omega, \omega \rangle + \alpha^2 \langle J_t | y|^{2a} |d\Omega|^2 \omega, \omega \rangle + \alpha \langle J_t | y|^{2a} A \omega, \omega \rangle. \tag{A.3.8}$$

First we put $a = 0$ in (A.3.8). The scalar product $\langle J_t | y|^{2a} \Box \omega, \omega \rangle$ is non-negative. Moreover, the operator $A$ is bounded and there exists $c > 0$ such that $|d\Omega|^2 > c|y|^2$. Thus there exists $C = C(\alpha)$, such that for all $t \geq 0$,

$$c||J_t^{1/2} y||^2 \leq \alpha^2 \langle J_t | y|^{2a} |d\Omega|^2 \omega, \omega \rangle \leq \lambda(\alpha) - \alpha \langle J_t A \omega, \omega \rangle \leq C(\alpha).$$

Taking the limit as $t \to \infty$ in the inequality above, we conclude that $\omega \in S_{0,1}$.

Next we integrate $\langle J_t | y|^{2a} \Box \omega, \omega \rangle$ by parts to get

$$\lambda(\alpha) = \|J_t^{1/2} | y|^{a} d\omega \|^2 + \|J_t^{1/2} | y|^{a} d^* \omega \|^2 + \langle J_t \wedge | y|^{2a} \omega, d\omega \rangle + \langle J_t \wedge | y|^{2a} \omega, d^* \omega \rangle + 2\alpha \langle J_t | y|^{2a-1} |d(|y|)\omega, d\omega \rangle + 2\alpha \langle J_t | y|^{2a-1} |d(|y|)\omega, d^* \omega \rangle$$

$$+ \alpha^2 \langle J_t | y|^{2a} |d\Omega|^2 \omega, \omega \rangle \tag{A.3.9}$$

After differentiating $J_t | y|^{2a}$ and writing the result of the substitution of (A.3.9) into (A.3.8) in the scalar product notation we will get

$$\lambda(\alpha) = \|J_t^{1/2} | y|^{a} d\omega \|^2 + \|J_t^{1/2} | y|^{a} d^* \omega \|^2 + \alpha^2 \langle J_t^{1/2} | y|^{a+1} \omega \rangle \geq \alpha^2 \langle J_t | y|^{2a+2} \omega, \omega \rangle. \tag{A.3.11}$$

Since there exists $c > 0$ such that $|d\Omega|^2 > c|y|^2$, we have

$$\lambda(\alpha) = \|J_t^{1/2} | y|^{a} d\omega \|^2 + \|J_t^{1/2} | y|^{a} d^* \omega \|^2 + \alpha^2 \langle J_t^{1/2} | y|^{a+1} \omega \rangle \geq \alpha^2 \langle J_t | y|^{2a+2} \omega, \omega \rangle. \tag{A.3.12}$$

After substituting (A.3.11) into (A.3.10) and taking into consideration the boundedness of $J_t$, $dJ_t$ and $d|y|$, we conclude that there exists $C = C(\alpha, a)$, such that for all $t > 0$, $\alpha \geq 1/2$, we will have

$$\|J_t^{1/2} | y|^{a} d\omega \|^2 + \|J_t^{1/2} | y|^{a} d^* \omega \|^2 + \alpha^2 \|J_t^{1/2} | y|^{a+1} \omega \|^2 \leq C(\|J_t^{1/2} | y|^{a} \omega \|^2 + \|y|^{a+1/2} d\omega \|_{supp(dJ_t)} \|y|^{a+1/2} \omega \|_{supp(dJ_t)}$$

$$+ \|y|^{a-1/2} d^* \omega \|_{supp(dJ_t)} \|y|^{a+1/2} \omega \|_{supp(dJ_t)}$$

$$+ \|J_t^{1/4} | y|^{a-1/2} \omega \|_{supp(dJ_t)} \|J_t^{1/4} | y|^{a-1/2} d\omega \| + \|J_t^{1/4} | y|^{a-1/2} \omega \|_{supp(dJ_t)} \|J_t^{1/4} | y|^{a-1/2} d^* \omega \|. \tag{A.3.12}$$
Now we use (A.3.12) to prove the lemma. First, we assume that \( a = 1/2 \). Since we already know that \( \omega \in S_{1,0} \), \( d\omega \in S_{0,0} \), and \( d^*\omega \in S_{0,0} \), we see that the right-hand side of (A.3.7) is bounded by a constant, which is independent on \( t \). After taking a limit as \( t \to \infty \), we conclude that \( \omega \in S_{3/2,0} \), \( d\omega \in S_{1/2,0} \), and \( d^*\omega \in S_{1/2,0} \). At each step we can conclude that if \( \omega \in S_{a+1/2,0} \), \( d\omega \in S_{a-1/2,0} \), and \( d^*\omega \in S_{a-1/2,0} \), then \( \omega \in S_{a+1,0} \), \( d\omega \in S_{a,0} \), and \( d^*\omega \in S_{a,0} \). By iterating this process we deduce the statement of the lemma. ■

Our next goal is to get an estimate of norms of covariant derivatives with respect to the Bismut connection in terms of the Witten Laplacian.

**Lemma A.3.4.** For large enough \( \alpha > 0 \) and for any form \( \omega \) in the domain of \( \Box(\alpha) \), there exists \( C = C(\alpha) > 0 \) such that the following elliptic estimate holds:

(A.3.13) \[
\sum_{i=1}^{n} \|\nabla_{a_i}\omega\|^2 + \sum_{j=1}^{n} \|\nabla_{b_j}\omega\|^2 \leq C(\Box(\alpha)\omega, \omega + \|\omega\|^2)
\]

**Proof.** We assume for simplicity that a form \( \omega \) has compact support. If \( \omega \) does not have compact support, then all calculations should be done for \( J_t\omega \). In this case as in Lemma A.3.3, we can take a limit as \( t \to \infty \).

After substituting \( \Box(\alpha) = \Box + \alpha^2|h|^2 + \alpha A \) into \( \langle \Box(\alpha)\omega, \omega \rangle \) and integrating by parts we get

(A.3.14) \[
\langle d\omega, d\omega \rangle + \langle d^*\omega, d^*\omega \rangle + \alpha^2 \langle |d|^2\omega, \omega \rangle + \alpha \langle A\omega, \omega \rangle = \langle \Box(\alpha)\omega, \omega \rangle
\]

We substitute \( d^{1,0} + d^{0,1} + d^{-1,2} \) for \( d \) and \( (d^{1,0})^* + (d^{0,1})^* + (d^{-1,2})^* \) for \( d^* \) in (A.3.14). After integrating by parts, we have

(A.3.15) \[
\langle \Box^{1,0}\omega, \omega \rangle + \langle \Box^{0,1}\omega, \omega \rangle + \langle \Box^{-1,2}\omega, \omega \rangle + \text{cross-terms} + \alpha^2 \langle |d|^2\omega, \omega \rangle + \alpha \langle A\omega, \omega \rangle = \langle \Box(\alpha)\omega, \omega \rangle
\]

We observe that according to Lemma A.2.5 the cross-terms which contain \( d^{1,0} \) with \( d^{0,1} \) and \( (d^{1,0})^* \) with \( (d^{0,1})^* \) will disappear. We will now estimate the rest of the cross-terms in terms of \( \langle \Box^{1,0}\omega, \omega \rangle \), \( \langle \Box^{0,1}\omega, \omega \rangle \) and \( \langle \Box^{-1,2}\omega, \omega \rangle \). Namely,

(A.3.16) \[
2\langle (d^{1,0}\omega, d^{-1,2}\omega) \rangle + 2\langle (d^{1,0})^*\omega, (d^{-1,2})^*\omega \rangle \leq 1/2 \langle \Box^{1,0}\omega, \omega \rangle + 2\langle \Box^{-1,2}\omega, \omega \rangle.
\]
Similarly,
(A.3.17)
\[2|\langle d^{0.1}_\omega, d^{-1.2}_\omega \rangle| + 2|\langle (d^{0.1})^\ast_\omega, (d^{-1.2})^\ast_\omega \rangle| \leq 1/2\langle \square^{0.1}_\omega, \omega \rangle + 2\langle \square^{-1.2}_\omega, \omega \rangle \]

We can use (A.3.16), (A.3.17) in (A.3.15) (estimating the cross-terms from below) to get the following estimate
(A.3.18)
\[\langle \square(\alpha)_\omega, \omega \rangle \geq 1/2(\langle \square^{1.0}_\omega, \omega \rangle + \langle \square^{0.1}_\omega, \omega \rangle) - 3\langle \square^{-1.2}_\omega, \omega \rangle + \alpha^2 \langle |dh|^2 \omega, \omega \rangle + \alpha \langle A\omega, \omega \rangle.\]

From the choice of the basis in the fiber (see Section A.2.1) we conclude that
(A.3.19)
\[\langle \square^{1.0}_\omega, \omega \rangle = \sum_{i=1}^{n} ||\nabla_{a_i} \omega||^2.\]

On the other hand, for \(\langle \square^{0.1}_\omega, \omega \rangle\) we have (after calculating \(\square^{0.1}\) in the coordinates \(\{b_j\}\))
(A.3.20)
\[\langle \square^{0.1}_\omega, \omega \rangle = \sum_{j=1}^{n} ||\nabla_{b_j} \omega||^2 + \sum_{k,l} \langle (b_k \wedge i(b_l)\bar{L}(b_k, b_l) - \nabla_{[L(b_k, b_l)y]} \rangle \omega, \omega \rangle.\]

It follows from the description in Lemma A.2.1 that the operator
\[\sum_{k,l} \langle (b_k \wedge i(b_l)\bar{L}(b_k, b_l) - \nabla_{[L(b_k, b_l)y]} \rangle \]

is a first order operator in \(\nabla_{a_i}\) and \(\nabla_{b_j}\) with the coefficients which are at most linear in \(y\). Therefore, it can be estimated in terms of operators \(\square^{1.0}, \square^{0.1}\) and \(1 + |y|^2\). That is, there is a constant \(C_1 > 0\) such that
(A.3.21)
\[|\sum_{k,l} \langle (b_k \wedge i(b_l)\bar{L}(b_k, b_l) - \nabla_{[L(b_k, b_l)y]} \rangle \omega, \omega \rangle| \leq 1/4 \langle \square^{1.0}_\omega, \omega \rangle + 1/4 \langle \square^{0.1}_\omega, \omega \rangle\]
\[+ C_1(||\omega||^2 + |||y|^2\omega, \omega||)\]

After estimating by using (A.3.21) the first order part on the right-hand side of (A.3.20) from below, we get
\[\langle \square^{0.1}_\omega, \omega \rangle \geq \sum_{j=1}^{n} ||\nabla_{b_j} \omega||^2 - 1/4 \langle \square^{1.0}_\omega, \omega \rangle - 1/4 \langle \square^{0.1}_\omega, \omega \rangle - C_1(||\omega||^2 + |||y|^2\omega, \omega||).\]
The inequality above is equivalent to

\[(A.3.22)\]

\[
\langle \Box^{1,0} \omega, \omega \rangle \geq 4/5 \sum_{j=1}^{n} \| \tilde{\nabla}_j \omega \|^2 - 1/5 \langle \Box^{1,0} \omega, \omega \rangle - 4/5 C_1(\| \omega \|^2 + \langle |y|^2 \omega, \omega \rangle).
\]

Next we substitute (A.3.22) into (A.3.18) to get

\[(A.3.23)\]

\[
\langle \Box(\alpha) \omega, \omega \rangle \geq 1/2 \langle \Box^{1,0} \omega, \omega \rangle + 2/5 \sum_{j=1}^{n} \| \tilde{\nabla}_j \omega \|^2
\]

\[
- 1/10 \langle \Box^{1,0} \omega, \omega \rangle - 2/5 C_1(\| \omega \|^2 + \langle |y|^2 \omega, \omega \rangle) - 3 \langle \Box^{-1,2} \omega, \omega \rangle
\]

\[
+ \alpha^2 \langle |dh|^2 \omega, \omega \rangle + \alpha \langle A \omega, \omega \rangle.
\]

We recall that \( |dh|^2 \geq c |y|^2 \) for some positive \( c \geq 0 \), and that \( A \) is a bounded zeroth order operator. Therefore, for large enough \( \alpha \) there exists \( C_2 = C_2(\alpha) \), such that

\[(A.3.24)\]

\[-3 \langle \Box^{-1,2} \omega, \omega \rangle - 2/5 C_1(\| \omega \|^2 + \langle |y|^2 \omega, \omega \rangle) + \alpha^2 (\langle |dh|^2 \omega, \omega \rangle + \alpha \langle A \omega, \omega \rangle) \geq -C_2 \| \omega \|^2.\]

Finally, we combine estimates (A.3.23) and (A.3.24) (substituting \( \sum_{i=1}^{n} \| \tilde{\nabla}_i \omega \|^2 \) for \( \langle \Box^{1,0} \omega, \omega \rangle \)) to get

\[(A.3.25)\]

\[
\langle \Box(\alpha) \omega, \omega \rangle \geq 2/5 \sum_{i=1}^{n} \| \tilde{\nabla}_i \omega \|^2 + 2/5 \sum_{j=1}^{n} \| \tilde{\nabla}_j \omega \|^2 - C_2 \| \omega \|^2.
\]

The statement of the lemma now easily follows from A.3.25. \( \square \)

**Corollary A.3.5.** For large enough \( \alpha \) and for any form \( \omega \), such that \( \Box(\alpha) \omega = \lambda(\alpha) \omega \), we have \( \omega \in S_{0,1} \cap S_{\infty,0} \).

The rest of the proof of Theorem 1.4 proceeds by induction in \( a \) and \( l \).

For \( 0 \leq a < \infty \) and \( l = 0, 1, 2, \ldots \), we denote by \( \| \omega \|_{a,l} \) the norm on \( S_{a,l} \) defined by

\[(3.26)\]

\[
\| \omega \|_{a,l}^2 = \sum_{\kappa, |\kappa| \leq l} \| (1 + |y|)^a (\tilde{\nabla})^\kappa \omega \|^2.
\]
Let \( \omega \) be such that \( \Box(\alpha)\omega = \lambda(\alpha)\omega \). Then by Corollary A.3.5 \( \omega \in S_{0,1} \cap S_{\infty,0} \). This starts our induction.

Next by substituting \( J_t|y|^a\omega \) instead of \( \omega \) into the elliptic estimate (A.3.13) and by commuting \( |y|^a \) with \( \nabla \) and \( \Box(\alpha) \) we can get the following estimate

\[
(A.3.2) \quad \|\omega\|_{a+1,t}^2 \leq C(a,\alpha)(\|\omega\|_{a+2,0}^2 + \|\omega\|_{a,t}^2).
\]

From (A.3.2) we consequently conclude that \( \omega \in S_{a,1} \cap S_{\infty,0} \) for \( a = 1,2,\ldots \).

Therefore, \( \omega \in S_{\infty,1} \).

By substituting \( J_t\Box(\alpha)\omega = \lambda(\alpha)J_t\omega \) instead of \( \omega \) into the elliptic estimate (A.3.13) and by commuting \( \Box(\alpha) \) with \( \nabla \), we deduce that \( \omega \in S_{0,2} \cap S_{\infty,1} \). Next we use induction to conclude that \( \omega \in S_{\infty,2} \).

In this fashion we see that \( \omega \in S_{\infty,k} \) for \( k = 0,1,2,\ldots \), which is the statement of Theorem 1.4.
APPENDIX 4

PROOF OF THEOREM 8.5

A.4.0. Introduction. The goal of this appendix is to prove Theorem 8.5.

Theorem 8.5. For any large enough \( \alpha > 0 \) and any \( p \) such that \( 0 \leq p \leq \dim N \), the number of \( \alpha \)-small eigenvalues of \( L^p(\alpha) \) is equal to the number \( \hat{k}^p(\alpha) \) of \( \alpha \)-small eigenvalues of \( \square^p(\alpha) \). Moreover, there exists \( C > 0 \), such that for \( \alpha \) large enough and for all \( 1 \leq j \leq \hat{k}^p(\alpha) \) we have

\[
|\lambda_j^p(\alpha) - \nu_j^p(\alpha)| \leq C\alpha^{-1/2}
\]

(8.7)

The idea of the proof is to use the classical variational approach to eigenvalues of \([\text{Du-Sc}].\)

The statement of Theorem 8.5 easily follows from the following two inequalities:

\[
\lambda_j^p(\alpha) \leq \nu_j^p(\alpha) + C\alpha^{-1/2}, \quad 1 \leq j \leq k^p(\alpha)
\]

(A.4.1)

and

\[
\nu_j^p(\alpha) \leq \lambda_j^p(\alpha) + C\alpha^{-1/2}, \quad 1 \leq j \leq \hat{k}^p(\alpha).
\]

We will prove only inequality (A.4.1). The proof of the second inequality is very similar.

In Section A.4.1. we introduce a variational characterization of the eigenvalues of \( \square(\alpha) \) and we define the test forms for the variational approach. In sections A.4.2. through A.4.6. we complete our estimates.

A.4.1. The test forms and the min-max principle. We start by defining our test forms. First, we need to introduce a smooth family of partitions of unity \( \{\chi_{R,j}\}_{j=0,\ldots,N} \) on \( N \). This family will depend on a parameter \( R \). We define

\[
\tilde{E}_{R,j} := \{ y \in E_j | |y| < R \}, \quad \tilde{E}_R = \tilde{E}_{R,1} \cup \cdots \cup \tilde{E}_{R,N},
\]

\[
\tilde{U}_{R,j} := f_j(\tilde{E}_{R,j}), \quad \tilde{U}_R = \tilde{U}_{R,1} \cup \cdots \cup \tilde{U}_{R,N}.
\]
Finally, we define the non-negative smooth functions \( \chi_{R,j} \), \( j = 1, \ldots, \Lambda \), to be 1 on \( \tilde{U}_{R,j} \) and to be 0 on \( M - \tilde{U}_{2R,j} \). We further define:

\[
\chi_{R,0} = 1 - \sum_{j=1}^{\Lambda} \chi_{R,j} = 1 - \chi_R.
\]

As our test forms we pick \( \psi_{R,i}^p(\alpha) \), \( i = 1, \ldots, j \), defined by

\[
\psi_{R,i}^p(\alpha) := \sum_{l=1}^{\Lambda} f_l^*(\phi_l^p(\alpha)\chi_R)
= f^*(\phi_i^p(\alpha)\chi_R) \in \Omega^p(E, V, \alpha).
\]

Then, provided that the test forms \{\( \psi_{R,i}^p \)\} are linearly independent for \( i = 1, \ldots, j \), we can use the min-max principle in the form of Rayleigh quotient ([Cha, Chapter 1] or [Da-Sc]) For all small enough \( R \) from the min-max principle we have

\[
(A.4.2) \quad \lambda_j^p(\alpha) \leq \frac{\langle \square^p(\alpha)\psi_{R,i}^p(\alpha), \psi_{R,i}^p(\alpha) \rangle_E}{\langle \psi_{R,i}^p(\alpha), \psi_{R,i}^p(\alpha) \rangle_E}, \quad i = 1, \ldots, j.
\]

For small enough \( R \) the metric on \( \tilde{U}_R \) is the pullback of the metric on \( E \) under the diffeomorphism \( f = (f_1, \ldots, f_\Lambda) \). Therefore, for such \( R \)

\[
(A.4.3) \quad \langle \square^p(\alpha)\psi_{R,i}^p(\alpha), \psi_{R,i}^p(\alpha) \rangle_E = \langle \square^p(\alpha)f^*(\phi_i^p(\alpha)\chi_R), f^*(\phi_i^p(\alpha)\chi_R) \rangle_E
= \langle L^p(\alpha)(\phi_i^p(\alpha)\chi_R), \phi_i^p(\alpha)\chi_R \rangle_N.
\]

Similarly,

\[
(A.4.4) \quad \langle \psi_{R,i}^p(\alpha), \psi_{R,i}^p(\alpha) \rangle_E = \langle \phi_i^p(\alpha)\chi_R, \phi_i^p(\alpha)\chi_R \rangle_N.
\]

Therefore, inequalities (A.4.2) become

\[
(A.4.5) \quad \lambda_j^p(\alpha) \leq \max_{1 \leq i \leq j} \frac{\langle L^p(\alpha)(\phi_i^p(\alpha)\chi_R), (\phi_i^p(\alpha)\chi_R) \rangle}{\langle \phi_i^p(\alpha)\chi_R, \phi_i^p(\alpha)\chi_R \rangle}.
\]
A.4.2. The computation of the variational quotient. It is easy to compute ([CFKS, Proposition 11.13]) that

\[ L(\alpha) = d^*d + dd^* + \alpha^2|dh|^2 + \alpha A, \]

where \( A \) is a zeroth order operator.

Since \( d^*d + dd^* = *d * d + d * d * \), where \( * \) denotes the Hodge \(*\)-operator on \( N \), we have

\[ (d^*d + dd^*)(\phi_i^p(\alpha)\chi) = *d * d(\phi_i\chi) + d * d *(\phi_i\chi) \]
\[ = *d *(d\chi \wedge \phi_i + \chi d\phi_i) + d * (d\chi \wedge *\phi_i + \chi d*(\phi_i)) \]
\[ = d^*(d\chi \wedge \phi_i) + *(d\chi \wedge *d\phi_i) + \chi d^*d\phi_i \]
\[ + d * (d\chi \wedge *\phi_i) + d\chi \wedge d^*\phi_i + \chi dd^*\phi_i. \]

After multiplying (A.4.7) by \( \chi \phi_i^p(\alpha) \) on both sides, integrating by parts, and combining terms we have

\[ (L\chi \phi_i, \chi \phi_i) = (L\chi \phi_i, \chi \phi_i) + (d\chi \wedge \phi_i, d(\chi \phi_i)) \]
\[ + (*(d\chi \wedge *d\phi_i), \chi \phi_i) + (*(d\chi \wedge *\phi_i), d * (\chi \phi_i)) + (d\chi \wedge d^*\phi_i, \chi \phi_i). \]

Recall that \( L\phi_i = \nu_i \phi_i \). We rewrite the formula above as

\[ (L\chi \phi_i, \chi \phi_i) = \nu_i \|\chi \phi_i\|^2 + \|d\chi \wedge \phi_i\|^2 \]
\[ + (d\chi \wedge \phi_i, \chi d\phi_i) + *(d\chi \wedge *\phi_i, \chi \phi_i) \]
\[ + *(d\chi \wedge *\phi_i, *(d\chi \wedge *d\phi_i), \chi \phi_i) \]
\[ + (d\chi \wedge d^*\phi_i, \chi \phi_i), \]

where \( \chi = \chi_R \).

We want to estimate the right hand side of this formula. We represent it as

\[ (L\chi \phi_i, \chi \phi_i) = \nu_i \|\chi R \phi_i\|^2 + I_1 + I_2, \]

where

\[ I_1 = \|d\chi R \wedge \phi_i\|^2 + *(d\chi R \wedge *d\phi_i), \chi R \phi_i) + \| * (d\chi R \wedge *\phi_i)\|^2; \]
\[ I_2 = \langle d\chi R R \phi_i + *(d\chi R R R \phi_i), \chi R d^* \phi_i \rangle + \langle d\chi R R \phi_i, \chi R d^* \phi_i \rangle. \]
A.4.3. An estimate of $I_1$. Since supp $(d\chi_R) \subset \bar{U}_{2R} \setminus \bar{U}_R$, we have

\[
|I_1| \leq C_1 \|(\phi_i)|_{\bar{U}_{2R} \setminus \bar{U}_R}\|^2.
\]

Now we need the following

**Lemma A.4.1.** There exists a constant $c = c(R)$ such that

\[
\|((\phi_i)|_{\bar{U}_{2R} \setminus \bar{U}_R}\|^2 \leq c\alpha^{-1}.
\]

**Proof.** After integrating by parts in (A.4.6) we have

\[
\nu_i(\alpha) = \langle L^p(\alpha)\phi_i, \phi_i \rangle
\]

\[
= \|d\phi_i\|^2 + \|d^*\phi_i\|^2 + \alpha^2 \langle |d\phi_i|^2 \phi_i, \phi_i \rangle + \alpha \langle A\phi_i, \phi_i \rangle.
\]

Since the term $\|d\phi_i\|^2 + \|d^*\phi_i\|^2$ is non-negative and $A$ is a bounded operator, we have

\[
\alpha^2 \langle |d\phi_i|^2 \phi_i, \phi_i \rangle \leq \nu_i(\alpha) + c_1 \alpha,
\]

Therefore, we also have a similar bound for the restriction:

\[
\alpha^2 \langle |d\phi_i|^2 \phi_i|_{\bar{U}_{2R} \setminus \bar{U}_R}, \phi_i \rangle \leq \nu_i(\alpha) + c_1 \alpha.
\]

On the other hand $\langle |d\phi_i|^2 \phi_i|_{\bar{U}_{2R} \setminus \bar{U}_R} \geq c_2$ for some positive $c_2 = c_2(R)$. Thus

\[
\alpha^2 c_2 \|(\phi_i)|_{\bar{U}_{2R} \setminus \bar{U}_R}\|^2 \leq \alpha^2 \langle |d\phi_i|^2 \phi_i|_{\bar{U}_{2R} \setminus \bar{U}_R}, \phi_i \rangle
\]

\[
\leq \nu_i(\alpha) + c_1 \alpha.
\]

After dividing both parts of (A.4.16) by $\alpha^2 c_2$ we have

\[
\|(\phi_i)|_{\bar{U}_{2R} \setminus \bar{U}_R}\|^2 \leq \frac{\nu_i(\alpha) + c_1 \alpha}{\alpha^2 c_2} \leq \alpha^{-1}.
\]

In the last inequality we assumed that $\nu_i(\alpha)$ is $\alpha$-small. ■

We can now apply the lemma above to the inequality (A.4.11) to get

\[
|I_1| \leq C_2 \alpha^{-1}.
\]
A.4.4. An estimate of \( I_2 \). Since \( \text{supp}(d\chi_R) \subset \bar{U}_{2R} \setminus \bar{U}_R \), we have

\[
|I_2| \leq C_3 \| (\phi_i)_{|\bar{U}_{2R} \setminus \bar{U}_R} \| \left( \| (d\phi_i)_{|\bar{U}_{2R} \setminus \bar{U}_R} \| + \| (d*\phi_i)_{|\bar{U}_{2R} \setminus \bar{U}_R} \| \right).
\]

To estimate the right hand side of (A.4.19) we need the following

**Lemma A.4.2.** There exists a constant \( c = c(R) \) such that

\[
\| (d\phi_i)_{|\bar{U}_{2R} \setminus \bar{U}_R} \| \leq c, \quad \| (d*\phi_i)_{|\bar{U}_{2R} \setminus \bar{U}_R} \| \leq c.
\]

**Proof.** We start with the equality (A.4.13):

\[
\nu_i(\alpha) = \| d\phi_i \|^2 + \| d^*\phi_i \|^2 + \alpha^2 \langle \| dh \|^2 \phi_i, \phi_i \rangle + \alpha \langle A\phi_i, \phi_i \rangle.
\]

Since the terms \( \| d\phi_i \|^2, \| d^*\phi_i \|^2, \) and \( \alpha^2 \langle \| dh \|^2 \phi_i, \phi_i \rangle \) are all non-negative, \( A \) is a bounded operator, and \( \nu_i(\alpha) \) is \( \alpha \)-small, we conclude that for some \( c_1 > 0 \),

\[
\| d\phi_i \|^2 \leq c_1 \alpha, \quad \| d^*\phi_i \|^2 \leq c_1 \alpha.
\]

Unfortunately, estimates (A.4.22) are not good enough, so we need to work a bit harder. We define a non-negative smooth characteristic function \( \chi_{[R,2R]} \) by

\[
\chi_{[R,2R]} = 1 \text{ on } \bar{U}_{2R} \setminus \bar{U}_R, \quad \text{supp}(\chi_{[R,2R]}) \subset \bar{U}_{4R} \setminus \bar{U}_{R/2}.
\]

Then, for \( \chi = \chi_{[R,2R]} \),

\[
B = \| (d\phi_i)_{|\bar{U}_{2R} \setminus \bar{U}_R} \|^2 + \| (d*\phi_i)_{|\bar{U}_{2R} \setminus \bar{U}_R} \|^2
\]

\[
\leq (\chi d\phi_i, d\phi_i) + (\chi d^*\phi_i, d^*\phi_i)
\]

\[
\leq (d* (\chi d\phi_i), \phi_i) + (d (\chi d^*\phi_i), \phi_i)
\]

\[
\leq (\chi (d^*d + dd^*)\phi_i, \phi_i) + (\chi d^*d^* \phi_i, \phi_i) + (\chi d\phi_i, \phi_i).
\]

Next, since \( \phi_i \) is an eigenform for the eigenvalue \( \nu_i \), we observe that we have the equality:

\[
\chi \nu_i \phi_i = \chi (d^*d + dd^*)\phi_i + \alpha^2 \chi |dh|^2 \phi_i + \alpha \chi A\phi_i.
\]
By expressing $\chi(d^*d + dd^*)\phi_i$ from (A.4.25) and substituting it into (A.4.24) we get

$$B \leq -\nu_i(\chi\phi_i, \phi_i) - \alpha^2 \langle \chi|d\phi|^2\phi_i, \phi_i \rangle - \alpha \langle \chi A\phi_i, \phi_i \rangle$$

(A.4.26)

$$+ \langle (*d\chi \wedge *d\phi_i), \phi_i \rangle + \langle d\chi \wedge d^*\phi_i, \phi_i \rangle.$$ Now we observe that, since $\chi|d\phi|^2$ is bounded from below by some positive constant, the term

$$-\alpha^2 \langle \chi|d\phi|^2\phi_i, \phi_i \rangle - \alpha \langle \chi A\phi_i, \phi_i \rangle$$
is non-positive for large $\alpha$. Therefore,

$$B \leq -\nu_i(\chi\phi_i, \phi_i) + \langle (*d\chi \wedge *d\phi_i), \phi_i \rangle + \langle d\chi \wedge d^*\phi_i, \phi_i \rangle$$

(A.4.27)

$$\leq \nu_i(\alpha)\|\phi_i\|_{U_{4R} \setminus U_{R/2}}^2 + c(\|\phi_i\|_{U_{4R} \setminus U_{R/2}}\|d\phi_i\|_{U_{4R} \setminus U_{R/2}}).$$

We use (A.4.22) and (A.4.12) to deduce from (A.4.26) the estimate in the statement of the lemma. ■

It follows from Lemma A.4.1 and Lemma A.4.2 that for some $C_3 > 0$,

$$\|I_2\| \leq C_3 \alpha^{-1/2}.$$  

(A.4.28)

Now we would like to estimate the denominator in (A.4.2). To do so we need the following lemma:

**Lemma A.4.3.** There exists a constant $c = c(R)$ such that for $\alpha$ large enough we have

$$\|\phi_i^p(\alpha)\chi_R, \phi_k^p(\alpha)\chi_R \rangle - \delta_{ik} \leq c\alpha^{-1}, \quad 1 \leq i, k \leq k^p(\alpha)$$

(A.4.29)

where $\delta_{ik}$ is a Kronecker symbol.

**Proof.** We write $\chi_R$ as $\chi_R = 1 + \chi_R - 1$. Then

$$\langle \phi_i^p(\alpha)\chi_R, \phi_k^p(\alpha)\chi_R \rangle = \langle \phi_i^p(\alpha) + (\chi_R - 1)\phi_i^p(\alpha), \phi_k^p(\alpha) + (\chi_R - 1)\phi_k^p(\alpha) \rangle$$

$$= \delta_{ik} + 2((\chi_R - 1)\phi_i^p(\alpha), \phi_k^p(\alpha)\chi_R)$$

$$+ \langle (\chi_R - 1)\phi_i^p(\alpha), (\chi_R - 1)\phi_k^p(\alpha) \rangle.$$
Thus since \( \text{supp}(\chi_R - 1) \subset N \setminus \tilde{U}_R \),

\[
|\langle \phi_i^p(\alpha)\chi_R, \phi_k^p(\alpha)\chi_R \rangle - \delta_{ik}| \leq c\|\phi_i^p(\alpha)|_{N \setminus \tilde{U}_R}\|\|\phi_k^p(\alpha)|_{N \setminus \tilde{U}_R}\| \leq c\alpha^{-1}.
\]

In the formula above the last inequality on the right follows from the appropriately modified proof of Lemma A.4.1. ■

**Corollary A.4.4.** For large enough \( \alpha \) the test functions in (A.4.2) are linearly independent.

Now we are ready to finish the proof of the Theorem 8.5. From (A.4.5), (A.4.18), and (A.4.28) it follows that there exists \( C_4 > 0 \), such that

\[
\lambda_j^p(\alpha) \leq \frac{\nu_i^p(\alpha)\|\chi_R \phi_i^p(\alpha)\|^2 + I_1 + I_2}{\|\chi_R \phi_i^p(\alpha)\|^2}
\leq \frac{\max_{1 \leq i \leq j}(\nu_i^p(\alpha) + C_4\alpha^{-1/2})}{1 - c/\alpha}
\]

Thus, there exists \( C > 0 \), such that

\[
\lambda_j^p(\alpha) - \nu_i^p(\alpha) \leq C\alpha^{-1/2}, i = 1, \ldots, j.
\]

In particular, we have (A.4.1).
REFERENCES


